Stability analysis of linear systems with time-varying delay via a quadratic function negative-definiteness determination method

Fei Long[1,2], Wen-Juan Lin[1,2], Yong He[1,2] ✉, Lin Jiang[3], Min Wu[1,2]

[1]School of Automation, China University of Geosciences, Wuhan 430074, People's Republic of China

[2]Hubei Key Laboratory of Advanced Control and Intelligent Automation for Complex Systems, Wuhan 430074, People's Republic of China

[3]Department of Electrical Engineering & Electronics, University of Liverpool, Liverpool L69 3GJ, UK

✉ *E-mail: heyong08@cug.edu.cn*

Abstract: This study aims to carry out the stability analysis of linear systems with a time-varying delay. It is known that the negative-definite condition of the derivative of a Lyapunov–Krasovskii functional (LKF) can be determined using the convex combination method if the convexity requirement is satisfied by the derivative of the LKF. However, this method is not feasible in cases where the LKF's derivative is a quadratic function. To address this problem, this study proposes a novel negative-definiteness determination lemma that encompasses the previous lemmas as its special cases and shows less conservatism. Then, this lemma is employed to derive a stability criterion, and its superiority is demonstrated using three examples.

1 Introduction

Time delays inevitably occur in various systems and are often considered to have negative impact on system stability [1–3]. Investigating the relationship between the size of time delay and the stability is a vital issue, which has attracted much attention [4–7]. In the early research, the frequency-domain approach opened the door for the stability analysis of time-delay systems, judging the stability by computing characteristic roots of the differential equations [2]. However, this approach is much difficult for the calculation because time-delay systems have transcendental characteristic equations. Currently, the research focuses on the time-domain stability analysis which does not require the information of characteristic roots and simplifies the analysis especially for the system with time-varying delays. For the time-domain stability analysis, the Lyapunov–Krasovskii functional (LKF) method combined with the linear matrix inequality (LMI) description technique is the most efficient approach [8, 9]. In this approach, three important procedures are required, namely (i) the construction of a proper LKF; (ii) the estimation of the LKF's derivative; and (iii) the acquisition of the negative-definite condition (NDC) for the LKF's derivative. These steps are strangely related to conservatism, and therefore, many approaches have been reported for reducing the conservatism when addressing those procedures.

For the primary step of the stability analysis, researchers have proposed various LKFs under the consideration of utilising more state-related information, for example the augmented terms [10–14], the multiple integral terms [15, 16], the matrix-refined-function-based terms [17, 18] and the delay-product-type terms [19, 20] were introduced into LKFs, and relaxing the conditions of Lyapunov matrices, for example the relaxed-type LKF [21]. For the estimation of the LKF's derivative, the popular bounding approach is to use inequalities which usually contain two steps, estimate the integral function as delay-reciprocal-related quadratic terms by integral inequalities, such as Jensen inequality [1], Wirtinger based inequality [22] and other polynomials-based inequalities [23–28], and then estimate the delay-reciprocal-related terms as delay-reciprocal-free quadratic terms by the reciprocally convex matrix inequalities (RCMIs), such as the classical RCMI [29], the extended RCMIs [30–32], the delay-dependent RCMI [33] and the generalised RCMI [34]. Moreover, some inequalities also combine the two steps into a single one, such as the relaxed integral inequality [8] and the free-matrix-based inequalities [35–37]. Through the above procedures, the derivative of the LKF is already derived as tractable quadratic terms and the NDC, that guarantees the negative definiteness of the LKF's derivative, can be obtained easily by the convex combination method without leading to any conservatism if the convexity of the LKF's derivative is confirmed.

However, for some cases, the convexity or concavity of the derivative of the LKF is unknown, such as the usages of the augmented LKFs [12, 13, 37, 38], the matrix-refined-function-based LKF [18] and the multiple integral-type LKF [39]. In [12, 13, 18, 37–39], the derivative of the LKF was derived as a quadratic function given by

$$f(d(t)) = \phi_0 + d(t)\phi_1 + d^2(t)\phi_2, \tag{1}$$

where $d(t)$ is the time-varying delay and $\phi_i, i = 1, 2, 3$, are scalars. In the case where it is impossible to judge whether $f(d(t))$ is convex or concave, the NDC that ensures the negative definiteness of $f(d(t))$ cannot be obtained by the convex combination method only. To overcome the difficulty, several methods have been proposed to provide some sufficient NDCs. In [39], the term ϕ_2 was forced to be positive, and thus the convex combination method was directly used to produce the NDC. In [38], a lemma containing an extra condition was introduced so that the NDC was acquired by this lemma, regardless of whether $f(d(t))$ is convex or concave. In [37], a lemma was reported to discuss several NDCs with respect to different cases of $\phi_j, j = 0, 2$ (ϕ_j is positive or negative). Among these three methods, the latter two have the same conservatism and are less conservative than the first one. The method of Kim [38] was also used in [12, 13]. To conclude, the above analysis reveals two important things, namely that if the derived LKF's derivative is not convex, then the determination process of the NDC may introduce extra conservatism, and that the existing methods are still conservative to different extents, so that a further study of the negative-definiteness determination method is necessary.

This paper concentrates on developing a novel negative-definiteness determination method for the stability analysis of linear systems with a time-varying delay. To reduce on the conservatism when giving out the NDC, a new negative-definiteness determination lemma with more flexibility is proposed, which is more general in comparison to the lemmas in [37–39]. Then, a less conservative stability criterion is derived by employing the proposed lemma, the auxiliary-based inequality and the generalised RCMI. Finally, the advantages of the developed lemma and the derived criterion are tested and illustrated by three examples.

Notations: Throughout this paper, $\mathscr{R}^n$ represents the n-dimensional Euclidean space; $Q > 0$ (≥ 0) indicates that Q is a positive definite (semi-positive definite) and symmetric matrix; the superscripts -1 and T mean the inverse and the transpose of a matrix, respectively; $\mathrm{Sym}\{Y\} = Y + Y^{\mathrm{T}}$; $\mathrm{col}\{y_1, y_2, \ldots, y_n\} = [y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \ldots, y_n^{\mathrm{T}}]^{\mathrm{T}}$; $\mathrm{diag}\{\cdot\}$ stands for a block-diagonal matrix; 0 is a zero matrix; the notation $*$ represents the symmetric term in a symmetric matrix; and $\|\cdot\|$ means the Euclidean vector norm.

2 Preliminaries

Consider the following linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)),\ t \geq 0, \\ x(t) = \phi(t),\ t \in [-h_2, 0], \end{cases} \tag{2}$$

where $x(t) \in \mathscr{R}^n$ is the system state, A and A_d are known real constant matrices, $\phi(t)$ is the initial condition to be continuous in $[-h_2, 0]$, and the time-varying delay $d(t)$ satisfies

$$0 \leq h_1 \leq d(t) \leq h_2, \tag{3}$$

where h_1 and h_2 are constants with $h_{12} = h_2 - h_1$.

The integral inequalities [24] and the reciprocally convex lemmas [31, 34] are recalled which will be used to derive stability criteria for system (2).

