# Invariant manifolds of homoclinic orbits: SUPER-HOMOCLINICS AND MULTI-PULSE HOMOCLINIC LOOPS 

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I certify that this thesis and the research to which it refers are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Sajjad Bakrani Balani

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## Abstract

Consider a Hamiltonian flow on $\mathbb{R}^{4}$ with a hyperbolic equilibrium $O$ and a transverse homoclinic orbit $\Gamma$. In this thesis, we study the dynamics near $\Gamma$ in its energy level when it leaves and enters $O$ along strong unstable and strong stable directions, respectively. In particular, we provide necessary and sufficient conditions for the existence of the local stable and unstable invariant manifolds of $\Gamma$. We then consider the case in which both of these manifolds exist. We globalize them and assume they intersect transversely. We prove that near any orbit of this intersection, called super-homoclinic, there exist infinitely many multi-pulse homoclinic loops.

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## Contents

1 Introduction ..... 8
1.1 Background ..... 8
1.2 Problem setting and results ..... 10
1.2.1 Dynamics near a single homoclinic orbit ..... 11
1.2.2 Dynamics near a homoclinic figure-eight ..... 13
1.2.3 Dynamics near a super-homoclinic orbit ..... 14
1.3 Motivations and applications ..... 16
1.3.1 Coupled nonlinear Schrödinger equations ..... 16
1.3.2 Liesegang rings ..... 19
1.4 Organization of the thesis ..... 20
2 Preliminaries ..... 22
2.1 Basic concepts and definitions ..... 22
2.2 Boundary value problems ..... 23
2.3 Theory of invariant manifolds ..... 25
2.3.1 Condition of the invariance of a manifold ..... 25
2.3.2 A family of invariant manifolds ..... 26
2.3.3 Cross-forms and a theorem on the existence of invariant manifolds ..... 29
3 Analysis near the equilibrium state $O$ ..... 31
3.1 Set-up and notations ..... 31
3.2 Choice of coordinates near the equilibrium state $O$ ..... 33
3.2.1 Choice of coordinates on the cross-sections ..... 35
3.2.2 Proofs of Lemmas 3.5, 3.6 and 3.8 ..... 37
3.3 Trajectories near the equilibrium state $O$ ..... 49
3.3.1 Our computational scheme ..... 50
3.3.2 Proofs of Lemmas 3.20, 3.22 and 3.24 ..... 52
3.4 Local maps and their properties ..... 57
3.4.1 Case $\lambda_{1}=\lambda_{2}$ ..... 58
3.4.2 Case $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$ ..... 59
3.4.3 Case $2 \lambda_{1}<\lambda_{2}$ ..... 60
4 Analysis near homoclinics and super-homoclinics ..... 76
4.1 Set-up and notations ..... 76
4.2 Dynamics near the homoclinic orbit $\Gamma$ : case $\lambda_{2}<2 \lambda_{1}$ ..... 77
4.3 Dynamics near the homoclinic orbit $\Gamma$ : case $2 \lambda_{1}<\lambda_{2}$ ..... 78
4.3.1 Proof of Lemma 4.5 ..... 82
4.3.2 Proof of Lemma 4.6 ..... 88
4.4 Dynamics near the homoclinic figure-eight ..... 89
4.5 Dynamics near super-homoclinic orbits ..... 96
Bibliography ..... 98
A Nondegenerate quadratic forms ..... 102

## Chapter 1

## Introduction

### 1.1. Background

Consider a $\mathcal{C}^{\infty}$-smooth Hamiltonian system

$$
\dot{x}=X(x)
$$

defined on $\mathbb{R}^{2 n}(n \geq 2)$ with a Hamiltonian $H$, and a hyperbolic equilibrium $O$ at the origin. An orbit $\Gamma=\{x(t): t \in \mathbb{R}\}$ of this system is said to be 'homoclinic to $O$ ' or simply 'homoclinic' if it belongs to both stable and unstable invariant manifolds of $O$, or equivalently, it is bi-asymptotic to the equilibrium $O$ (i.e. $x(t) \rightarrow O$ as $t \rightarrow \pm \infty$ ). Existence of homoclinic orbits for Hamiltonian systems is known to be a robust phenomenon. This is due to the fact that the $n$-dimensional stable and unstable invariant manifolds of $O$ lie in the same $(2 n-1)$-dimensional energy level of the Hamiltonian $H$, and they may intersect transversely in that level along the homoclinic orbits. Therefore, a natural question which arises here is the possible dynamics near homoclinic orbits in their energy level.

Let

$$
\mathbb{R}^{2 n}=E^{s s} \oplus E^{s L} \oplus E^{u L} \oplus E^{u u}
$$

be the $d X(O)$ invariant splitting of $\mathbb{R}^{2 n}$ into strong stable, leading stable, leading unstable and strong unstable subspaces. Since $X$ is Hamiltonian, we have $\operatorname{dim}\left(E^{s L}\right)=\operatorname{dim}\left(E^{u L}\right)$ and $\operatorname{dim}\left(E^{s s}\right)=$ $\operatorname{dim}\left(E^{u u}\right)$. Correspondingly, the equilibrium $O$ possesses strong stable $W^{s s}(O)$, leading stable $W^{s L}(O)$, leading unstable $W^{u L}(O)$ and strong unstable $W^{u u}(O)$ invariant manifolds which are tangent to $E^{s s}$, $E^{s L}, E^{u L}$ and $E^{u u}$ at $O$, respectively. Then, a homoclinic orbit $\Gamma$ can be classified as one of the following four types:

Type 1. $\Gamma \not \subset W^{u u}(O)$ and $\Gamma \not \subset W^{s s}(O)$,
Type 2. $\Gamma \subset W^{u u}(O)$ and $\Gamma \not \subset W^{s s}(O)$,
Type 3. $\Gamma \not \subset W^{u u}(O)$ and $\Gamma \subset W^{s s}(O)$,
Type 4. $\Gamma \subset W^{u u}(O)$ and $\Gamma \subset W^{s s}(O)$.
What generically happens for a homoclinic orbit is the first scenario. This is simply because if $\Gamma$ is of any other types, then it must lie in either $W^{u u}(O)$ or $W^{s s}(O)$ which are submanifolds with positive codimension of $W^{u}(O)$ or $W^{s}(O)$, respectively. This generic case has been studied by different authors. Turaev and Shilnikov [TS89] (see also [Tur14]) considered the case $\operatorname{dim}\left(E^{s L}\right)=$ $\operatorname{dim}\left(E^{u L}\right)=1$, i.e. the leading eigenvalues are real with multiplicity 1 , and assumed that the system has finitely many homoclinic orbits of the first type. They proved that except for this bunch of homoclinics and the equilibrium $O$, any other orbit leaves a small neighborhood of these homoclinic
orbits inside their energy level for both forward and backward times. In [Dev76], Devaney studied a case of $\operatorname{dim}\left(E^{s L}\right)=\operatorname{dim}\left(E^{u L}\right)=2$. He considered a 4-dimensional Hamiltonian system $(n=2)$ with a saddle focus equilibrium $O$, i.e. the eigenvalues of $d X(O)$ are $\pm(\alpha \pm i \omega)$ for some $\alpha, \omega>0$. He showed that the set of the orbits which entirely lie in a small neighborhood of a homoclinic orbit of the first type in the energy level of $O$ can be described in terms of symbolic dynamics with countably many symbols. Further results on the homoclinics to saddle-foci have been obtained by other authors too (see [BS90], [Ler91], [Ler00], [Ler97] and [BS96]). In [ST97], Shilnikov and Turaev studied the case $\operatorname{dim}\left(E^{s L}\right)=\operatorname{dim}\left(E^{u L}\right)=2$ in which the leading eigenvalues are real with multiplicity 2 . They considered a 4 -dimensional symmetric Hamiltonian system $(n=2)$ with a saddle equilibrium $O$, i.e. the spectrum of $d X(O)$ is $\{-\lambda,-\lambda, \lambda, \lambda\}$ for $\lambda \in \mathbb{R}^{+}$, which has a pair of homoclinic figure-eights (four homoclinic orbits). To state their result, we first define:

Definition 1.1. Let $\mathcal{A}=\Gamma_{1} \cup \Gamma_{2} \cup \cdots \cup \Gamma_{m}$, where $\Gamma_{i}$ are homoclinic to an equilibrium $O$, and $m \geq 1$. Consider a sufficiently small open neighborhood $\mathcal{U}$ of $\mathcal{A} \cup\{O\}$ in the energy level of $O$. The local stable (resp. unstable) set of $\mathcal{A}$, denoted by $W_{\text {loc }}^{s}(\mathcal{A}, \mathcal{U})$ (resp. $W_{\text {loc }}^{u}(\mathcal{A}, \mathcal{U})$ ), is the union of $\mathcal{A}$ itself and the set of the points in $\mathcal{U}$ whose forward (resp. backward) orbits lie in $\mathcal{U}$ and their $\omega$-limit sets (resp. $\alpha$-limit sets) coincide with $\mathcal{A} \cup\{O\}$. We may use the notations $W_{\text {loc }}^{s}(\mathcal{A})$ and $W_{\text {loc }}^{u}(\mathcal{A})$ for the stable and unstable sets of $\mathcal{A}$ when the neighborhood $\mathcal{U}$ is clear form the context.

It was proved in [ST97] that, under certain assumptions, the local stable and unstable sets of the pair of homoclinic figure-eights (the union of four homoclinic orbits) considered in that paper are 2-dimensional smooth invariant manifolds. Moreover, any orbit outside of these two manifolds, except the equilibrium $O$, leaves a small neighborhood of the homoclinic orbits in the level set of $O$ for both forward and backward times.

The cases of homoclinic orbits of the second and the third types were studied by Turaev [Tur01]. He considered the case where the spectrum of $d X(O)$ is $\left\{\lambda_{i},-\lambda_{2},-\lambda_{1}, \lambda_{1}, \lambda_{2}, \lambda_{j}\right\}_{i, j=3 \cdots n}$ for $\lambda_{1}, \lambda_{2} \in \mathbb{R}^{+}$ and $\operatorname{Re}\left(\lambda_{i}\right)<-\lambda_{2}<-\lambda_{1}<\lambda_{1}<\lambda_{2}<\operatorname{Re}\left(\lambda_{j}\right)$. Let $\Gamma$ be a homoclinic orbit which enters $O$ along the leading direction and leaves $O$ along the direction corresponding to the eigenvalue $\lambda_{2}$. It is proved in [Tur01] that the unstable set of $\Gamma$ is an $n$-dimensional invariant manifold. On the other hand, the stable set of this orbit is trivial (coincides with $\Gamma$ itself). Moreover, any orbit outside of this manifold, except the equilibrium $O$, leaves a small neighborhood of $\Gamma$ in its level for both forward and backward times. The homoclinic orbits studied in [Tur01] are of the second type, however, one can get the analogous results for the third type homoclinics from [Tur01] by a time reversion.

In this thesis, we focus on the dynamics near homoclinic orbits of the last type, i.e. those which leave and enter $O$ along strong directions. With the setting provided latter, we describe the dynamics near (a single or a pair of) these homoclinic orbits and provide necessary and sufficient conditions for the existence of non-trivial stable and unstable sets.

A distinguishing feature of the paper of Shilnikov and Turaev [ST97] is the coexistence of both local stable and unstable manifolds of a bunch of homoclinic orbits. This feature allowed them to consider the scenario in which both these manifolds are globalized by the flow of the system and intersect each other transversely. They referred to the orbits which lie in this intersection as 'superhomoclinic'. They studied the dynamics near the super-homoclinic orbits and in particular showed that the existence of such orbits implies the existence of infinitely many multi-pulse homoclinic loops.

Apart from the work of Shilnikov and Turaev, super-homoclinic orbits as the orbits whose $\omega$ and $\alpha$-limit sets have nonempty intersection have been taken into account in other literature as well. Turaev [Tur01] studied the case of a super-homoclinic orbit whose $\omega$-limit set is an equilibrium $O$ and the $\alpha$-limit set coincides with the union of a homoclinic orbit and the equilibrium $O$. The existence of multi-pulse homoclinics, as a result of the presence of super-homoclinic orbits, was also established in that paper. While the work of Turaev is in Hamiltonian context, Homburg [Hom96] studied the same type of super-homoclinics for general systems. Eleonsky et al. [EKTS89] spotted
a super-homoclinic orbit in their numerical investigation of an electromagnetic field in a nonlinear medium. Barrientos et al. [BRR19] studied the super-homoclinics as the orbits that are homoclinic to a network of homoclinic orbits in the context of reversible systems. Chawanya and Ashwin [CA10] built an example of a heteroclinic network that possesses a super-homoclinic in the sense of an orbit which connects sub-networks. In a broader sense, as [BRR19] and [CA10] suggest, super-homoclinic orbits may potentially appear in heteroclinic networks, especially if the network undergoes a chaotic behavior (see e.g. [NADP20]).

In the setting that we provide here, the non-trivial local stable and unstable invariant manifolds of a single homoclinic orbit (as well as homoclinic figure-eight) may coexist. This enables us to consider the case in which a transverse super-homoclinic orbit exists. We prove that in such a scenario, we have infinitely many multi-pulse homoclinic loops near the super-homoclinic orbit.

### 1.2. Problem setting and results

Consider a $\mathcal{C}^{\infty}$-smooth 4-dimensional system of differential equations

$$
\begin{equation*}
\dot{x}=X(x), \quad x \in \mathbb{R}^{4} \tag{1.2.1}
\end{equation*}
$$

with a $\mathcal{C}^{\infty}{ }_{\text {-smooth }}$ first integral $H: \mathbb{R}^{4} \rightarrow \mathbb{R}$, i.e.

$$
\begin{equation*}
H^{\prime}(x) X(x) \equiv 0 \tag{1.2.2}
\end{equation*}
$$

We assume that

## Assumption 1. $X$ has a hyperbolic equilibrium state $O$ at the origin.

By (1.2.2), we have $H^{\prime}(0) X^{\prime}(0) \equiv 0$. Since $X^{\prime}(0)$ is nonsingular by Assumption 1, the linear part of $H$ at $O$ vanishes. Assume that

Assumption 2. The quadratic part of $H$ at $O$ is a nondegenerate quadratic form.
Assumptions 1 and 2 imply that system (1.2.1) near $O$ can be brought to the following form by a linear transformation:

$$
\begin{equation*}
\dot{u}=-A u+o(|u|,|v|), \quad \dot{v}=A^{T} v+o(|u|,|v|) \tag{1.2.3}
\end{equation*}
$$

where $u=\left(u_{1}, u_{2}\right) \in \mathbb{R}^{2}, v=\left(v_{1}, v_{2}\right) \in \mathbb{R}^{2}$ and $A$ is a matrix whose eigenvalues have positive real parts. Moreover, the first integral takes the form:

$$
\begin{equation*}
H=\langle v, A u\rangle+o\left(u^{2}+v^{2}\right) \tag{1.2.4}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ is the standard inner product on $\mathbb{R}^{2}$ (see Appendix A). We also assume that
Assumption 3. System (1.2.3) is invariant with respect to the symmetry

$$
\begin{equation*}
\left(u_{1}, v_{1}\right) \leftrightarrow\left(-u_{1},-v_{1}\right) \tag{1.2.5}
\end{equation*}
$$

Assumption 3 implies that the plane $\left\{u_{1}=v_{1}=0\right\}$ is invariant with respect to the flow of system (1.2.3). Since the action of this symmetry commutes with the linear part of system (1.2.3), the matrix $A$ is diagonal and takes the form

$$
A=\left(\begin{array}{cc}
\lambda_{1} & 0 \\
0 & \lambda_{2}
\end{array}\right)
$$

for some positive real numbers $\lambda_{1}$ and $\lambda_{2}$. Without loss of generality, let $\lambda_{1} \leq \lambda_{2}$. We further assume

Assumption 4. $\lambda_{2} \neq 2 \lambda_{1}$.
This is not a technical assumption. Indeed, we will see that the cases $\lambda_{2}<2 \lambda_{1}$ and $2 \lambda_{1}<\lambda_{2}$ are dynamically different.

We can always assume that $H$ is invariant with respect to symmetry (1.2.5), i.e.

$$
\begin{equation*}
H\left(-u_{1}, u_{2},-v_{1}, v_{2}\right)=H\left(u_{1}, u_{2}, v_{1}, v_{2}\right) \tag{1.2.6}
\end{equation*}
$$

Otherwise, $\tilde{H}\left(u_{1}, u_{2}, v_{1}, v_{2}\right):=\frac{1}{2}\left[H\left(u_{1}, u_{2}, v_{1}, v_{2}\right)+H\left(-u_{1}, u_{2},-v_{1}, v_{2}\right)\right]$ can be taken as the first integral.

The equilibrium state $O$ is a saddle with 2-dimensional stable and unstable invariant manifolds $W^{s}(O)$ and $W^{u}(O)$ which are tangent at $O$ to the $u$-plane and $v$-plane respectively. Both the invariant manifolds lie in the 3-dimensional level $\{H=0\}$ and may intersect transversely in that level, producing a number of homoclinic loops. We consider the following specific case:

Assumption 5. There exists a homoclinic loop $\Gamma$ of the transverse intersection of $W^{s}(O)$ and $W^{u}(O)$ in the invariant plane $\left\{u_{1}=v_{1}=0\right\}$ (see Figure 1.1).


Figure 1.1: Existence of the transverse homoclinic loop $\Gamma$ in the invariant plane $\left\{u_{1}=v_{1}=0\right\}$.
Let $\mathcal{U}$ be a sufficiently small neighborhood of $\Gamma \cup\{O\}$ in the zero level-set $\{H=0\}$. The main issue which is addressed in this thesis is giving a complete description of dynamics in $\mathcal{U}$. Recall Definition 1.1. By this definition, the local stable and unstable sets of $\Gamma$ always contain $\Gamma$. Note that since $H$ is continuous, these sets lie in the zero-level set $\{H=0\}$. Denote by $W_{\mathcal{U}}^{s}(O)$ (resp. $\left.W_{\mathcal{U}}^{u}(O)\right)$ the set of the points in $W_{\text {glo }}^{s}(O)$ (resp. $\left.W_{\text {glo }}^{u}(O)\right)$ whose forward (backward) orbits lie entirely in $\mathcal{U}$. Obviously,

$$
W_{\mathcal{U}}^{s} \cap W_{\mathrm{loc}}^{s}(\Gamma)=W_{\mathcal{U}}^{u} \cap W_{\mathrm{loc}}^{u}(\Gamma)=\Gamma
$$

### 1.2.1. Dynamics near a single homoclinic orbit

Our first result is the following:
Theorem A1. Under Assumptions 1-5, there exists an open neighborhood $\mathcal{U}$ of $\Gamma \cup\{O\}$ in the energy level of $O$ such that if the forward (or backward) orbit of a point in $\mathcal{U}$ lies entirely in $\mathcal{U}$, then it must converge either to the equilibrium $O$ or to the set $\Gamma \cup\{O\}$. In other words, the forward (resp. backward) orbit of a point in $\mathcal{U}$ lies entirely in $\mathcal{U}$ if and only if it belongs to $W_{\mathcal{U}}^{s}(O) \cup W_{\text {loc }}^{s}(\Gamma)$ (resp. $\left.W_{\mathcal{U}}^{u}(O) \cup W_{l o c}^{u}(\Gamma)\right)$.

The next theorem concerns the dynamics near the homoclinic orbit $\Gamma$ when $\lambda_{2}<2 \lambda_{1}$.

Theorem A2. If $\lambda_{2}<2 \lambda_{1}$ and Assumptions 1-5 hold, then there exists an open neighborhood $\mathcal{U}$ of $\Gamma \cup\{O\}$ in the energy level of $O$ such that every point in $\mathcal{U}$ whose forward orbit (resp. backward orbit) lies in $\mathcal{U}$ belongs to $W_{\mathcal{U}}^{s}(O)$ (resp. $W_{\mathcal{U}}^{u}(O)$ ).

In other words, according to Theorem A1, Theorem A2 states that when $\lambda_{2}<2 \lambda_{1}$, we have $W_{\text {loc }}^{s}(\Gamma)=W_{\text {loc }}^{u}(\Gamma)=\Gamma$.

The next theorem describes the shapes of the local stable and local unstable sets of $\Gamma$ when $2 \lambda_{1}<\lambda_{2}$. The formulation of this theorem is based on a specific choice of coordinates near the equilibrium $O$. We introduce this coordinate system in Chapter 3 (see normal form (3.2.9)). For now, keep in mind that in this choice of coordinates, system (1.2.3) keeps its form and its invariance with respect to symmetry (1.2.5). Moreover, the first integral takes the form

$$
\begin{equation*}
H\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=\lambda_{1} u_{1} v_{1}-\lambda_{2} u_{2} v_{2}+o\left(u^{2}+v^{2}\right), \tag{1.2.7}
\end{equation*}
$$

and still satisfies (1.2.6) (note that, by (1.2.4), the quadratic part of $H$ must be $\lambda_{1} u_{1} v_{1}+\lambda_{2} u_{2} v_{2}$, however, by changing the signs of some of the coordinates, we can always write it as in (1.2.7)). The local stable and unstable as well as local strong stable and strong unstable invariant manifolds of $O$ are straightened (i.e. $W_{\text {loc }}^{s}(O)=\left\{v_{1}=v_{2}=0\right\}, W_{\text {loc }}^{u}(O)=\left\{u_{1}=u_{2}=0\right\}, W_{\text {loc }}^{s s}(O)=\left\{u_{1}=v_{1}=\right.$ $\left.v_{2}=0\right\}, W_{\text {loc }}^{u u}(O)=\left\{u_{1}=u_{2}=v_{1}=0\right\}$ ), and the loop $\Gamma$ leaves $O$ along $v_{2}$-axis toward positive $v_{2}$ and enters $O$ along $u_{2}$-axis toward positive $u_{2}$ (see Figure 1.2).

Take a small $\delta>0$ and consider the following two small 2-dimensional cross-sections to the loop $\Gamma$ inside the level $\{H=0\}$ :

$$
\Pi^{s}=\left\{u_{2}=\delta\right\} \cap\{H=0\} \quad \text { and } \quad \Pi^{u}=\left\{v_{2}=\delta\right\} \cap\{H=0\}
$$

(see Figure 1.2). On each of the cross-sections $\Pi^{s}$ and $\Pi^{u}$, the variables $u_{2}$ and $v_{2}$ are uniquely determined by ( $u_{1}, v_{1}$ ) (see Corollary 3.15). This allows us to choose ( $u_{1}, v_{1}$ )-coordinates on each of $\Pi^{s}$ and $\Pi^{u}$.


Figure 1.2: This figure shows the positions of cross-sections $\Pi^{s}$ and $\Pi^{u}$ on the homoclinic loop $\Gamma$. The green and blue curves correspond to $T^{\text {loc }}$ and $T^{\text {glo }}$, respectively. Namely, $T^{\text {loc }}$ maps the green point on $\Pi^{s}$ to the blue point on $\Pi^{u}$ and then $T^{\text {glo }}$ maps the blue point to the red point on $\Pi^{s}$. The Poincaré map $T$ maps the green point to the red one.

Orbits which lie in $\mathcal{U}$ define a Poincaré map $T$ from a subset of $\Pi^{s}$ to $\Pi^{s}$. This map can be written as a composition of a local map $T^{\text {loc }}$ from a subset of $\Pi^{s}$ to $\Pi^{u}$ which corresponds to the flow inside the $\delta$-neighborhood of $O$, and a global map $T^{\text {glo }}$ from $\Pi^{u}$ to $\Pi^{s}$ which corresponds to the flow near
the global piece of $\Gamma$ outside the $\delta$-neighborhood of $O$, i.e. $T=T^{\text {glo }} \circ T^{\text {loc }}$ (see Figure 1.2). Since the flight time from $\Pi^{u}$ to $\Pi^{s}$ is bounded, the global map $T^{\text {glo }}$ is a diffeomorphism. Define $M^{s, u}=\Gamma \cap \Pi^{s, u}$ (note that both points correspond to $(0,0)$ in $\Pi^{s, u}$ ). The Taylor expansion of $T^{\text {glo }}$ at $M^{s}$ has the form

$$
\begin{equation*}
T^{\mathrm{glo}}\left(u_{1}, v_{1}\right)=\left(a u_{1}+b v_{1}+o\left(u_{1}, v_{1}\right), c u_{1}+d v_{1}+o\left(u_{1}, v_{1}\right)\right), \tag{1.2.8}
\end{equation*}
$$

for some $a, b, c, d \in \mathbb{R}$. We have
Theorem A3. Let $2 \lambda_{1}<\lambda_{2}$ and Assumptions 1-5 hold. Suppose that system (1.2.3) near the equilibrium $O$ is brought to form (3.2.9) and let $a, b, c$ and $d$ in (1.2.8) be all non-zero. Then there exists an open neighborhood $\mathcal{U}$ of $\Gamma \cup\{O\}$ in the energy level of $O$ such that
I. If $0<c d$, then $W_{\text {loc }}^{s}(\Gamma)=\Gamma$. If $c d<0$, then $W_{\text {loc }}^{s}(\Gamma)$ is a $\mathcal{C}^{1}$-smooth 2-dimensional invariant manifold which is tangent to $W_{\text {glo }}^{s}(O)$ at every point of $\Gamma$.
II. If bd $<0$, then $W_{\text {loc }}^{u}(\Gamma)=\Gamma$. If $0<b d$, then $W_{\text {loc }}^{u}(\Gamma)$ is a $\mathcal{C}^{1}$-smooth 2-dimensional invariant manifold which is tangent to $W_{\text {glo }}^{u}(O)$ at every point of $\Gamma$.
It follows from the above theorems that the sets $W_{\text {loc }}^{s}(\Gamma, \mathcal{U})$ and $W_{\text {loc }}^{u}(\Gamma, \mathcal{U})$ (or simply $W_{\text {loc }}^{s}(\Gamma)$ and $\left.W_{\text {loc }}^{u}(\Gamma)\right)$ are smooth manifolds.

Definition 1.2. We call the sets $W_{\text {loc }}^{s}(\Gamma)$ and $W_{\text {loc }}^{u}(\Gamma)$ local stable and local unstable invariant manifolds of $\Gamma$, respectively.

### 1.2.2. Dynamics near a homoclinic figure-eight

A counterpart scenario of the existence of a single homoclinic loop is the existence of a pair of it, i.e. a homoclinic figure-8, in the invariant plane $\left\{u_{1}=v_{1}=0\right\}$ :

Assumption 6. There exist two homoclinic loops $\Gamma_{1}$ and $\Gamma_{2}$ of transverse intersection of $W^{s}(O)$ and $W^{u}(O)$ in the invariant plane $\left\{u_{1}=v_{1}=0\right\}$ such that they leave and enter $O$ along opposite directions (see Figure 1.3).


Figure 1.3: Existence of a pair of transverse homoclinic loops $\Gamma_{1}$ and $\Gamma_{2}$ in the invariant plane $\left\{u_{1}=v_{1}=0\right\}$.
Such scenario happens generically, when the level-set $\{H=0\}$ is compact. Let $\mathcal{V}$ be a small neighborhood of $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$ in the level-set $\{H=0\}$ and denote by $W_{\mathcal{V}}^{s}\left(W_{\mathcal{V}}^{u}\right)$ the set of the points in $W_{\text {glo }}^{s}(O)\left(W_{\text {glo }}^{u}(O)\right)$ whose forward (backward) orbits lie entirely in $\mathcal{V}$. Then

Theorem B1. Under Assumptions 1-4 and Assumption 6, there exists an open neighborhood $\mathcal{V}$ of $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$ in the energy level of $O$ such that if the forward (or backward) orbit of a point in $\mathcal{V}$ lies entirely in $\mathcal{V}$, then it must converge to one (and exactly one) of the following four objects: $\Gamma_{1}, \Gamma_{2}, O$, and $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$. In other words, the forward (resp. backward) orbit of a point in $\mathcal{V}$ lies entirely in $\mathcal{V}$ if and only if it belongs to $W_{\mathcal{V}}^{s}(O) \cup W_{\text {loc }}^{s}\left(\Gamma_{1}\right) \cup W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right) \cup W_{\text {loc }}^{s}\left(\Gamma_{2}\right)$ (resp. $\left.W_{\mathcal{V}}^{u}(O) \cup W_{\text {loc }}^{u}\left(\Gamma_{1}\right) \cup W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right) \cup W_{\text {loc }}^{u}\left(\Gamma_{2}\right)\right)$.

The next two theorems are analogous to Theorems A2 and A3 for the case of homoclinic figureeight.

Theorem B2. If $\lambda_{2}<2 \lambda_{1}$, and Assumptions 1-4 and 6 hold, then there exists an open neighborhood $\mathcal{V}$ of $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$ in the energy level of $O$ such that every point in $\mathcal{V}$ whose forward orbit (resp. backward orbit) lies in $\mathcal{V}$ belongs to $W_{\mathcal{V}}^{s}(O)$ (resp. $W_{\mathcal{U}}^{u}(O)$ ).

This theorem together with Theorems B1 and A2 implies

$$
W_{\mathrm{loc}}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)=W_{\mathrm{loc}}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)=\Gamma_{1} \cup \Gamma_{2} .
$$

Suppose that the coordinate system discussed above (see (3.2.9)) is chosen near the equilibrium $O$. Consider the cross-sections $\Pi_{1}^{s}=\left\{u_{2}=\delta\right\} \cap\{H=0\}$ and $\Pi_{1}^{u}=\left\{v_{2}=\delta\right\} \cap\{H=0\}$ on $\Gamma_{1}$, and $\Pi_{2}^{s}=\left\{u_{2}=-\delta\right\} \cap\{H=0\}$ and $\Pi_{2}^{u}=\left\{v_{2}=-\delta\right\} \cap\{H=0\}$ on $\Gamma_{2}$ (see Figure 3.3). We can choose ( $u_{1}, v_{1}$ )-coordinates on each of these cross-sections (see Corollary 3.15). Let $T_{i}, T_{i}^{\text {loc }}$ and $T_{i}^{\text {glo }}$ be the associated maps along $\Gamma_{i}$, and set $M_{i}^{s, u}=\Gamma_{i} \cap \Pi_{i}^{s, u}$ for $i=1,2$, and $a_{i}, b_{i}, c_{i}$ and $d_{i}$ be the corresponding coefficients in (1.2.8).

Theorem B3. Assume $2 \lambda_{1}<\lambda_{2}$ and Assumptions $1-4$ and 6. Suppose that system (1.2.3) near the equilibrium $O$ is brought to form (3.2.9) and let $a_{i}, b_{i}, c_{i}$ and $d_{i}(i=1,2)$ be all non-zero. Then there exists an open neighborhood $\mathcal{V}$ of $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$ in the energy level of $O$ such that
(i) If $c_{1} d_{1}>0$ and $c_{2} d_{2}>0$, then $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ is a $\mathcal{C}^{1}$-smooth 2-dimensional invariant manifold which is tangent to $W_{\text {glo }}^{s}(O)$ at every point of $\Gamma_{1} \cup \Gamma_{2}$.
(ii) If $b_{1} d_{1}<0$ and $b_{2} d_{2}<0$, then $W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ is a $\mathcal{C}^{1}$-smooth 2-dimensional invariant manifold which is tangent to $W_{\text {glo }}^{u}(O)$ at every point of $\Gamma_{1} \cup \Gamma_{2}$.
(iii) Otherwise, we have $W_{l o c}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)=W_{l o c}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)=\Gamma_{1} \cup \Gamma_{2}$.

It follows from the above theorems that the sets $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}, \mathcal{V}\right)$ and $W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}, \mathcal{V}\right)$ (or simply $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ and $\left.W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)\right)$ are smooth manifolds.

Definition 1.3. We call the sets $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ and $W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ local stable and local unstable invariant manifolds of the homoclinic figure-eight $\Gamma_{1} \cup \Gamma_{2}$, respectively.

### 1.2.3. Dynamics near a super-homoclinic orbit

Coming back to the case of the single homoclinic loop $\Gamma$, we consider the case in which both $W_{\text {loc }}^{s}(\Gamma)$ and $W_{\text {loc }}^{u}(\Gamma)$ exist. Continuing these two local manifolds by the flow of the system gives the global stable and unstable invariant manifolds of $\Gamma$, denoted by $W_{\mathrm{glo}}^{s}(\Gamma)$ and $W_{\mathrm{glo}}^{u}(\Gamma)$, respectively. These manifolds lie in the 3 -dimensional level $\{H=0\}$ which means that it would be quite reasonable if we assume that they intersect transversely in that level. Any orbit at this intersection is bi-asymptotic, or in other words, homoclinic to the union of homoclinic orbit $\Gamma$ and the equilibrium $O$, i.e. it converges to $\Gamma \cup\{O\}$ as $t \rightarrow \pm \infty$. We refer to such an orbit as 'homoclinic to homoclinic' or 'super-homoclinic' orbit.


Figure 1.4: The homoclinic orbit $\Gamma$ is shown by brown color. It is homoclinic to the saddle equilibrium $O$. The super-homoclinic orbit $\mathcal{S}$ is shown by blue color. This orbit is homoclinic to $\Gamma \cup\{O\}$.

Assumption 7. There exists a super-homoclinic orbit $\mathcal{S}$ of the transverse intersection of $W_{\text {glo }}^{s}(\Gamma)$ and $W_{g l o}^{u}(\Gamma)$.

Theorem C1. Under Assumptions 1-5 and 7, there exist infinitely many multi-pulse homoclinic loops in a small neighborhood of the closure of $\mathcal{S}$.

According to Theorem B3, the stable and unstable invariant manifolds of the homoclinic figureeight may coexist. This leads us to consider the scenario of the existence of homoclinic to homoclinic figure-eight:

Assumption 8. There exists a super-homoclinic orbit $\mathcal{S}$ of the transverse intersection of $W_{\text {glo }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ and $W_{\text {glo }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)$.

ThEOREM C2. Under Assumptions 1-4, 6 and 8, there exist infinitely many multi-pulse homoclinic loops in a small neighborhood of the closure of $\mathcal{S}$.

The multi-pulse homoclinic orbits in Theorem C1 (resp. Theorem C2) refer to homoclinic orbits $\Omega=\{x(t): t \in \mathbb{R}\}$, where $x=(u, v)$ and $\lim _{t \rightarrow \pm \infty} x(t)=O$, for which there exist $t_{1}, t_{2} \in \mathbb{R}\left(t_{1}<t_{2}\right)$ such that the connected pieces $\left\{x(t): t \in\left(-\infty, t_{1}\right]\right\}$ and $\left\{x(t): t \in\left[t_{2},+\infty\right)\right\}$ of $\Omega$ lie entirely in $\mathcal{U}$ (resp. $\mathcal{V}$ ), where $\mathcal{U}($ resp. $\mathcal{V})$ is the neighborhood given by Theorem A3 (resp. B3), and intersect the cross-section $\Pi^{s}$ (resp. $\Pi_{1}^{s}$ ) at $n_{1}$ and $n_{2}$ points $\left(n_{1}, n_{2} \in \mathbb{N}\right)$, respectively, such that $n_{1}+n_{2}>2$. We call such orbits $\left(n_{1}+n_{2}-1\right)$-pulse homoclinic or simply multi-pulse homoclinic. We prove that the existence of super-homoclinic orbits implies the existence of $n$-pulse homoclinic orbits for unboundedly large $n$.

### 1.3. Motivations and applications

### 1.3.1. Coupled nonlinear Schrödinger equations

The coupled nonlinear Schrödinger equations (CNLSE) is a system of coupled nonlinear PDEs which is one of the basic models for light propagation. This equation has also various applications in engineering and different branches of physics including optics, quantum physics, biophysics, plasma physics and hydrodynamics (see e.g. [Tod18], [Waz20], [Nak00], [NPF89], [RKL99], [ZMX ${ }^{+}$07], $\left[S G Y^{+} 09\right]$ and the references there). Apart from these applications that CNLSE has, it is also an interesting equation from mathematical point of view since it appears in the study of systems near a threshold of instability (see e.g. [KSM91], [Sch98], [FP98], [Wei85], [FRW09], [Wri95], [DM08] and [Sch97]).

Consider the following formulation of the CNLSE

$$
\begin{align*}
& i \Psi_{t}+\Psi_{x x}+2\left(\alpha|\Psi|^{2}+|\Phi|^{2}\right) \Psi=0 \\
& i \Phi_{t}+\Phi_{x x}+2\left(|\Psi|^{2}+\beta|\Phi|^{2}\right) \Phi=0 \tag{1.3.1}
\end{align*}
$$

where $\alpha$ and $\beta$ are some complex constants, $i=\sqrt{-1}$ and $\Psi$ and $\Phi$ are complex-valued functions of $(t, x)$. We consider the steady-state solutions of (1.3.1) which are of the form

$$
\Psi(t, x)=e^{i \omega_{1}^{2} t} \psi(x), \quad \Phi(t, x)=e^{i \omega_{2}^{2} t} \phi(x)
$$

for some real valued functions $\psi$ and $\phi$. By a rescaling, we can assume $\omega_{1}=1$ and $\omega_{2}=\omega(\omega>0)$. Thus, the stationary solutions of CNLSE satisfy

$$
\begin{aligned}
& \psi_{x x}=\psi-2\left(\alpha \psi^{2}+\phi^{2}\right) \psi \\
& \phi_{x x}=\omega^{2} \phi-2\left(\psi^{2}+\beta \phi^{2}\right) \phi
\end{aligned}
$$

Let $\nu=\omega^{2}$ and define $\psi_{1}(x)=\psi, \psi_{2}(x)=\psi_{x}, \phi_{1}(x)=\phi$ and $\phi_{2}(x)=\phi_{x}$. We have

$$
\begin{array}{ll}
\dot{\psi}_{1}=\psi_{2}, & \dot{\psi}_{2}=\psi_{1}-2\left(\alpha \psi_{1}^{2}+\phi_{1}^{2}\right) \psi_{1} \\
\dot{\phi}_{1}=\phi_{2}, & \dot{\phi}_{2}=\nu \phi_{1}-2\left(\psi_{1}^{2}+\beta \phi_{1}^{2}\right) \phi_{1} \tag{1.3.2}
\end{array}
$$

This system is Hamiltonian with two degrees of freedom, i.e.

$$
\dot{\psi}_{1}=\frac{\partial H}{\partial \psi_{2}}, \quad \dot{\phi}_{1}=\frac{\partial H}{\partial \phi_{2}}, \quad \dot{\psi}_{2}=-\frac{\partial H}{\partial \psi_{1}}, \quad \dot{\phi}_{2}=-\frac{\partial H}{\partial \phi_{1}}
$$

where $H=\frac{1}{2}\left[\psi_{2}^{2}+\phi_{2}^{2}-\psi_{1}^{2}-\nu \phi_{1}^{2}+\alpha \psi_{1}^{4}+2 \psi_{1}^{2} \phi_{1}^{2}+\beta \phi_{1}^{4}\right]$. Making a linear change of coordinates of the form

$$
\begin{array}{ll}
u_{1} & =\psi_{2}-\psi_{1},
\end{array} u_{2}=\phi_{1}-\frac{1}{\omega} \phi_{2}, ~ 子 ~ v_{2}=\frac{1}{2}\left(\omega \phi_{1}+\phi_{2}\right), ~ l
$$

reduces system (1.3.2) to

$$
\begin{align*}
\dot{u}_{1} & =-u_{1}+E_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) \\
\dot{u}_{2} & =-\omega u_{2}+E_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) \\
\dot{v}_{1} & =+v_{1}+\frac{1}{2} E_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)  \tag{1.3.3}\\
\dot{v}_{2} & =+\omega v_{2}-\frac{\omega}{2} E_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)
\end{align*}
$$

where

$$
\begin{aligned}
& E_{1}=\frac{\alpha}{4}\left(u_{1}^{3}-6 u_{1}^{2} v_{1}+12 u_{1} v_{1}^{2}-8 v_{1}^{3}\right)+\frac{u_{1} u_{2}^{2}}{4}-\frac{u_{2}^{2} v_{1}}{2}+\frac{u_{2} v_{2} u_{1}}{\omega}-\frac{2 u_{2} v_{2} v_{1}}{\omega}+\frac{u_{1} v_{2}^{2}}{\nu}-\frac{2 v_{2}^{2} v_{1}}{\nu} \\
& E_{2}=\frac{1}{\omega}\left[\frac{\beta}{4}\left(u_{2}^{3}+\frac{6 u_{2}^{2} v_{2}}{\omega}+\frac{12 u_{2} v_{2}^{2}}{\nu}+\frac{8 v_{2}^{3}}{\omega^{3}}\right)+\frac{u_{1}^{2} u_{2}}{4}+\frac{u_{1}^{2} v_{2}}{2 \omega}-u_{1} v_{1} u_{2}-\frac{2 u_{1} v_{1} v_{2}}{\omega}+v_{1}^{2} u_{2}+\frac{2 v_{1}^{2} v_{2}}{\omega}\right]
\end{aligned}
$$

Moreover, this change of coordinates transforms Hamiltonian $H$ to

$$
H=u_{1} v_{1}-\omega u_{2} v_{2}+\frac{\alpha}{2}\left(v_{1}-\frac{u_{1}}{2}\right)^{4}+\left(v_{1}-\frac{u_{1}}{2}\right)^{2}\left(\frac{u_{2}}{2}+\frac{v_{2}}{\omega}\right)^{2}+\frac{\beta}{2}\left(\frac{u_{2}}{2}+\frac{v_{2}}{\omega}\right)^{4}
$$

System (1.3.3) meets all Assumptions 1-4. In addition, this system possesses a pair of homoclinic solutions (homoclinic figure-eight):

$$
\begin{equation*}
u_{1}(x)=0, \quad u_{2}(x)=\frac{\kappa \omega e^{\omega x}}{\sqrt{\beta} \cosh ^{2}(\omega x)}, \quad v_{1}(x)=0, \quad v_{2}(x)=\frac{\kappa \nu e^{-\omega x}}{2 \sqrt{\beta} \cosh ^{2}(\omega x)} \tag{1.3.4}
\end{equation*}
$$

for $\kappa= \pm 1$. These solutions correspond to the following solutions of the original system (1.3.1):

$$
\Psi(t, x)=0, \quad \Phi(t, x)= \pm \frac{\sqrt{\nu \beta^{-1}} e^{i \nu t}}{\cosh (\sqrt{\nu} x)}
$$

Existence of homoclinic figure-eight (1.3.4) means that Assumption 6 is met too when $\omega \geq 1$. Therefore, the dynamics near this homoclinic figure-eight in the level $\{H=0\}$ can be analyzed by Theorems B2 and B3. For $\omega<2$, Theorem B2 guarantees that both forward and backward orbits of any point close to the homoclinic figure-eight leave a small neighborhood of it (in the level $\{H=0\}$ ) unless it lies on the stable or unstable invariant manifolds of $O$. For the case of $\omega>2$, in order
to apply Theorem B3, one needs to reduce system (1.3.3) to normal form (3.2.9) and compute the coefficients $a_{i}, b_{i}, c_{i}$ and $d_{i}(i=1,2)$.

System (1.3.3) is reversible with respect to the linear involution

$$
\begin{equation*}
\left(u_{1}, u_{2}, v_{1}, v_{2}\right) \mapsto\left(2 v_{1}, \frac{2}{\omega} v_{2}, \frac{1}{2} u_{1}, \frac{\omega}{2} u_{2}\right) \tag{1.3.5}
\end{equation*}
$$

In general, a system $\dot{x}=f(x)$ on $\mathbb{R}^{n}$ is said to be reversible with respect to an involution $R$, i.e. a diffeomorphism on $\mathbb{R}^{n}$ with the property $R^{2}=\mathrm{id}$, if $d R \circ f=-f \circ R$. It is easily seen that when $x(t)$ is a solution, so does $R \circ x(-t)$. The reversibility feature of system (1.3.3) implies

Proposition 1.4. For $A=\Gamma_{1}, \Gamma_{2}, \Gamma_{1} \cup \Gamma_{2}$, the manifold $W_{\text {loc }}^{s}(A)$ is non-trivial if and only if $W_{\text {loc }}^{u}(A)$ is non-trivial.

In addition to symmetry (1.2.5), system (1.3.3) is invariant with respect to the symmetry $\left(u_{2}, v_{2}\right) \leftrightarrow$ $\left(-u_{2},-v_{2}\right)$ too. Reducing this system to normal form (3.2.9) also preserves this symmetry. This implies that the loops $\Gamma_{1}$ and $\Gamma_{2}$ are symmetric, and $a_{1}=a_{2}, b_{1}=b_{2}, c_{1}=c_{2}$ and $d_{1}=d_{2}$. This symmetric structure together with Proposition 1.4 imply

Proposition 1.5. Simultaneously, all the manifolds $W_{l o c}^{u}\left(\Gamma_{1}\right), W_{l o c}^{s}\left(\Gamma_{1}\right), W_{l o c}^{u}\left(\Gamma_{2}\right)$ and $W_{l o c}^{s}\left(\Gamma_{2}\right)$ are either non-trivial or trivial.

It follows from Theorem B3 that if $b_{1} d_{1}>0$, then the local unstable invariant manifold of each of the loops $\Gamma_{1}$ and $\Gamma_{2}$ is non-trivial, while the local unstable invariant manifold of the homoclinic figure-eight $\Gamma_{1} \cup \Gamma_{2}$ is trivial (i.e. coincides with $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$ ). In contrast, when $b_{1} d_{1}<0$, the local unstable invariant manifold of the homoclinic figure-eight is non-trivial, while the local unstable invariant manifold of each of the loops $\Gamma_{1}$ and $\Gamma_{2}$ is trivial. The same conclusion holds for the corresponding stable manifolds. This analysis together with Propositions 1.4 and 1.5 give

Proposition 1.6. Let $\omega>2$ and suppose all the coefficients $b_{1}, c_{1}$ and $d_{1}$ are non-zero. Then, one (and only one) of the following two scenarios holds:
(i) The manifolds $W_{l o c}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ and $W_{l o c}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ are non-trivial, i.e. $b_{1} d_{1}=b_{2} d_{2}<0$ and $c_{1} d_{1}=c_{2} d_{2}>0$.
(ii) All the manifolds $W_{l o c}^{u}\left(\Gamma_{1}\right), W_{l o c}^{s}\left(\Gamma_{1}\right), W_{l o c}^{u}\left(\Gamma_{2}\right)$ and $W_{l o c}^{s}\left(\Gamma_{2}\right)$ are non-trivial, i.e. $b_{1} d_{1}=b_{2} d_{2}>$ 0 and $c_{1} d_{1}=c_{2} d_{2}<0$.

To figure out which of the scenarios above happens for CNLSE, one needs to find the corresponding coefficients $a_{1}, b_{1}, c_{1}$ and $d_{1}$. This can be done numerically for any particular values of $\alpha, \beta$ and $\omega$. Regardless of what these coefficients are (provided they are non-zero), Proposition 1.6 states that there are always non-trivial local stable and unstable invariant manifolds of homoclinic orbits in the CLNSE. Globalizing these manifolds, we conjecture that they intersect transversely along some superhomoclinic orbits:

Conjecture 1.7. The coupled nonlinear Schrödinger equations given by (1.3.1) possesses transverse super-homoclinic orbits.

Intuitively, one would expect this conjecture to be true since the stable and unstable manifolds of the homoclinic loops are 2-dimensional lying in the same 3-dimensional energy level, and hence they may intersect transversely along super-homoclinic orbits. Moreover, numerical evidence (see [EKKS93], [EK96], [Yan97] and [Yan98]) points to the existence of infinitely many multi-pulse homoclinic orbits in the CNLSE. This supports our conjecture since, by Theorems C1 and C2, the existence of these multi-pulse homoclinics might be a bi-product of the existence of transverse superhomoclinic orbits.


Figure 1.5: (Left) Classical Liesegang experiment with diffusing silver ions in silicagel, 24 hours after application of solution. (Right) double Liesegang experiment. Both figures are taken from [Jac].


Figure 1.6: (Left) Liesegang patterns grown in gels, for a number of sparingly soluble salts [SAR $\left.{ }^{+} 13\right]$. (Right) Liesegang patterns appearing in the reaction of NaOH and $\mathrm{MgCl}_{2}$ in polyvinylalcohol gel [Rác99].

### 1.3.2. Liesegang rings

Liesegang rings, also known as Liesegang patterns or Liesegang bands, refer to repeating patterns of two types of zones (rings) in which one of the zones grows (see Figures 1.5 and 1.6). These patterns appear commonly in chemical systems undergoing a precipitation reaction. This phenomenon was first observed in 1896 by German chemist Raphael E. Liesegang [Lie96] when he dropped a solution of silver nitrate on a glass plate covered by a thin layer of gel containing potassium dichromate. He noticed that after a few hours, some patterns of concentric rings were formed by insoluble silver dichromate. These patterns are now named after him. Since then, both theoretical and experimental aspects of these patterns have been studied widely (see e.g. [Rác99], [PM94], [KR81], [Hen05], [Ste54], [Ste67], [DFMO17], $\left[\mathrm{RCS}^{+} 19\right],[\mathrm{DFMO19}],\left[\mathrm{NPS}^{+} 19\right]$ and [Die19]). However, the mechanism of the formation of these patterns is still unclear.

It was pointed out by Scheel [Sch09] that these patterns can be seen in systems with homoclinic solutions when there exist initial points converging to the homoclinic loops. To clarify this further, consider a system $\dot{x}=f(x)$ with an equilibrium $O$ of saddle type at the origin, and let $\Gamma=\{x(t)$ :
$t \in \mathbb{R}\}$ be a homoclinic orbit of this system. The graph of $\|x(t)\|$ as a function of $t$ is shown in Figure 1.7. Since $x(t) \rightarrow O$ as $t \rightarrow \pm \infty$, the tails of this graph decay.


Figure 1.7: The graph of $\|x(t)\|$ as a function of $t$ for a typical homoclinic orbit $\Gamma=\{x(t): t \in \mathbb{R}\}$ which is homoclinic to a saddle equilibrium.

Now, suppose that there exists an orbit $\Gamma^{*}=\left\{x^{*}(t): t \in \mathbb{R}\right\}$ close to $\Gamma$ which converges to $\Gamma \cup\{O\}$ as a set. Let $\mathcal{B}$ be a sufficiently small closed ball centered at $O$. The orbit $\Gamma^{*}$ goes along $\Gamma$, enters $\mathcal{B}$, and after a finite time $T_{1}$, it leaves $\mathcal{B}$. Then, it keeps going along $\Gamma$ until it enters and leaves $\mathcal{B}$ again. Let $T_{2}$ be the time that $\Gamma^{*}$ spends in $\mathcal{B}$ for the second time. This pattern continues and gives the sequence $T_{3}, T_{4}, \ldots$ of the times that $\Gamma^{*}$ spends in $\mathcal{B}$. Since $\Gamma^{*}$ converges to $\Gamma \cup\{O\}$ as a set, we have

$$
T_{1}<T_{2}<T_{3}<\cdots
$$

The period of time from when $\Gamma^{*}$ leaves $\mathcal{B}$ until it enters $\mathcal{B}$ again is more or less the same as the time that $\Gamma$ spends outside of $\mathcal{B}$. Denote this time by $T$. Tracking the forward orbit of a point of $\Gamma^{*}$ leads to the following sequence

$$
\begin{equation*}
T_{1}, T, T_{2}, T, T_{3}, T, T_{4}, T, T_{5}, T, \cdots \tag{1.3.6}
\end{equation*}
$$

However, this sequence resembles a Liesegang pattern: it consists of a repeating patterns of numbers $T$ and $T_{i}(i=1 \ldots \infty)$, where $T_{i}$ s grow. We can visualize this pattern by plotting $\left\|x^{*}(t)\right\|$ as a function of $t$ (see Figure 1.8). In Figure 1.8, the spikes correspond to time period that $\Gamma^{*}$ spends near $\Gamma$ outside of $\mathcal{B}$ (corresponding to $T$ s in the above sequence) and the decaying parts correspond to the time periods that $\Gamma^{*}$ spends in $\mathcal{B}$ (corresponding to $T_{i} \mathrm{~s}$ in the above sequence).


Figure 1.8: The graph of $\left\|x^{*}(t)\right\|$ as a function of $t$. Here, $\Gamma^{*}=\left\{x^{*}(t): t \in \mathbb{R}\right\}$ is an orbit which converges to $\Gamma \cup\{O\}$ as a set for a homoclinic orbit $\Gamma$. The spikes and decaying parts correspond to the time periods that $\Gamma^{*}$ spends outside of or in a small neighborhood of the origin $O$, respectively

Following the above discussion, orbits on the non-trivial local stable invariant manifolds of homoclinics to saddle equilibria produce Liesegang rings. Therefore, Theorem A3 which provides necessary and sufficient conditions for the existence of these manifolds suggests a mechanism for detecting Liesegang rings in those phenomena which are modelled by 4 -dimensional conservative ODEs.

### 1.4. Organization of the thesis

Our approach for investigating the dynamics near a homoclinic orbit $\Gamma$ is to study the Poincaré map along this orbit. In contrast to the case of a periodic orbit in which the corresponding Poincaré map is a diffeomorphism defined on an open subset of some cross-section, the Poincaré map along the homoclinic orbit $\Gamma$ is a singular map defined on a non-trivial subset of some cross-section. Denote this
cross-section by $\Sigma$. The domain of this map is non-trivial because not every orbit starting from $\Sigma$ comes back to it. Such an orbit may deviate and go along other branches of the unstable manifold of the equilibrium when it gets close to the equilibrium state (e.g. in Figure 1.2, not every orbit starting from $\Pi^{s}$ goes along $\Gamma$ and intersects $\Pi^{u}$, some of them may go along the negative branch of $v_{2}$-axis). The singularity of the Poincaré map comes from the fact that for an orbit starting from $\Sigma$, the closer it is to the homoclinic orbit, the longer it takes to pass a small neighborhood of the equilibrium and come back to $\Sigma$. These observations suggest that to understand the behavior of a Poincaré map along a homoclinic orbit, we first need to understand the local dynamics near the equilibrium state. In other words, if we consider two cross-sections $\Pi^{s}$ and $\Pi^{u}$ on $\Gamma$ (as in Section 1.2) and write the Poincaré map $T$ as the composition of a global map $T^{\text {glo }}$ and a local map $T^{\text {loc }}$ (e.g. see Figure 1.2), then our first step to study the Poincare map $T$ would be the understanding the map $T^{\text {loc }}$.

Chapter 3 is dedicated to the study of the local map $T^{\text {loc }}$. In Section 3.1, we define this map and its domain precisely. Then, in Section 3.2, we bring our system near the equilibrium state $O$ to a normal form. Notice that our system is not necessarily linearizable. Indeed, since the spectrum of the linear part of the system is $\left\{-\lambda_{2},-\lambda_{1}, \lambda_{1}, \lambda_{2}\right\}$, for some $0<\lambda_{1} \leq \lambda_{2}$, there are always resonant terms appearing in the nonlinear part of the system. Once the system is brought to a normal form, we need to investigate the behavior of the orbits near the equilibrium state $O$. This is done in Section 3.3 by solving some boundary value problems. Finally, in Section 3.4, we use this result to analyze the domain and the behavior of the local map.

In Chapter 4, we use the results of Chapter 3 to study the dynamics near homoclinic orbits. In Section 4.1, we introduce some notations. In Section 4.2, we study the dynamics near the homoclinic orbit $\Gamma$ when $\lambda_{2}<2 \lambda_{1}$. Theorems A2 is proved in this section. The dynamics near $\Gamma$ when $2 \lambda_{1}<\lambda_{2}$ is studied in Section 4.3. We prove Theorem A3 in this section. Theorem A1 is also proved in these two sections (case $\lambda_{2}<2 \lambda_{1}$ in Section 4.2 and case $2 \lambda_{1}<\lambda_{2}$ in Section 4.3). The case of homoclinic figure-eight is studied in Section 4.4. The proofs of Theorems B1, B2 and B3 are provided in this section. Finally, we discuss the case of superhomoclinics in Section 4.5. We prove Theorems C1 and C2 in this section.

## Chapter 2

## Preliminaries

### 2.1. Basic concepts and definitions

Consider a $\mathcal{C}^{r}$-smooth $(r \geq 1)$ system of differential equations

$$
\begin{equation*}
\dot{x}=f(x), \tag{2.1.1}
\end{equation*}
$$

defined on $\mathbb{R}^{n}(n \geq 2)$. Denote the flow of this system by $\phi(t, x)$.
Definition 2.1. (first integral) A scalar-valued function $H: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is called a first integral for system (2.1.1) if
(i) the restriction of $H$ to any open subset of $\mathbb{R}^{n}$ is a non-constant function, and
(ii) $H$ is constant along any orbit of system (2.1.1), i.e. For any $x_{0} \in \mathbb{R}^{n}$ we have $H\left(\phi\left(t, x_{0}\right)\right)=$ $H\left(x_{0}\right)$ for all $t \in \mathbb{R}$.

Definition 2.2. (symmetry) Let $S: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a diffeomorphism. We say $S$ is a symmetry of system (2.1.1) if for any arbitrary solution $x(t)$ of system (2.1.1), we have

$$
\begin{equation*}
\frac{d}{d t} S(x(t))=f(S(x(t))) \tag{2.1.2}
\end{equation*}
$$

We say $S$ is a linear symmetry of system (2.1.1) if it is a linear map and satisfies (2.1.2). Moreover, if system (2.1.1) has a first integral $H$, we say $H$ is invariant with respect to symmetry $S$ if $H(S(x))=$ $H(x)$.

