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Experimental and numerical studies on internal solitary waves with a free surface

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1 Large-amplitude internal solitary waves in a two-layer fluid system with a free surface are
2 investigated in this paper. Experiments on strongly nonlinear internal solitary waves with
3 a free surface for a deep configuration are conducted. After comparing the experimental
4 data with the results of the model derived by Miyata, Choi & Camassa that includes
5 the free-surface effects (MCC-FS model), we find that the MCC-FS model does not
6 calculate accurately the internal solitary waves with a free surface. Thus, we develop
7 a strongly nonlinear model for a deep configuration, namely the two-layer High-Level
8 Green-Naghdi model that includes the free-surface effects (HLGN-FS model). Numerical
9 results of the HLGN-FS model, including the wave profile, velocity field and wave speed,
10 are presented for three cases. The first case is a shallow configuration with $\rho_2/\rho_1 = 0.977$
11 and $h_2/h_1 = 1/4.13$, where ρ_2 and ρ_1 are the densities of the upper-fluid layer and
12 the lower-fluid layer, respectively, and h_2 and h_1 are the depths of the upper-fluid layer
13 and the lower-fluid layer, respectively. The second case is also a shallow configuration
14 where $h_2/h_1 = 1/5$ while $\rho_2/\rho_1 = 0.859$. The third case is related to the present physical
15 experiments, where $\rho_2/\rho_1 = 0.869$ and $h_2/h_1 = 1/15$. It is shown that the MCC-FS model
16 can provide accurate results for the shallow configurations. Meanwhile, the HLGN-FS
17 model is shown to be accurate for describing the internal solitary waves for both shallow
18 and deep configurations.

19 **Key words:** internal solitary waves; High-Level Green-Naghdi model; two-layer fluid
20 flow

21 1. Introduction

22 Internal solitary waves in the oceans are often caused by currents propagating over
23 an uneven bottom or sea ridge. Internal solitary waves are observed in the oceans, such
24 as the one with amplitude as large as 240m observed in the South China Sea by Huang
25 *et al.* (2016). They can cause serious threats to submarines, offshore platforms, and
26 marine risers, among others. Hence, it is of interest to investigate internal solitary waves,
27 including the wave profile, velocity field and wave speed.

28 Large-amplitude internal solitary waves play an important role in nonlinear water

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29 waves. Both observations of Stanton & Ostrovsky (1998) (at northern Oregon Continental
 30 Shelf) and the theoretical models of Ostrovsky & Grue (2003) have shown that internal
 31 solitary waves with very large amplitudes (with respect to the upper layer depth) may
 32 be formed and remain stable. A similar wave was observed by Duda *et al.* (2004) in the
 33 South China Sea, where the wave amplitude was about four times the thickness of the
 34 mixed upper layer. Stable internal solitary waves with larger amplitudes are generated in
 35 laboratory environments, see e.g. Fructus *et al.* (2009). Large internal solitary waves may
 36 eventually become unstable and break due to either convective instability or shear-driven
 37 stresses, see e.g. Fructus *et al.* (2009), Lamb & Farmer (2011) and Carr *et al.* (2011). In
 38 this study, the breaking internal solitary waves are not considered.

39 Several laboratory experiments have been conducted on internal solitary waves in a
 40 two-layer fluid system. The common method for internal solitary wave generation at
 41 a laboratory is the gravity collapse method, used for example, by Kao *et al.* (1985),
 42 Michallet & Barthelemy (1997, 1998), Grue *et al.* (1999) and Kodaira *et al.* (2016).
 43 Among these experimental studies, Michallet & Barthelemy (1997, 1998) used ultrasonic
 44 probes to measure the wave profile at the interface. Grue *et al.* (1999) used particle
 45 tracking velocimetry (PTV) to measure the wave speed, wave profile and horizontal
 46 velocity distribution along the fluid column. Kodaira *et al.* (2016) used several wave
 47 probes to determine the profiles and speeds of internal solitary waves.

48 Numerical analysis of internal solitary waves is often carried out by use of the Korteweg-
 49 de Vries (KdV) equation. However, at such large amplitudes, internal solitary waves
 50 differ remarkably from the prediction of the KdV equation, see e.g. Miles (1980), Grue
 51 *et al.* (1999) and Kodaira *et al.* (2016). For more details on the KdV equation and its
 52 application to this problem, we refer the reader to Ostrovsky & Stepanyants (2005) and
 53 Helfrich & Melville (2006).

54 Strongly nonlinear models are required to study the generation and propagation of
 55 such nonlinear wave motions, see Grue (2006). Some efficient theoretical or numerical
 56 studies have been developed under the assumption that the free surface is a rigid lid,
 57 i.e. the rigid-lid (RL) assumption. For the two-layer fluid system, it is discussed by
 58 Camassa *et al.* (2006) that when the wavelength is long compared with the fluid layer
 59 depths (i.e. $h_2/\lambda \ll 1$ and $h_1/\lambda \ll 1$, where h_2, h_1 are the depths of the upper fluid
 60 layer and the lower fluid layer, respectively, and λ is the characteristic wavelength),
 61 it belongs to the ‘shallow configuration’. On the other hand, when the depth of one
 62 fluid layer is much larger than the other, while being comparable to or larger than the
 63 wavelength (e.g. $h_2/\lambda \ll 1$ and $h_1/\lambda = O(1)$), it belongs to the ‘deep configuration’.
 64 Miyata (1985, 1988) and Choi & Camassa (1999) derived a strongly nonlinear model for a
 65 shallow configuration. In their model, the velocity field was described by use of the depth-
 66 averaged horizontal velocities. This model is called the MCC (Miyata, Choi, Camassa)
 67 model. Here we refer to it as the MCC-RL model since the rigid-lid assumption was used.
 68 Because of its simple form and release of the assumption of small-amplitude motions,
 69 the MCC-RL model is widely used to study relatively large-amplitude internal solitary
 70 waves for a shallow configuration. Grue *et al.* (1999) obtained the internal solitary-wave
 71 solutions by solving Euler’s equations subject to the rigid-lid assumption, i.e. Euler-RL
 72 solution. The numerical results of Grue *et al.* (1999) showed very good agreement with
 73 the experimental data on the wave profile and velocity field for the case of $h_2/h_1 = 1/4.13$
 74 and $\rho_2/\rho_1 = 0.977$, where ρ_2, ρ_1 are the densities of the upper-fluid layer and the lower-
 75 fluid layer, respectively.

76 Meanwhile, some other models under the rigid-lid assumption for a deep configuration
 77 have been employed. Choi & Camassa (1999) developed a strongly nonlinear model for a
 78 deep configuration. In this model, the depth-averaged velocity approximation was applied

79 to describe the upper-layer velocity field, and the linear theory was applied to describe
 80 the lower-layer velocity field. Results showed some differences between this model and
 81 Euler-RL solution for a deep-configuration case. Debsarma *et al.* (2010) improved the
 82 deep-water model of Choi & Camassa (1999) and Camassa *et al.* (2006), and increased
 83 the approximation to $O(\epsilon^2)$ terms, where $\epsilon = h_2/\lambda$. Some differences between Euler-RL
 84 solution and the results given by Debsarma *et al.* (2010) for the deep-configuration case
 85 were observed. Recently, Zhao *et al.* (2016) developed the two-layer High-Level Green-
 86 Naghdi model under the rigid-lid assumption (HLGN-RL model). The HLGN-RL results
 87 showed very good agreement with Euler-RL solution on the wave profile, velocity field
 88 and wave speed for strongly nonlinear internal solitary waves for a deep configuration.

89 For relatively large density differences between the two fluids, e.g. $\rho_2/\rho_1 = 0.859$
 90 that was tested by Kodaira *et al.* (2016), the free-surface effects are not negligible and
 91 should be considered. The presence of the free surface introduces further challenges to
 92 the problem due to (i) the interaction between the surface and internal waves, and (ii)
 93 the difference in motion scale of the surface and internal waves. Choi & Camassa (1996)
 94 derived the MCC model that included the free-surface effects (MCC-FS model). Kodaira
 95 *et al.* (2016) showed that the wave profiles obtained by the MCC-FS model matched the
 96 experimental data very well, even for the large-amplitude internal solitary waves. Also,
 97 a comparative study between the MCC-RL results and the MCC-FS results on the wave
 98 profiles and wave speeds were presented by Kodaira *et al.* (2016). They showed that the
 99 wave profiles obtained by the MCC-RL model were wider and the wave speeds were larger
 100 than the results obtained by the MCC-FS model. Forgia & Sciotino (2019) conducted
 101 experiments on internal solitary waves with a free surface and measured the maximum
 102 surface elevation right above the trough of the internal solitary wave. They showed that
 103 the maximum surface elevation increased (with respect to the upper layer depth) with
 104 smaller density ratio ρ_2/ρ_1 , and smaller depth ratio h_2/h_1 .

105 To our knowledge, comparative studies between experiments and numerical results
 106 on internal solitary waves with a free surface for deep-configuration cases has not been
 107 performed before. The goals of this study are (i) to conduct laboratory experiments on
 108 strongly nonlinear internal solitary waves with a free surface for a deep configuration, (ii)
 109 to compare the results provided by the MCC-FS model with the present experimental
 110 data to test its capability, and (iii) to develop a strongly nonlinear internal solitary-wave
 111 model for a deep configuration and test its capability through some test cases.

112 In Section 2, the laboratory experiments and the MCC-FS results are presented.
 113 The two-layer High-Level Green-Naghdi equations that include the free-surface effects
 114 (HLGN-FS equations) are derived in Section 3. The algorithm to solve the HLGN-FS
 115 equations is shown in Section 4. The HLGN-FS results, MCC-FS results, experimental
 116 data from literature and present experimental data are presented and discussed in Section
 117 5. Conclusions are reached in Section 6.

118 2. Laboratory experiments

119 Laboratory experiments are conducted to study the strongly nonlinear internal solitary
 120 waves with a free surface for a deep configuration, and when the density ratio between
 121 the two fluids is not necessarily close to 1. The laboratory experiments are conducted
 122 at the Harbin Engineering University (HEU) of China and details are provided below.
 123 Internal solitary waves propagate as a depression when the interface between the two
 124 fluids is closer to the free surface than the bed. In this study, attention is confined to
 125 internal solitary waves of depression form.

2.1. Experimental facility and set-up

The laboratory experiments are conducted in a wave tank designed and built specifically for this study. The tank is 400cm long, 15cm wide, with a total depth of 50cm. The tank walls are made of glass. In all laboratory tests, the lower-fluid layer is fresh water ($\rho_1 = 997\text{kg/m}^3$) and the upper-fluid layer is silicone oil ($\rho_2 = 866.5\text{kg/m}^3$). To add the two fluids in the tank, first fresh water is added as the lower fluid. A polystyrene foam sheet, with mass density smaller than the fresh water, is used to reduce the disturbance when adding the silicone oil to the tank. The polystyrene foam is distributed uniformly over the fresh water and the silicone oil is added gradually through the foam sheet. The sheet is removed before the start of the experiments. Overall, using this approach, little to no mixing is observed. Figure 1(a) shows the physical wave tank and Figure 1(b) shows the schematic of the wave tank. The top layer is open to the atmosphere.

