Breuil–Mézard conjectures for central division algebras

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

We give a parametrization of the inertial classes of smooth representations of inner forms of GL_n over a p-adic field, based on type-theoretic invariants. Then we give a complete description of the behaviour of this parametrization under the Jacquet–Langlands correspondence, proving a conjecture of Broussous, Sécherre and Stevens on preservation of endo-classes. As an application of this result, we construct a Jacquet–Langlands transfer of types and Serre weights for central division algebras, and use it to deduce a form of the Breuil–Mézard conjecture, for discrete series Galois deformation rings and types of central division algebras, from the conjectural statement for GL_n.

A Mario Dotto, Giuseppina Ghiglione e Marina Giraudo

che hanno molta pazienza.

Acknowledgments

This thesis collects half of the results of research undertaken in 2016–2019 at Imperial College London. The other half, treating a question of Breuil on diagrams for $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ (joint work with Daniel Le), and a p-adic analogue of the inertial Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, will appear elsewhere.

Thanks are due to my supervisor Toby Gee for suggesting the problem of studying the Breuil–Mézard conjecture for $D^\times$, and for advice in various places along the way. Other people I have learned from in these years include Colin Bushnell, Ana Caraiani, Matthew Emerton, David Helm, Guy Henniart, Daniel Le, Vytautas Paškūnas, Vincent Sécherre, Shaun Stevens.

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Contents

1

Introduction.

This thesis is about the geometry of deformation spaces of local Galois representations and its relationship with the representation theory of p-adic groups. The paradigm for studying this is provided by the Breuil–Mézard conjecture, which aims to describing intersections amongst certain special loci in the universal deformation space for a representation

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$$

where $F/\mathbf{Q}_p$ is a finite extension. These special loci are Kisin's potentially semistable deformation rings, which depend on two parameters: a Hodge type λ, and an inertial type τ. For the most part we will be concerned with τ, which has a direct interpretation in terms of the representation theory of $\mathrm{GL}_n(F)$. Namely, it singles out a Bernstein component of the category $\mathrm{Rep}_{\overline{\mathbf{Q}}_p} \mathrm{GL}_n(F)$ of smooth representations with coefficients in $\overline{\mathbf{Q}}_p$. (We will be more precise about coefficients later in this work.)

The Breuil–Mézard conjecture, which is described precisely in chapter 9 (and generalized slightly), links the special fibre of the potentially semistable deformation ring of type τ to the mod p reduction of the Bushnell–Kutzko type of this component. This, together with a version of the Serre weight conjecture, turns out to be the central input in proving automorphy lifting theorems via Kisin's extension of the Taylor–Wiles method. A conceptual explanation of this phenomenon has been given in [Paš15] and [CEG+16], where it is presented as a consequence of a strong form of the p-adic Langlands correspondence (which is available for $\mathrm{GL}_2(\mathbf{Q}_p)$, but so far not for any group of higher rank).

One can therefore wonder if the principle of functoriality provides analogues of the Breuil–Mézard conjecture for groups other than $G = \mathrm{GL}_n(F)$. In this thesis, we treat this question for the Jacquet–Langlands correspondence between G and the unit group $H = D^\times$

of a central division algebra (see remark 1.0.1 for a comment about more general inner forms). This has a precedent in [GG15], which treated the case of quaternion algebras: although we follow a similar strategy, we remark that the techniques that we apply are completely different, due to the more complicated behaviour of the representation theory of $\mathrm{GL}_n(F)$ when n is not a prime number. (When n is prime, analogues of the arguments in [GG15] go through verbatim.) In our situation, we need to use the full theory of Bushnell and Kutzko, as well as its refinements due to Bushnell and Henniart, and its extensions to inner forms due to Broussous, Sécherre and Stevens. In addition, we need complete information regarding the behaviour of types under the Jacquet–Langlands correspondence, which implies as a particular case a positive answer to [BSS12, Conjecture 9.5].

Main results. To state our main results, let $G = \mathrm{GL}_n(F)$ and fix an inner form $H = \mathrm{GL}_m(D)$ of G, where we allow $m > 1$. Recall that the Jacquet–Langlands correspondence is a bijection

$$\mathrm{JL} : \mathbf{D}(H) \to \mathbf{D}(G)$$

between the sets of essentially square-integrable representations (or discrete series representations) of these groups, characterized by the equality

$$(-1)^m \mathrm{tr}(\pi) = (-1)^n \mathrm{tr}(\mathrm{JL}\,\pi)$$

on matching regular elliptic elements of G and H. Here, $\mathrm{tr}(\pi)$ denotes the Harish-Chandra character of π, identified with a function on regular semisimple elements. The category of smooth representations of the groups G and H, as for any other connected reductive group over F, decomposes according to the action of the Bernstein centre. Two discrete series representations are in the same block in the Bernstein decomposition if and only if they are unramified twists of each other. Since the Jacquet–Langlands correspondence commutes with twisting by characters, it yields a bijection on the sets of blocks for G and H containing discrete series representations.

Our first main result describes this bijection in explicit terms, through the parametrization that we construct in chapter 4. Although we expected this construction to be a routine application of type theory, we have found it to be subtler than anticipated, and in need of a more careful treatment than usual: we employ a technique that we learned from [BH14], where a similar situation has been observed in a different context. We refer the reader to the first sections of chapter 4 for a discussion of this point, together with an explicit example of what can go wrong. Here we just give a sketch of the construction.

Given a simple inertial class $\mathfrak{s}$ for $\mathrm{GL}_m(D)$, we denote by $\mathrm{cl}(\mathfrak{s})$ the endo-class of simple characters attached to $\mathfrak{s}$ in [BSS12], which coincides with the endo-class of maximal simple characters contained in any factor of the supercuspidal support of $\mathfrak{s}$. Fixing a lift of $\mathrm{cl}(\mathfrak{s})$

to its unramified parameter field, and a conjugacy class $\kappa(\mathrm{cl}\,\mathfrak{s})$ of maximal β-extensions in G of endo-class $\mathrm{cl}(\mathfrak{s})$, we construct a second invariant of the simple inertial classes of G, which we denote by $\mathfrak{s} \mapsto \Lambda_{\kappa(\mathrm{cl}\,\mathfrak{s})}(\mathfrak{s})$. It consists of a set of characters of the multiplicative group of a finite field, corresponding to a representation of a finite general linear group. Our parametrization is given by the fact that these two invariants $\mathrm{cl}(\mathfrak{s})$ and $\Lambda_{\kappa(\mathrm{cl}\,\mathfrak{s})}(\mathfrak{s})$ determine the inertial class $\mathfrak{s}$ uniquely, by theorem 4.5.4. (To emphasize the fact that it depends on the choice of a lift Θ_E, we will write its inverse in a slightly different manner, keeping track of the chosen lift of $\mathrm{cl}(\mathfrak{s})$ in the notation.)

Theorem A (Theorem 5.3.1)**.** Let $\mathfrak{s}_G$ and $\mathfrak{s}_H$ be simple inertial classes of complex representations for the groups G and H respectively, and assume that $\mathfrak{s}_G = \mathrm{JL}(\mathfrak{s}_H)$. Then the equality $\mathrm{cl}(\mathfrak{s}_G) = \mathrm{cl}(\mathfrak{s}_H)$ holds.

Since $\mathrm{cl}(\mathfrak{s})$ coincides with the endo-class attached to a simple inertial class in [BSS12] (see remark 4.5.8) this theorem implies [BSS12, Conjecture 9.5], the "endo-class invariance conjecture". We emphasize that we make heavy use of the techniques developed in [BH11, SS16b], which also prove special cases of our theorems A and B in the context of essentially tame endo-classes. In our next theorem, we study the behaviour of the second invariant in our parametrization.

Theorem B (Theorems 5.3.3, 5.3.4)**.** Let $\Theta_F = \mathrm{cl}(\mathfrak{s}_G) = \mathrm{cl}(\mathfrak{s}_H)$, and let ϵ^1_G and ϵ^1_H be the symplectic sign characters attached to any maximal simple character in G and H of endo-class Θ_F. Let κ_G and κ_H be the p-primary conjugacy classes of maximal β-extensions in G and H of endo-class Θ_F. Then

$$\Lambda_{\epsilon^1_G \kappa_G}(\mathfrak{s}_G) = \Lambda_{\epsilon^1_H \kappa_H}(\mathfrak{s}_H).$$

Now we state our version of the Breuil–Mézard conjecture for $D^\times$. Let τ be a discrete series inertial type and λ a Hodge type for $\overline{\rho}$. In section 9.1 we construct a quotient $R^\square_{\overline{\rho}}(\tau, \lambda)_{\mathfrak{P}_{\min}}$ of the universal framed deformation ring $R^\square_{\overline{\rho}}$ whose characteristic zero points correspond to potentially semistable lifts of the representation $\overline{\rho}$ whose Weil–Deligne representation is the Langlands parameter of an essentially square-integrable representation. These can be transferred to $D^\times$: the Jacquet–Langlands correspondence provides an inertial class $\mathfrak{s}_D(\tau) = \mathrm{JL}^{-1}\mathfrak{s}(\tau)$ of representations of $D^\times$, which admits types on the maximal compact subgroup $\mathcal{O}_D^\times$. In contrast with the case of $\mathrm{GL}_n(F)$, they are not uniquely determined, and we write $\sigma_D(\tau)$ for an arbitrarily chosen one: our results apply to all possible choices of $\sigma_D(\tau)$. The weight λ also determines a representation of $\mathcal{O}_D^\times$, and we write $\sigma_D(\tau, \lambda)$ for the tensor product of the two.

Theorem C (Breuil–Mézard conjecture for $D^\times$, section 9.2)**.** If the geometric Breuil–Mézard conjecture holds for $\mathrm{GL}_n(F)$, then there exists a group homomorphism

$$R_{\overline{\mathbf{F}}_p}(\mathcal{O}_D^\times) \to Z(R^\square_{\overline{\rho}}/\pi)$$

which for any (τ, λ) sends the semisimplified mod p reduction $\overline{\sigma}_D(\tau, \lambda)$ of $\sigma_D(\tau, \lambda)$ to $Z(R^{\square}_{\overline{\rho}}(\tau, \lambda)_{\mathfrak{P}_{\min}}/\pi)$. Here $R_{\overline{\mathbf{F}}_p}(\mathcal{O}_D^\times)$ denoted the Grothendieck group of the category of smooth finite-dimensional $\overline{\mathbf{F}}_p[\mathcal{O}_D^\times]$-modules, and $Z(R^{\square}_{\overline{\rho}}/\pi)$ is the group of cycles on the special fibre of $R^{\square}_{\overline{\rho}}$.

To prove this theorem, we construct a group homomorphism

$$\mathrm{JL}_p : R_{\overline{\mathbf{F}}_p}(\mathcal{O}_D^\times) \to R_{\overline{\mathbf{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$$

via Deligne–Lusztig induction, and we describe it in terms of the combinatorics of parabolic induction. Our main technical result is theorem 8.0.4, stating the equality

$$\mathrm{JL}_p(\overline{\sigma}_D(\tau, \lambda)) = \overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau, \lambda).$$

Granting this, one transfers the conjecture from $\mathrm{GL}_n(F)$ to $D^\times$ by composing with JL_p.

A JACQUET–LANGLANDS TRANSFER FOR MAXIMAL COMPACT SUBGROUPS. Since $F^\times \mathcal{O}_D^\times$ is a normal subgroup of $D^\times$ with finite cyclic quotient, one proves that every smooth irreducible representation of $\overline{\mathbf{Q}}_p[\mathcal{O}_D^\times]$ is a type for a Bernstein component of $\mathrm{Rep}_{\overline{\mathbf{Q}}_p} D^\times$. It follows that our constructions give rise to a natural group homomorphism

$$\mathrm{JL}_{\mathbf{K}} : R_{\overline{\mathbf{Q}}_p}(\mathcal{O}_D^\times) \to R_{\overline{\mathbf{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$$

and our main results imply that the following diagram commutes. See section 9.2 for details.

$$\begin{array}{ccc} R_{\overline{\mathbf{Q}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\mathbf{K}}} & R_{\overline{\mathbf{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \downarrow{\mathbf{r}_p} & & \downarrow{\mathbf{r}_p} \\ R_{\overline{\mathbf{F}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_p} & R_{\overline{\mathbf{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{1.0.1}$$

After having obtained our results, we have been notified of work in preparation of Zijian Yao that makes the following equivalent construction. Consider the abelian group $\bigoplus_{(\tau,N)} \mathbf{Z}$ where the sum is indexed by Galois inertial types τ with monodromy operator N. There is a map $R_{\overline{\mathbf{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \to \bigoplus_{(\tau,N)} \mathbf{Z}$, sending a representation σ to

$$\Big(\dim_{\overline{\mathbf{Q}}_p} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma, \pi_{\tau,N})\Big)_{(\tau,N)}$$

for any generic irreducible representation $\pi_{\tau,N}$ such that $\mathrm{rec}(\pi_{\tau,N})$ has inertial type τ and monodromy operator N. By definition, our representations $\sigma^+_{\mathfrak{P}}(\tau)$ yield a section of this map. There is an analogous map defined for $\mathcal{O}_D^\times$, whose image is contained in the direct sum of the factors indexed by discrete series inertial parameters. Yao defines $\mathrm{JL}_{\mathbf{K}}$ as the map making the following diagram commute

$$\begin{array}{ccc} R_{\overline{\mathbf{Q}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\mathbf{K}}} & R_{\overline{\mathbf{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \\ & \searrow \quad \nearrow_{\sigma^+_{\mathfrak{P}}(\tau)} & \\ & \bigoplus_{(\tau,N)} \mathbf{Z} & \end{array} \tag{1.0.2}$$

and goes on to conjecture the existence of a map JL_p making diagram (1.0.1) commute. Our results therefore provide a proof of this conjecture.

Remark 1.0.1. Yao makes similar conjectures in the case of more general inner forms, as this definition of $\mathrm{JL}_{\mathbf{K}}$ makes sense for $\mathrm{GL}_m(D)$ when formulated for those inertial types (τ, N) extending to a Langlands parameter for $\mathrm{GL}_m(D)$. At least in the case of discrete series parameters, it seems that our methods extend to this situation without too much trouble: our theorems A and B work in full generality, and there is a natural candidate for the JL_p map, namely Lusztig induction for the twisted Levi subgroup $\mathrm{GL}_m(\mathbf{d})$ of $\mathrm{GL}_n(\mathbf{f})$. We have chosen to focus on the simpler case of $D^\times$ because here we are mainly concerned with the Jacquet–Langlands correspondence, and outside of this special case we would rather have a general framework for the transfer of Serre weights between inner forms of unramified groups (see the end of this introduction for some remarks in this direction). In addition, we point out that from the viewpoint of a Jacquet–Langlands correspondence for maximal compact subgroups one expects somewhat weaker results for $\mathrm{GL}_m(D)$. For instance, not every irreducible representation of $\mathrm{GL}_m(\mathcal{O}_D)$ is a type for $\mathrm{GL}_m(D)$, as $\mathrm{GL}_m(D)$ has infinite-dimensional representations, and $\mathrm{JL}_{\mathbf{K}}$ does not see any information about non-typical representations of $\mathrm{GL}_m(\mathcal{O}_D)$, except their multiplicities in restrictions of $\mathrm{GL}_m(D)$-representations.