Lemma 1: For a matrix $R > 0$ and a vector $x\colon [a, b] \to \mathscr{R}^n$ that is differentiable in $[a, b]$, where a and b are constants satisfying $b > a$, the following inequalities hold:

$$(b-a)\int_a^b \dot{x}^{\mathrm{T}}(s) R \dot{x}(s)\,\mathrm{d}s \geq \sum_{i=1}^{3} (2i-1)\lambda_i^{\mathrm{T}} R \lambda_i, \tag{4}$$

$$(b-a)\int_a^b x^{\mathrm{T}}(s) R x(s)\,\mathrm{d}s \geq \lambda_4^{\mathrm{T}} R \lambda_4, \tag{5}$$

where

$$\begin{aligned} \lambda_1 &= x(b) - x(a), \\ \lambda_2 &= x(b) + x(a) - 2\int_a^b \frac{x(s)}{b-a}\,\mathrm{d}s, \\ \lambda_3 &= x(b) - x(a) + 6\int_a^b \frac{x(s)}{b-a}\,\mathrm{d}s - 12\int_a^b\int_\theta^b \frac{x(s)}{(b-a)^2}\,\mathrm{d}s\,\mathrm{d}\theta, \\ \lambda_4 &= \int_a^b x(s)\,\mathrm{d}s. \end{aligned}$$

Lemma 2: For a scalar $\alpha \in (0, 1)$, a matrix $R > 0$, vectors Γ_1 and Γ_2, and any matrix S, the following inequality holds:

$$\begin{aligned} \frac{1}{\alpha}\Gamma_1^{\mathrm{T}} R \Gamma_1 + \frac{1}{1-\alpha}\Gamma_2^{\mathrm{T}} R \Gamma_2 &\geq (2-\alpha)\Gamma_1^{\mathrm{T}} R \Gamma_1 + (1+\alpha)\Gamma_2^{\mathrm{T}} R \Gamma_2 \\ &+ \mathrm{Sym}\{\Gamma_1^{\mathrm{T}} S \Gamma_2\} - (1-\alpha)\Gamma_1^{\mathrm{T}} S R^{-1} S^{\mathrm{T}} \Gamma_1 \\ &- \alpha \Gamma_2^{\mathrm{T}} S^{\mathrm{T}} R^{-1} S \Gamma_2. \end{aligned} \tag{6}$$

Lemma 3: For a scalar $\alpha \in (0, 1)$, vectors Γ and $\bar{\Gamma}$, a matrix $R > 0$, symmetric matrices Y_1 and Y_2, and any matrices X_1, X_2 and Z, the following inequality holds:

$$\begin{aligned} \Gamma^{\mathrm{T}} \begin{bmatrix} \frac{1}{\alpha} R & 0 \\ 0 & \frac{1}{1-\alpha} R \end{bmatrix} \Gamma &\geq \Gamma^{\mathrm{T}} R_{XY} \Gamma - \alpha \Gamma_{XY1}^{\mathrm{T}} R^{-1} \Gamma_{XY1} \\ &- (1-\alpha)\Gamma_{XY2}^{\mathrm{T}} R^{-1} \Gamma_{XY2} \\ &+ \mathrm{Sym}\{\bar{\Gamma}^{\mathrm{T}} [(1-\alpha) Z^{\mathrm{T}} \ -\alpha Z^{\mathrm{T}}] \Gamma\}, \end{aligned} \tag{7}$$

where

$$R_{XY} = \begin{bmatrix} (2-\alpha)R + 2(1-\alpha)Y_1 & (1-\alpha)X_1 + \alpha X_2 \\ * & (1+\alpha)R + 2\alpha Y_2 \end{bmatrix},$$

$$\Gamma_{XY1} = [Y_1\ X_1]\Gamma + Z\bar{\Gamma},\ \Gamma_{XY2} = [X_2^{\mathrm{T}}\ Y_2]\Gamma - Z\bar{\Gamma}.$$

3 Main results

In this section, a new negative-definiteness determination lemma is proposed, and then, a less conservative stability criterion is developed based on the proposed lemma.

3.1 A new quadratic function negative-definiteness determination lemma

Lemma 4: Let $f(d(t)) = \phi_0 + d(t)\phi_1 + d^2(t)\phi_2$, where $\phi_i \in \mathscr{R}, i = 0, 1, 2$, and $d(t) \in [h_1, h_2]$ with $0 \leq h_1 \leq h_2$. Then, the NDCs of $f(d(t))$ are given for the following three cases:

• If $\phi_2 \geq 0$, then $f(d(t)) < 0, \forall d(t) \in [h_1, h_2]$, is ensured by

$$C_1\colon f(h_1) < 0,\ C_2\colon f(h_2) < 0. \tag{8}$$

• If $\phi_2 < 0$, then $f(d(t)) < 0, \forall d(t) \in [h_1, h_2]$, is ensured by

$$C_1,\ C_3\colon -h_{12}^2\phi_2 + f(h_2) < 0 \tag{9}$$

or

$$C_2,\ C_4\colon -h_{12}^2\phi_2 + f(h_1) < 0. \tag{10}$$

• If the convexity or concavity of ϕ_2 is unknown, then $f(d(t)) < 0$, $\forall d(t) \in [h_1, h_2]$, is ensured by

$$C_1,\ C_2,\ C_3 \text{ or } C_4. \tag{11}$$

Proof: First, when $\phi_2 \geq 0$, $f(d(t))$ is convex in $[h_1, h_2]$. Thus, if conditions C_1 and C_2 hold, then $f(d(t)) < 0$, $\forall d(t) \in [h_1, h_2]$, is guaranteed.

Second, when $\phi_2 < 0$, $f(d(t))$ is concave. Thus, based on the slope characteristic of the concave function, the following inequalities hold for $d(t) \in [h_1, h_2]$:

$$\begin{aligned} f(d(t)) &\leq (d(t) - h_1)\dot{f}(h_1) + f(h_1) \\ &:= K_1(d(t)), \end{aligned} \tag{12}$$

$$\begin{aligned} f(d(t)) &\leq -(h_2 - d(t))\dot{f}(h_2) + f(h_2) \\ &:= K_2(d(t)). \end{aligned} \tag{13}$$

It follows from inequality (12) that

$$K_1(h_1) = f(h_1), \tag{14}$$

$$K_1(h_2) = f(h_2) - (h_2 - h_1)^2\phi_2. \tag{15}$$

Since $K_1(d(t))$ is a first-order function, $K_1(h_1) < 0$ and $K_1(h_2) < 0$ ensure the negative definiteness of $f(d(t))$, and therefore, conditions C_1 and C_3 guarantee $f(d(t)) < 0, \forall d(t) \in [h_1, h_2]$.

Similarly, based on inequality (13), it is confirmed that conditions C_2 and C_4 also guarantee $f(d(t)) < 0, \forall d(t) \in [h_1, h_2]$.