To generate the internal solitary waves, the gravity collapse method (see Kao *et al.* (1985)) is used. Once the two fluid layers are filled and settled, a gate is placed at 25cm away from the left wall of the tank. A prescribed volume of silicone oil with depth d is added behind the gate, see Figure 1.

A pulley system is designed and used to remove the gate automatically and rapidly. One side of the pulley system is a weight that is connected by an electromagnetic relay, and the other side is the gate. Figure 2 shows the schematic of the pulley system. At the beginning of each test (corresponding to $t = 0\text{s}$), the power supply of the electromagnet relay is cut off. The weight then goes into free fall and it removes the gate quickly, resulting in generation of a single solitary wave of depression form. The wave then propagates to the right in the main section of the tank. Figure 3 shows a snapshot of an internal solitary wave propagating in the tank.

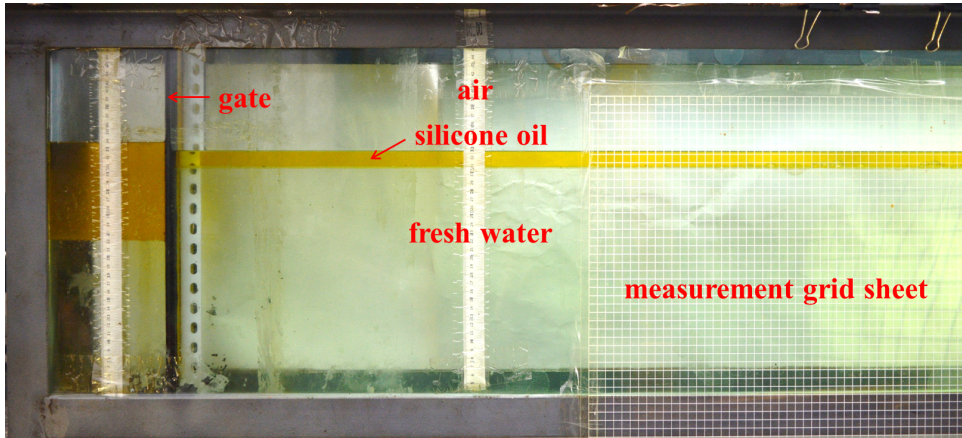
2.2. Measurements

The internal solitary waves generated by the removal of the gate, propagate along the interface of the two fluids to the right. Measurements focus on the internal-wave profile. A digital camera capable of recording videos at 60 fps is used. A transparent reference grid sheet, 120cm long, 1cm \times 1cm size per grid, is fixed to the tank wall at 55cm away from the gate. The location of the transparent grid sheet is chosen such that it is far from the right boundary to avoid any reflections during the measurements. The vertical location of the grid sheet is chosen carefully to cover the entire wave. The digital camera is located right in front of the transparent grid sheet and level with the interface of the fluids and recorded the motion of the internal solitary wave.

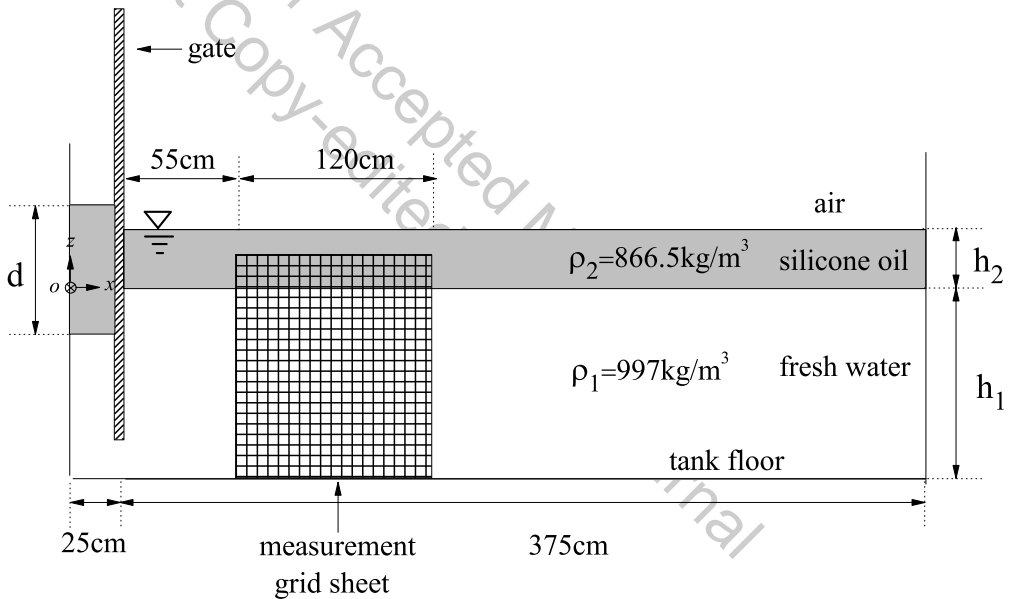
Recordings of the camera as the wave passed through the grid sheet are used to determine the wave profile. A sample recording of the internal solitary wave passing behind the sheet is shown in Figure 4, where the ox axis is set at the undisturbed interface and the internal solitary-wave amplitude a and profile $z = \eta_1(x, t)$ are shown. The *GetData Graph Digitizer* software is used to obtain the front half profile of the internal solitary wave manually. Careful attention is given to read the profiles of all cases at about the same location across the grid sheet, when the wave is right in front of the lens. The measurement error is $\varepsilon \approx \pm 0.06\text{cm}$ along the vertical direction due to the resolution of the recording picture and *GetData Graph Digitizer* software. Each test is conducted twice to assess the repeatability of the experiments.

2.3. Experimental cases and the MCC-FS results

The upper and lower fluids (silicone oil and water, respectively) are kept the same in all laboratory experiments resulting in the relative density ratio of $\rho_2/\rho_1 = 0.869$.



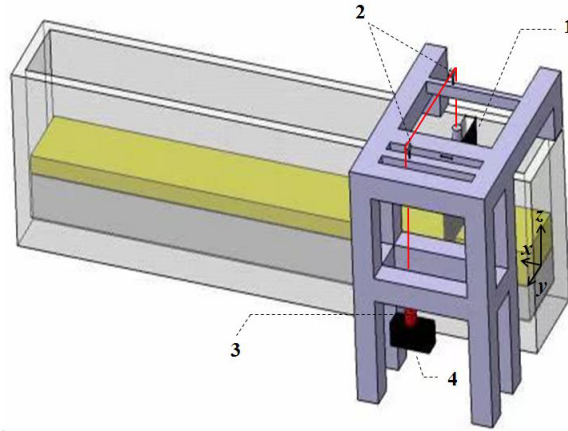
(a) The physical wave tank.



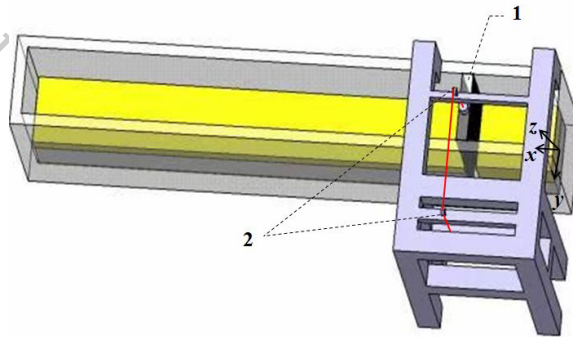
(b) Schematic of the wave tank. Figure not to scale.

Figure 1: Set-up of the experiments on internal solitary waves in a two-layer fluid system with a free surface.

173 In all laboratory experiments, the depths of the lower and upper layers are fixed at
 174 $h_1 = 30\text{cm}$ and $h_2 = 2\text{cm}$, respectively, corresponding to the depth ratio of $h_2/h_1 = 1/15$.
 175 Three initial depths (volumes in three dimensions) of the silicone oil behind the gate are
 176 considered, namely $d = 8\text{cm}$, $d = 10\text{cm}$ and $d = 12\text{cm}$. As a result, we obtained the
 177 internal solitary waves with the amplitudes of $a/h_2 = -1.41, -1.91$ and -2.35 . The
 178 parameters of the physical experiments are given in Table 1.



(a) Side view.



(b) Plan view.

Figure 2: Sketch of the pulley system, showing: 1. gate, 2. pulley, 3. electromagnetic relay, and 4. weight.

Case	h_2 (cm)	h_1 (cm)	d (cm)	a (cm)	h_2/h_1	a/h_2
a	2	30	8	-2.83	1/15	-1.41
b	2	30	10	-3.83	1/15	-1.91
c	2	30	12	-4.71	1/15	-2.35

Table 1: Parameters and results of the physical experiments conducted at the internal wave flume of HEU.

179 It was found that for a given collapse height, solitary waves generated by complete
 180 removal of the gate have larger amplitudes than those generated by partial removal of the
 181 gate. In either methods of partial or complete removal of the gate, solitary waves with
 182 desired amplitudes can be generated by carefully adjusting the initial collapse height.
 183 We observed no differences between the profile of solitary waves of the same amplitude,
 184 whether generated by complete or partial removal of the gate. Internal solitary waves in
 185 this paper are generated by partial removal of the gate. Results of the two repeats of the
 186 laboratory measurements are shown in Figure 5. Also shown in the figure is the results

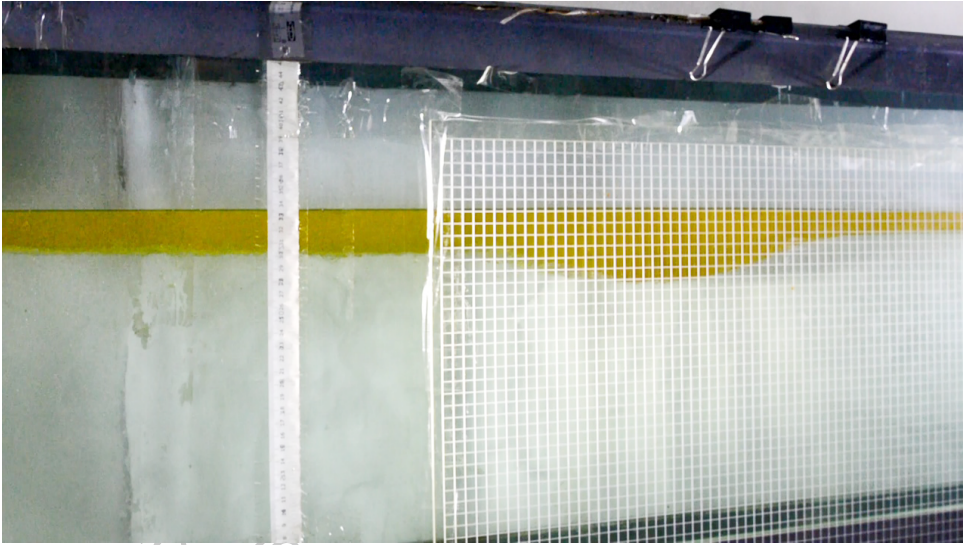


Figure 3: Snapshot of an internal solitary wave propagating in the tank approximately at $t = 4$ s (the gate is removed completely at $t = 0$ s).