THE CASE $l \neq p$. Working with l-adic coefficients when $l \neq p$, we can prove that there exists a (necessarily unique) morphism JL_l making the following diagram commute.

$$\begin{array}{ccc} R_{\overline{\mathbf{Q}}_l}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\mathbf{K}}} & R_{\overline{\mathbf{Q}}_l}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \downarrow \mathbf{r}_l & & \downarrow \mathbf{r}_l \\ R_{\overline{\mathbf{F}}_l}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_l} & R_{\overline{\mathbf{F}}_l}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{1.0.3}$$

The uniqueness statement follows from the fact that the reduction mod l map for $\mathcal{O}_D^\times$ is surjective. In fact we can give an explicit description of JL_l in terms of $\mathrm{JL}_{\mathbf{K}}$, and (as usual) the theorem is a tautology when l does not divide the pro-order of $\mathrm{GL}_n(\mathcal{O}_F)$, since in this case both vertical arrows are isomorphisms. It is worth stating explicitly a difference with the case $l = p$. The mod p irreducible representations of $\mathcal{O}_D^\times$ are characters, and they lift to level zero types for $D^\times$. It follows that compatibility with the Jacquet–Langlands transfer of level zero types already determines the JL_p map uniquely, and compatibility for all types imposes a strong constraint on their mod p reductions. When $l \neq p$, there are a lot more irreducible $\overline{\mathbf{F}}_l$-representations of $\mathcal{O}_D^\times$, and the only congruences arise between types with the same endo-class. This allows us to construct JL_l by fixing the endo-class and studying the mod l reduction of the level zero part, which is what is done in the proof of theorem 8.0.6.

Together with [Sho18, Theorem 4.6] (which requires the assumption that $p \neq 2$), this implies that a form of the geometric Breuil–Mézard conjecture holds for $D^\times$ and l-adic coefficients, expressing the fact that congruences between the special fibres of discrete series deformation rings are described by mod l congruences between types on the maximal compact subgroup of $D^\times$. See theorem 9.2.1.

Endoscopic transfer of Serre weights. We point out that maps like JL_p are likely to exist for all inner forms of unramified groups, and to enjoy favourable properties with respect to the inertial Langlands correspondence. In more detail, one has the Kazhdan–Varshavsky endoscopic transfer of Deligne–Lusztig functions [KV12], which could be enough to transfer all Serre weights because the characteristic functions of semisimple conjugacy classes are contained in the $\mathbf{Q}$-linear span of the Deligne–Lusztig functions. While a complete theory of types is not available for all groups, the analogue of theorem 8.0.4 for the depth zero Langlands correspondence of deBacker–Reeder and Kaletha should be within reach (see for instance the discussion at the end of the introduction to [Kal11]). As an application, one would be able to formulate Serre weight conjectures for non-quasisplit groups, and prove at least the "weight elimination" direction. In the case of $D^\times$, for example, this is a simple consequence of recent work of Le, Le Hung and Levin [LLHL], and in this case it is even possible to prove the full weight conjecture for $D^\times$ (in the global context of unitary groups). We intend to elaborate on these topics in future work.

The structure of this work. We conclude this introduction by describing the content of each chapter of this thesis.

First we recall some salient aspects of the representation theory of $\mathrm{GL}_m(D)$ and the local Langlands correspondence for these groups, and we provide some complements. Some of our results here will be known to experts, but we have not been able to find them in the literature in the form we need, and we have preferred to give complete proofs. Others, such as theorem 2.5.13, seem to be genuinely new.

In chapter 4, we give our parametrization of the inertial classes of smooth representations of inner forms of $\mathrm{GL}_n(F)$. Chapter 5 contains the proofs of theorems A and B. It employs a new argument for identifying an endo-class, using base change functoriality to pass to a split form, and it builds upon a technique of Sécherre and Stevens for studying congruences between the level zero parts of representations. We have found that this technique can also be applied to the local Langlands correspondence, producing a canonical normalization of the "β-extensions" of type theory: this is carried through in chapter 6. A need for canonical normalizations of this kind has been observed, for example, in [BHS].

As mentioned above, the main ingredient in our treatment of the Breuil–Mézard conjecture for $D^\times$ is a Jacquet–Langlands transfer of weights and types. The types that we need have been constructed by Schneider and Zink for $\mathrm{GL}_n(F)$: we review their construction in chapter 7 and extend it to $D^\times$. Then we construct our transfer and prove its basic properties in chapter 8. The applications to the Breuil–Mézard conjecture are given in chapter 9.

Notation and conventions. Fix a finite extension $F/\mathbf{Q}_p$ and an algebraic closure $\overline{F}/F$, and write $\mathcal{O}_F$ for the ring of integers and π_F for a fixed uniformizer. The residue field will be denoted by k_F or $\mathbf{f}$ (we have tried to preserve the notation of the references we have quoted most often), and its cardinality by q. Similar notation will be used for other local fields and central division algebras over them (so, for instance, $\mathbf{e}$ is the residue field of E). We remark that our theorems A and B are also valid for an equicharacteristic local field F, with the same proofs.

We write F_d for the unramified extension of F of degree d in $\overline{F}$, and $\mathbf{f}_d$ for the extension of $\mathbf{f}$ of degree d in the algebraic closure of $\mathbf{f}$ given by the residue field of the maximal unramified extension of F in $\overline{F}$. The group of Teichmüller roots of unity in F is denoted μ_F, and the absolute value on F is normalized so that $|\pi_F| = q^{-1}$.

Representations of a locally profinite group like $\mathrm{GL}_m(D)$ will be tacitly assumed to be smooth. The coefficient field R will change in the course of this thesis, but it will always be an algebraically closed field of characteristic different from p, and we will specify it explicitly when needed. Characters are not assumed to be unitary, and whenever a character χ of a group G and a representation π of a subgroup $H \subseteq G$ are given, the representation $\pi \otimes \chi|_H$ will be called a *twist* of π; when G is a p-adic reductive group and χ is an unramified character, this will be called an *unramified twist*. We write unramified characters of $F^\times$ as

$$\mathrm{nr}_\lambda : F^\times \to R^\times, \pi_F \mapsto \lambda.$$

Hence the absolute value character is $\mathrm{nr}_{q^{-1}}$. We fix an additive character $\psi_F : F \to R^\times$, which will be implicit in every discussion of simple characters. If E/F is a finite extension, we will put $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$.

If $H, K \subset G$ are an open, respectively a closed subgroup of a locally profinite group G we write c-Ind_H^G and Ind_K^G for the functors of compactly supported smooth induction and smooth induction, so that we have adjoint pairs $(\text{c-Ind}_H^G, \mathrm{Res}_H^G)$ and $(\mathrm{Res}_K^G, \mathrm{Ind}_K^G)$.

Parabolic induction from a standard Levi subgroup of $\mathrm{GL}_m(D)$ is always taken along the upper-triangular parabolic, and normalized. This requires us to fix a square root of q in $R^\times$, but changing it does not modify the inertial class of the supercuspidal support of any given irreducible representation, hence the choice will not affect any of our results that are concerned with inertial classes. We write $\pi_1 \times \cdots \times \pi_n$ for the parabolic induction of $\pi_1 \otimes \cdots \otimes \pi_n$.

For a central simple algebra A over F and E/F a field extension in A, the commutant of E in A will be denoted $Z_A(E)$, and the centralizer and normalizer of E in $G = A^\times$ will be denoted $Z_G(E) = Z_A(E)^\times$ and $N_G(E)$ respectively. Throughout this work, the reduced degree of a central division algebra D over F (positive square root of the F-dimension) is denoted by d. Usually $\mathrm{GL}_m(D)$ will denote an inner form of $\mathrm{GL}_n(F)$, so that $n = md$. The character "absolute value of the reduced norm" is an unramified character of $\mathrm{GL}_m(D)$, denoted ν. An unramified twist $\pi \otimes (\chi \circ \nu)$ will usually be written $\chi\pi$, and we reserve the notation $\pi(i)$ for $\pi \otimes \nu^i$. For $x \in G$, we write $\mathrm{ad}(x)$ for the automorphism $z \mapsto xzx^{-1}$ of A. In general, pullback by an automorphism φ will be denoted φ^*.

For an extension $\mathbf{l}/\mathbf{k}$ of finite fields, an element $x \in \mathbf{l}$ is $\mathbf{k}$-*regular* if it has $[\mathbf{l} : \mathbf{k}]$ different conjugates under $\mathrm{Gal}(\mathbf{l}/\mathbf{k})$. A $\mathbf{k}$-regular character of $\mathbf{l}^\times$ is defined similarly, via the right action $g : \chi \mapsto g^*\chi = \chi^g = \chi \circ g$ of $\mathrm{Gal}(\mathbf{l}/\mathbf{k})$ on characters. Notice that $x \in \mathbf{l}^\times$ can be $\mathbf{k}$-regular and still generate a proper subgroup of $\mathbf{l}^\times$ (consider, for instance, an extension of prime degree). For any character α of $\mathbf{l}^\times$, define the *stabilizer field* $\mathbf{k}[\alpha]$ as the fixed field of $\mathrm{Stab}_{\mathrm{Gal}(\mathbf{l}/\mathbf{k})}(\alpha)$. It only depends on the orbit of α under $\mathrm{Gal}(\mathbf{l}/\mathbf{k})$, which will be denoted $[\alpha]$. Similarly, if l is a prime number then the character α decomposes uniquely as a product $\alpha = \alpha_{(l)}\alpha^{(l)}$ in which $\alpha_{(l)}$ has order a power of l and $\alpha^{(l)}$ has order coprime to l. Since this decomposition is unique, the orbit $[\alpha^{(l)}]$ is independent of the representative $[\alpha]$, and similarly for $[\alpha_{(l)}]$. The orbit $[\alpha^{(l)}]$ is the l-*regular* part of $[\alpha]$. We will often apply the following lemma.

Lemma 1.0.2. If $\mathbf{l}/\mathbf{k}$ is an extension of finite fields, and χ is a character of $\mathbf{l}^\times$, then there exists a unique $\mathbf{k}$-regular character χ^{reg} of $\mathbf{k}[\chi]^\times$ such that $\chi = \chi^{\mathrm{reg}} \circ N_{\mathbf{l}/\mathbf{k}[\chi]}$.

Proof. Since the norm map $N_{\mathbf{l}/\mathbf{k}[\chi]}$ is surjective, for the existence part it suffices to prove that if $N_{\mathbf{l}/\mathbf{k}[\chi]}(x) = 1$ then $\chi(x) = 1$. But by Hilbert 90, $N_{\mathbf{l}/\mathbf{k}[\chi]}(x) = 1$ if and only if $x = \frac{g(y)}{y}$ for some $g \in \mathrm{Gal}(\mathbf{l}/\mathbf{k}[\chi])$ and some $y \in \mathbf{l}^\times$, and then $\chi(x) = 1$ as χ is $\mathrm{Gal}(\mathbf{l}/\mathbf{k}[\chi])$-stable. Uniqueness holds because $N_{\mathbf{l}/\mathbf{k}[\chi]}$ is surjective, and regularity holds because the stabilizer of χ in $\mathrm{Gal}(\mathbf{l}/\mathbf{k})$ is $\mathrm{Gal}(\mathbf{l}/\mathbf{k}[\chi])$. □

We consider partitions of a positive integer n as functions $\mathfrak{P} : \mathbf{Z}_{>0} \to \mathbf{Z}_{\geq 0}$ with finite support, such that $\sum_{i \in \mathbf{Z}_{>0}} i\mathfrak{P}(i) = n$. If n is an integer and l a prime number, we write n_l for the highest power of l dividing n and $n_{l'} = n/n_l$.

We conclude by introducing our notation for the supercuspidal support of representations, which will be defined in section 2.1. We write $\operatorname{cusp}_R D$, resp. $\operatorname{scusp}_R D$, for the set of isomorphism classes of irreducible cuspidal, resp. supercuspidal representations of $R[\mathrm{GL}_m(D)]$ as m varies amongst positive integers. Recall that an effective divisor on a set X is a function $X \to \mathbf{Z}_{\geq 0}$ with finite support. We will sometimes refer to effective divisors on X as multisets on X.

If $\pi \in \operatorname{Irr}_R \mathrm{GL}_m(D)$, its supercuspidal support admits a representative (L, σ) such that L is a standard Levi subgroup, and then σ is well-defined up to the action of the Weyl group. It follows that the supercuspidal support of π defines an element of $\mathrm{Div}^+(\operatorname{scusp}_R D)$, and we write

$$sc : \operatorname{Irr}_R \mathrm{GL}_m(D) \to \mathrm{Div}^+(\operatorname{scusp}_R D)$$

for the map there results. We write $\mathfrak{B}_R(\mathrm{GL}_m(D))$ for the set of inertial equivalence classes of supercuspidal pairs over R. We will sometimes drop R from the notation if it is clear from the context.

2

Representation theory of $\mathrm{GL}_m(D)$.

In this chapter, R is an algebraically closed field of characteristic different from p unless stated otherwise.

2.1 The Bernstein decomposition.

The category $\mathrm{Rep}_R \mathrm{GL}_m(D)$ of smooth representations of $\mathrm{GL}_m(D)$ with complex coefficients admits a block decomposition, according to the action of its centre and the cuspidal support of representations. In this section, we briefly review this theory due to Bernstein and Deligne [Ber84]. We will adopt the notation and viewpoint of [BK98], hence we work more generally with a connected reductive group G/F and an F-rational parabolic $P \subseteq G$ with an F-rational Levi decomposition $P = L \ltimes U$; then L is also a connected reductive group. We will usually denote algebraic groups over F and their groups of F-points by the same symbol.

Cuspidal support. Throughout this thesis, we will work with the normalized parabolic induction and restriction functors

$$i_P^G : \mathrm{Rep}(L) \to \mathrm{Rep}(G),\ r_P^G : \mathrm{Rep}(G) \to \mathrm{Rep}(L).$$

When necessary, we will fix a square root of q in R, which is real and positive when $R = \mathbf{C}$.

In the case of $\mathrm{GL}_n(F)$ and the upper-triangular Borel $P = B_n(F)$, the normalization means that $i_P^G(V)$ is the smooth induction from P to G of the twist $\delta_P^{1/2} V$, where U acts trivially and

$$\delta_P : L \cong \prod_{i=1}^{n} F^\times \to \mathbf{C}^\times,\ g \mapsto |g_1|^{n-1} \cdots |g_n|^{1-n}.$$

Recall that a smooth representation V of G is cuspidal if it is admissible, and $r_P^G(V) = 0$ for all proper parabolic subgroups $P \subset G$. If V is irreducible, then it is admissible, and it is cuspidal if and only if it is not a subspace of a proper parabolic induction $i_P^G(V_L)$ with V_L irreducible. If V is cuspidal, and it is not a subquotient of any proper parabolic induction, then we say that V is supercuspidal. These two conditions are equivalent if R has characteristic zero, but not in general.

Definition 2.1.1. A cuspidal, resp. supercuspidal pair (L, σ) over R consists of a Levi subgroup of $\mathrm{GL}_n(F)$ together with an irreducible cuspidal, resp. supercuspidal R-representation of L.

The group G acts on the set of cuspidal pairs by conjugacy

$$g : (L, \sigma) \mapsto g^*(L, \sigma) = (gLg^{-1}, \mathrm{ad}(g^{-1})^*\sigma),$$

and the orbits are called *cuspidal supports*. For $G = \mathrm{GL}_n(F)$, the cuspidal supports are in bijection with effective divisors of degree n on the set of isomorphism classes of irreducible cuspidal representations of some $\mathrm{GL}_m(F)$.

The cuspidal supports give rise to a partition of $\mathrm{Irr}(G)$, because the semisimplification of $i_P^G(\sigma_L)$ does not depend on the choice of a parabolic subgroup P with Levi factor L, by [Ren10, Théorème VI.5.4] and [MS14a, Proposition 2.2]. In addition, given $\pi \in \mathrm{Irr}_R(G)$, one proves that the cuspidal pairs such that $\pi \in \mathrm{JH}(i_P^G(\sigma))$ form a G-orbit: this is the cuspidal support of π.