Finally, if the positivity or negativity of ϕ_2 is unknown, then the conditions for $\phi_2 \geq 0$ and $\phi_2 < 0$ should be considered together, because ϕ_2 can only be positive, negative or zero. From the above analysis, when $\phi_2 \geq 0$, conditions C_1 and C_2 ensure that $f(d(t)) < 0$, $\forall d(t) \in [h_1, h_2]$. When $\phi_2 < 0$, the NDCs are $C_1 - C_3$ or C_1, C_2 and C_4. By considering these two types of NDCs, $f(d(t)) < 0$, $\forall d(t) \in [h_1, h_2]$, holds regardless of its convexity or concavity, if conditions $C_1 - C_3$ or conditions C_1, C_2 and C_4 hold. The proof is completed. □

Remark 1: Lemma 4 gives several NDCs for quadratic functions with respect to different cases of ϕ_2 in more details, and it is suitable for three facts that ϕ_2 is positive or negative definite or unable to be confirmed. That is Lemma 4 can be employed to address the negative-definiteness requirement for all the cases of $f(d(t))$, which is one of the advantages. However, Lemma 5 of [39] can only be used to handle the determination problem of $f(d(t))$ when ϕ_2 is positive definite.

Remark 2: The methods in [37–39] are involved in the proposed lemma. Take for instance, for Lemma 4, if $\phi_2 \geq 0$ is added, Lemma 4 reduces to Lemma 5 of [39], and if only conditions C_1, C_2 and C_4 are introduced, then Lemma 4 reduces to Lemma 2 of [38] or Lemma 4 of [37].

Remark 3: The significant difference from Lemma 2 of [38] is the introduced condition C_3 for the cases where the negativity of ϕ_2 is identified or unknown. This provides alternative choices to determine the NDCs of the LKF's derivative, that is conditions $C_1 - C_3$ or conditions C_1, C_2 and C_4 (equal to Lemma 2 of [38] can be used to produce the stability criteria. Therefore, Lemma 4 has potential to reduce the conservatism, and at least can lead to the same results as Lemma 2 of [38] does without introducing any additional decision variable.

3.2 A stability criterion

Theorem 1: For given constants h_1 and h_2 with $h_2 \geq h_1 \geq 0$, system (2) with the delay satisfying (3) is asymptotically stable, if there exist a $5n \times 5n$ matrix $P > 0$, $n \times n$ matrices $Q_1 > 0$ and $R_i > 0$, i=1,2, a $3n \times 3n$ matrix $Q_2 > 0$, a $2n \times 2n$ matrix $R_3 > 0$, a $2n \times 2n$ any matrix S, $3n \times 3n$ any matrices X_1 and X_2, $3n \times 3n$ symmetric matrices Y_1 and Y_2, and $3n \times 2n$ any matrix Z, such that the following LMIs hold:

$$\begin{bmatrix} \Xi(h_1, h_1^2) & * & * \\ E_{XY2} & -\hat{R}_2 & * \\ S^{\mathrm{T}} E_4 & 0_{2n \times 3n} & -R_3 \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} \Xi(h_2, h_2^2) & * & * \\ E_{XY1} & -\hat{R}_2 & * \\ S E_5 & 0_{2n \times 3n} & -R_3 \end{bmatrix} < 0, \tag{17}$$

$$\begin{cases} \begin{bmatrix} -h_{12}^2 \Upsilon + \Xi(h_2, h_2^2) & * & * \\ E_{XY1} & -\hat{R}_2 & * \\ S E_5 & 0_{2n \times 3n} & -R_3 \end{bmatrix} < 0 \\ \textbf{or} \\ \begin{bmatrix} -h_{12}^2 \Upsilon + \Xi(h_1, h_1^2) & * & * \\ E_{XY2} & -\hat{R}_2 & * \\ S^{\mathrm{T}} E_4 & 0_{2n \times 3n} & -R_3 \end{bmatrix} < 0, \end{cases} \tag{18}$$

where

$$\begin{aligned} \Xi(d(t), d^2(t)) &= \Xi_1(d(t), d^2(t)) + \Xi_2(d(t), d^2(t)) \\ &\quad + \Xi_3(d(t)) + \Xi_4(d(t)), \end{aligned} \tag{19}$$

$$\Xi_1(d(t), d^2(t)) = \mathrm{Sym}\{\Pi_1^{\mathrm{T}} P \Pi_2\}, \tag{20}$$

$$\begin{aligned} \Xi_2(d(t), d^2(t)) &= e_1^{\mathrm{T}} Q_1 e_1 - e_2^{\mathrm{T}} Q_1 e_2 \\ &\quad + \Pi_3^{\mathrm{T}} Q_2 \Pi_3 - \Pi_4^{\mathrm{T}} Q_2 \Pi_4 + \mathrm{Sym}\{\Pi_5 Q_2 \Pi_6\}, \end{aligned} \tag{21}$$

$$\begin{aligned} \Xi_3(d(t)) &= e_s^{\mathrm{T}} (h_1^2 R_1 + h_{12}^2 R_2) e_s - E_1^{\mathrm{T}} \hat{R}_1 E_1 - E_{23}^{\mathrm{T}} R_{XY}(d(t)) E_{23} \\ &\quad - \mathrm{Sym}\{\bar{E}^{\mathrm{T}} [(1 - \alpha_{d(t)}) Z^{\mathrm{T}} - \alpha_{d(t)} Z^{\mathrm{T}}] E_{23}\}, \end{aligned} \tag{22}$$

$$\begin{aligned} \Xi_4(d(t)) &= h_{12}^2 \begin{bmatrix} e_1 \\ e_s \end{bmatrix}^{\mathrm{T}} R_3 \begin{bmatrix} e_1 \\ e_s \end{bmatrix} - \mathrm{Sym}\{E_4^{\mathrm{T}} S E_5\} \\ &\quad - (2 - \alpha_{d(t)}) E_4^{\mathrm{T}} R_3 E_4 - (1 + \alpha_{d(t)}) E_5^{\mathrm{T}} R_3 E_5, \end{aligned} \tag{23}$$

$$\Upsilon = \mathrm{Sym}\{\Pi_7^{\mathrm{T}} P \Pi_2 + \hat{\Pi}_7^{\mathrm{T}} Q_2 \Pi_6\}, \tag{24}$$

$$\begin{aligned} \Pi_1 &= \mathrm{col}\{e_1, h_1 e_5, (d(t) - h_1) e_6 + (h_2 - d(t)) e_7, h_1^2 e_8, \\ &\qquad (d(t) - h_1)^2 e_9 + (h_2 - d(t))^2 e_{10} + (h_2 - d(t)) e_{11}\}, \\ \Pi_2 &= \mathrm{col}\{e_s, e_1 - e_2, e_2 - e_4, h_1(e_1 - e_5), h_{12} e_2 - e_{11} - e_{12}\}, \\ \Pi_3 &= \mathrm{col}\{e_2, e_1, 0\}, \\ \Pi_4 &= \mathrm{col}\{e_4, e_1, e_{11} + e_{12}\}, \\ \Pi_5 &= \mathrm{col}\{e_{11} + e_{12}, h_{12} e_1, \\ &\qquad (d(t) - h_1)^2 e_9 + (h_2 - d(t))^2 e_{10} + (h_2 - d(t)) e_{11}\}, \\ \Pi_6 &= \mathrm{col}\{0, e_s, e_2\}, \\ \Pi_7 &= \mathrm{col}\{0, 0, \hat{\Pi}_7\}, \\ \hat{\Pi}_7 &= \mathrm{col}\{0, 0, e_9 + e_{10}\}, \\ E_i &= \mathrm{col}\{e_i - e_{i+1}, e_i + e_{i+1} - 2 e_{i+4}, \\ &\qquad e_i - e_{i+1} + 6 e_{i+4} - 12 e_{i+7}\},\ i = 1, 2, 3, \end{aligned} \tag{25}$$