A change of coordinates of the form $\widetilde{x}=h(x)$ reduces system (2.1.1) to

$$
\begin{equation*}
\dot{\widetilde{x}}=h^{\prime}\left(h^{-1}(\widetilde{x})\right) f\left(h^{-1}(\widetilde{x})\right) . \tag{2.1.3}
\end{equation*}
$$

Assuming system (2.1.1) has a first integral $H$, this change of coordinates transforms $H$ to $\widetilde{H}(\widetilde{x})=$ $H\left(h^{-1}(\widetilde{x})\right)$. It is easy to see that

Proposition 2.3. If $S$ is a symmetry of system (2.1.1), and the diffeomorphism $h$ commutes with $S$, i.e. $h(S(x))=S(h(x))$, then $S$ is a symmetry of system (2.1.3) as well. Moreover, if system (2.1.1) has a first integral $H$ that is invariant with respect to $S$, then first integral $\widetilde{H}$ is invariant with respect to $S$ too.

Definition 2.4. ( $\omega$ - and $\alpha$-limit points and sets) Consider the orbit $\Lambda=\left\{\phi\left(t, x_{0}\right): t \in \mathbb{R}\right\}$ of system (2.1.1) for some $x_{0} \in \mathbb{R}^{n}$.
(i) A point $p \in \mathbb{R}^{n}$ is an $\omega$-limit point of $\Lambda$ if there exists a sequence $\left\{t_{k}\right\}_{k=1}^{\infty} \subset \mathbb{R}$ such that $t_{k} \rightarrow \infty$ as $k \rightarrow \infty$ and $\lim _{k \rightarrow \infty} \phi\left(t_{k}, x_{0}\right)=p$.
(ii) A point $p \in \mathbb{R}^{n}$ is an $\alpha$-limit point of $\Lambda$ if there exists a sequence $\left\{t_{k}\right\}_{k=1}^{\infty} \subset \mathbb{R}$ such that $t_{k} \rightarrow-\infty$ as $k \rightarrow \infty$ and $\lim _{k \rightarrow \infty} \phi\left(t_{k}, x_{0}\right)=p$.
(iii) The set of all $\omega$-limit points of $\Lambda$ is called $\omega$-limit set of $\Lambda$ and is denoted by $\omega(\Gamma)$.
(iv) The set of all $\alpha$-limit points of $\Lambda$ is called $\alpha$-limit set of $\Lambda$ and is denoted by $\alpha(\Gamma)$.

Definition 2.5. (homoclinic orbit) Let $O$ be an equilibrium point of system (2.1.1) and suppose $\Gamma=\left\{\phi\left(t, x_{0}\right): t \in \mathbb{R}\right\}$ for some $x_{0} \in \mathbb{R}^{n}$ is an orbit of this system. The orbit $\Gamma$ is called 'homoclonic to $O^{\prime}$ or simply 'homoclinic' if $\lim _{t \rightarrow \infty} \phi\left(t, x_{0}\right)=\lim _{t \rightarrow-\infty} \phi\left(t, x_{0}\right)=O$.

We can think of the concept of homoclinic orbits in a more general context:
Definition 2.6. (homoclinic to a non-empty set) Let $\mathcal{A}$ be a non-empty closed subset of $\mathbb{R}^{n}$. We say the orbit $\Gamma=\{x(t): t \in \mathbb{R}\}$ of system (2.1.1) is 'homoclinic to $\mathcal{A}$ ' if it converges to the set $\mathcal{A}$ as $t \rightarrow \pm \infty$.

Replacing the set $\mathcal{A}$ in the above definition by the union of a homoclinic orbit and its associated equilibrium (or a homoclinic figure-eight with its associated equilibrium) gives an alternative definition for super-homoclinic orbits which were introduced earlier.

### 2.2. Boundary value problems

Consider a $\mathcal{C}^{r}(r \geq 1)$ system of differential equations

$$
\begin{align*}
& \dot{x}=A x+f(x, y)  \tag{2.2.1}\\
& \dot{y}=B y+g(x, y)
\end{align*}, \quad x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}(n, m \geq 1)
$$

where $f$ and $g$ as well as their first derivatives vanish at the origin, and the eigenvalues of $A$ and $B$ have negative and positive real parts, respectively. In this section, we address the following boundary value problem:

$$
\begin{align*}
& \text { for given } \tau \geq 0, x_{0} \in \mathbb{R}^{n} \text { and } y_{1} \in \mathbb{R}^{m} \text {, does there exist any solution } \\
& (x(t), y(t)) \text { of system (2.2.1) such that } \\
& (2.2 .2) \quad x(0)=x_{0} \text { and } y(\tau)=y_{1} \text { ? } \tag{2.2.2}
\end{align*}
$$

The following theorem gives an affirmative answer to this boundary value problem:
Theorem 2.7 ([SSTC98], Theorems 2.9 and 2.10). Let $\epsilon>0$ be sufficiently small and assume $\max \left\{\left\|x_{0}\right\|,\left\|y_{1}\right\|\right\}<\epsilon$. Then, there exists a unique solution $\varphi\left(t, \tau, x_{0}, y_{1}\right)$ to the above boundary value problem. Moreover, this solution depends $\mathcal{C}^{r}$-smoothly on $\left(x_{0}, y_{1}, t, \tau\right)$.

Here, we only provide a sketch of the proof of this theorem and refer the reader to [SSTC98] for further detail.

Sketch of proof. Observe that $\varphi\left(t, \tau, x_{0}, y_{1}\right)=(x(t), y(t))$ is a solution to the boundary value problem if and only if it satisfies the following integral equations:

$$
\begin{align*}
& x(t)=e^{A t} x_{0}+\int_{0}^{t} e^{A(t-s)} f(x(s), y(s)) d s  \tag{2.2.3}\\
& y(t)=e^{-B(\tau-t)} y_{1}-\int_{t}^{\tau} e^{-B(s-t)} g(x(s), y(s)) d s
\end{align*}
$$

Consider the sequence $\left\{\left(x^{(n)}(t), y^{(n)}(t)\right)\right\}_{n \geq 1}$ defined for $t \in[0, \tau]$, where

$$
\begin{equation*}
\left(x^{(1)}(t), y^{(1)}(t)\right)=\left(e^{A t} x_{0}, e^{-B(\tau-t)} y_{1}\right) \tag{2.2.4}
\end{equation*}
$$

and

$$
\begin{align*}
& x^{(n+1)}(t)=e^{A t} x_{0}+\int_{0}^{t} e^{A(t-s)} f\left(x^{(n)}(s), y^{(n)}(s)\right) d s  \tag{2.2.5}\\
& y^{(n+1)}(t)=e^{-B(\tau-t)} y_{1}-\int_{t}^{\tau} e^{-B(s-t)} g\left(x^{(n)}(s), y^{(n)}(s)\right) d s
\end{align*}
$$

It is shown in [SSTC98] that this sequence converges uniformly to some limit function $\left(x^{*}(t), y^{*}(t)\right)$. The main idea here is to show that

$$
\begin{equation*}
\max _{0 \leq t \leq \tau}\left\|x^{(n+1)}(t)-x^{(n)}(t), y^{(n+1)}(t)-y^{(n)}(t)\right\| \leq \frac{1}{2} \max _{0 \leq s \leq \tau}\left\|x^{(n)}(s)-x^{(n-1)}(s), y^{(n)}(s)-y^{(n-1)}(s)\right\| \tag{2.2.6}
\end{equation*}
$$

where $\|x, y\|$ denotes $\max \{\|x\|,\|y\|\}$, holds for any $n \geq 2$. By virtue of this relation, we have that the series

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(x^{(n+1)}(t)-x^{(n)}(t), y^{(n+1)}(t)-y^{(n)}(t)\right) \tag{2.2.7}
\end{equation*}
$$

is majorized by a geometric progression with the coefficient $\frac{1}{2}$. This proves the uniform convergence of the sequence $\left(x^{(n)}(t), y^{(n)}(t)\right)$.

By taking the limit $n \rightarrow \infty$ in (2.2.5), it is seen that $\left(x^{*}(t), y^{*}(t)\right)$ is a solution of (2.2.3), i.e. a solution of the boundary value problem. Moreover, since the convergence is uniform, the dependence of $\left(x^{*}(t), y^{*}(t)\right)$ on $\left(x_{0}, y_{1}, \tau\right)$ is continuous. To prove the uniqueness of this solution, it is remarked in [SSTC98] that if $\left(x^{* *}(t), y^{* *}(t)\right)$ is another solution of (2.2.3), then the same procedure which yields (2.2.6) also gives

$$
\begin{equation*}
\max _{0 \leq t \leq \tau}\left\|x^{* *}(t)-x^{*}(t), y^{* *}(t)-y^{*}(t)\right\| \leq \frac{1}{2} \max _{0 \leq s \leq \tau}\left\|x^{* *}(s)-x^{*}(s), y^{* *}(s)-y^{*}(s)\right\| \tag{2.2.8}
\end{equation*}
$$

However, this immediately implies $x^{* *}=x^{*}$ and $y^{* *}=y^{*}$, as desired.
To finish the proof, one needs to show that $\left(x^{*}(t), y^{*}(t)\right)$ depends $\mathcal{C}^{r}$-smoothly on $\left(x_{0}, y_{1}, t, \tau\right)$. Let $y_{0}=y^{*}(0)$. The orbit $\left(x^{*}, y^{*}\right)$ depends $\mathcal{C}^{r}$-smoothly on $\left(x_{0}, y_{0}, t, \tau\right)$. Therefore, we are done once we show that $y_{0}$ depends $\mathcal{C}^{r}$-smoothly on $\left(x_{0}, y_{1}, t, \tau\right)$. Since $y_{1}=y^{*}(t=\tau)$ is a $\mathcal{C}^{r}$-smooth function of $\left(x_{0}, y_{0}, t, \tau\right)$, it is sufficient to prove that $\frac{\partial y_{1}}{\partial y_{0}}=\left.\frac{\partial y^{*}}{\partial y_{0}}\right|_{t=\tau}$ is invertible.

Define $X=\frac{\partial x^{*}}{\partial y_{0}}$ and $Y=\frac{\partial y^{*}}{\partial y_{0}}$. Then, $(X(t), Y(t))$ is the solution of the variational equations

$$
\begin{align*}
\dot{X} & =A X+f_{x}\left(x^{*}(t), y^{*}(t)\right) X+f_{y}\left(x^{*}(t), y^{*}(t)\right) Y \\
\dot{Y} & =B Y+g_{x}\left(x^{*}(t), y^{*}(t)\right) X+g_{y}\left(x^{*}(t), y^{*}(t)\right) Y \tag{2.2.9}
\end{align*}
$$

with the initial conditions $X(0)=0$ and $Y(0)=I_{m}$. The invertibility of $\frac{\partial y_{1}}{\partial y_{0}}=Y(\tau)$ is equivalent to the existence of a matrix $Q$ such that $Y(\tau) Q=I_{m}$. It is easily seen that $\widetilde{X}=X Q$ and $\widetilde{Y}=Y Q$ satisfy (2.2.9) with the boundary conditions

$$
\begin{equation*}
\widetilde{X}(0)=0, \quad \widetilde{Y}(\tau)=I_{m} \tag{2.2.10}
\end{equation*}
$$

Thus, $\frac{\partial y_{1}}{\partial y_{0}}=Y(\tau)$ is invertible if and only if this boundary value problem has a solution. However, as it is remarked in [SSTC98], the same proof for the existence of the solution of boundary value problem (2.2.2) also shows that boundary value problem (2.2.10) has a unique solution. This ends the sketch of proof of Theorem 2.7.

### 2.3. Theory of invariant manifolds

In this section, we briefly discuss some materials from the theory of invariant manifolds that we need throughout this thesis. Most of the statements in this section are either without any proof or only a sketch of proof is provided, and instead we refer the reader to [KH95], [SSTC98] for further details.

We start with definitions of different notions of the invariance of a set. Let $\mathcal{A}$ be a non-empty subset of $\mathbb{R}^{n}$. Then

Definition 2.8. (positively invariance) We say $\mathcal{A}$ is positively (or forward) invariant with respect to the flow of system (2.1.1) if $\phi(t, \mathcal{A}) \subset \mathcal{A}$ for all $t \geq 0$.

Definition 2.9. (negatively invariance) We say $\mathcal{A}$ is negatively (or backward) invariant with respect to the flow of system (2.1.1) if $\phi(t, \mathcal{A}) \subset \mathcal{A}$ for all $t \leq 0$.

Definition 2.10. (invariance) We say $\mathcal{A}$ is invariant with respect to the flow of system (2.1.1) if it is both positively and negatively invariant, i.e. $\phi(t, \mathcal{A}) \subset \mathcal{A}$ for all $t \in \mathbb{R}$.

Definition 2.11. (local invariance) We say $\mathcal{A}$ is locally invariant with respect to the flow of system (2.1.1) if there exists an open neighborhood $\mathcal{E}$ of $\mathcal{A}$ such that for any $x_{0} \in \mathcal{A}$ the property $\phi\left(t, x_{0}\right) \in \mathcal{E}$ implies $\phi\left(t, x_{0}\right) \in \mathcal{A}$.

In this thesis, we are mainly interested in the case which the set $\mathcal{A}$ has the structure of a smooth manifold.

A useful procedure that we use several times in this thesis to simplify our systems is straightening an invariant manifold. Consider the system

$$
\begin{align*}
& \dot{x}=f(x, y) \\
& \dot{y}=g(x, y) \tag{2.3.1}
\end{align*}
$$

where $x \in \mathbb{R}^{m}, y \in \mathbb{R}^{n}$ and $f(0,0)=g(0,0)=0$. Let $\varphi: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ be a smooth mapping such that $\varphi(0)=0$ and $\varphi^{\prime}(0)=0$. Assume the manifold $\mathcal{M}=\{(x, y): y=\varphi(x)\}$ is invariant with respect to the flow of system (2.3.1).

Definition 2.12. (straightening an invariant manifold) By straightening the invariant manifold $\mathcal{M}$, we mean applying a change of coordinates of the form

$$
\begin{equation*}
\widetilde{x}=x, \quad \widetilde{y}=y-\varphi(x) \tag{2.3.2}
\end{equation*}
$$

Making this change of coordinates reduces system (2.3.1) to

$$
\begin{align*}
\dot{x} & =f(x, y+\varphi(x)) \\
\dot{y} & =g(x, y+\varphi(x))-\varphi^{\prime}(x) f(x, y+\varphi(x)) \tag{2.3.3}
\end{align*}
$$

and transforms the manifold $\mathcal{M}$ to the linear subspace $\{(x, y): y=0\}$. Straightening an invariant manifold of the type $\{(x, y): x=\varphi(y)\}$, where $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ is a smooth mapping such that $\varphi(0)=0$ and $\varphi^{\prime}(0)=0$, is defined analogously.

### 2.3.1. Condition of the invariance of a manifold

Consider system (2.3.1) and let $\varphi: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ be a smooth mapping such that $\varphi(0)=0$. Then
Proposition 2.13. The manifold $\mathcal{M}=\{(x, y): y=\varphi(x)\}$ is invariant with respect to the flow of system (2.3.1) if and only if

$$
\begin{equation*}
g(x, \varphi(x))=\varphi^{\prime}(x) \cdot f(x, \varphi(x)) \tag{2.3.4}
\end{equation*}
$$

Proof. Let $\mathcal{M}$ be invariant with respect to the flow of system (2.3.1). Consider a point $(x, y) \in \mathcal{M}$ and let $(\mathrm{x}(t), \mathrm{y}(t))$ be an orbit of system (2.3.1) such that $(\mathrm{x}(0), \mathrm{y}(0))=(x, y)$. Since $\mathcal{M}$ is invariant, we have $\mathrm{y}(t)=\varphi(\mathrm{x}(t))$ for all $t$. Differentiating this relation with respect to $t$, gives

$$
\begin{equation*}
g(\mathrm{x}(t), \varphi(\mathrm{x}(t)))=\frac{d \mathrm{y}(t)}{d t}=\varphi^{\prime}(\mathrm{x}(t)) \cdot \frac{d \mathrm{x}(t)}{d t}=\varphi^{\prime}(\mathrm{x}(t)) \cdot f(\mathrm{x}(t), \varphi(\mathrm{x}(t))) \tag{2.3.5}
\end{equation*}
$$

which holds for any $t \in \mathbb{R}$. Thus, we can get (2.3.4) by evaluating (2.3.5) at $t=0$.
Regarding the other direction, suppose (2.3.4) holds and let $V$ be the vector field defined by the right-hand side of (2.3.1). Consider $(x, y) \in \mathcal{M}$. The tangent space of $\mathcal{M}$ at $(x, y)$ is the set of the vectors $(u, v) \in \mathbb{R}^{m} \times \mathbb{R}^{n}$ that satisfy

$$
v-y=\varphi^{\prime}(x) \cdot(u-x) .
$$

By (2.3.4), the vector $(x+f(x, y), y+g(x, y))$ is tangent to $\mathcal{M}$ at the point $(x, y)$. Therefore, we can restrict the vector field $V$ to the manifold $\mathcal{M}$. Denote this restricted vector field by $\left.V\right|_{\mathcal{M}}$. The vector field $\left.V\right|_{\mathcal{M}}$ generates a unique flow on $\mathcal{M}$ (see e.g. fundamental theorem on flows in [Lee13]). Any integral curve of $\left.V\right|_{\mathcal{M}}$ starting at a point $p \in \mathcal{M}$ lies in $\mathcal{M}$ and due to the uniqueness property, must coincide with the integral curve of $V$ which starts at $p$. This means that $\mathcal{M}$ is invariant with respect to the flow generated by $V$, i.e. the flow of system (2.3.1).

A similar statement holds for manifolds of the form $x=\psi(y)$, i.e. the manifold $\mathcal{N}=\{(x, y): x=$ $\psi(y)\}$ is invariant with respect to the flow of system (2.3.1) if and only if

$$
\begin{equation*}
f(\psi(y), y)=\psi^{\prime}(y) \cdot g(\psi(y), y) . \tag{2.3.6}
\end{equation*}
$$

Definition 2.14. (condition of the invariance of a manifold) We refer to relation (2.3.4) (relation (2.3.6)) as the condition of the invariance of the manifold $\mathcal{M}$ (the manifold $\mathcal{N}$ ) with respect to the flow of system (2.3.1).

### 2.3.2. A family of invariant manifolds

In this section, we briefly discuss an important family of invariant manifolds and their relations with symmetries. Consider the system

$$
\begin{align*}
& \dot{x}=A x+f(x, y), \\
& \dot{y}=B y+g(x, y),
\end{align*} \quad f, g \in \mathcal{C}^{r}(r \geq 1)
$$

where $f$ and $g$ as well as their first derivatives vanish at the origin, and
(i) the system is defined globally on whole $\mathbb{R}^{n}\left(x \in \mathbb{R}^{k}\right.$ and $\left.y \in \mathbb{R}^{n-k}\right)$, and
(ii) there exists $\gamma \in \mathbb{R}$ such that, in the complex plane, all the eigenvalues of $A$ lie at the left side of the line $\operatorname{Re}(\cdot)=\gamma$ and all the eigenvalues of $B$ lie at the right side of this line, and
(iii) all of the derivatives of $f$ and $g$ are bounded uniformly for all $(x, y) \in \mathbb{R}^{n}$. In particular,

$$
\begin{equation*}
\left\|\frac{\partial(f, g)}{\partial(x, y)}\right\|<\xi \tag{2.3.8}
\end{equation*}
$$

for some sufficiently small constant $\xi$.
Definition 2.15. (globally dichotomic systems) Any system of form (2.3.7) that satisfies the three conditions above is called globally dichotomic.

The second condition of this definition simply states that there is a gap between the real parts of the eigenvalues of the matrices $A$ and $B$. More precisely, define

$$
\begin{equation*}
\alpha=\max \{\operatorname{Re}(\lambda): \lambda \in \operatorname{spect}(A)\} \tag{2.3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta=\min \{\operatorname{Re}(\lambda): \lambda \in \operatorname{spect}(B)\} . \tag{2.3.10}
\end{equation*}
$$

Then $\alpha<\gamma<\beta$.
Let $\phi(t, x, y)$ be the flow of system (2.3.7). Fix $\gamma \in \mathbb{R}$ and let $W_{\gamma}^{s}$ be the set of all points $\left(x_{0}, y_{0}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{n-k}$ for which there exists a constant $C>0$ such that

$$
\left\|\phi\left(t, x_{0}, y_{0}\right)\right\| \leq C e^{\gamma t}, \quad \forall t \geq 0
$$

ThEOREM 2.16. Let $\alpha \leq \gamma<\beta$, and $q$ be the largest integer such that $q \alpha<\beta$ and $q \leq r$. The set $W_{\gamma}^{s}$ is a $\mathcal{C}^{q}$-smooth manifold. In particular, it is the graph of some $\mathcal{C}^{q}$-smooth map $\varphi: \mathbb{R}^{k} \rightarrow \mathbb{R}^{n-k}$, i.e. $W_{\gamma}^{s}=\{(x, y): y=\varphi(x)\}$.

Remark 2.17. The map $\varphi$ in the above theorem is independent of $\gamma$. In other words, $W_{\gamma_{1}}^{s}=W_{\gamma_{2}}^{s}$ for any two arbitrary $\alpha \leq \gamma_{1}, \gamma_{2}<\beta$.

Analogously, for any $\gamma \in \mathbb{R}$, we define $W_{\gamma}^{u}$ to be the set of all points $\left(x_{0}, y_{0}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{n-k}$ for which there exists a constant $C>0$ such that

$$
\left\|\phi\left(t, x_{0}, y_{0}\right)\right\| \leq C e^{\gamma t}, \quad \forall t \leq 0
$$

By applying Theorems 2.16 to the system which is derived from system (2.3.7) by a reversion of time, we obtain

THEOREM 2.18. Let $\alpha<\gamma \leq \beta$, and $q$ be the largest integer such that $\alpha<q \beta$ and $q \leq r$. The set $W_{\gamma}^{u}$ is a $\mathcal{C}^{q}$-smooth manifold. In particular, it is the graph of some $\mathcal{C}^{q}$-smooth map $\psi: \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{k}$, i.e. $W_{\gamma}^{u}=\{(x, y): x=\psi(y)\}$. Moreover, $W_{\gamma_{1}}^{u}=W_{\gamma_{2}}^{u}$ for any two arbitrary $\alpha<\gamma_{1}, \gamma_{2} \leq \beta$.

A restrictive feature of the globally dichotomic systems is that they are defined globally on whole $\mathbb{R}^{n}$. However, most of the systems studied in the literature are defined locally on an open neighborhood of $\mathbb{R}^{n}$ (usually near equilibria). The following lemma is a simple method for extending a local system to the global version:

Lemma 2.19. Consider the $\mathcal{C}^{r}$-system

$$
\begin{align*}
\dot{x} & =A x+f(x, y), \\
\dot{y} & =B y+g(x, y), \tag{2.3.11}
\end{align*}
$$

where $f$ and $g$ are defined on a neighborhood $\mathcal{U}$ of the origin $O$ such that $f, g$ and $\frac{\partial(f, g)}{\partial(x, y)}$ all vanish at $O$. Then for some open ball $\mathcal{B}_{\rho}(O) \subset \mathcal{U}$ around the origin, there exist functions $\widetilde{f}$ and $\widetilde{g}$ such that they coincide with $f$ and $g$, respectively, on $\mathcal{B}_{\frac{\rho}{2}}(O)$ and the system

$$
\begin{align*}
\dot{x} & =A x+\widetilde{f}(x, y)  \tag{2.3.12}\\
\dot{y} & =B y+\widetilde{g}(x, y)
\end{align*}
$$

is globally dichotomic.

Proof. Let $\chi:[0, \infty] \rightarrow[0,1]$ be a $\mathcal{C}^{\infty}$-smooth bump function such that

$$
\chi(z)=\left\{\begin{array}{ll}
1 & \text { for } z \leq \frac{1}{2}, \\
0 & \text { for } z \geq 1,
\end{array} \quad \text { and }-3 \leq \chi^{\prime} \leq 0\right.
$$

The existence of such a function is a well-known fact (see e.g. [Lee13]). Then, for a sufficiently small $\rho$,

$$
\tilde{f}(x, y)=f\left(x \chi\left(\frac{\|(x, y)\|}{\rho}\right), y \chi\left(\frac{\|(x, y)\|}{\rho}\right)\right) \quad \text { and } \quad \tilde{g}(x, y)=g\left(x \chi\left(\frac{\|(x, y)\|}{\rho}\right), y \chi\left(\frac{\|(x, y)\|}{\rho}\right)\right)
$$

are the desired functions.
By virtue of the preceding lemma, any locally defined system coincides with a globally dichotomic system on a neighborhood of the equilibrium $O$. Therefore, restricting the invariant manifolds $W_{\gamma}^{s}$ and $W_{\gamma}^{u}$ to that neighborhood gives invariant manifolds for the local system. The following two theorems from [SSTC98] give more information on these manifolds:

THEOREM 2.20. If $\alpha<0$, system (2.3.11) has a uniquely defined $\mathcal{C}^{r}$-smooth invariant manifold which is tangent to $\{y=0\}$ at $O$ and contains all the trajectories that tend to $O$ as $t \rightarrow \infty$ at a rate faster than $e^{\gamma t}$ for any $\alpha<\gamma<0$. We call this manifold 'strong stable' and denote it by $W^{s s}$.

THEOREM 2.21. If $\alpha>0$, system (2.3.11) has a $\mathcal{C}^{q}$-smooth invariant manifold ( $q$ is the largest integer that $q \alpha<\beta$ and $q \leq r$ ) which is tangent to $\{y=0\}$ at $O$ and contains the set $\mathcal{N}^{+}$of all trajectories which stay in a small neighborhood of $O$ for all positive times. This manifold is not necessarily unique, however, any two of them have the same tangent at each point of $\mathcal{N}^{+}$. Moreover, when this manifold is written as $\{y=\varphi(x)\}$, all derivatives of $\varphi$ (up to order $q$ ) are uniquely defined at all points of $\mathcal{N}^{+}$. We call such a manifold 'extended stable' and denote it by $W^{s E}$.

Remark 2.22. Strong unstable invariant manifolds, denoted by $W^{u u}$, and extended unstable invariant manifolds, denoted by $W^{u E}$, are defined analogously.

Remark 2.23. The non-uniqueness of extended stable (and extended unstable) is a bi-product of the extension process. Indeed, regardless of how we extend a local system to a global one, the strong stable invariant manifold is the set of all orbits which converge to $O$ faster than the rate $e^{\gamma t}$ and hence is unique by definition. However, the extended stable manifold, i.e. the set of the orbits which diverge from $O$ (and possibly leave a small neighborhood of $O$ ) slower than the rate $e^{\gamma} t$, depends on how we extend the local system outside a small neighborhood of $O$.

Let $J$ be a linear symmetry of the locally defined system (2.3.11). Suppose $\mathcal{W}$ is an invariant manifold of this system of one of the four types introduced above (strong stable, strong unstable, extended stable or extended unstable). Thus, $\mathcal{W}$ is described by $\{y=\varphi(x)\}$ (stable case) or $\{x=$ $\psi(y)\}$ (unstable case) for some functions $\varphi$ or $\psi$. Then

Proposition 2.24. The maps $(x, y) \mapsto(x, \varphi(x))$ and $(x, y) \mapsto(\psi(y), y)$ commute with $J$.
Proof. Suppose that system (2.3.11) is extended to a globally dichotomic system and $\mathcal{W}$ is extended to an invariant manifold $W_{\gamma}^{s}$ of this system (the proof of the unstable case $W_{\gamma}^{u}$ is the same). Thus, $W_{\gamma}^{s}=$ $\left\{(x, y): y=\varphi^{\dagger}(x)\right\}$, where $\varphi^{\dagger}$ is a smooth mapping that coincides with $\varphi$ on a small neighborhood of $O$. It follows from the proof of Lemma 2.19 (by choosing an appropriate norm) that the global system has the symmetry $J$ too. Let $z(t)=(x(t), y(t))$ be an orbit which belongs to $W_{\gamma}^{s}$. Since $J z(t)$ is also an orbit of the system, we have

$$
\begin{equation*}
\|J z(t)\| \leq\|J\|\|z(t)\| \leq\|J\| C e^{\gamma t}, \quad \forall t \geq 0 \tag{2.3.13}
\end{equation*}
$$

Due to the uniqueness of $W_{\gamma}^{s}$, this implies $J z(t) \in W_{\gamma}^{s}$. Write $J=\left(\begin{array}{ll}J_{1} & J_{2} \\ J_{3} & J_{4}\end{array}\right)$. For any $x$, we have

$$
\left(\begin{array}{ll}
J_{1} & J_{2} \\
J_{3} & J_{4}
\end{array}\right)\binom{x}{\varphi^{\dagger}(x)}=\binom{J_{1} x+J_{2} \varphi^{\dagger}(x)}{J_{3} x+J_{4} \varphi^{\dagger}(x)} \in W_{\gamma}^{s} \Longrightarrow J_{3} x+J_{4} \varphi^{\dagger}(x)=\varphi^{\dagger}\left(J_{1} x+J_{2} \varphi^{\dagger}(x)\right)
$$

Therefore, $(x, y) \mapsto\left(x, \varphi^{\dagger}(x)\right)$ commutes with $J$ and so does $(x, y) \mapsto(x, \varphi(x))$.
Corollary 2.25. It follows from this proposition and Proposition 2.3 that $J$ is a symmetry of the system which derives from system (2.3.11) by straightening the invariant manifold $\mathcal{W}$. Moreover, if system (2.3.11) has a first integral $H$ which is invariant with respect to $J$, then straightening $\mathcal{W}$ transforms $H$ to a first integral which is invariant with respect to $J$ as well.

### 2.3.3. Cross-forms and a theorem on the existence of invariant manifolds

Definition 2.26. Let $\left(X^{*},\|\cdot\|_{X^{*}}\right)$ and $\left(Y^{*},\|\cdot\|_{Y^{*}}\right)$ be two Banach spaces, and $\mathcal{U}$ be a subset of $X^{*} \times Y^{*}$. Let

$$
\begin{align*}
& T: \mathcal{U} \rightarrow T(\mathcal{U}) \\
&(x, y) \mapsto(\bar{x}, \bar{y}) \tag{2.3.14}
\end{align*}
$$

be a map. We say $T$ can be written in cross-form if and only if

$$
\begin{align*}
\bar{x} & =F(x, \bar{y}),  \tag{2.3.15}\\
y & =G(x, \bar{y})
\end{align*}
$$

holds for some functions $F$ and $G$. The map defined by (2.3.15) (which maps $(x, \bar{y})$ to $(\bar{x}, y)$ ), is called the cross-map of $T$ and denoted by $T^{\times}$.

In general, the composition of two maps which each of them can be written in cross-form cannot necessarily be written in cross-form. Here we provide a specific setting in which the property of 'being written in cross-form' can transfer to the composition map: let $(\mathcal{B},\|\cdot\|)$ be a Banach space, and $X_{1}, X_{2}, Y_{1}$ and $Y_{2}$ be convex subsets of $\mathcal{B}$. Consider the maps $T_{1}: X_{1} \times Y_{1} \rightarrow X_{2} \times Y_{2}$ and $T_{2}: X_{2} \times Y_{2} \rightarrow X_{1} \times Y_{1}$ and suppose that both of them can be written in cross-form in the following way:

$$
\begin{equation*}
(\bar{x}, \bar{y})=T_{1}(x, y) \quad \text { if and only if } \quad \bar{x}=p_{1}(x, \bar{y}) \quad \text { and } y=q_{1}(x, \bar{y}) \tag{2.3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
(\hat{x}, \hat{y})=T_{2}(\bar{x}, \bar{y}) \quad \text { if and only if } \quad \hat{x}=p_{2}(\bar{x}, \hat{y}) \quad \text { and } \bar{y}=q_{2}(\bar{x}, \hat{y}) \tag{2.3.17}
\end{equation*}
$$

where $p_{1}: X_{1} \times Y_{2} \rightarrow X_{2}, q_{1}: X_{1} \times Y_{2} \rightarrow Y_{1}, p_{2}: X_{2} \times Y_{1} \rightarrow X_{1}$ and $q_{2}: X_{2} \times Y_{1} \rightarrow Y_{2}$ are some smooth functions. Let

$$
\begin{align*}
& \max \left\{\left\|\frac{\partial p_{1}}{\partial x}\right\|,\left\|\frac{\partial p_{1}}{\partial \bar{y}}\right\|,\left\|\frac{\partial q_{1}}{\partial x}\right\|,\left\|\frac{\partial q_{1}}{\partial \bar{y}}\right\|\right\} \leq K_{1} \\
& \max \left\{\left\|\frac{\partial p_{2}}{\partial \bar{x}}\right\|,\left\|\frac{\partial p_{2}}{\partial \hat{y}}\right\|,\left\|\frac{\partial q_{2}}{\partial \bar{x}}\right\|,\left\|\frac{\partial q_{2}}{\partial \hat{y}}\right\|\right\} \leq K_{2} \tag{2.3.18}
\end{align*}
$$

for some constants $K_{1}$ and $K_{2}$.
Lemma 2.27. ([Tur14], Lemma 4) Define $T:=T_{2} \circ T_{1}: X_{1} \times Y_{1} \rightarrow X_{1} \times Y_{1}$. If $K_{1} K_{2}<1$, then
(i) the map $T$ can be written in cross-form, i.e. there exist functions $p$ and $q$ such that

$$
\begin{equation*}
(\hat{x}, \hat{y})=T(x, y) \quad \text { if and only if } \quad \hat{x}=p(x, \hat{y}) \quad \text { and } y=q(x, \hat{y}) \tag{2.3.19}
\end{equation*}
$$

Moreover, the functions $p$ and $q$ are smooth and defined everywhere on $X_{1} \times Y_{1}$.
(ii) Equip $X_{1} \times Y_{1}$ with the norm $\|(x, y)\|_{*}=\max \left\{\sqrt{K_{1}}\|x\|, \sqrt{K_{2}}\|y\|\right\}$. We have

$$
\begin{equation*}
\left\|\frac{\partial(p, q)}{\partial(x, \hat{y})}\right\|_{*} \leq \frac{\sqrt{K_{1} K_{2}}}{1-\sqrt{K_{1} K_{2}}} \tag{2.3.20}
\end{equation*}
$$

The next theorem provides a setting in which if a map $T$ possesses a cross-map $T^{\times}$which satisfies certain properties, then it has an invariant manifold that contains $\omega$-limit points of every forward orbit of the domain. This proposition becomes powerful when one is looking for the invariant manifolds of a non-smooth map whose cross-map is smooth. This result was first obtained by Afraimovich and Shilnikov in [AS77] for maps defined on an annulus (Cartesian product of a $n$-dimensional cube and a $m$-dimensional torus $\mathbb{T}^{m}$ ). The following formulation of this result which holds for arbitrary Banach spaces is stated in [SSTC98].

THEOREM 2.28. ([SSTC98], Theorem 4.3) With the setting in Definition 2.26, let $X$ and $Y$ be two convex closed subsets of $X^{*}$ and $Y^{*}$, respectively, such that $\mathcal{R}=X \times Y \subset \mathcal{U}, T^{\times}$is defined on $\mathcal{R}$ and $T^{\times}(\mathcal{R}) \subset \mathcal{R}$. Let $F$ and $G$ in (2.3.15) be $\mathcal{C}^{1}$-smooth and satisfy

$$
\sqrt{\sup _{(x, \bar{y}) \in X \times Y}\left\{\left\|\frac{\partial F}{\partial x}\right\| \cdot\left\|\frac{\partial G}{\partial \bar{y}}\right\|\right\}}+\sqrt{\left\|\frac{\partial F}{\partial \bar{y}}\right\|_{0} \cdot\left\|\frac{\partial G}{\partial x}\right\|_{0}}<1
$$

and

$$
\left\|\frac{\partial F}{\partial x}\right\|_{0}+\sqrt{\left\|\frac{\partial F}{\partial \bar{y}}\right\|_{0} \cdot\left\|\frac{\partial G}{\partial x}\right\|_{0}}<1
$$

where $\|\varphi(x, \bar{y})\|_{\circ}=\sup _{(x, \bar{y}) \in X \times Y}\|\varphi(x, \bar{y})\|$ for any vector-valued or matrix valued function $\varphi$. Then
(i) the map $T$ has a $\mathcal{C}^{1}$-smooth invariant manifold $M^{*}=\left\{(x, y) \in \mathcal{R}: x=h^{*}(y)\right\}$, where $h^{*}: Y \rightarrow$ $X$ is a Lipschitz function with the Lipschitz constant

$$
\mathcal{L}=\sqrt{\left\|\frac{\partial F}{\partial \bar{y}}\right\|_{0}\left(\left\|\frac{\partial G}{\partial x}\right\|_{0}\right)^{-1}}
$$

(ii) for any $\mathrm{x}=(x, y) \in \mathcal{R}$ and any arbitrary $\epsilon>0$, there exists an integer $\mathcal{N}_{\epsilon}^{\mathrm{x}} \in \mathbb{N}$ such that for any $n>\mathcal{N}_{\epsilon}^{\mathrm{x}}$ if $\left\{T^{i}(\mathrm{x})\right\}_{i=0}^{i=n} \subset \mathcal{R}$, then $\operatorname{dist}\left(T^{n}(\mathrm{x}), M^{*}\right)<\epsilon$. In particular, $M^{*}$ contains the $\omega$-limit set of any point of $\mathcal{R}$ whose forward orbit lies entirely in $\mathcal{R}$.
(iii) if $\mathcal{R}$ is bounded, then the integer $\mathcal{N}_{\mathrm{x}}^{\epsilon}$ given above can be chosen independent of x , i.e. for any arbitrary $\epsilon>0$, there exists an integer $\mathcal{N}_{\epsilon} \in \mathbb{N}$ such that for any $n>\mathcal{N}_{\epsilon}$ and any $\mathrm{x} \in \mathcal{R}$ if $\left\{T^{i}(\mathrm{x})\right\}_{i=0}^{i=n} \subset \mathcal{R}$, then $\operatorname{dist}\left(T^{n}(\mathrm{x}), M^{*}\right)<\epsilon$.
(iv) let $M$ be a $\mathcal{L}$-surface (i.e. $M$ is the graph of some $\mathcal{L}$-Lipschitz function $h: Y \rightarrow X$ ). Then $\left.T(M)\right|_{X \times Y}$ is a $\mathcal{L}$-surface as well. Moreover, the sequence $\left\{\left.T^{n}(M)\right|_{X \times Y}\right\}$ converges to $M^{*}$.
Proof. See [SSTC98], Theorem 4.3 as well as Theorem 4.2 and its proof.
Proposition 2.29. With the setting of Theorem 2.28, if $\mathcal{R}$ is bounded, $T^{-1}$ exists and the backward orbit of a point $\mathrm{x} \in \mathcal{R}$ lies entirely in $\mathcal{R}$ then $\mathrm{x} \in M^{*}$.
Proof. The proof is by contradiction. Assume $\mathrm{x} \notin M^{*}$. This implies $\operatorname{dist}\left(\mathrm{x}, M^{*}\right)>0$. Choose an $0<\epsilon<\operatorname{dist}\left(\mathrm{x}, M^{*}\right)$ and consider $\mathcal{N}_{\epsilon}$ given by Theorem 2.28. We have $\operatorname{dist}\left(M^{*}, T^{\mathcal{N}_{\epsilon}}\left(T^{-\mathcal{N}_{\epsilon}}(\mathrm{x})\right)\right)<\epsilon$ and therefore $\operatorname{dist}\left(M^{*}, \mathrm{x}\right)<\epsilon$, which is a contradiction.

## Chapter 3

## Analysis near the equilibrium state $O$

### 3.1. Set-up and notations

Our approach for studying the dynamics near the homoclinic loop $\Gamma$ (and homoclinic figure-eight $\Gamma_{1} \cup \Gamma_{2}$ ) is based on the studying the behavior of the corresponding Poincaré map(s). As was mentioned earlier, the Poincaré map $T$ along the homoclinic loop $\Gamma$ can be written as the composition of a global and a local map. This section is dedicated to the study of the behavior of the local map $T^{l o c}$. To do this, we first need to choose appropriate coordinates near the equilibrium state $O$ of system (1.2.3). This is done in Section 3.2 below. We consider three different cases of $\lambda_{1}=\lambda_{2}, \lambda_{1}<\lambda_{2}<2 \lambda_{1}$ and $2 \lambda_{1}<\lambda_{2}$ and introduce a specific normal form for each case. In Section 3.3, we employ the Shilnikov technique for solving boundary value problems to compute the flow near the equilibrium $O$. This allows us to find an approximation for the local map. Finally, in Section 3.4, we study the behavior of this map and investigate some of its properties.

In comparison to the global map, the local map has more complicated behavior. Indeed, $T^{g l o}$ is a diffeomorphism and can be approximated by its Taylor polynomial while the local map $T^{l o c}$ is a singular map with a non-trivial domain.

Let us now give a more precise meaning to the above terminologies. Recall the cross-sections $\Pi^{s}$ and $\Pi^{u}$. In all of the normal forms considered in Section 3.2, the local stable and local unstable as well as the local strong stable and local strong unstable invariant manifolds of $O$ are straightened. Therefore, the homoclinic loop $\Gamma$ intersects $\Pi^{s}$ and $\Pi^{u}$ at $M^{s}=(0, \delta, 0,0)$ and $M^{u}=(0,0,0, \delta)$, respectively. As it is proved later (see Section 3.2.1), we can choose a ( $u_{1}, v_{1}$ ) coordinate-system on each of these cross-sections. Both $M^{s}$ and $M^{u}$ correspond to $(0,0)$ in this coordinate-system.

Consider a point ( $u_{10}, v_{10}$ ) on $\Pi^{s}$ close to $M^{s}$ (e.g. the green point in Figure 1.2) whose forward orbit goes along the homoclinic loop $\Gamma$, after a certain time $\tau$ it crosses $\Pi^{u}$ at a point ( $u_{1 \tau}, v_{1 \tau}$ ) (e.g. the blue point in Figure 1.2), and after a finite time it comes back to $\Pi^{s}$ at a point $\left(\bar{u}_{10}, \bar{v}_{10}\right)$ (e.g. the red point in Figure 1.2). Obviously, $\tau \rightarrow \infty$ as $\left(u_{10}, v_{10}\right) \rightarrow M^{s}$. Let $\mathcal{D} \subset \Pi^{s}$ be the set of all such points ( $u_{10}, v_{10}$ ) that satisfy

$$
\begin{equation*}
\left\|\left(u_{10}, v_{10}\right)\right\|<\epsilon \quad \text { and } \quad\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\|<\epsilon_{u} \tag{3.1.1}
\end{equation*}
$$

for some sufficiently small constants $0<\epsilon \leq \epsilon_{u}<\delta$ (see Figure 3.1). It is trivial that $M^{s} \notin \mathcal{D}$. When $\mathcal{D} \neq \emptyset$, we define the Poincaré map $T: \mathcal{D} \rightarrow \Pi^{s}$ by

$$
\left(u_{10}, v_{10}\right) \longmapsto\left(\bar{u}_{10}, \bar{v}_{10}\right) .
$$

The local map $T^{l o c}: \mathcal{D} \rightarrow \Pi^{u}$ is defined by

$$
\begin{equation*}
\left(u_{10}, v_{10}\right) \longmapsto\left(u_{1 \tau}, v_{1 \tau}\right) . \tag{3.1.2}
\end{equation*}
$$

The global map is defined on the $\epsilon_{u}$-ball $\mathcal{B}_{\epsilon_{u}}$ in $\Pi^{u}$ centered at $M^{u}$, i.e. $T^{g l o}: \mathcal{B}_{\epsilon_{u}} \rightarrow \Pi^{s}$, and its restriction to $T^{l o c}(\mathcal{D}) \subset \mathcal{B}_{\epsilon_{u}}$ is

$$
\begin{equation*}
\left(u_{1 \tau}, v_{1 \tau}\right) \longmapsto\left(\bar{u}_{10}, \bar{v}_{10}\right) \tag{3.1.3}
\end{equation*}
$$

Obviously, $T=T^{g l o} \circ T^{l o c}$.


Figure 3.1: These two figures are derived from Figure 1.2 by zooming in on a small neighborhood of $O$. The $\epsilon$-ball around $M^{s}$ in $\Pi^{s}$ and the $\epsilon_{u}$-ball around $M^{u}$ in $\Pi^{u}$ are shown by green and red colors, respectively. The domain $\mathcal{D}$ of the Poincaré map $T$ is the set of the points $\left(u_{10}, v_{10}\right)$ in the green ball whose forward orbits intersect $\Pi^{u}$ at $\left(u_{1 \tau}, v_{1 \tau}\right)$ in the red ball (see relation (3.1.1)).

Not every orbit starting from $\Pi^{s}$ goes along $\Gamma$ and intersects the cross-section $\Pi^{u}$. Trivial examples are the orbits that start at $W_{\text {loc }}^{s}(O) \cap \Pi^{s}$. Other examples are the orbits that go along the other branch of $W_{\text {loc }}^{u}(O)$ (negative side of $v_{2}$-axis). Consider a cross-section $\Sigma=\left\{v_{2}=-\delta\right\} \cap\{H=0\}$ to the negative branch of $W_{\text {loc }}^{u}(O)$. It will be shown that $\left(u_{1}, v_{1}\right)$-coordinates can be chosen on this cross-section. Then

Definition 3.1. We denote by $\mathbb{D}$ the set of the points $\left(u_{10}, v_{10}\right)$ on $\Pi^{s}$ close to $M^{s}$ whose forward orbits go along the negative branch of $W_{\text {loc }}^{u}(O)$, and after a certain time $\tau$ they cross $\Sigma$ at $\left(u_{1 \tau}, v_{1 \tau}\right)$ such that (3.1.1) holds (see Figure 3.2).

For the case of homoclinic figure-eight, we define the domains $\mathcal{D}$ and $\mathbb{D}$ for each loop. Namely,
Notation 3.2. For $i=1,2$, we denote by $\mathcal{D}^{i}$ and $\mathbb{D}^{i}$ the corresponding domains $\mathcal{D} \subset \Pi_{i}^{s}$ and $\mathbb{D} \subset \Pi_{i}^{s}$ of the loop $\Gamma_{i}$, respectively.

An orbit staring from $\mathcal{D}^{1} \subset \Pi_{1}^{s}\left(\mathcal{D}^{2} \subset \Pi_{2}^{s}\right)$ goes along $\Gamma_{1}\left(\Gamma_{2}\right)$ and intersects $\Pi_{1}^{u}\left(\Pi_{2}^{u}\right)$, while an orbit which starts from $\mathbb{D}^{1} \subset \Pi_{1}^{s}\left(\mathbb{D}^{2} \subset \Pi_{2}^{s}\right)$ goes along the negative (positive) side of $v_{2}$-axis and intersects $\Pi_{2}^{u}\left(\Pi_{1}^{u}\right)$.

We introduced the Poincaré, local and global maps along a single homoclinic loop above. For the case of homoclinic figure eight, we also define these maps for each loop:

Notation 3.3. We denote by $T_{i}, T_{i}^{l o c}$ and $T_{i}^{g l o}$ the Poincaré, local and global maps along $\Gamma_{i}(i=1,2)$, respectively (see Figure 3.3).

In order to analyze the dynamics near a homoclinic figure-eight, we need to consider two extra local maps:

Definition 3.4. We define the map $T_{12}^{l o c}: \mathbb{D}^{1} \subset \Pi_{1}^{s} \rightarrow \Pi_{2}^{u}\left(T_{21}^{l o c}: \mathbb{D}^{2} \subset \Pi_{2}^{s} \rightarrow \Pi_{1}^{u}\right)$ by $\left(u_{10}, v_{10}\right) \mapsto$ $\left(u_{1 \tau}, v_{1 \tau}\right)$ where $\left(u_{10}, v_{10}\right) \in \mathbb{D}^{1}\left(\in \mathbb{D}^{2}\right)$ and $\left(u_{1 \tau}, v_{1 \tau}\right) \in \Pi_{2}^{u}\left(\in \Pi_{1}^{u}\right)$ (see Figure 3.3).


Figure 3.2: We defined the domain $\mathcal{D}$ of the Poincaré map as the set of the points on $\Pi^{s}$ close to $M^{s}$ that go along the homoclinic loop $\Gamma$ and intersect $\Pi^{u}=\left\{v_{2}=\delta\right\} \cap\{H=0\}$ at points close to $M^{u}$. For instance, the blue point on $\Pi^{s}$ belongs to $\mathcal{D}$. Similarly, we define $\mathbb{D}$ as the set of the points on $\Pi^{s}$ close to $M^{s}$ that go along $\Gamma$ until they get close to $O$ and then go along the negative side of $v_{2}$-axis and intersect the cross-section $\Sigma=\left\{v_{2}=-\delta\right\} \cap\{H=0\}$ at points close to the point of the intersection of $\Sigma$ and $v_{2}$-axis. For example, the pink point on $\Pi^{s}$ belongs to $\mathbb{D}$.

### 3.2. Choice of coordinates near the equilibrium state $O$

This section is dedicated to finding suitable coordinate systems near the equilibrium state $O$. As it was mentioned above, we consider three different cases of $\lambda_{1}=\lambda_{2}, \lambda_{1}<\lambda_{2}<2 \lambda_{1}$ and $2 \lambda_{1}<\lambda_{2}$, and for each case we bring system (1.2.3) into a particular normal form. The proofs of the results stated below are postponed to Section 3.2.2. We start with the following:

Lemma 3.5. Consider system (1.2.3) and first integral (1.2.4). There exists a $\mathcal{C}^{\infty}$-smooth change of coordinates which brings system (1.2.3) to the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+f_{11}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{1}+f_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+f_{21}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{1}+f_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{2} \\
& \dot{v}_{1}=+\lambda_{1} v_{1}+g_{11}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{1}+g_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{2}  \tag{3.2.1}\\
& \dot{v}_{2}=+\lambda_{2} v_{2}+g_{21}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{1}+g_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{2}
\end{align*}
$$

where the functions $f_{i j}, g_{i j}$ are $\mathcal{C}^{\infty}-s m o o t h ~ a n d ~ v a n i s h ~ a t ~ t h e ~ o r i g i n, ~ i . e . ~$

$$
\begin{equation*}
f_{i j}(0,0,0,0)=g_{i j}(0,0,0,0)=0 \tag{3.2.2}
\end{equation*}
$$

and transforms first integral (1.2.4) to

$$
\begin{equation*}
H=\lambda_{1} u_{1} v_{1}-\lambda_{2} u_{2} v_{2} \tag{3.2.3}
\end{equation*}
$$

Moreover, system (3.2.1) remains invariant with respect to symmetry (1.2.5). In particular, we have

$$
\begin{equation*}
f_{12}\left(0, u_{2}, 0, v_{2}\right) \equiv 0, \quad g_{12}\left(0, u_{2}, 0, v_{2}\right) \equiv 0 \tag{3.2.4}
\end{equation*}
$$



Figure 3.3: The blue, brown, green, yellow, red and pink curves correspond to the maps $T_{1}^{\text {glo }}, T_{12}^{\text {loc }}, T_{2}^{\text {glo }}, T_{21}^{\text {loc }}$, $T_{1}^{\text {loc }}$ and $T_{2}^{\text {loc }}$, respectively. The Poincaré maps $T_{1}$ (along the homoclinic orbit $\Gamma_{1}$ ) and $T_{2}$ (along the homoclinic orbit $\Gamma_{2}$ ) are defined by $T_{1}=T_{1}^{\text {glo }} \circ T_{1}^{\text {loc }}$ and $T_{2}=T_{2}^{\text {glo }} \circ T_{2}^{\text {loc }}$, respectively.

The statement of Lemma 3.5 holds for arbitrary $\lambda_{1}$ and $\lambda_{2}$. However, we will particularly use this for analyzing the case $\lambda_{1}=\lambda_{2}$.

Lemma 3.6. Consider system (1.2.3) and first integral (1.2.4), and assume $\lambda_{1}<\lambda_{2}$. There exists $a \mathcal{C}^{\infty}$-smooth change of coordinates which brings system (1.2.3) to the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, u_{2}, v\right) u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+f_{21}\left(u_{1}, v\right) u_{1}+f_{22}\left(u_{1}, u_{2}, v\right) u_{2} \\
& \dot{v}_{1}=+\lambda_{1} v_{1}+g_{11}\left(u, v_{1}\right) v_{1}+g_{12}\left(u, v_{1}, v_{2}\right) v_{2}  \tag{3.2.5}\\
& \dot{v}_{2}=+\lambda_{2} v_{2}+g_{21}\left(u, v_{1}\right) v_{1}+g_{22}\left(u, v_{1}, v_{2}\right) v_{2}
\end{align*}
$$

where the functions $f_{i j}, g_{i j}$ are $\mathcal{C}^{\infty}$-smooth and satisfy the identities

$$
\begin{gather*}
f_{11}(0, v) \equiv 0, \quad f_{11}\left(u_{1}, 0\right) \equiv 0, \quad f_{12}(u, 0) \equiv 0 \\
f_{21}(0, v) \equiv 0, \quad f_{22}(0, v) \equiv 0 \\
g_{11}(u, 0) \equiv 0, \quad g_{11}\left(0, v_{1}\right) \equiv 0, \quad g_{12}(0, v) \equiv 0  \tag{3.2.6}\\
g_{21}(u, 0) \equiv 0, \quad g_{22}(u, 0) \equiv 0
\end{gather*}
$$

This change of coordinates transforms first integral (1.2.4) to

$$
\begin{equation*}
H=\lambda_{1} u_{1} v_{1}\left[1+H_{1}(u, v)\right]-\lambda_{2} u_{2} v_{2}\left[1+H_{2}(u, v)\right] \tag{3.2.7}
\end{equation*}
$$

where $H_{1}$ and $H_{2}$ are $\mathcal{C}^{\infty}$-smooth functions such that $H_{1}(O)=H_{2}(O)=0$. Moreover, normal form (3.2.5) and first integral (3.2.7) remain invariant with respect to symmetry (1.2.5). In particular, (3.2.4) holds.

The statement of Lemma 3.6 holds for arbitrary $\lambda_{1}<\lambda_{2}$. However, we will particularly use this to analyze the local dynamics near $O$ in the case of $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$.

Remark 3.7. For simplicity, we can write (3.2.7) as

$$
\begin{equation*}
H=\lambda_{1} u_{1} v_{1}[1+o(1)]-\lambda_{2} u_{2} v_{2}[1+o(1)] \tag{3.2.8}
\end{equation*}
$$

The normal form that we will use later to analyze the case $2 \lambda_{1}<\lambda_{2}$ is given by the following lemma:

Lemma 3.8. Consider system (1.2.3) and first integral (1.2.4) and assume $2 \lambda_{1}<\lambda_{2}$. Let $q$ be the largest integer such that $q \lambda_{1}<\lambda_{2}$. There exists a $\mathcal{C}^{q}$-smooth change of coordinates which brings system (1.2.3) to the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, u_{2}, v\right) u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+f_{22}\left(u_{1}, u_{2}, v\right) u_{2} \\
& \dot{v}_{1}=+\lambda_{1} v_{1}+g_{11}\left(u, v_{1}\right) v_{1}+g_{12}\left(u, v_{1}, v_{2}\right) v_{2}  \tag{3.2.9}\\
& \dot{v}_{2}=+\lambda_{2} v_{2}+g_{22}\left(u, v_{1}, v_{2}\right) v_{2}
\end{align*}
$$

where $f_{i j}$ and $g_{i j}$ are $\mathcal{C}^{q-1}{ }_{-}$smooth and satisfy identities (3.2.6). This change of coordinates transforms first integral (1.2.4) to

$$
\begin{equation*}
H=\lambda_{1} u_{1} v_{1}\left[1+H_{1}(u, v)\right]-\lambda_{2} u_{2} v_{2}\left[1+H_{2}(u, v)\right]+u_{2} v_{1}^{2} H_{3}(u, v)+v_{2} u_{1}^{2} H_{4}(u, v) \tag{3.2.10}
\end{equation*}
$$

where $H$ is $\mathcal{C}^{q}$-smooth, and $H_{1}, H_{2}, H_{3}$ and $H_{4}$ are some $\mathcal{C}^{q-1}, \mathcal{C}^{q}, \mathcal{C}^{q-2}$ and $\mathcal{C}^{q-2}$ functions, respectively, such that $H_{1}(O)=H_{2}(O)=0$. Moreover, system (3.2.9) and first integral (3.2.10) remain invariant with respect to symmetry (1.2.5). In particular, $f_{12}$ and $g_{12}$ satisfy (3.2.4).

Remark 3.9. For simplicity, we can write (3.2.10) as

$$
H=\lambda_{1} u_{1} v_{1}[1+o(1)]-\lambda_{2} u_{2} v_{2}[1+o(1)]+u_{2} v_{1}^{2} O(1)+v_{2} u_{1}^{2} O(1)
$$

A common structure of all of normal forms (3.2.1), (3.2.5) and (3.2.9) is that the local stable and unstable as well as the local strong stable and strong unstable invariant manifolds of the equilibrium $O$ are straightened, i.e. $W_{\text {loc }}^{s}=\{v=0\}, W_{\text {loc }}^{u}=\{u=0\}, W_{\text {loc }}^{s s}=\left\{u_{1}=v_{1}=v_{2}=0\right\}$ and $W_{\text {loc }}^{u u}=\left\{u_{1}=u_{2}=v_{1}=0\right\}$. For the particular case of normal form (3.2.9), the local extended stable and extended unstable invariant manifolds of $O$ are straightened too, i.e. $W_{\text {loc }}^{s E}=\left\{v_{2}=0\right\}$ and $W_{\text {loc }}^{u E}=\left\{u_{2}=0\right\}$.