Definition 2.1.2. Two cuspidal, resp. supercuspidal pairs (L_i, σ_i) are *inertially equivalent* if there exists $g \in G$ and an unramified character $\chi : L_1 \to R^\times$ such that $\sigma_1 \cong \chi \otimes \mathrm{ad}(g)^*\sigma_2$. We write $\mathfrak{B}_R(G)$ for the set of inertial equivalence classes $[L, \sigma]$ of supercuspidal pairs over R.

Bernstein decomposition in characteristic zero. Assume that R has characteristic zero. Then we have a decomposition

$$\mathrm{Rep}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}^{\mathfrak{s}}(G)$$

where the subcategory $\mathrm{Rep}^{\mathfrak{s}}(G)$ consists of smooth representations all of whose irreducible subquotients have cuspidal (or, equivalently, supercuspidal) support contained in $\mathfrak{s}$. Hence every smooth representation V of G has a canonical subspace $V^{\mathfrak{s}} \in \mathrm{Rep}^{\mathfrak{s}}(G)$ for each $\mathfrak{s} \in \mathfrak{B}(G)$, and $V = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} V^{\mathfrak{s}}$.

Remark 2.1.3. Fix an inertial class $\mathfrak{s} = [L, \sigma]$ in G. Let us write $\mathfrak{s}_L = [L, \sigma]_L$ for the inertial equivalence class of (L, σ) viewed as a cuspidal support for L: it identifies with the set of twists of σ by unramified characters of L. Hence, $X_{\mathrm{nr}}(L)$ acts transitively on $\mathfrak{s}_L$, and $\mathfrak{s}_L$ is a principal homogeneous space over a complex torus (a quotient of $X_{\mathrm{nr}}(L)$ by a finite subgroup). By an argument using Clifford theory and the theory of the Heisenberg group, one proves that the centre of the category $\mathrm{Rep}^{\mathfrak{s}_L}(L)$ is canonically isomorphic to the ring of regular functions on the algebraic variety $\mathfrak{s}_L$: evaluation of z at a point $\chi\sigma$ corresponds to the scalar by which z acts on $\chi\sigma$ (guaranteed to exist by Schur's lemma). See [Ber84, 1.12—1.15].

Now $\mathfrak{s}$ identifies with the quotient of $\mathfrak{s}_L$ by a finite group, namely the stabilizer $W(L, \mathfrak{s}_L)$ of $\mathfrak{s}_L$ in the conjugation action of $N_G(L)/L$. Hence $\mathfrak{s}$ is also a complex algebraic variety (singular in general). The centre of $\mathrm{Rep}^{\mathfrak{s}}(G)$ acts on $\mathfrak{s}$, and the resulting map

$$\iota : \mathcal{Z}\,\mathrm{Rep}^{\mathfrak{s}}(G) \to (\mathcal{Z}\,\mathrm{Rep}^{\mathfrak{s}_L}(L))^{W(L,\mathfrak{s}_L)}$$

is an isomorphism. It can be described as follows [Ber84, Proposition 2.11]: if $\pi_L \in \mathfrak{s}_L$ and $z \in \mathcal{Z}\,\mathrm{Rep}^{\mathfrak{s}}(G)$, then z acts on $i_P^G(\pi_L)$ by the scalar $\iota(z)(\pi_L)$.

Remark 2.1.4. It is expected that the Bernstein component of an irreducible $\pi \in \mathrm{Rep}(G)$ is strongly related to the restriction of its Langlands parameter to the inertia subgroup of F. In the case of $G = \mathrm{GL}_n$, this expectation is confirmed in proposition 3.1.10. However, we point out that such a one-to-one correspondence between inertial parameters and Bernstein components is specific to GL_n: in general, unramified Langlands parameters (that is, trivial on inertia) correspond to unipotent representations, see for instance [Lus95], and already for $\mathrm{Sp}_4(F)$ these contain all the irreducible representations in the block of unramified principal series, as well as a cuspidal block. See [Dat17] for more on this.

Bernstein decomposition in positive characteristic. Assume $\mathrm{char}\, R = l \neq p$. In contrast with the discussion so far, complete results on the existence of a Bernstein decomposition are only known for groups related to $\mathrm{GL}_m(D)$. While the uniqueness of the cuspidal support still holds, its analogue for supercuspidal supports can fail: there can be $\pi \in \mathrm{Irr}_R(G)$ which are Jordan–Hölder factors of parabolic inductions $\mathrm{Ind}_{L_i}^G \sigma_i$ for non-conjugate supercuspidal pairs (L_i, σ_i). See [Dat18]. For this reason, we restrict our discussion the case of $\mathrm{GL}_m(D)$.

Uniqueness of the supercuspidal support up to conjugacy holds for $\mathrm{Rep}_R\,\mathrm{GL}_m(D)$ by [MS14a, Section6.2]. By [Vig96, SS16a], an analogue of the Bernstein decomposition holds for $\mathrm{Rep}_R\,\mathrm{GL}_m(D)$, and the blocks are in bijection with inertial equivalence classes of supercuspidal supports. Again, the block corresponding to $[L, \sigma]$ consists of those smooth representations all of whose irreducible subquotients have supercuspidal support in $[L, \sigma]$ [SS16a, Section 10.1].

Simple inertial classes. In this thesis, we will mostly be concerned with the simple inertial classes, which are the ones that take part in the Jacquet–Langlands correspondence. They are defined as follows.

Definition 2.1.5. An inertial class $\mathfrak{s} \in \mathfrak{B}_R(\mathrm{GL}_m(D))$ is *simple* if it has a representative of the form $(\mathrm{GL}_{m/r}(D), \pi_0^{\otimes r})$ for some $r|d$. An irreducible representation $\pi \in \mathrm{Irr}_R(D)$ is simple if (π) is contained in a simple inertial class.

We remark that it is often possible to generalize a result from simple inertial classes to the whole of $\mathfrak{B}_R(\mathrm{GL}_m(D))$: see the discussion after proposition 6.2.4 for an example, and [MS14a, Théorème 6.18] for a general theorem to this effect.

2.2 The Bernstein–Zelevinsky classification.

In this section, we write $G_n = \mathrm{GL}_n(F)$. Let $D = \pi_1 + \cdots + \pi_r \in \mathrm{Div}^+(\mathrm{scusp}_R F)$ be a supercuspidal support for G. The irreducible representations π with $sc(\pi) = D$ have been classified in [BZ77, Zel80] when $R = \mathbf{C}$ and in [Vig96, MS14a] in general. We will follow the notation of [SZ99], in order to be able to refer to their results later in this document, and we will work with $R = \mathbf{C}$ until the next section.

Remark 2.2.1. The results in this section have analogues for the groups $\mathrm{GL}_m(D)$, due to Tadić and Mínguez–Sécherre. Since we will not need them, we will restrict to the case of $\mathrm{GL}_n(F)$.

Fix a set $\mathcal{C}$ of representatives of the isomorphism classes of irreducible preunitary supercuspidal representations of G_n (for variable n) up to unramified twist. Every irreducible supercuspidal π admits a twist that is preunitary: it suffices to twist so that the central character of π is trivial, because then π is a compact representation of $\mathrm{GL}_n(F)/Z(\mathrm{GL}_n(F))$.

Definition 2.2.2. Given a divisor $D \in \mathrm{Div}^+(\mathcal{C})$, a *partition on* D is a partition-valued function $\mathcal{P}$ on $\mathcal{C}$, such that the divisor

$$D(\mathcal{P}) : \pi \mapsto \sum_{x>0} x\mathcal{P}_\pi(x)$$

is equal to D.

Example 2.2.3. If $D = \sum_{i=0}^m n_i\pi_i$, a partition on D is the choice of a partition of each of the n_i.

The Bernstein components of G_n are in bijection with the set of divisors on $\mathcal{C}$ of degree n, where $\deg(D) = \sum_{\sigma\in\mathcal{C}} D(\sigma)\deg(\sigma)$. To see that this is a parametrization of the inertial conjugacy classes of cuspidal pairs, consider all G_n at once and attach to each effective divisor D on $\mathcal{C}$ the inertial class $[M_D, \sigma_D]$ in the group $G_{\deg D}$, where

$$M_D = \prod_{\sigma \in \mathcal{C}} G_{\deg(\sigma)}^{\times D(\sigma)} \text{ and } \sigma_D = \bigotimes_{\sigma \in \mathcal{C}} \sigma^{\otimes D(\sigma)}$$

for some ordering of the factors (which doesn't affect $[M_D, \sigma_D]$). This defines a bijection

$$\mathrm{Div}^+(\mathcal{C}) \to (\text{inertial conjugacy classes of cuspidal pairs of some } G_n), \quad D \mapsto [M_D, \sigma_D].$$

The Bernstein component associated to $[M_D, \sigma_D]$ will sometimes be denoted by Ω_D.

THE DISCRETE SERIES. We use the term "discrete series" as synonymous with "essentially square-integrable", so that $\pi \in \mathrm{Irr}_R\, G_b$ is discrete series if the matrix coefficients of an unramified twist of π are square-integrable on $G_n/Z(G_n)$. Recall the absolute value character

$$|\cdot| = \mathrm{nr}_{1/q} : F^\times \to \mathbf{C}^\times.$$

If $x \in \mathbf{Q}$ and $\sigma \in \mathcal{C}$, we write $\sigma(x) = \sigma \otimes |\cdot|^x$ (with roots taken in $\mathbf{R}_{>0}^\times$), and

$$\Delta(\sigma, s) = \left\{\sigma(\frac{1-s}{2} + t), 0 \le t \le s-1\right\}$$

for the segment of representations of length s centered at σ. The parabolic induction

$$\sigma(\frac{1-s}{2}) \times \cdots \times \sigma(\frac{s-1}{2})$$

(which is the unnormalized parabolic induction of $\sigma^{\otimes s}$) has a unique irreducible quotient, denoted $L\Delta(\sigma, s)$, which is a discrete series representation of $G_{s \deg \sigma}$ contained in $\Omega_{s\sigma}$. The representations $L\Delta(\sigma, s)$ form a set of representatives for the discrete series of any G_n, up to unramified twist.

THE CLASSIFICATION. Let $\mathcal{P}$ be a partition-valued function on $\mathcal{C}$. We can attach to $\mathcal{P}$ the multiset of segments

$$\{\mathcal{P}_\sigma(i)\Delta(\sigma, i) : \sigma \in \mathcal{C}, i > 0\}$$

containing $\Delta(\sigma, i)$ with multiplicity $\mathcal{P}_\sigma(i)$. There is a corresponding discrete series representation

$$\tau_{\mathcal{P}} = \bigotimes_{(\sigma,i)} L\Delta(\sigma, i)^{\otimes \mathcal{P}_\sigma(i)}$$

of a Levi subgroup $N_{\mathcal{P}}$. More generally, we can enlarge $\mathcal{C}$ to the set $\mathrm{cusp}_R\, G_n$ (which consists of representatives of the *isomorphism classes* of supercuspidal representations of any G_n, rather than the twist classes), and attach a representation $\tau_{\mathfrak{P}}$ of $N_{\mathfrak{P}}$ to any partition-valued function $\mathfrak{P}$ on $\operatorname{cusp}_R G_n$. This is defined via the same formulas as $\tau_{\mathcal{P}}$. Similarly, to $\mathfrak{P}$ we can attach a multiset of segments of cuspidal irreducible representations, and we have a bijection from the partition-valued functions on $\operatorname{cusp}_R G_n$ to the set of such multisets of segments.

The Bernstein–Zelevinsky classification states that if $\mathfrak{P}$ is a partition-valued function on $\operatorname{cusp}_R G_n$, corresponding to the multiset $\{\Delta_1, \dots, \Delta_r\}$, then the representation parabolically induced from $\tau_{\mathfrak{P}}$ has a unique irreducible quotient $L(\Delta_1, \dots, \Delta_r)$, *provided* that the factors in the induction are appropriately ordered. More precisely, if Δ_i precedes Δ_j (meaning that their union is a segment that properly contains both, and the left extreme of the union is in Δ_i) then we require that $i > j$. Furthermore, $L(\Delta_1, \dots, \Delta_r)$ only depends on $\mathfrak{P}$, and every irreducible representation of any G_n has this form for a unique partition-valued function $\mathfrak{P}$ on $\operatorname{cusp}_R G_n$.

Example 2.2.4. Consider $\mathfrak{P}_\sigma(i) = 0$ unless $(\sigma, i) = (1_{\mathbb{G}_m}, n)$, in which case it is equal to one. Then $L(\mathfrak{P})$ is $L\Delta(1, n)$, which is the irreducible quotient of

$$|\cdot|^{(1-n)/2} \times \cdots \times |\cdot|^{(n-1)/2}.$$

This is the Steinberg representation of $\mathrm{GL}_n(F)$.

On the other hand, consider $\mathfrak{P}_\sigma(i) = 0$ unless $(\sigma, i) = (|\cdot|^{\frac{1-n}{2}+t}, 1)$ (for $0 \leq t \leq n-1$), in which case it is equal to 1. Then $L(\mathfrak{P})$ is the only irreducible quotient of

$$|\cdot|^{(n-1)/2} \times \cdots \times |\cdot|^{(1-n)/2}$$

by the ordering condition. Since normalized parabolic induction commutes with smooth duality, this is the trivial character of $\mathrm{GL}_n(F)$.

Since one of the main concerns in [SZ99] is a parametrization of the tempered dual, they use partitions on $\mathcal{C}$ rather than cusp, and write the classification in the following way. Given a partition-valued function $\mathcal{P}$ on $\mathcal{C}$ and an unramified character α of $N_{\mathcal{P}}$, they define

$$L(\alpha\tau_{\mathcal{P}}) = L(\alpha_1\Delta_1, \dots, \alpha_r\Delta_r),$$

where $\mathcal{P}$ corresponds to the multiset $\{\Delta_1, \dots, \Delta_r\}$. This gives rise to a map

$$Q_{\mathcal{P}} : X_{\mathrm{nr}}(N_{\mathcal{P}}) \to \operatorname{Irr}(\Omega_{D(\mathcal{P})}),\ \alpha \mapsto L(\alpha\tau_{\mathcal{P}})$$

which is, in general, not injective. (This can be seen already for supercuspidal inertial classes.)

Since the $\sigma \in \mathcal{C}$ are representatives of the supercuspidal twist classes, and $\mathrm{nr}_x \cdot L\Delta(\sigma, s) \cong L\Delta(\mathrm{nr}_x\sigma, s)$ for all $x \in R^\times$, the two classifications we have described are compatible. In more detail, given a cuspidal support $D \in \operatorname{Div}^+(\operatorname{cusp}_R G_n)$, and a representation π

supported in D, the multiset of segments such that $\pi \cong L(\Delta_1, \dots, \Delta_r)$ identifies with a partition-valued function on $\mathrm{cusp}_R G_n$. By twisting, we obtain a unique partition-valued function $\mathcal{P}$ on $\mathcal{C}$, and we have that $\pi \in \mathrm{im}(Q_{\mathcal{P}})$.

2.3 Type theory.

In this section we assume that R has characteristic zero, and for simplicity we take $R = \mathbf{C}$. By work of Bushnell–Kutzko, the Bernstein components of $\mathrm{GL}_n(F)$ can be distinguished by representations of certain compact open subgroups, called *types*. Here we recall the basic principles of this theory, which can be formulated for an arbitrary connected reductive group G/F and aims to constructing Morita-type equivalences of $\mathrm{Rep}^{\mathfrak{s}}(G)$ with the category of modules over an *explicit* $\mathbf{C}$-algebra $A_{\mathfrak{s}}$, for an arbitrary connected reductive group G over F and inertial class $\mathfrak{s} \in \mathfrak{B}(G)$. The theory is essentially complete for $\mathrm{GL}_m(D)$, and a lot is known for classical groups by work of Stevens and collaborators. We will review the construction of types for $\mathrm{GL}_m(D)$, and give some complements, in the next section.