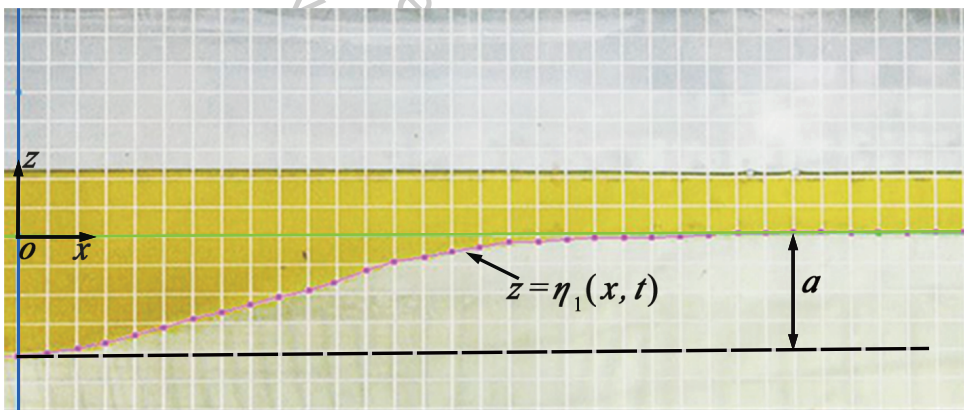
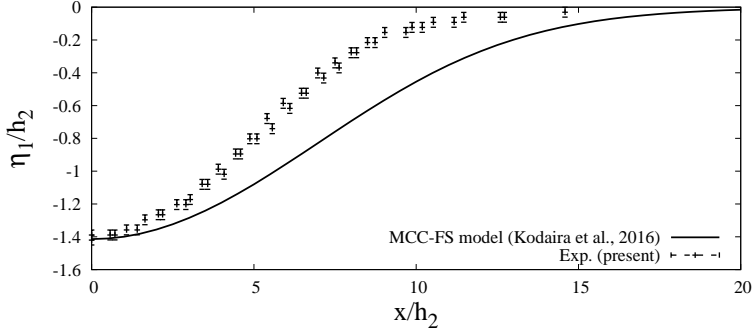
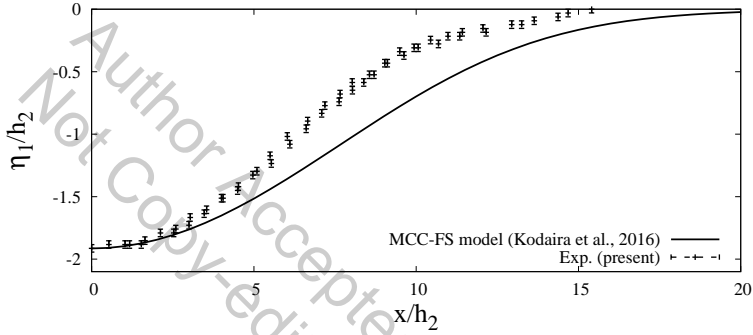
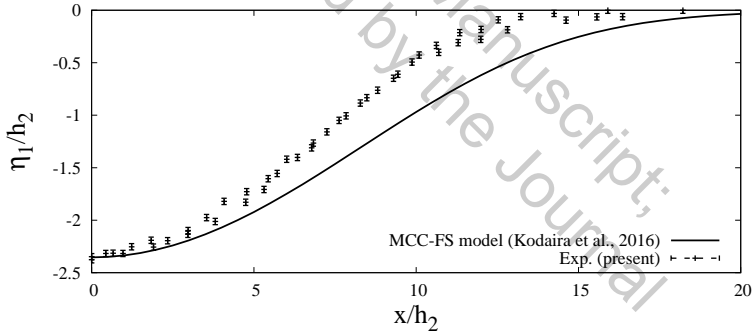


Figure 4: A sample recording of the internal solitary wave behind the transparent sheet.

187 of the MCC-FS model of Kodaira *et al.* (2016), included for comparisons. We find that
 188 the physical experiments are repeatable. From the comparison, it is clear that the wave
 189 profiles obtained by the MCC-FS model are much wider than the physical experimental
 190 measurements. This is not surprising given that the case of $h_2/h_1 = 1/15$ belongs to the
 191 deep configuration, and the MCC model is valid only for a shallow configuration, see
 192 Camassa *et al.* (2006).

193 Hence, it is concluded that for the present case of the density ratio $\rho_2/\rho_1 = 0.869$
 194 and depth ratio $h_2/\rho_1 = 1/15$, the MCC-FS model does not accurately describe the
 195 internal solitary waves with a free surface. Instead, a strongly nonlinear model for a
 196 deep configuration is required to solve such a problem accurately.

(a) $a/h_2 = -1.41$.(b) $a/h_2 = -1.91$.(c) $a/h_2 = -2.35$.Figure 5: Profiles of internal solitary waves, $\rho_2/\rho_1 = 0.869$, $h_2/h_1 = 1/15$.

197 3. Two-layer HLGN-FS equations

198 In this section, we will develop the two-layer High-Level Green-Naghdi (HLGN) model
 199 that includes the free-surface effects (HLGN-FS model). Here the two-dimensional model
 200 is considered, although this is not a requirement in general. The fluid is assumed inviscid
 201 and incompressible. The coordinate origin is set at the still interface surface. x is the
 202 horizontal axis, positive to the right, and z is the vertical axis, positive up.

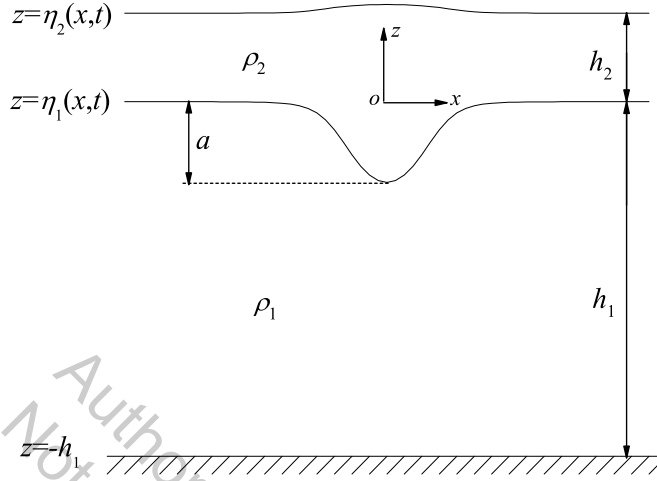


Figure 6: Setup of the theoretical tank of internal solitary wave generation and propagation in a two-layer fluid system with a free surface.

203 For both the upper layer and the lower layer, the continuity equation is written as

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (3.1)$$

204 where u and w are the velocity components in the x and z directions, respectively.

The Euler equations are written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (3.2a)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial z} + \rho g \right), \quad (3.2b)$$

205 where t is the time, ρ is the mass density of the fluid, p is the pressure and g is the
206 gravitational acceleration.

207 In this paper, the free surface, interface and bottom boundary are expressed by $z =$
208 $\eta_2(x, t)$, $z = \eta_1(x, t)$ and $z = -h_1$ respectively, where h_1 is constant. ρ_2 is the mass
209 density of the upper layer fluid and ρ_1 is the mass density of the lower layer fluid, also
210 shown in Figure 6.

The kinematic boundary conditions are expressed as

$$w^U = \frac{\partial \eta_2}{\partial t} + u^U \frac{\partial \eta_2}{\partial x} \quad z = \eta_2(x, t), \quad (3.3a)$$

$$w^U = \frac{\partial \eta_1}{\partial t} + u^U \frac{\partial \eta_1}{\partial x} \quad z = \eta_1(x, t), \quad (3.3b)$$

$$w^L = \frac{\partial \eta_1}{\partial t} + u^L \frac{\partial \eta_1}{\partial x} \quad z = \eta_1(x, t), \quad (3.3c)$$

$$w^L = 0 \quad z = -h_1, \quad (3.3d)$$

211 where the superscripts ‘ U ’ and ‘ L ’ indicate the variable in the upper layer and lower
212 layer, respectively.

In the HLGN model, e.g. Zhao *et al.* (2016), the velocity field is given as

$$u^U(x, z, t) = \sum_{n=0}^{K^U} u_n^U(x, t) z^n, \quad w^U(x, z, t) = \sum_{n=0}^{K^U} w_n^U(x, t) z^n, \quad (3.4a)$$

$$u^L(x, z, t) = \sum_{n=0}^{K^L} u_n^L(x, t) z^n, \quad w^L(x, z, t) = \sum_{n=0}^{K^L} w_n^L(x, t) z^n, \quad (3.4b)$$

213 where K^U is the level of the HLGN model applied for the upper layer and K^L is the level
214 of the HLGN model applied for the lower layer, respectively. K^U and K^L can be chosen
215 independent of each other. We use HLGN- K^U - K^L to indicate which level we use for each
216 layer (in this paper, we select $K^U = K^L$ to obtain the converged HLGN results but this
217 is not necessary in general). $u_n^U, w_n^U, u_n^L, w_n^L$ are the unknown velocity coefficients that
218 are determined as part of the solution.

Substituting Eqs. (3.4) into Eq. (3.1) results in the following relations:

$$u_{K^U}^U = 0, \quad (3.5a)$$

$$w_n^U = -\frac{1}{n} \frac{\partial u_{n-1}^U}{\partial x} \quad n = 1, 2, \dots, K^U, \quad (3.5b)$$

$$u_{K^L}^L = 0, \quad (3.5c)$$

$$w_n^L = -\frac{1}{n} \frac{\partial u_{n-1}^L}{\partial x} \quad n = 1, 2, \dots, K^L. \quad (3.5d)$$

Substituting Eqs. (3.4) into Eq (3.2), and multiplying each term by z^n and integrating
from η_1 to η_2 for the upper layer, and from $-h_1$ to η_1 for the lower layer, will result in

$$\begin{aligned} & \frac{\partial}{\partial x} (G_n^U + gS1_n^U) + nE_{n-1}^U - \eta_1^n \frac{\partial}{\partial x} (G_0^U + gS1_0^U) \\ & + (\eta_2^n - \eta_1^n) \frac{\partial}{\partial x} \left(\frac{\hat{p}^U}{\rho_2} \right) = 0 \quad n = 1, 2, \dots, K^U, \end{aligned} \quad (3.6a)$$

$$\begin{aligned} & \frac{\partial}{\partial x} (G_n^L + gS1_n^L) + nE_{n-1}^L - (-h_1)^n \frac{\partial}{\partial x} (G_0^L + gS1_0^L) \\ & + (\eta_1^n - (-h_1)^n) \frac{\partial}{\partial x} \left(\frac{\hat{p}^L}{\rho_1} \right) = 0 \quad n = 1, 2, \dots, K^L, \end{aligned} \quad (3.6b)$$