$$E_4 = \mathrm{col}\{e_{11}, e_2 - e_3\},\ E_5 = \mathrm{col}\{e_{12}, e_3 - e_4\}, \tag{26}$$

$$\hat{R}_j = \mathrm{diag}\{R_j, 3R_j, 5R_j\},\ j = 1, 2, \tag{27}$$

$$E_{23} = \mathrm{col}\{E_2, E_3\}, \tag{28}$$

$$\bar{E} = \mathrm{col}\{e_1, e_2\}, \tag{29}$$

$$\begin{aligned} &R_{XY}(d(t)) \\ &= \begin{bmatrix} (2 - \alpha_{d(t)}) \hat{R}_{23} + 2(1 - \alpha_{d(t)}) Y_1 & (1 - \alpha_{d(t)}) X_1 + \alpha_{d(t)} X_2 \\ * & (1 + \alpha_{d(t)}) \hat{R}_{23} + 2 \alpha_{d(t)} Y_2 \end{bmatrix}, \end{aligned} \tag{30}$$

$$E_{XY1} = [Y_1\ X_1] E_{23} + Z \bar{E}, \tag{31}$$

$$E_{XY2} = [X_2^{\mathrm{T}}\ Y_2] E_{23} - Z \bar{E}, \tag{32}$$

$$\alpha_{d(t)} = \frac{d(t) - h_1}{h_{12}}, \tag{33}$$

$$\begin{aligned} e_s &= A e_1 + A_d e_3, \\ e_i &= [0_{n \times (i-1)n}, I, 0_{n \times (12-i)n}], \quad i = 1, 2, \ldots, 12. \end{aligned}$$

Proof: Consider the following as the LKF candidate:

$$V(t) = \sum_{i=1}^{4} V_i(t), \tag{34}$$

where

$$V_1(t) = \xi_1^{\mathrm{T}}(t) P \xi_1(t), \tag{35}$$

$$V_2(t) = \int_{t-h_1}^{t} x^{\mathrm{T}}(s) Q_1 x(s)\,\mathrm{d}s + \int_{t-h_2}^{t-h_1} \xi_2^{\mathrm{T}}(s,t) Q_2 \xi_2(s,t)\,\mathrm{d}s, \tag{36}$$

$$V_3(t) = h_1\int_{-h_1}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta + h_{12}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta, \tag{37}$$

$$V_4(t) = h_{12}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^{\mathrm{T}}R_3\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}\mathrm{d}s\,\mathrm{d}\theta, \tag{38}$$

$$\begin{aligned}\xi_1(t) = \mathrm{col}\Big\{&x(t), \int_{t-h_1}^{t}x(s)\,\mathrm{d}s, \int_{t-h_2}^{t-h_1}x(s)\,\mathrm{d}s,\\ &\int_{-h_1}^{0}\int_{t+\theta}^{t}x(s)\,\mathrm{d}s\,\mathrm{d}\theta, \int_{-h_2}^{-h_1}\int_{t+\theta}^{t-h_1}x(s)\,\mathrm{d}s\,\mathrm{d}\theta\Big\},\\ \xi_2(s,t) = \mathrm{col}\Big\{&x(s), x(t), \int_{s}^{t-h_1}x(u)\,\mathrm{d}u\Big\},\end{aligned} \tag{39}$$

and these Lyapunov matrices in $V(t)$ satisfy $P > 0$, $Q_1 > 0$, $Q_2 > 0$ and $R_i > 0, i = 1,2,3$, and then $V(t) \geq \varepsilon_1 \parallel x(t) \parallel^2$ for a sufficiently small scalar $\varepsilon_1 > 0$.

Then, differentiating the LKF $V(t)$ along the solution of (2) produces

$$\dot{V}(t) = \sum_{i=1}^{4}\dot{V}_i(t), \tag{40}$$

where $\dot{V}_i(t), i = 1,2,\ldots,4$, are computed as follows:

$$\begin{aligned}\dot{V}_1(t) &= 2\xi_1^{\mathrm{T}}(t)P\dot{\xi}_1(t)\\ &= \xi^{\mathrm{T}}(t)\Xi_1(d(t),d^2(t))\xi(t),\end{aligned} \tag{41}$$

$$\begin{aligned}\dot{V}_2(t) = {}& x^{\mathrm{T}}(t)Q_1x(t) - x^{\mathrm{T}}(t-h_1)Q_1x(t-h_1)\\ &+\xi_2^{\mathrm{T}}(t-h_1)Q_2\xi_2(t-h_1) - \xi_2^{\mathrm{T}}(t-h_2)Q_2\xi_2(t-h_2)\\ &+2\int_{t-h_2}^{t-h_1}\xi_2^{\mathrm{T}}(s,t)\,\mathrm{d}sQ_2\frac{\partial\xi_2(s,t)}{\partial t}\\ = {}&\xi^{\mathrm{T}}(t)\Xi_2(d(t),d^2(t))\xi(t),\end{aligned} \tag{42}$$

$$\dot{V}_3(t) = \dot{x}^{\mathrm{T}}(t)\big(h_1^2R_1 + h_{12}^2R_2\big)\dot{x}(t) + \delta_1 + \delta_2, \tag{43}$$

$$\dot{V}_4(t) = h_{12}^2\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}^{\mathrm{T}}R_3\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + \delta_3 \tag{44}$$

and

$$\delta_1 = -h_1\int_{t-h_1}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\,\mathrm{d}s, \tag{45}$$

$$\delta_2 = -h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s, \tag{46}$$

$$\delta_3 = -h_{12}\int_{t-h_2}^{t-h_1}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^{\mathrm{T}}R_3\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}\mathrm{d}s, \tag{47}$$

$$\begin{aligned}\dot{\xi}_1(t) = \mathrm{col}\Big\{&\dot{x}(t), x(t) - x(t-h_1), x(t-h_1) - x(t-h_2),\\ &h_1x(t) - \int_{t-h_1}^{t}x(s)\,\mathrm{d}s, h_{12}x(t-h_1) - \int_{t-h_2}^{t-h_1}x(s)\,\mathrm{d}s\Big\},\end{aligned} \tag{48}$$

$$\xi_2(t-h_1) = \mathrm{col}\{x(t-h_1), x(t), 0\}, \tag{49}$$

$$\begin{aligned}\xi_2(t-h_2) = \mathrm{col}\Big\{&x(t-h_2), x(t),\\ &\int_{t-d(t)}^{t-h_1}x(s)\,\mathrm{d}s + \int_{t-h_2}^{t-d(t)}x(s)\,\mathrm{d}s\Big\},\end{aligned} \tag{50}$$

$$\begin{aligned}\int_{t-h_2}^{t-h_1}\xi_2(s,t)\,\mathrm{d}s = \mathrm{col}\Big\{&\int_{t-d(t)}^{t-h_1}x(s)\,\mathrm{d}s + \int_{t-h_2}^{t-d(t)}x(s)\,\mathrm{d}s,\\ &h_{12}x(t), \int_{t-d(t)}^{t-h_1}\int_{s}^{t-h_1}x(u)\,\mathrm{d}u\,\mathrm{d}s\\ &+\int_{t-h_2}^{t-d(t)}\int_{s}^{t-d(t)}x(u)\,\mathrm{d}u\,\mathrm{d}s\\ &+(h_2 - d(t))\int_{t-d(t)}^{t-h_1}x(s)\,\mathrm{d}s\Big\},\end{aligned} \tag{51}$$

$$\frac{\partial\xi_2(s,t)}{\partial t} = \mathrm{col}\{0, \dot{x}(t), x(t-h_1)\}, \tag{52}$$

with $\Xi_1(d(t),d^2(t))$, $\Xi_2(d(t),d^2(t))$ and $\xi(t)$ are defined in (20) and (21) and (see (53)) respectively.