### 3.2.1. Choice of coordinates on the cross-sections

Our goal here is to show that in any of the cases $\lambda_{1}=\lambda_{2}, \lambda_{2}<2 \lambda_{1}$ and $2 \lambda_{1}<\lambda_{2}$, we can always choose $\left(u_{1}, v_{1}\right)$ - coordinate system on each of the cross-sections $\Pi^{s}, \Pi^{u}, \Pi_{1}^{s}, \Pi_{1}^{u}, \Pi_{2}^{s}$ and $\Pi_{2}^{u}$. This is done by showing that on each of these cross-sections, the variables $u_{2}$ and $v_{2}$ are uniquely determined by $\left(u_{1}, v_{1}\right)$.

As a direct consequence of Lemma 3.5, we have
Corollary 3.10. Consider system (3.2.1) and let $\left(u_{10}, \delta, v_{10}, v_{20}\right)$ and $\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, \delta\right)$ be two points on $\Pi^{s}$ and $\Pi^{u}$, respectively. According to Lemma 3.5, the variable $v_{20}$ is uniquely determined by $u_{10}$ and $v_{10}$. Analogously, the variable $u_{2 \tau}$ is uniquely determined by $u_{1 \tau}$ and $v_{1 \tau}$. More precisely,

$$
\begin{equation*}
v_{20}=\frac{\lambda_{1}}{\lambda_{2} \delta} \cdot u_{10} v_{10} \quad \text { and } \quad u_{2 \tau}=\frac{\lambda_{1}}{\lambda_{2} \delta} \cdot u_{1 \tau} v_{1 \tau} \tag{3.2.11}
\end{equation*}
$$

With Lemma 3.6, we can prove the following proposition:
Proposition 3.11. Consider first integral (3.2.7). We have

$$
\begin{equation*}
H_{v_{2}}(0, \delta, 0,0) \neq 0 \quad \text { and } \quad H_{u_{2}}(0,0,0, \delta) \neq 0 \tag{3.2.12}
\end{equation*}
$$



Figure 3.4: We can always choose $\left(u_{1}, v_{1}\right)$-coordinate system on each of the cross-sections $\Pi^{s}, \Pi^{u}, \Pi_{1}^{s}, \Pi_{1}^{u}, \Pi_{2}^{s}$, $\Pi_{2}^{u}$ and $\Sigma$. This figure shows $\left(u_{1}, v_{1}\right)$-coordinate system on $\Pi_{1}^{s}, \Pi_{1}^{u}, \Pi_{2}^{s}$ and $\Pi_{2}^{u}$.

Proof. Differentiating (3.2.7) with respect to $v_{2}$ gives

$$
H_{v_{2}}=\lambda_{1} u_{1} v_{1} H_{v_{2}}(u, v)-\lambda_{2} u_{2}\left[1+H_{2}(u, v)\right]-\lambda_{2} u_{2} v_{2} H_{2 v_{2}}(u, v) .
$$

Note that $H_{1 v_{2}}(u, v)=O(1), H_{2}(u, v)=o(1)$ and $H_{2 v_{2}}(u, v)=O(1)$. Thus,

$$
\begin{equation*}
H_{v_{2}}=u_{1} v_{1} O(1)-\lambda_{2} u_{2}[1+o(1)] . \tag{3.2.13}
\end{equation*}
$$

Evaluating this relation at $(0, \delta, 0,0)$ gives $H_{v_{2}}(0, \delta, 0,0)=-\lambda_{2} \delta[1+o(1)] \neq 0$. Similarly, one can easily show that

$$
\begin{equation*}
H_{u_{2}}=u_{1} v_{1} O(1)-\lambda_{2} v_{2}[1+o(1)] \tag{3.2.14}
\end{equation*}
$$

which implies $H_{u_{2}}(0,0,0, \delta)=-\lambda_{2} \delta[1+o(1)] \neq 0$. This ends the proof.
Remark 3.12. Together with the implicit function theorem, this proposition implies that when system (3.2.5) is given, locally, the variables $v_{2}$ on $\Pi^{s}$ and $u_{2}$ on $\Pi^{u}$ can be written as functions of $\left(u_{1}, v_{1}\right)$.

With Lemma 3.8, we can prove the following proposition:
Proposition 3.13. Consider first integral (3.2.10). We have

$$
\begin{equation*}
H_{v_{2}}(0,0, \delta, 0) \neq 0 \quad \text { and } \quad H_{u_{2}}(0,0,0, \delta) \neq 0 . \tag{3.2.15}
\end{equation*}
$$

Proof. Denote first integral (3.2.7) by $H^{\circ}$ and let $x:=v_{2}+\phi^{s E}\left(u_{1}, u_{2}, v_{1}\right)$ and $y:=u_{2}+\phi^{u E}\left(u_{1}, v_{1}, x\right)$. We have $H=H^{\circ}\left(u_{1}, y, v_{1}, x\right)$. Differentiating this relation with respect to $v_{2}$ gives

$$
H_{v_{2}}=\left.\left.\frac{\partial H^{\circ}}{\partial u_{2}}\right|_{\left(u_{1}, y, v_{1}, x\right)} \cdot \frac{\partial \phi^{u E}}{\partial v_{2}}\right|_{\left(u_{1}, v_{1}, x\right)}+\left.\frac{\partial H^{\circ}}{\partial v_{2}}\right|_{\left(u_{1}, y, v_{1}, x\right)}
$$

By (3.2.13), (3.2.14) and Corollary 3.18 (this corollary is stated later), at $u_{2}=\delta$, we have

$$
\begin{aligned}
\left.\frac{\partial H^{\circ}}{\partial u_{2}}\right|_{\left(u_{1}, y, v_{1}, x\right)} & =u_{1} v_{1} O(1)+O\left(v_{1}^{2}\right)-\lambda_{2} v_{2}[1+o(1)] \\
\left.\frac{\partial \phi^{u E}}{\partial v_{2}}\right|_{\left(u_{1}, v_{1}, x\right)} & =O\left(u_{1}\right) \\
\left.\frac{\partial H^{\circ}}{\partial v_{2}}\right|_{\left(u_{1}, y, v_{1}, x\right)} & =u_{1} v_{1} O(1)+O\left(u_{1}^{2}\right)-\lambda_{2} \delta[1+o(1)]
\end{aligned}
$$

Thus,

$$
H_{v_{2}}=u_{1} v_{1} O(1)+O\left(u_{1} v_{2}\right)+O\left(u_{1}^{2}\right)-\lambda_{2} \delta[1+o(1)]=-\lambda_{2} \delta[1+O(\delta)]
$$

and therefore $H_{v_{2}}(0,0, \delta, 0)=-\lambda_{2} \delta[1+O(\delta)] \neq 0$. The proof of the other relation in (3.2.15) is the same. This ends the proof.

Remark 3.14. Together with the implicit function theorem, this proposition implies that when system (3.2.5) is given, locally, the variables $v_{2}$ on $\Pi^{s}$ and $u_{2}$ on $\Pi^{u}$ can be written as $\mathcal{C}^{q}$-smooth functions of $\left(u_{1}, v_{1}\right)$.

In all the cases $\lambda_{1}=\lambda_{2}, \lambda_{1}<\lambda_{2}<2 \lambda_{1}$ and $2 \lambda_{1}<\lambda_{2}$, the same statements hold for the crosssections over $\Gamma_{1}$ and $\Gamma_{2}$ (as well as the auxiliary cross-section $\Sigma$ introduced in Chapter 3). This leads to the following:

Corollary 3.15. We can always choose $\left(u_{1}, v_{1}\right)$-coordinate system on each of the cross-sections $\Pi^{s}, \Pi^{u}, \Pi_{1}^{s}, \Pi_{1}^{u}, \Pi_{2}^{s}$ and $\Pi_{2}^{u}$.

### 3.2.2. Proofs of Lemmas 3.5, 3.6 and 3.8

Proof of Lemma 3.5. To reduce system (1.2.3) to the form (3.2.1), we straighten the local stable and local unstable invariant manifolds of the equilibrium state $O$, i.e. we apply a change of coordinates

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}-\varphi_{1 s}\left(v_{1}, v_{2}\right), & \tilde{u}_{2}=u_{2}-\varphi_{2 s}\left(v_{1}, v_{2}\right)  \tag{3.2.16}\\
\tilde{v}_{1}=v_{1}-\psi_{1 u}\left(u_{1}, u_{2}\right), & \tilde{v}_{2}=v_{2}-\psi_{2 u}\left(u_{1}, u_{2}\right)
\end{array}
$$

where $\left\{u_{1}=\varphi_{1 s}\left(v_{1}, v_{2}\right), u_{2}=\varphi_{2 s}\left(v_{1}, v_{2}\right)\right\}$ and $\left\{v_{1}=\psi_{1 u}\left(u_{1}, u_{2}\right), v_{2}=\psi_{2 u}\left(u_{1}, u_{2}\right)\right\}$ are the equations of the local stable and the local unstable invariant manifolds of $O$, respectively. Thus, after applying (3.2.16), the equations of the local stable and the local unstable manifolds of $O$ become $\left\{v_{1}=v_{2}=0\right\}$ and $\left\{u_{1}=u_{2}=0\right\}$, respectively. This implies that system (1.2.3) can be written in the form (3.2.1) such that (3.2.2) is satisfied. Notice that change of coordinates (3.2.16) does not affect the quadratic part of (1.2.4). Therefore, the updated first integral $H$ keeps the form (1.2.4).

Since $H$ vanishes at every point of the local unstable invariant manifold $\left\{u_{1}=u_{2}=0\right\}$, it can be written as

$$
\begin{equation*}
H\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=\lambda_{1} u_{1}\left[v_{1}+H_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)\right]-\lambda_{2} u_{2}\left[v_{2}+H_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)\right] \tag{3.2.17}
\end{equation*}
$$

for some $\mathcal{C}^{\infty}$-smooth $H_{1}, H_{2}: \mathbb{R}^{4} \rightarrow \mathbb{R}$ such that $H_{1}$ and $H_{2}$ and their first derivatives vanish at $O$. On the other hand, $H$ vanishes at every point of the local stable invariant manifold $\left\{v_{1}=v_{2}=0\right\}$. This implies

$$
\begin{equation*}
0=H\left(u_{1}, u_{2}, 0,0\right)=\lambda_{1} u_{1} H_{1}\left(u_{1}, u_{2}, 0,0\right)-\lambda_{2} u_{2} H_{2}\left(u_{1}, u_{2}, 0,0\right) \tag{3.2.18}
\end{equation*}
$$

This yields

$$
\begin{aligned}
H\left(u_{1}, u_{2}, v_{1}, v_{2}\right)= & \lambda_{1} u_{1}\left[v_{1}+H_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)-H_{1}\left(u_{1}, u_{2}, 0,0\right)\right] \\
& -\lambda_{2} u_{2}\left[v_{2}+H_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)-H_{2}\left(u_{1}, u_{2}, 0,0\right)\right]
\end{aligned}
$$

This suggests that, without loss of generality, we can assume that $H_{1}$ and $H_{2}$ vanish at $\left\{v_{1}=v_{2}=0\right\}$. Now, consider the change of coordinates

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}, & \tilde{u}_{2}=u_{2} \\
\tilde{v}_{1}=v_{1}+H_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right), & \tilde{v}_{2}=v_{2}+H_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) \tag{3.2.19}
\end{array}
$$

Since $H_{1}\left(u_{1}, u_{2}, 0,0\right)=H_{2}\left(u_{1}, u_{2}, 0,0\right)=0$, applying this change of coordinates on system (3.2.1) keeps the local stable and local unstable invariant manifolds straightened and therefore keeps the form (3.2.1) of the system such that (3.2.2) still holds. However, this change of coordinates reduces the first integral $H$ to the form (3.2.3).

It is a direct consequence of Corollary 2.25 that change of coordinates (3.2.16) preserves the symmetric structure of the system and the first integral. Concerning the change of coordinates (3.2.19), note that since $H$ in (3.2.17) satisfies (1.2.6), we have

$$
H_{1}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right)=-H_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) \quad \text { and } \quad H_{2}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right)=H_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)
$$

This implies that the change of coordinates (3.2.19) commutes with symmetry (1.2.5). Therefore, by Proposition 2.3, this change of coordinates preserves the invariance of the system with respect to symmetry (1.2.5). This ends the proof of Lemma 3.5.

Our proof of Lemma 3.6 is based on a theorem in [SSTC98] (Theorem A.1). A special case of this theorem that we need for the proof of that lemma is stated below:

Lemma 3.16. ( [SSTC98], Theorem A.1) Consider system (3.2.1) and assume $\lambda_{1}<\lambda_{2}$. There exists a $\mathcal{C}^{\infty}$-smooth change of coordinates which brings system (3.2.1) to the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+f_{11}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{1}+f_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+f_{21}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{1}+f_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{2}  \tag{3.2.20}\\
& \dot{v}_{1}=+\lambda_{1} v_{1}+g_{11}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{1}+g_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{2} \\
& \dot{v}_{2}=+\lambda_{2} v_{2}+g_{21}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{1}+g_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{2}
\end{align*}
$$

where the functions $f_{i j}, g_{i j}$ are $\mathcal{C}^{\infty}$-smooth and

$$
\begin{align*}
f_{i j}(0,0,0,0) & =0, \quad g_{i j}(0,0,0,0)=0 \\
f_{1 i}\left(u_{1}, u_{2}, 0,0\right) & \equiv 0, \quad g_{1 i}\left(0,0, v_{1}, v_{2}\right) \equiv 0  \tag{3.2.21}\\
f_{j 1}\left(0,0, v_{1}, v_{2}\right) & \equiv 0, \quad g_{j 1}\left(u_{1}, u_{2}, 0,0\right) \equiv 0, \quad(i, j=1,2)
\end{align*}
$$

As a matter of comparison between this lemma and Lemma 3.6, the functions $f_{i 1}$ and $g_{i 1}(i=1,2)$ in (3.2.5) do not depend on $u_{2}$ and $v_{2}$, respectively, and (3.2.6) includes all conditions (3.2.21) as well as two extra constraints

$$
\begin{equation*}
f_{22}(0, v) \equiv 0 \tag{3.2.22}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{22}(u, 0) \equiv 0 \tag{3.2.23}
\end{equation*}
$$

To obtain the statement of Lemma 3.16 from Theorem A. 1 in [SSTC98], it is sufficient to ignore the dependence of the system in that theorem on the parameter $\mu$ and put $m_{1}=m_{2}=n_{1}=n_{2}=1$, $x=u_{1}, u=u_{2}, y=v_{1}, v=v_{2}, A_{1}=-\lambda_{1}, A_{2}=-\lambda_{2}, B_{1}=\lambda_{1}$ and $B_{2}=\lambda_{2}$. Here, we sketch a proof of this lemma (mainly those parts of the proof that we need later in this thesis) and refer the reader for further details to Appendix A in [SSTC98].

Sketch of proof of Lemma 3.16. We can recast system (3.2.1) into the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+\underline{R_{1}\left(u_{1}, u_{2}\right)}+\underline{\varphi_{1}\left(v_{1}, v_{2}\right) u_{1}}+\varphi_{2}\left(v_{1}, v_{2}\right) u_{2}+\ldots \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+R_{2}\left(u_{1}, u_{2}\right)+\underline{\varphi_{3}\left(v_{1}, v_{2}\right) u_{1}}+\varphi_{4}\left(v_{1}, v_{2}\right) u_{2}+\ldots \\
& \dot{v}_{1}=+\lambda_{1} v_{1}+\underline{P_{1}\left(v_{1}, v_{2}\right)}+\underline{\psi_{1}\left(u_{1}, u_{2}\right) v_{1}}+\psi_{2}\left(u_{1}, u_{2}\right) v_{2}+\ldots  \tag{3.2.24}\\
& \dot{v}_{2}=+\lambda_{2} v_{2}+P_{2}\left(v_{1}, v_{2}\right)+\underline{\psi_{3}\left(u_{1}, u_{2}\right) v_{1}}+\psi_{4}\left(u_{1}, u_{2}\right) v_{2}+\ldots,
\end{align*}
$$

where

$$
\begin{array}{cc}
R_{i}=f_{i 1}\left(u_{1}, u_{2}, 0,0\right) u_{1}+f_{i 2}\left(u_{1}, u_{2}, 0,0\right) u_{2}, \\
P_{i}=g_{i 1}\left(0,0, v_{1}, v_{2}\right) v_{1}+g_{i 2}\left(0,0, v_{1}, v_{2}\right) v_{2}, \\
\varphi_{1}=f_{11}\left(0,0, v_{1}, v_{2}\right), & \varphi_{2}=f_{12}\left(0,0, v_{1}, v_{2}\right),  \tag{3.2.25}\\
\varphi_{3}=f_{21}\left(0,0, v_{1}, v_{2}\right), & \varphi_{4}=f_{22}\left(0,0, v_{1}, v_{2}\right), \\
\psi_{1}=g_{11}\left(u_{1}, u_{2}, 0,0\right), & \psi_{2}=g_{12}\left(u_{1}, u_{2}, 0,0\right), \\
\psi_{3}=g_{21}\left(u_{1}, u_{2}, 0,0\right), & \psi_{4}=g_{22}\left(u_{1}, u_{2}, 0,0\right),
\end{array}
$$

and

$$
\begin{gather*}
R_{i}\left(u_{1}, u_{2}\right)=\tilde{R}_{i 1}\left(u_{1}, u_{2}\right) u_{1}+\tilde{R}_{i 2}\left(u_{1}, u_{2}\right) u_{2} \\
P_{i}\left(v_{1}, v_{2}\right)=\tilde{P}_{i 1}\left(v_{1}, v_{2}\right) v_{1}+\tilde{P}_{i 2}\left(v_{1}, v_{2}\right) v_{2}  \tag{3.2.26}\\
\tilde{R}_{i j}(0,0) \equiv 0, \quad \tilde{P}_{i j}(0,0) \equiv 0 \\
\varphi_{j}(0,0) \equiv 0, \quad \psi_{j}(0,0) \equiv 0
\end{gather*}
$$

and the dots stand for some terms which we will hereafter call negligible: in the first two equations these are the terms of the form $\tilde{f}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{1}$ and $\tilde{f}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) u_{2}$ such that

$$
\begin{equation*}
\tilde{f}\left(0,0, v_{1}, v_{2}\right) \equiv 0, \quad \tilde{f}\left(u_{1}, u_{2}, 0,0\right) \equiv 0 \tag{3.2.27}
\end{equation*}
$$

and in the last two equations these are the terms of the form $\tilde{g}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{1}$ and $\tilde{g}\left(u_{1}, u_{2}, v_{1}, v_{2}\right) v_{2}$ such that

$$
\begin{equation*}
\tilde{g}\left(0,0, v_{1}, v_{2}\right) \equiv 0, \quad \tilde{g}\left(u_{1}, u_{2}, 0,0\right) \equiv 0 \tag{3.2.28}
\end{equation*}
$$

The proof of this lemma is reduced to eliminating the underlined terms in (3.2.24). To kill these terms we will carry out a series of consecutive changes of variables

$$
\begin{array}{ll}
\xi_{1}=u_{1}+h_{1}\left(v_{1}, v_{2}\right) u_{1}, & \xi_{2}=u_{2}+h_{2}\left(v_{1}, v_{2}\right) u_{1}  \tag{3.2.29}\\
\eta_{1}=v_{1}, & \eta_{2}=v_{2}
\end{array}
$$

where $h_{i}(0,0)=0$;

$$
\begin{array}{ll}
\xi_{1}=u_{1}, & \xi_{2}=u_{2} \\
\eta_{1}=v_{1}+s_{1}\left(u_{1}, u_{2}\right) v_{1}, & \eta_{2}=v_{2}+s_{2}\left(u_{1}, u_{2}\right) v_{1} \tag{3.2.30}
\end{array}
$$

where $s_{i}(0,0)=0$;

$$
\begin{array}{ll}
\xi_{1}=u_{1}+r_{1}\left(u_{1}, u_{2}\right) u_{1}+r_{2}\left(u_{1}, u_{2}\right) u_{2}, & \xi_{2}=u_{2}  \tag{3.2.31}\\
\eta_{1}=v_{1}, & \eta_{2}=v_{2}
\end{array}
$$

where $r_{i}(0,0)=0$; and

$$
\begin{array}{ll}
\xi_{1}=u_{1}, & \xi_{2}=u_{2}  \tag{3.2.32}\\
\eta_{1}=v_{1}+p_{1}\left(v_{1}, v_{2}\right) v_{1}+p_{2}\left(v_{1}, v_{2}\right) v_{2}, & \eta_{2}=v_{2}
\end{array}
$$

where $p_{i}(0,0)=0$.
The change of variables (3.2.29) removes terms $\varphi_{1}$ and $\varphi_{3}$ in system (3.2.24). By a change of variables (3.2.30) we eliminate the terms $\psi_{1}$ and $\psi_{3}$. By a change of variables (3.2.31) we eliminate the terms $R_{1}$. Finally, by a change of variables $(3.2 .32)$ we eliminate the terms $P_{1}$, thereby reducing the original system to the desired form.

Consider the system

$$
\begin{align*}
\dot{U}_{1} & =-\varphi_{1}+\varphi_{2} U_{2}-\varphi_{1} U_{1}+\varphi_{2} U_{1} U_{2} \\
\dot{U}_{2} & =\left(\lambda_{1}-\lambda_{2}\right) U_{2}-\varphi_{3}+\left(\varphi_{4}-\varphi_{1}\right) U_{2}+\varphi_{2} U_{2}^{2}  \tag{3.2.33}\\
\dot{v}_{1} & =\lambda_{1} v_{1}+P_{1} \\
\dot{v}_{2} & =\lambda_{2} v_{2}+P_{2}
\end{align*}
$$

where $\left(U_{1}, U_{2}, v_{1}, v_{2}\right) \in \mathbb{R}^{4}$. This system possesses a two dimensional strongly unstable invariant manifold defined by the equation $\left\{U_{1}=h_{1}\left(v_{1}, v_{2}\right), U_{2}=h_{2}\left(v_{1}, v_{2}\right)\right\}$ for some functions $h_{1}$ and $h_{2}$. As it is shown in [SSTC98], these $h_{1}$ and $h_{2}$ can be chosen as the desired $h_{1}$ and $h_{2}$ in change of coordinates (3.2.29). After making this change of coordinates, our system takes the form (3.2.24), where $\varphi_{1} \equiv 0$ and $\varphi_{3} \equiv 0$.

To show that $s_{1}$ and $s_{2}$ in (3.2.30) exists, consider the system

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+R_{1} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+R_{2} \\
& \dot{V}_{1}=-\psi_{1}+\psi_{2} V_{2}-V_{1} \psi_{1}+\psi_{2} V_{1} V_{2}  \tag{3.2.34}\\
& \dot{V}_{2}=\left(\lambda_{2}-\lambda_{1}\right) V_{2}-\psi_{3}+\left(\psi_{4}-\psi_{1}\right) V_{2}+\psi_{2} V_{2}^{2}
\end{align*}
$$

where $\left(u_{1}, u_{2}, V_{1}, V_{2}\right) \in \mathbb{R}^{4}$. This system possesses a two dimensional strongly stable invariant manifold defined by the equation $\left\{V_{1}=s_{1}\left(u_{1}, u_{2}\right), V_{2}=s_{2}\left(u_{1}, u_{2}\right)\right\}$. As it is shown in [SSTC98], the functions $s_{1}$ and $s_{2}$ in this equation can be chosen as the desired $s_{1}$ and $s_{2}$ in change of coordinates (3.2.30).

After making changes of coordinates (3.2.29) and (3.2.30), system (3.2.1) takes the form (3.2.24), where $\varphi_{1} \equiv 0, \varphi_{3} \equiv 0, \psi_{1} \equiv 0$ and $\psi_{3} \equiv 0$.

Now, consider the system

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+R_{1}\left(u_{1}, u_{2}\right) \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+R_{2}\left(u_{1}, u_{2}\right) \\
& \dot{V}_{1}=-\left(1+V_{1}\right) \tilde{R}_{11}-V_{2} \tilde{R}_{21}  \tag{3.2.35}\\
& \dot{V}_{2}=\left(\lambda_{2}-\lambda_{1}\right) V_{2}-\left(1+V_{1}\right) \tilde{R}_{12}-\tilde{R}_{22} V_{2}
\end{align*}
$$

where $\left(u_{1}, u_{2}, V_{1}, V_{2}\right) \in \mathbb{R}^{4}$. This system possesses a two dimensional strongly stable invariant manifold defined by the equation $\left\{V_{1}=r_{1}\left(u_{1}, u_{2}\right), V_{2}=r_{2}\left(u_{1}, u_{2}\right)\right\}$ for some functions $r_{1}$ and $r_{2}$. As it is shown in [SSTC98], these $r_{1}$ and $r_{2}$ can be chosen as the desired $r_{1}$ and $r_{2}$ in change of coordinates (3.2.31).

So far, changes of coordinates (3.2.29), (3.2.30) and (3.2.31) have reduced system (3.2.1) to the form (3.2.24), where $\varphi_{1} \equiv 0, \varphi_{3} \equiv 0, \psi_{1} \equiv 0, \psi_{3} \equiv 0$ and $R_{1} \equiv 0$. Now, consider the system

$$
\begin{align*}
\dot{U}_{1} & =-\left(1+U_{1}\right) \tilde{P}_{11}-\tilde{P}_{21} U_{2} \\
\dot{U}_{2} & =\left(\lambda_{1}-\lambda_{2}\right) U_{2}-\left(1+U_{1}\right) \tilde{P}_{12}-\tilde{P}_{22} U_{2}  \tag{3.2.36}\\
\dot{v}_{1} & =\lambda_{1} u_{1}+P_{1}\left(v_{1}, v_{2}\right) \\
\dot{v}_{2} & =\lambda_{2} u_{2}+P_{2}\left(v_{1}, v_{2}\right)
\end{align*}
$$

where $\left(U_{1}, U_{2}, v_{1}, v_{2}\right) \in \mathbb{R}^{4}$. This system possesses a two dimensional strongly unstable invariant manifold defined by the equation $\left\{U_{1}=p_{1}\left(v_{1}, v_{2}\right), U_{2}=p_{2}\left(v_{1}, v_{2}\right)\right\}$ for some functions $p_{1}$ and $p_{2}$.

As it is shown in [SSTC98], these $p_{1}$ and $p_{2}$ can be chosen as the desired $p_{1}$ and $p_{2}$ in change of coordinates (3.2.32).

Applying changes of coordinates (3.2.29), (3.2.30), (3.2.31) and (3.2.30) reduces system (3.2.1) to the form (3.2.24), where $\varphi_{1} \equiv 0, \varphi_{3} \equiv 0, \psi_{1} \equiv 0, \psi_{3} \equiv 0, R_{1} \equiv 0$ and $P_{1} \equiv 0$. This ends the sketch of proof of Lemma 3.16.

As it was mentioned above, the desired change of coordinates in Lemma 3.16 is in fact a composition of several changes of coordinates, each describing some invariant manifolds. We do not explain it in detail and refer the reader to [SSTC98] for further information. However, the technique which is used in [SSTC98] to derive these changes of coordinates is also used here explicitly in the proof of Lemma 3.6 (specifically, changes of coordinates (3.2.39) and (3.2.40)).

Proof of Lemma 3.6. According to Lemmas 3.5 and 3.16, there exists a change of coordinates which brings system (1.2.3) to system (3.2.20) where the functions $f_{i j}, g_{i j}$ are $\mathcal{C}^{\infty}$-smooth and satisfy (3.2.21). We show that there exists a change of coordinates which brings system (3.2.20) into the form (3.2.5), where $f_{i j}, g_{i j}$ are $\mathcal{C}^{\infty}$-smooth and satisfy (3.2.6).

Consider system (3.2.20) and let

$$
\begin{aligned}
f_{i 1}^{\text {new }}\left(u_{1}, v\right) & =f_{i 1}\left(u_{1}, 0, v_{1}, v_{2}\right), \\
f_{i 2}^{\text {new }}\left(u_{1}, u_{2}, v\right) & =\left[\frac{f_{i 1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)-f_{i 1}\left(u_{1}, 0, v_{1}, v_{2}\right)}{u_{2}}\right] u_{1}+f_{i 2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right), \\
g_{i 1}^{\text {new }}\left(u, v_{1}\right) & =g_{i 1}\left(u_{1}, u_{2}, v_{1}, 0\right), \\
g_{i 2}^{\text {new }}\left(u, v_{1}, v_{2}\right) & =\left[\frac{g_{i 1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)-g_{i 1}\left(u_{1}, u_{2}, v_{1}, 0\right)}{v_{2}}\right] v_{1}+g_{i 2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right),
\end{aligned}
$$

for $i=1,2$. It is easily seen that $\left\{f_{i j}^{\text {new }}\right\}$ and $\left\{g_{i j}^{\text {new }}\right\}$ satisfy (3.2.21). Thus, by rewriting system (3.2.20) with $\left\{f_{i j}^{\text {new }}\right\}$ and $\left\{g_{i j}^{\text {new }}\right\}$, this system takes the form (3.2.5) such that (3.2.21) holds.

From now on, we assume that system (3.2.5) is given such that $f_{i j}$ and $g_{i j}$ satisfy (3.2.21). Recast this system in the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, u_{2}, v\right) u_{2}, \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+f_{21}\left(u_{1}, v\right) u_{1}+J_{1}(u, v) u_{2}+\underline{J_{2}(v) u_{2}}, \\
& \dot{v}_{1}=+\lambda_{1} v_{1}+g_{11}\left(u, v_{1}\right) v_{1}+g_{12}\left(u, v_{1}, v_{2}\right) v_{2},  \tag{3.2.37}\\
& \dot{v}_{2}=+\lambda_{2} v_{2}+g_{21}\left(u, v_{1}\right) v_{1}+J_{3}(u, v) v_{2}+\underline{J_{4}(u) v_{2}},
\end{align*}
$$

where

$$
\begin{array}{ll}
J_{1}(u, v)=f_{22}(u, v)-f_{22}(0, v), & J_{2}(v)=f_{22}(0, v) \\
J_{3}(u, v)=g_{22}(u, v)-g_{22}(u, 0), & J_{4}(u)=g_{22}(u, 0) . \tag{3.2.38}
\end{array}
$$

In order to obtain conditions (3.2.22) and (3.2.23), we need to find a change of coordinates which eliminates the underlined terms in (3.2.37). We claim that this is possible by applying two consecutive $\mathcal{C}^{\infty}$-smooth changes of coordinates of the form

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}, & \tilde{u}_{2}=u_{2}+q_{1}\left(v_{1}, v_{2}\right) u_{2}, \\
\tilde{v}_{1}=v_{1}, & \tilde{v}_{2}=v_{2}, \tag{3.2.39}
\end{array}
$$

and

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}, & \tilde{u}_{2}=u_{2}, \\
\tilde{v}_{1}=v_{1}, & \tilde{v}_{2}=v_{2}+q_{2}\left(u_{1}, u_{2}\right) v_{2}, \tag{3.2.40}
\end{array}
$$

where $q_{1}$ and $q_{2}$ are some functions such that $q_{1}(0)=q_{2}(0)=0$. We show that the underlined terms $J_{2}(v) u_{2}$ and $J_{4}(u) v_{2}$ can be eliminated by applying a change of coordinates of forms (3.2.39) and (3.2.40), respectively.

Applying the change of coordinates (3.2.39) brings system (3.2.37) to the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+\left[f_{11}\left(u_{1}, v\right)\right] u_{1}+\left[\frac{f_{12}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v\right)}{1+q_{1}(v)}\right] u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+\left[\left(1+q_{1}(v)\right) f_{21}\left(u_{1}, v\right)\right] u_{1}+Q_{1}(u, v) u_{2}+Q_{2}(v) u_{2}  \tag{3.2.41}\\
& \dot{v}_{1}=\lambda_{1} v_{1}+\left[g_{11}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v_{1}\right)\right] v_{1}+\left[g_{12}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v_{1}, v_{2}\right)\right] v_{2}, \\
& \dot{v}_{2}=\lambda_{2} v_{2}+\left[g_{21}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v_{1}\right)\right] v_{1}+\left[g_{22}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v\right) v_{2}\right] v_{2},
\end{align*}
$$

where

$$
\begin{aligned}
Q_{1}(u, v)= & J_{1}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v\right)+\frac{q_{1 v_{1}}(v)}{1+q_{1}(v)} \cdot\left[g_{11}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v_{1}\right) v_{1}\right. \\
& \left.+g_{12}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v\right) v_{2}\right]+\frac{q_{1 v_{2}}(v)}{1+q_{1}(v)} \cdot\left[g_{21}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v_{1}\right) v_{1}\right. \\
& \left.+g_{22}\left(u_{1}, \frac{u_{2}}{1+q_{1}(v)}, v\right) v_{2}-g_{21}\left(0, v_{1}\right) v_{1}-g_{22}(0, v) v_{2}\right]
\end{aligned}
$$

and

$$
\begin{equation*}
Q_{2}(v)=J_{2}(v)+\frac{\lambda_{1} q_{1 v_{1}}(v) v_{1}+q_{1 v_{2}}(v)\left(\lambda_{2} v_{2}+g_{21}\left(0, v_{1}\right) v_{1}+g_{22}(0, v) v_{2}\right)}{1+q_{1}(v)} \tag{3.2.42}
\end{equation*}
$$

It is easy to see that $Q_{1}$ vanishes at $u=0$ and also the updated $f_{i j}$ and $g_{i j}$ in system (3.2.41) satisfy all the conditions (3.2.6) except for (3.2.22) and (3.2.23). In order to get (3.2.22), it is sufficient to find $q_{1}(v)$ such that $Q_{2}(v) \equiv 0$, i.e. $q_{1}(v)$ satisfies the relation

$$
\begin{equation*}
-\left(1+q_{1}(v)\right) J_{2}(v)=q_{1_{1}}(v) \cdot\left[\lambda_{1} v_{1}\right]+q_{v_{2}}(v) \cdot\left[\lambda_{2} v_{2}+g_{21}\left(0, v_{1}\right) v_{1}+g_{22}(0, v) v_{2}\right] \tag{3.2.43}
\end{equation*}
$$

Consider the $\mathcal{C}^{\infty}$-smooth system

$$
\begin{align*}
\dot{U} & =-(1+U) J_{2}(v) \\
\dot{v}_{1} & =\lambda_{1} v_{1}  \tag{3.2.44}\\
\dot{v}_{2} & =\lambda_{2} v_{2}+g_{21}\left(0, v_{1}\right) v_{1}+g_{22}(0, v) v_{2}
\end{align*}
$$

where $\left(U, v_{1}, v_{2}\right) \in \mathbb{R}^{3}$. The linear part of this system at the origin is

$$
\left(\begin{array}{ccc}
0 & \frac{\partial J_{2}}{\partial v_{1}}(0) & \frac{\partial J_{2}}{\partial v_{2}}(0) \\
0 & \lambda_{1} & 0 \\
0 & 0 & \lambda_{2}
\end{array}\right)
$$

with the spectrum $\left\{0, \lambda_{1}, \lambda_{2}\right\}$. Therefore, this system has a $\mathcal{C}^{\infty}$-smooth 2-dimensional local unstable invariant manifold defined by the equation $\left\{U=q_{1}\left(v_{1}, v_{2}\right)\right\}$ for some function $q_{1}$. Moreover, this function satisfies (3.2.43) because this relation is nothing but the condition of the invariance of the local unstable invariant manifold with respect to the flow of system (3.2.44) (see Definition 2.14). Thereby, as we required, a $\mathcal{C}^{\infty}$-smooth function $q_{1}\left(v_{1}, v_{2}\right)$ that fulfills (3.2.43) exists.

Now, consider system (3.2.37) such that $J_{2} \equiv 0$. Applying change of coordinates (3.2.40) reduces this system to

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+\left[f_{11}\left(u_{1}, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right)\right] u_{1}+\left[f_{12}\left(u, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right)\right] u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+\left[f_{21}\left(u_{1}, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right)\right] u_{1}+\left[J_{1}\left(u, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right)\right] u_{2}  \tag{3.2.45}\\
& \dot{v}_{1}=+\lambda_{1} v_{1}+\left[g_{11}\left(u, v_{1}\right)\right] v_{1}+\left[g_{12}\left(u, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right)\right] v_{2} \\
& \dot{v}_{2}=+\lambda_{2} v_{2}+\left[\left(1+q_{2}(u)\right) g_{21}\left(u, v_{1}\right)\right] v_{1}+Q_{3}(u, v) v_{2}+Q_{4}(u) v_{2}
\end{align*}
$$

where,

$$
\begin{aligned}
Q_{3}(u, v)= & J_{3}\left(u, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right)+\frac{q_{2 u_{1}}(u)}{1+q_{2}(u)} \cdot\left[f_{11}\left(u_{1}, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right) u_{1}\right. \\
& \left.+f_{12}\left(u_{1}, u_{2}, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right) u_{2}\right]+\frac{q_{2 u_{2}}(u)}{1+q_{2}(u)} \cdot\left[f_{21}\left(u_{1}, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right) u_{1}\right. \\
& \left.+J_{1}\left(u, v_{1}, \frac{v_{2}}{1+q_{2}(u)}\right) u_{2}-f_{21}\left(u_{1}, 0\right) u_{1}-f_{22}(u, 0) u_{2}\right]
\end{aligned}
$$

and

$$
\begin{equation*}
Q_{4}(u)=J_{4}(u)+\frac{-\lambda_{1} q_{2 u_{1}}(u) u_{1}+q_{2 u_{2}}(u)\left(-\lambda_{2} u_{2}+f_{21}\left(u_{1}, 0\right) u_{1}+J_{1}(u, 0) u_{2}\right)}{1+q_{2}(u)} \tag{3.2.46}
\end{equation*}
$$

It is easy to see that $Q_{3}$ vanishes at $v=0$ and also the updated $f_{i j}$ and $g_{i j}$ in system (3.2.45) satisfy all the conditions in (3.2.6) except identity (3.2.23). In order to get (3.2.23), it is sufficient to find $q_{2}(u)$ such that $Q_{4}(v) \equiv 0$, i.e. $q_{2}(u)$ satisfies the relation

$$
\begin{equation*}
-\left(1+q_{2}(u)\right) J_{4}(u)=q_{2 u_{1}}(u) \cdot\left[-\lambda_{1} u_{1}\right]+q_{2 u_{2}}(u) \cdot\left[-\lambda_{2} u_{2}+f_{21}\left(u_{1}, 0\right) u_{1}+J_{1}(u, 0) u_{2}\right] \tag{3.2.47}
\end{equation*}
$$

Consider the $\mathcal{C}^{\infty}$-smooth system

$$
\begin{align*}
\dot{u}_{1} & =-\lambda_{1} u_{1} \\
\dot{u}_{2} & =-\lambda_{2} u_{2}+f_{21}\left(u_{1}, 0\right) u_{1}+J_{1}(u, 0) u_{2}  \tag{3.2.48}\\
\dot{V} & =-(1+V) J_{4}(u)
\end{align*}
$$

where $\left(u_{1}, u_{2}, V\right) \in \mathbb{R}^{3}$. The linear part of this system at the origin is

$$
\left(\begin{array}{ccc}
-\lambda_{1} & 0 & 0 \\
0 & -\lambda_{2} & 0 \\
\frac{\partial J_{4}}{\partial u_{1}}(0) & \frac{\partial J_{4}}{\partial u_{2}}(0) & 0
\end{array}\right)
$$

with the spectrum $\left\{-\lambda_{2},-\lambda_{1}, 0\right\}$. Therefore, this system has a $\mathcal{C}^{\infty}$-smooth two dimensional local stable invariant manifold defined by the equation $\left\{V=q_{2}\left(u_{1}, u_{2}\right)\right\}$ for some function $q_{2}$. Moreover, this function satisfies (3.2.47) because this relation is nothing but the condition of the invariance of the local stable invariant manifold with respect to the flow of system (3.2.48) (see Definition 2.14). Thereby, as we required, a $\mathcal{C}^{\infty}$-smooth function $q_{2}\left(u_{1}, u_{2}\right)$ that fulfills (3.2.47) exists.

So far, we have shown that applying the series of consecutive changes of coordinates (3.2.29), (3.2.30), (3.2.31), (3.2.32), (3.2.39) and (3.2.40) reduces system (3.2.1) to system (3.2.5) such that
(3.2.6) holds. Let us now prove that each of theses changes of coordinates commutes with symmetry (1.2.5). According to Proposition 2.3, this implies that system (3.2.5) and the corresponding first integral $H$ are invariant with respect to this symmetry.

The invariance of system (3.2.1) with respect to symmetry (1.2.5) implies

$$
\begin{aligned}
& f_{11}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=f_{11}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right) \\
& f_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=-f_{12}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right) \\
& f_{21}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=-f_{21}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right) \\
& f_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=f_{22}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
& g_{11}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=g_{11}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right) \\
& g_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=-g_{12}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right) \\
& g_{21}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=-g_{21}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right) \\
& g_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=g_{22}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right)
\end{aligned}
$$

Therefore, we have

$$
\begin{align*}
& R_{1}\left(u_{1}, u_{2}\right)=-R_{1}\left(-u_{1}, u_{2}\right), \quad R_{2}\left(u_{1}, u_{2}\right)=R_{2}\left(-u_{1}, u_{2}\right), \\
& P_{1}\left(v_{1}, v_{2}\right)=-P_{1}\left(-v_{1}, v_{2}\right), \quad P_{2}\left(v_{1}, v_{2}\right)=P_{2}\left(-v_{1}, v_{2}\right),  \tag{3.2.49}\\
& \varphi_{i}\left(v_{1}, v_{2}\right)=\varphi_{i}\left(-v_{1}, v_{2}\right), \quad \varphi_{j}\left(v_{1}, v_{2}\right)=-\varphi_{j}\left(-v_{1}, v_{2}\right), \\
& \psi_{i}\left(u_{1}, u_{2}\right)=\psi_{i}\left(-u_{1}, u_{2}\right), \quad \psi_{j}\left(u_{1}, u_{2}\right)=-\psi_{j}\left(-u_{1}, u_{2}\right),
\end{align*}
$$

for $i=1,4$ and $j=2,3$, and

$$
\begin{gather*}
\tilde{R}_{11}\left(u_{1}, u_{2}\right)=\tilde{R}_{11}\left(-u_{1}, u_{2}\right), \quad \tilde{R}_{12}\left(u_{1}, u_{2}\right)=-\tilde{R}_{12}\left(-u_{1}, u_{2}\right) \\
\tilde{R}_{21}\left(u_{1}, u_{2}\right)=-\tilde{R}_{21}\left(-u_{1}, u_{2}\right), \quad \tilde{R}_{22}\left(u_{1}, u_{2}\right)=\tilde{R}_{22}\left(-u_{1}, u_{2}\right) \\
\tilde{P}_{11}\left(v_{1}, v_{2}\right)=\tilde{P}_{11}\left(-v_{1}, v_{2}\right), \quad \tilde{P}_{12}\left(v_{1}, v_{2}\right)=-\tilde{P}_{12}\left(-v_{1}, v_{2}\right)  \tag{3.2.50}\\
\tilde{P}_{21}\left(v_{1}, v_{2}\right)=-\tilde{P}_{21}\left(-v_{1}, v_{2}\right), \quad \tilde{P}_{22}\left(v_{1}, v_{2}\right)=\tilde{P}_{22}\left(-v_{1}, v_{2}\right)
\end{gather*}
$$

According to (3.2.49), system (3.2.33) is invariant with respect to the symmetry $\left(U_{2}, v_{1}\right) \leftrightarrow\left(-U_{2},-v_{1}\right)$. Therefore, by Proposition 2.24, we have

$$
\begin{equation*}
h_{1}\left(v_{1}, v_{2}\right)=h_{1}\left(-v_{1}, v_{2}\right), \quad h_{2}\left(v_{1}, v_{2}\right)=-h_{2}\left(-v_{1}, v_{2}\right) \tag{3.2.51}
\end{equation*}
$$

This implies that the change of coordinates (3.2.29) commutes with symmetry (1.2.5).
According to (3.2.49), system (3.2.34) is invariant with respect to the symmetry $\left(u_{1}, V_{2}\right) \leftrightarrow$ $\left(-u_{1},-V_{2}\right)$. Therefore, by Proposition 2.24 , we have

$$
\begin{equation*}
s_{1}\left(u_{1}, u_{2}\right)=s_{1}\left(-u_{1}, u_{2}\right), \quad s_{2}\left(u_{1}, u_{2}\right)=-s_{2}\left(-u_{1}, u_{2}\right) \tag{3.2.52}
\end{equation*}
$$

This implies that change of coordinates (3.2.30) commutes with symmetry (1.2.5).
According to (3.2.49) and (3.2.50), system (3.2.35) is invariant with respect to the symmetry $\left(u_{1}, V_{2}\right) \leftrightarrow\left(-u_{1},-V_{2}\right)$. Therefore, by Proposition 2.24 , we have

$$
\begin{equation*}
r_{1}\left(u_{1}, u_{2}\right)=r_{1}\left(-u_{1}, u_{2}\right), \quad r_{2}\left(u_{1}, u_{2}\right)=-r_{2}\left(-u_{1}, u_{2}\right) \tag{3.2.53}
\end{equation*}
$$

This implies that change of coordinates (3.2.31) commutes with symmetry (1.2.5).
According to (3.2.49) and (3.2.50), system (3.2.36) is invariant with respect to the symmetry $\left(U_{2}, v_{1}\right) \leftrightarrow\left(-U_{2},-v_{1}\right)$. Therefore, by Proposition 2.24 , we have

$$
\begin{equation*}
p_{1}\left(v_{1}, v_{2}\right)=p_{1}\left(-v_{1}, v_{2}\right), \quad p_{2}\left(v_{1}, v_{2}\right)=-p_{2}\left(-v_{1}, v_{2}\right) \tag{3.2.54}
\end{equation*}
$$

This implies that change of coordinates (3.2.32) commutes with symmetry (1.2.5).
Since the changes of coordinates listed above commute with symmetry (1.2.5), we have that system (3.2.37) is invariant with respect to this symmetry. This yields

$$
\begin{array}{cc}
f_{11}\left(u_{1}, v_{1}, v_{2}\right)=f_{11}\left(-u_{1},-v_{1}, v_{2}\right), & f_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=-f_{12}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right), \\
f_{21}\left(u_{1}, v_{1}, v_{2}\right)=-f_{21}\left(-u_{1},-v_{1}, v_{2}\right), & f_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=f_{22}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right), \\
g_{11}\left(u_{1}, u_{2}, v_{1}\right)=g_{11}\left(-u_{1}, u_{2},-v_{1}\right), & g_{12}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)-g_{12}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right), \\
g_{21}\left(u_{1}, u_{2}, v_{1}\right)=-g_{21}\left(-u_{1}, u_{2},-v_{1}\right), & g_{22}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=g_{22}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right),  \tag{3.2.55}\\
J_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=J_{1}\left(-u_{1}, u_{2},-v_{1}, v_{2}\right), & J_{3}\left(u_{1}, u_{2}, v_{1}, v_{2}\right)=J_{3}\left(-u_{1}, u_{2},-v_{1}, v_{2},\right. \\
J_{2}\left(v_{1}, v_{2}\right)=J_{2}\left(-v_{1}, v_{2}\right), & J_{4}\left(u_{1}, u_{2}\right)=J_{4}\left(-u_{1}, u_{2}\right) .
\end{array}
$$

This implies that system (3.2.44) is invariant with respect to symmetry $v_{1} \leftrightarrow-v_{1}$. Therefore, by Proposition 2.24, we have $q_{1}\left(v_{1}, v_{2}\right)=q_{1}\left(-v_{1}, v_{2}\right)$. This means that change of coordinates (3.2.39) commutes with symmetry (1.2.5).

According to (3.2.55), system (3.2.48) is invariant with respect to the symmetry $u_{1} \leftrightarrow-u_{1}$. Therefore, Proposition 2.24 implies $q_{2}\left(u_{1}, u_{2}\right)=q_{2}\left(-u_{1}, u_{2}\right)$. Consequently, change of coordinates (3.2.40) commutes with symmetry (1.2.5).

Note that by $(3.2 .51),(3.2 .52),(3.2 .53)$ and (3.2.54), we have

$$
\begin{equation*}
h_{2}\left(0, v_{2}\right)=0, \quad s_{2}\left(0, u_{2}\right)=0, \quad r_{2}\left(0, u_{2}\right)=0, \quad p_{2}\left(0, v_{2}\right)=0 \tag{3.2.56}
\end{equation*}
$$

We have proved that each of the changes of coordinates we made to obtain system (3.2.5) from system (1.2.3) commutes with symmetry (1.2.5). This implies that system (3.2.5) and its corresponding first integral $H$ are invariant with respect to that symmetry. To finish the proof of Lemma 3.6, we need to show that making changes of coordinates (3.2.29), (3.2.30), (3.2.31), (3.2.32), (3.2.39) and (3.2.40) transforms first integral (3.2.3) to the form (3.2.7). To do this, first, observe that according to (3.2.56), each of these changes of coordinates can be written in the form

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}[1+o(1)], & \tilde{u}_{2}=u_{2}[1+o(1)]+u_{1} v_{1} O(1), \\
\tilde{v}_{1}=v_{1}[1+o(1)], & \tilde{v}_{2}=v_{2}[1+o(1)]+u_{1} v_{1} O(1), \tag{3.2.57}
\end{array}
$$

where $o(1)$ and $O(1)$ stand for $\mathcal{C}^{\infty}$-smooth functions of $(u, v)$ which converge to zero and are bounded above by a constant, respectively, as $(u, v) \rightarrow O$. On the other hand, the form (3.2.3) of the first integral $H$ is already of the form (3.2.7). Thus, we will be done once we show that making a change of coordinates of the form (3.2.57) preserves the form (3.2.7) of the first integral. According to (3.2.57),

$$
\begin{array}{ll}
u_{1}=\tilde{u}_{1}[1+o(1)], & u_{2}=\tilde{u}_{2}[1+o(1)]+\tilde{u}_{1} \tilde{v}_{1} O(1) \\
v_{1}=\tilde{v}_{1}[1+o(1)], & v_{2}=\tilde{v}_{2}[1+o(1)]+\tilde{u}_{1} \tilde{v}_{1} O(1) \tag{3.2.58}
\end{array}
$$

Substituting (3.2.58) into $H(u, v)=\lambda_{1} u_{1} v_{1}[1+o(1)]-\lambda_{2} u_{2} v_{2}[1+o(1)]$ gives

$$
\begin{aligned}
H(\tilde{u}, \tilde{v}) & =\lambda_{1} \tilde{u}_{1} \tilde{v}_{1}[1+o(1)]-\lambda_{2} \tilde{u}_{2} \tilde{v}_{2}[1+o(1)]+\tilde{u}_{1} \tilde{v}_{1} O\left(\left|\tilde{u}_{2}\right|+\left|\tilde{v}_{2}\right|\right)+\tilde{u}_{1}^{2} \tilde{v}_{1}^{2} O(1) \\
& =\lambda_{1} \tilde{u}_{1} \tilde{v}_{1}[1+o(1)]-\lambda_{2} \tilde{u}_{2} \tilde{v}_{2}[1+o(1)]
\end{aligned}
$$

which is of the form (3.2.7). This ends the proof of Lemma 3.6.
Proof of Lemma 3.8. By Lemma 3.6, there exists a change of coordinates which commutes with symmetry (1.2.5) and brings system (1.2.3) and first integral (1.2.4) to (3.2.5) and (3.2.7), respectively. According to Section 2.3 (see Theorem 2.16), system (3.2.5) possesses a $\mathcal{C}^{q}$-smooth three dimensional extended unstable invariant manifold $W^{u E}$ defined by

$$
\begin{equation*}
\left\{(u, v): \quad u_{2}=\phi^{u E}\left(u_{1}, v_{1}, v_{2}\right)\right\} \tag{3.2.59}
\end{equation*}
$$

and a $\mathcal{C}^{q}$-smooth extended stable invariant manifold $W^{s E}$ defined by

$$
\begin{equation*}
\left\{(u, v): \quad v_{2}=\phi^{s E}\left(u_{1}, u_{2}, v_{1}\right)\right\} \tag{3.2.60}
\end{equation*}
$$

We claim that, straightening $W^{u E}$, i.e. applying the $\mathcal{C}^{q}$-smooth change of coordinates

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}, & \tilde{u}_{2}=u_{2}-\phi^{u E}\left(u_{1}, v_{1}, v_{2}\right)  \tag{3.2.61}\\
\tilde{v}_{1}=v_{1}, & \tilde{v}_{2}=v_{2}
\end{array}
$$

and straightening $W^{s E}$, i.e. applying the $\mathcal{C}^{q}$-smooth change of coordinates

$$
\begin{array}{ll}
\tilde{u}_{1}=u_{1}, & \tilde{u}_{2}=u_{2} \\
\tilde{v}_{1}=v_{1}, & \tilde{v}_{2}=v_{2}-\phi^{s E}\left(u_{1}, u_{2}, v_{1}\right), \tag{3.2.62}
\end{array}
$$

reduce system (3.2.5) to system (3.2.9), where (3.2.6) is satisfied, and transforms first integral (3.2.7) to (3.2.10). On the other hand, thanks to Corollary 2.25 , straightening these manifolds keeps the invariance of system (3.2.5) and first integral (3.2.7) with respect to symmetry (1.2.5). Thus, we will be done as soon as we prove this claim. To this end, we use the following lemma

Lemma 3.17. The following hold for the $\mathcal{C}^{q}$-smooth functions $\phi^{u E}$ and $\phi^{s E}$ :
(i) $\phi^{u E}\left(0, v_{1}, v_{2}\right) \equiv \phi_{u_{1}}^{u E}\left(0, v_{1}, v_{2}\right) \equiv 0$,
(ii) $\phi^{s E}\left(u_{1}, u_{2}, 0\right) \equiv \phi_{v_{1}}^{s E}\left(u_{1}, u_{2}, 0\right) \equiv 0$.

The following are immediate consequences of this lemma:
Corollary 3.18. We can write $\phi^{u E}$ and $\phi^{s E}$ as

$$
\begin{aligned}
& \phi^{u E}\left(u_{1}, v_{1}, v_{2}\right)=u_{1} p_{1}^{u E}\left(u_{1}, v_{1}, v_{2}\right)=u_{1}^{2} p_{2}^{u E}\left(u_{1}, v_{1}, v_{2}\right) \\
& \phi^{s E}\left(u_{1}, u_{2}, v_{1}\right)=v_{1} p_{1}^{s E}\left(u_{1}, u_{2}, v_{1}\right)=v_{1}^{2} p_{2}^{s E}\left(u_{1}, u_{2}, v_{1}\right)
\end{aligned}
$$

where $p_{1}^{s E}$ and $p_{1}^{u E}$ are some $\mathcal{C}^{q-1}$-smooth functions and $p_{2}^{s E}$ and $p_{2}^{u E}$ are some $\mathcal{C}^{q-2}$-smooth functions such that $p_{1}^{u E}=u_{1} p_{2}^{u E}$ and $p_{1}^{s E}=v_{1} p_{2}^{s E}$.

Corollary 3.19. We have
(i) $\phi_{v_{1}}^{u E}\left(0, v_{1}, v_{2}\right) \equiv \phi_{v_{2}}^{u E}\left(0, v_{1}, v_{2}\right) \equiv 0$,
(ii) $\phi_{u_{1}}^{s E}\left(u_{1}, u_{2}, 0\right) \equiv \phi_{u_{2}}^{s E}\left(u_{1}, u_{2}, 0\right) \equiv 0$.

We prove Lemma 3.17 later. For now, let us see how change of coordinates (3.2.61) affects system (3.2.5): this change of coordinates reduces system (3.2.5) to

$$
\begin{align*}
\dot{u}_{1}= & -\lambda_{1} u_{1}+f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)\left(u_{2}+\phi^{u E}\left(u_{1}, v\right)\right)  \tag{3.2.63}\\
\dot{u}_{2}= & -\lambda_{2}\left(u_{2}+\phi^{u E}\left(u_{1}, v\right)\right)+f_{21}\left(u_{1}, v\right) u_{1}+f_{22}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)\left(u_{2}+\phi^{u E}\left(u_{1}, v\right)\right) \\
& -\phi_{u_{1}}^{u E}\left(u_{1}, v\right) \cdot\left[-\lambda_{1} u_{1}+f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)\left(u_{2}+\phi^{u E}\left(u_{1}, v\right)\right)\right] \\
& -\phi_{v_{1}}^{u E}\left(u_{1}, v\right) \cdot\left[\lambda_{1} v_{1}+g_{11}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right) v_{1}+g_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right) v_{2}\right] \\
& -\phi_{v_{2}}^{u E}\left(u_{1}, v\right) \cdot\left[\lambda_{2} v_{2}+g_{21}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right) v_{1}+g_{22}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right) v_{2}\right], \\
\dot{v}_{1}= & +\lambda_{1} v_{1}+g_{11}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right) v_{1}+g_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}, v_{2}\right) v_{2}, \\
\dot{v}_{2}= & +\lambda_{2} v_{2}+g_{21}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right) v_{1}+g_{22}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}, v_{2}\right) v_{2},
\end{align*}
$$

where, $f_{i j}$ and $g_{i j}$ are as in (3.2.5).