We will be considering pairs (K, ρ) consisting of a compact open subgroup of G and an irreducible smooth representation of K. Since our coefficient field has characteristic zero, the category of smooth representations of K is semisimple, and ρ gives rise to an idempotent e_ρ in the Bernstein centre $\mathcal{Z}\,\mathrm{Rep}(G)$ as follows. Given any $V \in \mathrm{Rep}(G)$, $e_\rho : V \to V$ is the projection to the ρ-isotypic component of $V|_K$, which is a direct summand in V (it is the sum of all K-subspaces of V isomorphic to ρ). This idempotent is contained in the image of $\mathcal{H}(G) \to \mathcal{Z}\,\mathrm{Rep}(G)$, and represented by the function e_ρ supported in K with

$$e_\rho(k) = \frac{\dim \rho}{\mathrm{vol}(K)} \mathrm{tr}(\rho(k^{-1})).$$

On the other hand, the spherical Hecke algebra $\mathcal{H}(G, \rho) \cong \mathrm{End}(\text{c-Ind}_K^G(\rho))$ acts on the right on $\mathrm{Hom}_K(\rho, \pi)$ for each $\pi \in \mathrm{Rep}(G)$, by Frobenius reciprocity. Hence we obtain an exact functor

$$\mathbf{M}_\rho : \mathrm{Rep}(G) \to \mathrm{Mod}(\mathcal{H}(G, \rho)^{\mathrm{op}}),\ \pi \mapsto \mathrm{Hom}_K(\rho, \pi).$$

Type theory is concerned with situations in which $\mathcal{H}(G, \rho)$ is explicit and $\mathbf{M}_\rho$ is an equivalence.

Remark 2.3.1. We define $\mathcal{H}(G, \rho)$ as the convolution algebra of K-biinvariant, compactly supported functions $f : G \to \mathrm{End}_{\mathbf{C}}(\rho)$, so that $f(k_1 g k_2) = \rho(k_1) \circ f(g) \circ \rho(k_2)$. In [BK98, BK93], they consider instead functions $G \to \mathrm{End}_{\mathbf{C}}(\rho^\vee)$ to the dual, so our spherical Hecke algebra identifies canonically with what they denote $\mathcal{H}(G, \rho^\vee)$. However, there is an anti-isomorphism $f \mapsto \check{f}$, where $\check{f} : g \mapsto f(g^{-1})^\vee$, which allows us to identify the Bushnell–Kutzko $\mathcal{H}(G, \rho)$ with the opposite of $\mathrm{End}_G(\text{c-Ind}_K^G(\rho))$. Accordingly, they regard $\mathbf{M}_\rho$ a functor to left $\mathcal{H}(G,\rho)$-modules (for their version of $\mathcal{H}(G,\rho)$). We won't do this.

By [BK93, Proposition 4.2.4], there is a canonical $\mathbf{C}$-linear ring isomorphism

$$\Upsilon : \mathcal{H}(G,\rho)^{\mathrm{op}} \otimes_{\mathbf{C}} \mathrm{End}_{\mathbf{C}}(\rho) \to e_\rho \mathcal{H}(G) e_\rho$$

and one checks that $\Upsilon_*(\mathrm{Hom}_K(\rho,\pi) \otimes_{\mathbf{C}} \rho)$ is isomorphic to the ρ-isotypic component π^ρ, by evaluation. In the case that ρ is the trivial representation, we know that the action of $e_\rho \mathcal{H}(G) e_\rho = \mathcal{H}(G,K)$ on $\pi^\rho = \pi^K$ determines the isomorphism class of π when π is irreducible: when $K = \mathrm{GL}_n(\mathcal{O}_F)$, this is the familiar fact that the Hecke eigenvalues determine an unramified representation. This stays true in general, and $\pi \mapsto \pi^\rho$ induces a bijection from irreducible representations π generated over G by π^ρ, to simple $e_\rho \mathcal{H}(G) e_\rho$-modules. By Morita equivalence, the same is true for $\mathbf{M}_\rho$.

What is *not* true in general is that $\mathbf{M}_\rho$ is an equivalence, even when restricted to the category $\mathrm{Rep}_\rho(G)$ of representations generated by their ρ-isotypic component, as can already be seen for unramified representations.

Example 2.3.2. In more details, let $G = GL_n(F)$. The smooth dual $\pi = i_B^G(\mathrm{triv}_B)^\vee$ has irreducible G-cosocle, isomorphic to the trivial character of G, and no other unramified subquotient. Since the coefficient field has characteristic zero, this K-invariant lifts to π, and it generates π over G by the cosocle condition. However, $\mathbf{M}_{\mathrm{triv}_K}(\pi) \cong \mathbf{M}_{\mathrm{triv}_K}(\mathrm{triv}_G)$, and triv_G is also generated by its K-invariants over G. So $\mathbf{M}_{\mathrm{triv}_K}$ is not an equivalence.

Definition 2.3.3. Given (K,ρ), the category $\mathrm{Rep}_\rho(G)$ is the full subcategory of $\mathrm{Rep}(G)$ whose objects are the representations generated over $\mathbf{C}[G]$ by their ρ-isotypic component. The pair (K,ρ) is called a *type* in G if $\mathrm{Rep}_\rho(G)$ is closed under taking subquotients.

In the terminology of [BK98], (K,ρ) is a type if and only if $e_\rho \in \mathcal{H}(G)$ is a *special idempotent*, and properties of special idempotents imply the following result.

Theorem 2.3.4. [BK98, Propositions 3.5, 3.6, Theorem 4.3] The pair (K,ρ) is a type if and only if there exists a finite subset $\mathfrak{S} \subset \mathfrak{B}(G)$ such that $\mathrm{Rep}_\rho(G) = \mathrm{Rep}^{\mathfrak{S}}(G)$: equivalently, $\pi \in \mathrm{Irr}(G)$ contains ρ if and only if π has cuspidal support in $\mathfrak{S}$. If (K,ρ) is a type, then $\mathbf{M}_\rho$ is an equivalence of categories.

Remark 2.3.5. Let $\mathfrak{s} \in \mathfrak{B}(G)$. As noted in [BK98, Section 3], it was already known to Bernstein that the theory of special idempotents allows one to prove directly the existence of an associative unital $\mathbf{C}$-algebra $A_\mathfrak{s}$ and an equivalence

$$\mathrm{Rep}^\mathfrak{s}(G) \to \mathrm{Mod}(A_\mathfrak{s}).$$

See for instance [Ber84, Corollaire 3.9] for a related result (the trivial character of a compact open subgroup K is *always* a type if K satisfies certain properties).

2.4 Construction of types for $\mathrm{GL}_m(D)$.

In this section we recall some basic properties of the objects which go into the definition of types for cuspidal representations of $\mathrm{GL}_m(D)$. A lot of this material is standard, but we need generalizations to the non-split case of certain well-known properties of simple characters of $\mathrm{GL}_n(F)$, and we could not find these in the literature. While in the previous section we have described type theory in characteristic zero, here we will allow R to be an algebraically closed field of positive characteristic different from p. Even if one cannot expect results as complete as those in theorem 2.3.4, these constructions are fundamental to the classification of irreducible modular representations of $\mathrm{GL}_m(D)$ in [Vig96, MS14a], and our study of certain mod l congruences will be based on these results.

Throughout this section, we write $G = \mathrm{GL}_m(D)$ and regard it as the group of units in the central simple F-algebra $A = M_m(D)$. We regard F as a central subfield of A via the given embedding, and we fix a simple A-module $V = D^{\oplus m}$.

Lattice sequences. Consider *lattice sequences* in V, which are decreasing functions

$$\Lambda : \mathbf{Z} \to \ (\mathcal{O}_D\text{-lattices in } V)$$

where the right hand side is ordered by inclusion, such that there exists a positive integer e with $\Lambda_{k+e} = \Lambda_k \mathfrak{p}_D$ for all k. The number e is called the $\mathcal{O}_D$-period of the sequence, and the sequence is called a *chain* or a *strict sequence* if it is strictly decreasing. A sequence is called *uniform* if it is a chain and the dimension of Λ_k/Λ_{k+1} over the residue field $\mathbf{d}$ of D is constant as k varies [Frö87, 1.7].

Every sequence defines a hereditary $\mathcal{O}_F$-order $\mathfrak{A} = \mathfrak{P}_0(\Lambda)$ in A equipped with a filtration by $\mathcal{O}_F$-lattices $\mathfrak{P}_n(\Lambda)$, via

$$\mathfrak{P}_n(\Lambda) = \{a \in A : a\Lambda_k \subseteq \Lambda_{k+n} \text{ for all } k \in \mathbf{Z}\}.$$

The Jacobson radical $\mathfrak{P}(\mathfrak{A})$ of $\mathfrak{A}$ then equals $\mathfrak{P}_1(\Lambda)$ (see [Séc04, 1.2]), and we write $U^n(\Lambda)$ for $1 + \mathfrak{P}_n(\Lambda)$. The *normalizer* of a sequence is defined as

$$\mathfrak{K}(\Lambda) = \{g \in G : \text{ there exists } n \in \mathbf{Z} \text{ such that } g(\Lambda_k) = \Lambda_{k+n} \text{ for all } k\}.$$

Such an integer n is unique and denoted $v_\Lambda(g)$. This defines a group homomorphism $v_\Lambda : \mathfrak{K}(\Lambda) \to \mathbf{Z}$ whose kernel $U(\mathfrak{A})$ is the unit group of $\mathfrak{A}$. The unit groups of hereditary $\mathcal{O}_F$-orders in A are precisely the parahoric subgroups of G. As in [Séc04, 1.2], this defines a bijection $\Lambda \mapsto \mathfrak{P}_0(\Lambda)$ from lattice *chains* up to $\mathbf{Z}$-translation, to hereditary orders in A. It follows that the normalizer of a lattice chain coincides with the normalizer in G of the corresponding hereditary order.

Let E/F be a field extension in A. An $\mathcal{O}_D$-lattice sequence Λ in V is called *E-pure* if $E^\times \subseteq \mathfrak{K}(\Lambda)$. This condition is equivalent to Λ being an $\mathcal{O}_E$-lattice sequence in V viewed as an E-vector space. Denote by $B = Z_A(E)$ the commutant of E in A. This is a central simple algebra over E, and we write n'^2 for its E-dimension. Then B is E-isomorphic to $M_{m'}(D_E)$ for some central division E-algebra D_E of E-dimension d'^2, and we have the identities*

$$n' = \frac{n}{[E:F]},\ d' = \frac{d}{(d,[E:F])},\ m'd' = n'$$

as in [BH11, 2.1.1].

The need of considering general lattice sequences instead of focusing on chains, which can be done in the split case, arises from the behaviour of filtrations of hereditary orders attached to E-pure sequences under intersection $\mathfrak{A} \mapsto \mathfrak{A} \cap B$. Upon fixing a simple left B-module V_E, one has the following result.

Theorem 2.4.1. [SS08, Theorem 1.4] Given an E-pure lattice sequence Λ in V, there exists an $\mathcal{O}_{D_E}$-lattice sequence Γ in V_E such that

$$\mathfrak{P}_k(\Lambda) \cap B = \mathfrak{P}_k(\Gamma) \text{ for all } k \in \mathbf{Z},$$

and the normalizer $\mathfrak{K}(\Gamma)$ equals $\mathfrak{K}(\Lambda) \cap B^\times$. The sequence Γ is unique up to translation.

The sequence $\Gamma = \mathrm{tr}_B\Lambda$ is called the *trace* of the lattice sequence Λ, and Λ is called the *continuation* of Γ. Notice that the theorem does not say that every $\mathcal{O}_{D_E}$-lattice sequence has a continuation: this doesn't necessarily hold (see [SS08, Exemple 1.6]). Usually, $\mathfrak{B}$ will denote the hereditary order $\mathfrak{A} \cap B = \mathfrak{P}_0(\Gamma)$.

When $a, b \in \mathbf{Z}$, we can rescale a lattice sequence Λ to

$$a\Lambda + b : k \mapsto \Lambda_{\lceil \frac{k-b}{a} \rceil},$$

and the set of these sequences is called the *affine class* of Λ. If $\Lambda = a\Lambda_0$ for a lattice chain Λ_0, the sequence Λ will be called a *multiple* of Λ_0: in this case we have $\mathfrak{K}(\Lambda) = \mathfrak{K}(\Lambda_0)$, and what changes is the filtration on this group. The map $\Lambda \mapsto \mathrm{tr}_B(\Lambda)$ preserves affine classes. One can't say much about the trace of an arbitrary sequence—for instance, the trace of a chain needn't be a chain, see [BL02, Section 6]—but the following result on preimages holds.

Proposition 2.4.2. Assume Λ is an E-pure lattice sequence in V whose trace $\Gamma = a\Gamma_0$ is a multiple of a uniform chain Γ_0 of $\mathcal{O}_E$-period r. Then Λ is a multiple of a uniform chain of $\mathcal{O}_D$-period $\frac{re(E/F)}{(d,re(E/F))}$.

Proof. By [BL02, Proposition II.5.4], if Γ is a multiple of a uniform chain then so is Λ.

*The notation $(d,[E:F])$ stands for the highest common factor of d and $[E:F]$.

By [SS08, Théorème 1.7] and its proof, there exists a unique chain Λ_0 in V whose trace is a multiple of Γ_0, and the $\mathcal{O}_D$-period of Λ_0 is $re(E/F)/(d, re(E/F))$ (see also [BSS12, Lemma 4.18]). The claim now follows as Λ is a multiple of some chain, which must be Λ_0, since tr_B commutes with scaling. □

Simple characters. We only discuss simple characters attached to simple strata of the form $[\mathfrak{A}, \beta]$, consisting of a principal $\mathcal{O}_F$-order $\mathfrak{A}$ in A attached to a lattice chain Λ in V, and an element $\beta \in A$ generating a field $E = F[\beta]$, such that $E^\times \subseteq \mathfrak{K}(\Lambda)$ and the condition

$$k_0(\beta, \mathfrak{A}) < 0$$

on the critical exponent holds (see for instance [Séc04] for an exposition). We follow [BH14] in shortening notation to $[\mathfrak{A}, \beta]$ for what is otherwise denoted $[\mathfrak{A}, -v_\Lambda(\beta), 0, \beta]$, as these are the only strata that will show up in what follows.

As in [Séc04, Proposition 3.42] and [BH11, Section 2.5], there exist $\mathcal{O}_F$-orders $\mathfrak{h}(\beta, \mathfrak{A}) \subseteq \mathfrak{j}(\beta, \mathfrak{A}) \subseteq \mathfrak{A}$ attached to a simple stratum $[\mathfrak{A}, \beta]$ in A, with a filtration by ideals $\mathfrak{h}^k(\beta, \mathfrak{A})$ and $\mathfrak{j}^k(\beta, \mathfrak{A})$. There are compact open subgroups $H(\beta, \mathfrak{A}) = \mathfrak{h}(\beta, \mathfrak{A})^\times$ and $J(\beta, \mathfrak{A}) = \mathfrak{j}(\beta, \mathfrak{A})^\times$, with filtrations by subgroups

$$J^k(\beta, \mathfrak{A}) = J(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A}) = 1 + \mathfrak{j}^k(\beta, \mathfrak{A})$$
$$H^k(\beta, \mathfrak{A}) = H(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A}) = 1 + \mathfrak{h}^k(\beta, \mathfrak{A}).$$

These groups are normalized by $J(\beta, \mathfrak{A})$ and by $\mathfrak{K}(\mathfrak{A}) \cap B^\times$, H^k is normal in J^k, and the quotients J^k/H^k are finite-dimensional vector spaces over $\mathbf{F}_p$ (see [Séc04, Proposition 4.3]). There are isomorphisms $\mathfrak{B}/\mathfrak{P}_1(\mathfrak{B}) \to \mathfrak{j}(\beta, \mathfrak{A})/\mathfrak{j}^1(\beta, \mathfrak{A})$ and $U(\mathfrak{B})/U^1(\mathfrak{B}) \to J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A})$ induced by the inclusions.