Employing inequality (4) to estimate terms δ_1 and δ_2 appearing in (43) obtains

$$\delta_1 \leq -\xi^{\mathrm{T}}(t)E_1^{\mathrm{T}}\hat{R}_1E_1\xi(t), \tag{54}$$

$$\delta_2 \leq -\xi^{\mathrm{T}}(t)\Big(\frac{h_{12}}{d(t)-h_1}E_2^{\mathrm{T}}\hat{R}_2E_2 + \frac{h_{12}}{h_2-d(t)}E_3^{\mathrm{T}}\hat{R}_2E_3\Big)\xi(t), \tag{55}$$

where $E_i, i = 1,2,3$, are defined in (25) and $\hat{R}_j, j = 1,2$, are given in (27).

After the estimation that is made in (55), we employ Lemma 3 to estimate these delay-reciprocal-related terms as delay-reciprocal-free quadratic terms. For matrices X_i, $Y_i, i = 1,2$, and Z, the following inequality holds:

$$\begin{aligned}\delta_2 \leq -\xi(t)\big\{&E_{23}^{\mathrm{T}}R_{XY}(d(t))E_{23} - \Omega_1(d(t))\\ &+\mathrm{Sym}\{\bar{E}^{\mathrm{T}}[(1-\alpha_{d(t)})Z^{\mathrm{T}} - \alpha_{d(t)}Z^{\mathrm{T}}]E_{23}\}\big\}\xi(t),\end{aligned} \tag{56}$$

where

$$\Omega_1(d(t)) = \alpha_{d(t)}E_{XY1}^{\mathrm{T}}\hat{R}_2^{-1}E_{XY1} + (1-\alpha_{d(t)})E_{XY2}^{\mathrm{T}}\hat{R}_2^{-1}E_{XY2}$$

with $E_{23}, \bar{E}, R_{XY}(d(t)), E_{XY1}, E_{XY2}$ and $\alpha_{d(t)}$ defined in (28)–(33), respectively.

Through the procedures in (43), (55) and (56), $\dot{V}_3(t)$ is enlarged as

$$\begin{aligned}\xi(t) = \mathrm{col}\Big\{&x(t), x(t-h_1), x(t-d(t)), x(t-h_2), \int_{t-h_1}^{t}\frac{x(s)}{h_1}\,\mathrm{d}s,\\ &\int_{t-d(t)}^{t-h_1}\frac{x(s)}{d(t)-h_1}\,\mathrm{d}s, \int_{t-h_2}^{t-d(t)}\frac{x(s)}{h_2-d(t)}\,\mathrm{d}s, \int_{t-h_1}^{t}\int_{\theta}^{t}\frac{x(s)}{h_1^2}\,\mathrm{d}s\,\mathrm{d}\theta,\\ &\int_{-d(t)}^{-h_1}\int_{t+\theta}^{t-h_1}\frac{x(s)}{(d(t)-h_1)^2}\,\mathrm{d}s\,\mathrm{d}\theta, \int_{-h_2}^{-d(t)}\int_{t+\theta}^{t-d(t)}\frac{x(s)}{(h_2-d(t))^2}\,\mathrm{d}s\,\mathrm{d}\theta,\\ &\int_{t-d(t)}^{t-h_1}x(s)\,\mathrm{d}s, \int_{t-h_2}^{t-d(t)}x(s)\,\mathrm{d}s\Big\},\end{aligned} \tag{53}$$

$$\dot{V}_3(t) \le \xi^{\mathrm{T}}(t)(\Xi_3(d(t)) + \Omega_1(d(t)))\xi(t), \tag{57}$$

where $\Xi_3(d(t))$ is shown in (22).

Then, employing inequality (5) to estimate terms δ_3 in (44) obtains

$$\delta_3 \le -\xi^{\mathrm{T}}(t)\left(\frac{h_{12}}{d(t)-h_1}E_4^{\mathrm{T}}R_3E_4 + \frac{h_{12}}{h_2-d(t)}E_5^{\mathrm{T}}R_3E_5\right)\xi(t), \tag{58}$$

where $E_i, i = 4, 5$, are defined in (26).

Thus, by using Lemma 2, the second enlargement makes these delay-reciprocal-related terms in (58) become delay-reciprocal-free quadratic terms. For a slack matrix S, the following holds:

$$\begin{aligned}\delta_3 \le\ &-\xi^{\mathrm{T}}(t)\big((2-\alpha_{d(t)})E_4^{\mathrm{T}}R_3E_4 + (1+\alpha_{d(t)})E_5^{\mathrm{T}}R_3E_5 \\ &+\mathrm{Sym}\{E_4^{\mathrm{T}}SE_5\} - \Omega_2(d(t))\big)\xi(t),\end{aligned} \tag{59}$$

where

$$\Omega_2(d(t)) = (1-\alpha_{d(t)})E_4^{\mathrm{T}}SR_3^{-1}S^{\mathrm{T}}E_4 + \alpha_{d(t)}E_5^{\mathrm{T}}S^{\mathrm{T}}R_3^{-1}SE_5. \tag{60}$$

Through the procedures in (44), (58) and (59), $\dot{V}_4(t)$ is estimated as

$$\dot{V}_4(t) \le \xi^{\mathrm{T}}(t)(\Xi_4(d(t)) + \Omega_2(d(t)))\xi(t), \tag{61}$$

where $\Xi_4(d(t))$ is shown in (23).