Denote the right-hand side of the second equation of $(3.2 .63)$ by $Q\left(u_{1}, u_{2}, v_{1}, v_{2}\right)$. Since $Q$ vanishes at $\left\{u_{2}=0\right\}$, we can rewrite

$$
\dot{u}_{2}=Q\left(u_{1}, u_{2}, v_{1}, v_{2}\right)-Q\left(u_{1}, 0, v_{1}, v_{2}\right)
$$

We recast system (3.2.63) in the form

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+\tilde{f}_{11}\left(u_{1}, v\right) u_{1}+\tilde{f}_{12}\left(u_{1}, u_{2}, v\right) u_{2} \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+\tilde{f}_{22}\left(u_{1}, u_{2}, v\right) u_{2}  \tag{3.2.64}\\
& \dot{v}_{1}=+\lambda_{1} v_{1}+\tilde{g}_{11}\left(u, v_{1}\right) v_{1}+\tilde{g}_{12}\left(u, v_{1}, v_{2}\right) v_{2} \\
& \dot{v}_{2}=+\lambda_{2} v_{2}+\tilde{g}_{21}\left(u, v_{1}\right) v_{1}+\tilde{g}_{22}\left(u, v_{1}, v_{2}\right) v_{2}
\end{align*}
$$

where

$$
\begin{aligned}
\tilde{f}_{11}\left(u_{1}, v\right)= & f_{11}\left(u_{1}, v\right)+f_{12}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) p_{1}^{u E}\left(u_{1}, v\right) \\
\tilde{f}_{12}(u, v)= & f_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)+P_{1}(u, v) \phi^{u E}\left(u_{1}, v\right) \\
\tilde{f}_{22}(u, v)= & f_{22}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)+P_{2}(u, v) \phi^{u E}\left(u_{1}, v\right) \\
& -\phi_{u_{1}}^{u E}\left(u_{1}, v\right)\left[f_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)+P_{3}(u, v) \phi^{u E}\left(u_{1}, v\right)\right] \\
& -\phi_{v_{1}}^{u E}\left(u_{1}, v\right)\left[P_{4}(u, v) v_{1}+P_{5}(u, v) v_{2}\right] \\
& -\phi_{v_{2}}^{u E}\left(u_{1}, v\right)\left[P_{6}(u, v) v_{1}+P_{7}(u, v) v_{2}\right] \\
\tilde{g}_{i 1}\left(u, v_{1}\right)= & g_{i 1}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right), \quad(i=1,2) \\
\tilde{g}_{i 2}(u, v)= & g_{i 2}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right), \quad(i=1,2)
\end{aligned}
$$

and

$$
\begin{aligned}
& u_{2} P_{1}(u, v)=f_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)-f_{12}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) \\
& u_{2} P_{2}(u, v)=f_{22}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)-f_{22}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) \\
& u_{2} P_{3}(u, v)=f_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)-f_{12}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) \\
& u_{2} P_{4}(u, v)=g_{11}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right)-g_{11}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v_{1}\right) \\
& u_{2} P_{5}(u, v)=g_{12}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)-g_{12}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) \\
& u_{2} P_{6}(u, v)=g_{21}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v_{1}\right)-g_{21}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v_{1}\right) \\
& u_{2} P_{7}(u, v)=g_{22}\left(u_{1}, u_{2}+\phi^{u E}\left(u_{1}, v\right), v\right)-g_{22}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right)
\end{aligned}
$$

Here $\tilde{f}_{i j}$ are $\mathcal{C}^{q-1}{ }_{\text {-smooth }}$ and $\tilde{g}_{i j}$ are $\mathcal{C}^{q}$-smooth. Using Lemma 3.17 and Corollaries 3.18 and 3.19 and taking into account that the expression $u_{2}+\phi^{u E}\left(u_{1}, v\right)$ vanish at $u=0$ and also the functions $f_{i j}$ and $g_{i j}$ satisfy (3.2.6), one can easily show that $\tilde{f}_{i j}$ and $\tilde{g}_{i j}$ satisfy (3.2.6) as well.

System (3.2.64) is of the form (3.2.5) where $f_{21}\left(u_{1}, v\right) \equiv 0$. Similar to the case of straightening the extended unstable manifold, one can use Lemma 3.17 and Corollaries 3.18 and 3.19 and show that making change of coordinates (3.2.62) reduces system (3.2.64) to system (3.2.9) where the correspond$\operatorname{ing} f_{i j}$ and $g_{i j}$ are $\mathcal{C}^{q-1}$-smooth and satisfy (3.2.6). This ends the proof of the first part of Lemma 3.8.

Denote the $H_{1}$ and $H_{2}$ in (3.2.7) by $H_{1}^{\circ}$ and $H_{2}^{\circ}$, respectively, and let $x:=\left(u_{1}, u_{2}, v_{1}\right), y:=$ $\left(u_{1}, v_{1}, v_{2}+\phi^{s E}(x)\right)$ and $z:=\left(u_{1}, u_{2}+\phi^{u E}(y), v_{1}, v_{2}+\phi^{s E}(x)\right)$. Applying changes of coordinates (3.2.61) and (3.2.62) brings (3.2.7) to

$$
H=\lambda_{1} u_{1} v_{1}\left[1+H_{1}^{\circ}(z)\right]-\lambda_{2}\left(u_{2}+\phi^{u E}(y)\right)\left(v_{2}+\phi^{s E}(x)\right)\left[1+H_{2}^{\circ}(z)\right]
$$

which by Corollary 3.18, can be written in the form (3.2.10), for

$$
\begin{array}{ll}
H_{1}=H_{1}^{\circ}(z)+\lambda_{2} \lambda_{1}^{-1} p_{1}^{u E}(y) p_{1}^{s E}(x)\left[1+H_{2}^{\circ}(z)\right], & H_{2}=H_{2}^{\circ} \\
H_{3}=p_{2}^{s E}(x)\left[1+H_{2}^{\circ}(z)\right], & H_{4}=p_{2}^{u E}(y)\left[1+H_{2}^{\circ}(z)\right]
\end{array}
$$

This proves the second part of Lemma 3.8.
All that remains to finish the proof of Lemma 3.8 is proving Lemma 3.17. We only prove part (i) of this lemma; the proof of part (ii) is the same. The first identity

$$
\begin{equation*}
\phi^{u E}\left(0, v_{1}, v_{2}\right) \equiv 0 \tag{3.2.65}
\end{equation*}
$$

is an immediate consequence of the fact that the extended unstable invariant manifold $W^{u E}$ contains the unstable invariant manifold $\left\{u_{1}=u_{2}=0\right\}$ (see Section 2.3.2). Indeed,

$$
\begin{aligned}
\forall v_{1}, v_{2}, & \left(0,0, v_{1}, v_{2}\right) \in\left\{(u, v): u_{2}=\phi^{u E}\left(u_{1}, v_{1}, v_{2}\right)\right\} \\
\Longrightarrow & \forall v_{1}, v_{2}, \quad \phi^{u E}\left(0, v_{1}, v_{2}\right)=0 \Longrightarrow \phi^{u E}\left(0, v_{1}, v_{2}\right) \equiv 0
\end{aligned}
$$

It is important to notice that relation (3.2.65) is sufficient to obtain the statement of part I of Corollary 3.19. In other words, (3.2.65) implies part I of Corollary 3.19.

To prove the identity

$$
\begin{equation*}
\phi_{u_{1}}^{u E}\left(0, v_{1}, v_{2}\right) \equiv 0 \tag{3.2.66}
\end{equation*}
$$

we consider the condition of the invariance of the manifold $W^{u E}$ with respect to the flow of system (3.2.5) (see Definition 2.14), i.e.

$$
\begin{aligned}
-\lambda_{2} \phi^{u E} & \left(u_{1}, v\right)+f_{21}\left(u_{1}, v\right) u_{1}+f_{22}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) \phi^{u E}\left(u_{1}, v\right)= \\
& \phi_{u_{1}}^{u E}\left(u_{1}, v\right)\left[-\lambda_{1} u_{1}+f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) \phi^{u E}\left(u_{1}, v\right)\right] \\
& +\phi_{v_{1}}^{u E}\left(u_{1}, v\right)\left[\lambda_{1} v_{1}+g_{11}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v_{1}\right) v_{1}+g_{12}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) v_{2}\right] \\
& +\phi_{v_{2}}^{u E}\left(u_{1}, v\right)\left[\lambda_{2} v_{2}+g_{21}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v_{1}\right) v_{1}+g_{22}\left(u_{1}, \phi^{u E}\left(u_{1}, v\right), v\right) v_{2}\right]
\end{aligned}
$$

Both sides of this relation are $\mathcal{C}^{q-1}$-smooth $\left(q \geq 2\right.$ because $\left.2 \lambda_{1}<\lambda_{2}\right)$ functions of $u_{1}, v_{1}$ and $v_{2}$. Taking (3.2.65) as well as conditions (3.2.6) and Corollary 3.19 into account, we can differentiate this relation with respect to $u_{1}$ at $u_{1}=0$ and obtain

$$
\begin{align*}
0= & {\left[\left(\lambda_{2}-\lambda_{1}\right) \phi_{u_{1}}^{u E}(0, v)+f_{12}(0, v)\left(\phi_{u_{1}}^{u E}(0, v)\right)^{2}\right]+\left[\lambda_{1} v_{1}\right] \phi_{u_{1} v_{1}}^{u E}(0, v) }  \tag{3.2.67}\\
& +\left[\lambda_{2} v_{2}+g_{21}\left(0, v_{1}\right) v_{1}+g_{22}(0, v) v_{2}\right] \phi_{u_{1} v_{2}}^{u E}(0, v)
\end{align*}
$$

We introduce the notation $z=z(v)=\phi_{u_{1}}^{u E}(0, v)$. Using this notation, (3.2.67) can be written as

$$
\begin{align*}
0= & {\left[\left(\lambda_{2}-\lambda_{1}\right) z+f_{12}(0, v) z^{2}\right]+\left[\lambda_{1} v_{1}\right] \cdot \frac{\partial}{\partial v_{1}} z(v) } \\
& +\left[\lambda_{2} v_{2}+g_{21}\left(0, v_{1}\right) v_{1}+g_{22}(0, v) v_{2}\right] \cdot \frac{\partial}{\partial v_{2}} z(v) \tag{3.2.68}
\end{align*}
$$

where $z(0)=0$ (note that $\left.\phi_{u_{1}}^{u E}(0,0,0)=0\right)$.
To get (3.2.66), we need to show $z(v) \equiv 0$. First, note that $z(v) \equiv 0$ satisfies (3.2.68). Thus, (3.2.66) holds if we show that $z \equiv 0$ is the unique solution of (3.2.68). Note that, according to Proposition $2.13, z(v)$ satisfies (3.2.68) if and only if the two dimensional manifold

$$
\begin{equation*}
\{(v, z): z=z(v) \text { and } z(0)=0\} \tag{3.2.69}
\end{equation*}
$$

be invariant with respect to the flow of the $\mathcal{C}^{q-1}$-smooth system

$$
\begin{align*}
\dot{v}_{1} & =-\lambda_{1} v_{1} \\
\dot{v}_{2} & =-\lambda_{2} v_{2}-g_{21}\left(0, v_{1}\right) v_{1}-g_{22}(0, v) v_{2},  \tag{3.2.70}\\
\dot{z} & =\left(\lambda_{2}-\lambda_{1}\right) z+f_{12}(0, v) z^{2}
\end{align*}
$$

which is defined on a small neighborhood of the origin in $\mathbb{R}^{3}$. (Indeed, relation (3.2.68) is the condition of the invariance of (3.2.69) with respect to the flow of system (3.2.70).) Therefore, the uniqueness of the solution of (3.2.68) can be proved by showing that system (3.2.70) has a unique invariant manifold of the form (3.2.69). To do this, first, notice that this system possesses a unique two dimensional stable invariant manifold of the form (3.2.69). Second, we observe that any orbit on manifold (3.2.69) converges to the origin of system (3.2.70): the first two equations in (3.2.70) are independent of $z$ and have $\left(v_{1}, v_{2}\right)=(0,0)$ as an asymptotically stable equilibrium. Therefore, as $t \rightarrow \infty$, an orbit $(v(t), z(v(t)))$ of system (3.2.70) which belongs to invariant manifold (3.2.69) converges to ( $0, z(0)$ ). Since $z(0)=0$, this means that any invariant manifold of the form (3.2.70) must be a subset of the stable manifold of system (3.2.70). However, since both manifolds are 2-dimensional, they must be the same. Therefore, system (3.2.70) has a unique invariant manifold of the form (3.2.69) which is in fact its stable invariant manifold. This ends the proof of Lemma 3.17 and hence the proof of Lemma 3.8 .

### 3.3. Trajectories near the equilibrium state $O$

In this section, we will estimate solutions of systems (3.2.1), (3.2.5) and (3.2.9) near the equilibrium state $O$ by using the technique of successive approximations (see Section 2.2). Consider the system

$$
\begin{align*}
& \dot{u}_{1}=-\lambda_{1} u_{1}+F_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right), \\
& \dot{u}_{2}=-\lambda_{2} u_{2}+F_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right),  \tag{3.3.1}\\
& \dot{v}_{1}=+\lambda_{1} v_{1}+G_{1}\left(u_{1}, u_{2}, v_{1}, v_{2}\right), \\
& \dot{v}_{2}=+\lambda_{2} v_{2}+G_{2}\left(u_{1}, u_{2}, v_{1}, v_{2}\right),
\end{align*}
$$

where $F_{1}, F_{2}, G_{1}$ and $G_{2}$ and their first derivatives vanish at the origin. According to Theorem 2.7, for given $\tau \geq 0$ and sufficiently small $u_{10}, u_{20}, v_{1 \tau}$ and $v_{2 \tau}$ there exists a unique solution ( $\left.u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right)$ of system (3.3.1) such that

$$
\begin{equation*}
u_{1}(0)=u_{10}, \quad u_{2}(0)=u_{20}, \quad v_{1}(\tau)=v_{1 \tau}, \quad v_{2}(\tau)=v_{2 \tau} \tag{3.3.2}
\end{equation*}
$$

The dependence of this solution on each of the variables $\tau, u_{10}, u_{20}, v_{1 \tau}$ and $v_{2 \tau}$ is as smooth as the original system (3.3.1).

The following lemmas estimate solutions of systems (3.2.1), (3.2.5) and (3.2.9) that satisfy boundary condition (3.3.2).

Lemma 3.20. Let $\lambda=\lambda_{1}=\lambda_{2}$. There exists $M>0$ such that for any sufficiently small $\delta>0$, and any $u_{10}, u_{20}, v_{1 \tau}$ and $v_{2 \tau}$, where $\max \left\{\left|u_{10}\right|,\left|u_{20}\right|,\left|v_{1 \tau}\right|,\left|v_{2 \tau}\right|\right\} \leq \delta$, the solution $(u(t), v(t))$ of system (3.2.1) that satisfies boundary condition (3.3.2) can be written as

$$
\begin{align*}
u_{1}(t) & =e^{-\lambda t} u_{10}+\xi_{1}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right), \\
u_{2}(t) & =e^{-\lambda t} u_{20}+\xi_{2}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right), \\
v_{1}(t) & =e^{-\lambda(\tau-t)} v_{1 \tau}+\zeta_{1}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right),  \tag{3.3.3}\\
v_{2}(t) & =e^{-\lambda(\tau-t)} v_{2 \tau}+\zeta_{2}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right),
\end{align*}
$$

where $t \in[0, \tau], \max \left\{\left|\xi_{1}\right|,\left|\xi_{2}\right|\right\} \leq M e^{-\lambda t} \delta^{2}$ and $\max \left\{\left|\zeta_{1}\right|,\left|\zeta_{2}\right|\right\} \leq M e^{-\lambda(\tau-t)} \delta^{2}$.

Remark 3.21. For simplicity, we can write (3.3.3) as

$$
\begin{align*}
u_{1}(t) & =e^{-\lambda t}\left[u_{10}+O\left(\delta^{2}\right)\right], & & u_{2}(t)=e^{-\lambda t}\left[u_{20}+O\left(\delta^{2}\right)\right], \\
v_{1}(t) & =e^{-\lambda(\tau-t)}\left[v_{1 \tau}+O\left(\delta^{2}\right)\right], & & v_{2}(t)=e^{-\lambda(\tau-t)}\left[v_{2 \tau}+O\left(\delta^{2}\right)\right] . \tag{3.3.4}
\end{align*}
$$

Lemma 3.22. There exists $M>0$ such that for any sufficiently small $\delta>0$, and any $u_{10}, u_{20}$, $v_{1 \tau}$ and $v_{2 \tau}$, where $\max \left\{\left|u_{10}\right|,\left|u_{20}\right|,\left|v_{1 \tau}\right|,\left|v_{2 \tau}\right|\right\} \leq \delta$, the solution $(u(t), v(t))$ of system (3.2.5) that satisfies boundary condition (3.3.2) can be written as

$$
\begin{align*}
& u_{1}(t)=e^{-\lambda_{1} t} u_{10}+\xi_{1}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right), \\
& u_{2}(t)=e^{-\lambda_{2} t} u_{20}+\xi_{2}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right), \\
& v_{1}(t)=e^{-\lambda_{1}(\tau-t)} v_{1 \tau}+\zeta_{1}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right),  \tag{3.3.5}\\
& v_{2}(t)=e^{-\lambda_{2}(\tau-t)} v_{2 \tau}+\zeta_{2}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right),
\end{align*}
$$

where $t \in[0, \tau]$ and

$$
\begin{array}{ll}
\left|\xi_{1}\right| \leq M\left[e^{-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|\right], & \left|\xi_{2}\right| \leq M e^{-\lambda_{2} t} \delta^{2} \\
\left|\zeta_{1}\right| \leq M\left[e^{-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right|\right], & \left|\zeta_{2}\right| \leq M e^{-\lambda_{2}(\tau-t)} \delta^{2}
\end{array}
$$

Remark 3.23. For simplicity, we can write (3.3.5) as

$$
\begin{align*}
u_{1}(t) & =e^{-\lambda_{1} t} u_{10}[1+O(\delta)]+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} O\left(\delta v_{1 \tau}\right), & u_{2}(t)=e^{-\lambda_{2} t}\left[u_{20}+O\left(\delta^{2}\right)\right], \\
v_{1}(t) & =e^{-\lambda_{1}(\tau-t)} v_{1 \tau}[1+O(\delta)]+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} O\left(\delta u_{10}\right), & v_{2}(t)=e^{-\lambda_{2}(\tau-t)}\left[v_{2 \tau}+O\left(\delta^{2}\right)\right] . \tag{3.3.6}
\end{align*}
$$

Lemma 3.24. There exists $M>0$ such that for any sufficiently small $\delta>0$, and any $u_{10}, u_{20}$, $v_{1 \tau}$ and $v_{2 \tau}$, where $\max \left\{\left|u_{10}\right|,\left|u_{20}\right|,\left|v_{1 \tau}\right|,\left|v_{2 \tau}\right|\right\} \leq \delta$, the solution $(u(t), v(t))$ of system (3.2.9) that satisfies boundary condition (3.3.2) can be written in the form (3.3.5), where $t \in[0, \tau]$ and

$$
\begin{array}{ll}
\left|\xi_{1}\right| \leq M\left[e^{-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|\right], & \left|\xi_{2}\right| \leq M e^{-\lambda_{2} t} \delta^{2} \\
\left|\zeta_{1}\right| \leq M\left[e^{-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{1}(2 \tau+t)} \delta\left|u_{10}\right|\right], & \left|\zeta_{2}\right| \leq M e^{-\lambda_{2}(\tau-t)} \delta^{2}
\end{array}
$$

Remark 3.25. For simplicity, we can write the solution given by Lemma 3.24 as

$$
\begin{align*}
u_{1}(t) & =e^{-\lambda_{1} t} u_{10}[1+O(\delta)]+e^{-\lambda_{1}(\tau+t)} O\left(\delta v_{1 \tau}\right), & & u_{2}(t)=e^{-\lambda_{2} t}\left[u_{20}+O\left(\delta^{2}\right)\right],  \tag{3.3.7}\\
v_{1}(t) & =e^{-\lambda_{1}(\tau-t)} v_{1 \tau}[1+O(\delta)]+e^{-\lambda_{1}(2 \tau-t)} O\left(\delta u_{10}\right), & & v_{2}(t)=e^{-\lambda_{2}(\tau-t)}\left[v_{2 \tau}+O\left(\delta^{2}\right)\right] .
\end{align*}
$$

### 3.3.1. Our computational scheme

Here, we present the technique which is used in the proofs of preceding lemmas and Lemma 3.34 which is stated later. Consider system (3.3.1) and denote its unique solution that satisfies boundary condition (3.3.2) by $\left(u^{*}, v^{*}\right)$, where $u^{*}=\left(u_{1}^{*}, u_{2}^{*}\right)$ and $v^{*}=\left(v_{1}^{*}, v_{2}^{*}\right)$. We may also write this as

$$
\begin{equation*}
\left(u^{*}, v^{*}\right)=\left(u^{*}(t), v^{*}(t)\right)=\left(u^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right), v^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)\right), \tag{3.3.8}
\end{equation*}
$$

to emphasise that in addition to time variable $t$, this solution explicitly depends on $\tau$ and the boundary conditions $u_{10}, u_{20}, v_{1 \tau}$ and $v_{2 \tau}$ as well. Following the proof of Theorem 2.7, $\left(u^{*}(t), v^{*}(t)\right)$ is a solution
of this system with boundary conditions (3.3.2) if and only if

$$
\begin{align*}
& u_{1}^{*}(t)=e^{-\lambda_{1} t} u_{10}+\int_{0}^{t} e^{\lambda_{1}(s-t)} F_{1}\left(u^{*}(s), v^{*}(s)\right) d s \\
& u_{2}^{*}(t)=e^{-\lambda_{2} t} u_{20}+\int_{0}^{t} e^{\lambda_{2}(s-t)} F_{2}\left(u^{*}(s), v^{*}(s)\right) d s  \tag{3.3.9}\\
& v_{1}^{*}(t)=e^{-\lambda_{1}(\tau-t)} v_{1 \tau}-\int_{t}^{\tau} e^{-\lambda_{1}(s-t)} G_{1}\left(u^{*}(s), v^{*}(s)\right) d s \\
& v_{2}^{*}(t)=e^{-\lambda_{2}(\tau-t)} v_{2 \tau}-\int_{t}^{\tau} e^{-\lambda_{2}(s-t)} G_{2}\left(u^{*}(s), v^{*}(s)\right) d s
\end{align*}
$$

For a given $\tau$, we denote by $\mathcal{I}$ the set of all vector valued functions $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right)$ defined for $t \in[0, \tau]$ on some small neighborhood of the origin in $\mathbb{R}^{4}$. Then, the right-hand side of (3.3.9) defines an integral operator on $\mathcal{I}$, denote it by $\mathfrak{T}$, as follows:

$$
\mathfrak{T}:\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right) \mapsto\left(\bar{u}_{1}(t), \bar{u}_{2}(t), \bar{v}_{1}(t), \bar{v}_{2}(t)\right)
$$

where

$$
\begin{aligned}
& \bar{u}_{1}(t)=e^{-\lambda_{1} t} u_{10}+\int_{0}^{t} e^{\lambda_{1}(s-t)} F_{1}(u(s), v(s)) d s \\
& \bar{u}_{2}(t)=e^{-\lambda_{2} t} u_{20}+\int_{0}^{t} e^{\lambda_{2}(s-t)} F_{2}(u(s), v(s)) d s \\
& \bar{v}_{1}(t)=e^{-\lambda_{1}(\tau-t)} v_{1 \tau}-\int_{t}^{\tau} e^{-\lambda_{1}(s-t)} G_{1}(u(s), v(s)) d s \\
& \bar{v}_{2}(t)=e^{-\lambda_{2}(\tau-t)} v_{2 \tau}-\int_{t}^{\tau} e^{-\lambda_{2}(s-t)} G_{2}(u(s), v(s)) d s
\end{aligned}
$$

The solution $\left(u^{*}(t), v^{*}(t)\right)$ is in fact the fixed point of this integral operator. It follows from the proof of Theorem 2.7 that this integral operator is a contraction and its fixed point is the limit of the sequence of successive approximations

$$
\left\{\left(u^{(n)}(t), v^{(n)}(t)\right)=\left(u_{1}^{(n)}(t), u_{2}^{(n)}(t), v_{1}^{(n)}(t), v_{2}^{(n)}(t)\right)\right\}_{n=0}^{n=\infty}
$$

where $\left(u^{(0)}, v^{(0)}\right) \equiv(0,0)$ and

$$
\left(u^{(n+1)}(t), v^{(n+1)}(t)\right)=\mathfrak{T}\left(u^{(n)}(t), v^{(n)}(t)\right), \quad \forall n \geq 0
$$

Let $\mathcal{A}$ be a closed subset of $\mathcal{I}$ such that $(u(t), v(t)) \equiv(0,0) \in \mathcal{A}$ and $\mathfrak{T}(\mathcal{A}) \subset \mathcal{A}$. Since $\left(u^{(0)}, v^{(0)}\right) \equiv$ $(0,0) \in \mathcal{A}$, the invariance of $\mathcal{A}$ implies that $\left(u^{(n)}(t), v^{(n)}(t)\right)$ belongs to $\mathcal{A}$ for all $n>0$, and so does the solution $\left(u^{*}(t), v^{*}(t)\right)$.

Remark 3.26. Assume that there exists a 'certain estimate' which for any arbitrary $(u(t), v(t)) \in$ $\mathcal{A}$, its image $\mathfrak{T}(u(t), v(t))$ satisfies. Therefore, since $\mathfrak{T}\left(u^{*}, v^{*}\right)=\left(u^{*}, v^{*}\right)$, the solution $\left(u^{*}, v^{*}\right)$ itself satisfies that certain estimate as well.

Our approach for proving Lemmas 3.20, 3.22 and 3.24 (and Lemma 3.34) is based on this remark. We construct the integral operator, introduce the invariant set $\mathcal{A}$ and find an estimate for the image of the elements of this set under $\mathfrak{T}$. Then, this estimate holds for the solution $\left(u^{*}, v^{*}\right)$ too.

### 3.3.2. Proofs of Lemmas 3.20, 3.22 and 3.24

Proof of Lemma 3.20. We recast system (3.2.1) into the form (3.3.1), where

$$
\begin{align*}
F_{1}(u, v)= & \mathbf{f}_{11}(u, v) u_{1}^{2}+\mathbf{f}_{12}(u, v) u_{1} u_{2}+\mathbf{f}_{13}(u, v) u_{1} v_{1}+\mathbf{f}_{14}(u, v) u_{1} v_{2} \\
& +\mathbf{f}_{15}(u, v) u_{2}^{2}+\mathbf{f}_{16}(u, v) u_{2} v_{1}+\mathbf{f}_{17}(u, v) u_{2} v_{2} \\
F_{2}(u, v)= & \mathbf{f}_{21}(u, v) u_{1}^{2}+\mathbf{f}_{22}(u, v) u_{1} u_{2}+\mathbf{f}_{23}(u, v) u_{1} v_{1}+\mathbf{f}_{24}(u, v) u_{1} v_{2} \\
& +\mathbf{f}_{25}(u, v) u_{2}^{2}+\mathbf{f}_{26}(u, v) u_{2} v_{1}+\mathbf{f}_{27}(u, v) u_{2} v_{2} \\
G_{1}(u, v)= & \mathbf{g}_{11}(u, v) v_{1}^{2}+\mathbf{g}_{12}(u, v) v_{1} v_{2}+\mathrm{g}_{13}(u, v) v_{1} u_{1}+\mathrm{g}_{14}(u, v) v_{1} u_{2}  \tag{3.3.10}\\
& +\mathrm{g}_{15}(u, v) v_{2}^{2}+\mathrm{g}_{16}(u, v) v_{2} u_{1}+\mathrm{g}_{17}(u, v) v_{2} u_{2} \\
G_{2}(u, v)= & \mathbf{g}_{21}(u, v) v_{1}^{2}+\mathbf{g}_{22}(u, v) v_{1} v_{2}+\mathbf{g}_{23}(u, v) v_{1} u_{1}+\mathrm{g}_{24}(u, v) v_{1} u_{2} \\
& +\mathrm{g}_{25}(u, v) v_{2}^{2}+\mathrm{g}_{26}(u, v) v_{2} u_{1}+\mathrm{g}_{27}(u, v) v_{2} u_{2}
\end{align*}
$$

for some continuous functions $\mathrm{f}_{i j}$ and $\mathrm{g}_{i j}$. Let $\Omega$ be a small compact neighborhood of $O$ and define

$$
\begin{equation*}
M^{*}:=\sup _{(u, v) \in \Omega}\left\{\left|\mathrm{f}_{i j}(u, v)\right|,\left|\mathrm{g}_{i j}(u, v)\right|\right\} \tag{3.3.11}
\end{equation*}
$$

Let $\delta>0$ be small and consider the set

$$
\begin{align*}
\mathcal{A}=\left\{\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right): \quad\right. & \left|u_{1}(t)\right|,\left|u_{2}(t)\right| \leq 2 e^{-\lambda t} \delta  \tag{3.3.12}\\
& \left.\left|v_{1}(t)\right|,\left|v_{2}(t)\right| \leq 2 e^{-\lambda(\tau-t)} \delta\right\}
\end{align*}
$$

where $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right)$ is any continuous function defined on $\Omega$ for $t \in[0, \tau]$.
We will first show that $\mathcal{A}$ is invariant with respect to the integral operator $\mathfrak{T}$, i.e. $\mathfrak{T}(\mathcal{A}) \subseteq \mathcal{A}$. By $(3.3 .10),(3.3 .11)$ and $(3.3 .12)$, for any $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right)$ in $\mathcal{A}$, we have

$$
\begin{align*}
& \max \left\{\left|F_{1}(u(t), v(t))\right|,\left|F_{2}(u(t), v(t))\right|\right\} \leq M^{*}\left(12 e^{-2 \lambda t} \delta^{2}+16 e^{-\lambda \tau} \delta^{2}\right) \\
& \max \left\{\left|G_{1}(u(t), v(t))\right|,\left|G_{2}(u(t), v(t))\right|\right\} \leq M^{*}\left(12 e^{-2 \lambda(\tau-t)} \delta^{2}+16 e^{-\lambda \tau} \delta^{2}\right) \tag{3.3.13}
\end{align*}
$$

Therefore

$$
\begin{aligned}
\left|\bar{u}_{1}(t)-e^{-\lambda t} u_{10}\right| & =\left|\int_{0}^{t} e^{\lambda(s-t)} F_{1}(u(s), v(s)) d s\right| \leq \int_{0}^{t} e^{\lambda(s-t)}\left|F_{1}(u(s), v(s))\right| d s \\
& \leq 16 M^{*} \delta^{2} \int_{0}^{t} e^{\lambda(s-t)}\left(e^{-2 \lambda s}+e^{-\lambda \tau}\right) d s \\
& =16 M^{*} \delta^{2} \lambda^{-1}\left[e^{-\lambda t}\left(1-e^{-\lambda t}\right)+e^{-\lambda(t+\tau)}\left(e^{\lambda t}-1\right)\right] \\
& \leq 16 M^{*} \delta^{2} \lambda^{-1}\left(e^{-\lambda t}+e^{-\lambda \tau}\right) \leq 32 M^{*} \lambda^{-1} e^{-\lambda t} \delta^{2}
\end{aligned}
$$

The same holds for $\bar{u}_{2}(t)$, i.e. $\left|\bar{u}_{2}(t)-e^{-\lambda t} u_{20}\right| \leq 32 M^{*} \lambda^{-1} e^{-\lambda t} \delta^{2}$.
Concerning $\bar{v}_{1}(t)$, we have

$$
\begin{aligned}
\left|\bar{v}_{1}(t)-e^{-\lambda(\tau-t)} v_{1 \tau}\right| & =\left|\int_{t}^{\tau} e^{\lambda(t-s)} G_{1}(u(s), v(s)) d s\right| \leq \int_{t}^{\tau} e^{\lambda(t-s)}\left|G_{1}(u(s), v(s))\right| d s \\
& \leq 16 M^{*} \delta^{2} \int_{t}^{\tau} e^{\lambda(t-s)}\left(e^{-2 \lambda(\tau-s)}+e^{-\lambda \tau}\right) \\
& =16 M^{*} \delta^{2} \lambda^{-1}\left[e^{\lambda(t-2 \tau)}\left(e^{\lambda \tau}-e^{\lambda t}\right)+e^{-\lambda(\tau-t)}\left(e^{-\lambda t}-e^{-\lambda \tau}\right)\right] \\
& \leq 16 M^{*} \delta^{2} \lambda^{-1}\left(e^{-\lambda(\tau-t)}+e^{-\lambda \tau}\right) \leq 32 M^{*} \lambda^{-1} e^{-\lambda(\tau-t)} \delta^{2}
\end{aligned}
$$

and the same holds for $\bar{v}_{2}(t)$, i.e. $\left|\bar{v}_{2}(t)-e^{-\lambda(\tau-t)} v_{2 \tau}\right| \leq 32 M^{*} \lambda^{-1} e^{-\lambda(\tau-t)} \delta^{2}$.
Let $M=32 M^{*} \lambda^{-1}$ and choose $\delta$ sufficiently small such that $M \delta<1$. Taking into account that $\max \left\{\left|u_{10}\right|,\left|u_{20}\right|,\left|v_{1 \tau}\right|,\left|v_{2 \tau}\right|\right\} \leq \delta$, we have

$$
\begin{aligned}
& \left|\bar{u}_{1}(t)\right| \leq e^{-\lambda t} u_{10}+M e^{-\lambda t} \delta^{2} \leq 2 e^{-\lambda t} \delta \\
& \left|\bar{u}_{2}(t)\right| \leq e^{-\lambda t} u_{20}+M e^{-\lambda t} \delta^{2} \leq 2 e^{-\lambda t} \delta \\
& \left|\bar{v}_{1}(t)\right| \leq e^{-\lambda(\tau-t)} v_{1 \tau}+M e^{-\lambda(\tau-t)} \delta^{2} \leq 2 e^{-\lambda(\tau-t)} \delta, \\
& \left|\bar{v}_{2}(t)\right| \leq e^{-\lambda(\tau-t)} v_{2 \tau}+M e^{-\lambda(\tau-t)} \delta^{2} \leq 2 e^{-\lambda(\tau-t)} \delta .
\end{aligned}
$$

This implies $\left(\bar{u}_{1}(t), \bar{u}_{2}(t), \bar{v}_{1}(t), \bar{v}_{2}(t)\right) \in \mathcal{A}$ as desired.
Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ can be written in the form (3.3.3) such that the corresponding $\xi_{1}, \xi_{2}, \zeta_{1}$ and $\zeta_{2}$ satisfy the estimates given in the statement of the lemma. However, since $\left(u^{(0)}, v^{(0)}\right)=(0,0) \in \mathcal{A}$, it follows from Remark 3.26 that the same holds for the solution $(u(t), v(t))$ that satisfies boundary condition (3.3.2). This ends the proof of Lemma 3.20 .

Proof of Lemma 3.22. By (3.2.6) and (3.2.4), we can write system (3.2.5) in the form (3.3.1), where

$$
\begin{align*}
& F_{1}(u, v)=\mathbf{f}_{11}(u, v) u_{1}^{2}+\mathbf{f}_{12}(u, v) u_{1} u_{2}+\mathbf{f}_{13}(u, v) v_{1} u_{2} \\
& F_{2}(u, v)=\mathbf{f}_{21}(u, v) u_{1}^{2}+\mathbf{f}_{22}(u, v) u_{1} u_{2}+\mathbf{f}_{23}(u, v) u_{2}^{2} \\
& G_{1}(u, v)=\mathrm{g}_{11}(u, v) v_{1}^{2}+\mathrm{g}_{12}(u, v) v_{1} v_{2}+\mathrm{g}_{13}(u, v) u_{1} v_{2}  \tag{3.3.14}\\
& G_{2}(u, v)=\mathrm{g}_{21}(u, v) v_{1}^{2}+\mathrm{g}_{22}(u, v) v_{1} v_{2}+\mathrm{g}_{23}(u, v) v_{2}^{2}
\end{align*}
$$

for some continuous functions $\mathrm{f}_{i j}$ and $\mathrm{g}_{i j}$. Let $\Omega$ be a small compact neighborhood of $O$ and define $M^{*}$ as in (3.3.11). Let $\delta>0$ be small and consider the set

$$
\begin{align*}
& \mathcal{A}=\left\{\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right):\left|u_{1}(t)\right| \leq 2 e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t}\left|v_{1 \tau}\right|,\right. \\
& \left|u_{2}(t)\right| \leq 2 e^{-\lambda_{2} t} \delta,  \tag{3.3.15}\\
& \left|v_{1}(t)\right| \leq 2 e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right|, \\
& \left.\left|v_{2}(t)\right| \leq 2 e^{-\lambda_{2}(\tau-t)} \delta\right\},
\end{align*}
$$

where $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right)$ is any continuous function defined on $\Omega$ for $t \in[0, \tau]$.
We first show that $\mathcal{A}$ is invariant with respect to the integral operator $\mathfrak{T}$, i.e. $\mathfrak{T}(\mathcal{A}) \subseteq \mathcal{A}$. By (3.3.14), (3.3.11) and (3.3.15) and taking into account that $\lambda_{2}<2 \lambda_{1}$ and $\max \left\{\left|u_{10}\right|,\left|u_{20}\right|,\left|v_{1 \tau}\right|,\left|v_{2 \tau}\right|\right\} \leq$ $\delta$, we have

$$
\begin{aligned}
\left|F_{1}(u(t), v(t))\right| \leq & M^{*} \delta\left[4 e^{-2 \lambda_{1} t}\left|u_{10}\right|+4 e^{-\lambda_{1} \tau-\lambda_{2} t}\left|u_{10}\right|+e^{-2 \lambda_{1}(\tau-t)-2 \lambda_{2} t}\left|v_{1 \tau}\right|+4 e^{-\left(\lambda_{1}+\lambda_{2}\right) t}\left|u_{10}\right|\right. \\
& \left.+2 e^{-\lambda_{1}(\tau-t)-2 \lambda_{2} t}\left|v_{1 \tau}\right|+4 e^{-\lambda_{1}(\tau-t)-\lambda_{2} t}\left|v_{1 \tau}\right|+2 e^{-\lambda_{2} \tau-\lambda_{1} t}\left|u_{10}\right|\right] \\
\leq & M^{*} \delta\left[14 e^{-2 \lambda_{1} t}\left|u_{10}\right|+7 e^{-\lambda_{1}(\tau-t)-\lambda_{2} t}\left|v_{1 \tau}\right|\right] \\
\left|F_{2}(u(t), v(t))\right| \leq & M^{*}\left[4 e^{-2 \lambda_{1} t}+e^{-2 \lambda_{1}(\tau-t)-2 \lambda_{2} t}+4 e^{-\lambda_{1} \tau-\lambda_{2} t}+4 e^{-\left(\lambda_{1}+\lambda_{2}\right) t}+2 e^{-\lambda_{1}(\tau-t)-2 \lambda_{2} t}\right. \\
& \left.+4 e^{-2 \lambda_{2} t}\right] \delta^{2} \leq 19 M^{*} e^{-2 \lambda_{1} t} \delta^{2}
\end{aligned}
$$

and concerning $G_{1}$ and $G_{2}$, we have

$$
\begin{aligned}
\left|G_{1}(u(t), v(t))\right| \leq & M^{*} \delta\left[4 e^{-2 \lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)-\lambda_{1} t}\left|v_{1 \tau}\right|+e^{-2 \lambda_{2}(\tau-t)-2 \lambda_{1} t}\left|u_{10}\right|\right. \\
& +4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\left|v_{1 \tau}\right|+2 e^{-2 \lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right|+4 e^{-\lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right| \\
& \left.+2 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)-\lambda_{2} t}\left|v_{1 \tau}\right|\right] \\
\leq & M^{*} \delta\left(14 e^{-2 \lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+7 e^{-\lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right|\right) \\
\left|G_{2}(u(t), v(t))\right| \leq & M^{*}\left[4 e^{-2 \lambda_{1}(\tau-t)}+4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)-\lambda_{1} t}+e^{-2 \lambda_{2}(\tau-t)-2 \lambda_{1} t}+4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\right. \\
& \left.+2 e^{-2 \lambda_{2}(\tau-t)-\lambda_{1} t}+4 e^{-2 \lambda_{2}(\tau-t)}\right] \delta^{2} \leq 19 M^{*} e^{-2 \lambda_{1}(\tau-t)} \delta^{2},
\end{aligned}
$$

which hold for any $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right) \in \mathcal{A}$. Therefore, concerning $\bar{u}_{1}(t)$, we have

$$
\begin{aligned}
\left|\bar{u}_{1}(t)-e^{-\lambda_{1} t} u_{10}\right| & =\left|\int_{0}^{t} e^{\lambda_{1}(s-t)} F_{1}(u(s), v(s)) d s\right| \leq \int_{0}^{t} e^{\lambda_{1}(s-t)}\left|F_{1}(u(s), v(s))\right| d s \\
& \leq 14 M^{*} \delta \int_{0}^{t} e^{\lambda_{1}(s-t)}\left[e^{-2 \lambda_{1} s}\left|u_{10}\right|+e^{-\lambda_{1}(\tau-s)-\lambda_{2} s}\left|v_{1 \tau}\right|\right] d s \\
& =14 M^{*} e^{-\lambda_{1} t} \delta\left[\frac{\left|u_{10}\right|}{\lambda_{1}} \cdot\left(1-e^{-\lambda_{1} t}\right)+\frac{e^{-\lambda_{1} \tau}\left|v_{1 \tau}\right|}{2 \lambda_{1}-\lambda_{2}} \cdot\left(e^{\left(2 \lambda_{1}-\lambda_{2}\right) t}-1\right)\right] \\
& \leq 14 M^{*}\left(2 \lambda_{1}-\lambda_{2}\right)^{-1}\left(e^{-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|\right)
\end{aligned}
$$

concerning $\bar{u}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{u}_{2}(t)-e^{-\lambda_{2} t} u_{20}\right| & =\left|\int_{0}^{t} e^{\lambda_{2}(s-t)} F_{2}(u(s), v(s)) d s\right| \leq \int_{0}^{t} e^{\lambda_{2}(s-t)}\left|F_{2}(u(s), v(s))\right| d s \\
& \leq 19 M^{*} e^{-\lambda_{2} t} \delta^{2} \int_{0}^{t} e^{-\left(2 \lambda_{1}-\lambda_{2}\right) s} d s=\frac{19 M^{*} \delta^{2}}{2 \lambda_{1}-\lambda_{2}} \cdot e^{-\lambda_{2} t}\left(1-e^{-\left(2 \lambda_{1}-\lambda_{2}\right) t}\right) \\
& \leq 19 M^{*}\left(2 \lambda_{1}-\lambda_{2}\right)^{-1} e^{-\lambda_{2} t} \delta^{2}
\end{aligned}
$$

concerning $\bar{v}_{1}(t)$, we have

$$
\begin{aligned}
\mid \bar{v}_{1}(t)-e^{-\lambda_{1}(\tau-t)} & v_{1 \tau}\left|=\left|\int_{t}^{\tau} e^{\lambda_{1}(t-s)} G_{1}(u(s), v(s)) d s\right| \leq \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\right| G_{1}(u(s), v(s)) \mid d s \\
& \leq 14 M^{*} \delta \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\left[e^{-2 \lambda_{1}(\tau-s)}\left|v_{1 \tau}\right|+e^{-\lambda_{2}(\tau-s)-\lambda_{1} s}\left|u_{10}\right|\right] d s \\
& =14 M^{*} e^{\lambda_{1} t} \delta\left[\frac{e^{-2 \lambda_{1} \tau}\left|v_{1 \tau}\right|}{\lambda_{1}}\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)+\frac{e^{-\lambda_{2} \tau}\left|u_{10}\right|}{2 \lambda_{1}-\lambda_{2}}\left(e^{-\left(2 \lambda_{1}-\lambda_{2}\right) t}-e^{-\left(2 \lambda_{1}-\lambda_{2}\right) \tau}\right)\right] \\
& \leq 14 M^{*}\left(2 \lambda_{1}-\lambda_{2}\right)^{-1}\left(e^{-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right|\right)
\end{aligned}
$$

and concerning $\bar{v}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{v}_{2}(t)-e^{-\lambda_{2}(\tau-t)} v_{2 \tau}\right| & =\left|\int_{t}^{\tau} e^{\lambda_{2}(t-s)} G_{2}(u(s), v(s)) d s\right| \leq \int_{t}^{\tau} e^{\lambda_{2}(t-s)}\left|G_{2}(u(s), v(s))\right| d s \\
& \leq 19 M^{*} e^{\lambda_{2} t-2 \lambda_{1} \tau} \delta^{2} \int_{t}^{\tau} e^{\left(2 \lambda_{1}-\lambda_{2}\right) s} d s \\
& =19 M^{*}\left(2 \lambda_{1}-\lambda_{2}\right)^{-1} e^{\lambda_{2} t-2 \lambda_{1} \tau} \delta^{2}\left(e^{\left(2 \lambda_{1}-\lambda_{2}\right) \tau}-e^{\left(2 \lambda_{1}-\lambda_{2}\right) t}\right) \\
& \leq 19 M^{*}\left(2 \lambda_{1}-\lambda_{2}\right)^{-1} e^{-\lambda_{2}(\tau-t)} \delta^{2}
\end{aligned}
$$

Let $M=19 M^{*}\left(2 \lambda_{1}-\lambda_{2}\right)^{-1}$ and choose $\delta$ sufficiently small such that $M \delta<1$. Then

$$
\begin{aligned}
& \left|\bar{u}_{1}(t)\right| \leq(1+M \delta) e^{-\lambda_{1} t}\left|u_{10}\right|+M \delta e^{-\lambda_{1}(\tau-t)-\lambda_{2} t}\left|v_{1 \tau}\right| \leq 2 e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t}\left|v_{1 \tau}\right| \\
& \left|\bar{u}_{2}(t)\right| \leq(1+M \delta) e^{-\lambda_{2} t} \delta \leq 2 e^{-\lambda_{2} t} \delta \\
& \left|\bar{v}_{1}(t)\right| \leq(1+M \delta) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+M \delta e^{-\lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right| \leq 2 e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right| \\
& \left|\bar{v}_{2}(t)\right| \leq(1+M \delta) e^{-\lambda_{2}(\tau-t)} \delta \leq 2 e^{-\lambda_{2}(\tau-t)} \delta
\end{aligned}
$$

which implies $\left(\bar{u}_{1}(t), \bar{u}_{2}(t), \bar{v}_{1}(t), \bar{v}_{2}(t)\right) \in \mathcal{A}$ as desired.
Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ can be written in the form (3.3.5) such that the corresponding $\xi_{1}, \xi_{2}, \zeta_{1}$ and $\zeta_{2}$ satisfy the estimates given in the statement of the lemma. However, since $\left(u^{(0)}, v^{(0)}\right)=(0,0) \in \mathcal{A}$, it follows from Remark 3.26 that the same holds for the solution $(u(t), v(t))$ that satisfies boundary condition (3.3.2). This ends the proof of Lemma 3.22 .

Proof of Lemma 3.24. By (3.2.6) and (3.2.4), we can write system (3.2.9) in the form (3.3.1), where

$$
\begin{align*}
& F_{1}(u, v)=\mathrm{f}_{11}(u, v) u_{1}^{2}+\mathrm{f}_{12}(u, v) u_{1} u_{2}+\mathrm{f}_{13}(u, v) v_{1} u_{2} \\
& F_{2}(u, v)=\mathrm{f}_{21}(u, v) u_{1} u_{2}+\mathrm{f}_{22}(u, v) u_{2}^{2} \\
& G_{1}(u, v)=\mathrm{g}_{11}(u, v) v_{1}^{2}+\mathrm{g}_{12}(u, v) v_{1} v_{2}+\mathrm{g}_{13}(u, v) u_{1} v_{2}  \tag{3.3.16}\\
& G_{2}(u, v)=\mathrm{g}_{21}(u, v) v_{1} v_{2}+\mathrm{g}_{22}(u, v) v_{2}^{2}
\end{align*}
$$

for some continuous functions $\mathbf{f}_{i j}$ and $\boldsymbol{g}_{i j}$. Let $\Omega$ be a small compact neighborhood of $O$ and define $M^{*}$ as in (3.3.11). Let $\delta>0$ be small and consider the set

$$
\begin{align*}
\mathcal{A}=\left\{\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right):\right. & \left|u_{1}(t)\right| \leq 2 e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)}\left|v_{1 \tau}\right| \\
& \left|u_{2}(t)\right| \leq 2 e^{-\lambda_{2} t} \delta, \\
& \left|v_{1}(t)\right| \leq 2 e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+e^{-\lambda_{1}(2 \tau-t)}\left|u_{10}\right|,  \tag{3.3.17}\\
& \left.\left|v_{2}(t)\right| \leq 2 e^{-\lambda_{2}(\tau-t)} \delta\right\}
\end{align*}
$$

where $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right)$ is any continuous function defined on $\Omega$ for $t \in[0, \tau]$.
We will first show that $\mathcal{A}$ is invariant with respect to the integral operator $\mathfrak{T}$, i.e. $\mathfrak{T}(\mathcal{A}) \subseteq \mathcal{A}$. By (3.3.14), (3.3.11) and (3.3.15) and taking into account that $2 \lambda_{1}<\lambda_{2}$ and $\max \left\{\left|u_{10}\right|,\left|u_{20}\right|,\left|v_{1 \tau}\right|,\left|v_{2 \tau}\right|\right\} \leq$
$\delta$, we have

$$
\begin{align*}
\left|F_{1}(u(t), v(t))\right| \leq & M^{*}\left[4 e^{-2 \lambda_{1} t}\left|u_{10}\right|^{2}+4 e^{-\lambda_{1} \tau-2 \lambda_{1} t}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1}(\tau+t)}\left|v_{1 \tau}\right|^{2}\right. \\
& +4 e^{-\left(\lambda_{1}+\lambda_{2}\right) t} \delta\left|u_{10}\right|+2 e^{-\lambda_{1}(\tau+t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|+4 e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right| \\
& \left.+2 e^{-2 \lambda_{1} \tau+\left(\lambda_{1}-\lambda_{2}\right) t} \delta\left|u_{10}\right|\right] \\
\leq & M^{*}\left[4 e^{-2 \lambda_{1} t}\left|u_{10}\right|^{2}+4 e^{-\lambda_{1} \tau-2 \lambda_{1} t}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1}(\tau+t)}\left|v_{1 \tau}\right|^{2}\right. \\
& \left.+6 e^{-\left(\lambda_{1}+\lambda_{2}\right) t} \delta\left|u_{10}\right|+6 e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|\right], \\
\left|F_{2}(u(t), v(t))\right| \leq & M^{*}\left[4 e^{-\left(\lambda_{1}+\lambda_{2}\right) t} \delta\left|u_{10}\right|+2 e^{-\lambda_{1}(\tau+t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|+4 e^{-2 \lambda_{2} t} \delta^{2}\right],  \tag{3.3.18}\\
\left|G_{1}(u(t), v(t))\right| \leq & M^{*}\left[4 e^{-2 \lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|^{2}+4 e^{-2 \lambda_{1}(\tau-t)-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1}(\tau-t)-2 \lambda_{1} \tau}\left|u_{10}\right|^{2}\right. \\
& +4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)} \delta\left|v_{1 \tau}\right|+2 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)-\lambda_{1} \tau} \delta\left|u_{10}\right|+4 e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right| \\
& \left.+2 e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau+\left(\lambda_{2}-\lambda_{1}\right) t} \delta\left|v_{1 \tau}\right|\right] \\
\leq & M^{*}\left[4 e^{-2 \lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|^{2}+4 e^{-2 \lambda_{1}(\tau-t)-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1}(\tau-t)-2 \lambda_{1} \tau}\left|u_{10}\right|^{2}\right. \\
& \left.+6 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)} \delta\left|v_{1 \tau}\right|+6 e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right|\right] \\
\left|G_{2}(u(t), v(t))\right| \leq & M^{*}\left[4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)} \delta\left|v_{1 \tau}\right|+2 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)-\lambda_{1} \tau} \delta\left|u_{10}\right|+4 e^{-2 \lambda_{2}(\tau-t)} \delta^{2}\right],
\end{align*}
$$

which holds for any $\left(u_{1}(t), u_{2}(t), v_{1}(t), v_{2}(t)\right) \in \mathcal{A}$. Therefore, concerning $\bar{u}_{1}(t)$, we have

$$
\begin{aligned}
\mid \bar{u}_{1}(t) & -e^{-\lambda_{1} t} u_{10}\left|=\left|\int_{0}^{t} e^{\lambda_{1}(s-t)} F_{1}(u(s), v(s)) d s\right| \leq \int_{0}^{t} e^{\lambda_{1}(s-t)}\right| F_{1}(u(s), v(s)) \mid d s \\
\leq & 6 M^{*} \int_{0}^{t} e^{\lambda_{1}(s-t)}\left[e^{-2 \lambda_{1} s}\left|u_{10}\right|^{2}+e^{-\lambda_{1} \tau-2 \lambda_{1} s}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1}(\tau+s)}\left|v_{1 \tau}\right|^{2}+e^{-\left(\lambda_{1}+\lambda_{2}\right) s} \delta\left|u_{10}\right|\right. \\
& \left.+e^{-\lambda_{1}(\tau-s)-\lambda_{2} s} \delta\left|v_{1 \tau}\right|\right] d s=6 M^{*} e^{-\lambda_{1} t}\left[\left(\left|u_{10}\right|^{2}+e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau}\left|v_{1 \tau}\right|^{2}\right) \lambda_{1}^{-1}\left(1-e^{-\lambda_{1} t}\right)\right. \\
& \left.+\delta\left|u_{10}\right| \lambda_{2}^{-1}\left(1-e^{-\lambda_{2} t}\right)+\frac{e^{-\lambda_{1} \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{2}-2 \lambda_{1}}\left(1-e^{\left(2 \lambda_{1}-\lambda_{2}\right) t}\right)\right] \\
\leq & 6 M^{*}\left[\lambda_{1}^{-1}\left(e^{-\lambda_{1} t}\left|u_{10}\right|^{2}+e^{-\lambda_{1}(\tau+t)}\left|u_{10} v_{1 \tau}\right|+e^{-\lambda_{1}(2 \tau+t)}\left|v_{1 \tau}\right|^{2}\right)+\frac{e^{-\lambda_{1} t} \delta\left|u_{10}\right|}{\lambda_{2}}+\frac{e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|}{\lambda_{2}-2 \lambda_{1}}\right] \\
\leq & \leq 18 M^{*} \lambda_{1}^{-1} e^{-\lambda_{1} t} \delta\left|u_{10}\right|+12 M^{*}\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1} e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right| \\
\leq & 18 M^{*}\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1}\left(e^{-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|\right)
\end{aligned}
$$

concerning $\bar{u}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{u}_{2}(t)-e^{-\lambda_{2} t} u_{20}\right| & =\left|\int_{0}^{t} e^{\lambda_{2}(s-t)} F_{2}(u(s), v(s)) d s\right| \leq \int_{0}^{t} e^{\lambda_{2}(s-t)}\left|F_{2}(u(s), v(s))\right| d s \\
& \leq 4 M^{*} \delta \int_{0}^{t} e^{\lambda_{2}(s-t)}\left[e^{-\left(\lambda_{1}+\lambda_{2}\right) s}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+s)-\lambda_{2} s}\left|v_{1 \tau}\right|+e^{-2 \lambda_{2} s} \delta\right] d s \\
& =4 M^{*} \delta e^{-\lambda_{2} t}\left[\lambda_{1}^{-1}\left(\left|u_{10}\right|+e^{-\lambda_{1} \tau}\left|v_{1 \tau}\right|\right)\left(1-e^{-\lambda_{1} t}\right)+\lambda_{2}^{-1} \delta\left(1-e^{-\lambda_{2} t}\right)\right] \\
& \leq 4 M^{*} \lambda_{1}^{-1} \delta e^{-\lambda_{2} t}\left[\left|u_{10}\right|+e^{-\lambda_{1} \tau}\left|v_{1 \tau}\right|+\delta\right] \leq 12 M^{*} \lambda_{1}^{-1} e^{-\lambda_{2} t} \delta^{2}
\end{aligned}
$$

concerning $\bar{v}_{1}(t)$, we have

$$
\begin{aligned}
\mid \bar{v}_{1}(t)- & e^{-\lambda_{1}(\tau-t)} v_{1 \tau}\left|=\left|\int_{t}^{\tau} e^{\lambda_{1}(t-s)} G_{1}(u(s), v(s)) d s\right| \leq \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\right| G_{1}(u(s), v(s)) \mid d s \\
\leq & 6 M^{*} \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\left[e^{-2 \lambda_{1}(\tau-s)}\left|v_{1 \tau}\right|^{2}+e^{-2 \lambda_{1}(\tau-s)-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1}(\tau-s)-2 \lambda_{1} \tau}\left|u_{10}\right|^{2}\right. \\
& \left.+e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-s)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2}(\tau-s)-\lambda_{1} s} \delta\left|u_{10}\right|\right] d s \\
= & 6 M^{*} e^{\lambda_{1} t}\left[\left(\frac{e^{-2 \lambda_{1} \tau}\left|v_{1 \tau}\right|^{2}}{\lambda_{1}}+\frac{e^{-3 \lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|}{\lambda_{1}}+\frac{e^{-4 \lambda_{1} \tau}\left|u_{10}\right|^{2}}{\lambda_{1}}\right)\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)\right. \\
& \left.+\frac{e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{2}}\left(e^{\lambda_{2} \tau}-e^{\lambda_{2} t}\right)+\frac{e^{-\lambda_{2} \tau} \delta\left|u_{10}\right|}{\lambda_{2}-2 \lambda_{1}}\left(e^{\left(\lambda_{2}-2 \lambda_{1}\right) \tau}-e^{\left(\lambda_{2}-2 \lambda_{1}\right) t}\right)\right] \\
\leq & 6 M^{*} e^{\lambda_{1} t}\left[\frac{e^{-\lambda_{1} \tau}\left|v_{1 \tau}\right|^{2}}{\lambda_{1}}+\frac{e^{-2 \lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|}{\lambda_{1}}+\frac{e^{-3 \lambda_{1} \tau}\left|u_{10}\right|^{2}}{\lambda_{1}}+\frac{e^{-\lambda_{1} \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{2}}+\frac{e^{-2 \lambda_{1} \tau} \delta\left|u_{10}\right|}{\lambda_{2}-2 \lambda_{1}}\right] \\
\leq & 18 M^{*} \lambda_{1}^{-1} e^{-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|+12 M^{*}\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1} e^{-2 \lambda_{1} \tau+\lambda_{1} t} \delta\left|u_{10}\right| \\
\leq & 18 M^{*}\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1}\left(e^{-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau+\lambda_{1} t} \delta\left|u_{10}\right|\right),
\end{aligned}
$$

and concerning $\bar{v}_{2}(t)$, we have

$$
\begin{aligned}
\mid \bar{v}_{2}(t) & -e^{-\lambda_{2}(\tau-t)} v_{2 \tau}\left|=\left|\int_{t}^{\tau} e^{\lambda_{2}(t-s)} G_{2}(u(s), v(s)) d s\right| \leq \int_{t}^{\tau} e^{\lambda_{2}(t-s)}\right| G_{2}(u(s), v(s)) \mid d s \\
& \leq 4 M^{*} \delta \int_{t}^{\tau} e^{\lambda_{2}(t-s)}\left[e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-s)}\left|v_{1 \tau}\right|+e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-s)-\lambda_{1} \tau}\left|u_{10}\right|+e^{-2 \lambda_{2}(\tau-s)} \delta\right] d s \\
& =4 M^{*} \delta e^{\lambda_{2} t}\left[\left(\frac{e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau}\left|v_{1 \tau}\right|+e^{-\left(2 \lambda_{1}+\lambda_{2}\right) \tau}\left|u_{10}\right|}{\lambda_{1}}\right)\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)+\frac{e^{-2 \lambda_{2} \tau} \delta}{\lambda_{2}}\left(e^{\lambda_{2} \tau}-e^{\lambda_{2} t}\right)\right] \\
& \leq 4 M^{*} \delta \lambda_{1}^{-1}\left[e^{-\lambda_{2}(\tau-t)}\left|v_{1 \tau}\right|+e^{-\lambda_{1}-\lambda_{2}(\tau-t)}\left|u_{10}\right|+e^{-\lambda_{2}(\tau-t)} \delta\right] \leq 12 M^{*} \lambda_{1}^{-1} e^{-\lambda_{2}(\tau-t)} \delta^{2} .
\end{aligned}
$$

Let $M=18 M^{*}\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1}$ and choose $\delta$ sufficiently small such that $M \delta<1$. We have

$$
\begin{aligned}
& \left|\bar{u}_{1}(t)\right| \leq(1+M \delta) e^{-\lambda_{1} t}\left|u_{10}\right|+M \delta e^{-\lambda_{1}(\tau+t)}\left|v_{1 \tau}\right| \leq 2 e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)}\left|v_{1 \tau}\right|, \\
& \left|\bar{u}_{2}(t)\right| \leq(1+M \delta) e^{-\lambda_{2} t} \delta \leq 2 e^{-\lambda_{2} t} \delta, \\
& \left|\bar{v}_{1}(t)\right| \leq(1+M \delta) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+M \delta e^{-2 \lambda_{1} \tau+\lambda_{1} t}\left|u_{10}\right| \leq 2 e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau-\lambda_{1} t}\left|u_{10}\right|, \\
& \left|\bar{v}_{2}(t)\right| \leq(1+M \delta) e^{-\lambda_{2}(\tau-t)} \delta \leq 2 e^{-\lambda_{2}(\tau-t)} \delta,
\end{aligned}
$$

which implies $\left(\bar{u}_{1}(t), \bar{u}_{2}(t), \bar{v}_{1}(t), \bar{v}_{2}(t)\right) \in \mathcal{A}$ as desired.
Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ can be written in the form (3.3.5) such that the corresponding $\xi_{1}, \xi_{2}, \zeta_{1}$ and $\zeta_{2}$ satisfy the estimates given in the statement of the lemma. However, since $\left(u^{(0)}, v^{(0)}\right)=(0,0) \in \mathcal{A}$, it follows from Remark 3.26 that the same holds for the solution $(u(t), v(t))$ that satisfies boundary condition (3.3.2). This ends the proof of Lemma 3.24 .