The group $H^1(\beta, \mathfrak{A})$ carries a distinguished finite set $C(\mathfrak{A}, \beta)$ of *simple characters*, which is fundamental for the construction of types, and is defined and studied in [Séc04] and [SS08, Section 2]. These references treat the more general case of simple characters of positive level, which form a set $C(\mathfrak{A}, m, \beta)$: one has $C(\mathfrak{A}, \beta) = C(\mathfrak{A}, 0, \beta)$. The definition of $C(\mathfrak{A}, \beta)$ also depends on the choice of an additive character ψ of F, which is fixed throughout. Since the group $H^1(\beta, \mathfrak{A})$ is a pro-p group, simple characters are valued in $\mu_{p^\infty}(R)$. Hence there is a canonical bijection from the simple characters over $\overline{\mathbf{Q}}_l$ to those over $\overline{\mathbf{F}}_l$, given by reduction mod l, whenever $l \neq p$ is a prime number.

Simple characters satisfy the "intertwining implies conjugacy" property to various degrees; in full generality, one has the following result, which in the split case can be strengthened (see [BK93, Theorem 3.5.11] and [BH14, 2.6]). In order to state it, we need the notion of an *embedding* in A, which is defined as a pair (E, Λ), where E is a field extension of F in A, and Λ is an E-pure $\mathcal{O}_D$-lattice sequence in V. Two embeddings are *equivalent* if there exists $g \in A^\times$ such that Λ_1 and $g\Lambda_2$ coincide up to translation, the maximal unramified extensions $E_i^{\mathrm{ur},d}$ of F in E_1 and E_2 of degree dividing d are isomorphic, and $\mathrm{ad}(g)E_2^{\mathrm{ur},d} = E_1^{\mathrm{ur},d}$. Two simple strata $[\mathfrak{A}_i, \beta_i]$ have the same *embedding type* if the embeddings $(F[\beta_i], \Lambda_i)$ are equivalent, where Λ_i is the chain attached to $\mathfrak{A}_i$.

Theorem 2.4.3. [BSS12, Theorem 1.12] Given two simple strata $[\mathfrak{A}, \beta_i]$ with the same embedding type, and two simple characters $\theta_i \in C(\mathfrak{A}, \beta_i)$ which intertwine in $A^\times$, let K_i be the maximal unramified extension of F in $F[\beta_i]$. Then there exists $u \in \mathfrak{K}(\mathfrak{A})$ such that

1. $K_2 = uK_1u^{-1}$
2. $H^1(\beta_2, \mathfrak{A}) = uH^1(\beta_1, \mathfrak{A})u^{-1}$ and $\theta_1 = \mathrm{ad}(u)^*\theta_2$.

Endo-classes. Consider now all the groups $\mathrm{GL}_n(F)$ and their inner forms $\mathrm{GL}_m(D)$ for varying n, and the set of all simple characters of these groups. There is an equivalence relation on this set, called *endo-equivalence*, which is discussed in [BH96] in the split case and [BSS12] in general. An *endo-class* of simple characters over F is an equivalence class for this equivalence relation. The endo-class of a simple character θ will be denoted $\mathrm{cl}(\theta)$. Again, we identify endo-classes of simple $\overline{\mathbf{Q}}_l$-characters and simple $\overline{\mathbf{F}}_l$-characters.

It is important to notice that we might have two endo-equivalent simple characters θ_i of endo-class Θ_F, defined by simple strata $[\mathfrak{A}_i, \beta_i]$, in which the extensions $F[\beta_i]$ of F are not isomorphic. However, by [BH96, 8.11] and [BSS12, Lemma 4.7], they will have the same ramification index and residue class degree. The degrees $F[\beta_i]/F$ therefore also coincide. These are invariants of Θ_F, which which will be denoted $e(\Theta_F)$, $f(\Theta_F)$ and $\delta(\Theta_F)$ respectively.

A simple character in A is *maximal* if it can be defined by a stratum $[\mathfrak{A}, \beta]$ such that $\mathfrak{B} = \mathfrak{A} \cap Z_A(F[\beta])$ is a maximal $\mathcal{O}_{F[\beta]}$-order in $Z_A(F[\beta])$. Such a stratum will be called a *maximal simple stratum*. By proposition 2.4.6 below, maximality does not depend on the stratum defining θ. Recall (see [BSS12, Definition 1.14]) that a simple stratum $[\mathfrak{A}, \beta]$ is *sound* if $\mathfrak{B}$ is a principal $\mathcal{O}_F$-order and $\mathfrak{K}(\mathfrak{A}) \cap B^\times = \mathfrak{K}(\mathfrak{B})$.

Proposition 2.4.4. Maximal simple strata are sound.

Proof. Let $[\mathfrak{A}, \beta]$ be a maximal simple stratum, corresponding to a lattice chain Λ in V. By definition, $\mathfrak{A} \cap B = \mathfrak{B}$ is a maximal order in B. Consider the trace $\Gamma = \mathrm{tr}_B(\Lambda)$. Then $\mathfrak{P}_0(\Gamma) = \mathfrak{B}$, and it follows that the chain Γ_0 associated to Γ is principal of period 1. So necessarily $\Gamma = t\Gamma_0$ for some positive integer t. It follows that $\mathfrak{K}(\mathfrak{B}) = \mathfrak{K}(\Gamma_0)$ is actually equal to $\mathfrak{K}(\Gamma)$ (but the filtration on it changes). Since $\mathfrak{K}(\Gamma) = \mathfrak{K}(\Lambda) \cap B^\times$ by definition, we have $\mathfrak{K}(\mathfrak{A}) \cap B^\times = \mathfrak{K}(\mathfrak{B})$, that is, the stratum $[\mathfrak{A}, \beta]$ is sound. □

The relation of endo-equivalence between maximal simple characters in the same group takes on a simple form: it coincides with conjugacy.

Proposition 2.4.5. Maximal simple strata $[\mathfrak{A}_i, \beta_i]$ in the same central simple algebra A over F, defining endo-equivalent maximal simple characters θ_i, have the same embedding type. Endo-equivalent maximal simple characters in the same group are conjugate.

Proof. Write $B_{\beta_i} = Z_A(F[\beta_i])$, and let Λ_i be the lattice chains in V corresponding to the $\mathfrak{A}_i$. By the Skolem–Noether theorem there exists $x \in A^\times$ conjugating the maximal unramified extensions of F in $F[\beta_i]$, as they have the same degree $f(\Theta_F)$ over F, so we can assume that they both coincide with a subfield E of A. Because the orders $\mathfrak{B}_i = \mathfrak{A}_i \cap B_{\beta_i}$ are maximal, there are extensions of $F[\beta_i]$ in B_{β_i} which have maximal degree, are unramified, and normalize the $\mathfrak{A}_i$. To see this, observe that $\mathfrak{B}_i^\times$ is a maximal compact subgroup of $B_{\beta_i}^\times$. Choose any maximal unramified extension L_i of $F[\beta_i]$ in B_{β_i}, so that $\mathcal{O}_{L_i}^\times$ is contained up to conjugacy in $\mathfrak{B}_i^\times$. Since $L_i^\times = \pi_{F[\beta_i]}^{\mathbf{Z}} \times \mathcal{O}_{L_i}^\times$, we have $L_i^\times \subseteq \mathfrak{K}(\mathfrak{B}_i)$. By proposition 2.4.4, we have $\mathfrak{K}(\mathfrak{A}_i) \cap B_{\beta_i}^\times = \mathfrak{K}(\mathfrak{B}_i)$ and so $L_i^\times \subseteq \mathfrak{K}(\mathfrak{A}_i)$.

To prove that $[\mathfrak{A}_i, \beta_i]$ have the same embedding type, it is enough to prove that $\mathrm{tr}_{Z_A(E)}(\Lambda_i)$ are conjugate under $Z_A(E)^\times$ (up to translation), as then the same will hold for their continuations Λ_i by the uniqueness statement in theorem 2.4.1. The sequences $\Delta_i = \mathrm{tr}_{Z_A(L_i)}(\Lambda_i)$ are both multiples of a chain of period 1, since $Z_A(L_i) = L_i$ and so there are no other lattice sequences for $Z_A(L_i)$. By proposition 2.4.2, the sequence $\mathrm{tr}_{Z_A(E)}(\Lambda_i)$ is a multiple of a uniform chain Γ_i, as its trace to L_i is Δ_i. Write $\mathrm{tr}_{Z_A(E)}(\Lambda_i) = a_i\Gamma_i$. The $\mathcal{O}_{D_E}$-period of Γ_i is equal to $\frac{e(L_i/E)}{(d_{Z_A(E)}, e(L_i/E))}$, but $e(L_i/E) = e(F[\beta_i]/F)$ is independent of i, hence Γ_1 and Γ_2 have the same $\mathcal{O}_{D_E}$-period and the same $\mathcal{O}_E$-period, which we denote by t. By the proof of [SS08, Theorem 1.7], the integer a_i then equals $\frac{d}{(d, e(E/F)t)} = \frac{d}{(d,t)}$ and is independent of i, and so the sequences $\mathrm{tr}_{Z_A(E)}(\Lambda_i)$ are conjugate under $Z_A(E)^\times$ up to translation. (By [SS08, Remark 1.8] and the fact that the Λ_i are *chains*, a_i is equal to what is denoted ρ in that reference, and we are applying the formula for ρ that is given there.)

That θ_1 and θ_2 are conjugate now follows from theorem 2.4.3. □

Proposition 2.4.6. If $[\mathfrak{A}_1, \beta_1]$ and $[\mathfrak{A}_2, \beta_2]$ are maximal simple strata in A defining the simple character θ, then $\mathfrak{A}_1 = \mathfrak{A}_2$ and $J^i(\beta_1, \mathfrak{A}_1) = J^i(\beta_2, \mathfrak{A}_2)$ for $i = 0, 1$.

Proof. To see that $J^i(\beta_1, \mathfrak{A}_1) = J^i(\beta_2, \mathfrak{A}_2)$ we argue as in [BH14, 2.1.1]. Namely, first we compute the normalizer $\mathbf{J}(\theta)$ of θ in G, as follows. We know from [Séc05, Proposition 2.3] that the intertwining of θ in G is $J(\beta, \mathfrak{A})B_\beta^\times J(\beta, \mathfrak{A})$, for any maximal simple stratum $[\mathfrak{A}, \beta]$ defining θ, and that $\mathfrak{K}(\mathfrak{B}_\beta)J(\beta, \mathfrak{A})$ normalizes θ (using that $\mathfrak{K}(\mathfrak{B}_\beta) = \mathfrak{K}(\mathfrak{A}) \cap B_\beta$ by proposition 2.4.4). Now assume that $g \in B_\beta^\times$ normalizes θ. Then it normalizes $H^1(\beta, \mathfrak{A}) \cap B_\beta^\times = U^1(\mathfrak{B}_\beta)$ (this equality is claimed in [Séc05] after Remarque 2.4). But the normalizer of $U^1(\mathfrak{B}_\beta)$ in $B_\beta^\times$ equals $\mathfrak{K}(\mathfrak{B}_\beta)$, by the argument in [BK93, 1.1], hence $g \in \mathfrak{K}(\mathfrak{B}_\beta)$ and so the normalizer $\mathbf{J}(\theta)$ equals $\mathfrak{K}(\mathfrak{B}_\beta)J(\beta, \mathfrak{A})$.

Since $\mathfrak{K}(\mathfrak{B}_\beta) = D_{F[\beta]}^\times \mathfrak{B}_\beta^\times$, we see that $\mathbf{J}(\theta)$ has a unique maximal compact subgroup J_θ, which equals $J(\beta, \mathfrak{A})$ for any maximal simple stratum $[\mathfrak{A}, \beta]$ defining θ, and J_θ has a unique subgroup J_θ^1 that is maximal amongst its normal pro-p subgroups, which is then equal to $J^1(\beta, \mathfrak{A})$. This recovers the groups H^1, J^1 and J intrinsically to θ.

By proposition 2.4.5, the strata $[\mathfrak{A}_i, \beta_i]$ have the same embedding type, hence for some $g \in G$ we have $g\mathfrak{A}_1 g^{-1} = \mathfrak{A}_2$. The characters $g^*\theta$ and θ intertwine, hence there exists $u \in \mathfrak{K}(\mathfrak{A}_1)$ conjugating them, by theorem 2.4.3. So $(gu)^*\theta = \theta$, and then $gu \in \mathbf{J}(\theta)$ and conjugates $\mathfrak{A}_1$ to $\mathfrak{A}_2$. But $\mathbf{J}(\theta)$ normalizes $\mathfrak{A}_1$, as it equals $(\mathfrak{K}(\mathfrak{A}_1) \cap B_{\beta_1})J_\theta$, and so $\mathfrak{A}_1 = \mathfrak{A}_2$, as gu normalizes $\mathfrak{A}_1$ and at the same time it conjugates it to $\mathfrak{A}_2$. □

Definition 2.4.7. By proposition 2.4.6, the groups $H^1(\beta, \mathfrak{A})$, $J^1(\beta, \mathfrak{A})$ and $J(\beta, \mathfrak{A})$ for a simple stratum $[\mathfrak{A}, \beta]$ defining a maximal simple character θ only depend on θ, and will be denoted H_θ^1, J_θ^1 and J_θ.

The endo-classes of F can be lifted and restricted through tamely ramified field extensions E/F. In this context there exists a restriction map

$$\mathrm{Res}_{E/F} : \mathcal{E}(E) \to \mathcal{E}(F)$$

from the set of endo-classes of simple characters of E to those for F. It is surjective, and its fiber over a given endo-class Θ_F consists by definition of the set of E-lifts of Θ_F. We'll need some details as to how the lifting can be performed in practice, in the unramified case.

Proposition 2.4.8. ([BH96, Section 7] and [BSS12, Section 5, 6]) Let θ be a maximal simple character in A defined by the simple stratum $[\mathfrak{A}, \beta]$, with endo-class Θ_F. Let K be an unramified extension of F in A such that β commutes with K and generates a field extension of K in $A_K = Z_A(K)$, and assume that $K[\beta]^\times \subseteq \mathfrak{K}(\mathfrak{A})$. Then $\theta_K = \theta | H_\theta^1 \cap A_K$ is a simple character, with

$$\begin{aligned} H^1_{\theta_K} &= H^1_\theta \cap A_K \\ J^1_{\theta_K} &= J^1_\theta \cap A_K \\ J_{\theta_K} &= J_\theta \cap A_K. \end{aligned}$$

These groups will be denoted H_K^1, J_K^1 and J_K respectively. The character θ_K is called the *interior K-lift* of θ. Its endo-class $\Theta_K = \mathrm{cl}(\theta_K)$ is a K-lift of Θ_F.

Finally, we state a compatibility of endo-classes with automorphisms of the base field. More generally, if $\alpha : F_1 \to F_2$ is a continuous isomorphism between local fields, it induces a pullback

$$\alpha^* : \mathcal{E}(F_2) \to \mathcal{E}(F_1) \qquad (2.4.1)$$

on the sets of endo-classes. When a central simple algebra A over F_2 is given, together with a simple character θ in $A^\times$, one can regard A as a central simple F_1-algebra via α, and then $\mathrm{cl}_{F_1}(\theta)$, the endo-class of θ as a simple character over F_1, is equal to $\alpha^* \mathrm{cl}_{F_2}(\theta)$. The functoriality property

$$(\alpha_1\alpha_2)^* = \alpha_2^*\alpha_1^*$$

also holds. It follows that the group of continuous automorphisms of F acts to the right on the set $\mathcal{E}(F)$ of endo-classes of F. The action will be denoted $g : \Theta_F \mapsto \Theta_F^g = g^*\Theta_F$.