219 where $\hat{p}^U = 0$ is the pressure at the upper surface of the upper layer (without loss in
220 generality), \hat{p}^L is the pressure at the upper surface of the lower layer that equals the
221 pressure at the lower surface of the upper layer \bar{p}^U , which is written as

$$\hat{p}^L = \bar{p}^U = \rho_2 G_0^U + \rho_2 g S1_0^U, \quad (3.7)$$

and

$$E_n^U = \sum_{m=0}^{K^U} \left(\frac{\partial u_m^U}{\partial t} S2_{mn}^U + \frac{\partial w_m^U}{\partial x} Q_{mn}^U + u_m^U H_{mn}^U \right), \quad (3.8a)$$

$$G_n^U = \sum_{m=0}^{K^U} \left(\frac{\partial w_m^U}{\partial t} S2_{mn}^U + \frac{\partial w_m^U}{\partial x} Q_{mn}^U + w_m^U H_{mn}^U \right), \quad (3.8b)$$

$$Q_{mn}^U = \sum_{r=0}^{K^U} u_r^U S3_{mrn}^U, \quad H_{mn}^U = \sum_{r=0}^{K^U} w_r^U S4_{mrn}^U, \quad (3.8c)$$

$$S1_n^U = \int_{\eta_1}^{\eta_2} z^n dz, \quad S2_{mn}^U = \int_{\eta_1}^{\eta_2} z^{m+n} dz, \quad (3.8d)$$

$$S3_{mrn}^U = \int_{\eta_1}^{\eta_2} z^{m+r+n} dz, \quad S4_{mrn}^U = m \int_{\eta_1}^{\eta_2} z^{m+r+n-1} dz. \quad (3.8e)$$

222 We note that the expression for the lower layer is obtained by changing ‘ U ’ to ‘ L ’, η_1 to
223 $-h_1$ and η_2 to η_1 in Eq. (3.8).

Substituting Eq. (3.4) into Eq. (3.3), we obtain

$$\frac{\partial \eta_2}{\partial t} = \sum_{n=0}^{K^U} \eta_2^n \left(w_n^U - \frac{\partial \eta_2}{\partial x} u_n^U \right), \quad (3.9a)$$

$$w_0^U = \frac{\partial \eta_1}{\partial t} + \frac{\partial \eta_1}{\partial x} u_0^U + \sum_{n=1}^{K^U} \eta_1^n \left(\frac{1}{n} \frac{\partial u_{n-1}^U}{\partial x} + \frac{\partial \eta_1}{\partial x} u_n^U \right), \quad (3.9b)$$

$$\frac{\partial \eta_1}{\partial t} = \sum_{n=0}^{K^L} \eta_1^n \left(w_n^L - \frac{\partial \eta_1}{\partial x} u_n^L \right), \quad (3.9c)$$

$$w_0^L = \sum_{n=1}^{K^L} \frac{(-h_1)^n}{n} \frac{\partial u_{n-1}^L}{\partial x}. \quad (3.9d)$$

224 There is no gap at the interface between the two fluids at any time by assumption.
225 Hence, solutions of the two layers are coupled at the interface. Therefore, we can eliminate
226 w_n^U and w_n^L by using Eqs. (3.5b), (3.5d), (3.9b) and (3.9d). As a result, the unknowns are
227 η_2 , η_1 , u_n^U ($n = 0, 1, \dots, K^U - 1$) and u_n^L ($n = 0, 1, \dots, K^L - 1$). The number of unknowns
228 is $K^U + K^L + 2$. On the other hand, the number of equations, including Eqs. (3.6), (3.9a)
229 and (3.9c), is also $K^U + K^L + 2$. Hence, the problem is closed.

230 4. Solution algorithm

231 In the HLGN-FS equations, Eq. (3.6) is expressed by

$$\mathbf{A} \dot{\xi}_{,xx} + \mathbf{B} \dot{\xi}_{,x} + \mathbf{C} \dot{\xi} = \mathbf{f}, \quad (4.1)$$

232 where \mathbf{A} , \mathbf{B} and \mathbf{C} are $(K^U + K^L) \times (K^U + K^L)$ matrices, \mathbf{f} is a $(K^U + K^L)$ vector and

$$\dot{\xi}(x, t) = [\dot{u}_0^U, \dot{u}_1^U, \dots, \dot{u}_{K^U-1}^U, \dot{u}_0^L, \dot{u}_1^L, \dots, \dot{u}_{K^L-1}^L]^T. \quad (4.2)$$

233 The dot over a variable indicates time derivative, i.e. $\dot{\xi} = \partial \xi / \partial t$, and the subscript
234 after comma is differentiation with respect to the indicated variable. \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{f} are
235 functions of $\eta_1(x, t)$, $\eta_2(x, t)$, $\xi(x, t)$ and their spatial derivatives. For simplification, this
236 dependence will not be shown here.

237 The above system of differential equations are solved by use of the spatial finite-
 238 difference discretization. The domain of x over which a solution to the equations is desired
 239 is assumed to have a uniform grid of x values, spaced a distance of Δx apart. The i^{th}
 240 point on the grid is denoted by $x_i = i\Delta x$ for $i = 1, 2, \dots, nx$. Time is discretized with
 241 intervals Δt , with $t_j = j\Delta t$. The value of the solution vector $\xi(x_i, t_j)$ will be denoted by
 242 $\xi^{(i)}$ (j is omitted in the following part, because we refer to the same j time) and similar
 243 superscripts will be used for other vectors and matrices. The spatial derivatives $\dot{\xi}_{,x}$ and
 244 $\dot{\xi}_{,xx}$ are approximated by the five-point central difference scheme as

$$\dot{\xi}_{,x}^{(i)} = \frac{1}{12\Delta x}(\dot{\xi}^{(i-2)} - 8\dot{\xi}^{(i-1)} + 8\dot{\xi}^{(i+1)} - \dot{\xi}^{(i+2)}), \quad (4.3a)$$

$$\dot{\xi}_{,xx}^{(i)} = \frac{1}{12\Delta x^2}(-\dot{\xi}^{(i-2)} + 16\dot{\xi}^{(i-1)} - 30\dot{\xi}^{(i)} + 16\dot{\xi}^{(i+1)} - \dot{\xi}^{(i+2)}). \quad (4.3b)$$

245 The five-point central difference scheme provides fourth-order accuracy for the first
 246 and second derivatives, and second-order accuracy for the third derivative. With these
 247 approximations, Eq. (4.1) can now be written as

$$\tilde{\mathbf{A}}^{(i)}\dot{\zeta}^{(i-2)} + \tilde{\mathbf{B}}^{(i)}\dot{\zeta}^{(i-1)} + \tilde{\mathbf{C}}^{(i)}\dot{\zeta}^{(i)} + \tilde{\mathbf{D}}^{(i)}\dot{\zeta}^{(i+1)} + \tilde{\mathbf{E}}^{(i)}\dot{\zeta}^{(i+2)} = \mathbf{f}^{(i)}, \quad (4.4)$$

where

$$\tilde{\mathbf{A}}^{(i)} = -\mathbf{A}^{(i)}\frac{1}{12\Delta x^2} + \mathbf{B}^{(i)}\frac{1}{12\Delta x}, \quad (4.5a)$$

$$\tilde{\mathbf{B}}^{(i)} = \mathbf{A}^{(i)}\frac{16}{12\Delta x^2} - \mathbf{B}^{(i)}\frac{8}{12\Delta x}, \quad (4.5b)$$

$$\tilde{\mathbf{C}}^{(i)} = -\mathbf{A}^{(i)}\frac{30}{12\Delta x^2} + \mathbf{C}^{(i)}, \quad (4.5c)$$

$$\tilde{\mathbf{D}}^{(i)} = \mathbf{A}^{(i)}\frac{16}{12\Delta x^2} + \mathbf{B}^{(i)}\frac{8}{12\Delta x}, \quad (4.5d)$$

$$\tilde{\mathbf{E}}^{(i)} = -\mathbf{A}^{(i)}\frac{1}{12\Delta x^2} - \mathbf{B}^{(i)}\frac{1}{12\Delta x}. \quad (4.5e)$$

248 The algorithm to solve Eq. (4.4) can be found in Zhao *et al.* (2014). The algorithm
 249 obtains the values of $\dot{u}_0^U, \dot{u}_1^U, \dots, \dot{u}_{K^U-1}^U, \dot{u}_0^L, \dot{u}_1^L, \dots, \dot{u}_{K^L-1}^L$. In addition, $\eta_1(x, t)$ and
 250 $\eta_2(x, t)$ can be calculated by means of Eqs. (3.9a) and (3.9c). We then use the fourth-
 251 order Adams predictor-corrector scheme for time marching.

252 The initial values are given by the steady solution of the internal solitary waves with a
 253 free surface provided by the HLGN-FS model. For a similar method to obtain the steady
 254 solution, see Zhao *et al.* (2016) for details.

255 As an example, we show an internal solitary wave propagating at different times in
 256 Figure 7. The parameters are: $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$ and $a/h_2 = -1.21$. From
 257 $t = 0$ to $t = 60s$, we find that the internal solitary wave propagates steadily. **Meanwhile,**
 258 **since we use the pressure continuity condition at the interface given in Eq. (3.7) (rather**
 259 **than velocity continuity), velocity jump across the interface is allowed. This, however,**
 260 **does not have any affect on the numerical simulation. Numerical instability was observed**
 261 **for the MCC-RL model, shown by Jo & Choi (2008).**

262 For comparison, in Figure 8 we plot the internal solitary wave (Figure 8(b)) and the
 263 free surface elevation (Figure 8(a)) of different times on top of each other. We observe
 264 that at $t = 0s, 20s, 40s$ and $60s$, the wave profiles show very good agreement. Thus, we
 265 have obtained an accurate solution of the internal solitary waves with a free surface.

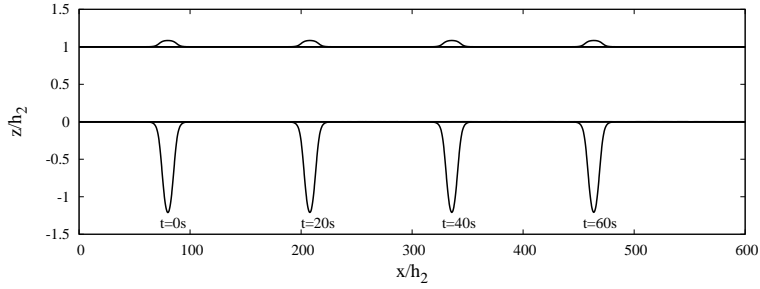
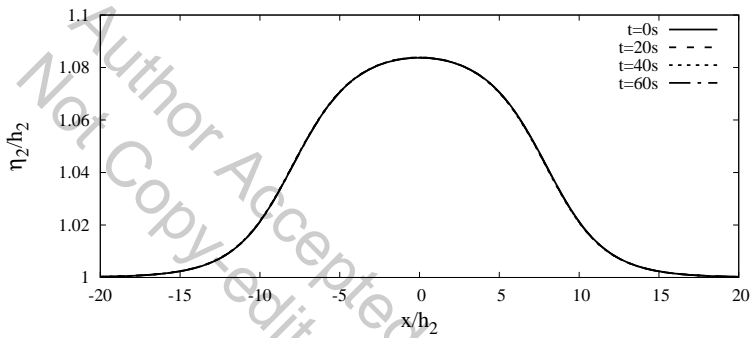
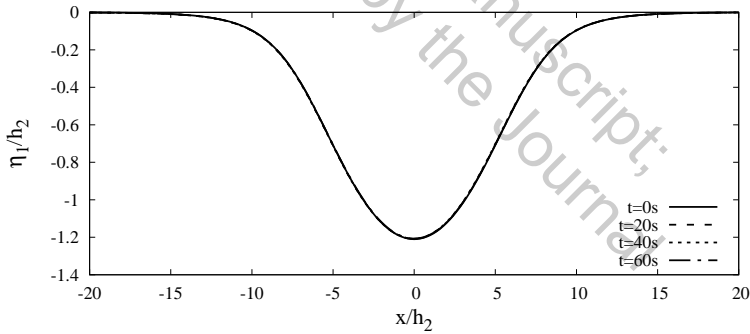


Figure 7: Snapshots of the theoretical internal solitary wave with a free surface at different times, $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$, $a/h_2 = -1.21$.