### 3.4. Local maps and their properties

In this section, we use the results of the previous two sections to study the local maps for each of systems (3.2.1), (3.2.5) and (3.2.9). Recall (3.1.2) and write

$$
\begin{equation*}
T^{\mathrm{loc}}\left(u_{10}, v_{10}\right)=\left(u_{1 \tau}, v_{1 \tau}\right)=\left(\eta_{1}\left(u_{10}, v_{10}\right), \eta_{2}\left(u_{10}, v_{10}\right)\right) \tag{3.4.1}
\end{equation*}
$$

where $\eta_{1}$ and $\eta_{2}$ are some functions. In the previous section, for each of systems (3.2.1), (3.2.5) and (3.2.9), we have approximated the unique solution $\left(u^{*}, v^{*}\right)$ which satisfies boundary conditions (3.3.2)
(see Lemmas 3.20, 3.22 and 3.24). We write this solution as

$$
\begin{align*}
u_{1}^{*}(t) & =u_{1}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right) \\
u_{2}^{*}(t) & =u_{2}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right) \\
v_{1}^{*}(t) & =v_{1}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)  \tag{3.4.2}\\
v_{2}^{*}(t) & =v_{2}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)
\end{align*}
$$

to point out that it explicitly depends on $t, \tau, u_{10}, u_{20}, v_{1 \tau}$ and $v_{2 \tau}$. This solution represents an orbit which at $t=0$ is at the point $\left(u_{10}, u_{20}, v_{10}, v_{20}\right)$ and at $t=\tau$ is at the point $\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, v_{2 \tau}\right)$.

To study the map $T^{\mathrm{loc}}$, we consider the case in which $u_{20}=v_{2 \tau}=\delta$, i.e. $\left(u_{10}, u_{20}, v_{10}, v_{20}\right)$ and $\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, v_{2 \tau}\right)$ belong to $\Pi^{s}$ and $\Pi^{u}$, respectively. Evaluating the first equation of (3.4.2) at $t=\tau$ and the last two equations of (3.4.2) at $t=0$ gives

$$
\begin{align*}
u_{1 \tau} & =u_{1}^{*}\left(\tau, \tau, u_{10}, \delta, v_{1 \tau}, \delta\right), \\
v_{10} & =v_{1}^{*}\left(0, \tau, u_{10}, \delta, v_{1 \tau}, \delta\right),  \tag{3.4.3}\\
v_{20} & =v_{2}^{*}\left(0, \tau, u_{10}, \delta, v_{1 \tau}, \delta\right),
\end{align*}
$$

which is an implicit relation between $u_{10}, v_{10}, v_{20}, u_{1 \tau}=\eta_{1}\left(u_{10}, v_{10}\right), v_{1 \tau}=\eta_{2}\left(u_{10}, v_{10}\right)$ and $\tau$. On the other hand, $\tau$ and $v_{20}$ can be expressed as functions of $\left(u_{10}, v_{10}\right)$. This allows us to approximate the functions $\eta_{1}$ and $\eta_{2}$.

For our purposes, approximating $T^{\mathrm{loc}}$ is enough when $\lambda_{2}<2 \lambda_{1}$. For the case of $2 \lambda_{1}<\lambda_{2}$, in addition to the approximation of the local map, we also need to approximate the derivatives of this map with respect to $u_{10}$ and $v_{10}$. This is given by Lemma 3.34 below.

In the rest of this thesis, we will use the following notation
Notation 3.27. We denote the quantity $\frac{\lambda_{1}}{\lambda_{2}}$ by $\gamma\left(\lambda_{1}, \lambda_{2}\right)$ or simply $\gamma$, i.e. $\gamma=\frac{\lambda_{1}}{\lambda_{2}}$.

### 3.4.1. Case $\lambda_{1}=\lambda_{2}$

We prove $\mathcal{D}=\emptyset$ by showing that (3.1.1) never holds. This implies that the Poincaé map along $\Gamma$ cannot be defined when $\lambda_{1}=\lambda_{2}$. This also proves Theorem A2 for the particular case of $\lambda_{1}=\lambda_{2}$.

Let $\lambda=\lambda_{1}=\lambda_{2}$. Evaluating the first two equations of (3.3.4) at $t=\tau$ and the last two equations at $t=0$ gives

$$
\begin{align*}
u_{1 \tau} & =e^{-\lambda \tau}\left[u_{10}+O\left(\delta^{2}\right)\right], & & u_{2 \tau}=e^{-\lambda \tau}\left[u_{20}+O\left(\delta^{2}\right)\right] \\
v_{10} & =e^{-\lambda \tau}\left[v_{1 \tau}+O\left(\delta^{2}\right)\right], & & v_{20}=e^{-\lambda \tau}\left[v_{2 \tau}+O\left(\delta^{2}\right)\right] \tag{3.4.4}
\end{align*}
$$

For the particular case of $u_{20}=v_{2 \tau}=\delta$, we have

$$
\begin{align*}
u_{1 \tau} & =e^{-\lambda \tau}\left[u_{10}+O\left(\delta^{2}\right)\right], & & u_{2 \tau}=e^{-\lambda \tau} \delta[1+O(\delta)]  \tag{3.4.5}\\
v_{10} & =e^{-\lambda \tau}\left[v_{1 \tau}+O\left(\delta^{2}\right)\right], & & v_{20}=e^{-\lambda \tau} \delta[1+O(\delta)]
\end{align*}
$$

It follows from (3.2.11) that $v_{20}=\delta^{-1} u_{10} v_{10}$. Substituting this into (3.4.5) gives $e^{-\lambda \tau}=\frac{u_{10} v_{10}}{\delta^{2}}[1+O(\delta)]$. This relation and relation (3.4.5) imply

$$
\begin{equation*}
v_{1 \tau}=e^{\lambda \tau} v_{10}+O\left(\delta^{2}\right)=\delta^{2} u_{10}^{-1}[1+O(\delta)]+O\left(\delta^{2}\right)=u_{10}^{-1} \delta^{2}[1+O(\delta)] \tag{3.4.6}
\end{equation*}
$$

For a given sufficiently small $\delta$, we have

$$
\begin{equation*}
\lim _{u_{10} \rightarrow 0}\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\| \geq \lim _{u_{10} \rightarrow 0}\left|v_{1 \tau}\right|=\lim _{u_{10} \rightarrow 0}\left|u_{10}\right|^{-1} \delta^{2}[1+O(\delta)]=\infty \tag{3.4.7}
\end{equation*}
$$

This means that (3.1.1) does not hold when $\epsilon$ and $\epsilon_{u}$ are chosen sufficiently small. On the other hand, it is easily seen that the same happens for the points $\left(u_{10}, v_{10}\right)$ in $\mathbb{D}$. The same also holds for the case of homoclinic figure-eight. Therefore,

Proposition 3.28. When $\lambda_{1}=\lambda_{2}$, we have $\mathcal{D}=\mathbb{D}=\mathcal{D}^{1}=\mathbb{D}^{1}=\mathcal{D}^{2}=\mathbb{D}^{2}=\emptyset$.

### 3.4.2. Case $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$

Suppose $\lambda<\lambda_{2}<2 \lambda_{1}$. Evaluating the first two equations of (3.3.6) at $t=\tau$ and the last two equations at $t=0$ gives

$$
\begin{aligned}
u_{1 \tau} & =e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-\lambda_{2} \tau} O\left(\delta v_{1 \tau}\right), & & u_{2 \tau}=e^{-\lambda_{2} \tau}\left[u_{20}+O\left(\delta^{2}\right)\right], \\
v_{10} & =e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)]+e^{-\lambda_{2} \tau} O\left(\delta u_{10}\right), & & v_{20}=e^{-\lambda_{2} \tau}\left[v_{2 \tau}+O\left(\delta^{2}\right)\right] .
\end{aligned}
$$

For the particular case of $u_{20}=v_{2 \tau}=\delta$, we have

$$
\begin{align*}
u_{1 \tau} & =e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-\lambda_{2} \tau} O\left(\delta v_{1 \tau}\right), & & u_{2 \tau}=e^{-\lambda_{2} \tau} \delta[1+O(\delta)],  \tag{3.4.8}\\
v_{10} & =e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)]+e^{-\lambda_{2} \tau} O\left(\delta u_{10}\right), & & v_{20}=e^{-\lambda_{2} \tau} \delta[1+O(\delta)] .
\end{align*}
$$

This, in particular, implies

$$
\begin{equation*}
v_{1 \tau}=e^{\lambda_{1} \tau} v_{10}[1+O(\delta)]+e^{\left(\lambda_{1}-\lambda_{2}\right) \tau} O\left(\delta u_{10}\right) . \tag{3.4.9}
\end{equation*}
$$

Since first integral (3.2.7) vanishes at $\left(u_{10}, \delta, v_{10}, v_{20}\right) \in \Pi^{s}$, we have $v_{20}=\frac{\gamma}{\delta} \cdot u_{10} v_{10}[1+o(1)]$. Thus, (3.4.8) implies

$$
\begin{equation*}
e^{-\lambda_{2} \tau}=\frac{\gamma}{\delta^{2}} \cdot u_{10} v_{10}[1+O(\delta)], \tag{3.4.10}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
e^{-\lambda_{1} \tau}=\left(\frac{\gamma}{\delta^{2}} \cdot u_{10} v_{10}\right)^{\gamma}[1+O(\delta)] . \tag{3.4.11}
\end{equation*}
$$

By these relations, we rewrite (3.4.9) as

$$
\begin{equation*}
v_{1 \tau}=e^{\lambda_{1} \tau} v_{10}[1+O(\delta)] . \tag{3.4.12}
\end{equation*}
$$

Substituting this into the equation of $u_{1 \tau}$ in (3.4.8) gives

$$
\begin{equation*}
u_{1 \tau}=e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{\left(\lambda_{1}-\lambda_{2}\right) \tau} O\left(\delta v_{10}\right) . \tag{3.4.13}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
& \eta_{1}\left(u_{10}, v_{10}\right)=u_{1 \tau}=\left(\frac{\gamma}{\delta^{2}} \cdot u_{10} v_{10}\right)^{\gamma} u_{10}[1+O(\delta)]+\left(\frac{\gamma}{\delta^{2}} \cdot u_{10} v_{10}\right)^{1-\gamma} O\left(\delta\left|v_{10}\right|\right),  \tag{3.4.14}\\
& \eta_{2}\left(u_{10}, v_{10}\right)=v_{1 \tau}=\left(\frac{\gamma}{\delta^{2}} \cdot u_{10} v_{10}\right)^{-\gamma} v_{10}[1+O(\delta)] .
\end{align*}
$$

Let us now explore the domain $\mathcal{D}$ of the map $T^{\text {loc }}$. We have

$$
\frac{u_{1 \tau}}{v_{1 \tau}}=e^{-2 \lambda_{1} \tau} \frac{u_{10}}{v_{10}}[1+O(\delta)]+e^{-\lambda_{2} \tau} O(\delta) .
$$

Choosing $\delta$ sufficiently small such that $|O(\delta)| \leq 1$ yields

$$
\left|\frac{u_{1 \tau}}{v_{1 \tau}}\right| \leq 2 e^{-\lambda_{2} \tau} \frac{\left|u_{10}\right|}{\left|v_{10}\right|}+e^{-\lambda_{2} \tau} \leq \frac{4 \gamma}{\delta^{2}}\left(u_{10}^{2}+\left|u_{10} v_{10}\right|\right) \leq \frac{8 \gamma}{\delta^{2}} \epsilon^{2} .
$$

Therefore, for any given (fixed) sufficiently small $\delta$, we have

$$
\begin{equation*}
u_{1 \tau}=v_{1 \tau} O\left(\epsilon^{2}\right) \tag{3.4.15}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\| & =\sqrt{u_{1 \tau}^{2}+v_{1 \tau}^{2}}=\left|v_{1 \tau}\right| \sqrt{1+\left|\frac{u_{1 \tau}}{v_{1 \tau}}\right|^{2}}=\left|v_{1 \tau}\right|\left[1+O\left(\epsilon^{2}\right)\right]  \tag{3.4.16}\\
& =\left(\gamma \delta^{-2}\right)^{-\gamma}\left|u_{10}\right|^{-\gamma}\left|v_{10}\right|^{1-\gamma}[1+O(\delta)] .
\end{align*}
$$

Therefore, $\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\|<\epsilon_{u}$ if and only if

$$
\begin{equation*}
\left|v_{10}\right|<\epsilon_{u} \frac{1}{1-\gamma}\left(\gamma \delta^{-2}\right)^{\frac{\gamma}{1-\gamma}}\left|u_{10}\right|^{\frac{\gamma}{1-\gamma}}[1+O(\delta)], \quad\left(\gamma=\lambda_{1} \lambda_{2}^{-1}>0.5\right) . \tag{3.4.17}
\end{equation*}
$$

By virtue of (3.4.10), we see that if $\left(u_{10}, v_{10}\right) \in \mathcal{D}$, then $u_{10} v_{10}>0$. It is also easy to see that analogous statements hold for the points in $\mathbb{D}$. This gives:

Proposition 3.29. Let $\lambda<\lambda_{2}<2 \lambda_{1}$. For a given sufficiently small $\delta$, we can choose $\epsilon$ and $\epsilon_{u}$ so that the domain $\mathcal{D}(\mathbb{D})$ becomes the set of all points $\left(u_{10}, v_{10}\right)$ in $\Pi^{s}$ such that $0<u_{10} v_{10}\left(u_{10} v_{10}<0\right)$, $\left\|\left(u_{10}, v_{10}\right)\right\|<\epsilon$ and (3.4.17) holds (see Figure 3.5).


Figure 3.5: The regions $\mathcal{D}$ and $\mathbb{D}$ for the case $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$ are shown in green and blue, respectively. They are surrounded by horizontal axis, $\epsilon$-ball $\mathcal{B}_{\epsilon}$ and the curves characterized by (3.4.17). Note that since $\gamma=\lambda_{1} \lambda_{2}{ }^{-1}>0.5$, these curves are tangent to the horizontal axis at ( $u_{10}, v_{10}$ ) $=(0,0)$.

Remark 3.30. The case of homoclinic figure-eight is the same. Relation (3.4.15) holds for any $\left(u_{10}, v_{10}\right)$ on $\Pi_{i}^{s}$ and the domains $\mathcal{D}^{i}$ and $\mathbb{D}^{i}$ are given by Proposition $3.29(i=1,2)$.

### 3.4.3. Case $2 \lambda_{1}<\lambda_{2}$

Suppose $2 \lambda_{1}<\lambda_{2}$. Evaluating the first two equations of (3.3.7) at $t=\tau$ and the last two equations at $t=0$ gives

$$
\begin{align*}
u_{1 \tau} & =e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta v_{1 \tau}\right), & & u_{2 \tau}=e^{-\lambda_{2} \tau}\left[u_{20}+O\left(\delta^{2}\right)\right], \\
v_{10} & =e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta u_{10}\right), & & v_{20}=e^{-\lambda_{2} \tau}\left[v_{2 \tau}+O\left(\delta^{2}\right)\right] . \tag{3.4.18}
\end{align*}
$$

For the particular case of $u_{20}=v_{2 \tau}=\delta$, we have

$$
\begin{align*}
u_{1 \tau} & =e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta\left|v_{1 \tau}\right|\right), & & u_{2 \tau}=e^{-\lambda_{2} \tau} \delta[1+O(\delta)], \\
v_{10} & =e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta\left|u_{10}\right|\right), & & v_{20}=e^{-\lambda_{2} \tau} \delta[1+O(\delta)] . \tag{3.4.19}
\end{align*}
$$

This, in particular, implies

$$
\begin{equation*}
v_{1 \tau}=e^{\lambda_{1} \tau} v_{10}[1+O(\delta)]+e^{-\lambda_{1} \tau} O\left(\delta\left|u_{10}\right|\right) . \tag{3.4.20}
\end{equation*}
$$

Substituting this into the equation of $u_{1 \tau}$ in (3.4.19) gives

$$
\begin{equation*}
u_{1 \tau}=e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-\lambda_{1} \tau} O\left(\delta\left|v_{10}\right|\right) . \tag{3.4.21}
\end{equation*}
$$

Then, local map (3.4.1) maps $\left(u_{10}, v_{10}\right)$ to $\left(u_{1 \tau}, v_{1 \tau}\right)$, where $u_{1 \tau}$ and $v_{1 \tau}$ are as in (3.4.21) and (3.4.20), respectively, and $\tau$ is a function of $\left(u_{10}, v_{10}\right)$. It is not as straightforward as the previous two cases to express $\tau$ as a function of $\left(u_{10}, v_{10}\right)$. This is not straightforward either to find the domain $\mathcal{D}$ of $T^{\text {loc }}$. Below, we divide $\mathcal{D}$ into three regions (it is shown that $\mathcal{D} \neq \emptyset$ ) and study each region separately.

Let $\mathcal{B}_{\epsilon}$ be the $\epsilon$-ball in $\Pi^{s}$ centered at $M^{s}$. For a given $m>1$ define

$$
\begin{gather*}
Y_{1}^{m}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}:\left|v_{10}\right|<m^{-1}\left|u_{10}\right|\right\},  \tag{3.4.22}\\
Y_{2}^{m}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}: m^{-1}\left|u_{10}\right| \leq\left|v_{10}\right| \leq m\left|u_{10}\right|\right\}
\end{gather*}
$$

and

$$
\begin{equation*}
Y_{3}^{m}:=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}: m\left|u_{10}\right|<\left|v_{10}\right|\right\} \tag{3.4.23}
\end{equation*}
$$

(see Figure 3.6). Obviously, $\mathcal{B}_{\epsilon}=Y_{1}^{m} \cup Y_{2}^{m} \cup Y_{3}^{m}$. We define

$$
\begin{equation*}
\mathcal{D}_{1}^{\epsilon}:=\mathcal{D} \cap Y_{1}^{m}, \quad \mathcal{D}_{2}^{\epsilon}:=\mathcal{D} \cap Y_{2}^{m}, \quad \mathcal{D}_{3}^{\epsilon}:=\mathcal{D} \cap Y_{3}^{m} . \tag{3.4.24}
\end{equation*}
$$

Analogously, we define

$$
\mathbb{D}_{1}^{\epsilon}:=\mathbb{D} \cap Y_{1}^{m}, \quad \mathbb{D}_{2}^{\epsilon}:=\mathbb{D} \cap Y_{2}^{m}, \quad \mathbb{D}_{3}^{\epsilon}:=\mathbb{D} \cap Y_{3}^{m} .
$$

We may drop the subscript $\epsilon$ and $m$, when no confusion arises.
For $\left(u_{10}, v_{10}\right) \in Y_{1}^{m} \cup Y_{2}^{m}$, we have $\left|v_{10}\right| \leq m\left|u_{10}\right|$ and therefore

$$
\begin{equation*}
v_{10}=O\left(u_{10}\right) \tag{3.4.25}
\end{equation*}
$$

On the other hand, first integral (3.2.10) vanishes at $\left(u_{10}, \delta, v_{10}, v_{20}\right) \in \Pi^{s}$. Thus, by (3.4.25), we have

$$
\begin{aligned}
0 & =\lambda_{1} u_{10} v_{10}[1+o(1)]-\lambda_{2} v_{20} \delta[1+o(1)]+v_{10}^{2} O(\delta)+v_{20} u_{10}^{2} O(1) \\
& =\lambda_{1} u_{10} v_{10}[1+o(1)]-\lambda_{2} v_{20} \delta[1+o(1)]+u_{10} v_{10} O(\delta)+v_{20} O\left(\epsilon^{2}\right) \\
& =\lambda_{1} u_{10} v_{10}[1+O(\delta)]-\lambda_{2} v_{20} \delta[1+O(\delta)],
\end{aligned}
$$

which implies

$$
\begin{equation*}
v_{20}=\frac{\gamma}{\delta} \cdot u_{10} v_{10}[1+O(\delta)] . \tag{3.4.26}
\end{equation*}
$$

This relation together with (3.4.19) imply that any point $\left(u_{10}, v_{10}\right) \in Y_{1}^{m} \cup Y_{2}^{m}$ reaches $\Pi^{u}$ if $u_{10} v_{10}>0$, and reaches $\Sigma$ if $u_{10} v_{10}<0$. Therefore, to find $\mathcal{D}_{1} \cup \mathcal{D}_{2}\left(\mathbb{D}_{1} \cup \mathbb{D}_{2}\right)$ it is sufficient to find the points in $Y_{1}^{m} \cup Y_{2}^{m}$ for which $\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\|<\epsilon_{u}$.

Like the preceding two cases, relation (3.4.26) yields (3.4.10) and (3.4.11). Let $\delta$ be sufficiently small. By (3.4.20) and (3.4.21), we have

$$
\begin{align*}
\left|v_{1 \tau}\right| & \leq 2 e^{\lambda_{1} \tau}\left|v_{10}\right|+e^{-\lambda_{1} \tau} \delta\left|u_{10}\right| \leq 4\left(\delta^{2} \gamma^{-1}\right)^{\gamma}\left|u_{10}\right|^{-\gamma}\left|v_{10}\right|^{1-\gamma}+e^{-\lambda_{1} \tau} \delta\left|u_{10}\right|  \tag{3.4.27}\\
& \leq 4 m^{1-\gamma}\left(\delta^{2} \gamma^{-1}\right)^{\gamma}\left|u_{10}\right|^{1-2 \gamma}+e^{-\lambda_{1} \tau} \delta\left|u_{10}\right| \leq\left[4 m^{1-\gamma}\left(\delta^{2} \gamma^{-1}\right)^{\gamma}+1\right] \epsilon^{1-2 \gamma},
\end{align*}
$$

and

$$
\begin{equation*}
\left|u_{1 \tau}\right| \leq 2 e^{-\lambda_{1} \tau}\left|u_{10}\right|+e^{-\lambda_{1} \tau} \delta\left|v_{10}\right| \leq(2+\delta) e^{-\lambda_{1} \tau} \epsilon . \tag{3.4.28}
\end{equation*}
$$

Proposition 3.31. For given $m$, sufficiently small $\delta$ and sufficiently small $\epsilon_{u}$, we can choose $\epsilon$ sufficiently small such that for $i=1,2$ we have

$$
\mathcal{D}_{i}=\left\{\left(u_{10}, v_{10}\right) \in Y_{i}^{m}, \quad u_{10} v_{10}>0\right\}, \quad \mathbb{D}_{i}=\left\{\left(u_{10}, v_{10}\right) \in Y_{i}^{m}, \quad u_{10} v_{10}<0\right\}
$$



Figure 3.6: We divide the $\epsilon$-ball in $\Pi^{s}$ centered at $M^{s}$ into three disjoint regions: $Y_{1}, Y_{2}$ and $Y_{3}$. They are shown by blue, green and pink colors, respectively, in the left figure. To investigate the sets $\mathcal{D}$ and $\mathbb{D}$ when $\lambda_{2}>2 \lambda_{1}$, we consider the intersection of each of these sets with the regions $Y_{1}, Y_{2}$ and $Y_{3}$. In this direction, we define $\mathcal{D}_{i}=\mathcal{D} \cap Y_{i}$ and $\mathbb{D}_{i}=\mathbb{D} \cap Y_{i}$. The regions $\mathcal{D}_{1}, \mathcal{D}_{2}, \mathbb{D}_{1}$ and $\mathbb{D}_{2}$ are shown by blue, green, yellow and gray, respectively in the right figure. The sets $\mathcal{D}_{3}$ and $\mathbb{D}_{3}$ are subsets of the pink region.

Now, consider $\left(u_{10}, v_{10}\right) \in Y_{2}^{m} \cup Y_{3}^{m}$. We have $\left|u_{10}\right| \leq m\left|v_{10}\right|$ and hence

$$
\begin{equation*}
u_{10}=O\left(v_{10}\right) \tag{3.4.29}
\end{equation*}
$$

By virtue of this relation and relation (3.4.18), we write

$$
v_{10}=e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta\left|u_{10}\right|\right)=e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta\left|v_{10}\right|\right)
$$

which gives

$$
\begin{equation*}
v_{10}=e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)] \quad \text { and } \quad v_{1 \tau}=e^{\lambda_{1} \tau} v_{10}[1+O(\delta)] \tag{3.4.30}
\end{equation*}
$$

This relation together with (3.4.18) give

$$
\begin{equation*}
\frac{u_{1 \tau}}{v_{1 \tau}}=\frac{e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-2 \lambda_{1} \tau} O\left(\delta v_{1 \tau}\right)}{v_{1 \tau}}=e^{-2 \lambda_{1} \tau}\left[\frac{u_{10}}{v_{10}}+O(\delta)\right]=o(1) \tag{3.4.31}
\end{equation*}
$$

which implies $u_{1 \tau}=o\left(v_{1 \tau}\right)$. Thus,

$$
\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\|=\sqrt{u_{1 \tau}^{2}+v_{1 \tau}^{2}}=\left|v_{1 \tau}\right| \sqrt{1+\left|\frac{u_{1 \tau}}{v_{1 \tau}}\right|^{2}}=\left|v_{1 \tau}\right|[1+o(1)]
$$

Note that it was relation (3.4.26) that enabled us to, first, identify the points in $Y_{i}^{m}$ for which $\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\|<\epsilon_{u}$ holds, and second, distinguish $\mathcal{D}_{i}$ from $\mathbb{D}_{i}$ for $i=1,2$. For the case of $i=3$, we cannot deduce such a relation from first integral (3.2.10). However, as we see later, the dynamics on $Y_{3}^{m}$ is quite simple and can be analyzed without knowing $\mathcal{D}_{3}$ and $\mathbb{D}_{3}$ precisely.

Meanwhile, we have shown the following

Corollary 3.32. If $\left(u_{10}, v_{10}\right) \in Y_{2}^{m}$ then (3.4.26), (3.4.10), (3.4.11), (3.4.25), (3.4.29) and (3.4.30) hold.

Corollary 3.33. If $\left(u_{10}, v_{10}\right) \in Y_{3}^{m}$ then (3.4.31) and therefore $u_{1 \tau}=o\left(v_{1 \tau}\right)$ hold.
We now proceed to deal with the differential of the local map at the points in $\mathcal{D}_{2}$ :
Lemma 3.34. Let (3.4.1) be the local map of system (3.2.9) and suppose $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$. We have

$$
\begin{array}{lll}
\left.\frac{\partial \eta_{1}}{\partial u_{1}}\right|_{\left(u_{10}, v_{10}\right)} & =(1+\gamma) e^{-\lambda_{1} \tau}[1+O(\delta)], & \left.\frac{\partial \eta_{1}}{\partial v_{1}}\right|_{\left(u_{10}, v_{10}\right)} \\
=\gamma \frac{u_{10}}{v_{10}} \cdot e^{-\lambda_{1} \tau}[1+O(\delta)]  \tag{3.4.32}\\
\left.\frac{\partial \eta_{2}}{\partial u_{1}}\right|_{\left(u_{10}, v_{10}\right)} & =-\gamma \frac{v_{10}}{u_{10}} \cdot e^{\lambda_{1} \tau}[1+O(\delta)], & \left.\frac{\partial \eta_{2}}{\partial v_{1}}\right|_{\left(u_{10}, v_{10}\right)}=(1-\gamma) e^{\lambda_{1} \tau}[1+O(\delta)]
\end{array}
$$

Proof. Let (3.4.2) be the solution of system (3.2.9) that satisfies boundary conditions (3.3.2), where $u_{20}=v_{2 \tau}=\delta$. When the point $\left(u_{10}, \delta, v_{10}, v_{20}\right)$ on $\Pi^{s}$ reaches the cross-section $\Pi^{u}$ at $\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, \delta\right)$, the corresponding flight time $\tau$ is uniquely determined by $u_{10}$ and $v_{10}$, i.e. $\tau=\tau\left(u_{10}, v_{10}\right)$, for some function $\tau$. Thus, by (3.4.3), we have

$$
\begin{gather*}
v_{10}=v_{1}^{*}\left(0, \tau\left(u_{10}, v_{10}\right), u_{10}, \delta, \eta_{2}\left(u_{10}, v_{10}\right), \delta\right)  \tag{3.4.33}\\
v_{20}=v_{2}^{*}\left(0, \tau\left(u_{10}, v_{10}\right), u_{10}, \delta, \eta_{2}\left(u_{10}, v_{10}\right), \delta\right)  \tag{3.4.34}\\
\eta_{1}\left(u_{10}, v_{10}\right)=u_{1}^{*}\left(\tau\left(u_{10}, v_{10}\right), \tau\left(u_{10}, v_{10}\right), u_{10}, \delta, \eta_{2}\left(u_{10}, v_{10}\right), \delta\right) \tag{3.4.35}
\end{gather*}
$$

Recall that, by Remark $3.14, v_{20}$ is a function of $u_{10}$ and $v_{10}$ which we denote it by $\kappa\left(u_{10}, v_{10}\right)$. Both sides of (3.4.33), (3.4.34) and (3.4.35) are functions of $u_{10}$ and $v_{10}$. Differentiating (3.4.33) with respect to $u_{10}$ and $v_{10}$ give

$$
\begin{equation*}
0=\left.\frac{\partial v_{1}^{*}}{\partial \tau}\right|_{t=0} \cdot \frac{\partial \tau}{\partial u_{10}}+\left.\frac{\partial v_{1}^{*}}{\partial u_{10}}\right|_{t=0}+\left.\frac{\partial v_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=0} \cdot \frac{\partial \eta_{2}}{\partial u_{10}} \tag{3.4.36}
\end{equation*}
$$

and

$$
\begin{equation*}
1=\left.\frac{\partial v_{1}^{*}}{\partial \tau}\right|_{t=0} \cdot \frac{\partial \tau}{\partial v_{10}}+\left.\frac{\partial v_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=0} \cdot \frac{\partial \eta_{2}}{\partial v_{10}} \tag{3.4.37}
\end{equation*}
$$

Differentiating (3.4.34) with respect to $u_{10}$ and $v_{10}$ give

$$
\begin{equation*}
\frac{\partial \kappa}{\partial u_{10}}=\left.\frac{\partial v_{2}^{*}}{\partial \tau}\right|_{t=0} \cdot \frac{\partial \tau}{\partial u_{10}}+\left.\frac{\partial v_{2}^{*}}{\partial u_{10}}\right|_{t=0}+\left.\frac{\partial v_{2}^{*}}{\partial v_{1 \tau}}\right|_{t=0} \cdot \frac{\partial \eta_{2}}{\partial u_{10}} \tag{3.4.38}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial \kappa}{\partial v_{10}}=\left.\frac{\partial v_{2}^{*}}{\partial \tau}\right|_{t=0} \cdot \frac{\partial \tau}{\partial v_{10}}+\left.\frac{\partial v_{2}^{*}}{\partial v_{1 \tau}}\right|_{t=0} \cdot \frac{\partial \eta_{2}}{\partial v_{10}} \tag{3.4.39}
\end{equation*}
$$

Differentiating (3.4.35) with respect to $u_{10}$ and $v_{10}$ give

$$
\begin{equation*}
\frac{\partial \eta_{1}}{\partial u_{10}}=\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau} \cdot \frac{\partial \tau}{\partial u_{10}}+\left.\frac{\partial u_{1}^{*}}{\partial \tau}\right|_{t=\tau} \cdot \frac{\partial \tau}{\partial u_{10}}+\left.\frac{\partial u_{1}^{*}}{\partial u_{10}}\right|_{t=\tau}+\left.\frac{\partial u_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=\tau} \cdot \frac{\partial \eta_{2}}{\partial u_{10}} \tag{3.4.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial \eta_{1}}{\partial v_{10}}=\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau} \cdot \frac{\partial \tau}{\partial v_{10}}+\left.\frac{\partial u_{1}^{*}}{\partial \tau}\right|_{t=\tau} \cdot \frac{\partial \tau}{\partial v_{10}}+\left.\frac{\partial u_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=\tau} \cdot \frac{\partial \eta_{2}}{\partial v_{10}} \tag{3.4.41}
\end{equation*}
$$

Here is our strategy to get the estimates in (3.4.32): we first estimate the expressions
(i) $\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau}$,
(ii) $\frac{\partial \kappa}{\partial u_{10}}, \frac{\partial \kappa}{\partial v_{10}}$
(iii) $\left.\frac{\partial u_{1}^{*}}{\partial u_{10}}\right|_{t=\tau},\left.\quad \frac{\partial v_{1}^{*}}{\partial u_{10}}\right|_{t=0},\left.\quad \frac{\partial v_{2}^{*}}{\partial u_{10}}\right|_{t=0}$,
(iv) $\left.\frac{\partial u_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=\tau},\left.\quad \frac{\partial v_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=0},\left.\quad \frac{\partial v_{2}^{*}}{\partial v_{1 \tau}}\right|_{t=0}$,
(v) $\left.\frac{\partial u_{1}^{*}}{\partial \tau}\right|_{t=\tau},\left.\quad \frac{\partial v_{1}^{*}}{\partial \tau}\right|_{t=0},\left.\quad \frac{\partial v_{2}^{*}}{\partial \tau}\right|_{t=0}$.

We then substitute these estimates into (3.4.37) and (3.4.39). This gives us two independent equations with two unknowns: $\frac{\partial \tau}{\partial v_{10}}$ and $\frac{\partial \eta_{2}}{\partial v_{10}}$. We will solve these equations and obtain some estimates for the unknown expressions. With the same method, we will obtain estimates for $\frac{\partial \tau}{\partial u_{10}}$ and $\frac{\partial \eta_{2}}{\partial u_{10}}$ from equations (3.4.36) and (3.4.38). Finally, substituting these into (3.4.40) and (3.4.41) will give us estimates for $\frac{\partial \eta_{1}}{\partial u_{10}}$ and $\frac{\partial \eta_{1}}{\partial v_{10}}$.
(i) Estimate for $\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau}$ : Recall that we can use (3.2.6) and (3.2.4) to write the first equation of (3.2.9) as

$$
\dot{u}_{1}=-\lambda_{1} u_{1}+\mathbf{f}_{11}(u, v) u_{1}^{2}+\mathbf{f}_{12}(u, v) u_{1} u_{2}+\mathbf{f}_{13}(u, v) v_{1} u_{2}
$$

for some continuous functions $\mathbf{f}_{1 j}$ (see relation (3.3.16)). Thus,

$$
\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau}=-\lambda_{1} u_{1 \tau}+O\left(u_{1 \tau}^{2}\right)+O\left(u_{1 \tau} u_{2 \tau}\right)+O\left(v_{1 \tau} u_{2 \tau}\right)
$$

and by (3.4.19), (3.4.20) and (3.4.21), we have

$$
\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau}=-\lambda_{1} e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)]+e^{-\lambda_{1} \tau} O\left(\delta v_{10}\right)
$$

In particular, for $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{1} \cup \mathcal{D}_{2}$, we have

$$
\begin{equation*}
\left.\frac{\partial u_{1}^{*}}{\partial t}\right|_{t=\tau}=-\lambda_{1} e^{-\lambda_{1} \tau} u_{10}[1+O(\delta)] \tag{3.4.42}
\end{equation*}
$$

(ii) Estimates for $\frac{\partial \kappa}{\partial u_{10}}$ and $\frac{\partial \kappa}{\partial v_{10}}$ : Following Remark 3.14, $\kappa$ is a $\mathcal{C}^{q}$-smooth $(q \geq 2)$ function of $\left(u_{10}, v_{10}\right)$ which is defined on an open neighborhood of $M^{s} \in \Pi^{s}$. Since its restriction to $\mathcal{D}_{1} \cup \mathcal{D}_{2}$ is of the form (3.4.26), we have

$$
\kappa(0,0)=\frac{\partial \kappa}{\partial u_{10}}(0,0)=\frac{\partial \kappa}{\partial v_{10}}(0,0)=\frac{\partial^{2} \kappa}{\partial u_{10}^{2}}(0,0)=\frac{\partial^{2} \kappa}{\partial v_{10}^{2}}(0,0)=0, \quad \frac{\partial^{2} \kappa}{\partial u_{10} v_{10}}(0,0)=\frac{\gamma}{\delta}
$$

Thus, by Taylor theorem, we have

$$
\begin{aligned}
\frac{\partial \kappa}{\partial u_{10}}\left(u_{10}, v_{10}\right) & =\frac{\gamma}{\delta} v_{10}+o\left(u_{10}\right)+o\left(v_{10}\right) \\
\frac{\partial \kappa}{\partial v_{10}}\left(u_{10}, v_{10}\right) & =\frac{\gamma}{\delta} u_{10}+o\left(u_{10}\right)+o\left(v_{10}\right) .
\end{aligned}
$$

By (3.4.25) and (3.4.29), for any $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, we have

$$
\frac{\partial \kappa}{\partial u_{10}}\left(u_{10}, v_{10}\right)=\frac{\gamma}{\delta} v_{10}[1+o(1)], \quad \frac{\partial \kappa}{\partial v_{10}}\left(u_{10}, v_{10}\right)=\frac{\gamma}{\delta} u_{10}[1+o(1)]
$$

In order to get estimates for $\frac{\partial u_{1}^{*}}{\partial \theta}, \frac{\partial v_{1}^{*}}{\partial \theta}$ and $\frac{\partial v_{2}^{*}}{\partial \theta}$, where $\theta=u_{10}, v_{1 \tau}$ and $\tau$, we will solve some boundary value problems. Let (3.4.2) be the solution of system (3.2.9) which satisfies boundary conditions (3.3.2). By writing system (3.2.9) in the form (3.3.1), i.e.

$$
\begin{align*}
F_{1}(u, v) & =f_{11}\left(u_{1}, v\right) u_{1}+f_{12}\left(u_{1}, u_{2}, v\right) u_{2}, \\
F_{2}(u, v) & =f_{22}\left(u_{1}, u_{2}, v\right) u_{2}, \\
G_{1}(u, v) & =g_{11}\left(u, v_{1}\right) v_{1}+g_{12}\left(u, v_{1}, v_{2}\right) v_{2},  \tag{3.4.43}\\
G_{2}(u, v) & =g_{22}\left(u, v_{1}, v_{2}\right) v_{2}
\end{align*}
$$

where $f_{i j}$ and $g_{i j}$ satisfy (3.2.6) and (3.2.4), we have

$$
\begin{align*}
& \dot{u}_{1}^{*}=-\lambda_{1} u_{1}^{*}+F_{1}\left(u_{1}^{*}, u_{2}^{*}, v_{1}^{*}, v_{2}^{*}\right), \\
& \dot{u}_{2}^{*}=-\lambda_{2} u_{2}^{*}+F_{2}\left(u_{1}^{*}, u_{2}^{*}, v_{1}^{*}, v_{2}^{*}\right), \\
& \dot{v}_{1}^{*}=+\lambda_{1} v_{1}^{*}+G_{1}\left(u_{1}^{*}, u_{2}^{*}, v_{1}^{*}, v_{2}^{*}\right),  \tag{3.4.44}\\
& \dot{v}_{2}^{*}=+\lambda_{2} v_{2}^{*}+G_{2}\left(u_{1}^{*}, u_{2}^{*}, v_{1}^{*}, v_{2}^{*}\right) .
\end{align*}
$$

Differentiating (3.4.44) with respect to $\theta$, where $\theta=u_{10}, v_{1 \tau}$ and $\tau$, gives

$$
\begin{equation*}
\binom{\dot{U}}{\dot{V}}=\operatorname{diagonal}\left(-\lambda_{1},-\lambda_{2}, \lambda_{1}, \lambda_{2}\right) \cdot\binom{U}{V}+\mathbf{M}(t) \cdot\binom{U}{V} \tag{3.4.45}
\end{equation*}
$$

where

$$
\begin{gathered}
U=\left(U_{1}, U_{2}\right), \quad V=\left(V_{1}, V_{2}\right), \\
\mathbf{M}(t)=\left(\begin{array}{cccc}
F_{1 u_{1}}\left(u^{*}, v^{*}\right) & F_{1 u_{2}}\left(u^{*}, v^{*}\right) & F_{1 v_{1}}\left(u^{*}, v^{*}\right) & F_{1 v_{2}}\left(u^{*}, v^{*}\right) \\
F_{2 u_{1}}\left(u^{*}, v^{*}\right) & F_{2 u_{2}}\left(u^{*}, v^{*}\right) & F_{2 v_{1}}\left(u^{*}, v^{*}\right) & F_{2 v_{2}}\left(u^{*}, v^{*}\right) \\
G_{1 u_{1}}\left(u^{*}, v^{*}\right) & G_{1 u_{2}}\left(u^{*}, v^{*}\right) & G_{1 v_{1}}\left(u^{*}, v^{*}\right) & G_{1 v_{2}}\left(u^{*}, v^{*}\right) \\
G_{2 u_{1}}\left(u^{*}, v^{*}\right) & G_{2 u_{2}}\left(u^{*}, v^{*}\right) & G_{2 v_{1}}\left(u^{*}, v^{*}\right) & G_{2 v_{2}}\left(u^{*}, v^{*}\right)
\end{array}\right),
\end{gathered}
$$

and

$$
\begin{align*}
U_{i}(t) & =\frac{\partial u_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial \theta}  \tag{3.4.46}\\
V_{i}(t) & =\frac{\partial v_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial \theta}
\end{align*}
$$

for $(i=1,2)$. It follows from the proof of Theorem 2.7 that the solution $(U(t), V(t))$ of system (3.4.45) that satisfies boundary conditions

$$
\begin{equation*}
U_{1}(0)=U_{10}, \quad U_{2}(0)=U_{20}, \quad V_{1}(\tau)=V_{1 \tau}, \quad V_{2}(\tau)=V_{2 \tau} \tag{3.4.47}
\end{equation*}
$$

is in fact the fixed point of the integral operator

$$
\begin{equation*}
\mathfrak{T}:\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right) \mapsto\left(\bar{U}_{1}(t), \bar{U}_{2}(t), \bar{V}_{1}(t), \bar{V}_{2}(t)\right) \tag{3.4.48}
\end{equation*}
$$

such that

$$
\begin{aligned}
& \bar{U}_{1}(t)=e^{-\lambda_{1} t} U_{10}+\int_{0}^{t} e^{\lambda_{1}(s-t)} P_{1}(s) d s \\
& \bar{U}_{2}(t)=e^{-\lambda_{2} t} U_{20}+\int_{0}^{t} e^{\lambda_{2}(s-t)} P_{2}(s) d s \\
& \bar{V}_{1}(t)=e^{-\lambda_{1}(\tau-t)} V_{1 \tau}+\int_{t}^{\tau} e^{\lambda_{1}(t-s)} Q_{1}(s) d s \\
& \bar{V}_{2}(t)=e^{-\lambda_{2}(\tau-t)} V_{2 \tau}+\int_{t}^{\tau} e^{\lambda_{2}(t-s)} Q_{2}(s) d s
\end{aligned}
$$

where

$$
\begin{aligned}
P_{1}(t)= & F_{1 u_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{1}(t)+F_{1 u_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{2}(t) \\
& +F_{1 v_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{1}(t)+F_{1 v_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{2}(t), \\
P_{2}(t)= & F_{2 u_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{1}(t)+F_{2 u_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{2}(t) \\
& +F_{2 v_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{1}(t)+F_{2 v_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{2}(t), \\
Q_{1}(t)= & G_{1 u_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{1}(t)+G_{1 u_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{2}(t) \\
& +G_{1 v_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{1}(t)+G_{1 v_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{2}(t), \\
Q_{2}(t)= & G_{2 u_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{1}(t)+G_{2 u_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot U_{2}(t) \\
& +G_{2 v_{1}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{1}(t)+G_{2 v_{2}}\left(u^{*}(t), v^{*}(t)\right) \cdot V_{2}(t) .
\end{aligned}
$$

Moreover, this integral operator is a contraction and the fixed point $(U(t), V(t))$ of this operator is the limit of the sequence of successive approximations

$$
\left\{\left(U^{(n)}(t), V^{(n)}(t)\right)=\left(U_{1}^{(n)}(t), U_{2}^{(n)}(t), V_{1}^{(n)}(t), V_{2}^{(n)}(t)\right)\right\}_{n=0}^{n=\infty}
$$

where $\left(U^{(0)}, V^{(0)}\right)=(0,0)$ and

$$
\left(U^{(n+1)}(t), V^{(n+1)}(t)\right)=\mathfrak{T}\left(U^{(n)}(t), V^{(n)}(t)\right), \quad \forall n \geq 0 .
$$

Below, we will solve such a boundary value problem for each of the cases $\theta=u_{10}, v_{1 \tau}$ and $\tau$.
(iii) Estimates for $\left.\frac{\partial u_{1}^{*}}{\partial u_{10}}\right|_{t=\tau},\left.\frac{\partial v_{1}^{*}}{\partial u_{10}}\right|_{t=0}$ and $\left.\frac{\partial v_{2}^{*}}{\partial u_{10}}\right|_{t=0}$ : Let $\left(U_{1}, U_{2}, V_{1}, V_{2}\right)$ be the solution of system (3.4.45), i.e. the fixed point of (3.4.48), where

$$
\begin{aligned}
& U_{i}(t)=\frac{\partial u_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial u_{10}}, \\
& V_{i}(t)=\frac{\partial v_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial u_{10}}, \quad(i=1,2) .
\end{aligned}
$$

Taking into account that (3.3.9) holds for the solution $\left(u^{*}, v^{*}\right)$ of system (3.2.9), we have

$$
\begin{equation*}
U_{1}(0)=U_{10}=1, \quad U_{2}(0)=U_{20}=0, \quad V_{1}(\tau)=V_{1 \tau}=0, \quad V_{2}(\tau)=V_{2 \tau}=0 . \tag{3.4.49}
\end{equation*}
$$

We claim that the solution $(U, V)$ that satisfies boundary conditions (3.4.49) is of the form

$$
\begin{array}{ll}
U_{1}(t)=e^{-\lambda_{1} t}[1+O(\delta)], & U_{2}(t)=e^{-\lambda_{2} t} O(\delta), \\
V_{1}(t)=e^{-\lambda_{1}(\tau-t)} O(\delta), & V_{2}(t)=e^{-\lambda_{2}(\tau-t)} O(\delta) . \tag{3.4.50}
\end{array}
$$

To prove the claim, let us first show that the set

$$
\begin{aligned}
\mathcal{A}=\left\{\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right):\right. & \left|U_{1}(t)\right| \leq 2 e^{-\lambda_{1} t}, \quad\left|U_{2}(t)\right| \leq e^{-\lambda_{2} t}, \\
& \left.\left|V_{1}(t)\right| \leq e^{-\lambda_{1}(\tau-t)}, \quad\left|V_{2}(t)\right| \leq e^{-\lambda_{2}(\tau-t)}\right\},
\end{aligned}
$$

where $\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right)$ is any continuous function defined on $t \in[0, \tau]$, is invariant with respect to integral operator (3.4.48). Notice that since $f_{i j}$ and $g_{i j}$ in (3.2.9) are $\mathcal{C}^{q-1}$-smooth ( $q \geq 2$ )
and satisfy (3.2.6) and (3.2.4), we can write the first derivatives of $F_{i}$ and $G_{i}$ as

$$
\begin{array}{ll}
F_{1 u_{1}}(u, v)=\mathrm{f}_{11}^{1}(u, v) u_{1}+\mathrm{f}_{12}^{1}(u, v) u_{2}, & F_{1 u_{2}}(u, v)=\mathrm{f}_{11}^{2}(u, v) v_{1}+\mathrm{f}_{12}^{2}(u, v) v_{2}, \\
F_{1 v_{1}}(u, v)=\mathrm{f}_{11}^{3}(u, v) u_{1}+\mathrm{f}_{12}^{3}(u, v) u_{2}, & F_{1 v_{2}}(u, v)=\mathrm{f}_{11}^{4}(u, v) u_{1}+\mathrm{f}_{12}^{4}(u, v) u_{2}, \\
F_{2 u_{1}}(u, v)=\mathrm{f}_{21}^{1}(u, v) u_{2}, & F_{2 u_{2}}(u, v)=\mathrm{f}_{21}^{2}(u, v) u_{1}+\mathrm{f}_{22}^{2}(u, v) u_{2}, \\
F_{2 v_{1}}(u, v)=\mathrm{f}_{21}^{3}(u, v) u_{2}, & F_{2 v_{2}}(u, v)=\mathrm{f}_{21}^{4}(u, v) u_{2}, \\
G_{1 u_{1}}(u, v)=\mathrm{g}_{11}^{1}(u, v) v_{1}+\mathrm{g}_{12}^{1}(u, v) v_{2}, & G_{1 u_{2}}(u, v)=\mathrm{g}_{11}^{2}(u, v) v_{1}+\mathrm{g}_{12}^{2}(u, v) v_{2}, \\
G_{1 v_{1}}(u, v)=\mathrm{g}_{11}^{3}(u, v) v_{1}+\mathrm{g}_{12}^{3}(u, v) v_{2}, & G_{1 v_{2}}(u, v)=\mathrm{g}_{11}^{4}(u, v) u_{1}+\mathrm{g}_{12}^{4}(u, v) u_{2}, \\
G_{2 u_{1}}(u, v)=\mathrm{g}_{21}^{1}(u, v) v_{2}, \quad G_{2 u 2}(u, v)=\mathrm{g}_{21}^{2}(u, v) v_{2}, \quad G_{2 v_{1}}(u, v)=\mathrm{g}_{21}^{3}(u, v) v_{2}, \\
G_{2 v_{2}}(u, v)=\mathrm{g}_{21}^{4}(u, v) v_{1}+\mathrm{g}_{22}^{4}(u, v) v_{2}, &
\end{array}
$$

where $\mathrm{f}_{i j}^{k}$ and $\mathrm{g}_{i j}^{k}$ are some continuous functions. Consider the constant $M$ given by Lemma 3.24 and let $M^{\dagger}=\max \{3,3 M\}$. For the solution $\left(u^{*}(t), v^{*}(t)\right)$ of system (3.2.9), we have

$$
\begin{aligned}
& \left|u_{1}^{*}(t)\right| \leq(1+M \delta) e^{-\lambda_{1} t}\left|u_{10}\right|+M e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right| \leq M^{\dagger} e^{-\lambda_{1} t} \delta \\
& \left|u_{2}^{*}(t)\right| \leq e^{-\lambda_{2} t}\left(\left|u_{20}\right|+M \delta^{2}\right) \leq M^{\dagger} e^{-\lambda_{2} t} \delta \\
& \left|v_{1}^{*}(t)\right| \leq(1+M \delta) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+M e^{-2 \lambda_{1} \tau+\lambda_{1} t} \delta\left|u_{10}\right| \leq M^{\dagger} e^{-\lambda_{1}(\tau-t)} \delta \\
& \left|v_{2}^{*}(t)\right|=e^{-\lambda_{2}(\tau-t)}\left(\left|v_{2 \tau}\right|+M \delta^{2}\right) \leq M^{\dagger} e^{-\lambda_{2}(\tau-t)} \delta
\end{aligned}
$$

Let $\Omega$ be a small compact neighborhood of the equilibrium $O$ of system (3.2.9) and define

$$
\begin{equation*}
M^{*}:=\sup _{(u, v) \in \Omega}\left\{\left|f_{i j}^{k}(u, v)\right|,\left|g_{i j}^{k}(u, v)\right|\right\}, \tag{3.4.51}
\end{equation*}
$$

and $M^{\ddagger}:=M^{*} M^{\dagger}$. We have

$$
\begin{align*}
& \left|F_{1 u_{1}}\left(u^{*}, v^{*}\right)\right|,\left|F_{1 v_{1}}\left(u^{*}, v^{*}\right)\right|,\left|F_{1 v_{2}}\left(u^{*}, v^{*}\right)\right|,\left|F_{2 u_{2}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 v_{2}}\left(u^{*}, v^{*}\right)\right| \leq M^{\ddagger} e^{-\lambda_{1} t} \delta \\
& \left|F_{2 u_{1}}\left(u^{*}, v^{*}\right)\right|,\left|F_{2 v_{1}}\left(u^{*}, v^{*}\right)\right|,\left|F_{2 v_{2}}\left(u^{*}, v^{*}\right)\right| \leq M^{\ddagger} e^{-\lambda_{2} t} \delta \\
& \left|F_{1 u_{2}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 u_{1}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 u_{2}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 v_{1}}\left(u^{*}, v^{*}\right)\right|,\left|G_{2 v_{2}}\left(u^{*}, v^{*}\right)\right| \leq M^{\ddagger} e^{-\lambda_{1}(\tau-t)} \delta  \tag{3.4.52}\\
& \left|G_{2 u_{1}}\left(u^{*}, v^{*}\right)\right|,\left|G_{2 u_{2}}\left(u^{*}, v^{*}\right)\right|,\left|G_{2 v_{1}}\left(u^{*}, v^{*}\right)\right| \leq M^{\ddagger} e^{-\lambda_{2}(\tau-t)} \delta .
\end{align*}
$$

This implies

$$
\begin{aligned}
& \left|P_{1}(t)\right| \leq M^{\ddagger} \delta\left[2 e^{-2 \lambda_{1} t}+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t}+e^{-\lambda_{1} \tau}+e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}\right] \leq M^{\ddagger} \delta\left[3 e^{-2 \lambda_{1} t}+2 e^{-\lambda_{1} \tau}\right], \\
& \left|P_{2}(t)\right| \leq M^{\ddagger} \delta\left[3 e^{-\left(\lambda_{1}+\lambda_{2}\right) t}+e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}+e^{-\lambda_{2} \tau}\right] \leq M^{\ddagger} \delta\left[3 e^{-\left(\lambda_{1}+\lambda_{2}\right) t}+2 e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}\right], \\
& \left|Q_{1}(t)\right| \leq M^{\ddagger} \delta\left[2 e^{-\lambda_{1} \tau}+e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}+e^{-2 \lambda_{1}(\tau-t)}+e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}\right] \leq M^{\ddagger} \delta\left[3 e^{-\lambda_{1} \tau}+2 e^{-2 \lambda_{1}(\tau-t)}\right], \\
& \left|Q_{2}(t)\right| \leq M^{\ddagger} \delta\left[2 e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}+e^{-\lambda_{2} \tau}+2 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\right] \leq M^{\ddagger} \delta\left[3 e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}+2 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\right],
\end{aligned}
$$

Using these relations, we have

$$
\begin{aligned}
& \left|\bar{U}_{1}(t)-e^{-\lambda_{1} t}\right|=\left|\int_{0}^{t} e^{\lambda_{1}(s-t)} P_{1}(s) d s\right| \leq \int_{0}^{t}\left|e^{\lambda_{1}(s-t)} P_{1}(s)\right| d s \\
& \quad \leq 3 M^{\ddagger} \delta \int_{0}^{t} e^{\lambda_{1}(s-t)}\left[e^{-2 \lambda_{1} s}+e^{-\lambda_{1} \tau}\right] d s=\frac{3 M^{\ddagger} e^{-\lambda_{1} t} \delta}{\lambda_{1}}\left[\left(1-e^{-\lambda_{1} t}\right)+e^{-\lambda_{1} \tau}\left(e^{\lambda_{1} t}-1\right)\right] \\
& \quad \leq 3 M^{\ddagger} \lambda_{1}^{-1} e^{-\lambda_{1} t} \delta\left[1+e^{-\lambda_{1}(\tau-t)}\right] \leq 6 M^{\ddagger} \lambda_{1}^{-1} e^{-\lambda_{1} t} \delta .
\end{aligned}
$$

Concerning $\bar{U}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{U}_{2}(t)\right| & =\left|\int_{0}^{t} e^{\lambda_{2}(s-t)} P_{2}(s) d s\right| \leq \int_{0}^{t}\left|e^{\lambda_{2}(s-t)} P_{2}(s)\right| d s \leq 3 M^{\ddagger} \delta \int_{0}^{t} e^{\lambda_{2}(s-t)}\left[e^{-\left(\lambda_{1}+\lambda_{2}\right) s}+e^{-\lambda_{2} s-\lambda_{1}(\tau-s)}\right] d s \\
& =\frac{3 M^{\ddagger} e^{-\lambda_{2} t} \delta}{\lambda_{1}}\left[\left(1-e^{-\lambda_{1} t}\right)+e^{-\lambda_{1} \tau}\left(e^{\lambda_{1} t}-1\right)\right] \leq \frac{3 M^{\ddagger} e^{-\lambda_{2} t} \delta}{\lambda_{1}}\left[1+e^{-\lambda_{1}(\tau-t)}\right] \leq 6 M^{\ddagger} \lambda_{1}^{-1} e^{-\lambda_{2} t} \delta .
\end{aligned}
$$

Concerning $\bar{V}_{1}(t)$, we have

$$
\begin{aligned}
& \left|\bar{V}_{1}(t)\right|=\left|\int_{t}^{\tau} e^{\lambda_{1}(t-s)} Q_{1}(s) d s\right| \leq \int_{t}^{\tau}\left|e^{\lambda_{1}(t-s)} Q_{1}(s)\right| d s \leq 3 M^{\ddagger} \delta \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\left[e^{-\lambda_{1} \tau}+e^{-2 \lambda_{1}(\tau-s)}\right] d s \\
& \quad=3 M^{\ddagger} e^{\lambda_{1} t} \delta \lambda_{1}^{-1}\left[e^{-\lambda_{1} \tau}\left(e^{-\lambda_{1} t}-e^{-\lambda_{1} \tau}\right)+e^{-2 \lambda_{1} \tau}\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)\right] \leq 3 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{1}(\tau-t)} \delta\left[e^{-\lambda_{1} t}+1\right] \\
& \quad \leq 6 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{1}(\tau-t)} \delta,
\end{aligned}
$$

and concerning $\bar{V}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{V}_{2}(t)\right| & =\left|\int_{t}^{\tau} e^{\lambda_{2}(t-s)} Q_{2}(s) d s\right| \leq \int_{t}^{\tau}\left|e^{\lambda_{2}(t-s)} Q_{2}(s)\right| d s \\
& \leq 3 M^{\ddagger} \delta \int_{t}^{\tau} e^{\lambda_{2}(t-s)}\left[e^{-\lambda_{1} s-\lambda_{2}(\tau-s)}+e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-s)}\right] d s \\
& =3 M^{\ddagger} e^{\lambda_{2} t} \delta \lambda_{1}-1\left[e^{-\lambda_{2} \tau}\left(e^{-\lambda_{1} t}-e^{-\lambda_{1} \tau}\right)+e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau}\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)\right] \\
& \leq 3 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{2}(\tau-t)} \delta\left[e^{-\lambda_{1} t}+1\right] \leq 6 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{2}(\tau-t)} \delta .
\end{aligned}
$$

Let $M=6 M^{\ddagger} \lambda_{1}{ }^{-1}$ and choose $\delta$ sufficiently small such that $M \delta<1$. We have

$$
\begin{array}{ll}
\left|\bar{U}_{1}(t)\right| \leq(1+M \delta) e^{-\lambda_{1} t} \leq 2 e^{-\lambda_{1} t}, & \left|\bar{U}_{2}(t)\right| \leq M e^{-\lambda_{2} t} \delta \leq e^{-\lambda_{2} t}, \\
\left|\bar{V}_{1}(t)\right| \leq M e^{-\lambda_{1}(\tau-t)} \delta \leq e^{-\lambda_{1}(\tau-t)}, & \left|\bar{V}_{2}(t)\right| \leq M e^{-\lambda_{2}(\tau-t)} \delta \leq e^{-\lambda_{2}(\tau-t)},
\end{array}
$$

which implies $\left(\bar{U}_{1}(t), \bar{U}_{2}(t), \bar{V}_{1}(t), \bar{V}_{2}(t)\right) \in \mathcal{A}$ as desired.
Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ is of the form (3.4.50). However, since $\left(U^{(0)}, V^{(0)}\right) \equiv(0,0) \in \mathcal{A}$, it follows from Remark 3.26 that the same holds for the solution $(U(t), V(t))$ that satisfies boundary condition (3.4.47). This proves the claim.