Moving from H to J. Let θ be a maximal simple character in G, defined by a simple stratum $[\mathfrak{A}, \beta]$. There exists a unique irreducible representation $\eta = \eta(\theta)$ of J^1_θ which contains θ, called the *Heisenberg representation* attached to θ (we will give more details in what follows). The dimension of θ is a power of p and the restriction $\eta|H^1_\theta$ is a multiple of θ, and θ and $\eta(\theta)$ have the same G-intertwining. By [MS14b, Section 2.4], there exists an extension of η to J_θ with the same G-intertwining as θ and η, called a *β-extension* of η. By [Séc05, Théorème 2.28] and [MS14b, 2.2] we know that the group of characters of $\mathbf{e}^\times$ is transitive on the set of β-extensions of η, by the twisting action

$$\chi : \kappa \mapsto \kappa \otimes (\chi \circ \nu_B)$$

where $\chi : \mathbf{e}^\times \to R^\times$ has been inflated to $\mathcal{O}_E^\times$, and $\nu_B : \mathfrak{B}^\times \to \mathcal{O}_E^\times$ is the reduced norm.

Proposition 2.4.9. There exists exactly one β-extension κ of η to J_θ such that the determinant character of κ has order a power of p. We will refer to κ as a *p-primary β-extension.*

Proof. Write $E = F[\beta]$. Fix an E-linear isomorphism

$$\Phi : B \to M_{m'}(D')$$

where D' is a central division algebra of reduced degree d' over E, such that the order $\mathfrak{B}$ gets mapped to $M_{m'}(\mathcal{O}_{D'})$. We then get an isomorphism $\Phi : J_\theta/J^1_\theta \to \mathfrak{B}^\times/U^1(\mathfrak{B}) \to \mathrm{GL}_{m'}(\mathbf{d}')$, for $\mathbf{d}'$ the residue field of D', via Φ above and the inverse of the isomorphism $\mathfrak{B}^\times/U^1(\mathfrak{B}) \to J_\theta/J^1_\theta$ induced by the inclusion.

Let κ be a β-extension of η. The determinant character $\det\kappa$ has prime-to-p part $(\det\kappa)^{(p)}$ that is trivial on the pro-p group J^1_θ, hence $(\det\kappa)^{(p)}$ is the inflation to J_θ of a character γ of $\mathbf{d}'^\times$ through the determinant of $\mathrm{GL}_{m'}(\mathbf{d}')$ and the isomorphism Φ. Assume that γ is norm-inflated from $\mathbf{e}^\times$. Observe that $\det(\kappa \otimes (\chi \circ \nu_B)) = \det(\kappa)(\chi^{\dim\kappa} \circ \nu_B)$. Now since $\dim\kappa$ is a power of p and the character group of $\mathbf{e}^\times$ has order prime to p, there exists a unique χ such that $\chi^{-\dim\kappa} \circ \nu_B|_J = \det\kappa^{(p)}$, and the claim follows.

So it's enough to prove that γ is norm-inflated from $\mathbf{e}^\times$, which happens if and only if γ is stable under $\mathrm{Gal}(\mathbf{d}'/\mathbf{e})$. If $\pi_{D'}$ is a uniformizer of D', its conjugacy action on $\mathfrak{B}^\times$ induces on $\Phi(\mathfrak{B}^\times)$ the Frobenius automorphism on matrix entries, so it's enough to prove that the restriction of $(\det \kappa)^{(p)}$ to $\Phi(\mathfrak{B}^\times)$ is normalized by $\pi_{D'}$, and this is true because $B^\times$ intertwines κ, hence it intertwines both $\det \kappa$ and $(\det \kappa)^{(p)}$ with themselves. □

Maximal simple types. Fix a maximal simple character θ in G, with corresponding Heisenberg representation η. Let κ be a β-extension of η to J_θ. Let σ be a cuspidal irreducible representation of J_θ/J_θ^1, and define $\lambda = \sigma \otimes \kappa$. A pair (J_θ, λ) arising thus is called a *maximal simple type* in G. Over the complex numbers, these are types for the supercuspidal inertial classes of G. The modular case is different, as for instance there may exist cuspidal non-supercuspidal representations, which will contain maximal simple types but won't exhaust an inertial class. However, the following result holds, for which we refer to [MS14b] and the references therein.

Theorem 2.4.10. Let ρ be an irreducible cuspidal R-representation of G. Then

1. ρ contains a unique G-conjugacy class of maximal simple types.
2. if (J, λ) is a maximal simple type contained in ρ, then λ admits extensions to its normalizer $\mathbf{J}(\lambda)$, and for precisely one such extension Λ the compact induction $\pi(\Lambda) = \text{c-Ind}_{\mathbf{J}(\lambda)}^G \Lambda$ is isomorphic to ρ.
3. two irreducible cuspidal representations ρ_i contain the same maximal simple types if and only if they are unramified twists of one another.

We will need some information on the structure of the normalizers $\mathbf{J}(\lambda)$. If (J_θ, λ) is any maximal simple type arising from θ and σ, and $[\mathfrak{A}, \beta]$ is a maximal simple stratum defining θ, by [MS14b, Paragraph 3.4] the order $s(\sigma)$ of the stabilizer of σ in $\text{Gal}(\mathbf{d}'/\mathbf{e})$ equals the index of $F[\beta]^\times J_\theta$ in $\mathbf{J}(\lambda)$. Fixing an isomorphism $B \to M_{m'}(D')$, we have $\mathbf{J}(\theta) = \mathfrak{K}(\mathfrak{B})J_\theta = \pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$ for any uniformizer $\pi_{D'}$ of D', and so the index of $\mathbf{J}(\lambda)$ in $\mathbf{J}(\theta)$ equals the size $b(\sigma)$ of the orbit of σ under $\text{Gal}(\mathbf{d}'/\mathbf{e})$.

Symplectic invariants. From now until the end of this section, we will work with $R = \mathbf{C}$. Fix a maximal simple character θ in G. One has a well-defined map

$$J_\theta^1/H_\theta^1 \times J_\theta^1/H_\theta^1 \to \mu_p(\mathbf{C}),\ (x, y) \mapsto \theta[x, y],$$

where $\mu_p(\mathbf{C})$ is the group of complex roots of unity of order p and $[x, y] = xyx^{-1}y^{-1}$. By [Séc05, Proposition 2.3], this map is a symplectic form on the $\mathbf{F}_p$-vector space J_θ^1/H_θ^1: it is alternating, $\mathbf{F}_p$-bilinear and nondegenerate.

This is a special case of the following situation, which is described in [BF83, Section 8], and [BH10, Section 3]. Consider triples (G, N, θ) where G is a group with a normal subgroup N such that the quotient $V = G/N$ is a finite-dimensional $\mathbf{F}_p$-vector space, and θ is a faithful character $\theta : N \to \mathbf{C}^\times$ such that θ is stable under conjugation by G and $(gN, hN) \mapsto \theta[g, h]$ is a symplectic form on V. In the above, we have $G = J^1_\theta / \ker(\theta)$ and $N = H^1_\theta / \ker(\theta)$.

Proposition 2.4.11. [BF83, 8.3.3] There exists a unique irreducible representation $\eta = \eta(\theta)$ of G which contains θ, called the *Heisenberg representation* attached to θ. The dimension of θ is a power of p and the restriction $\eta|N$ is a multiple of θ.

Let now Γ be a finite cyclic subgroup of the automorphism group $\mathrm{Aut}(G)$, of order prime to p, preserving N and the character θ, so that V is a symplectic $\mathbf{F}_p$-representation of Γ. Because Γ is cyclic, η extends to $\Gamma \ltimes G$. All extensions are twists of each other by characters inflated from Γ. Since the dimension of η is a power of p and the order of Γ is prime to p, there exists a unique extension $\widetilde{\eta}$ such that $\widetilde{\eta}|\Gamma$ has trivial determinant character (as in the proof of proposition 2.4.9).

We will be interested in cases where the trace of Γ on V is valued in $\mathbf{Z}$, and we will need an explicit formula for its sign. In order to find one, we follow [BH10, Section 3] and attach invariants to every symplectic $\mathbf{F}_p[\Gamma]$-representation (V, h), which will specialize to this sign when V arises as G/N from (G, N, θ) in the previous set-up. We give some details since we'll need the full construction in order to prove proposition 2.4.22.

The isomorphism classes of simple $\mathbf{F}_p[\Gamma]$-modules are in bijection with the Galois orbits on $\mathrm{Hom}(\Gamma, \overline{\mathbf{F}}_p^\times)$. Given a character $\chi : \Gamma \to \overline{\mathbf{F}}_p^\times$, the corresponding simple module V_χ is isomorphic to the field generated over $\mathbf{F}_p$ by the values of χ, with the Γ-action through χ.

Proposition 2.4.12. [BH10, Proposition 3] If (V, h) is indecomposable, then exactly one of the following holds:

1. (V, h) is Γ-isometric to a hyperbolic space $\mathcal{H}(V_\chi)$ for $\chi^2 = 1$ or $G_{\mathbf{F}_p}\chi \neq G_{\mathbf{F}_p}\chi^{-1}$.
2. $V \cong V_\chi$ for a character such that $\chi^2 \neq 1$ and $G_{\mathbf{F}_p}\chi^{-1} = G_{\mathbf{F}_p}\chi$. This is the anisotropic case.

The Γ-isometry class of any symplectic module (V, h) is uniquely determined by the Γ-isomorphism class of V.

Proposition 2.4.13. [BH10, Proposition 4] Assume $\chi^2 \neq 1, G_{\mathbf{F}_p}\chi = G_{\mathbf{F}_p}\chi^{-1}$. Then $\mathbf{k} = \mathbf{F}_p[\chi]$ has p^{2d} elements, for some integer d. Let $\mathcal{T}_\mathbf{k} \subset \mathbf{k}^\times$ be the only subgroup of order $p^d + 1$, namely the kernel of the norm $\mathbf{k}^\times \to \mathbf{k}_0 = \mathbf{F}_{p^2}$. Then $\chi(\Gamma) \subset \mathcal{T}_\mathbf{k}$.

Now let V be a finite $\mathbf{F}_p[\Gamma]$-module and fix $\delta \in \Gamma$. We write $s_V(\delta)$ for the signature of the permutation of V issuing from the action of δ (it has a fixed point, namely $0 \in V$). This gives rise to a character $\Gamma \to \{\pm 1\}$. Now assume V is symplectic. We are going to define a constant $t^0_\Gamma(V) \in \{\pm 1\}$ and another character $t^1_\Gamma(V) : \Gamma \to \{\pm 1\}$.

Definition 2.4.14. Assume V is an indecomposable symplectic $\mathbf{F}_p[\Gamma]$-module. If $V \cong \mathcal{H}(U)$, then
$$t^0_\Gamma(V) = +1 \text{ and } t^1_\Gamma(V) = s_U.$$
If $V \cong V_\chi$ is anisotropic, let $|V| = p^{2d}$, and let $\mathcal{T}_\chi$ be defined as in the above. Then we put
$$t^0_\Gamma(V) = -1 \text{ and } t^1_\Gamma(V, \delta) = \begin{cases} +1 \text{ if } \chi(\delta) \in \mathcal{T}_\chi^2 \\ -1 \text{ otherwise.} \end{cases}$$

We extend this definition to all symplectic modules by taking the product of the invariants of their indecomposable summands.

Definition 2.4.15. Let V be a finite symplectic $\mathbf{F}_p[\Gamma]$-module. Then we define an invariant
$$t_\Gamma(V) = t^0_\Gamma(V) t^1_\Gamma(V, \gamma)$$
for any generator γ of Γ.

Since $t^1_\Gamma(V, -)$ is a quadratic character of Γ, and Γ has at most two quadratic characters, we see that this definition is independent of the choice of γ, and $t^1_\Gamma(V, \gamma)$ measures whether $t^1_\Gamma(V, -)$ is trivial or not. These refined invariants will usually be considered in conjunction with the following proposition.

Proposition 2.4.16. [BH10, Proposition 5] Let $\Delta \subset \Gamma$ be a subgroup such that $V^\Delta = V^\Gamma$. Then
$$t_\Delta(V) = t^0_\Gamma(V) t^1_\Gamma(V, \delta)$$
for any generator δ of Δ.

Now fix a maximal simple character θ in G, defined on a simple stratum $[\mathfrak{A}, \beta]$ in A. Let $B = Z_A(F[\beta])$ and $\mathfrak{B} = B \cap \mathfrak{A}$. Write η for the corresponding Heisenberg representation of $J^1(\beta, \mathfrak{A})$, and let K be a maximal unramified extension of $F[\beta]$ in B, normalizing $\mathfrak{A}$: this exists by the argument in proposition 2.4.5. Let $F \subset L \subset K$ be an intermediate field extension, unramified over F, write $A_L = Z_A(L)$, and recall from proposition 2.4.8 that we have equalities
$$J^1(\beta, \mathfrak{A}_L) = J^1(\beta, \mathfrak{A}) \cap A_L^\times \text{ and } H^1(\beta, \mathfrak{A}_L) = H^1(\beta, \mathfrak{A}) \cap A_L^\times,$$
and $\theta_L = \theta | H^1(\beta, \mathfrak{A}_L)$ is a simple character.

Lemma 2.4.17. θ_L is a maximal simple character.

Proof. Let Λ be the chain corresponding to $\mathfrak{A}$. It suffices to prove that the trace $\mathrm{tr}_{L[\beta]}\Lambda$ is a multiple of a uniform chain of $\mathcal{O}_{D_{L[\beta]}}$-period one. But $\mathrm{tr}_{L[\beta]}\Lambda$ is the continuation of $\mathrm{tr}_K\Lambda$, which is a multiple of a uniform chain of $\mathcal{O}_K$-period one, since $K \subset A$ is a maximal subfield. The claim follows by proposition 2.4.2 since $K/L[\beta]$ is unramified. $\square$

We introduce notation

$$N = \ker(\theta), Q = J^1_\theta/N, Z = H^1_\theta/N, V = J^1_\theta/H^1_\theta.$$

as well as

$$Q_L = J^1_{\theta_L}/\ker(\theta_L) \text{ and } V_L = J^1_{\theta_L}/H^1_{\theta_L}.$$

These are subgroups of Q and V respectively. The group $K^\times$ acts on Q and V, and centralizes Z.

Proposition 2.4.18. [BH10, Proposition 6] The fixed point subgroup of $L^\times$ on Q, resp. V is Q_L, resp. V_L.

Introduce the group $\Psi_{K/F} = K^\times/F^\times U^1_K$. The action of $K^\times$ on Q factors through $\Psi_{K/F}$, and V is a symplectic $\mathbf{F}_p[\Psi_{K/F}]$-module via the pairing

$$(x, y) \mapsto \theta(xyx^{-1}y^{-1}) \text{ for } x, y \in J^1.$$

Proposition 2.4.19 (Glauberman correspondence)**.** [BH10, Lemma 10] View η as a representation of Q. Then

1. there exists a unique irreducible representation $\widetilde{\eta}$ of $\Psi_{K/F} \ltimes Q$ such that

$$\widetilde{\eta}|Q \cong \eta \text{ and } \det \widetilde{\eta}|\Psi_{K/F} = 1.$$

2. let $\Gamma \subset \Psi_{K/F}$ be a cyclic subgroup. There is a unique irreducible representation η^Γ of Q^Γ such that $\eta^\Gamma|Z$ contains θ. There exists a constant $\epsilon_\theta(\Gamma) \in \{\pm 1\}$ such that

$$\mathrm{tr}\widetilde{\eta}(\gamma x) = \epsilon_\theta(\Gamma)\mathrm{tr}\eta^\Gamma(x)$$

for all $x \in Q^\Gamma$ and all generators γ of Γ.