(a) Profiles of the waves on the free surface (lines are on top of each other).



(b) Profiles of internal solitary waves (lines are on top of each other).

Figure 8: Profiles of the internal solitary wave with a free surface at different times, $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$, $a/h_2 = -1.21$.

266 5. Results and discussion

267 In this section, three numerical cases are considered as shown in Table 2 and the
 268 solutions of internal solitary waves obtained by the HLGN-FS model are presented,
 269 including the wave profile, velocity field and wave speed. We note that for each case,
 270 we have performed the HLGN-FS self-convergence tests by using different K^U and K^L
 271 given in Eq. (3.4). We refer the reader to Zhao *et al.* (2014) and Zhao *et al.* (2016)

Case	Experiments by	ρ_2/ρ_1	h_2/h_1	a/h_2
1	Grue <i>et al.</i> (1999)	0.977	1/4.13	-0.36, -0.91, -1.23
2	Kodaira <i>et al.</i> (2016)	0.859	1/5	-0.50, -0.77, -1.21
3	present	0.869	1/15	-1.41, -1.91, -2.35

Table 2: Parameters of the numerical cases, and the laboratory experiments used for comparisons.

for the HLGN convergence tests for details. The results presented in this paper are the converged HLGN-FS results, which can be regarded as Euler's solution that includes the free-surface effects (Euler-FS solution).

5.1. $\rho_2/\rho_1 = 0.977, h_2/h_1 = 1/4.13$

Grue *et al.* (1999) conducted experiments on internal waves in a two-layer fluid system with a free surface. The rigid lid was set up on the free surface in their numerical simulations. Following the experiments of Grue *et al.* (1999), we select the parameters as $h_1 = 0.62m$, $h_2 = 0.15m$, $\rho_1 = 1022kg/m^3$ and $\rho_2 = 999kg/m^3$.

Three different amplitudes are considered here; they are $a/h_2 = -0.36, -0.91$ and -1.23 . We compare the converged results of the HLGN-FS model (HLGN-3-3-FS results for this case) with the results of the HLGN-RL model of Zhao *et al.* (2016) and the experimental data of Grue *et al.* (1999) as shown in Figure 9.

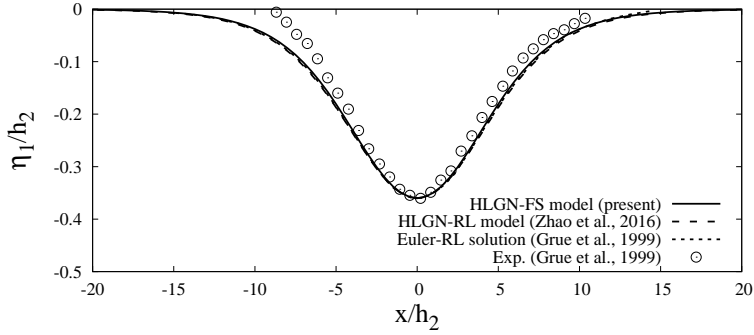
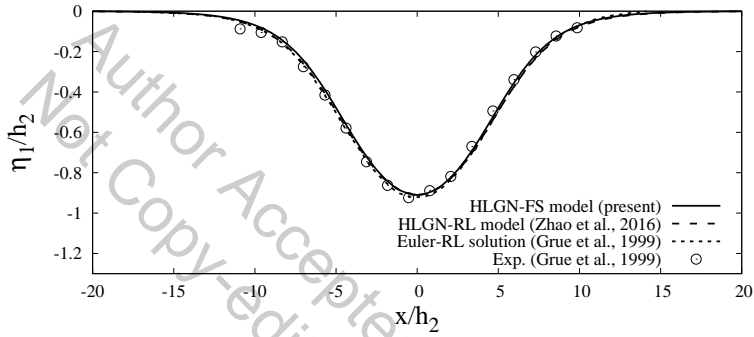
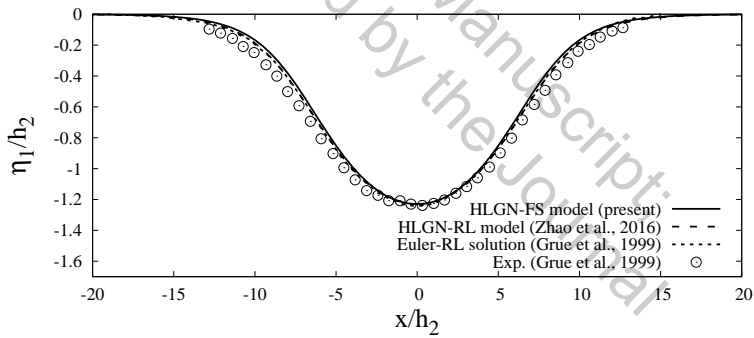
From Figure 9, we observe that the HLGN-FS results and the HLGN-RL results show very good agreement, both match Euler-RL solution and experimental data well, even for the strongly nonlinear cases.

We next show the wave profiles on the free surface obtained by the HLGN-FS model and the MCC-FS model in Figure 10. From Figure 10, we find that a surface elevation exists at the free surface in each case. For the cases of internal solitary waves of $a/h_2 = -0.36, -0.91$ and -1.23 , the amplitudes of the surface waves, b , are $b/h_2 = 0.5\%, 1.0\%$ and 1.1% , respectively. Thus, it is demonstrated here that the disturbance on the free surface is quite small for the cases of Grue *et al.* (1999). Meanwhile, we observe that the amplitudes of the surface wave predicted by the MCC-FS model show good agreement with the HLGN-FS model, while the wave profiles are slightly wider for the MCC-FS model than these of the HLGN-FS model.

Next, we focus on the horizontal velocity along the fluid column at the maximal displacement. The results of the HLGN-FS model and the results of the HLGN-RL model are shown in Figure 11, where the reference speed $c_0 = \sqrt{\frac{gh_1h_2(\rho_1-\rho_2)}{\rho_2h_1-\rho_1h_2}}$ is the linear long wave speed. We find that the horizontal velocity predicted by the HLGN-FS model shows good agreement with that predicted by the HLGN-RL model. They both match Euler-RL solution of Grue *et al.* (1999) and experimental data very well as shown in Figure 11.

The relationship between the internal solitary wave amplitude $|a|/h_2$ and exceedance wave speed $c/c_0 - 1$ obtained by the HLGN-FS model and HLGN-RL model is shown in Figure 12. Figure 12 shows that the results of the HLGN-FS model are close to the HLGN-RL results and Euler-RL solution of Grue *et al.* (1999).

In general, in the cases with $\rho_2/\rho_1 = 0.977$ and $h_2/h_1 = 1/4.13$, the results predicted by the HLGN-FS model and the HLGN-RL model agree very well with each other. This is because the mass densities between the upper-fluid layer and lower-fluid layer are very

(a) $a/h_2 = -0.36$.(b) $a/h_2 = -0.91$.(c) $a/h_2 = -1.23$.Figure 9: Profiles of internal solitary waves, $\rho_2/\rho_1 = 0.977$, $h_2/h_1 = 1/4.13$.

310 close to each other, and hence the free surface disturbance is very small. Thus, it can
 311 be regarded as a rigid lid. This conclusion has been discussed previously, see e.g. Lamb
 312 (1932) and Kodaira *et al.* (2016).

313 5.2. $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$

314 Kodaira *et al.* (2016) conducted experiments on internal waves in a two-layer fluid
 315 system. The mass densities of the lower-fluid layer and the upper-fluid layer are $\rho_1 =$
 316 996kg/m^3 and $\rho_2 = 856\text{kg/m}^3$, respectively. The depths are $h_1 = 0.25\text{m}$ and $h_2 = 0.05\text{m}$,
 317 respectively. The gravity collapse method was used to generate internal solitary waves in

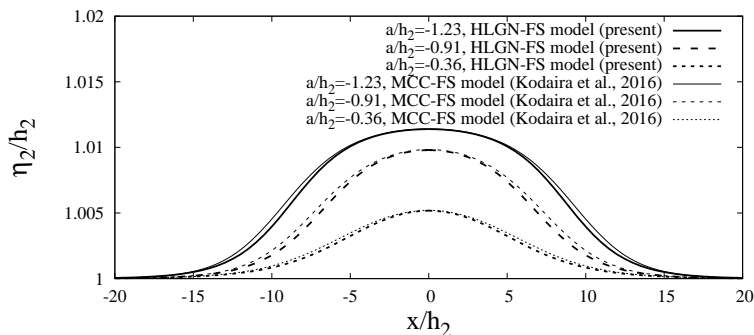


Figure 10: Profiles of waves on the free surface, $\rho_2/\rho_1 = 0.977$, $h_2/h_1 = 1/4.13$.

318 the experiments. By adjusting the interface displacement behind the gate, five internal
 319 solitary waves with different amplitudes were generated; they are: $a/h_2 = -0.24, -0.50,$
 320 $-0.77, -0.99$ and -1.21 . In the numerical study of Kodaira *et al.* (2016), the MCC-RL
 321 model and the MCC-FS model were used to obtain the wave profiles and wave speed.

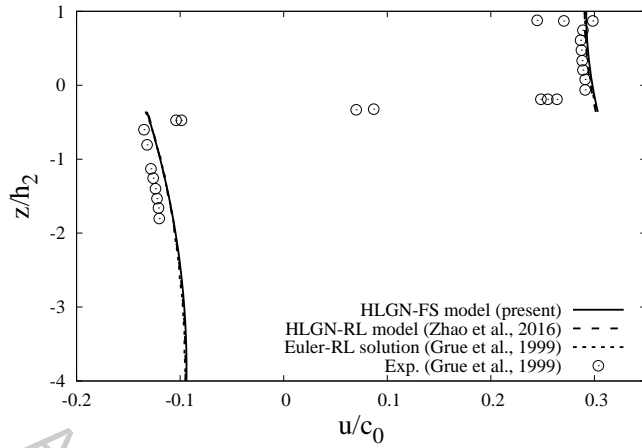
322 For simplification, we will show the results for three cases, namely the internal solitary
 323 waves with $a/h_2 = -0.50, -0.77$ and -1.21 . The wave profiles obtained by the HLGN-RL
 324 model and the HLGN-FS model are shown in Figs. 13-15, where the MCC-RL results and
 325 the MCC-FS results are also shown for comparison. The results provided by the models
 326 that are based on the rigid-lid assumption are not as accurate since the wave profiles they
 327 predict are wider than the experimental data, see Figures 13(a), 14(a) and 15(a). When
 328 we consider the free-surface effects, we find that both the converged HLGN-FS results
 329 (HLGN-3-3 results for this case) and the MCC-FS results show very good agreement
 330 with the experimental data, see Figures 13(b), 14(b) and 15(b). Thus, the effect of free
 331 surface is important in these cases.