Combining formulas (40)–(42), (57) and (61), $\dot{V}(t)$ is estimated as

$$\dot{V}(t) \le \xi^{\mathrm{T}}(t)\hat{\Xi}(d(t), d^2(t))\xi(t), \tag{62}$$

where

$$\hat{\Xi}(d(t), d^2(t)) = \Xi(d(t), d^2(t)) + \Omega_1(d(t)) + \Omega_2(d(t)), \tag{63}$$

and $\Xi(d(t), d^2(t))$ is shown in (19). It is clear that $\dot{V}(t)$ has the same form of $f(d(t))$ shown in Lemma 4, with

$$f(d(t)) := \xi^{\mathrm{T}}(t)\hat{\Xi}(d(t), d^2(t))\xi(t), \tag{64}$$

$$\phi_2 := \xi^{\mathrm{T}}(t)\Upsilon\xi(t), \tag{65}$$

where Υ is shown in (24), and its convexity or concavity is misty. Thus, by Lemma 4, the validity of $\dot{V}(t) < 0$ for $d(t) \in [h_1, h_2]$ is ensured by

$$\begin{cases}\hat{\Xi}(h_1, h_1^2) < 0,\\ \hat{\Xi}(h_2, h_2^2) < 0,\\ -h_{12}^2\Upsilon + \hat{\Xi}(h_2, h_2^2) < 0\\ \text{or}\\ -h_{12}^2\Upsilon + \hat{\Xi}(h_1, h_1^2) < 0.\end{cases} \tag{66}$$

Based on the Schur complement, it is found that the validity of the conditions in (66) is guaranteed by the LMIs in (16)–(18). Therefore, if the LMIs in (16)–(18) hold, we claim that $\dot{V}(t) \le -\varepsilon_2 \| x(t) \|^2$ for a sufficiently small scalar $\varepsilon_2 > 0$, and system (2) with delay satisfying (3) is asymptotically stable. The proof is finished. □

As stated in Remark 2, if only conditions $C1$, $C2$ and $C4$ are considered, then Lemma 4 becomes either Lemma 2 of [38] or Lemma 4 of [37]. Then, the following criterion is obtained using either Lemma 2 of [38] or Lemma 4 of [37], based on (64) and (65).

Corollary 1: For given constants h_1 and h_2 with $h_2 \ge h_1 \ge 0$, system (2) with the delay satisfying (3) is asymptotically stable, if there exist a $5n \times 5n$ matrix $P > 0$, $n \times n$ matrices $Q_1 > 0$ and $R_i > 0$, i=1,2, a $3n \times 3n$ matrix $Q_2 > 0$, a $2n \times 2n$ matrix $R_3 > 0$, a $2n \times 2n$ any matrix S, $3n \times 3n$ any matrices X_1 and X_2, $3n \times 3n$ symmetric matrices Y_1 and Y_2, and $3n \times 2n$ any matrix Z, such that the following LMIs hold:

$$\begin{bmatrix}\Xi(h_1, h_1^2) & * & * \\ E_{XY2} & -\hat{R}_2 & * \\ S^{\mathrm{T}}E_4 & 0_{2n\times 3n} & -R_3\end{bmatrix} < 0, \tag{67}$$

$$\begin{bmatrix}\Xi(h_2, h_2^2) & * & * \\ E_{XY1} & -\hat{R}_2 & * \\ SE_5 & 0_{2n\times 3n} & -R_3\end{bmatrix} < 0, \tag{68}$$

$$\begin{bmatrix}-h_{12}^2\Upsilon + \Xi(h_1, h_1^2) & * & * \\ E_{XY2} & -\hat{R}_2 & * \\ S^{\mathrm{T}}E_4 & 0_{2n\times 3n} & -R_3\end{bmatrix} < 0, \tag{69}$$

where the notations used here are defined in Theorem 1.

Remark 4: Corollary 1 is derived by using the same LKF and inequalities as those used for Theorem 1, but using the different negative-definiteness determination method. From the conditions of Theorem 1 and Corollary 1, it is found that the LMIs of Corollary 1 in (67) and (68) are same as those of Theorem 1 in (16) and (17), and the LMI in (69) is the same as the second LMI in (18). However, the alternative condition corresponding to the first LMI in (18) is not introduced in Corollary 1. That is Theorem 1 is with more flexibility and has the less conservatism, and at least has the same conservatism, compared with Corollary 1.

Remark 5: In the LKF $V(t)$, the terms $V_i(t), i = 1, 2, 3$, are constructed by following the LKF in [38], and are useful in deriving a delay-dependent stability criterion. The introduction of $V_4(t)$ in (38) is helpful for producing the NDC. From (53), it is found that $\int_{t-d(t)}^{t-h_1} x(s)\,\mathrm{d}s$ and $\int_{t-h_2}^{t-d(t)} x(s)\,\mathrm{d}s$ are taken as state variables. While these terms $V_i(t), i = 1, 2, 3$, do not produce any quadratic terms related to the above two state variables, they do produce some quadratic terms related to the other state variables in (53). After the introduction of $V_4(t)$, the estimation result in (59) contains the quadratic terms that are associated with $\int_{t-d(t)}^{t-h_1} x(s)\,\mathrm{d}s$ and $\int_{t-h_2}^{t-d(t)} x(s)\,\mathrm{d}s$, and these terms are negative definite. That is only $V_i(t)$, $i = 1, 2, 3$, cannot give the NDC, because the quadratic terms related to the two integral-type state variables are not produced, and these terms in $V(t)$ are necessary to derive a stability criterion.

Remark 6: The maximal delay margin provided by Theorem 1 means that system (2) is stable if $d(t)$ varies within the maximal delay margin. While if $d(t)$ is very large and bigger than the maximal upper bound of the delay margin, then the LMIs of Theorem 1 would not hold and system (2) will become unstable. In this situation, the stabilisation control is needed to stabilise the time-delay system, and the development of stabilisation control by the proposed method will be performed in the future work.

4 Three examples

In this section, three examples are given to show the advantages of Lemma 4 and Theorem 1.

Example 1: Consider system (2) with

$$A = \begin{bmatrix}0 & 1\\ -10 & -1\end{bmatrix}, A_d = \begin{bmatrix}0 & 0.1\\ 0.1 & 0.2\end{bmatrix}.$$

In Table 1, the maximal upper bounds of delay (maximal values of h_2) computed by Theorem 1 and Corollary 1 are listed for

Table 1 Maximal values of h_2 for various h_1 (Example 1)

Method	h_1				
	0.0	0.3	0.7	1.0	2.0
[22]	1.59	2.01	2.41	2.62	3.59
[24]	1.64	2.13	2.70	2.96	3.63
[35]	1.80	2.19	2.58	2.79	3.68
[30]	1.790	2.232	2.618	2.812	3.763
[34]	1.862	2.288	2.695	2.895	3.849
Corollary 1	1.9534	2.3869	2.9989	3.2586	3.9164
Theorem 1	1.9534	2.4103	2.9989	3.2586	3.9164

Table 2 Maximal values of h_2 for various h_1 (Example 2)

Method	h_1				
	0.0	0.3	0.5	0.8	1.0
[7]	0.77	0.94	1.09	1.34	1.51
[29]	1.06	1.24	1.38	1.60	1.75
[24]	1.19	1.35	1.47	1.67	1.82
[33, 35]	1.20	1.35	1.47	1.67	1.82
Corollary 1	1.2246	1.3751	1.4942	1.6935	1.8375
Theorem 1	1.2288	1.3796	1.4980	1.6962	1.8394

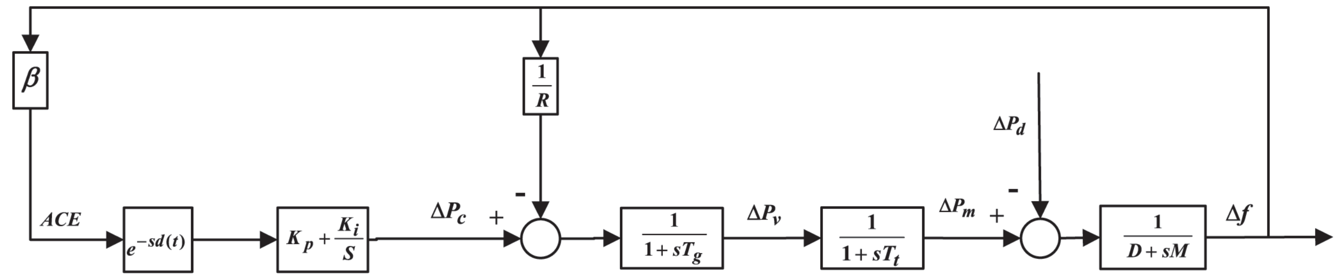

Fig. 1 *Dynamic model of one-area LFC scheme*

various values of h_1, and some latest results in [22, 24, 30, 34, 35] are also collected for full comparison.