By (3.4.50), we have the following estimates:

$$
\left.\frac{\partial u_{1}^{*}}{\partial u_{10}}\right|_{t=\tau}=e^{-\lambda_{1} \tau}[1+O(\delta)],\left.\quad \frac{\partial v_{1}^{*}}{\partial u_{10}}\right|_{t=0}=e^{-\lambda_{1} \tau} O(\delta),\left.\quad \frac{\partial v_{2}^{*}}{\partial u_{10}}\right|_{t=0}=e^{-\lambda_{2} \tau} O(\delta) .
$$

(iv) Estimates for $\left.\frac{\partial u_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=\tau},\left.\frac{\partial v_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=0}$ and $\left.\frac{\partial v_{2}^{*}}{\partial v_{1 \tau}}\right|_{t=0}$ : Let $\left(U_{1}, U_{2}, V_{1}, V_{2}\right)$ be the solution of system (3.4.45), i.e. the fixed point of (3.4.48), where

$$
\begin{aligned}
U_{i}(t) & =\frac{\partial u_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial v_{1 \tau}} \\
V_{i}(t) & =\frac{\partial v_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial v_{1 \tau}}, \quad(i=1,2) .
\end{aligned}
$$

Taking into account that (3.3.9) holds for the solution $\left(u^{*}, v^{*}\right)$ of system (3.2.9), we have

$$
\begin{equation*}
U_{1}(0)=U_{10}=0, \quad U_{2}(0)=U_{20}=0, \quad V_{1}(\tau)=V_{1 \tau}=1, \quad V_{2}(\tau)=V_{2 \tau}=0 . \tag{3.4.53}
\end{equation*}
$$

We claim that when $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, the solution $(U, V)$ that satisfies boundary conditions (3.4.53) is of form

$$
\begin{array}{ll}
U_{1}(t)=e^{-\lambda_{1}(\tau+t)} O(\delta), & U_{2}(t)=e^{-\lambda_{2} t} O(\delta), \\
V_{1}(t)=e^{-\lambda_{1}(\tau-t)}[1+O(\delta)], & V_{2}(t)=e^{-\lambda_{2}(\tau-t)} O(\delta) . \tag{3.4.54}
\end{array}
$$

To prove the claim, let us first show that the set

$$
\begin{array}{ll}
\mathcal{A}=\left\{\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right):\right. & \left|U_{1}(t)\right| \leq e^{-\lambda_{1}(\tau+t)}, \quad\left|U_{2}(t)\right| \leq e^{-\lambda_{2} t}, \\
& \left.\left|V_{1}(t)\right| \leq 2 e^{-\lambda_{1}(\tau-t)}, \quad\left|V_{2}(t)\right| \leq e^{-\lambda_{2}(\tau-t)}\right\},
\end{array}
$$

where $\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right)$ is any continuous function defined on $t \in[0, \tau]$, is invariant with respect to integral operator (3.4.48). To show this, we use the upper bounds given by (3.4.52), except the cases $\left|F_{1 v_{1}}\right|$ and $\left|F_{1 v_{2}}\right|$ which we use the following:

$$
\left|F_{1 v_{1}}\left(u^{*}, v^{*}\right)\right|,\left|F_{1 v_{2}}\left(u^{*}, v^{*}\right)\right| \leq M^{\ddagger}\left[e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2} t} \delta\right] .
$$

We have

$$
\begin{aligned}
\left|P_{1}(t)\right| \leq & M^{\ddagger}\left[e^{-\lambda_{1}(2 t+\tau)} \delta+3 e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta+2 e^{-\lambda_{1} \tau}\left|u_{10}\right|+2 e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|\right. \\
& \left.+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{2}(\tau-t)-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2} \tau} \delta\right] \\
\leq & M^{\ddagger}\left[e^{-\lambda_{1}(2 t+\tau)} \delta+4 e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta+3 e^{-\lambda_{1} \tau}\left|u_{10}\right|+3 e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|\right], \\
\left|P_{2}(t)\right| \leq & M^{\ddagger} \delta\left[e^{-\left(\lambda_{1}+\lambda_{2}\right) t-\lambda_{1} \tau}+e^{-\left(\lambda_{1}+\lambda_{2}\right) t}+2 e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}+e^{-\lambda_{2} \tau}\right] \\
\leq & M^{\ddagger} \delta\left[2 e^{-\left(\lambda_{1}+\lambda_{2}\right) t}+3 e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}\right],
\end{aligned}
$$

and concerning $Q_{1}$ and $Q_{2}$, we have

$$
\begin{aligned}
\left|Q_{1}(t)\right| & \leq M^{\ddagger} \delta\left[e^{-2 \lambda_{1} \tau}+e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}+2 e^{-2 \lambda_{1}(\tau-t)}+e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}\right] \\
& \leq M^{\ddagger} \delta\left[e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}+4 e^{-2 \lambda_{1}(\tau-t)}\right], \\
\left|Q_{2}(t)\right| & \leq M^{\ddagger} \delta\left[e^{-\lambda_{1}(\tau+t)-\lambda_{2}(\tau-t)}+e^{-\lambda_{2} \tau}+3 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\right] \\
& \leq M^{\ddagger} \delta\left[e^{-\lambda_{2} \tau}+4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\right] .
\end{aligned}
$$

Therefore, concerning $\bar{U}_{1}(t)$, we have

$$
\begin{align*}
\left|\bar{U}_{1}(t)\right|= & \left|\int_{0}^{t} e^{\lambda_{1}(s-t)} P_{1}(s) d s\right| \leq \int_{0}^{t}\left|e^{\lambda_{1}(s-t)} P_{1}(s)\right| d s \\
\leq & 4 M^{\ddagger} \int_{0}^{t} e^{\lambda_{1}(s-t)}\left[e^{-\lambda_{1}(2 s+\tau)} \delta+e^{-\lambda_{1}(\tau-s)-\lambda_{2} s} \delta+e^{-\lambda_{1} \tau}\left|u_{10}\right|+e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|\right] d s \\
= & 4 M^{\ddagger} e^{-\lambda_{1}(t+\tau)}\left[\frac{\delta}{\lambda_{1}}\left(1-e^{-\lambda_{1} t}\right)+\frac{\delta}{\lambda_{2}-2 \lambda_{1}}\left(1-e^{\left(2 \lambda_{1}-\lambda_{2}\right) t}\right)+\frac{\left|u_{10}\right|}{\lambda_{1}}\left(e^{\lambda_{1} t}-1\right)\right.  \tag{3.4.55}\\
& \left.+\frac{e^{-\lambda_{1} \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{1}}\left(e^{\lambda_{1} t}-1\right)\right] \leq \frac{4 M^{\ddagger} e^{-\lambda_{1} \tau}}{\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}}\left[2 e^{-\lambda_{1} t} \delta+\left|u_{10}\right|+e^{-\lambda_{1} \tau} \delta\left|v_{1 \tau}\right|\right] \\
\leq & 4\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1} M^{\ddagger}\left[3 e^{-\lambda_{1}(\tau+t)} \delta+e^{-\lambda_{1} \tau}\left|u_{10}\right|\right] .
\end{align*}
$$

Concerning $\bar{U}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{U}_{2}(t)\right|= & \left|\int_{0}^{t} e^{\lambda_{2}(s-t)} P_{2}(s) d s\right| \leq \int_{0}^{t}\left|e^{\lambda_{2}(s-t)} P_{2}(s)\right| d s \\
& \leq 3 M^{\ddagger} \delta \int_{0}^{t} e^{\lambda_{2}(s-t)}\left[e^{-\left(\lambda_{1}+\lambda_{2}\right) s}+e^{-\lambda_{2} s-\lambda_{1}(\tau-s)}\right] d s=3 M^{\ddagger} e^{-\lambda_{2} t} \delta\left[\frac{1}{\lambda_{1}}\left(1-e^{-\lambda_{1} t}\right)\right. \\
& \left.+\frac{e^{-\lambda_{1} \tau}}{\lambda_{1}}\left(e^{\lambda_{1} t}-1\right)\right] \leq 3 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{2} t} \delta\left[1+e^{-\lambda_{1}(\tau-t)}\right] \leq 6 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{2} t} \delta,
\end{aligned}
$$

concerning $\bar{V}_{1}(t)$, we have

$$
\begin{aligned}
& \left|\bar{V}_{1}(t)-e^{-\lambda_{1}(\tau-t)}\right|=\left|\int_{t}^{\tau} e^{\lambda_{1}(t-s)} Q_{1}(s) d s\right| \leq \int_{t}^{\tau}\left|e^{\lambda_{1}(t-s)} Q_{1}(s)\right| d s \\
& \quad \leq 4 M^{\ddagger} \delta \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\left[e^{-\lambda_{2} s-\lambda_{1}(\tau-s)}+e^{-2 \lambda_{1}(\tau-s)}\right] d s=4 M^{\ddagger} e^{\lambda_{1} t} \delta\left[\frac{e^{-\lambda_{1} \tau}}{\lambda_{2}}\left(e^{-\lambda_{2} t}-e^{-\lambda_{2} \tau}\right)\right. \\
& \left.\quad+\frac{e^{-2 \lambda_{1} \tau}}{\lambda_{1}}\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)\right] \leq \frac{4 M^{\ddagger}}{\lambda_{1}} e^{-\lambda_{1}(\tau-t)} \delta\left[e^{-\lambda_{2} t}+1\right] \leq 8 \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{1}(\tau-t)} \delta,
\end{aligned}
$$

and concerning $\bar{V}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{V}_{2}(t)\right|= & \left|\int_{t}^{\tau} e^{\lambda_{2}(t-s)} Q_{2}(s) d s\right| \leq \int_{t}^{\tau}\left|e^{\lambda_{2}(t-s)} Q_{2}(s)\right| d s \\
\leq & 4 M^{\ddagger} \delta \int_{t}^{\tau} e^{\lambda_{2}(t-s)}\left[e^{-\lambda_{2} \tau}+e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-s)}\right] d s=4 M^{\ddagger} e^{\lambda_{2} t} \delta\left[\frac{e^{-\lambda_{2} \tau}}{\lambda_{2}}\left(e^{-\lambda_{2} t}-e^{-\lambda_{2} \tau}\right)\right. \\
& \left.+\frac{e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau}}{\lambda_{1}}\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)\right] \leq \frac{4 M^{\ddagger}}{\lambda_{1}} e^{-\lambda_{2}(\tau-t)} \delta\left[e^{-\lambda_{2} t}+1\right] \leq 8 \lambda_{1}{ }^{-1} M^{\ddagger} e^{-\lambda_{2}(\tau-t)} \delta .
\end{aligned}
$$

Note that by Corollary 3.32, for $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, we have

$$
e^{-\lambda_{1} \tau}\left|u_{10}\right| \leq K e^{-\lambda_{1} \tau}\left|v_{10}\right| \leq K^{2} e^{-2 \lambda_{1} \tau}\left|v_{1 \tau}\right| \leq K^{2} e^{-2 \lambda_{1} \tau} \delta,
$$

for some constant $K>0$. Thus, we can rewrite (3.4.55) as $\left|\bar{U}_{1}(t)\right| \leq M_{1} e^{-\lambda_{1}(\tau+t)} \delta$, for a sufficiently large constant $M_{1}$. Let $M=\max \left\{M_{1}, 8\left(\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}\right)^{-1} M^{\ddagger}\right\}$ and choose $\delta$ sufficiently small such that $M \delta<1$. Then

$$
\begin{array}{ll}
\left|\bar{U}_{1}(t)\right| \leq M e^{-\lambda_{1}(\tau+t)} \delta \leq e^{-\lambda_{1}(\tau+t)}, & \left|\bar{U}_{2}(t)\right| \leq M e^{-\lambda_{2} t} \delta \leq e^{-\lambda_{2} t}, \\
\left|\bar{V}_{1}(t)\right| \leq(1+M \delta) e^{-\lambda_{1}(\tau-t)} \delta \leq 2 e^{-\lambda_{1}(\tau-t)}, & \left|\bar{V}_{2}(t)\right| \leq M e^{-\lambda_{2}(\tau-t)} \delta \leq e^{-\lambda_{2}(\tau-t)},
\end{array}
$$

which implies $\left(\bar{U}_{1}(t), \bar{U}_{2}(t), \bar{V}_{1}(t), \bar{V}_{2}(t)\right) \in \mathcal{A}$ as desired.
Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ is of the form (3.4.54). However, since $\left(U^{(0)}, V^{(0)}\right) \equiv(0,0) \in \mathcal{A}$, it follows from Remark 3.26 that the same holds for the solution $(U(t), V(t))$ that satisfies boundary condition (3.4.53). This proves the claim.

By (3.4.54), we have the following estimates:

$$
\left.\frac{\partial u_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=\tau}=e^{-2 \lambda_{1} \tau} O(\delta),\left.\quad \frac{\partial v_{1}^{*}}{\partial v_{1 \tau}}\right|_{t=0}=e^{-\lambda_{1} \tau}[1+O(\delta)],\left.\quad \frac{\partial v_{2}^{*}}{\partial v_{1 \tau}}\right|_{t=0}=e^{-\lambda_{2} \tau} O(\delta) .
$$

(v) Estimates for $\left.\frac{\partial u_{1}^{*}}{\partial \tau}\right|_{t=\tau},\left.\frac{\partial v_{1}^{*}}{\partial \tau}\right|_{t=0}$ and $\left.\frac{\partial v_{2}^{*}}{\partial \tau}\right|_{t=0}$ : Let $\left(U_{1}, U_{2}, V_{1}, V_{2}\right)$ be the solution of system (3.4.45), i.e. the fixed point of (3.4.48), where

$$
\begin{aligned}
& U_{i}(t)=\frac{\partial u_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial \tau}, \\
& V_{i}(t)=\frac{\partial v_{i}^{*}\left(t, \tau, u_{10}, u_{20}, v_{1 \tau}, v_{2 \tau}\right)}{\partial \tau}, \quad(i=1,2) .
\end{aligned}
$$

Taking into account that (3.3.9) holds for the solution $\left(u^{*}, v^{*}\right)$ of system (3.2.9), we have

$$
\begin{equation*}
U_{1}(0)=U_{10}=0, \quad U_{2}(0)=U_{20}=0 . \tag{3.4.56}
\end{equation*}
$$

Differentiating the third and the fourth equations of (3.3.9) with respect to $\tau$ (with using the Leibniz integral rule) give

$$
\begin{equation*}
V_{1 \tau}=\left.\frac{\partial v_{1}^{*}}{\partial \tau}\right|_{t=\tau}=-\lambda_{1} v_{1 \tau}+G_{1}\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, v_{2 \tau}\right) \tag{3.4.57}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{2 \tau}=\left.\frac{\partial v_{2}^{*}}{\partial \tau}\right|_{t=\tau}=-\lambda_{2} \delta+G_{2}\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, v_{2 \tau}\right) . \tag{3.4.58}
\end{equation*}
$$

We claim that, for the case $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, the solution $(U, V)$ that satisfies boundary conditions (3.4.56), (3.4.57) and (3.4.58) is of form

$$
\begin{array}{ll}
U_{1}(t)=e^{-\lambda_{1} t} O\left(\delta\left|u_{10}\right|\right), & U_{2}(t)=e^{-\lambda_{2} t} O\left(\delta^{2}\right), \\
V_{1}(t)=e^{-\lambda_{1}(\tau-t)} v_{1 \tau}\left[-\lambda_{1}+O(\delta)\right], & V_{2}(t)=e^{-\lambda_{2}(\tau-t)} \delta\left[-\lambda_{2}+O(\delta)\right] . \tag{3.4.59}
\end{array}
$$

To prove the claim, let us first show that the set

$$
\begin{aligned}
\mathcal{A}=\left\{\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right):\right. & \left|U_{1}(t)\right| \leq e^{-\lambda_{1} t}\left|u_{10}\right|, \quad\left|U_{2}(t)\right| \leq e^{-\lambda_{2} t} \delta, \\
& \left|V_{1}(t)\right| \leq\left(1+\lambda_{1}\right) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|, \\
& \left.\left|V_{2}(t)\right| \leq\left(1+\lambda_{2}\right) e^{-\lambda_{2}(\tau-t)} \delta\right\},
\end{aligned}
$$

where $\left(U_{1}(t), U_{2}(t), V_{1}(t), V_{2}(t)\right)$ is any continuous function defined on $t \in[0, \tau]$, is invariant with respect to integral operator (3.4.48). We use the upper bounds given by (3.4.52), except the cases $\left|F_{1 u_{2}}\right|,\left|F_{1 v_{1}}\right|,\left|F_{1 v_{2}}\right|,\left|G_{1 u_{1}}\right|,\left|G_{1 u_{2}}\right|$ and $\left|G_{1 v_{2}}\right|$ which we use the following:

$$
\begin{aligned}
& \max \left\{\left|F_{1 u_{2}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 u_{1}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 u_{2}}\left(u^{*}, v^{*}\right)\right|\right\} \\
& \leq M^{\ddagger}\left[e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau+\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{2}(\tau-t)} \delta\right], \\
& \max \left\{\left|F_{1 v_{1}}\left(u^{*}, v^{*}\right)\right|,\left|F_{1 v_{2}}\left(u^{*}, v^{*}\right)\right|,\left|G_{1 v_{2}}\left(u^{*}, v^{*}\right)\right|\right\} \\
& \leq M^{\ddagger}\left[e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2} t} \delta\right] .
\end{aligned}
$$

We have

$$
\begin{aligned}
\left|P_{1}(t)\right| \leq & M^{\ddagger}\left[e^{-2 \lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau+\left(\lambda_{1}-\lambda_{2}\right) t} \delta^{2}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta^{2}\right. \\
& +\left(1+\lambda_{1}\right) e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+\left(1+\lambda_{1}\right) e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|^{2}+\left(1+\lambda_{1}\right) e^{-\lambda_{2} t-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right| \\
& \left.+\left(1+\lambda_{2}\right) e^{-\lambda_{1} t-\lambda_{2}(\tau-t)} \delta\left|u_{10}\right|+\left(1+\lambda_{2}\right) e^{-\lambda_{1}(\tau+t)-\lambda_{2}(\tau-t)} \delta^{2}\left|v_{1 \tau}\right|+\left(1+\lambda_{2}\right) e^{-\lambda_{2} \tau} \delta^{2}\right] \\
\leq & M^{\ddagger}\left[2 e^{-2 \lambda_{1} t} \delta\left|u_{10}\right|+\left(2+\lambda_{1}\right) e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|+\left(2+\lambda_{2}\right) e^{-\lambda_{2} \tau} \delta^{2}+\left(1+\lambda_{1}\right) e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|\right. \\
& \left.+\left(1+\lambda_{1}\right) e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|^{2}+\left(1+\lambda_{2}\right) e^{-\lambda_{1} t-\lambda_{2}(\tau-t)} \delta\left|u_{10}\right|+\left(1+\lambda_{2}\right) e^{-\lambda_{1}(\tau+t)-\lambda_{2}(\tau-t)} \delta^{2}\left|v_{1 \tau}\right|\right], \\
\left|P_{2}(t)\right| \leq & M^{\ddagger} \delta\left[e^{-\left(\lambda_{1}+\lambda_{2}\right) t}\left|u_{10}\right|+e^{-\left(\lambda_{1}+\lambda_{2}\right) t} \delta+\left(1+\lambda_{1}\right) e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+\left(1+\lambda_{2}\right) e^{-\lambda_{2} \tau} \delta\right] \\
\leq & M^{\ddagger} \delta^{2}\left[2 e^{-\left(\lambda_{1}+\lambda_{2}\right) t}+\left(2+\lambda_{1}+\lambda_{2}\right) e^{-\lambda_{2} t-\lambda_{1}(\tau-t)}\right],
\end{aligned}
$$

and concerning $Q_{1}$ and $Q_{2}$, we have

$$
\begin{aligned}
\left|Q_{1}(t)\right| \leq & M^{\ddagger}\left[e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau} \delta\left|u_{10}\right|^{2}+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|\right. \\
& +e^{-2 \lambda_{1} \tau+\left(\lambda_{1}-\lambda_{2}\right) t} \delta^{2}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta^{2}+\left(1+\lambda_{1}\right) e^{-2 \lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right| \\
& \left.+\left(1+\lambda_{2}\right) e^{-\lambda_{1} t-\lambda_{2}(\tau-t)} \delta\left|u_{10}\right|+\left(1+\lambda_{2}\right) e^{-\lambda_{1}(\tau+t)-\lambda_{2}(\tau-t)} \delta^{2}\left|v_{1 \tau}\right|+\left(1+\lambda_{2}\right) e^{-\lambda_{2} \tau} \delta^{2}\right] \\
\leq & M^{\ddagger}\left[e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau} \delta\left|u_{10}\right|^{2}+\left(2+\lambda_{2}\right) e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|\right. \\
& \left.+e^{-2 \lambda_{1} \tau+\left(\lambda_{1}-\lambda_{2}\right) t} \delta^{2}\left|u_{10}\right|+\left(2+\lambda_{2}\right) e^{-\lambda_{2} \tau} \delta^{2}+\left(2+\lambda_{1}+\lambda_{2}\right) e^{-2 \lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|\right], \\
\left|Q_{2}(t)\right| \leq & M^{\ddagger} \delta\left[e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta+\left(2+\lambda_{1}+\lambda_{2}\right) e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\left|v_{1 \tau}\right|\right] \\
\leq & M^{\ddagger} \delta^{2}\left[2 e^{-\lambda_{1} t-\lambda_{2}(\tau-t)}+\left(2+\lambda_{1}+\lambda_{2}\right) e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-t)}\right] .
\end{aligned}
$$

Therefore, concerning $\bar{U}_{1}(t)$, we have

$$
\begin{aligned}
\left|\bar{U}_{1}(t)\right|= & \left|\int_{0}^{t} e^{\lambda_{1}(s-t)} P_{1}(s) d s\right| \leq \int_{0}^{t}\left|e^{\lambda_{1}(s-t)} P_{1}(s)\right| d s \\
\leq & \left(2+\lambda_{2}\right) M^{\ddagger} \int_{0}^{t} e^{\lambda_{1}(s-t)}\left[e^{-2 \lambda_{1} s} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau-s)-\lambda_{2} s} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2} \tau} \delta^{2}+e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|\right. \\
& \left.+e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|^{2}+e^{-\lambda_{1} s-\lambda_{2}(\tau-s)} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau+s)-\lambda_{2}(\tau-s)} \delta^{2}\left|v_{1 \tau}\right|\right] d s \\
= & \left(2+\lambda_{2}\right) M^{\ddagger} e^{-\lambda_{1} t}\left[\frac{\delta\left|u_{10}\right|}{\lambda_{1}}\left(1-e^{-\lambda_{1} t}\right)+\frac{e^{-\lambda_{1} \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{2}-2 \lambda_{1}}\left(1-e^{\left(2 \lambda_{1}-\lambda_{2}\right) t}\right)+\left(\frac{e^{-\lambda_{2} \tau} \delta^{2}}{\lambda_{1}}\right.\right. \\
& \left.\left.+\frac{e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|}{\lambda_{1}}+\frac{e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|^{2}}{\lambda_{1}}\right)\left(e^{\lambda_{1} t}-1\right)+\left(\frac{e^{-\lambda_{2} \tau} \delta\left|u_{10}\right|}{\lambda_{2}}+\frac{e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau} \delta^{2}\left|v_{1 \tau}\right|}{\lambda_{2}}\right)\left(e^{\lambda_{2} t}-1\right)\right] \\
\leq & \frac{\left(2+\lambda_{2}\right) M^{\ddagger}}{\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}}\left[e^{-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)} \delta\left|v_{1 \tau}\right|+e^{-\lambda_{2} \tau} \delta^{2}+e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|^{2}\right. \\
& \left.+e^{-\lambda_{2}(\tau-t)-\lambda_{1} t} \delta\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)-\lambda_{2}(\tau-t)} \delta^{2}\left|v_{1 \tau}\right|\right] \\
\leq & \frac{3\left(2+\lambda_{2}\right) M^{\ddagger} \delta}{\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}}\left[e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)}\left|v_{1 \tau}\right|+e^{-\lambda_{2} \tau} \delta\right] .
\end{aligned}
$$

Concerning $\bar{U}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{U}_{2}(t)\right| & =\left|\int_{0}^{t} e^{\lambda_{2}(s-t)} P_{2}(s) d s\right| \leq \int_{0}^{t}\left|e^{\lambda_{2}(s-t)} P_{2}(s)\right| d s \\
& \leq 2\left(1+\lambda_{2}\right) M^{\ddagger} \delta^{2} \int_{0}^{t} e^{\lambda_{2}(s-t)}\left[e^{-\left(\lambda_{1}+\lambda_{2}\right) s}+e^{-\lambda_{2} s-\lambda_{1}(\tau-s)}\right] d s \\
& =2\left(1+\lambda_{2}\right) \lambda_{1}^{-1} M^{\ddagger} e^{-\lambda_{2} t} \delta^{2}\left[\left(1-e^{-\lambda_{1} t}\right)+e^{-\lambda_{1} \tau}\left(e^{\lambda_{1} t}-1\right)\right] \\
& \leq 2 \lambda_{1}^{-1}\left(1+\lambda_{2}\right) M^{\ddagger} e^{-\lambda_{2} t} \delta^{2}\left[1+e^{-\lambda_{1}(\tau-t)}\right] \leq 4 \lambda_{1}^{-1}\left(1+\lambda_{2}\right) M^{\ddagger} e^{-\lambda_{2} t} \delta^{2}
\end{aligned}
$$

Concerning $\bar{V}_{1}(t)$, we have

$$
\begin{aligned}
\mid \bar{V}_{1}(t)- & e^{-\lambda_{1}(\tau-t)} V_{1 \tau}\left|=\left|\int_{t}^{\tau} e^{\lambda_{1}(t-s)} Q_{1}(s) d s\right| \leq \int_{t}^{\tau}\right| e^{\lambda_{1}(t-s)} Q_{1}(s) \mid d s \\
\leq & 2\left(1+\lambda_{2}\right) M^{\ddagger} \int_{t}^{\tau} e^{\lambda_{1}(t-s)}\left[e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau} \delta\left|u_{10}\right|^{2}+e^{-\lambda_{2}(\tau-s)-\lambda_{1} s} \delta\left|u_{10}\right|\right. \\
& \left.+e^{-\lambda_{1}(\tau-s)-\lambda_{2} s} \delta\left|v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau+\left(\lambda_{1}-\lambda_{2}\right) s} \delta^{2}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta^{2}+e^{-2 \lambda_{1}(\tau-s)} \delta\left|v_{1 \tau}\right|\right] d s \\
= & 2\left(1+\lambda_{2}\right) M^{\ddagger} e^{\lambda_{1} t}\left[\frac{e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|}{\lambda_{1}}\left(e^{-\lambda_{1} t}-e^{-\lambda_{1} \tau}\right)\right. \\
& +\frac{e^{-2 \lambda_{1} \tau} \delta\left|u_{10}\right|^{2}}{\lambda_{1}}\left(e^{-\lambda_{1} t}-e^{-\lambda_{1} \tau}\right)+\frac{e^{-\lambda_{2} \tau} \delta\left|u_{10}\right|}{\lambda_{2}-2 \lambda_{1}}\left(e^{\left(\lambda_{2}-2 \lambda_{1}\right) \tau}-e^{\left(\lambda_{2}-2 \lambda_{1}\right) t}\right) \\
& +\frac{e^{-\lambda_{1} \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{2}}\left(e^{-\lambda_{2} t}-e^{-\lambda_{2} \tau}\right)+\frac{e^{-2 \lambda_{1} \tau} \delta^{2}\left|u_{10}\right|}{\lambda_{2}}\left(e^{-\lambda_{2} t}-e^{-\lambda_{2} \tau}\right) \\
& \left.+\frac{e^{-\lambda_{2} \tau} \delta^{2}}{\lambda}\left(e^{-\lambda_{1} t}-e^{-\lambda_{1} \tau}\right)+\frac{e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|}{\lambda_{1}}\left(e^{\lambda_{1} \tau}-e^{-\lambda_{1} t}\right)\right] \\
\leq & \frac{2\left(1+\lambda_{2}\right) M^{\ddagger}}{\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}}\left[e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau} \delta\left|u_{10}\right|^{2}+e^{-\lambda_{1}(2 \tau-t)} \delta\left|u_{10}\right|\right. \\
& \left.+e^{-\lambda_{1}(\tau-t)-\lambda_{2} t} \delta\left|v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau+\left(\lambda_{1}-\lambda_{2}\right) t} \delta^{2}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta^{2}+e^{-\lambda_{1}(\tau-t)} \delta\left|v_{1 \tau}\right|\right] \\
\leq & \frac{6\left(1+\lambda_{2}\right) M^{\ddagger} \delta}{\min \left\{\lambda_{1}, \lambda_{2}-2 \lambda_{1}\right\}}\left[e^{-\lambda_{1}(2 \tau-t)}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta+e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|\right]
\end{aligned}
$$

and concerning $\bar{V}_{2}(t)$, we have

$$
\begin{aligned}
\left|\bar{V}_{2}(t)-e^{-\lambda_{2}(\tau-t)} V_{2 \tau}\right| & =\left|\int_{t}^{\tau} e^{\lambda_{2}(t-s)} Q_{2}(s) d s\right| \leq \int_{t}^{\tau}\left|e^{\lambda_{2}(t-s)} Q_{2}(s)\right| d s \\
& \leq 2\left(1+\lambda_{2}\right) M^{\ddagger} \delta^{2} \int_{t}^{\tau} e^{\lambda_{2}(t-s)}\left[e^{-\lambda_{1} s-\lambda_{2}(\tau-s)}+4 e^{-\left(\lambda_{1}+\lambda_{2}\right)(\tau-s)}\right] d s \\
& =2\left(1+\lambda_{2}\right) M^{\ddagger} e^{\lambda_{2} t} \delta^{2}\left[\frac{e^{-\lambda_{2} \tau}}{\lambda_{1}}\left(e^{-\lambda_{1} t}-e^{-\lambda_{1} \tau}\right)+\frac{e^{-\left(\lambda_{1}+\lambda_{2}\right) \tau}}{\lambda_{1}}\left(e^{\lambda_{1} \tau}-e^{\lambda_{1} t}\right)\right] \\
& \leq \frac{2\left(1+\lambda_{2}\right) M^{\ddagger}}{\lambda_{1}} e^{-\lambda_{2}(\tau-t)} \delta^{2}\left[e^{-\lambda_{1} t}+1\right] \leq \frac{4\left(1+\lambda_{2}\right) M^{\ddagger}}{\lambda_{1}} e^{-\lambda_{2}(\tau-t)} \delta^{2} .
\end{aligned}
$$

Evaluating the last two equations of (3.3.18) at $t=\tau$ gives

$$
\begin{aligned}
\left|G_{1}\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, v_{2 \tau}\right)\right| \leq & M^{*}\left[4\left|v_{1 \tau}\right|^{2}+4 e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|+e^{-2 \lambda_{1} \tau}\left|u_{10}\right|^{2}+4 \delta\left|v_{1 \tau}\right|+6 e^{-\lambda_{1} \tau} \delta\left|u_{10}\right|\right. \\
& \left.+2 e^{-2 \lambda_{1} \tau} \delta\left|v_{1 \tau}\right|\right] \leq M^{*}\left[14 \delta\left|v_{1 \tau}\right|+7 e^{-\lambda_{1} \tau} \delta\left|u_{10}\right|\right] \\
\left|G_{2}\left(u_{1 \tau}, u_{2 \tau}, v_{1 \tau}, v_{2 \tau}\right)\right| \leq & M^{*}\left[4 \delta\left|v_{1 \tau}\right|+2 e^{-\lambda_{1} \tau} \delta\left|u_{10}\right|+4 \delta^{2}\right] \leq 10 M^{*} \delta^{2}
\end{aligned}
$$

where $M^{*}$ is as in (3.4.51). Therefore, by (3.4.57), (3.4.58) and the other relations above, we have

$$
\begin{aligned}
& \left|\bar{U}_{1}(t)\right| \leq M_{0} \delta\left[e^{-\lambda_{1} t}\left|u_{10}\right|+e^{-\lambda_{1}(\tau+t)}\left|v_{1 \tau}\right|+e^{-\lambda_{2} \tau} \delta\right] \\
& \left|\bar{U}_{2}(t)\right| \leq M_{0} e^{-\lambda_{2} t} \delta^{2}, \\
& \left|\bar{V}_{1}(t)\right| \leq\left(\lambda_{1}+M_{0} \delta\right) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|+M_{0} \delta\left[e^{-\lambda_{1}(2 \tau-t)}\left|u_{10}\right|+e^{-\lambda_{2} \tau} \delta\right], \\
& \left|\bar{V}_{2}(t)\right| \leq\left(\lambda_{2}+M_{0} \delta\right) e^{-\lambda_{2}(\tau-t)} \delta,
\end{aligned}
$$

for some sufficiently large $M_{0}>0$. According to Corollary 3.32, we have

$$
\begin{gathered}
e^{-\lambda_{1}(\tau+t)}\left|v_{1 \tau}\right|<K e^{-\lambda_{1} t}\left|v_{10}\right|<K^{2} e^{-\lambda_{1} t}\left|u_{10}\right|, \\
e^{-\lambda_{2} \tau} \delta^{2}<K\left|u_{10} v_{10}\right|<K^{2} e^{-\lambda_{1} \tau}\left|u_{10} v_{1 \tau}\right|, \\
e^{-\lambda_{1}(2 \tau-t)}\left|u_{10}\right|<K e^{-\lambda_{1}(2 \tau-t)}\left|v_{10}\right|<K^{2} e^{-\lambda_{1}(3 \tau-t)}\left|v_{1 \tau}\right|,
\end{gathered}
$$

for a sufficiently large $K$. Therefore, for a sufficiently large $M>0$, we have

$$
\begin{array}{ll}
\left|\bar{U}_{1}(t)\right| \leq M e^{-\lambda_{1} t} \delta\left|u_{10}\right|, & \left|\bar{U}_{2}(t)\right| \leq M e^{-\lambda_{2} t} \delta^{2}, \\
\left|\bar{V}_{1}(t)\right| \leq\left(\lambda_{1}+M \delta\right) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|, & \left|\bar{V}_{2}(t)\right| \leq\left(\lambda_{2}+M \delta\right) e^{-\lambda_{2}(\tau-t)} \delta .
\end{array}
$$

Choose $\delta$ sufficiently small such that $M \delta<1$. Thus,

$$
\begin{array}{ll}
\left|\bar{U}_{1}(t)\right| \leq e^{-\lambda_{1} t}\left|u_{10}\right|, & \left|\bar{U}_{2}(t)\right| \leq e^{-\lambda_{2} t} \delta, \\
\left|\bar{V}_{1}(t)\right| \leq\left(\lambda_{1}+1\right) e^{-\lambda_{1}(\tau-t)}\left|v_{1 \tau}\right|, & \left|\bar{V}_{2}(t)\right| \leq\left(\lambda_{2}+1\right) e^{-\lambda_{2}(\tau-t)} \delta,
\end{array}
$$

which implies $\left(\bar{U}_{1}(t), \bar{U}_{2}(t), \bar{V}_{1}(t), \bar{V}_{2}(t)\right) \in \mathcal{A}$ as desired.
Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ is of the form (3.4.59). However, since $\left(U^{(0)}, V^{(0)}\right) \equiv(0,0) \in \mathcal{A}$, it follows from Remark 3.26 that the same holds for the solution $(U(t), V(t))$ that satisfies boundary conditions (3.4.56), (3.4.57) and (3.4.58). This proves the claim.

By (3.4.59), we have the following estimates:

$$
\begin{aligned}
& \left.\frac{\partial u_{1}^{*}}{\partial \tau}\right|_{t=\tau}=e^{-\lambda_{1} \tau} O\left(\delta\left|u_{10}\right|\right), \\
& \left.\frac{\partial v_{1}^{*}}{\partial \tau}\right|_{t=0}=-\lambda_{1} e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)], \\
& \left.\frac{\partial v_{2}^{*}}{\partial \tau}\right|_{t=0}=-\lambda_{2} e^{-\lambda_{2} \tau} \delta[1+O(\delta)]
\end{aligned}
$$

So far, we have obtained all estimates that we required. Substituting these estimates into the relations (3.4.37) and (3.4.39) gives

$$
\begin{equation*}
1=-\lambda_{1} e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)] \cdot \frac{\partial \tau}{\partial v_{10}}+e^{-\lambda_{1} \tau}[1+O(\delta)] \cdot \frac{\partial \eta_{2}}{\partial v_{10}} \tag{3.4.60}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\gamma u_{10}}{\delta}[1+O(\delta)]=-\lambda_{2} e^{-\lambda_{2} \tau} \delta[1+O(\delta)] \cdot \frac{\partial \tau}{\partial v_{10}}+e^{-\lambda_{2} \tau} O(\delta) \cdot \frac{\partial \eta_{2}}{\partial v_{10}} \tag{3.4.61}
\end{equation*}
$$

Relation (3.4.61) implies

$$
\begin{equation*}
\frac{\partial \tau}{\partial v_{10}}=\frac{-1}{\lambda_{2} \delta}\left(e^{\lambda_{2} \tau} \frac{\gamma u_{10}}{\delta}+O(\delta) \cdot \frac{\partial \eta_{2}}{\partial v_{10}}\right)[1+O(\delta)] \tag{3.4.62}
\end{equation*}
$$

Substituting this into (3.4.60) gives

$$
1=\frac{\gamma e^{-\lambda_{1} \tau} v_{1 \tau}}{\delta} \cdot\left(e^{\lambda_{2} \tau} \frac{\gamma u_{10}}{\delta}+O(\delta) \cdot \frac{\partial \eta_{2}}{\partial v_{10}}\right)[1+O(\delta)]+e^{-\lambda_{1} \tau}[1+O(\delta)] \cdot \frac{\partial \eta_{2}}{\partial v_{10}}
$$

and therefore

$$
\frac{\partial \eta_{2}}{\partial v_{10}}=\left(e^{\lambda_{1} \tau}-\frac{\gamma^{2}}{\delta^{2}} \cdot e^{\lambda_{2} \tau} u_{10} v_{1 \tau}\right)[1+O(\delta)]
$$

Thus, by Corollary 3.32, we have

$$
\begin{equation*}
\frac{\partial \eta_{2}}{\partial v_{10}}=(1-\gamma) e^{\lambda_{1} \tau}[1+O(\delta)] \tag{3.4.63}
\end{equation*}
$$

as desired in (3.4.32). By Corollary 3.32 , substituting (3.4.63) into (3.4.62) yields

$$
\begin{equation*}
\frac{\partial \tau}{\partial v_{10}}=-\frac{1}{\lambda_{2}} \cdot \frac{1}{v_{10}}\left[1+O\left(v_{1 \tau}\right)\right]=-\frac{1}{\lambda_{2}} \cdot \frac{1}{v_{10}}[1+O(\delta)] \tag{3.4.64}
\end{equation*}
$$

Similarly, we can estimate derivatives of $\tau$ and $\eta_{2}$ with respect to $u_{10}$. We rewrite (3.4.36) and (3.4.38) as

$$
\begin{equation*}
0=-\lambda_{1} e^{-\lambda_{1} \tau} v_{1 \tau}[1+O(\delta)] \cdot \frac{\partial \tau}{\partial u_{10}}+e^{-\lambda_{1} \tau} O(\delta)+e^{-\lambda_{1} \tau}[1+O(\delta)] \cdot \frac{\partial \eta_{2}}{\partial u_{10}} \tag{3.4.65}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\gamma v_{10}}{\delta}[1+O(\delta)]=-\lambda_{2} e^{-\lambda_{2} \tau} \delta[1+O(\delta)] \cdot \frac{\partial \tau}{\partial u_{10}}+e^{-\lambda_{2} \tau} O(\delta)+e^{-\lambda_{2} \tau} O(\delta) \cdot \frac{\partial \eta_{2}}{\partial u_{10}} \tag{3.4.66}
\end{equation*}
$$

From (3.4.66), we have

$$
\begin{equation*}
\frac{\partial \tau}{\partial u_{10}}=\frac{-1}{\lambda_{2} \delta}\left(\frac{\gamma}{\delta} e^{\lambda_{2} \tau} v_{10}+O(\delta)+O(\delta) \cdot \frac{\partial \eta_{2}}{\partial u_{10}}\right)[1+O(\delta)] \tag{3.4.67}
\end{equation*}
$$

Substituting this into (3.4.65) gives

$$
0=\frac{\gamma}{\delta} v_{1 \tau}\left(\frac{\gamma}{\delta} e^{\lambda_{2} \tau} v_{10}+O(\delta)+O(\delta) \cdot \frac{\partial \eta_{2}}{\partial u_{10}}\right)[1+O(\delta)]+O(\delta)+\frac{\partial \eta_{2}}{\partial u_{10}} \cdot[1+O(\delta)],
$$

and by Corollary 3.32, we have

$$
\begin{equation*}
\frac{\partial \eta_{2}}{\partial u_{10}}=-\gamma \frac{v_{1 \tau}}{u_{10}}[1+O(\delta)]=-\gamma e^{\lambda_{1} \tau} \frac{v_{10}}{u_{10}}[1+O(\delta)] \tag{3.4.68}
\end{equation*}
$$

as desired in (3.4.32). By Corollary 3.32 , substituting (3.4.68) into (3.4.67) gives

$$
\frac{\partial \tau}{\partial u_{10}}=-\frac{1}{\lambda_{2}} \cdot \frac{1}{u_{10}}[1+O(\delta)] .
$$

It is easily seen that substituting the estimates we have derived so far into (3.4.40) and (3.4.41) gives the estimates in (3.4.32) for $\frac{\partial \eta_{1}}{\partial u_{10}}$ and $\frac{\partial \eta_{1}}{\partial v_{10}}$. This ends the proof of Lemma 3.34.

Remark 3.35. In the case of homoclinic figure-eight, the estimates given by Lemma 3.34 also hold for the local maps $T_{1}^{\text {loc }}$ (on $\mathcal{D}_{2}^{1}$ ), $T_{12}^{\text {loc }}$ (on $\mathbb{D}_{2}^{1}$ ), $T_{21}^{\text {loc }}\left(\right.$ on $\mathbb{D}_{2}^{2}$ ) and $T_{2}^{\text {loc }}$ (on $\mathcal{D}_{2}^{2}$ ). For instance, applying Lemma 3.34 on the local map $T^{\text {loc }}$ on $\mathcal{D}_{2}$ of the system which is derived from system (3.2.9) by applying the linear change of coordinates $\left(\tilde{u}_{1}, \tilde{u}_{2}, \tilde{v}_{1}, \tilde{v}_{2}\right)=\left(u_{1}, u_{2},-v_{1},-v_{2}\right)$ gives the estimates in Lemma 3.34 for $T_{12}^{\text {loc }}$ on $\mathbb{D}_{2}^{1}$.

## Chapter 4

## Analysis near homoclinics and super-homoclinics

The purpose of this chapter is to study the dynamics near (single and figure-eight) homoclinic and super-homoclinic orbits. In particular, we prove in this chapter, all the theorems stated in the Introduction. In the first section, we introduce some concepts and notations which are used throughout the whole chapter. The second section is dedicated to study the dynamics near a single homoclinic orbit. We prove Theorems A1, A2 and A3 in this section. The ideas and techniques which are used to prove these theorems are also used in the third and fourth sections of this chapter. In the third section, we extend the results obtained for a single homoclinic to the case of the homoclinic figureeight. The proofs of Theorems B1, B2 and B3 are provided in this section. Finally, we study the case of a super-homoclinic and prove Theorems C1 and C2 in the fourth, and the last, section of this chapter.

### 4.1. Set-up and notations

Choose a sufficiently small $\delta>0$ such that all the statements of the previous chapters hold. Fix this $\delta$. According to (3.1.3) and (1.2.8), for $\left(u_{10}, v_{10}\right) \in \mathcal{D} \subset \Pi^{s}$, we have

$$
\begin{equation*}
\binom{\bar{u}_{10}}{\bar{v}_{10}}=T\binom{u_{10}}{v_{10}}=\binom{[a+o(1)] u_{1 \tau}+[b+o(1)] v_{1 \tau}}{[c+o(1)] u_{1 \tau}+[d+o(1)] v_{1 \tau}} \tag{4.1.1}
\end{equation*}
$$

where $a, b, c$ and $d$ are real constants (in fact, these coefficients are functions of $\delta$ but since $\delta$ is assumed to be fixed, we treat these coefficients as constants). Our job is to analyze this map for different values of $a, b, c$ and $d$, and for each of the cases $\lambda_{2}<2 \lambda_{1}$ and $2 \lambda_{1}<\lambda_{2}$. In this strand, we first introduce some notations:

Notation 4.1. Let $\mathcal{N} \subset \mathcal{M}$ be two arbitrary sets and $f: \mathcal{N} \rightarrow \mathcal{M}$ be an injective map. We denote the set of the points in $\mathcal{N}$ whose forward orbits lie entirely in $\mathcal{N}$ by $\Lambda_{\mathcal{N}, f}^{s}$ or $\Lambda_{\mathcal{N}}^{s}$, when no confusion arises. Indeed,

$$
\Lambda_{\mathcal{N}, f}^{s}=\Lambda_{\mathcal{N}}^{s}=\left\{x \in \mathcal{N}: f^{n}(x) \in \mathcal{N}, \quad \forall n \geq 0\right\}
$$

We denote the set of the points in $\mathcal{N}$ whose backward orbits lie entirely in $\mathcal{N}$ by $\Lambda_{\mathcal{N}, f}^{u}$ or $\Lambda_{\mathcal{N}}^{u}$, when no confusion arises. Indeed,

$$
\Lambda_{\mathcal{N}, f}^{u}=\Lambda_{\mathcal{N}}^{u}=\left\{x \in \mathcal{N}: \text { for all } n \geq 0, \quad f^{-n}(x) \text { exists and belongs to } \mathcal{N}\right\} .
$$

Notation 4.2. Given a point $\left(u_{10}, v_{10}\right)$ on a given cross-section, we denote the quantity $\frac{v_{10}}{u_{10}}$ (when $\left.u_{10} \neq 0\right)$ by $w\left(u_{10}, v_{10}\right)$ or $w$. Consider the case $\left(u_{10}, v_{10}\right) \in \mathcal{D}$ and let $\left(\bar{u}_{10}, \bar{v}_{10}\right) \in \Pi^{s}$ be its image under the Poincaré map T. We denote the quantity $\frac{\bar{v}_{10}}{\bar{u}_{10}}\left(w h e n \bar{u}_{10} \neq 0\right)$ by $\bar{w}\left(u_{10}, v_{10}\right)$ or $\bar{w}$.

Notation 4.3. We denote the straight line $\left\{v_{10}=\frac{d}{b} u_{10}\right\}$ in $\Pi^{s}$ by $\ell^{*}$.

### 4.2. Dynamics near the homoclinic orbit $\Gamma$ : case $\lambda_{2}<2 \lambda_{1}$

Here, we show that when $\lambda_{2}<2 \lambda_{1}$, any point in the domain $\mathcal{D}$ of the Poincaré map $T$ leaves $\mathcal{D}$ by both forward and backward iterations of the Poincaré map. The proof of the case $\lambda_{1}=\lambda_{2}$ directly follows from Proposition 3.28 in which we have shown that the domain $\mathcal{D}$ of the Poincaré map is empty. For the case of $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$, we prove that the image of the domain $\mathcal{D}$ under the Poincaré map $T$ has no intersection with $\mathcal{D}$. This is also illustrated in Figure 4.1 where $\mathcal{D}$ is shown by green color, and $T(\mathcal{D})$ is a subset of the region which is shown in gray. We formalize this discussion in the following lemma:

Lemma 4.4. When $\lambda_{1} \leq \lambda_{2}<2 \lambda_{1}$, we have $\Lambda_{\mathcal{D}, T}^{s}=\Lambda_{\mathcal{D}, T}^{u}=\emptyset$.
Proof. When $\lambda_{1}=\lambda_{2}$, the statement follows from Proposition 3.28.
Suppose $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$. By Proposition 3.29, the domain $\mathcal{D}$ of the Poincaré map is

$$
\left\{\left(u_{10}, v_{10}\right) \in \Pi^{s}: u_{10} v_{10}>0, \quad\left\|\left(u_{10}, v_{10}\right)\right\|<\epsilon, \quad\left|v_{10}\right|<K_{\epsilon_{u}}\left|u_{10}\right|^{\frac{\gamma}{1-\gamma}}[1+O(\delta)]\right\}
$$

where $\frac{1}{2}<\gamma=\frac{\lambda_{1}}{\lambda_{2}}<1$ and $K_{\epsilon_{u}}$ is some constant (see (3.4.17)). By (3.4.15), i.e. relation $u_{1 \tau}=$ $v_{1 \tau} O\left(\epsilon^{2}\right)$, Poincaré map (4.1.1) can be written as

$$
\left(\bar{u}_{10}, \bar{v}_{10}\right)=\left(\left[b+O\left(\epsilon^{2}\right)\right] v_{1 \tau},\left[d+O\left(\epsilon^{2}\right)\right] v_{1 \tau}\right)
$$

which implies $\bar{w}=\frac{d}{b}+O\left(\epsilon^{2}\right)$. This means that the images of the points in the domain $\mathcal{D}$ under the Poincaré map $T$ accumulate near $\ell^{*}$. However, for a fixed $\delta$ and a sufficiently small $\epsilon$, this line has no intersection with the domain $\mathcal{D}$ (see Figure 4.1). This implies $\Lambda_{\mathcal{D}, T}^{s}=\Lambda_{\mathcal{D}, T}^{u}=\emptyset$ as desired.