3. assume Γ is the image of $L^\times$ in $\Psi_{K/F}$. Then the pullback of η^Γ through the canonical map $J^1_L \to Q_L = Q^\Gamma$ is isomorphic to the Heisenberg representation attached to θ_L.

Then the symplectic invariants we have defined compute the signs $\epsilon_\theta(\Gamma)$, in the following way.

Proposition 2.4.20. [BH10, Proposition 7] If $\Gamma \subset \Psi_{K/F}$ is a cyclic subgroup, then $\epsilon_\theta(\Gamma) = t_\Gamma(V)$.

If $x \in \mu_K$, let us write $\epsilon_\theta(x) = \epsilon_\theta(\langle x \rangle) = t_{\langle x \rangle}(V_\theta)$. Let us assume that x generates the field extension L/F. Then it need not be true that $\langle x \rangle = \mu_L$. However, we have the following lemma.

Lemma 2.4.21. The fixed points of x on V_θ coincide with those of μ_L.

Proof. The same argument proving that V^{μ_L} is the image of $J^1_\theta \cap Z_A(L)$ in V also implies that V^x is the image of $J^1_\theta \cap Z_A(x)$ in V_θ (see the proof of [BH10, Proposition 6]). Since $L = F[x]$, the claim follows. □

By this lemma together with proposition 2.4.16, we deduce that

$$t_{\langle x \rangle}(V) = t^0_{\mu_L}(V) t^1_{\mu_L}(V, x). \tag{2.4.2}$$

Proposition 2.4.22. Let $x \in \mu_K$ generate the field extension L/F. Then we have the equality

$$\epsilon_\theta(x) = t^0_{\mu_K}(V_{\theta_L}) t^0_{\mu_K}(V_\theta) t^1_{\mu_K}(V_\theta, x).$$

Proof. We have an orthogonal decomposition

$$V_\theta = V_\theta^{\mu_L} \perp (V_\theta^{\mu_L})^\perp$$

as symplectic $\mathbf{F}_p[\mu_K]$-modules. Indeed, the restriction of the symplectic form to $V_\theta^{\mu_L}$ is nondegenerate because the order of μ_L is prime to p.

Since by construction the invariants are multiplicative in V with respect to orthogonal sums, we deduce by (2.4.2) that

$$\epsilon_\theta(x) = t^0_{\mu_L}(V_\theta^{\mu_L}) t^1_{\mu_L}(V_\theta^{\mu_L}, x) t^0_{\mu_L}(V_\theta^{\mu_L,\perp}) t^1_{\mu_L}(V_\theta^{\mu_L,\perp}, x).$$

Since μ_L acts trivially on $V_\theta^{\mu_L}$, we deduce

$$\epsilon_\theta(x) = t^0_{\mu_L}(V_\theta^{\mu_L,\perp}) t^1_{\mu_L}(V_\theta^{\mu_L,\perp}, x). \tag{2.4.3}$$

On the orthogonal complement $V_\theta^{\mu_L,\perp}$, the fixed subspaces of the groups $\langle x \rangle, \mu_L, \mu_K$ are all trivial. Hence proposition 2.4.16 implies that

$$t_{\langle x \rangle}(V_\theta^{\mu_L,\perp}) = t^0_{\mu_L}(V_\theta^{\mu_L,\perp}) t^1_{\mu_L}(V_\theta^{\mu_L,\perp}, x)$$
$$t_{\langle x \rangle}(V_\theta^{\mu_L,\perp}) = t^0_{\mu_K}(V_\theta^{\mu_L,\perp}) t^1_{\mu_K}(V_\theta^{\mu_L,\perp}, x).$$

This implies that

$$t^0_{\mu_L}(V_\theta^{\mu_L,\perp}) t^1_{\mu_L}(V_\theta^{\mu_L,\perp}, x) = t^0_{\mu_K}(V_\theta^{\mu_L,\perp}) t^1_{\mu_K}(V_\theta^{\mu_L,\perp}, x).$$

Multiply both sides of this equation by $t^0_{\mu_K}(V_{\theta_L})t^1_{\mu_K}(V_{\theta_L}, x)$, and recall that $V_{\theta_L} = V_\theta^{\mu_L}$. By (2.4.3), the resulting equality is

$$t^0_{\mu_K}(V_{\theta_L})t^1_{\mu_K}(V_{\theta_L}, x)\epsilon_\theta(x) = t^0_{\mu_K}(V_\theta)t^1_{\mu_K}(V_\theta, x).$$

There remains to notice that, by construction, the character $t^1_{\mu_K}(V_{\theta_L}, -)$ is the inflation to μ_K of $t^1_{\mu_K/\mu_L}(V^{\mu_L}, -)$, hence takes value 1 at $x \in \mu_L$. □

Definition 2.4.23. We write $\epsilon^0_\theta = \epsilon^0_{\mu_K}(V_\theta)$ and $\epsilon^1_\theta(-) = \epsilon^1_{\mu_K}(V_\theta, -)$. Hence we have an equality

$$\epsilon_\theta(x) = \epsilon^0_{\theta_L}\epsilon^0_\theta\epsilon^1_\theta(x)$$

whenever $x \in \mu_K$ generates L over F.

Remark 2.4.24. We will sometimes abuse notation and identify ϵ^1_θ with a character of any group isomorphic to μ_K. Since a cyclic group has at most one nontrivial quadratic character, there is no ambiguity in doing so.

2.5 Representation theory of $\mathrm{GL}_n(\mathbf{F}_q)$.

In this section we recall the combinatorial classification, in terms of partitions, of the irreducible R-representations of $\mathrm{GL}_n(\mathbf{F}_q)$ with simple supercuspidal support, following [SZ99, Sections 3, 4]. Then we give a construction, in terms of Deligne–Lusztig theory, of certain virtual representations which enjoy special properties with respect to this classification, and which will appear in the construction of the element of $R_{\overline{\mathbf{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$ corresponding under the Breuil–Mézard conjecture to the maximal stratum of a discrete series deformation ring.

Until the discussion of modular representations, we assume that R has characteristic zero. We write $G = \mathrm{GL}_n(\mathbf{F}_q)$.

Harish–Chandra series of simple representations. Every irreducible $R[G]$-module has a supercuspidal support, which is unique up to conjugacy. As in the previous section, we identify it with an effective divisor on the set of supercuspidal representations of $\mathrm{GL}_m(\mathbf{F}_q)$ for m varying in $\mathbf{Z}_{>0}$.

The *simple* supercuspidal supports are those of the form $r\pi_0$. Amongst the representations with $sc(\pi) = r\pi_0$, there exists a unique nondegenerate one, denoted $\mathrm{St}(\pi_0, r)$. To classify the others, we consider partitions $\mathfrak{P}$ of r, and to each $\mathfrak{P}$ we associate a block-diagonal Levi subgroup

$$L_\mathfrak{P}(\pi_0) = \prod_{i \in \mathbf{Z}_{>0}} \mathrm{GL}_{ni/r}(\mathbf{F}_q)^{\times \mathfrak{P}(i)}$$

and a parabolically induced representation of $\mathrm{GL}_n(\mathbf{F}_q)$

$$\pi_{\mathfrak{P}}(\pi_0) = \mathrm{Ind}_{P_{\mathfrak{P}}}^{G} \bigotimes_{i \in \mathbf{Z}_{>0}} \mathrm{St}(\pi_0, i)^{\otimes \mathfrak{P}(i)}.$$

The partition $\mathfrak{P}_{\max}$ sending 1 to r and every other positive integer to 0 corresponds to writing r as a sum of 1. The representation $\pi_{\mathfrak{P}_{\max}}(\pi_0)$ is the full parabolic induction of $\pi_0^{\otimes r}$.

Definition 2.5.1. The Harish-Chandra series corresponding to $r\pi_0$ is the set of irreducible representations of G with supercuspidal support $r\pi_0$. It coincides with the set of Jordan–Hölder factors of $\pi_{\mathfrak{P}_{\max}}(\pi_0)$.

Write $\mathfrak{P}' \leq \mathfrak{P}$ for the reverse of the dominance partial order on partitions, as in [SZ99]. Then $\mathfrak{P}_{\max}$ is the maximal element amongst partitions of r.

Proposition 2.5.2. [SZ99, Section 4] There is a bijection $\mathfrak{P} \mapsto \sigma_{\mathfrak{P}}(\pi_0)$ from the set of partitions of r to the Harish-Chandra series for $r\pi_0$, characterized by the fact that $\sigma_{\mathfrak{P}}(\pi_0)$ occurs in $\pi_{\mathfrak{P}'}(\pi_0)$ if and only if $\mathfrak{P} \leq \mathfrak{P}'$, and it occurs in $\pi_{\mathfrak{P}}(\pi_0)$ with multiplicity one.

The smallest element amongst partitions of r is $\mathfrak{P}_{\min}$ sending r to 1 and every other positive integer to 0. We have $\sigma_{\mathfrak{P}_{\min}}(\pi_0) = \pi_{\mathfrak{P}_{\min}}(\pi_0) = \mathrm{St}(\pi_0, r)$.

Definition 2.5.3. When $\mathfrak{P} \leq \mathfrak{P}'$, the multiplicity of $\sigma_{\mathfrak{P}}(\pi_0)$ in $\pi_{\mathfrak{P}'}(\pi_0)$ is the *Kostka number* $K_{\mathfrak{P},\mathfrak{P}'}$.

We remark that our normalization coincides with [Sho18, Definition 6.2], since the partial order there is the reverse of $\leq$.

Lusztig induction. We follow the presentation of Deligne–Lusztig theory in [DM91] and [Dud]. The material here is mostly standard, but we need certain results about products and Weil restriction of scalars that are probably well-known but we couldn't find in the literature (although [Lus77, 1.18] is closely related). So we have decided to provide the proofs.

Let $\mathbf{G}_0$ be a connected reductive group over $\mathbf{k} = \mathbf{F}_q$, fix an algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$, and write $\mathbf{G} = \mathbf{G}_0 \times_{\mathbf{k}} \overline{\mathbf{k}}$. The rational structure gives rise to a $\overline{\mathbf{k}}$-linear Frobenius endomorphism F of $\mathbf{G}$, the pullback of the absolute q-th power Frobenius morphism of $\mathbf{G}_0$. The Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts to the right on $\mathbf{G}$, via $\mathbf{F}_q$-linear automorphisms. We write φ for the geometric Frobenius element of the Galois group, acting as $x \mapsto x^{1/q}$. If $\mathbf{H}$ is a subgroup of $\mathbf{G}$ we will write $F\mathbf{H}$ for the parabolic subgroup $\varphi(\mathbf{H})$ of $\mathbf{G}$, whose group of $\overline{\mathbf{k}}$-points is $F(\mathbf{H}(\overline{\mathbf{k}}))$. We will say that $\mathbf{H}$ is *F-stable*, or *rational*, if $F\mathbf{H} = \mathbf{H}$. Recall from [DM91, Chapter 8] the invariant $\epsilon_{\mathbf{G}_0} = (-1)^{\eta(\mathbf{G}_0)}$, where $\eta(\mathbf{G}_0)$ is the dimension of the maximal split subtorus of any quasisplit maximal torus in $\mathbf{G}_0$ (the quasisplit maximal tori are those contained in a rational Borel subgroup).

Fix a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, with unipotent radical $\mathbf{U}$ and F-stable Levi factor $\mathbf{L}$ (without assuming that $\mathbf{P}$ is F-stable). The associated Deligne–Lusztig varieties can be defined in terms of the Lang isogeny

$$\mathcal{L} : \mathbf{G} \to \mathbf{G}, x \mapsto x^{-1}F(x)$$

by setting

$$\begin{aligned}\mathbf{X}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} &= \mathcal{L}^{-1}(F\mathbf{P})/(\mathbf{P}\cap F\mathbf{P})\\ \mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} &= \mathcal{L}^{-1}(F\mathbf{U})/(\mathbf{U}\cap F\mathbf{U}).\end{aligned}$$

Both varieties have an action of $\mathbf{G}^F \cong \mathbf{G}(\mathbf{k})$ by left multiplication, and $\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ has an action of $\mathbf{L}^F \cong \mathbf{L}(\mathbf{k})$ by right multiplication. We write $H^*_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}})$ for the alternating sum $\sum_{i\in\mathbf{Z}}(-1)^i[H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}, \overline{\mathbf{Q}}_l)]$ of compactly supported l-adic cohomology groups, for a prime number $l \neq p$. Each cohomology group carries a left action of $\mathbf{G}^F$ and a right action of $\mathbf{L}^F$. The associated Lusztig induction functor is

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : R_{\overline{\mathbf{Q}}_l}(\mathbf{L}^F) \to R_{\overline{\mathbf{Q}}_l}(\mathbf{G}^F),\ [V] \mapsto H^*_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}) \otimes_{\overline{\mathbf{Q}}_l[\mathbf{L}^F]} V.$$

On characters, we have the formula [DM91, Proposition 4.5]

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\theta)(g) = |\mathbf{L}^F|^{-1}\sum_{l\in\mathbf{L}^F}\sum_{i\in\mathbf{Z}}(-1)^i\mathrm{tr}\left((g,l)|H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}, \overline{\mathbf{Q}}_l)\right)\theta(l^{-1}).$$

Remark 2.5.4. Since $\mathbf{U}\cap F\mathbf{U}$ is an affine space we obtain the same induction functor via the bimodule $H^*_c(\mathcal{L}^{-1}(F\mathbf{U}))$. This is the functor denoted $R^{\mathbf{G}}_{\mathbf{L}\subset F\mathbf{P}}$ in [DM91], since their $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is constructed via $H^*_c(\mathcal{L}^{-1}(\mathbf{U}))$.

When $\mathbf{L}$ is a maximal torus, there is another description of Lusztig induction via the Bruhat decomposition of $\mathbf{G}$. Fix a pair $(\mathbf{T},\mathbf{B})$ consisting of an F-stable maximal torus and an F-stable Borel subgroup of $\mathbf{G}$. By [DL76, Lemma 1.13], there is a bijection between the $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{B}',\mathbf{T}')$ consisting of a Borel subgroup of $\mathbf{G}$ and a rational maximal torus of $\mathbf{B}'$, and the Weyl group $W(\mathbf{T})$, given by the map $(g\mathbf{B}g^{-1}, g\mathbf{T}g^{-1}) \mapsto g^{-1}F(g)$ (here $g \in \mathbf{G}(\overline{\mathbf{k}})$). The F-conjugacy classes in $W(\mathbf{T})$ are the equivalence classes for $x \sim gxF(g^{-1})$, and they classify $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori in $\mathbf{G}$ by [DL76, Corollary 1.14]. For w in $W(\mathbf{T})$, we write $\mathbf{T}_w$ for an F-stable maximal torus in $\mathbf{G}$ classified by the F-conjugacy class of w, and we say that w is the *type* of $\mathbf{T}_w$.

The Bruhat decomposition for $\mathbf{G}$ is $\mathbf{G} = \bigsqcup_{w\in W(\mathbf{T})}\mathbf{B}\dot{w}\mathbf{B}$ for any choice of representatives $\dot{w}$ of $W(\mathbf{T})$ in $\mathbf{G}$ (it is independent of the choice of $\dot{w}$). The quotient $\mathbf{B}w\mathbf{B}/\mathbf{B}$ is a Schubert cell in the flag variety $\mathbf{G}/\mathbf{B}$, and there is an associated Deligne–Lusztig variety

$$\mathbf{X}(w) = (\mathcal{L}^{-1}(\mathbf{B}w\mathbf{B}))/\mathbf{B}$$

together with a covering

$$\mathbf{Y}(\dot{w}) = (\mathcal{L}^{-1}(\mathbf{U}\dot{w}\mathbf{U}))/\mathbf{U}$$

induced by the canonical surjection $\mathbf{G}/\mathbf{U} \to \mathbf{G}/\mathbf{B}$. Both varieties have a left multiplication action by $\mathbf{G}^F$. If we equip $\mathbf{T}$ with the twisted Frobenius endomorphism $wF : t \mapsto wF(x)w^{-1}$, then the group of fixed points $\mathbf{T}^{wF}$ acts by right multiplication on $\mathbf{Y}(\dot{w})$. One checks as in [DL76, 1.8] that the isomorphism class of this covering, together with the action of $\mathbf{T}^{wF}$ and $\mathbf{G}^F$, is independent of the choice of $\dot{w}$.