Several observations are obtained from the numerical results:

- First, the results produced by Theorem 1 are greater than those of [22, 24, 30, 34, 35], for instance, for $h_1 = 0.0$, the maximal value of h_2 computed by Theorem 1 is 1.9534, while that of [34] is 1.862. This means that the delay margin computed by Theorem 1 is $[0.0, 1.9534]$, which is larger than $[0.0, 1.862]$. Therefore, Theorem 1 is less conservative than the criteria of [22, 24, 30, 34, 35].
- Second, Theorem 1 provides a bigger upper bound of delay than that provided by Corollary 1 when $h_1 = 0.3$, and provides the same results when $h_1 \in \{0, 0.7, 1.0, 2.0\}$. It is verified that Theorem 1 is less conservative than Corollary 1 or at least has the same conservatism. At the same time, Lemma 4 is proven to be less conservative than Lemma 2 of [38] and Lemma 4 of [37].

Overall, Lemma 4 proposed in this paper shows less conservatism than the existing methods, and then, Theorem 1 is superior to the criteria of [22, 24, 30, 34, 35].

Example 2: Consider system (2) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

The maximal values of h_2 calculated by Theorem 1 and Corollary 1 and some of the latest results are gathered together in Table 2.

According to the data presented in Table 2, Theorem 1 produces greater upper bounds of delay than Corollary 1 and the criteria of [7, 24, 29, 35] do. Therefore, the advantages of Lemma 4 and Theorem 1 are shown again.

Example 3: Consider the one-area load frequency control system [40] shown in Fig. 1.

Its nominal model is system (2) with

$$x(t) = \left[\triangle f\ \triangle P_{\mathrm{m}}\ \triangle P_{\mathrm{v}} \int \mathrm{ACE}\right]^{\mathrm{T}},$$

$$A = \begin{bmatrix} -\dfrac{D}{M} & \dfrac{1}{M} & 0 & 0 \\ 0 & -\dfrac{1}{T_{\mathrm{t}}} & \dfrac{1}{T_{\mathrm{t}}} & 0 \\ -\dfrac{1}{RT_{\mathrm{g}}} & 0 & -\dfrac{1}{T_{\mathrm{g}}} & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{K_{\mathrm{p}}\beta}{T_{\mathrm{g}}} & 0 & 0 & -\dfrac{K_{\mathrm{i}}}{T_{\mathrm{g}}} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\triangle f$, $\triangle P_{\mathrm{m}}$, $\triangle P_{\mathrm{v}}$ and $\int \mathrm{ACE}$ denote the deviation of frequency, the generator mechanical output, valve position and load, respectively. D and M are the generator damping coefficient and the moment of inertia of the generator, respectively, T_{g} and T_{t} are the time constants of the governor and the turbine, respectively, K_{p} and K_{i} are the gains of PI controller, R is the speed drop and β is the frequency bias factor.

The values of those parameters are given as $D = 1.0$, $M = 10$, $T_{\mathrm{g}} = 0.1$, $T_{\mathrm{t}} = 0.3$, $K_{\mathrm{p}} = 0.05$, $K_{\mathrm{i}} = 0.2$, $R = 0.05$ and $\beta = 21$. Then the maximal values of h_2 for $h_1 \in \{1.0, 2.0, 3.0, 3.5, 4.0\}$ computed by Theorem 1 and Corollary 1 are listed in Table 3.

Table 3 Maximal values of h_2 for various h_1 (Example 3)

Method	h_1				
	1.0	2.0	3.0	3.5	4.0
Corollary 1	6.6966	6.7084	6.7314	6.7601	6.8039
Theorem 1	6.6971	6.7098	6.7336	6.7620	6.8052

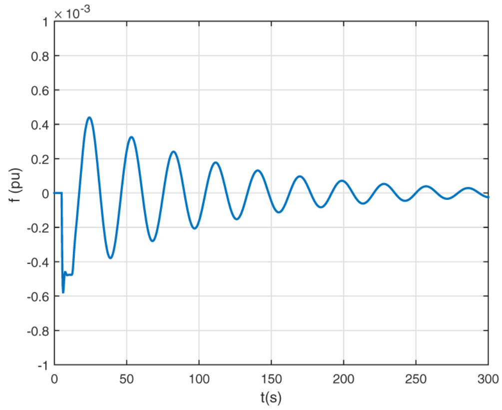

Fig. 2 *Response of the deviation of frequency (△f)*

An examination of the numerical results shows that the maximal values of h_2 provided by Theorem 1 provides are bigger than those of Corollary 1, demonstrating the superiority of the proposed method.

Finally, a simulation study is carried out to demonstrate the effectiveness of the proposed stability criterion and the method. With $d(t) = 6.8052$, the response of the deviation of frequency is shown in Fig. 2. It is found that the deviation of frequency decreases gradually, and thus, the system is asymptotically stable.

5 Conclusions

A novel negative-definiteness determination lemma has been derived to address the quadratic function that appears in the derivative of the LKF, and this lemma was proven to encompass the previous methods. The application of the proposed lemma led to a less conservative stability criterion for a system with a time-varying delay, which was verified using three examples.

6 Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under grant nos. 61973284 and 61873347, by the Hubei Provincial Natural Science Foundation of China under grant nos. 2019CFA040 and 2015CFA010, and by the 111 project under grant no. B17040.

7 References

[1] Fridman, E.: '*Introduction to time-delay systems: analysis and control*' (Birkhäuser, Boston, 2014)

[2] Gu, K., Chen, J., Kharitonov, V.: '*Stability of time-delay systems*' (Birkhäuser, Boston, 2003)

[3] Zhang, X.M., Han, Q.L., Seuret, A., *et al.*: 'Overview of recent advances in stability of linear systems with time-varying delays', *IET Control Theory Appl.*, 2019, **13**, (1), pp. 1–16

[4] Liu, Y., Park, J.H., Guo, B.Z.: 'Results on stability of linear systems with time varying delay', *IET Control Theory Appl.*, 2017, **11**, (1), pp. 129–134

[5] Zhao, T., Liang, W., Dian, S., *et al.*: 'Improved stability and stabilisation criteria for discrete time-delay systems via a novel double summation inequality', *IET Control Theory Appl.*, 2018, **12**, (3), pp. 327–337

[6] Briat, C.: '*Linear parameter-varying and time-delay systems: analysis, observation, filtering & control*' (Springer-Verlag, London, 2015)