332 Comparisons between the numerical results and the experimental data are not so good
 333 for the largest wave amplitude, $a/h_2 = -1.21$ shown in Figure 15 for $x/h_2 > 0$ part.
 334 This is similar to the case of internal solitary wave of $a/h_2 = -1.51$ in the experiments
 335 conducted by Grue *et al.* (1999). Kodaira *et al.* (2016) pointed out that this large internal
 336 solitary wave suffered from the Kelvin-Helmholtz instability excited by the large shear
 337 across the interface.

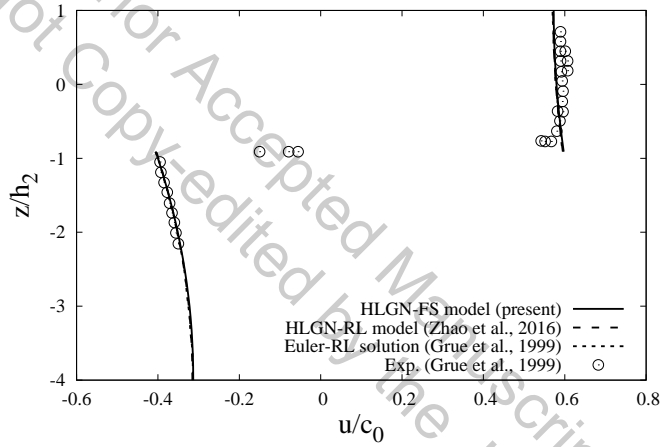
338 The wave profiles on the free surface obtained by the MCC-FS model and the HLGN-
 339 FS model for the cases of Kodaira *et al.* (2016) are shown in Figure 16. It is determined
 340 that for the cases of the internal solitary waves for $a/h_2 = -0.50, -0.77$ and -1.21 , the
 341 wave amplitudes on the free surface are $b/h_2 = 4.8\%, 6.4\%$ and 8.4% . Thus, there are
 342 obvious disturbances on the free surface, and therefore it could not be regarded as a rigid
 343 lid. Again, we find that the MCC-FS model and the HLGN-FS model predict the same
 344 amplitude of the surface wave. The profiles obtained by the MCC-FS model are wider
 345 than these obtained by the HLGN-FS model.

346 The velocity distribution along the fluid column at the maximal displacement for the
 347 three cases determined by the HLGN-FS model are shown in Figure 17. We also apply
 348 the modified velocity expression given by Camassa *et al.* (2006) for the MCC-FS model
 349 and obtain the velocity distribution shown in Figure 17 for comparison purposes. The
 350 MCC-FS results and the HLGN-FS results match very well.

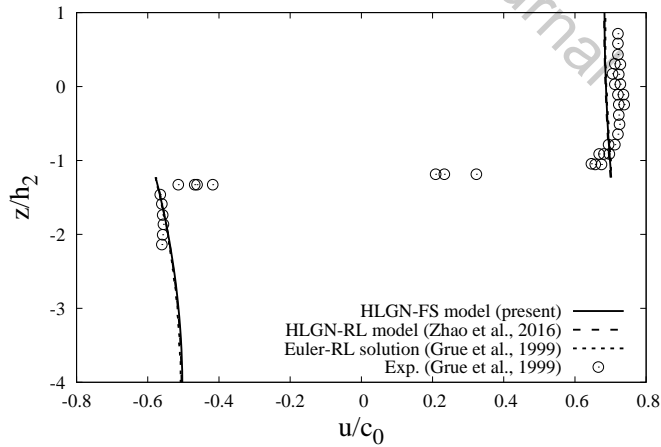
351 The relationship between the amplitude $|a|/h_2$ and the exceedance wave speed $c/c_0 - 1$



(a) $a/h_2 = -0.36$.



(b) $a/h_2 = -0.91$.



(c) $a/h_2 = -1.23$.

Figure 11: Horizontal velocity along the fluid column at maximal displacement, $\rho_2/\rho_1 = 0.977$, $h_2/h_1 = 1/4.13$.

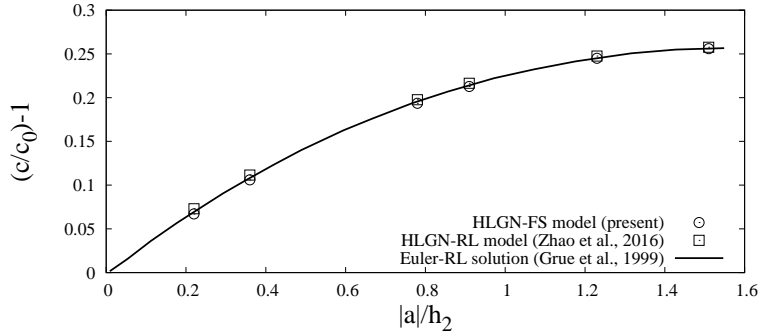
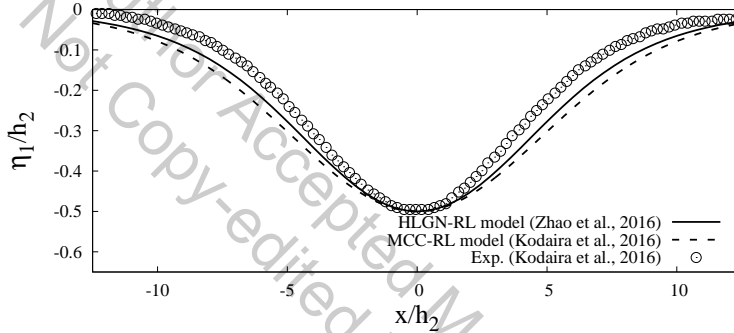
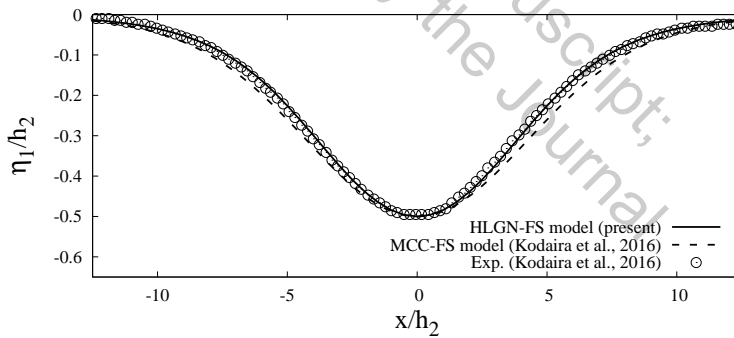


Figure 12: Excess propagation speed $(c/c_0) - 1$ versus amplitude, $\rho_2/\rho_1 = 0.977$, $h_2/h_1 = 1/4.13$.



(a) Rigid-lid assumption.

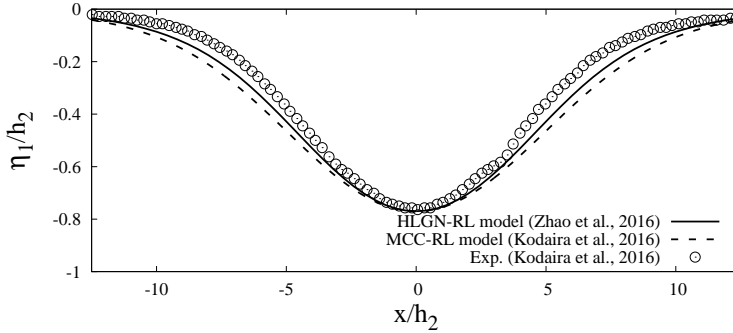


(b) With a free surface.

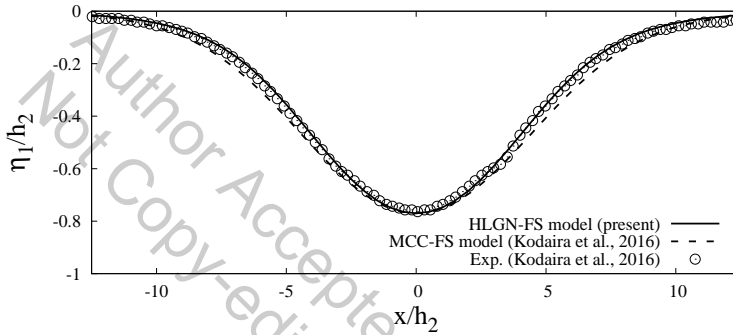
Figure 13: Profiles of internal solitary waves with $a/h_2 = -0.50$, $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$.

352 obtained by the HLG-N-FS model and the MCC-FS model is shown in Figure 18. From
 353 Figure 18, it is found that the two models are in good agreement in general.

354 In this case, we find that when the density ratio between the two fluids is not close
 355 to 1 (e.g. $\rho_2/\rho_1 = 0.859$), the rigid-lid assumption does not provide accurate description
 356 of the internal solitary waves. Instead, the models that include the free-surface effects



(a) Rigid-lid assumption.



(b) With a free surface.

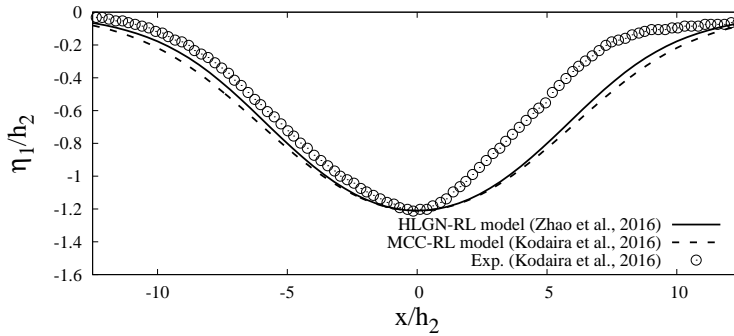
Figure 14: Profiles of internal solitary waves with $a/h_2 = -0.77$, $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$.

357 should be applied. The MCC-FS model and the HLG-FS model both describe the
 358 strongly nonlinear internal solitary waves with a free surface for the shallow configuration
 359 accurately.