Figure 4.1: Case $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$ : the domain $\mathcal{D}$ of the Poincaré map $T$ is shown in green. The images of the points in $\mathcal{D}$ under the Poincaré map $T$ accumulate near the straight line $\ell^{*}$ (the line whose slope is $\frac{d}{b}$ ) in the gray region. As it is seen, the green and the gray regions have no intersection which means $\mathcal{D} \cap T(\mathcal{D})=\emptyset$. This implies that the backward and forward orbits of any point of the domain $\mathcal{D}$ leaves $\mathcal{D}$. The figure in the left shows the case of $b d>0$ ( $\ell^{*}$ lies in the first and the third quadrants), and the figure in the right shows the case of $b d<0$ ( $\ell^{*}$ lies in the second and the fourth quadrants).

Theorem A2 is an immediate consequence of this lemma. We have

Proof of Theorem A2. Any orbit in $W_{\text {loc }}^{s}(\Gamma)$ other than $\Gamma$ must intersect $\Pi^{s}$ at $\Lambda_{\mathcal{D}, T}^{s}$. However, by Lemma 4.4, we have $\Lambda_{\mathcal{D}, T}^{s}=\emptyset$. This implies $W_{\text {loc }}^{s}(\Gamma)=\Gamma$. The proof of $W_{\text {loc }}^{u}(\Gamma)=\Gamma$ is the same.

The statement of Theorem A1 for the particular case of $\lambda_{2}<2 \lambda_{1}$ also follows from Lemma 4.4:
Proof of Theorem A1: case $\lambda_{2}<2 \lambda_{1}$. Consider a point in $\mathcal{U} \backslash W_{\mathcal{U}}^{s}$ whose forward orbit lies entirely in $\mathcal{U}$. The forward orbit of this point must intersect $\Pi^{s}$ at $\Lambda_{\mathcal{D}, T}^{s}$. However, by Lemma 4.4, $\Lambda_{\mathcal{D}, T}^{s}=\emptyset$. On the other hand, Theorem A2 implies $W_{l o c}^{s}(\Gamma)=\Gamma$ and therefore $W_{\text {loc }}^{s}(\Gamma) \subset W_{\mathcal{U}}^{s}$. Thus, when $\lambda_{2}<2 \lambda_{1}$, the forward orbit of a point in $\mathcal{U}$ lies entirely in $\mathcal{U}$ if and only if it belongs to $W_{\mathcal{U}}^{s}(O) \cup W_{\text {loc }}^{s}(\Gamma)$. The proof of the case of backward orbits is the same. This finishes the proof of Theorem A1 for the specific case of $\lambda_{2}<2 \lambda_{1}$.

The proof of Theorem A1 for the case of $2 \lambda_{1}<\lambda_{2}$ is provided in the next section.

### 4.3. Dynamics near the homoclinic orbit $\Gamma$ : case $2 \lambda_{1}<\lambda_{2}$

In this section, we study the dynamics near the homoclinic orbit $\Gamma$ for the case $2 \lambda_{1}<\lambda_{2}$, and prove Theorems A1 (the case $2 \lambda_{1}<\lambda_{2}$ ) and A3.

Recall from Section 3.4.3 that when $2 \lambda_{1}<\lambda_{2}$, we divide the domain $\mathcal{D}$ of the Poincaré map $T$ into three subsets $\mathcal{D}_{1}, \mathcal{D}_{2}$ and $\mathcal{D}_{3}$, i.e. $\mathcal{D}=\mathcal{D}_{1} \cup \mathcal{D}_{2} \cup \mathcal{D}_{3}$. Namely, for a given sufficiently large $m>0$, we have seen that:

- $\mathcal{D}_{1}^{\epsilon}=\mathcal{D}_{1}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}, \quad 0<\frac{v_{10}}{u_{10}}<\frac{1}{m}\right\}$,
- $\mathcal{D}_{2}^{\epsilon}=\mathcal{D}_{2}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}, \quad \frac{1}{m} \leq \frac{v_{10}}{u_{10}} \leq m\right\}$, and
- $\mathcal{D}_{3}^{\epsilon}=\mathcal{D}_{3} \subset\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}, \quad m<\left|\frac{v_{10}}{u_{10}}\right|\right\}$,
where $\mathcal{B}_{\epsilon}$ is the open $\epsilon$-ball in $\Pi^{s}$ centered at $M^{s}$ (see Figure 4.2).


Figure 4.2: When $2 \lambda_{1}<\lambda_{2}$, we write the domain $\mathcal{D}$ of the Poincaré map $T$ as the disjoint union of three subsets $\mathcal{D}_{1}, \mathcal{D}_{2}$ and $\mathcal{D}_{3}$, i.e. $\mathcal{D}=\mathcal{D}_{1} \cup \mathcal{D}_{2} \cup \mathcal{D}_{3}$. The subset $\mathcal{D}_{1}$ is shown in blue and $\mathcal{D}_{2}$ is shown in green. The set $\mathcal{D}_{3}$ is a subset of the purple region.

In order to understand the dynamics near the homoclinic loop $\Gamma$, we need to investigate the set of the points on the domain $\mathcal{D}$ whose forward or backward orbits (under the iterations of the Poincaré $\operatorname{map} T$ ) lie in $\mathcal{D}$, i.e. the sets $\Lambda_{\mathcal{D}, T}^{s}$ and $\Lambda_{\mathcal{D}, T}^{u}$ (see Notation 4.1). To this end, we take the following three steps:

- Step 1: Investigating the set of the points in $\mathcal{D}_{2} \cup \mathcal{D}_{3}$ whose forward or backward orbits lie entirely in $\mathcal{D}_{2} \cup \mathcal{D}_{3}$, i.e. the sets $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$ and $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$.
- Step 2: Investigating the set of the points in $\mathcal{D}_{1}$ whose forward or backward orbits lie entirely in $\mathcal{D}_{1}$, i.e. the sets $\Lambda_{\mathcal{D}_{1}, T}^{s}$ and $\Lambda_{\mathcal{D}_{1}, T}^{u}$.

Obviously, $\Lambda_{\mathcal{D}_{1}, T}^{s}$ and $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$ are subsets of $\Lambda_{\mathcal{D}, T}^{s}$. In addition, $\Lambda_{\mathcal{D}_{1}, T}^{u}$ and $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$ are subsets of $\Lambda_{\mathcal{D}, T}^{u}$. In the third step, we show that the reverse directions also hold: $\Lambda_{\mathcal{D}, T}^{s} \subset \Lambda_{\mathcal{D}_{1}, T}^{s} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$ and $\Lambda_{\mathcal{D}, T}^{u} \subset \Lambda_{\mathcal{D}_{1}, T}^{u} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$. Equivalently,

- Step 3: We show $\Lambda_{\mathcal{D}, T}^{s}=\Lambda_{\mathcal{D}_{1}, T}^{s} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$ and $\Lambda_{\mathcal{D}, T}^{u}=\Lambda_{\mathcal{D}_{1}, T}^{u} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$.

Notice that the statement of Step 3 is not trivial. In fact, at the first stage, one can consider the possibility of the existence of a point $x \in \mathcal{D}$ such that its forward orbit lies entirely in $\mathcal{D}$, i.e. $x \in \Lambda_{\mathcal{D}, T}^{s}$, but it does not lie entirely in only one of the sets $\mathcal{D}_{1}$ or $\mathcal{D}_{2} \cup \mathcal{D}_{3}$, i.e. $x \notin \Lambda_{\mathcal{D}_{1}, T}^{s}$ and $x \notin \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$. In other words, the forward orbit of $x$ stays in $\mathcal{D}$ but switches between $\mathcal{D}_{1}$ and $\mathcal{D}_{2} \cup \mathcal{D}_{3}$. In Step 3, we indeed show that this scenario does not happen.

We take Step 1 in the following lemma. This lemma helps us to understand the dynamics of the Poincaré map $T$ on the set $\mathcal{D}_{2} \cup \mathcal{D}_{3}$. We explore in this lemma that how $T$ behaves on this set, with which rate the orbits of this set grow, and how $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$ and $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$ look like. From technical point of view, part (vii) of this lemma which shows the existence of the unstable manifold of the Poincaré map $T$ is the main result of this section. The techniques which are used in the proof of this part is also used in Section 4.4 for the proof of the existence of the unstable manifold of the homoclinic figure-eight. We prove this lemma in Section 4.3.1.

Lemma 4.5. Let $w$ and $\ell^{*}$ be as in Notations 4.2 and 4.3, respectively. Assume $2 \lambda_{1}<\lambda_{2}$ and $b d \neq 0$. For $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2} \cup \mathcal{D}_{3}$, we have
(i) $\bar{w}=w\left(T\left(u_{10}, v_{10}\right)\right)=\frac{d}{b}+o(1)$, where $o(1)$ stands for a function of $\left(u_{10}, v_{10}\right)$ that converges to zero as $\left(u_{10}, v_{10}\right) \rightarrow(0,0)$.
(ii) There exists a constant $C>0$ such that $\left\|\left(u_{10}, v_{10}\right)\right\|^{1-2 \gamma}<C\left\|T\left(u_{10}, v_{10}\right)\right\|$ holds for arbitrary $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, where $\gamma=\lambda_{1} \lambda_{2}{ }^{-1}<0.5$.
(iii) if bd>0, then $T\left(u_{10}, v_{10}\right)$ lies in $\mathcal{D}_{2}$ unless it leaves $\mathcal{B}_{\epsilon}$.
(iv) $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}=\emptyset$.
(v) $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}=\Lambda_{\mathcal{D}_{2}, T}^{u}$
(vi) when bd $<0$ we have $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}=\emptyset$.
(vii) when $b d>0$, the set $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$ is a one-dimensional $\mathcal{C}^{1}$-manifold which is tangent to $\ell^{*}$ at $M^{s}$.

It follows from this lemma that the image of $\mathcal{D}_{2} \cup \mathcal{D}_{3}$ under the Poincaré map $T$ lies near $\ell^{*}$, and the Poincaré map increases the norm of any point of this set. Informally speaking, for the particular case of $b d>0$, this means that the Poincaré map $T$ preserves and expands the region $\mathcal{D}_{2} \cup \mathcal{D}_{3}$. A geometrical picture of this behavior is illustrated in Figure 4.3.

We now take the second step in the next lemma. In this lemma, we study the dynamics of $T^{-1}$ on the set $\mathcal{D}_{1}$. Most of the statements of the following lemma are analogous to the statements of the preceding lemma. This is not a coincidence. In fact, we see later in the proof of Lemma 4.6 that the dynamics of $T^{-1}$ on $\mathcal{D}_{1}$ can be obtained from the dynamics of $T$ on $\mathcal{D}_{2} \cup \mathcal{D}_{3}$ by a permutation and reversion of time. The proof of this lemma is postponed to Section 4.3.2.


Figure 4.3: The straight line whose slope is $\frac{d}{b}$ is denoted by $\ell^{*}$. The left figure corresponds to the case $b d<0$ and the right one corresponds to the case $b d>0$. The set $\mathcal{D}_{2}$ is shown in green. The set $\mathcal{D}_{3}$ is a subset of the purple region. the image of $\mathcal{D}_{2} \cup \mathcal{D}_{3}$ under the Poincaré map, i.e. $T\left(\mathcal{D}_{2} \cup \mathcal{D}_{3}\right)$, is a subset of the wavy region. Informally speaking, the Poincaré map $T$ preserves and expands the region $\mathcal{D}_{2} \cup \mathcal{D}_{3}$. We show in Lemma 4.5 that when $b d>0$, there exists an unstable invariant manifold for the Poincaré map $T$ in the vawy region, tangent to $\ell^{*}$ at $M^{s}$.

Lemma 4.6. Assume $2 \lambda_{1}<\lambda_{2}$ and $c d \neq 0$. For $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{1}$, we have
(i) if $c d>0$, then $T(\mathcal{D}) \cap \mathcal{D}_{1}=\emptyset$.
(ii) if $c d<0$, then $w\left(T^{-1}\left(u_{10}, v_{10}\right)\right)=o(1)$, where $o(1)$ stands for a function of $\left(u_{10}, v_{10}\right)$ that converges to zero as $\left(u_{10}, v_{10}\right) \rightarrow(0,0)$. In other words, $T^{-1}\left(\mathcal{D}_{1}\right)$ accumulates near the horizontal axis.
(iii) if $c d<0$, then $\left\|\left(u_{10}, v_{10}\right)\right\|^{1-2 \gamma}<C\left\|T^{-1}\left(u_{10}, v_{10}\right)\right\|$ for some constant $C>0$.
(iv) if $c d<0$, then $T^{-1}\left(u_{10}, v_{10}\right)$ remains in $\mathcal{D}_{1}$ unless it leaves $\mathcal{B}_{\epsilon}$.
(v) $\Lambda_{\mathcal{D}_{1}, T^{-1}}^{s}=\emptyset$. Equivalently, $\Lambda_{\mathcal{D}_{1}, T}^{u}=\emptyset$.
(vi) if $c d<0$, then the set $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{1}, T^{-1}}^{u}$ (equivalently, the set $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{1}, T}^{s}$ ) is a one-dimensional $\mathcal{C}^{1}$-manifold which is tangent to the horizontal axis at $M^{s}$.

In the preceding two lemmas, we have shown that the sets $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{s}$ and $\Lambda_{\mathcal{D}_{1}, T}^{u}$ are always empty. It was also shown that $\Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}=\Lambda_{\mathcal{D}_{2}, T}^{u}$. This allows us to reformulate Step 3 as in the following lemma:

Lemma 4.7. Assume $2 \lambda_{1}<\lambda_{2}$ and $b c d \neq 0$. We have
(i) $\Lambda_{\mathcal{D}, T}^{s}=\Lambda_{\mathcal{D}_{1}, T}^{s}$.
(ii) $\Lambda_{\mathcal{D}, T}^{u}=\Lambda_{\mathcal{D}_{2}, T}^{u}$.

Proof. Let $\mathrm{x} \in \mathcal{D}_{2} \cup \mathcal{D}_{3}$. It follows from parts (i), (ii) and (iii) of Lemma 4.5 that if $b d<0$, then $T(\mathrm{x}) \notin \mathcal{D}$, and if $b d>0$, then for some $k, T^{k}(\mathrm{x}) \notin \mathcal{B}_{\epsilon}$. Thus, any point in $\Lambda_{\mathcal{D}, T}^{s}$ must belong to $\mathcal{D}_{1}$. This proves part (i).

To prove part (ii), notice that if $\Lambda_{\mathcal{D}, T}^{u}=\emptyset$, then $\Lambda_{\mathcal{D}_{2}, T}^{u}=\emptyset$ and therefore $\Lambda_{\mathcal{D}, T}^{u}=\Lambda_{\mathcal{D}_{2}, T}^{u}$. So we assume that $\Lambda_{\mathcal{D}, T}^{u}$ is non-empty. Let $\mathrm{x} \in \Lambda_{\mathcal{D}, T}^{u}$. We need to show $\mathrm{x} \in \mathcal{D}_{2}$. To do this, we first prove
$\mathrm{x} \notin \mathcal{D}_{1}$. Assume the contrary, i.e. $\mathrm{x} \in \mathcal{D}_{1}$. It follows from parts (i) and (iii) of Lemma 4.5 that if $T^{-1}(\mathrm{x}) \in \mathcal{D}_{2} \cup \mathcal{D}_{3}$, then $\mathrm{x}=T\left(T^{-1}(\mathrm{x})\right)$ either belongs to $\mathcal{D}_{2}$ or lies outside the domain $\mathcal{D}$ which contradicts the assumption $\mathrm{x} \in \mathcal{D}_{1}$. Therefore, $T^{-1}(\mathrm{x}) \notin \mathcal{D}_{2} \cup \mathcal{D}_{3}$, and so $T^{-1}(\mathrm{x}) \in \mathcal{D}_{1}$. By virtue of part (i) of Lemma 4.6, this relation implies $c d<0$. On the other hand, when $c d<0$, it follows from parts (iii) and (iv) of Lemma 4.6 that there exists a $k>0$ such that $T^{-k}(\mathrm{x}) \notin \mathcal{B}_{\epsilon}$ and hence $T^{-k}(\mathrm{x}) \notin \mathcal{D}$. This contradicts the preliminary assumption $\mathrm{x} \in \Lambda_{\mathcal{D}, T}^{u}$. Therefore, if $\mathrm{x} \in \Lambda_{\mathcal{D}, T}^{u}$, then $\mathrm{x} \notin \mathcal{D}_{1}$, or equivalently, $T^{-n}(\mathrm{x}) \notin \mathcal{D}_{1}$ for all $n \geq 0$.

To finish the proof, it is sufficient to show that $\mathrm{x} \notin \mathcal{D}_{3}$. Assume the contrary, i.e. $\mathrm{x} \in \mathcal{D}_{3}$. Since $\mathrm{x} \in \Lambda_{\mathcal{D}, T}^{u}$ implies $T^{-n}(\mathrm{x}) \notin \mathcal{D}_{1}$ for all $n \geq 0$, we have $T^{-1}(\mathrm{x}) \notin \mathcal{D}_{1}$. On the other hand, parts (i) and (iii) of Lemma 4.5 imply that if $T^{-1}(\mathrm{x}) \in \mathcal{D}_{2} \cup \mathcal{D}_{3}$, then $\mathrm{x}=T\left(T^{-1}(\mathrm{x})\right)$ either belongs to $\mathcal{D}_{2}$ or lies outside the domain $\mathcal{D}$ which contradicts the assumption $\mathrm{x} \in \mathcal{D}_{3}$. Therefore, $\mathrm{x} \notin \mathcal{D}_{3}$ as desired.

Recall that the local stable (unstable) set of the homoclinic loop $\Gamma$, denoted by $W_{\text {loc }}^{s}(\Gamma)\left(W_{\text {loc }}^{u}(\Gamma)\right)$, is the union of $\Gamma$ itself and the set of the points in a sufficiently small neighborhood $\mathcal{U}$ of $\Gamma$ whose forward (backward) orbits lie in $\mathcal{U}$ and their $\omega$-limit sets ( $\alpha$-limit sets) coincide with $\Gamma \cup\{O\}$. By this definition, the intersection of $W_{\mathrm{loc}}^{s}(\Gamma)$ and $\Pi^{s}$ must belong to $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}}^{s}$, and the intersection of $W_{\text {loc }}^{u}(\Gamma)$ and $\Pi^{s}$ must belong to $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}}^{u}$. On the other hand, we have shown in the above lemmas that when $\Lambda_{\mathcal{D}}^{s}\left(\Lambda_{\mathcal{D}}^{u}\right)$ is non-empty, any point on this set converges to $M^{s}$ by the forward (backward) iterations of the Poincaré map $T$. This leads to the following:

Proposition 4.8. Let $\phi_{t}$ be the flow of system (3.2.9). Then

$$
W_{l o c}^{s}(\Gamma)=\Gamma \cup \phi_{t}\left(\Lambda_{\mathcal{D}, T}^{s}\right) \quad \text { for } t \geq 0, \quad \text { and } \quad W_{l o c}^{u}(\Gamma)=\Gamma \cup \phi_{t}\left(\Lambda_{\mathcal{D}, T}^{u}\right) \quad \text { for } t \leq 0
$$


(a)

(b)

Figure 4.4: The local unstable invariant manifold of the equilibrium $O$ intersects $\Pi^{u}$ at the $v_{1}$-axis. Thus, the blue curve ( $v_{1}$-axis restricted to a small neighborhood of $M^{u}$ in $\Pi^{u}$ ) lies at the intersection of the local unstable invariant manifold of $O$ and the cross-section $\Pi^{u}$. This curve is mapped to the blue curve on $\Pi^{s}$ by $T^{\text {glo }}$ which means that the blue curve on $\Pi^{s}$ lies in $W_{\text {glo }}^{u}(O) \cap \Pi^{s}$. Since $v_{1}$-axis on $\Pi^{u}$ is mapped to $\ell^{*}$ on $\Pi^{s}$ by $d T^{\text {glo }}$, the straight line $\ell^{*}$ is tangent to the blue curve on $\Pi^{s}$ at $M^{s}$.

In system (3.2.9), the local unstable invariant manifold of the equilibrium $O$ is straightened, i.e. $W_{\text {loc }}^{u}(O)=\{u=0\}$. Thus, the intersection of this manifold and the cross-section $\Pi^{u}=\left\{v_{2}=\right.$ $\delta\} \cap\{H=0\}$ is the straight line $\left\{u_{1}=0\right\}$, i.e. $v_{1}$-axis. Consider the restriction of this line to a small neighborhood of $M^{u}$ (in Figure 4.4, it is shown by blue color on $\Pi^{u}$ ). The global map $T^{\text {glo }}$ maps this restricted piece to a curve, denote it by $\gamma^{u}$, on $\Pi^{s}$ (shown by blue color on $\Pi^{s}$ in Figure 4.4). This curve is in fact at the intersection of the global unstable invariant manifold of $O$ and the cross-section $\Pi^{s}$. Since $T^{\text {glo }}$ is a diffeomorphism and the vector $\binom{0}{1}$ is tangent to $v_{1}$-axis at $M^{u}$, the vector $d T^{\text {glo }}\binom{0}{1}=\binom{b}{d}$ is tangent to $\gamma^{u}$ at $M^{s}$ in $\Pi^{s}$, i.e. $\gamma^{u}$ is tangent to $\ell^{*}$ at $M^{s}$ (recall that $\ell^{*}$ is the line in $\Pi^{s}$ whose slope is $\frac{d}{b}$ ). Therefore, it follows from Lemma 4.7 and part (vii) of Lemma 4.5 that when $b d>0, W_{\text {glo }}^{u}(O) \cap \Pi^{s}$ and $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}, T}^{u}$ are tangent at $M^{s}$. On the other hand, it follows from Lemma 4.7 and part (vi) of Lemma 4.6 that when $c d<0$ the intersection of the local stable manifold of $O$ and the cross-section $\Pi^{s}$, i.e. the horizontal axis, is tangent to $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}, T}^{s}$ at $M^{s}$. Moreover, by Assumption 5, the homoclinic orbit $\Gamma$ is at the transverse intersection of the global stable and unstable invariant manifolds of the equilibrium $O$. Therefore, the intersection of these two manifolds with the cross-section $\Pi^{s}$, i.e. the horizontal axis and the curve $\gamma^{u}$, intersect transversely at $M^{s}$. Since $\gamma^{u}$ is tangent to $\ell^{*}$ at $M^{s}$, we have that the intersection of $W_{\text {glo }}^{s}(O)$ and $W_{\text {glo }}^{u}(O)$ at $\Gamma$ is transverse if and only if the horizontal axis on $\Pi^{s}$ and the straight line $\ell^{*}$ are distinct. These statements give

Proposition 4.9. Assume $2 \lambda_{1}<\lambda_{2}$ and bcd $\neq 0$. We have
(i) When $b d>0$, the 2-dimensional $\mathcal{C}^{1}$-smooth invariant manifold $W_{\text {loc }}^{u}(\Gamma)$ is tangent to $W_{\text {glo }}^{u}(O)$ at every point of $\Gamma$.
(ii) When $c d<0$, the 2-dimensional $\mathcal{C}^{1}$-smooth invariant manifold $W_{\text {loc }}^{s}(\Gamma)$ is tangent to $W_{\text {glo }}^{s}(O)$ at every point of $\Gamma$.
(iii) The intersection of $W_{g l o}^{s}(O)$ and $W_{g l o}^{u}(O)$ at $\Gamma$ is transverse if and only if $d \neq 0$.

By virtue of the above results, we have:
Proof of Theorem A3. By Proposition 4.8 and the preceding Lemmas we have $W_{\text {loc }}^{u}(\Gamma)=\Gamma$ when $b d<0$, and $W_{\text {loc }}^{s}(\Gamma)=\Gamma$ when $c d>0$. The rest of the theorem is already proved (see Proposition 4.9).

We have proved Theorem A1 for the case of $\lambda_{2}<2 \lambda_{1}$ in the previous section. We now prove this theorem for the case of $2 \lambda_{1}<\lambda_{2}$ :

Proof of Theorem A1: case $2 \lambda_{1}<\lambda_{2}$. By definition, the forward orbit of any point on $W_{\mathcal{U}}^{s}(O) \cup$ $W_{\text {loc }}^{s}(\Gamma)$ lies in $\mathcal{U}$. Consider a point in $\mathcal{U} \backslash W_{\mathcal{U}}^{s}(O)$ whose forward orbit lies entirely in $\mathcal{U}$. The forward orbit of this point must intersect $\Pi^{s}$ at $\Lambda_{\mathcal{D}, T}^{s}$. Therefore, it follows from Proposition 4.8 that this point lies on $W_{\text {loc }}^{s}(\Gamma)$. This finishes the proof for the case of forward orbits. The proof of the case of backward orbits is the same. This finishes the proof of Theorem A1 for the case $2 \lambda_{1}<\lambda_{2}$.

### 4.3.1. Proof of Lemma 4.5

Proof of part (i). By (3.4.31) and Corollary 3.33, $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2} \cup \mathcal{D}_{3}$ implies $u_{1 \tau}=o\left(v_{1 \tau}\right)$. Thus, Poincaré map (4.1.1) takes the form

$$
\begin{equation*}
\left(\bar{u}_{10}, \bar{v}_{10}\right)=\left([b+o(1)] v_{1 \tau},[d+o(1)] v_{1 \tau}\right) \tag{4.3.1}
\end{equation*}
$$

which implies $\bar{w}=\frac{d}{b}+o(1)$.

Proof of part (ii). For $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, relations (3.4.11) and (4.3.1) imply

$$
\left\|T\left(u_{10}, v_{10}\right)\right\|=\left\|\left(\bar{u}_{10}, \bar{v}_{10}\right)\right\|=\left[b^{2}+d^{2}+o(1)\right]^{\frac{1}{2}}\left|v_{1 \tau}\right|=K\left|u_{10}\right|^{-\gamma}\left|v_{10}\right|^{1-\gamma}
$$

where $K=K\left(u_{10}, v_{10}\right)=\gamma^{-\gamma} \delta^{2 \gamma}\left[b^{2}+d^{2}+o(1)\right]^{\frac{1}{2}}$. Therefore, for $C>K^{-1} m^{\gamma}(1+m)^{\frac{1}{2}}$, we have

$$
\frac{\left\|\left(u_{10}, v_{10}\right)\right\|}{\left\|T\left(u_{10}, v_{10}\right)\right\|}=\frac{\left|v_{10}\right|\left(1+\frac{u_{10}}{v_{10}}\right)^{\frac{1}{2}}}{K\left|u_{10}\right|^{-\gamma}\left|v_{10}\right|^{1-\gamma}} \leq K^{-1} m^{\gamma}(1+m)^{\frac{1}{2}}\left|v_{10}\right|^{2 \gamma}<C\left\|\left(u_{10}, v_{10}\right)\right\|^{2 \gamma}
$$

Proof of part (iii). By part (i) of Lemma 4.5, T ( $\left.u_{10}, v_{10}\right)$ is somewhere close to the line $\ell^{*}$ and since, for $b d>0$, the restriction of $\ell^{*} \backslash\left\{M^{s}\right\}$ to $\mathcal{B}_{\epsilon}$ lies in $\mathcal{D}_{2}$ we have that if $T\left(u_{10}, v_{10}\right)$ lies in $\mathcal{B}_{\epsilon}$, then it must belong to $\mathcal{D}_{2}$.

Proofs of parts (iv), (v) and (vi). They are easy consequences of the previous parts.
Proof of part (vii). Recall the definition of the set $\mathcal{D}_{2}^{\epsilon}$ from (3.4.24) and consider $\mathcal{D}_{2}^{\epsilon_{1}}$ for a sufficiently small $\epsilon_{1}>0$. Choose $\epsilon_{2}<\epsilon_{1}$ such that $\mathcal{X} \subset \mathcal{D}_{2}^{\epsilon_{1}}$, where $\mathcal{X}=\left\{\left(u_{10}, v_{10}\right) \in \Pi^{s}: m^{-1} \leq \frac{v_{10}}{u_{10}} \leq m, u_{10} \neq\right.$ $\left.0,\left|v_{10}\right| \leq \epsilon_{2}\right\}$ and $m$ is as in (3.4.22) (see Figure 4.5). Recall $w$ in Notation 4.2 and define the new variable $z$ by

$$
\begin{equation*}
z=z\left(u_{10}, v_{10}\right)=\operatorname{sgn}\left(v_{10}\right)\left|v_{10}\right|^{\alpha}, \quad(0<\alpha \text { will be specefied later }) \tag{4.3.2}
\end{equation*}
$$



Figure 4.5: The set $\mathcal{X} \subset \mathcal{D}_{2}^{\epsilon_{1}}$ in $\left(u_{10}, v_{10}\right)$-plane is shown by green color.
Let $\mathcal{Y}$ be the set $\mathcal{X}$ equipped with $(w, z)$-coordinates. Thus, $\mathcal{Y}=\left[m^{-1}, m\right] \times\left(\left[-\epsilon_{2}{ }^{\alpha}, \epsilon_{2}{ }^{\alpha}\right] \backslash\{0\}\right)$ (see Figure 4.6). Consider the restriction of the Poincaré map $T$ to the set $\mathcal{X}$, i.e. $\left.T\right|_{\mathcal{X}}$, and denote the representation of this map in $(w, z)$-coordinates by $\mathcal{T}$. We write

$$
\begin{equation*}
\mathcal{T}:(w, z) \mapsto(\bar{w}, \bar{z})=(f(w, z), g(w, z)) \tag{4.3.3}
\end{equation*}
$$

for some smooth functions $f$ and $g$ defined on $\mathcal{Y}$. Note that by (3.4.11) and the relation $\bar{z}=g(w, z)=$ $\operatorname{sgn}\left(\bar{v}_{10}\right)\left|\bar{v}_{10}\right|^{\alpha}$, we can derive

$$
\begin{equation*}
\bar{z}=g(w, z)=\operatorname{sgn}(d z)|d|^{\alpha}\left(\frac{\gamma}{\delta^{2}}\right)^{-\gamma \alpha} w^{\gamma \alpha}|z|^{1-2 \gamma}[1+O(\delta)]=O\left(|z|^{1-2 \gamma}\right) \tag{4.3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
z=O\left(|\bar{z}|^{\frac{1}{1-2 \gamma}}\right) \tag{4.3.5}
\end{equation*}
$$

We now make a statement which is proved later:
Lemma 4.10. $g_{z}(w, z)$ is non-zero for any $(w, z) \in \mathcal{Y}$.


Figure 4.6: The set $\mathcal{Y}$ (the set $\mathcal{X}$ equipped with $(w, z)$-coordinates) is shown by green color. It contains two connected components (below and above horizontal axis).

According to this lemma and the implicit function theorem, the variable $z$ is a $\mathcal{C}^{q}$-smooth ( $q$ is as in Lemma 3.8) function of $(w, \bar{z})$ for $w \in\left[m^{-1}, m\right]$ and $\bar{z} \in g(\mathcal{Y})$. Denote this function by $G$. Regarding the domain of this function, note that not every $(w, \bar{z})$ necessarily belongs to the domain of $G$. In other words, for an arbitrary $(w, \bar{z})$, there might not exist $z \in\left[-\epsilon_{2}{ }^{\alpha}, \epsilon_{2}{ }^{\alpha}\right] \backslash\{0\}$ such that $z=G(w, \bar{z})$. However, by (4.3.4), this relation holds if $\bar{z}$ is chosen sufficiently small, i.e. for a sufficiently small $\theta>0$ we have

$$
\left[m^{-1}, m\right] \times([-\theta, \theta] \backslash\{0\}) \subset \operatorname{domain}(G) .
$$

Denote this set by $\mathcal{R}$, i.e. $\mathcal{R}=\left[m^{-1}, m\right] \times([-\theta, \theta] \backslash\{0\})$. Without loss of generality, assume $\theta<\epsilon_{2}{ }^{\alpha}$. Having the function $G$ in hand means that we can write the Poincaré map $\mathcal{T}$ in cross-form: we define the cross-map $\mathcal{T}^{\times}:(w, \bar{z}) \mapsto(\bar{w}, z)$ by

$$
\begin{equation*}
(\bar{w}, z)=(F(w, \bar{z}), G(w, \bar{z})), \quad \text { where } F(w, \bar{z})=f(w, G(w, \bar{z})), \tag{4.3.6}
\end{equation*}
$$

and $(w, \bar{z}) \in$ domain $(G)$. It follows from part (i) of Lemma 4.5 (proved earlier), relation (4.3.5) and the fact that $z=0$ if and only if $\bar{z}=0$ (follows from (4.3.4)) that $\mathcal{T}^{\times}(\mathcal{R}) \subset \mathcal{R}$. Hereafter, we focus on the restriction of $\mathcal{T}^{\times}$on $\mathcal{R}$. Our approach to prove the existence of the desired invariant manifold for the Poincaré map $\mathcal{T}$ is to apply Theorem 2.28 on the cross-map $\mathcal{T}^{\times}$. However, to do this, there are two issues that we need to take care of. The first is that the domain $\mathcal{R}$ does not satisfy the assumption of Theorem 2.28 (in that theorem, the domain must be written as a Cartesian product of two convex closed sets but $\mathcal{R}$ is not of this form since it does not contain the line $\bar{z}=0$ ). Second, we need to compute the partial derivatives of the cross-map $\mathcal{T}^{\times}$. The second issue is resolved by the following lemma:

Lemma 4.11. Let $\beta=\alpha^{-1} \min \{4 \gamma, 1-2 \gamma\}$. We have

$$
\begin{array}{ll}
F_{w}(w, \bar{z})=O\left(|\bar{z}|^{\frac{\beta}{1-2 \gamma}}\right), & F_{\bar{z}}(w, \bar{z})=O\left(|\bar{z}|^{\frac{\beta-1+2 \gamma}{1-2 \gamma}}\right), \\
G_{w}(w, \bar{z})=O\left(|\bar{z}|^{\frac{1}{1-2 \gamma}}\right), & G_{\bar{z}}(w, \bar{z})=O\left(|\bar{z}|^{\frac{2 \gamma}{1-2 \gamma}}\right) .
\end{array}
$$

This lemma is proved later. We now extend the domain $\mathcal{R}$ to $\widetilde{\mathcal{R}}$, where $\widetilde{\mathcal{R}}=\left[m^{-1}, m\right] \times[-\theta, \theta]$. We also extend the map $\mathcal{T}^{\times}$to the map $\widetilde{\mathcal{T}}^{\times}$defined on $\widetilde{\mathcal{R}}$ by

$$
\tilde{\mathcal{T}}^{\times}(w, \bar{z}):= \begin{cases}\mathcal{T}^{\times}(w, \bar{z})=(F(w, \bar{z}), G(w, \bar{z})) & \bar{z} \neq 0, \\ \left(\frac{d}{b}, 0\right) & \bar{z}=0,\end{cases}
$$

Lemma 4.11 implies that for a fixed sufficiently small $\alpha$, the map $\widetilde{\mathcal{T}}^{\times}: \widetilde{\mathcal{R}} \rightarrow \widetilde{\mathcal{R}}$ is a $\mathcal{C}^{1}$-smooth extension of $\mathcal{T}^{\times}$to $\widetilde{\mathcal{R}}$.

Now, let us come back to the Poincaré map $\mathcal{T}$ defined on $\mathcal{Y}$. We extend this map to

$$
\widetilde{\mathcal{T}}(w, z):= \begin{cases}\mathcal{T}(w, z) & (w, z) \in \mathcal{Y} \\ \left(\frac{d}{b}, 0\right) & z=0\end{cases}
$$

It is clear that the map $\widetilde{\mathcal{T}} \times$ is in fact the cross-map of $\widetilde{\mathcal{T}} \widetilde{\widetilde{\mathcal{T}}}$ on $\widetilde{\mathcal{R}}$. Note that since $\theta<\epsilon_{2}{ }^{\alpha}$, we have $\widetilde{\mathcal{R}} \subset \mathcal{Y}$. Thus, both of the maps $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}^{\times}$are defined on $\widetilde{\mathcal{R}}$. Therefore, for a sufficiently small $\theta$, the $\underset{\widetilde{\mathcal{T}}}{\operatorname{map}} \widetilde{\mathcal{T}}^{\times}$satisfies the assumptions of Theorem 2.28 and Proposition 2.29. This implies that the map $\widetilde{\mathcal{T}}$ possesses a $\mathcal{C}^{1}$-smooth invariant manifold

$$
M^{*}=\left\{(w, z): w=h^{*}(z)\right\} \subset \widetilde{\mathcal{R}}
$$

where $h^{*}$ is some $\mathcal{C}^{1}$-smooth function defined on $[-\theta, \theta]$. Moreover, by Proposition 2.29, if the backward orbit of a point in $\widetilde{\mathcal{R}}$ remains in $\widetilde{\mathcal{R}}$, then it must belong to $M^{*}$. Therefore, $\Lambda_{\widetilde{\mathcal{R}}, \widetilde{\mathcal{T}}}^{u} \subset M^{*}$. Removing the point $\left(\frac{d}{b}, 0\right)$ from $M^{*}$, we obtain a set which is invariant under the map $\mathcal{T}$. Moreover, we have $\Lambda_{\mathcal{R}, \mathcal{T}}^{u} \subset M^{*} \backslash\left\{\left(\frac{d}{b}, 0\right)\right\}$.

Let us now come back to $\left(u_{10}, v_{10}\right)$-coordinates and the Poincaré map $T$. Equip $\mathcal{R}$ with ( $u_{10}, v_{10}$ )coordinates and choose $0<\epsilon<\theta$. Thus, $\mathcal{D}_{2}^{\epsilon} \subset \mathcal{R}$. Consider the manifold $M^{*}$ in $\left(u_{10}, v_{10}\right)$-coordinates and restrict it to $\mathcal{D}_{2}^{\epsilon}$. Denote this restriction by $\mathcal{M}^{*}$. We have that $\mathcal{M}^{*} \backslash\left\{M^{s}\right\}$ is invariant under $T$, and $\Lambda_{\mathcal{D}_{2}, T}^{u} \subset \mathcal{M}^{*} \backslash\left\{M^{s}\right\}$. Choosing a sufficiently small $\epsilon$ also guarantees that $\mathcal{M}^{*}$ is a connected piece of $M^{*}$ and hence is a $\mathcal{C}^{1}$-manifold.

The manifold $\mathcal{M}^{*}$ is our desired manifold if we show $\Lambda_{\mathcal{D}_{2}^{e}, T}^{u}=\mathcal{M}^{*} \backslash\left\{M^{s}\right\}$. So far, we have shown that $\Lambda_{\mathcal{D}_{2}, T}^{u} \subset \mathcal{M}^{*} \backslash\left\{M^{s}\right\}$ and so it is sufficient to show $\mathcal{M}^{*} \backslash\left\{M^{s}\right\} \subset \Lambda_{\mathcal{D}_{e}, T}^{u}$. However, this is just a direct consequence of part (ii) of this lemma (proved earlier). The fact that $\mathcal{M}^{*}$ is tangent to $\ell^{*}$ at $M^{s}$ is also a direct consequence of part (i) of this lemma.

To finish the proof of part (vii), we need to prove Lemmas 4.10 and 4.11. Before we proceed to the proofs of these two lemmas, we first state and prove two auxiliary propositions that are used in the proofs of Lemmas 4.10 and 4.11.

Two auxiliary propositions: The absolute values of the terms of the form $O(\delta)$ in Lemma 3.22 are bounded by $K \delta$, for some constant $K>0$. It was mentioned earlier that in this chapter we assume that $\delta$ is fixed. Without loss of generality, suppose $\delta$ is chosen sufficiently small such that $K \delta \ll 1$. The first proposition is:

Proposition 4.12. For a sufficiently small $\epsilon>0$ and any $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, we have

$$
\begin{array}{ll}
\frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}}=-b \gamma \frac{v_{10}}{u_{10}} \cdot e^{\lambda_{1} \tau}[1+O(\delta)], & \frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}}=b(1-\gamma) e^{\lambda_{1} \tau}[1+O(\delta)], \\
\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}}=-d \gamma \frac{v_{10}}{u_{10}} \cdot e^{\lambda_{1} \tau}[1+O(\delta)], & \frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}}=d(1-\gamma) e^{\lambda_{1} \tau}[1+O(\delta)] .
\end{array}
$$

The proof is as follows: by the chain rule, we have

$$
\begin{align*}
\left(\begin{array}{ll}
\frac{\partial \bar{u}_{10}}{\partial u_{10}} & \frac{\partial \bar{u}_{10}}{\partial v_{10}} \\
\frac{\partial \bar{v}_{10}}{\partial u_{10}} & \frac{\partial \bar{v}_{10}}{\partial v_{10}}
\end{array}\right) & =\left.D T\right|_{\left(u_{10}, v_{10}\right)}=\left.\left.D T^{g l o}\right|_{\left(u_{1 \tau}, v_{1 \tau}\right)} \cdot D T^{l o c}\right|_{\left(u_{10}, v_{10}\right)} \\
& =\left(\begin{array}{cc}
a+o\left(v_{1 \tau}\right) & b+o\left(v_{1 \tau}\right) \\
c+o\left(v_{1 \tau}\right) & d+o\left(v_{1 \tau}\right)
\end{array}\right)\left(\begin{array}{cc}
\frac{\partial \eta_{1}}{\partial u_{10}} & \frac{\partial \eta_{1}}{\partial v_{10}} \\
\frac{\partial \eta_{2}}{\partial u_{10}} & \frac{\partial \eta_{2}}{\partial v_{10}}
\end{array}\right) \tag{4.3.7}
\end{align*}
$$

Substituting (3.4.32) into the above relation gives the desired estimates in Proposition 4.12.
As a consequence of Lemma 3.34, for $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, we have

$$
\frac{\partial \eta_{1}}{\partial u_{10}}=-\frac{1+\gamma}{\gamma w} \cdot e^{-2 \lambda_{1} \tau}[1+O(\delta)] \cdot \frac{\partial \eta_{2}}{\partial u_{10}}, \quad \frac{\partial \eta_{1}}{\partial v_{10}}=\frac{\gamma e^{-2 \lambda_{1} \tau}}{(1-\gamma) w}[1+O(\delta)] \cdot \frac{\partial \eta_{2}}{\partial v_{10}}
$$

and

$$
\begin{equation*}
\frac{\partial \eta_{2}}{\partial u_{10}}=\frac{-\gamma w}{1-\gamma}[1+O(\delta)] \cdot \frac{\partial \eta_{2}}{\partial v_{10}} \tag{4.3.8}
\end{equation*}
$$

where $O(\delta)$ stands for the terms whose absolute values are bounded by $K^{\prime} \delta$ for some $K^{\prime}>0$ such that $K^{\prime} \delta \ll 1$. Relation (4.3.7) implies

Proposition 4.13. For a sufficiently small $\epsilon>0$ and any $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, we have

$$
\begin{align*}
& \frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}}=\left[\frac{a(1+\gamma)}{-\gamma w} e^{-2 \lambda_{1} \tau}[1+O(\delta)]+b+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial u_{10}}=\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial u_{10}} \\
& \frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}}=\left[\frac{a \gamma}{(1-\gamma) w} e^{-2 \lambda_{1} \tau}[1+O(\delta)]+b+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial v_{10}}=\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial v_{10}},  \tag{4.3.9}\\
& \frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}}=\left[\frac{c(1+\gamma)}{-\gamma w} e^{-2 \lambda_{1} \tau}[1+O(\delta)]+d+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial u_{10}}=\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial u_{10}}, \\
& \frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}}=\left[\frac{c \gamma}{(1-\gamma) w} e^{-2 \lambda_{1} \tau}[1+O(\delta)]+d+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial v_{10}}=\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+o\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial v_{10}}
\end{align*}
$$

Proof of Lemma 4.10: Notice that

$$
u_{10}=u_{10}(w, z)=\operatorname{sgn}(z) w^{-1}|z|^{\frac{1}{\alpha}} \quad \text { and } \quad v_{10}=v_{10}(w, z)=\operatorname{sgn}(z)|z|^{\frac{1}{\alpha}}
$$

which implies

$$
\begin{equation*}
\frac{\partial u_{10}}{\partial w}=-\operatorname{sgn}(z) w^{-2}|z|^{\frac{1}{\alpha}}, \quad \frac{\partial v_{10}}{\partial w}=0, \quad \frac{\partial u_{10}}{\partial z}=\frac{1}{\alpha w}|z|^{\frac{1}{\alpha}-1}, \quad \frac{\partial v_{10}}{\partial z}=\frac{1}{\alpha}|z|^{\frac{1}{\alpha}-1} \tag{4.3.10}
\end{equation*}
$$

Differentiating $\bar{z}=g(w, z)=\operatorname{sgn}\left(\bar{v}_{10}\right)\left|\bar{v}_{10}\right|^{\alpha}$ with respect to $z$ gives

$$
\begin{equation*}
g_{z}(w, z)=\alpha\left|\bar{v}_{10}\right|^{\alpha-1}\left[\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}} \cdot \frac{\partial u_{10}(w, z)}{\partial z}+\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}} \cdot \frac{\partial v_{10}(w, z)}{\partial z}\right] \tag{4.3.11}
\end{equation*}
$$

Relation (4.3.10) and Proposition 4.12 imply

$$
\begin{equation*}
g_{z}(w, z)=\operatorname{sgn}(d)(1-2 \gamma)|d|^{\alpha} e^{\alpha \lambda_{1} \tau}[1+O(\delta)] \tag{4.3.12}
\end{equation*}
$$

which is non-zero for any $(w, z) \in \mathcal{Y}$. This finishes the proof of Lemma 4.10.
Proof of Lemma 4.11: By (4.3.12) and (3.4.11), we have

$$
\begin{equation*}
G_{\bar{z}}(w, \bar{z})=\left(g_{z}(w, z)\right)^{-1}=\frac{\operatorname{sgn}(d)|d|^{-\alpha}}{1-2 \gamma} \cdot e^{-\alpha \lambda_{1} \tau}[1+O(\delta)]=O\left(|\bar{z}|^{\frac{2 \gamma}{1-2 \gamma}}\right) \tag{4.3.13}
\end{equation*}
$$

Relation $\bar{z}=g(w, z)=g(w, G(w, \bar{z}))$ implies $G_{w}(w, \bar{z})=\left(g_{z}(w, z)\right)^{-1} g_{w}(w, z)$. By (4.3.10) and Proposition 4.12, one can differentiate $\bar{z}=g(w, z)=\operatorname{sgn}\left(\bar{v}_{10}\right)\left|\bar{v}_{10}\right|^{\alpha}$ with respect to $w$ and compute

$$
g_{w}(w, z)=\alpha\left|\bar{v}_{10}\right|^{\alpha-1}\left[\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}} \cdot \frac{\partial u_{10}(w, z)}{\partial w}\right]=\frac{\operatorname{sgn}(d z) \alpha \gamma|d|^{\alpha}}{w} \cdot e^{\alpha \lambda_{1} \tau}|z|[1+O(\delta)] .
$$

Therefore

$$
\begin{equation*}
G_{w}(w, \bar{z})=\frac{\operatorname{sgn}(z) \alpha \gamma}{(1-2 \gamma) w} \cdot|z|[1+O(\delta)]=O(|z|)=O\left(|\bar{z}|^{\frac{1}{1-2 \gamma}}\right) . \tag{4.3.14}
\end{equation*}
$$

Computations of $F_{w}$ and $F_{\bar{z}}$ are slightly tricky in the sense that instead of estimates given by Proposition 4.12, we use estimates (4.3.8) and (4.3.9). We have

$$
\begin{align*}
f_{z}(w, z)= & \frac{1}{\bar{u}_{10}}\left[\left(\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}} \cdot \bar{u}_{10}-\frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}} \cdot \bar{v}_{10}\right) \cdot \frac{\partial u_{10}(w, z)}{\partial z}\right. \\
& \left.+\left(\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}} \cdot \bar{u}_{10}-\frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial v_{10}} \cdot \bar{v}_{10}\right) \cdot \frac{\partial v_{10}(w, z)}{\partial z}\right] . \tag{4.3.15}
\end{align*}
$$

Since $T$ is at least two times differentiable, we can write (4.1.1) as

$$
\binom{\bar{u}_{10}}{\bar{v}_{10}}=\binom{a u_{1 \tau}+b v_{1 \tau}+O\left(u_{1 \tau}^{2}+\left|u_{1 \tau} v_{1 \tau}\right|+v_{1 \tau}^{2}\right)}{c u_{1 \tau}+d v_{1 \tau}+O\left(u_{1 \tau}^{2}+\left|u_{1 \tau} v_{1 \tau}\right|+v_{1 \tau}^{2}\right)} .
$$

However, for $\left(u_{10}, v_{10}\right) \in \mathcal{D}_{2}$, relation (3.4.31) implies $u_{1 \tau}=e^{-2 \lambda_{1} \tau} w^{-1} v_{1 \tau}[1+O(\delta)]$. Therefore

$$
\begin{aligned}
& \bar{u}_{10}=\left(a e^{-2 \lambda_{1} \tau} w^{-1}[1+O(\delta)]+b+O\left(v_{1 \tau}\right)\right) v_{1 \tau}=\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+O\left(v_{1 \tau}\right)\right] v_{1 \tau} \\
& \bar{v}_{10}=\left(c e^{-2 \lambda_{1} \tau} w^{-1}[1+O(\delta)]+d+O\left(v_{1 \tau}\right)\right) v_{1 \tau}=\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+O\left(v_{1 \tau}\right)\right] v_{1 \tau} .
\end{aligned}
$$

This yields

$$
\begin{aligned}
f_{z}(w, z)= & \frac{|z|^{\frac{1}{\alpha}-1} v_{1 \tau}}{\alpha \bar{u}_{10}^{2}}\left[\left(\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+o\left(v_{1 \tau}\right)\right]\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+O\left(v_{1 \tau}\right)\right]\right.\right. \\
& \left.-\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+o\left(v_{1 \tau}\right)\right]\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+O\left(v_{1 \tau}\right)\right]\right) w^{-1} \frac{\partial \eta_{2}}{\partial u_{10}} \\
& +\left(\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+o\left(v_{1 \tau}\right)\right]\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+O\left(v_{1 \tau}\right)\right]\right. \\
& \left.\left.-\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+o\left(v_{1 \tau}\right)\right]\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+O\left(v_{1 \tau}\right)\right]\right) \frac{\partial \eta_{2}}{\partial v_{10}}\right]
\end{aligned}
$$

which, by (4.3.8), can be simplified as

$$
\begin{aligned}
f_{z}(w, z) & =\frac{|z|^{\frac{1}{\alpha}-1} v_{1 \tau}}{\alpha \bar{u}_{10}{ }^{2}}\left[\left[O\left(e^{-2 \lambda_{1} \tau}\right)+O\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial u_{10}}+\left[O\left(e^{-2 \lambda_{1} \tau}\right)+O\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial v_{10}}\right] \\
& =\frac{|z|^{\frac{1}{\alpha}-1} v_{1 \tau}}{\alpha \bar{u}_{10}{ }^{2}}\left[O\left(e^{-2 \lambda_{1} \tau}\right)+O\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial v_{10}} .
\end{aligned}
$$

Note that by Corollary 3.32 , we can write (3.4.11) as

$$
\begin{equation*}
e^{-\lambda_{1} \tau}=\left(\frac{\gamma}{\delta^{2}}\right)^{\gamma} w^{-\gamma}|z|^{\frac{2 \gamma}{\alpha}}[1+O(\delta)], \tag{4.3.16}
\end{equation*}
$$

and therefore, by (3.4.30), we have $v_{1 \tau}=O\left(|z|^{\frac{1-2 \gamma}{\alpha}}\right)$. Using these relations as well as the estimates in Lemma 3.34, we obtain

$$
\begin{equation*}
f_{z}(w, z)=|z|^{-1}\left[O\left(|z|^{\frac{4 \gamma}{\alpha}}\right)+O\left(|z|^{\frac{1-2 \gamma}{\alpha}}\right)\right]=O\left(|z|^{\beta-1}\right), \quad \beta=\min \left\{\frac{4 \gamma}{\alpha}, \frac{1-2 \gamma}{\alpha}\right\} . \tag{4.3.17}
\end{equation*}
$$

Substituting this and (4.3.13) into $F_{\bar{z}}(w, \bar{z})=f_{z}(w, G(w, \bar{z})) \cdot G_{\bar{z}}(w, \bar{z})$ gives

$$
F_{\bar{z}}(w, \bar{z})=O\left(|\bar{z}|^{\frac{\beta-1+2 \gamma}{1-2 \gamma}}\right) .
$$

Concerning $f_{w}(w, z)$, we have

$$
f_{w}(w, z)=\frac{1}{\bar{u}_{10}^{2}} \cdot\left(\frac{\partial \bar{v}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}} \cdot \bar{u}_{10}-\frac{\partial \bar{u}_{10}\left(u_{10}, v_{10}\right)}{\partial u_{10}} \cdot \bar{v}_{10}\right) \cdot \frac{\partial u_{10}(w, z)}{\partial w} .
$$

Thus,

$$
\begin{aligned}
f_{w}(w, z)= & \frac{-\operatorname{sgn}(z)|z|^{\frac{1}{\alpha}} v_{1 \tau}}{w^{2} \bar{u}_{10}{ }^{2}}\left(\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+o\left(v_{1 \tau}\right)\right]\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+O\left(v_{1 \tau}\right)\right]\right. \\
& \left.-\left[O\left(e^{-2 \lambda_{1} \tau}\right)+b+o\left(v_{1 \tau}\right)\right]\left[O\left(e^{-2 \lambda_{1} \tau}\right)+d+O\left(v_{1 \tau}\right)\right]\right) \frac{\partial \eta_{2}}{\partial u_{10}} \\
= & \frac{|z|^{\frac{1}{\alpha}} v_{1 \tau}}{\bar{u}_{10}{ }^{2}}\left[O\left(e^{-2 \lambda_{1} \tau}\right)+O\left(v_{1 \tau}\right)\right] \frac{\partial \eta_{2}}{\partial u_{10}} .
\end{aligned}
$$

By (4.3.16), relation $v_{1 \tau}=O\left(|z|^{\frac{1-2 \gamma}{\alpha}}\right)$ and estimates in Lemma 3.34, we have

$$
f_{w}(w, z)=O\left(|z|^{\frac{4 \gamma}{\alpha}}\right)+O\left(|z|^{\frac{1-2 \gamma}{\alpha}}\right)=O\left(|z|^{\beta}\right),
$$

where $\beta$ is as in (4.3.17). Substituting this as well as (4.3.17) and (4.3.14) into

$$
F_{w}(w, \bar{z})=f_{w}(w, G(w, \bar{z}))+f_{z}(w, G(w, \bar{z})) \cdot G_{w}(w, \bar{z})
$$

gives

$$
F_{w}(w, \bar{z})=O\left(|z|^{\beta}\right)+O\left(|z|^{\beta-1}|\bar{z}|^{\frac{1}{1-2 \gamma}}\right)=O\left(|\bar{z}|^{\frac{\beta}{1-2 \gamma}}\right) .
$$

This finishes the proof of Lemma 4.11.
Remark 4.14. Lemma 4.5 states that when bd>0, the set $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$ is a $\mathcal{C}^{1}$-smooth curve which is tangent to $\ell^{*}$ at $M^{s}$, and any point on this curve converges to $M^{s}$ by the backward iterations of the Poincaré map T. It follows from part (iii) of Theorem 2.28 and the proof of Lemma 4.5 that, when bd $>0$, if we take a curve $\zeta$ in $\mathcal{D}_{2}$ passing through $M^{s}$, then $\left\{\left.T^{n}(\zeta)\right|_{\mathcal{D}_{2}}\right\}_{n=1}^{\infty}$ converges uniformly to the curve $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{2} \cup \mathcal{D}_{3}, T}^{u}$.