[7] He, Y., Wang, Q.G., Lin, C., *et al.*: 'Delay-range-dependent stability for systems with time-varying delay', *Automatica*, 2007, **43**, (2), pp. 371–376

[8] Zhang, C.K., He, Y., Jiang, L., *et al.*: 'Stability analysis of systems with time-varying delay via relaxed integral inequalities', *Syst. Control Lett.*, 2016, **92**, pp. 52–61

[9] Kim, J.H.: 'Note on stability of linear systems with time-varying delay', *Automatica*, 2011, **47**, pp. 2118–2121

[10] He, Y., Wang, Q.G., Lin, C., *et al.*: 'Augmented Lyapunov functional and delay-dependent stability criteria for neutral systems', *Int. J. Robust Nonlinear Control*, 2005, **15**, pp. 923–933

[11] Park, M.J., Kwon, O.M., Ryu, J.H.: 'Advanced stability criteria for linear systems with time-varying delays', *J. Franklin Inst.*, 2017, **355**, (1), pp. 520–543

[12] Zhang, X.M., Han, Q.L., Seuret, A., *et al.*: 'An improved reciprocally convex inequality and an augmented Lyapunov–Krasovskii functional for stability of linear systems with time-varying delay', *Automatica*, 2017, **84**, pp. 221–226

[13] Chen, J., Park, J.H., Xu, S.: 'Stability analysis of continuous-time systems with time-varying delay using new Lyapunov–Krasovskii functionals', *J. Franklin Inst.*, 2018, **355**, pp. 5957–5967

[14] Long, F., Jiang, L., He, Y., *et al.*: 'Stability analysis of systems with time-varying delay via novel augmented Lyapunov–Krasovskii functionals and an improved integral inequality', *Appl. Math. Comput.*, 2019, **357**, pp. 325–337

[15] Sun, J., Liu, G.P., Chen, J., *et al.*: 'Improved delay-range-dependent stability criteria for linear systems with time-varying delays', *Automatica*, 2010, **46**, (2), pp. 466–470

[16] Chen, J., Xu, S., Zhang, B.: 'Single/Multiple integral inequalities with applications to stability analysis of time-delay systems', *IEEE Trans. Autom. Control*, 2017, **62**, (7), pp. 3488–3493

[17] Lee, T.H., Park, J.H.: 'A novel Lyapunov functional for stability of time-varying delay systems via matrix-refined-function', *Automatica*, 2017, **80**, pp. 239–242

[18] Lee, T.H., Park, J.H.: 'Improved stability conditions of time-varying delay systems based on new Lyapunov functionals', *J. Franklin Inst.*, 2018, **355**, pp. 1176–1191

[19] Zhang, C.K., He, Y., Jiang, L., *et al.*: 'Notes on stability of time-delay systems: bounding inequalities and augmented Lyapunov–Krasovskii functionals', *IEEE Trans. Autom. Control*, 2017, **60**, (10), pp. 5331–5336

[20] Zhang, C.K., He, Y., Jiang, L., *et al.*: 'Delay-variation-dependent stability of delayed discrete-time systems', *IEEE Trans. Autom. Control*, 2016, **61**, (9), pp. 2663–2669

[21] Lee, T.H., Park, J.H., Xu, S.: 'Relaxed conditions for stability of time-varying delay systems', *Automatica*, 2017, **75**, pp. 11–15

[22] Seuret, A., Gouaisbaut, F.: 'Wirtinger-based integral inequality: application to time-delay systems', *Automatica*, 2013, **49**, (9), pp. 2860–2866

[23] Hien, L.V., Trinh, H.: 'Refined Jensen-based inequality approach to stability analysis of time-delay systems', *IET Control Theory Appl.*, 2015, **9**, (14), pp. 2188–2194

[24] Park, P., Lee, W., Lee, S.Y.: 'Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems', *J. Franklin Inst.*, 2015, **352**, (4), pp. 1378–1396

[25] Seuret, A., Gouaisbaut, F.: 'Hierarchy of LMI conditions for the stability of time delay systems', *Syst. Control Lett.*, 2015, **81**, pp. 1–7

[26] Liu, K., Seuret, A., Xia, Y.: 'Stability analysis of systems with time-varying delays via the second-order Bessel–Legendre inequality', *Automatica*, 2017, **76**, pp. 138–142

[27] Seuret, A., Gouaisbaut, F.: 'Stability of linear systems with time-varying delays using Bessel–Legendre inequalities', *IEEE Trans. Autom. Control*, 2018, **63**, (1), pp. 225–232

[28] Lee, W.I., Lee, S.Y., Park, P.G.: 'Affine Bessel–Legendre inequality: application to stability analysis for systems with time-varying delays', *Automatica*, 2018, **93**, pp. 535–539

[29] Park, P., Ko, J., Jeong, C.: 'Reciprocally convex approach to stability of systems with time-varying delays', *Automatica*, 2011, **47**, (1), pp. 235–238

[30] Zhang, C.K., He, Y., Jiang, L., *et al.*: 'An extended reciprocally convex matrix inequality for stability analysis of systems with time-varying delay', *Automatica*, 2017, **85**, pp. 481–485

[31] Zhang, C.K., He, Y., Jiang, L., *et al.*: 'Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality', *IEEE Trans. Cybern.*, 2017, **47**, (10), pp. 3040–3049

[32] Long, F., Zhang, C.K., He, Y., *et al.*: 'Stability analysis of Lur'e systems with additive delay components via a relaxed matrix inequality', *Appl. Math. Comput.*, 2018, **328**, pp. 224–242

[33] Seuret, A., Gouaisbaut, E.: 'Delay-dependent reciprocally convex combination lemma'. Rapport LAAS n16006, 2016

[34] Seuret, A., liu, K., Gouaisbaut, E.: 'Generalized reciprocally convex combination lemmas and its application to time-delay systems', *Automatica*, 2018, **95**, pp. 488–493

[35] Zeng, H.B., He, Y., Wu, M., *et al.*: 'Free-matrix-based integral inequality for stability analysis of systems with time-varying delay', *IEEE Trans. Autom. Control*, 2015, **60**, (10), pp. 2768–2772

[36] Zeng, H.B., He, Y., Wu, M., *et al.*: 'New results on stability analysis for systems with discrete distributed delay', *Automatica*, 2015, **60**, pp. 189–192

[37] Zhang, C.K., He, Y., Jiang, L., *et al.*: 'Delay-dependent stability analysis of neural networks with time-varying delay: a generalized free-weighting-matrix approach', *Appl. Math. Comput.*, 2017, **294**, pp. 102–120

[38] Kim, J.H.: 'Further improvement of Jensen inequality and application to stability of time-delayed systems', *Automatica*, 2016, **64**, pp. 121–125

[39] Zhang, X.M., Han, Q.L.: 'New stability criterion using a matrix-based quadratic convex approach and some novel integral inequalities', *IET Control Theory Appl.*, 2014, **8**, (12), pp. 1054–1061

[40] Jiang, L., Yao, W., Wu, Q.H., *et al.*: 'Delay-dependent stability for load frequency control with constant and time-varying delays', *IEEE Trans. Power Syst.*, 2012, **27**, (4), pp. 932–941