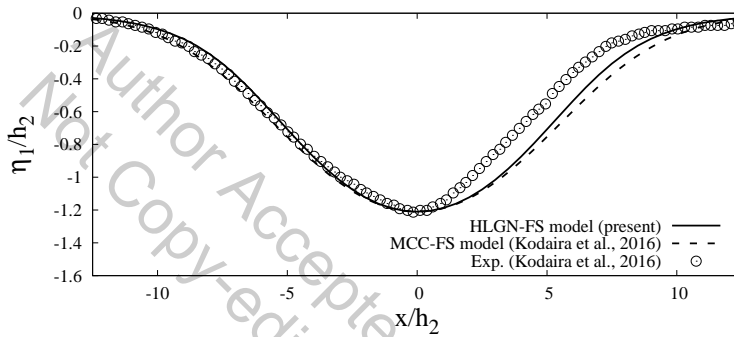
360 5.3. $\rho_2/\rho_1 = 0.869$, $h_2/h_1 = 1/15$

361 Next, we study a case where the depth ratio h_2/h_1 is significantly smaller than other
 362 cases. Here, we consider the case of $\rho_2/\rho_1 = 0.869$ and $h_2/h_1 = 1/15$. This is the same
 363 case as used in our laboratory experiments, see Table 1 for the parameters. Here, we
 364 use the HLG-FS model to calculate the strongly nonlinear internal solitary waves for
 365 deep configuration. The internal solitary wave profiles of this case are shown in Figure
 366 19. We also use the MCC-FS model in this case for comparison purposes. After having
 367 the self-convergence test of the HLG-FS model, it is determined that the HLG-5-5-FS
 368 model can provide converged HLG-FS results. We find that the converged HLG-FS
 369 results match the experimental data very well. However, the MCC-FS results show large
 370 errors compared with the experimental data.

371 We then use the MCC-FS model and the HLG-FS model to obtain the wave profiles
 372 on the free surface. Results are shown in Figure 20. We find that for the cases of $a/h_2 =$
 373 -1.41 , -1.91 and -2.35 , the wave amplitudes on the free surface are $b/h_2 = 11.6\%$, 14.4%
 374 and 16.6% obtained by the HLG-FS model. Meanwhile, for this deep-configuration case,
 375 we find that the surface-wave profiles obtained by the MCC-FS model are obviously



(a) Rigid-lid assumption.



(b) With a free surface.

Figure 15: Profiles of internal solitary waves with $a/h_2 = -1.21$, $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$.

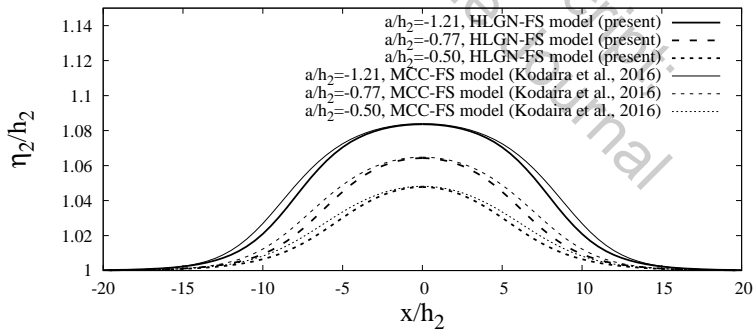
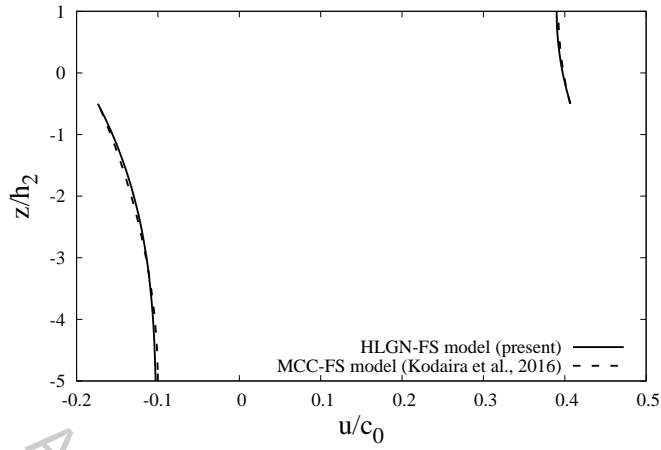


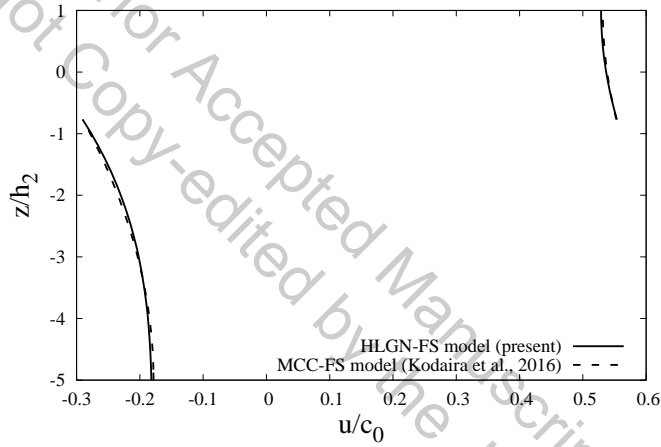
Figure 16: Wave profiles on the free surface.

376 higher and wider than these obtained by the HLGN-FS model. Meanwhile, similar to
 377 Forgia & Sciotino (2019), we measured the maximum surface elevation of the top fluid
 378 layer recorded in the laboratory experiments as shown in Figure 20. Good agreement is
 379 found between the numerical results and the experimental data.

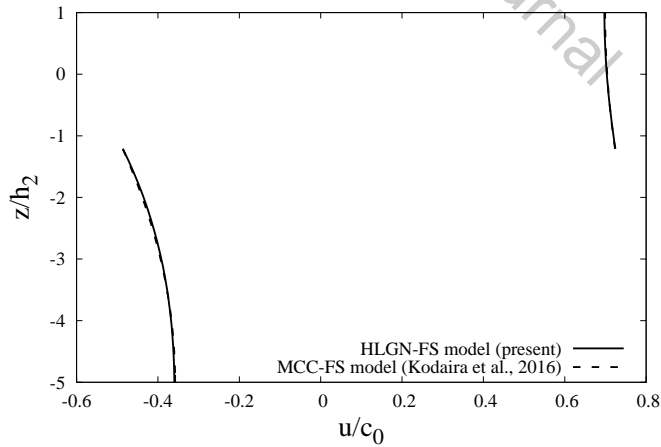
380 We note that in the experiments of Kodaira *et al.* (2016), surface disturbance is
 381 observed towards the front of the internal solitary wave, and not over the trough. This



(a) $a/h_2 = -0.50$.



(b) $a/h_2 = -0.77$.



(c) $a/h_2 = -1.21$.

Figure 17: Horizontal velocity along the fluid column at the maximal displacement, $\rho_2/\rho_1 = 0.859, h_2/h_1 = 1/5$.

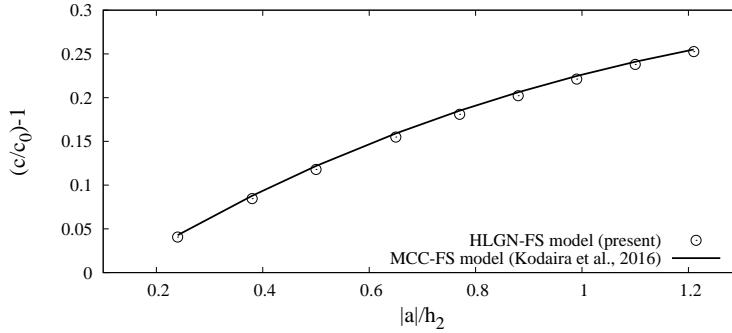


Figure 18: Excess propagation speed $(c/c_0) - 1$ versus amplitude, $\rho_2/\rho_1 = 0.859$, $h_2/h_1 = 1/5$.

surface disturbance is not observed in the present experiments. Further investigation is required to assess the existence of these surface disturbances.

The horizontal velocity along the fluid column at the maximal displacement is presented in Figure 21, where the MCC-FS results and the HLG-N-FS results are included. Some differences appear between the MCC-FS results and the HLG-N-FS results for this deep configuration case. For the upper-layer horizontal velocity, the horizontal velocity predicted by the MCC-FS model are larger than those obtained by the HLG-N-FS model. Meanwhile, for the lower-fluid layer near the maximal displacement, the horizontal velocity obtained by the MCC-FS model is larger than that obtained by the HLG-N-FS model. Near the bottom, the MCC-FS model predicts smaller horizontal velocity than that of the HLG-N-FS model.

Variations of the exceedance wave speed with the wave amplitude obtained by the HLG-N-FS model and the MCC-FS model are shown in Figure 22. The wave speed predicted by the MCC-FS model is slightly larger than that obtained by the HLG-N-FS model.

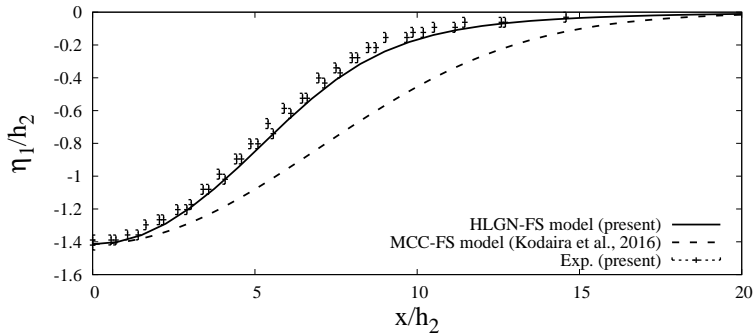
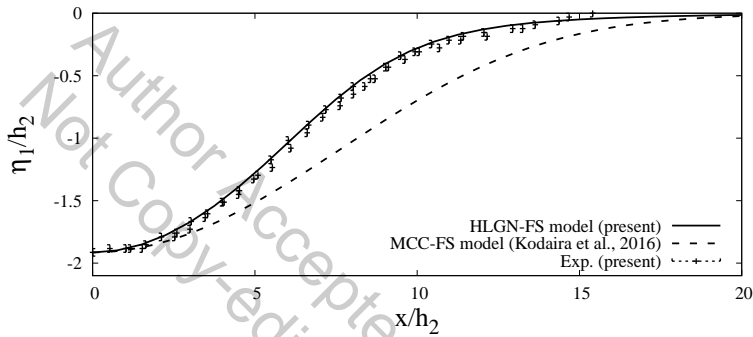
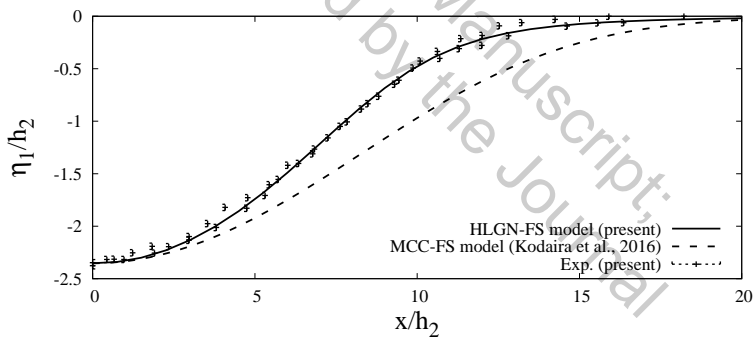
For this case, we find that the MCC-FS results do not agree well with the present experimental data. Meanwhile, the HLG-N-FS model is shown to be accurate to describe the internal solitary waves with a free surface in this case.