### 4.3.2. Proof of Lemma 4.6

Reverse the time direction in system (3.2.9) (i.e. $t \rightarrow-t$ ) and exchange the stable and unstable components, i.e. apply the linear change of coordinates

$$
\begin{equation*}
\left(\widetilde{u}_{1}, \widetilde{u}_{2}, \widetilde{v}_{1}, \widetilde{v}_{2}\right)=\left(v_{1}, v_{2}, u_{1}, u_{2}\right) \tag{4.3.18}
\end{equation*}
$$

This gives a system which is of the form of system (3.2.9), where all the assumptions of Lemma 3.8 are satisfied. The global map along $\Gamma$ for this system is $J\left(T^{\text {glo }}\right)^{-1} J^{-1}$, where $J=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ and $T^{\text {glo }}$ is the global map of system (3.2.9). Thus, the differential of this map at $M^{s}$ is

$$
J\left(d T^{g l o}\left(M^{s}\right)\right)^{-1} J^{-1}=J \cdot \frac{1}{a d-b c}\left(\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right) \cdot J^{-1}=\frac{1}{a d-b c}\left(\begin{array}{cc}
a & -c \\
-b & d
\end{array}\right) .
$$

This implies that if we replace conditions $b d>0$ and $b d<0$ in Lemma 4.5 by $c d<0$ and $c d>0$, respectively, and the line $\ell^{*}$ by the straight line whose slope is $\frac{-d}{c}$, then all the statements of Lemma 4.5 also hold for this system and the region $\mathcal{D}_{2} \cup \mathcal{D}_{3} \subset \Pi^{s}$. Consequently, by applying the inverse of change of coordinates (4.3.18), all the statements of Lemma 4.5 also hold for the system which is derived from system (3.2.9) by a reversion of time and the region $\left\{\left(u_{1}, v_{1}\right) \in \mathcal{B}_{\epsilon_{u}} \subset \Pi^{u}: 0<\frac{v_{1}}{u_{1}} \leq m, u_{1} \neq 0\right\} \subset \Pi^{u}$. In this case, the line $\ell^{*}$ is replaced by the straight line in $\Pi^{u}$ whose slope is $\frac{-c}{d}$. The homoclinic loop $\Gamma$ in this system leaves and enters $O$ along the positive sides of $u_{2}$ and $v_{2}$, respectively, and the corresponding Poincaré map, call it $\widetilde{T}$, is defined on $\Pi^{u}$. Therefore, the statements of Lemma 4.5 also hold for the map

$$
\begin{equation*}
T^{\mathrm{glo}} \circ \widetilde{T} \circ\left(T^{\mathrm{glo}}\right)^{-1} \tag{4.3.19}
\end{equation*}
$$

and the set

$$
\begin{equation*}
K=T^{\text {glo }}\left(\left\{\left(u_{1}, v_{1}\right) \in \mathcal{B}_{\epsilon_{u}} \subset \Pi^{u}: 0<\frac{v_{1}}{u_{1}} \leq m, u_{1} \neq 0\right\}\right) \tag{4.3.20}
\end{equation*}
$$

where the line $\ell^{*}$ is replaced by the horizontal axis in $\Pi^{s}$. The later one is simply because

$$
d T^{\mathrm{glo}}\left(M^{u}\right)\binom{d}{-c}=\binom{a d-b c}{0}
$$

Notice that map (4.3.19) is conjugate to the inverse of Poincaré map, $T^{-1}$.
The map (4.3.19) coincides with $T^{-1}$ on $T(\mathcal{D})$. Note that, for sufficiently large $m$, the set $T^{\text {glo }}\left(\left\{\left(u_{1}, v_{1}\right) \in \mathcal{B}_{\epsilon_{u}} \subset \Pi^{u}: u_{1} \neq 0, m<\frac{v_{1}}{u_{1}}\right\}\right)$ has no intersection with $\mathcal{D}_{1}$. Therefore, Lemma 4.6 will be proved once we show that $\mathcal{D}_{1} \subset K$ if $c d<0$, and $K \cap \mathcal{D}_{1}=\emptyset$ if $c d>0$. However, this is an immediate consequence of the discussion above. In fact, it follows from the above discussion that the line $\ell^{*}$ passes through $\mathcal{D}_{2} \cup \mathcal{D}_{3}$ if and only if the horizontal axis passes through $K$. The later case, for sufficiently large $m$, is equivalent to the condition $\mathcal{D}_{1} \subset K$ and happens if and only if $c d<0$. This ends the proof of Lemma 4.6.

Remark 4.15. Lemma 4.6 states that if $c d<0$, then the set $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{1}, T^{-1}}^{u}$ is a $\mathcal{C}^{1}$-smooth curve which is tangent to the horizontal axis at $M^{s}$. Moreover, any point on this curve converges to $M^{s}$ by the forward iterations of the Poincaré map T. It follows from Remark 4.14 and the proof of Lemma 4.6 that if we take a set $\zeta$ in $K \cap \mathcal{B}_{\epsilon}$ such that $\zeta \cup\left\{M^{s}\right\}$ be a curve, then $\left\{\left.T^{-n}(\zeta)\right|_{K \cap \mathcal{B}_{\epsilon}}\right\}_{n=1}^{\infty}$ converges uniformly to the curve $\left\{M^{s}\right\} \cup \Lambda_{\mathcal{D}_{1}, T^{-1}}^{u}$.

### 4.4. Dynamics near the homoclinic figure-eight

In this section, we study the dynamics near the homoclinic figure-eight $\Gamma_{1} \cup \Gamma_{2}$. In particular, we prove Theorems B1, B2 and B3 in this section. We start with recalling some definitions and notations from Section 3.1.

For $i=1,2$, we denote by $\mathcal{D}^{i}$ the set of the points $\left(u_{10}, v_{10}\right)$ on $\Pi_{i}^{s}$ whose forward orbits go along the homoclinic orbit $\Gamma_{i}$ and intersect $\Pi_{i}^{u}$ at $\left(u_{1 \tau}, v_{1 \tau}\right)$ such that

$$
\begin{equation*}
\left\|\left(u_{10}, v_{10}\right)\right\|<\epsilon \quad \text { and } \quad\left\|\left(u_{1 \tau}, v_{1 \tau}\right)\right\|<\epsilon_{u} \tag{4.4.1}
\end{equation*}
$$

for some sufficiently small constants $0<\epsilon \leq \epsilon_{u}<\delta$. We denote by $\mathbb{D}^{1}\left(\mathbb{D}^{2}\right)$ the set of the points $\left(u_{10}, v_{10}\right)$ on $\Pi_{1}^{s}\left(\Pi_{2}^{s}\right)$ whose forward orbits go along the negative (positive) side of $v_{2}$-axis and intersect $\Pi_{2}^{u}\left(\Pi_{1}^{u}\right)$ at $\left(u_{1 \tau}, v_{1 \tau}\right)$ such that (4.4.1) holds (see Figure 4.7). We also denote by $T_{i}, T_{i}^{\text {loc }}$ and $T_{i}^{\text {glo }}$ the Poincaré, local and global maps along $\Gamma_{i}(i=1,2)$, respectively (see Figure 4.8). The maps $T_{1}^{\text {glo }}$ and
$T_{2}^{\text {glo }}$ are defined on the open $\epsilon_{u}$-balls around $M_{1}^{s}$ and $M_{2}^{s}$, respectively. Regarding the other maps, we have domain $\left(T_{1}^{\text {loc }}\right)=$ domain $\left(T_{1}\right)=\mathcal{D}^{1}$ and domain $\left(T_{2}^{\text {loc }}\right)=$ domain $\left(T_{2}\right)=\mathcal{D}^{2}$. We also define the map $T_{12}^{\text {loc }}: \mathbb{D}^{1} \subset \Pi_{1}^{s} \rightarrow \Pi_{2}^{u}\left(T_{21}^{\text {loc }}: \mathbb{D}^{2} \subset \Pi_{2}^{s} \rightarrow \Pi_{1}^{u}\right)$ by $\left(u_{10}, v_{10}\right) \mapsto\left(u_{1 \tau}, v_{1 \tau}\right)$, where $\left(u_{10}, v_{10}\right) \in \mathbb{D}^{1}$ $\left(\in \mathbb{D}^{2}\right)$ and $\left(u_{1 \tau}, v_{1 \tau}\right) \in \Pi_{2}^{u}\left(\in \Pi_{1}^{u}\right)$ (see Figure 4.8).


Figure 4.7: This figure illustrates a small neighborhood of the equilibrium $O$ in the presence of the homoclinic figureeight $\left(\Gamma_{1} \cup \Gamma_{2}\right)$. Recall that the the cross-sections $\Pi_{1}^{s}, \Pi_{1}^{u}, \Pi_{2}^{s}$ and $\Pi_{2}^{u}$ are the intersection of the zero-level of the first integral $H$, i.e. $\{H=0\}$, and $\left\{u_{2}=\delta\right\},\left\{v_{2}=\delta\right\},\left\{u_{2}=-\delta\right\}$ and $\left\{v_{2}=-\delta\right\}$, respectively. We consider $\epsilon$-neighborhoods of $M_{1}^{s}$ and $M_{2}^{s}$ (green dashed circles) in $\Pi_{1}^{s}$ and $\Pi_{2}^{s}$, respectively, as well as $\epsilon_{u}$-neighborhoods of $M_{1}^{u}$ and $M_{2}^{u}$ (red dashed circles) in $\Pi_{1}^{u}$ and $\Pi_{2}^{u}$, respectively. The set $\mathcal{D}^{1}\left(\mathcal{D}^{2}\right)$ is the set of the points in the $\epsilon$-neighborhood in $\Pi_{1}^{s}\left(\Pi_{2}^{s}\right)$ whose forward orbits go along $\Gamma_{1}\left(\Gamma_{2}\right)$ and intersect the $\epsilon_{u}$-neighborhood in $\Pi_{1}^{u}\left(\Pi_{2}^{u}\right)$. The blue point on $\Pi_{1}^{s}$ and the brown point on $\Pi_{2}^{s}$ belong to $\mathcal{D}^{1}$ and $\mathcal{D}^{2}$, respectively. We denote by $\mathbb{D}^{1}\left(\mathbb{D}^{2}\right)$ the set of the points in the $\epsilon$-neighborhood in $\Pi_{1}^{s}$ $\left(\Pi_{2}^{s}\right)$ whose forward orbits go along the negative (positive) side of $v_{2}$-axis and intersect the $\epsilon_{u}$-neighborhood in $\Pi_{2}^{u}\left(\Pi_{1}^{u}\right)$.

Let $\mathcal{V}$ be a sufficiently small neighborhood of $\Gamma_{1} \cup \Gamma_{2}$ and define $\Xi=\mathcal{D}^{1} \cup \mathbb{D}^{1} \cup \mathcal{D}^{2} \cup \mathbb{D}^{2}$. For any $\mathrm{x} \in \Xi$, we correspond a (finite or infinite) sequence $\left\{\mathrm{x}_{k}\right\}$ to x in the following way: (i) $\mathrm{x}_{0}=\mathrm{x}$, (ii) if $\mathrm{x}_{k} \in \Xi(k \geq 0)$, we define $\mathrm{x}_{k+1}$ to be the first intersection point of the forward orbit of $\mathrm{x}_{k}$ and $\Pi_{1}^{s} \cup \Pi_{2}^{s}$. Similarly, if $\mathrm{x}_{k} \in \Xi(k \leq 0)$, we define $\mathrm{x}_{k-1}$ to be the first intersection point of the backward orbit of $\mathrm{x}_{k}$ and $\Pi_{1}^{s} \cup \Pi_{2}^{s}$. In order to understand the dynamics in $\mathcal{V}$, we need to find the set of the points whose forward or backward orbits lie entirely in $\mathcal{V}$, i.e. the set of the points $x \in \Xi$ for which the sequence $\left\{\mathrm{x}_{k}\right\}$ is well-defined for all $k \geq 0$ or $k \leq 0$.

When $\lambda_{2}<2 \lambda_{1}$, the dynamics near the homoclinic figure-eight is quite similar to the case of a single homoclinic loop: the forward and backward orbit of any arbitrary point in $\mathcal{V}$ leaves $\mathcal{V}$. When $\lambda_{1}=\lambda_{2}$, it follows from Proposition 3.28 that $\Xi=\emptyset$ and so there is no dynamics near the homoclinic figure-eight. For the case of $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$, we show in the next proof that for any $\mathrm{x} \in \Xi$ whose corresponding $\mathrm{x}_{1}$ is defined, the point $\mathrm{x}_{1}$ lies close to the straight lines with slope $\frac{d_{1}}{b_{1}}$ (if $\mathrm{x}_{1}$ lies in $\mathcal{D}^{1} \cup \mathbb{D}^{1}$ ) or $\frac{d_{2}}{b_{2}}\left(\right.$ if $\mathrm{x}_{1}$ lies in $\mathcal{D}^{2} \cup \mathbb{D}^{2}$ ), and hence, outside of the set $\Xi$ (see Figure 4.9).

Proof of Theorem B2. The proof for the case $\lambda_{1}=\lambda_{2}$ is an immediate consequence of Proposition 3.28 .


Figure 4.8: The blue, brown, green, yellow, red and pink curves correspond to the maps $T_{1}^{\text {glo }}, T_{12}^{\text {loc }}, T_{2}^{\text {glo }}, T_{21}^{\text {loc }}$, $T_{1}^{\text {loc }}$ and $T_{2}^{\text {loc }}$, respectively.

Suppose $\lambda_{1}<\lambda_{2}<2 \lambda_{1}$. By Proposition 3.29, we have

$$
\begin{equation*}
\Xi=\left\{\left(u_{10}, v_{10}\right): \quad\left\|\left(u_{10}, v_{10}\right)\right\|<\epsilon \quad \text { and } \quad 0<\left|v_{10}\right|<K_{\epsilon_{u}}\left|u_{10}\right|^{\frac{\gamma}{1-\gamma}}[1+O(\delta)]\right\} \tag{4.4.2}
\end{equation*}
$$

where $K_{\epsilon_{u}}>0$ is some constant and $\gamma=\lambda_{1} \lambda_{2}^{-1}>0.5$ (see Figure 4.9). Consider $\left(u_{10}, v_{10}\right) \in \Xi$. According to (3.4.15), i.e. relation $u_{1 \tau}=v_{1 \tau} O\left(\epsilon^{2}\right)$ (see also Remark 3.30), the forward orbit of this point intersects one of the cross-sections $\Pi_{1}^{u}$ or $\Pi_{2}^{u}$ at a point close to the vertical axis and then it ends up either in the cross-section $\Pi_{1}^{s}$ close to the straight line with the slope $\frac{d_{1}}{b_{1}}$ or in the cross-section $\Pi_{2}^{s}$ close to the straight line with the slope $\frac{d_{2}}{b_{2}}$. In both cases, this point is outside of the set $\Xi$ (see Figure 4.9). This proves Theorem B2.

$\Pi_{1}^{s}$

$\Pi_{2}^{s}$

Figure 4.9: This figure corresponds to the case $\lambda_{1}<\lambda_{2}<2 \lambda_{1}, b_{1} d_{1}>0$ and $b_{2} d_{2}>0$. The regions $\mathcal{D}^{1}, \mathbb{D}^{1}, \mathcal{D}^{2}$ and $\mathbb{D}^{2}$ are shown in green, blue, pink and yellow, respectively. We define $\Xi$ as the union of these four regions. Let $\mathrm{x}_{1}$ be the first intersection point of the forward orbit of $\mathrm{x} \in \Xi$ and $\Pi_{1}^{s} \cup \Pi_{2}^{s}$. It is shown in the proof of Theorem B2 that for any $\mathrm{x} \in \Xi$ the point $\mathrm{x}_{1}$ lies in one of the gray regions on $\Pi_{1}^{s}$ or $\Pi_{2}^{s}$

A point in $\Pi_{1}^{s} \cup \Pi_{2}^{s}$ whose forward orbit lies entirely in $\mathcal{V}$ and does not lie on the stable manifold of $O$ must belong to $\Xi$. We denote the set of these points by $\Lambda^{s}$. The same holds for backward orbits. We also define the set $\Lambda^{u}$ analogously. In order to understand the dynamics in $\mathcal{V}$, we need to investigate these two sets. For the case of $\lambda_{2}<2 \lambda_{1}$, Theorem B2 states that both of these sets are empty. Our approach to investigate $\Lambda^{s}$ and $\Lambda^{u}$ for the case $2 \lambda_{1}<\lambda_{2}$ is similar to what we have done in the previous section for the case of a single homoclinic loop. Recall from Section 3.4.3 that when $2 \lambda_{1}<\lambda_{2}$, we divide each of the sets $\mathcal{D}^{1}, \mathbb{D}^{1}, \mathcal{D}^{2}$ and $\mathbb{D}^{2}$ into three regions, i.e. for $i=1,2$, we write $\mathcal{D}^{i}=\mathcal{D}_{1}^{i} \cup \mathcal{D}_{2}^{i} \cup \mathcal{D}_{3}^{i}$ and $\mathbb{D}^{i}=\mathbb{D}_{1}^{i} \cup \mathbb{D}_{2}^{i} \cup \mathbb{D}_{3}^{i}$ where
$\mathcal{D}_{1}^{i}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}\left(M_{i}^{s}\right), \quad 0<\frac{v_{10}}{u_{10}}<\frac{1}{m}\right\}, \quad \mathbb{D}_{1}^{i}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}\left(M_{i}^{s}\right), \quad-\frac{1}{m}<\frac{v_{10}}{u_{10}}<0\right\}$,
$\mathcal{D}_{2}^{i}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}\left(M_{i}^{s}\right), \quad \frac{1}{m} \leq \frac{v_{10}}{u_{10}} \leq m\right\}, \quad \mathbb{D}_{2}^{i}=\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}\left(M_{i}^{s}\right), \quad-m \leq \frac{v_{10}}{u_{10}} \leq-\frac{1}{m}\right\}$,
$\mathcal{D}_{3}^{i} \subset\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}\left(M_{i}^{s}\right), \quad m<\left|\frac{v_{10}}{u_{10}}\right|\right\}$ and $\quad \mathbb{D}_{3}^{i} \subset\left\{\left(u_{10}, v_{10}\right) \in \mathcal{B}_{\epsilon}\left(M_{i}^{s}\right), \quad m<\left|\frac{v_{10}}{u_{10}}\right|\right\}$
(see Figure 4.10). Here, $m>0$ is some sufficiently large constant and $\mathcal{B}_{\epsilon}\left(M_{i}^{s}\right)$ is the open $\epsilon$-ball in $\Pi_{i}^{s}$ centered at $M_{i}^{s}$. It has been mentioned in Section 3.4.3 that for $i=1,2$, we are not able to distinguish the sets $\mathcal{D}_{3}^{i}$ and $\mathbb{D}_{3}^{i}$. However, as we see below (Lemma 4.17), the dynamics on both of these sets are quite trivial.


Figure 4.10: The left and right figures show $\Pi_{1}^{s}$ and $\Pi_{2}^{s}$, respectively. When $2 \lambda_{1}<\lambda_{2}$, we divide $\mathcal{D}^{i}$ into three subsets $\mathcal{D}_{1}^{i}, \mathcal{D}_{2}^{i}$ and $\mathcal{D}_{3}^{i}(i=1,2)$. Similarly, we divide $\mathbb{D}^{i}$ into three subsets $\mathbb{D}_{1}^{i}, \mathbb{D}_{2}^{i}$ and $\mathbb{D}_{3}^{i}(i=1,2)$. The sets $\mathcal{D}_{3}^{1}$ and $\mathbb{D}_{3}^{1}$ are subsets of the purple region, and the sets $\mathcal{D}_{3}^{2}$ and $\mathbb{D}_{3}^{2}$ are subsets of the yellow region (in Section 3.4.3, we have discussed that we are not able to find the exact shapes of $\mathcal{D}_{3}^{1}, \mathbb{D}_{3}^{1}, \mathcal{D}_{3}^{2}$ and $\mathbb{D}_{3}^{2}$ ).

Write $\Xi=\mathcal{I} \cup \mathcal{J} \subset \Pi_{1}^{s} \cup \Pi_{2}^{s}$, where

$$
\begin{equation*}
\mathcal{I}:=\bigcup_{i=1,2}\left(\mathcal{D}_{1}^{i} \cup \mathbb{D}_{1}^{i}\right) \quad \text { and } \quad \mathcal{J}:=\bigcup_{\substack{i=1,2 \\ j=2,3}}\left(\mathcal{D}_{j}^{i} \cup \mathbb{D}_{j}^{i}\right) \tag{4.4.3}
\end{equation*}
$$

Definition 4.16. We define $\Lambda_{\mathcal{I}}^{s}\left(\Lambda_{\mathcal{I}}^{u}\right)$ as the set of the points in $\mathcal{I}$ whose forward (backward) orbits intersect $\Xi$ infinitely many times and all the intersection points belong to $\mathcal{I}$. More precisely,

$$
\begin{equation*}
\Lambda_{\mathcal{I}}^{s}=\left\{\mathrm{x}=\mathrm{x}_{0}: \quad \mathrm{x}_{k} \in \mathcal{I} \text { for all } k \geq 0\right\} \quad \text { and } \quad \Lambda_{\mathcal{I}}^{u}=\left\{\mathrm{x}=\mathrm{x}_{0}: \quad \mathrm{x}_{k} \in \mathcal{I} \text { for all } k \leq 0\right\} \tag{4.4.4}
\end{equation*}
$$

The sets $\Lambda_{\mathcal{J}}^{s}$ and $\Lambda_{\mathcal{J}}^{u}$ are defined analogously.

Similar to the case of a single homoclinic, we take three steps to investigate the sets $\Lambda^{s}$ and $\Lambda^{u}$. In the first step, we investigate the sets $\Lambda_{\mathcal{J}}^{s}$ and $\Lambda_{\mathcal{J}}^{u}$. This is done in Lemma 4.17. From technical point of view, part (viii) of this lemma which proves the existence of an unstable invariant manifold of the homoclinic figure-eight is the main result of this section. The techniques which are used in the proof of this part rely on the proof of part (vii) of Lemma 4.5. In the second step, we investigate the sets $\Lambda_{\mathcal{I}}^{s}$ and $\Lambda_{\mathcal{I}}^{u}$. This is also done in Lemma 4.18. Finally, in Lemma 4.19, we clarify the relations between the sets $\Lambda_{\mathcal{J}}^{s}, \Lambda_{\mathcal{J}}^{u}, \Lambda_{\mathcal{I}}^{s}$ and $\Lambda_{\mathcal{I}}^{u}$, and the sets $\Lambda^{s}$ and $\Lambda^{u}$. This enables us to prove Theorem B2. We start with Lemma 4.17.

Lemma 4.17. Assume $2 \lambda_{1}<\lambda_{2}$ and let $w$ be as in Notation 4.2. Then, for $\mathrm{x} \in \mathcal{J}$, we have
(i) if $\mathrm{x} \in \mathcal{D}_{2}^{1} \cup \mathcal{D}_{3}^{1} \cup \mathbb{D}_{2}^{2} \cup \mathbb{D}_{3}^{2}$ (i.e. $\mathrm{x} \in \mathcal{J} \cap \Pi_{1}^{s}$ ), then $w\left(\mathrm{x}_{1}\right)=\frac{d_{1}}{b_{1}}+o(1)$. If $\mathrm{x} \in \mathcal{D}_{2}^{2} \cup \mathcal{D}_{3}^{2} \cup \mathbb{D}_{2}^{1} \cup \mathbb{D}_{3}^{1}$ (i.e. $\mathrm{x} \in \mathcal{J} \cap \Pi_{2}^{s}$ ), then $w\left(\mathbf{x}_{1}\right)=\frac{d_{2}}{b_{2}}+o(1)$. Here, o(1) stands for a function of x that converges to zero as $\mathrm{x} \rightarrow M_{1,2}^{s}$.
(ii) There exists a constant $C>0$ such that $\|\mathrm{x}\|^{1-2 \gamma}<C\left\|\mathrm{x}_{1}\right\|$ holds for arbitrary $\mathrm{x}\left(0<\gamma=\frac{\lambda_{1}}{\lambda_{2}}<\right.$ $0.5)$.
(iii) $\mathrm{x}_{1} \in \mathcal{B}_{\epsilon}$ implies $\mathrm{x}_{1} \in \mathcal{J}$.
(iv) $\Lambda_{\mathcal{J}}^{s}=\emptyset$.
(v) if $b_{1} d_{1}>0$ and $b_{2} d_{2}<0$, then $\Lambda_{\mathcal{J}}^{u}=W_{\text {loc }}^{u}\left(\Gamma_{1}\right) \cap \mathcal{D}_{2}^{1}$.
(vi) if $b_{1} d_{1}<0$ and $b_{2} d_{2}>0$, then $\Lambda_{\mathcal{J}}^{u}=W_{\text {loc }}^{u}\left(\Gamma_{2}\right) \cap \mathcal{D}_{2}^{2}$.
(vii) if $b_{1} d_{1}>0$ and $b_{2} d_{2}>0$, then $\Lambda_{\mathcal{J}}^{u}=\left[W_{\text {loc }}^{u}\left(\Gamma_{1}\right) \cap \mathcal{D}_{2}^{1}\right] \cup\left[W_{\text {loc }}^{u}\left(\Gamma_{2}\right) \cap \mathcal{D}_{2}^{2}\right]$.
(viii) if $b_{1} d_{1}<0$ and $b_{2} d_{2}<0$, then $\Lambda_{\mathcal{J}}^{u} \subset \mathbb{D}_{2}^{1} \cup \mathbb{D}_{2}^{2}$. More precisely, for each $i=1,2$, the union of $M_{i}^{s}$ and $\Lambda_{\mathcal{J}}^{u} \cap \mathbb{D}_{2}^{i}$ is a one-dimensional $\mathcal{C}^{1}$-manifold in $\Pi_{i}^{s}$ which at $M_{i}^{s}$ is tangent to the straight line with slope $\frac{d_{i}}{b_{i}}$. Moreover, the backward orbit of any point in $\Lambda_{\mathcal{J}}^{u}$ intersects these two manifolds alternately, i.e. for any $\mathrm{x} \in \Lambda_{\mathcal{J}}^{u}$, all the points $\mathrm{x}_{k}$ for even and negative $k$ s belong to only one of the manifolds and all the other $\mathrm{x}_{k}$ (odd and negative $k s$ ) belong to the other manifold.

Proof. The same techniques that were used in the proof of Lemma 4.5 also prove parts (i), (ii) and (iii).

Part (iv) is an immediate consequence of (ii) and (iii).
In the rest of the proof, we assume $\mathrm{x} \in \Lambda_{\mathcal{J}}^{u}$. Notice that $\mathrm{x}=\mathrm{x}_{0} \in \Lambda_{\mathcal{J}}^{u}$ implies that $\mathrm{x}_{k}$ is defined for all $k \leq 0$ and $\mathrm{x}_{k} \in \Lambda_{\mathcal{J}}^{u}$. Since $\Lambda_{\mathcal{J}}^{u} \subset \mathcal{J}$, we have two possibilities for $\mathrm{x}_{k}$ :

$$
\begin{equation*}
\mathrm{x}_{k} \in \mathbb{D}_{2}^{2} \cup \mathbb{D}_{3}^{2} \cup \mathcal{D}_{2}^{1} \cup \mathcal{D}_{3}^{1} \quad \text { or } \quad \mathrm{x}_{k} \in \mathcal{D}_{2}^{2} \cup \mathcal{D}_{3}^{2} \cup \mathbb{D}_{2}^{1} \cup \mathbb{D}_{3}^{1} . \tag{4.4.5}
\end{equation*}
$$

Our strategy for proving the rest of this lemma is to consider both of these possibilities and keep track of the sequence $\mathrm{x}_{k}, \mathrm{x}_{k+1}, \cdots, \mathrm{x}_{-1}, \mathrm{x}_{0}$. We analyze the behaviors and patterns of this sequence for arbitrary $\mathrm{x} \in \Lambda_{\mathcal{J}}^{u}$.

To prove part (v), suppose $b_{1} d_{1}>0$ and $b_{2} d_{2}<0$. By part (i), for $\mathrm{x}_{-2}$, we observe
(i) $\mathrm{x}_{-2} \in \mathbb{D}_{2}^{2} \cup \mathbb{D}_{3}^{2} \cup \mathcal{D}_{2}^{1} \cup \mathcal{D}_{3}^{1} \Longrightarrow \mathrm{x}_{-1} \in \mathcal{D}_{2}^{1} \Longrightarrow \mathrm{x} \in \mathcal{D}_{2}^{1}$, and
(ii) $\mathrm{x}_{-2} \in \mathcal{D}_{2}^{2} \cup \mathcal{D}_{3}^{2} \cup \mathbb{D}_{2}^{1} \cup \mathbb{D}_{3}^{1} \Longrightarrow \mathrm{x}_{-1} \in \mathbb{D}_{2}^{2} \Longrightarrow \mathrm{x} \in \mathcal{D}_{2}^{1}$.

According to this observation, $\mathrm{x} \in \Lambda_{\mathcal{J}}^{u}$ implies $\mathrm{x} \in \mathcal{D}_{2}^{1}$. In other words, $\Lambda_{\mathcal{J}}^{u}$ is in fact the set of all $\mathrm{x} \in \mathcal{D}_{2}^{1}$ whose backward orbits only intersect $\Pi_{1}^{s}$, and not $\Pi_{2}^{s}$, and all the intersection points belong to $\mathcal{D}_{2}^{1}$. It follows from Theorem A3 that this set is nothing but $W_{\text {loc }}^{u}\left(\Gamma_{1}\right) \cap \mathcal{D}_{2}^{1}$. This proves part (v).

The proof of part (vi) is analogous to the proof of part (v).
To prove part (vii), suppose $b_{1} d_{1}>0$ and $b_{2} d_{2}>0$. By (i), for $\mathrm{x}_{k-2}(k \leq 0)$ we observe
(i) $\mathrm{x}_{k-2} \in \mathbb{D}_{2}^{2} \cup \mathbb{D}_{3}^{2} \cup \mathcal{D}_{2}^{1} \cup \mathcal{D}_{3}^{1} \Longrightarrow \mathrm{x}_{k-1} \in \mathcal{D}_{2}^{1} \Longrightarrow \mathrm{x}_{k} \in \mathcal{D}_{2}^{1} \Longrightarrow \cdots \Longrightarrow \mathrm{x} \in \mathcal{D}_{2}^{1}$, and
(ii) $\mathrm{x}_{k-2} \in \mathcal{D}_{2}^{2} \cup \mathcal{D}_{3}^{2} \cup \mathbb{D}_{2}^{1} \cup \mathbb{D}_{3}^{1} \Longrightarrow \mathrm{x}_{k-1} \in \mathcal{D}_{2}^{2} \Longrightarrow \mathrm{x}_{k} \in \mathcal{D}_{2}^{2} \Longrightarrow \cdots \Longrightarrow \mathrm{x} \in \mathcal{D}_{2}^{2}$.

This observation holds for any arbitrary $k \leq 0$ which means that the set $\Lambda_{\mathcal{J}}^{u}$ consists of two disjoint sets: the first is the set of all $\mathrm{x} \in \mathcal{D}_{2}^{1}$ whose backward orbits intersect $\Xi$ infinitely many time and every time at $\mathcal{D}_{2}^{1}$, and the second is the set of all $\mathrm{x} \in \mathcal{D}_{2}^{2}$ whose backward orbits intersect $\Xi$ infinitely many time and every time at $\mathcal{D}_{2}^{2}$. According to Theorem A3, the first set is in fact $W_{\text {loc }}^{u}\left(\Gamma_{1}\right) \cap \mathcal{D}_{2}^{1}$ and the second one is $W_{\text {loc }}^{u}\left(\Gamma_{2}\right) \cap \mathcal{D}_{2}^{2}$. This proves part (vii).

To prove part (viii), assume $b_{1} d_{1}<0$ and $b_{2} d_{2}<0$. By (i), for $\mathrm{x}_{k-1}(k \leq-4)$, we observe
(i) $\mathrm{x}_{k-1} \in \mathbb{D}_{2}^{2} \cup \mathbb{D}_{3}^{2} \cup \mathcal{D}_{2}^{1} \cup \mathcal{D}_{3}^{1} \Longrightarrow \mathrm{x}_{k} \in \mathbb{D}_{2}^{1} \Longrightarrow \mathrm{x}_{k+1} \in \mathbb{D}_{2}^{2} \Longrightarrow \mathrm{x}_{k+2} \in \mathbb{D}_{2}^{1}$
$\Longrightarrow x_{k+3} \in \mathbb{D}_{2}^{2} \Longrightarrow \cdots \Longrightarrow x \in \mathbb{D}_{2}^{1}\left(\right.$ if $\left.x_{-1} \in \mathbb{D}_{2}^{2}\right)$ or $x \in \mathbb{D}_{2}^{2}\left(\right.$ if $x_{-1} \in \mathbb{D}_{2}^{1}$ ), and
(ii) $\mathrm{x}_{k-1} \in \mathcal{D}_{2}^{2} \cup \mathcal{D}_{3}^{2} \cup \mathbb{D}_{2}^{1} \cup \mathbb{D}_{3}^{1} \Longrightarrow \mathrm{x}_{k} \in \mathbb{D}_{2}^{2} \Longrightarrow \mathrm{x}_{k+1} \in \mathbb{D}_{2}^{1} \Longrightarrow \mathrm{x}_{k+2} \in \mathbb{D}_{2}^{2}$
$\Longrightarrow \mathrm{x}_{k+3} \in \mathbb{D}_{2}^{1} \Longrightarrow \cdots \Longrightarrow \mathrm{x} \in \mathbb{D}_{2}^{1}\left(\right.$ if $\mathrm{x}_{-1} \in \mathbb{D}_{2}^{2}$ ) or $\mathrm{x} \in \mathbb{D}_{2}^{2}\left(\right.$ if $\mathrm{x}_{-1} \in \mathbb{D}_{2}^{1}$ ).
This observation holds for any arbitrary $k \leq-4$ and means that the backward orbit of x intersects $\mathcal{J}$ at $\mathbb{D}_{2}^{1}$ and $\mathbb{D}_{2}^{2}$ alternately.

Define the maps $\mathbb{T}_{12}: \mathbb{D}^{1} \rightarrow \Pi_{2}^{s}$ and $\mathbb{T}_{21}: \mathbb{D}^{2} \rightarrow \Pi_{1}^{s}$ by $\mathbb{T}_{12}:=T_{2} \circ T_{12}$ and $\mathbb{T}_{21}:=T_{1} \circ T_{21}$. We then define $\mathbb{T}: \mathbb{D}^{1} \rightarrow \Pi_{1}^{s}$ by $\mathbb{T}:=\mathbb{T}_{21} \circ \mathbb{T}_{12}$. According to the above observation, the set $\Lambda_{\mathcal{J}}^{u}$ is in fact the set of the points $x \in \mathbb{D}_{2}^{1}$ such that $\mathbb{T}^{-n}(x) \in \mathbb{D}_{2}^{1}$ for all integers $n>0$.

Recall $(w, z)$ coordinate system and the map $\widetilde{T}$ introduced in the proof of Lemma 4.5. Similar to that proof, we equip $\mathbb{D}_{2}^{1}$ and $\mathbb{D}_{2}^{2}$ with $(w, z)$ coordinates and define the maps $\widetilde{\mathbb{T}}_{12}$ and $\widetilde{\mathbb{T}}_{21}$ by

$$
\widetilde{\mathbb{T}}_{12}(w, z):=\left\{\begin{array}{ll}
(\bar{w}, \bar{z}) & z \neq 0, \\
\left(\frac{d_{2}}{b_{2}}, 0\right) & z=0,
\end{array} \quad \text { for }(w, z) \in \widetilde{\mathcal{R}}_{1}\right.
$$

and

$$
\widetilde{T}_{21}(w, z):=\left\{\begin{array}{ll}
(\bar{w}, \bar{z}) & z \neq 0, \\
\left(\frac{d_{1}}{b_{1}}, 0\right) & z=0,
\end{array} \quad \text { for }(w, z) \in \widetilde{\mathcal{R}}_{2}\right.
$$

where $\widetilde{\mathcal{R}}_{1}$ and $\widetilde{\mathcal{R}}_{2}$ are some appropriate rectangles defined analogous to the proof of Lemma 4.5. According to Remark 3.35, the estimates given by Lemma 3.34 also hold for the local maps $T_{12}$ and ${\underset{\sim}{21}}^{T_{21}}$. Therefore, with exactly the same proof as the proof of Lemma 4.5, we see that both of the maps $\widetilde{\mathbb{T}}_{12}$ and $\widetilde{T}_{21}$ can be written in cross-form and the partial derivatives of the cross-map satisfies the estimates given by Lemma 4.11. Moreover, as it can be seen from the proof of Lemma 4.5, we can make the estimates in Lemma 4.11 sufficiently small by choosing $\theta$ small enough. This means that the maps $\widetilde{\mathbb{T}}_{12}$ and $\widetilde{\mathbb{T}}_{21}$ satisfy the assumptions of Lemma 2.27 for sufficiently small $K_{1}$ and $K_{2}$. Thus, Lemma 2.27 implies that by choosing an appropriate norm, the map $\widetilde{\mathbb{T}}:=\widetilde{\mathbb{T}}_{21} \circ \widetilde{\mathbb{T}}_{12}$ (which is in fact the representation of $\mathbb{T}$ in $(w, z)$ coordinates) can be written in cross-form and the cross-map has sufficiently small partial derivatives. Therefore, this cross-map satisfies the assumptions of Theorem 2.28. The rest of the proof follows from the proof of Lemma 4.5.

The following lemma is analogous to Lemma 4.17. The proof of this lemma is a simple modification of the proof of Lemma 4.6 for the case of homoclinic figure-eight.

Lemma 4.18. Assume $2 \lambda_{1}<\lambda_{2}$ and let $w$ be as in Notation 4.2. Then, for $\mathrm{x} \in \mathcal{I}$, we have
(i) $w\left(\mathrm{x}_{-1}\right)=o(1)$, where $o(1)$ stands for a function of x that converges to zero as $\mathrm{x} \rightarrow M_{1,2}^{s}$.
(ii) There exists a constant $C>0$ such that $\|\mathrm{x}\|^{1-2 \gamma}<C\left\|\mathrm{x}_{-1}\right\|$ holds for any $\mathrm{x}\left(0<\gamma=\frac{\lambda_{1}}{\lambda_{2}}<0.5\right)$.
(iii) $\mathrm{x}_{-1} \in \mathcal{B}_{\epsilon}$ implies $\mathrm{x}_{-1} \in \mathcal{I}$.
(iv) $\Lambda_{\mathcal{I}}^{u}=\emptyset$.
(v) if $c_{1} d_{1}<0$ and $c_{2} d_{2}>0$, then $\Lambda_{\mathcal{I}}^{s}=W_{\text {loc }}^{s}\left(\Gamma_{1}\right) \cap \mathcal{D}_{1}^{1}$.
(vi) if $c_{1} d_{1}>0$ and $c_{2} d_{2}<0$, then $\Lambda_{\mathcal{I}}^{s}=W_{\text {loc }}^{s}\left(\Gamma_{2}\right) \cap \mathcal{D}_{1}^{2}$.
(vii) if $c_{1} d_{1}<0$ and $c_{2} d_{2}<0$, then $\Lambda_{\mathcal{I}}^{s}=\left[W_{\text {loc }}^{u}\left(\Gamma_{1}\right) \cap \mathcal{D}_{1}^{1}\right] \cup\left[W_{\text {loc }}^{s}\left(\Gamma_{2}\right) \cap \mathcal{D}_{1}^{2}\right]$.
(viii) if $c_{1} d_{1}>0$ and $c_{2} d_{2}>0$, then $\Lambda_{\mathcal{I}}^{s} \subset \mathbb{D}_{1}^{1} \cup \mathbb{D}_{1}^{2}$. More precisely, for each $i=1,2$, the union of $M_{i}^{s}$ and $\Lambda_{\mathcal{I}}^{s} \cap \mathbb{D}_{1}^{i}$ is a one-dimensional $\mathcal{C}^{1}$-manifold in $\Pi_{i}^{s}$ which at $M_{i}^{s}$ is tangent to the horizontal axis. Moreover, the forward orbit of any point in $\Lambda_{\mathcal{I}}^{s}$ intersects these two manifolds alternately, i.e. for any $\mathrm{x} \in \Lambda_{\mathcal{I}}^{s}$, all the points $\mathrm{x}_{k}$ for even and negative $k s$ belong to only one of the manifolds and all the other $\mathrm{x}_{k}$ (odd and negative $k s$ ) belong to the other manifold.

The following Lemma states that the forward (backward) orbit of a point in $\mathcal{V}$ lies in $\mathcal{V}$ if and only if it intersects the cross-sections $\Pi_{1}^{s}$ and $\Pi_{2}^{s}$ only at $\mathcal{I}(\mathcal{J})$.

Lemma 4.19. We have $\Lambda^{u}=\Lambda_{\mathcal{J}}^{u}$ and $\Lambda^{s}=\Lambda_{\mathcal{I}}^{s}$.
Proof. It follows from parts (ii) and (iii) of Lemma 4.18 that if $\mathrm{x} \in \mathcal{I}$, then the sequence $\left\{\mathrm{x}_{k}\right\}$ is not defined for all $k \leq 0$. Indeed, For some $k_{0} \leq 0$, we have $\left\{\mathrm{x}_{k_{0}}, \cdots, x_{-1}\right\} \subset \mathcal{I}$ such that $\mathrm{x}_{k_{0}-1}$ lies outside the $\epsilon$-balls around $M_{1}^{s}$ or $M_{2}^{s}$. This means that if x belongs to $\Lambda^{u}$, then it must belong to $\mathcal{J}$. Therefore, $\mathrm{x} \in \Lambda^{u}$ implies $\mathrm{x} \in \Lambda_{\mathcal{J}}^{u}$. On the other hand, we know $\Lambda_{\mathcal{J}}^{u} \subset \Lambda^{u}$. This proves the first part of the lemma. The proof of the other part is the same.

By virtue of the preceding lemmas, we are now in a position to prove Theorem B3.
Proof of Theorem B3. The local stable (unstable) set of the homoclinic figure-eight $\Gamma_{1} \cup \Gamma_{2}$, denoted by $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)\left(W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)\right)$, is the union of $\Gamma_{1} \cup \Gamma_{2}$ itself and the set of the points in a sufficiently small neighborhood $\mathcal{V}$ of $\Gamma_{1} \cup \Gamma_{2}$ whose forward (backward) orbits lie in $\mathcal{V}$ and their $\omega$-limit sets ( $\alpha$-limit sets) coincide with $\Gamma_{1} \cup \Gamma_{2} \cup\{O\}$. By this definition, the intersection of $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ and any of the cross-sections $\Pi_{1}^{s}$ and $\Pi_{2}^{s}$ must belong to $\left\{M_{1}^{s}, M_{2}^{s}\right\} \cup \Lambda^{s}$. Similarly, the intersection of $W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ and the cross-sections $\Pi_{1}^{s}$ and $\Pi_{2}^{s}$ must belong to $\left\{M_{1}^{s}, M_{2}^{s}\right\} \cup \Lambda^{u}$.

It follows from Lemma 4.18 that in any cases except the case $c_{1} d_{1}>0$ and $c_{2} d_{2}>0$, the $\omega$-limit set of any orbit in $\Lambda^{s}$ coincides with either $\Gamma_{1} \cup\{O\}$ or $\Gamma_{2} \cup\{O\}$. Therefore, in all of these cases, we have $W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)=\Gamma_{1} \cup \Gamma_{2}$.

Denote the flow of system (3.2.9) by $\phi_{t}$. When $c_{1} d_{1}>0$ and $c_{2} d_{2}>0$, it follows from parts (ii) and (viii) of Lemma 4.18 that the set $\Gamma_{1} \cup \Gamma_{2} \cup \phi_{t}\left(\Lambda^{s}\right)$ for $t \geq 0$ is a 2-dimensional $\mathcal{C}^{1}$ manifold, and the forward orbit of any point on this manifold converges to $\Gamma_{1} \cup\{O\} \cup \Gamma_{2}$ as $t \rightarrow \infty$. This means that this manifold is in fact the local stable set of the homoclinic figure-eight $\Gamma_{1} \cup \Gamma_{2}$. The fact that this manifold is tangent to $W_{\text {glo }}^{s}(O)$ at every point of $\Gamma_{1} \cup \Gamma_{2}$ is an straightforward consequence of the discussion before Proposition 4.9.

The proof for the case of $W_{\text {loc }}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ is the same. This ends the proof of Theorem B3.
Corollary 4.20. Let $\phi_{t}$ be the flow of system (3.2.9). Then

$$
\begin{array}{ll}
W_{l o c}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)=\Gamma_{1} \cup \Gamma_{2} \cup \phi_{t}\left(\Lambda^{s}\right), & \text { for } t \geq 0, \\
W_{l o c}^{u}\left(\Gamma_{1} \cup \Gamma_{2}\right)=\Gamma_{1} \cup \Gamma_{2} \cup \phi_{t}\left(\Lambda^{u}\right), & \text { for } t \leq 0 .
\end{array}
$$

Finally, we prove

Proof of Theorem B1. Denote the set $W_{\text {loc }}^{s}\left(\Gamma_{1}\right) \cup W_{\text {loc }}^{s}\left(\Gamma_{2}\right) \cup W_{\text {loc }}^{s}\left(\Gamma_{1} \cup \Gamma_{2}\right)$ by $\mathcal{W}^{s}$. By definition, the forward orbit of any point on $W_{\mathcal{V}}^{\mathcal{S}}(O) \cup \mathcal{W}^{s}$ lies in $\mathcal{V}$. Consider a point in $\mathcal{V} \backslash W_{\mathcal{V}}^{s}(O)$ whose forward orbit lies entirely in $\mathcal{V}$. The forward orbit of this point must intersect $\Pi_{1}^{s} \cup \Pi_{2}^{s}$ at $\Lambda^{s}$. Therefore, it follows from the proof of Theorem B2 (for the case $\lambda_{2}<2 \lambda_{1}$ ) and Corollary 4.20 (for the case $2 \lambda_{1}<\lambda_{2}$ ) that this point lies on $\mathcal{W}^{s}$. This finishes the proof for the case of forward orbits.

The proof of the case of backward orbits is the same. This finishes the proof of Theorem B1.

### 4.5. Dynamics near super-homoclinic orbits

In this section, we prove Theorem C1. The idea of the proof is to show that there exist sequences of curves $\left\{l_{k}^{u}\right\}_{k=1}^{\infty} \subset W_{\text {glo }}^{u}(O) \cap \Pi^{s}$ and $\left\{l_{k}^{s}\right\}_{k=1}^{\infty} \subset W_{\text {glo }}^{s}(O) \cap \Pi^{s}$ that accumulate to $W_{\text {loc }}^{u}(\Gamma) \cap \Pi^{s}$ and $W_{\text {loc }}^{s}(\Gamma) \cap \Pi^{s}$, respectively (see Figure 4.11). Then, the flow near the super-homoclinic orbit defines a map which maps the first sequence to a sequence of curves, denoted by $\left\{m_{k}^{u}\right\}$ in Figure 4.11, such that each of the curves $\left\{m_{k}^{u}\right\}$ intersects each of the curves $\left\{l_{k}^{s}\right\}$ at a single point. Each of these intersection points correspond to a homoclinic orbit. The proof of Theorem C2 is exactly the same.

Proof of Theorem C1. Let $\mathcal{W}^{s}=W_{\text {loc }}^{s}(\Gamma) \cap \mathcal{D}_{1}$ and $\mathcal{W}^{u}=W_{\text {loc }}^{u}(\Gamma) \cap \mathcal{D}_{2}$. We have shown in Section 4.3 (after Proposition 4.8) that $T^{\text {glo }}\left(W_{\text {loc }}^{u}(O) \cap \Pi^{u}\right)$ intersects $\Pi^{s}$ at a curve which is tangent to $\ell^{*}$ at $M^{s}$. For a sufficiently small $\epsilon$, the restriction of this curve to $\mathcal{B}_{\epsilon} \backslash\left\{M^{s}\right\}$ lies in $\mathcal{D}_{2}$. Denote this curve by $L_{0}^{u}$, and let $L_{k}^{u}(k \geq 1)$ be the restriction of $T\left(L_{k-1}^{u}\right)$ to $\mathcal{B}_{\epsilon} \backslash\left\{M^{s}\right\}$. By Remark 4.14, the sequence $\left\{L_{k}^{u}\right\}_{k=1}^{\infty}$ converges to $\mathcal{W}^{u}$ uniformly.

Now, consider the restriction of $W_{\text {loc }}^{s}(O) \cap \Pi^{s}$ to $\mathcal{B}_{\epsilon} \backslash\left\{M^{s}\right\}$ and denote it by $L_{0}^{s}$. We have $L_{0}^{s} \subset K$, where $K$ is as in (4.3.20). Let $L_{k}^{s}(k \geq 1)$ be the restriction of $T^{-1}\left(L_{k-1}^{s}\right)$ to $\mathcal{B}_{\epsilon} \backslash\left\{M^{s}\right\}$. By Remark 4.15, the sequence $\left\{L_{k}^{s}\right\}_{k=1}^{\infty}$ converges to $\mathcal{W}^{s}$ uniformly.

The super-homoclinic orbit $\mathcal{S}$ intersects $\Pi^{s}$ at $\mathcal{W}^{u}$ and $\mathcal{W}^{s}$ infinitely many times. Denote the furthest points of $\mathcal{S} \cap \mathcal{W}^{u}$ and $\mathcal{S} \cap \mathcal{W}^{s}$ from $M^{s}$ by $q^{u}$ and $q^{s}$, respectively. Let $B^{u}$ be a sufficiently small open ball in $\mathcal{D}_{2}$ centered at $q^{u}$. The orbits starting from $B^{u}$ leave the small neighborhood $\mathcal{U}$ of $\Gamma$ and go along the super-homoclinic orbit $\mathcal{S}$, and after a finite time, they come back and intersect $\Pi^{s}$ at some points close to $q^{s}$. These orbits induce a global map

$$
\begin{equation*}
T_{\mathcal{S}}: B^{u} \subset \Pi^{s} \rightarrow B^{s} \subset \Pi^{s} \tag{4.5.1}
\end{equation*}
$$

along the super-homoclinic orbit $\mathcal{S}$, where $B^{s}=T_{\mathcal{S}}\left(B^{u}\right)$ and $T_{\mathcal{S}}\left(q^{u}\right)=q^{s}$. Since $B^{u}$ is sufficiently small and the map $T_{\mathcal{S}}$ is a diffeomorphism, the neighborhood $B^{s}$ is small, connected and convex.

Define $l^{u}=\mathcal{W}^{u} \cap B^{u}$ and $l^{s}=\mathcal{W}^{s} \cap B^{s}$. Since the sequence $\left\{L_{k}^{u}\right\}_{k=1}^{\infty}$ converges to $\mathcal{W}^{u}$ uniformly, there exists a sufficiently large $k_{s}$ such that for all $k \geq k_{s}$, the curve $L_{k}^{s}$ intersects $B^{s}$. Let $l_{k}^{s}=L_{k}^{s} \cap B^{s}$ for $k \geq k_{s}$. This implies that $l_{k}^{s} \xrightarrow{\text { unif }} l^{s}$. Similarly, for some sufficiently large $k_{u}$, all the curves $L_{k}^{u}$ for $k \geq k_{u}$ intersect $B^{u}$. Let $l_{k}^{u}=L_{k}^{u} \cap B^{u}$ for $k \geq k_{u}$. Therefore, $l_{k}^{u} \xrightarrow{\text { unif }} l^{u}$.

The map $T_{\mathcal{S}}$ maps $B^{u}$ to $B^{s}$. Thus, the curves $l^{u}$ and $l_{k}^{u}$ in $B^{u}$ are mapped to some curves in $B^{s}$ by $T_{\mathcal{S}}$. Let $m^{u}=T_{\mathcal{S}}\left(l^{u}\right)$ and $m_{k}^{u}=T_{\mathcal{S}}\left(l_{k}^{u}\right)$ for $k \geq k_{u}$. Since the super-homoclinic orbit $\mathcal{S}$ is at the transverse intersection of the stable and unstable invariant manifolds of the homoclinic orbit $\Gamma$, the curves $m^{u}$ and $l^{s}$ intersect each other transversely. On the other hand, the sequences of the curves $m_{k}^{u}$ and $l_{k}^{s}$ converge to $m^{u}$ and $l^{s}$, respectively. This implies that the curves $m_{k}^{u}$ intersect the curves $l_{k}^{s}$ transversely. Moreover, without loss of generality, we can assume that the integers $k_{u}$ and $k_{s}$ are large enough such that the curves $m_{i}^{u}$ and $l_{j}^{s}$ intersect each other at a unique point $p_{i, j}$ for any $i \geq k_{u}$ and $j \geq k_{s}$. The orbits passing through the points $p_{i, j}$ are the desired multi-pulse homoclinic orbits. This proves Theorem C1.


Figure 4.11: The blue and green curves belong to the intersection of $\Pi^{s}$ and the global unstable and stable invariant manifolds of the equilibrium $O$, respectively. The blue curves accumulate to $\mathcal{W}^{u}$ and the green curves accumulate to $\mathcal{W}^{s}$, where $\mathcal{W}^{u}=W_{\text {loc }}^{u}(\Gamma) \cap \Pi^{s}$ and $\mathcal{W}^{s}=W_{\text {loc }}^{s}(\Gamma) \cap \Pi^{s}$. Let $q^{u} \in \mathcal{W}^{u}$ and $q^{s} \in \mathcal{W}^{s}$ be at the intersection of the super-homoclinic orbit and the cross-section $\Pi^{s}$. The flow near the super-homoclinic orbit defines a map on a small neighborhood $B^{u}$ of $q^{u}$ onto a small neighborhood $B^{s}$ of $q^{s}$. This map maps the blue curves restricted to $B^{u}$ to the blue curves in $B^{s}$. The blue and green curves in $B^{s}$ intersect transversely. Any point of these intersections belongs to both stable and unstable invariant manifolds of $O$. Thus, the orbits passing through these points are homoclinic to $O$.

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## Appendix A

## Nondegenerate quadratic forms

A quadratic form in $n$ variables $(n \in \mathbb{N})$ over $\mathbb{R}$ is a homogeneous polynomial of degree 2 in $n$ variables with coefficients in $\mathbb{R}$ :

$$
q\left(x_{1}, \ldots, x_{n}\right)=\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i j} x_{i} x_{j}, \quad a_{i j} \in \mathbb{R}
$$

For a given quadratic form $q(x)$ there exists a unique symmetric matrix $A\left(A=A^{T}\right)$ such that

$$
q(x)=x^{T} A x
$$

where $x=\left(x_{1}, \ldots, x_{n}\right)$ is a column vector. A quadratic form $q(x)$ is called nondegenerate if $\operatorname{det}(A) \neq 0$.
Consider a system of differential equations

$$
\dot{\mathrm{x}}=X(\mathrm{x})
$$

defined on a neighborhood of the origin in $\mathbb{R}^{n+m}$ where the origin is a hyperbolic equilibrium. This system can be written as

$$
\begin{align*}
\dot{x} & =-B x+\cdots,  \tag{A.0.1}\\
\dot{y} & =C y+\cdots,
\end{align*}
$$

where $B$ and $C$ are square matrices of dimensions $n \times n$ and $m \times m$, respectively, and their eigenvalues have positive real parts. Assume that this system has a first integral $H(x, y)$ and suppose $H(0,0)=0$. Due to the hyperbolicity of the origin, we have $H^{\prime}(0,0)=0$. Thus, we can write $H$ as

$$
H(x, y)=q(x, y)+\cdots
$$

where $q(x, y)$ stands for quadratic terms and the dots stand for cubic and higher order terms. Assume that $q$ is nondegenerate. Then

Lemma A.1. (i) $n=m$.
(ii) There exists a linear change of coordinates which brings system (A.0.1) to the form

$$
\begin{equation*}
\dot{x}=-B x+\cdots, \quad \dot{y}=B^{T} y+\cdots \tag{A.0.2}
\end{equation*}
$$

and the first integral $H$ to the form

$$
\begin{equation*}
H=<y, B x>+\cdots \tag{A.0.3}
\end{equation*}
$$

The dots in (A.0.2) stand for quadratic and higher order terms and the dots in (A.0.3) stand for cubic and higher order terms.

Proof. The assumptions that we use to prove this lemma are the hyperbolicity of the origin and the nondegeneracy of $q$.

Let $J$ be the symmetric matrix that $q(x, y)=(x, y)^{T} \cdot J \cdot(x, y)$. We write $J$ in the block form

$$
J=\left(\begin{array}{cc}
D_{1} & D \\
D^{T} & D_{2}
\end{array}\right)
$$

where $D_{1}$ and $D_{2}$ are symmetric matrices of dimensions $n \times n$ and $m \times m$, respectively, and $D$ is a matrix of dimension $n \times m$. We have

$$
q(x, y)=x^{T} D_{1} x+y^{T} D^{T} x+x^{T} D y+y^{T} D_{2} y
$$

Equation (1.2.2) implies that

$$
-\frac{\partial}{\partial x} q(x, y) \cdot B x+\frac{\partial}{\partial y} q(x, y) \cdot C y \equiv 0, \quad \forall x, y
$$

or equivalently

$$
\begin{equation*}
-x^{T} D_{1} B x-y^{T} D^{T} B x+x^{T} D C y+y^{T} D_{2} C y \equiv 0, \quad \forall x, y \tag{A.0.4}
\end{equation*}
$$

Evaluating the above relation at $x=0$ gives $y^{T} D_{2} C y \equiv 0$ for every $y$ which implies $D_{2} C \equiv 0$. The matrix $C$ is invertible because of the hyperbolicity of the origin. Thus $D_{2}=0_{m \times m}$. Similarly, evaluating (A.0.4) at $y=0$ implies $D_{1}=0_{n \times n}$. Consequently, $J$ takes the form

$$
J=\left(\begin{array}{cc}
0 & D \\
D^{T} & 0
\end{array}\right)
$$

and relation (A.0.4) will be simplified as $y^{T} D^{T} B x=x^{T} D C y$ for all $x$ and $y$. Since $y^{T} D^{T} B x$ is scalar valued, we have $y^{T} D^{T} B x=\left(y^{T} D^{T} B x\right)^{T}=x^{T} B^{T} D y$ and therefore $B^{T} D y=x^{T} D C y$, which is valid for all $x$ and $y$. This implies $B^{T} D=D C$. Since $q$ is nondegenerate, the matrix $J$ is invertible and so the matrix $D_{n \times m}$ has right and left inverses. It is also easily seen by induction that for every integer $k \geq 0$ and every constant $\lambda$ we have

$$
D(C-\lambda I)^{k}=\left(B^{T}-\lambda I\right)^{k} D
$$

This equality as well as the fact that $D$ has right and left inverses imply that the matrices $C$ and $B^{T}$ have the same spectrum and the multiplicity of every eigenvalue of $C$ is the same as its multiplicity as the eigenvalue of $B^{T}$. Therefore $n=m$ and $D$ is an invertible square matrix. It is easy to see that

$$
\binom{x^{\text {new }}}{y^{\text {new }}}=\left(\begin{array}{cc}
B^{-1} & 0 \\
0 & D
\end{array}\right)\binom{x}{y}
$$

is the desired change of coordinates.