To further assess the differences between the HLG-N-FS and the MCC-FS results, the linear dispersion relation of the HLG-N-FS model with different levels is compared with the exact linear dispersion relation (Ten & Kashiwagi 2004), given as

$$K(k \sinh kh - K \cosh kh) + \varepsilon(K^2 - k^2) \sinh kh_1 \sinh kh_2 = 0, \quad (5.1)$$

where $K = \omega^2/g$, $h = h_1 + h_2$ and $\varepsilon = 1 - (\rho_2/\rho_1)$. The accuracy of the HLG-N-FS model with different levels is shown in Figure 23 for the case of $\rho_2/\rho_1 = 0.869$ and $h_2/h_1 = 1/15$.

Also shown in Figure 23, is the accuracy of the linear dispersion relation of the MCC-FS model. The MCC-FS and the HLG-N-1-1-FS results are on top of each other, and they predict the wave speed well for waves with $kh_1 < 3$. It is observed that higher level of the HLG-N-FS model provides better agreement with the exact results for larger values of kh_1 . The wave speed calculated by the HLG-N-5-5-FS model is within 2% error of the exact solution for waves with $kh_1 < 20$. Following the definition of half-amplitude point $\lambda_{0.5}$ by Koop & Butler (1981), and based on the present experimental results, we estimate that for the cases of $a/h_2 = -1.41, -1.91$ and -2.35 in this subsection, kh_1 is approximately 17, 14 and 13, respectively. Shown in Figure 23, the MCC-FS model

(a) $a/h_2 = -1.41$.(b) $a/h_2 = -1.91$.(c) $a/h_2 = -2.35$.Figure 19: Profiles of internal solitary waves, $\rho_2/\rho_1 = 0.869$, $h_2/h_1 = 1/15$.

414 and the HLGN-1-1-FS model do not predict accurately these internal solitary waves for
 415 deep configurations. The high level HLGN-FS models, for example HLGN-5-5-FS model,
 416 predict accurately the internal solitary waves for both shallow and deep configurations.

417 6. Conclusions

418 In this study, large-amplitude internal solitary waves in a two-layer fluid system
 419 with a free surface are investigated by use of experimental and theoretical approaches.
 420 Laboratory experiments are conducted for some strongly nonlinear, deep-configuration,

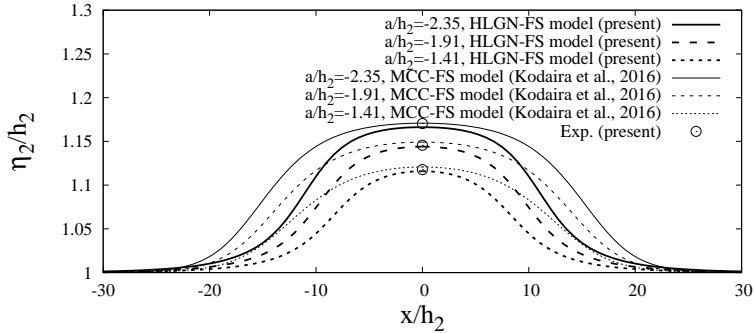


Figure 20: Wave profiles on the free surface.

421 internal solitary-wave cases. Upon comparison of existing numerical solution of the
 422 problem, including the MCC-FS model proposed by Kodaira *et al.* (2016), it was found
 423 that the existing numerical models do not provide accurate results under such conditions.
 424 This formed the motivation of developing a two-layer fluids model based on the High-
 425 Level Green-Naghdi equations, which can also include the free-surface effects, namely the
 426 HLG-N-FS model. Comparisons of the results of the HLG-N-FS model with the laboratory
 427 experiment show excellent agreement.

428 We apply the HLG-N-FS model to study the internal solitary waves for three cases, and
 429 consider the wave profile, velocity field and wave speed. The conclusions are as follows:

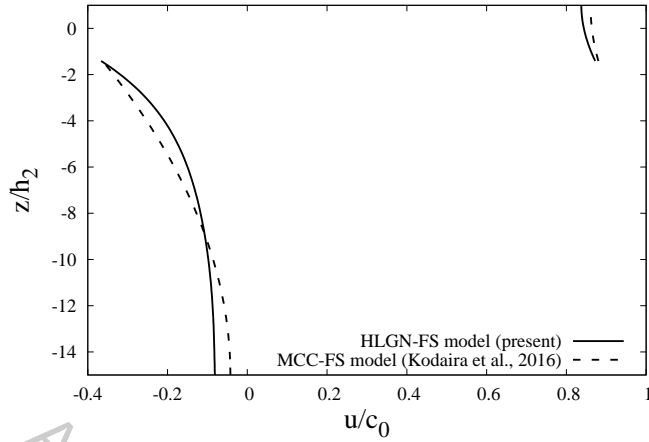
430 When the density ratio between the two fluids with constant densities is close to 1
 431 (e.g. $\rho_2/\rho_1 = 0.977$, Case 1), the rigid-lid assumption is reasonable. When the density
 432 ratio between the two fluids is not close to 1 (e.g. $\rho_2/\rho_1 = 0.859$, Case 2; $\rho_2/\rho_1 = 0.869$,
 433 Case 3), the free-surface effects should be included;

434 For the strongly nonlinear, shallow-configuration case (e.g. $\rho_2/\rho_1 = 0.859$ and $h_2/h_1 =$
 435 $1/5$, Case 2), the HLG-N-FS model and the MCC-FS model both provide accurate
 436 solutions of the internal solitary waves with a free surface;

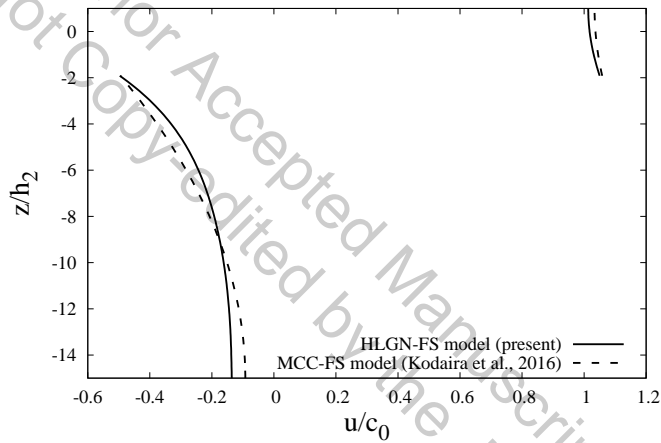
437 For the strongly nonlinear, deep-configuration case (e.g. $\rho_2/\rho_1 = 0.869$ and $h_2/h_1 =$
 438 $1/15$, Case 3), the HLG-N-FS results match the experimental data much better than those
 439 provided by the MCC-FS model. Thus, the HLG-N-FS model is shown to be accurate for
 440 describing the internal solitary waves for both shallow and deep configurations with free
 441 surface.

442 Acknowledgement

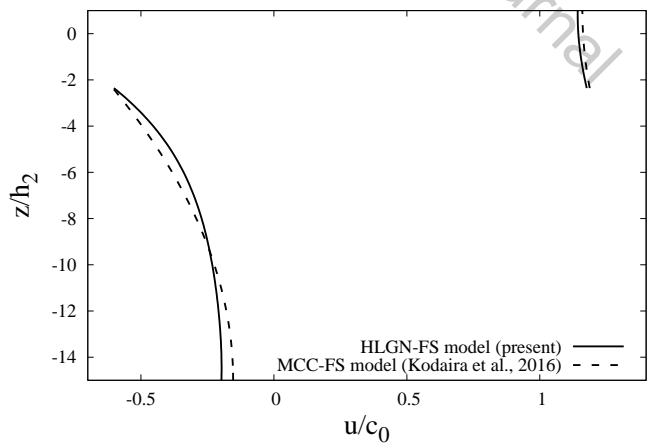
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 444 Natural Science Foundation of China (No. 51490671, 11572093), International Science
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 446 Technology of China (No. 2012DFA70420) and the Special Fund for Basic Scientific
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 448 to the anonymous referees for their comments that improved our paper.



(a) $a/h_2 = -1.41$.



(b) $a/h_2 = -1.91$.



(c) $a/h_2 = -2.35$.

Figure 21: Horizontal velocity along the fluid column at the maximal displacement, $\rho_2/\rho_1 = 0.869$, $h_2/h_1 = 1/15$.

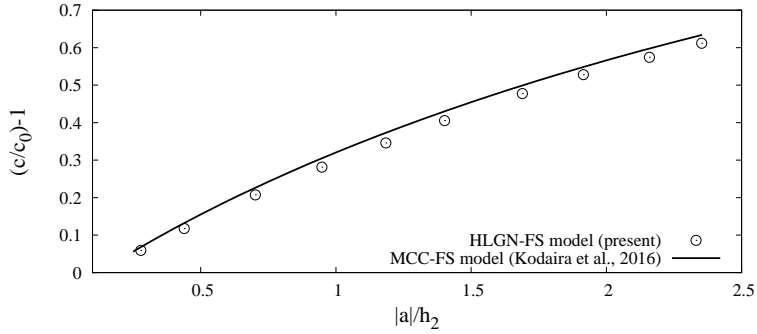


Figure 22: Excess propagation speed $(c/c_0) - 1$ versus amplitude, $\rho_2/\rho_1 = 0.869$, $h_2/h_1 = 1/15$.

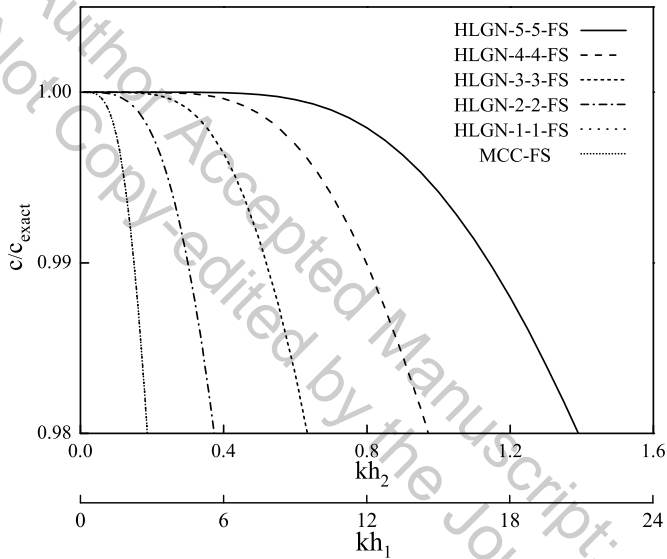


Figure 23: Accuracy of the linear dispersion relations of the HLGN-FS model and the MCC-FS model, compared with the exact linear dispersion relation (Ten & Kashiwagi 2004), $\rho_2/\rho_1 = 0.869$, $h_2/h_1 = 1/15$. The MCC-FS and the HLGN-1-1-FS results are on top of each other.

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