Now consider a pair $(\mathbf{B}', \mathbf{T}')$ consisting of a Borel subgroup of $\mathbf{G}$ and a rational maximal torus of $\mathbf{B}'$, classified as in the above by some $w \in W(\mathbf{T})$. By [DL76, Proposition 1.19] whenever we have $x \in \mathbf{G}$ with $(\mathbf{B}', \mathbf{T}') = x(\mathbf{B}, \mathbf{T})x^{-1}$ and $\mathcal{L}(x) = \dot{w}$, the map $g \mapsto gx^{-1}$ induces an isomorphism $\mathbf{Y}(\dot{w}) \to \mathbf{Y}^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}'}$ that is equivariant for the isomorphism $\mathrm{ad}(x) : \mathbf{T}^{wF} \to (\mathbf{T}')^F$ and $\mathbf{G}^F$-equivariant.

It follows that we can attach to each element $w \in W(\mathbf{T})$ an induction map

$$R_w : R_{\overline{\mathbf{Q}}_l}(\mathbf{T}^{wF}) \to R_{\overline{\mathbf{Q}}_l}(\mathbf{G}^F)$$

via the cohomology $H^*_c(\mathbf{Y}(\dot{w}))$ for any representative $\dot{w}$ of w.

Restriction of scalars and products. We study the behaviour of the maps R_w with respect to Weil restriction of scalars and products. Let $\mathbf{G}_0$ be a connected reductive group over $\mathbf{k}$, and let $\mathbf{G} = \mathbf{G}_0 \times_{\mathbf{k}} \overline{\mathbf{k}}$, $\mathbf{G}_n = \mathbf{G}_0 \times_{\mathbf{k}} \mathbf{k}_n$, and

$$\mathbf{G}_0^+ = \mathrm{Res}_{\mathbf{k}_n/\mathbf{k}}\left(\mathbf{G}_0 \times_{\mathbf{k}} \mathbf{k}_n\right).$$

The base change $\mathbf{G}^+ = \mathbf{G}_0^+ \times_{\mathbf{k}} \overline{\mathbf{k}}$ is isomorphic to a product $\prod_{i=1}^n \mathbf{G}$, and its Frobenius endomorphism acts (on R-points, for any $\overline{\mathbf{k}}$-algebra R) by

$$(g_1, \ldots, g_m) \mapsto (F(g_m), F(g_1), \ldots, F(g_{m-1}))$$

where the map $F : \mathbf{G}(R) \to \mathbf{G}(R)$ is the Frobenius endomorphism for the $\mathbf{k}$-structure $\mathbf{G}_0$ (hence the one for the $\mathbf{k}_n$-structure $\mathbf{G}_n$ is F^n). Notice that projection on the first factor $(\mathbf{G}^+)^F \to \mathbf{G}^{F^n}$ is an isomorphism. We fix an F^n-stable pair $(\mathbf{B}, \mathbf{T})$ in $\mathbf{G}$ and work with the F-stable pair $(\mathbf{B}^+, \mathbf{T}^+) = (\prod_{i=1}^n \mathbf{B}, \prod_{i=1}^n \mathbf{T})$ in $\mathbf{G}^+$. Then there is an inclusion $\iota : W(\mathbf{T}) \to W(\mathbf{T}^+), w \mapsto (w, 1, \ldots, 1)$, inducing a bijection on F-conjugacy classes. Indeed, we see that $(w, 1, \ldots, 1)$ and $(xwF^n(x^{-1}), 1, \ldots, 1)$ are F-conjugates by

$(x, F(x), \ldots, F^{n-1}(x))$, and given an arbitrary $x = (x_1, \ldots, x_n)$ we can always find $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha x \alpha^{-1}$ is in the image of ι: it suffices to choose α_1 arbitrarily and to solve the equations $\alpha_i x_i F(\alpha_{i-1})^{-1} = 1$ recursively, for $2 \leq i \leq n$.

Lemma 2.5.5. Let $w \in W(\mathbf{T})$. There is an isomorphism $(\mathbf{T}^+)^{\iota(w)F} \to \mathbf{T}^{wF^n}$ identifying R_w and $R_{\iota w}$.

Proof. The isomorphism is again projection on the first factor. Indeed, the fixed points are given by $(t, F(t), \ldots, F^{n-1}(t))$ with the property that $t = wF^n(t)w^{-1}$. For the identification of Lusztig functors, we have that the Bruhat cell $\mathbf{B}^+\iota(w)\mathbf{B}^+$ decomposes as a product $\mathbf{B}w\mathbf{B} \times \mathbf{B} \times \cdots \times \mathbf{B}$, and so the preimage $\mathcal{L}^{-1}(\mathbf{B}^+\iota(w)\mathbf{B}^+)$ is given on $\overline{\mathbf{k}}$-points by

$$(g, F(g)b_1, \ldots, F^{m-1}(g)b_{m-1})$$

for arbitrary $b_i \in \mathbf{B}(\overline{\mathbf{k}})$ and $g \in \mathbf{G}(\overline{\mathbf{k}})$ such that $g^{-1}F^m(g) \in \mathbf{B}w\mathbf{B}$. A similar calculation works for the unipotent groups, after choosing a representative $\dot{w}$ of w and the corresponding representative $(\dot{w}, 1 \ldots, 1)$ of $\iota(w)$. It follows that projection onto the first component induces an isomorphism $\mathbf{Y}(\dot{w}, 1, \ldots, 1) \to \mathbf{Y}(\dot{w})$, which is equivariant with respect to our isomorphisms $(\mathbf{G}^+)^F \to \mathbf{G}^{F^n}$ and $(\mathbf{T}^+)^{\iota(w)F} \to \mathbf{T}^{wF^n}$. □

Lemma 2.5.6. For $i \in \{1, \ldots, n\}$, fix connected reductive groups $\mathbf{G}_{0,i}$ over $\mathbf{k}$, pairs $(\mathbf{B}_i, \mathbf{T}_i)$ in $\mathbf{G}_i$, and elements $w_i \in W(\mathbf{T}_i)$. Let $\mathbf{G}_0 = \prod_i \mathbf{G}_{0,i}$ with $(\mathbf{B}, \mathbf{T}) = (\prod_i \mathbf{B}_i, \prod_i, \mathbf{T}_i)$, and $\dot{w} = \dot{w}_1 \times \cdots \times \dot{w}_n$. Then $R_w : R_{\overline{\mathbf{Q}}_l}(\prod_i \mathbf{T}_i^F) \to R_{\overline{\mathbf{Q}}_l}(\prod_i \mathbf{G}_i^F)$ sends a one-dimensional character $\chi_1 \cdots \chi_n$ to $R_{w_1}(\chi_1) \cdots R_{w_n}(\chi_n)$.

Proof. As in the the proof of lemma 2.5.5 we have an equivariant isomorphism $\mathbf{Y}(\dot{w}) \to \prod_i \mathbf{Y}(\dot{w}_i)$, and the claim follows from the Künneth formula for the cohomology of $\mathbf{Y}(\dot{w})$. □

A CHARACTER FORMULA. We now specialize to the case of $\mathbf{G}_0 = \mathrm{GL}_{n,\mathbf{k}}$, with $\mathbf{B}$ the upper triangular Borel subgroup and $\mathbf{T}$ the diagonal torus. The Weyl group $W(\mathbf{T})$ identifies with the symmetric group S_n, the F-conjugacy classes coincide with the conjugacy classes, and we normalize the lifts $\dot{w}$ via permutation matrices. We give a formula for the Lusztig induction map corresponding to the Weyl group element $w = (12 \ldots n)$, on semisimple conjugacy classes. The group $\mathbf{T}^{wF}$ is isomorphic to $\mathbf{k}_n^\times$. Choosing a basis of $\mathbf{k}_n$ as a $\mathbf{k}$-vector space yields an inclusion $\mathrm{Res}_{\mathbf{k}_n/\mathbf{k}}\mathbb{G}_m \to \mathrm{GL}_{n,\mathbf{k}}$ which represents the $\mathbf{G}^F$-conjugacy class of rational maximal tori classified by w. These tori all have the same ϵ-invariant, which we'll denote by ϵ_w. Notice that in our case the signs $\epsilon_{\mathbf{G}_0} = (-1)^n$ and $\epsilon_w = -1$. The following proposition is a very special case of the Lusztig classification we discuss in the following, but it is historically prior to it, and can be proved by different means. See for instance [BH10, 2.1].

Proposition 2.5.7. Assume that $\chi : \mathbf{k}_n^\times \to \overline{\mathbf{Q}}_l^\times$ is a $\mathrm{Gal}(\mathbf{k}_n/\mathbf{k})$-regular character. Then the function $(-1)^{n-1}R_w(\chi)$ is the character of an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbf{k})$, denoted $\sigma[\chi]$. The map $\chi \mapsto \sigma[\chi]$ induces a bijection from the set of orbits of $\mathrm{Gal}(\mathbf{k}_n/\mathbf{k})$ on the $\mathbf{k}$-regular characters of $\mathbf{k}_n^\times$, to the irreducible cuspidal representations of $\mathrm{GL}_n(\mathbf{k})$ over $\overline{\mathbf{Q}}_l$.

Remark 2.5.8. It is not always the case that $(-1)^{n-1}R_w(\chi)$ is effective, if χ is not regular. However, these virtual representations will be important for us, since they will give rise to the Breuil–Mézard cycles of discrete series deformation rings. We will sometimes abuse notation and write $\sigma[\chi]$ for $(-1)^{n-1}R_w(\chi)$, even if χ is not regular.

In the next proposition, we compute the character $(-1)^{n-1}R_w(\chi)$ on semisimple classes, generalizing a well-known calculation in the case of $\mathbf{k}$-regular χ.

Proposition 2.5.9. Let $\chi : \mathbf{k}_n^\times \to \overline{\mathbf{Q}}_l^\times$ be a character, and let $w = (12\dots n)$. Then $\sigma[\chi]$ vanishes on semisimple conjugacy classes of $\mathrm{GL}_n(\mathbf{k})$ not represented in $\mathbf{k}_n^\times$, and for $x \in \mathbf{k}_n^\times$ we have

$$\sigma[\chi](x) = (-1)^{n+n/\deg(x)}(\mathrm{GL}_{n/\deg(x)}(\mathbf{k}_{\deg(x)}) : \mathbf{k}_n^\times)_{p'} \sum_{\gamma \in \mathrm{Gal}(\mathbf{k}_{\deg(x)}/\mathbf{k})} \chi(\gamma x)$$

where $\deg(x)$ is the degree of x over $\mathbf{k}$.

Proof. By [Car85, Proposition 7.4], we have the equality

$$\epsilon_{\mathbf{G}_0}\epsilon_w R_w(\chi)\mathrm{St}_{\mathbf{G}_0} = \mathrm{Ind}_{\mathbf{k}_n^\times}^{\mathrm{GL}_n(\mathbf{k})}(\chi)$$

where $\mathrm{St}_{\mathbf{G}_0}$ is the Steinberg character and the induction is taken with respect to the embedding of $\mathbf{k}_n^\times$ in $\mathrm{GL}_n(\mathbf{k})$ corresponding to some $\mathbf{k}$-basis of $\mathbf{k}_n$. By [DM91, 9.3 corollary], the Steinberg character vanishes away from semisimple classes, and if x is a semisimple element of $\mathrm{GL}_n(\mathbf{k})$ then

$$\mathrm{St}_{\mathbf{G}_0}(x) = \epsilon_{\mathbf{G}_0}\epsilon_{Z^+_{\mathbf{G}}(x)}|Z^+_{\mathbf{G}^F}(x)|_p$$

where $Z^+_{\mathbf{G}}(x)$ is the centralizer of x, a connected reductive group over $\mathbf{k}$.

Hence $R_w(\chi)(x) = 0$ if x is a semisimple element with no conjugates in $\mathbf{k}_n^\times$. When $x \in \mathbf{k}_n^\times$, we compute the character of the induction as

$$\mathrm{Ind}_{\mathbf{k}_n^\times}^{\mathrm{GL}_n(\mathbf{k})}(\chi)(x) = |Z^+_{\mathbf{G}^F}(x)||\mathbf{k}_n^\times|^{-1} \sum_{\gamma \in \mathrm{Gal}(\mathbf{k}_{\deg(x)}/\mathbf{k})} \chi(\gamma x)$$

since the $\mathbf{G}^F$-conjugates of x in $\mathbf{k}_n^\times$ are precisely its Galois conjugates. The centralizer is isomorphic to $\mathrm{GL}_{n/\deg(x)}(\mathbf{k}_{\deg(x)})$. Then the claim follows since $\mathrm{Res}_{\mathbf{k}_{\deg(x)}/\mathbf{k}}\mathbb{G}_m^{\times n/\deg(x)}$ is a quasisplit maximal torus in $\mathrm{Res}_{\mathbf{k}_{\deg(x)}/\mathbf{k}}\mathrm{GL}_{n/\deg(x)}$ of rational rank $\frac{n}{\deg(x)}$. □

Remark 2.5.10. The formula in proposition 2.5.9 characterizes the map $\chi \mapsto \sigma[\chi]$, from $\overline{\mathbf{Q}}_l$-characters of $\mathbf{k}_n^\times$ to virtual $\overline{\mathbf{Q}}_l$representations of $\mathrm{GL}_n(\mathbf{k})$, because of [SZ00, Theorem 1.1]. It follows that for any field R of characteristic zero containing all roots of unity of order dividing the exponent of $\mathrm{GL}_n(\mathbf{k})$ there exists a unique map $\chi \mapsto \sigma[\chi]$, from R-characters of $\mathbf{k}_n^\times$ to R-representations of $\mathrm{GL}_n(\mathbf{k})$, that satisfies the same character identity. It induces a bijection from regular R-characters to irreducible supercuspidal R-representations. We will sometimes abuse notation and refer to $\sigma[\chi]$ as the Deligne–Lusztig induction of χ, even if strictly speaking, we are not working with cohomology with R-coefficients.

Unipotent characters. Let $\chi : W(\mathbf{T}) \to \overline{\mathbf{Q}}_l$ be the character of an irreducible representation. By [DM91, Theorem 15.8], the unipotent characters of $\mathbf{G}^F$ are the functions

$$A_\chi = |W(\mathbf{T})|^{-1} \sum_{w \in W(\mathbf{T})} \chi(w) R_w(1_{\mathbf{T}^{wF}})$$

for varying χ. Notice that the maps R_{w_i} for $w_2 = ww_1w^{-1}$ are intertwined by the isomorphism $\mathrm{ad}(\dot{w}) : \mathbf{T}^{w_1F} \to \mathbf{T}^{w_2F}$, since for an arbitrary F-stable maximal torus $\mathbf{T}$ the map $R^{\mathbf{G}}_{\mathbf{T}\subset\mathbf{B}}$ does not depend on the choice of Borel subgroup containing $\mathbf{T}$ (see [DM91, Corollary 11.15]). By orthogonality of Deligne–Lusztig characters we deduce that

$$(R_w(1_{\mathbf{T}^{wF}}), A_\chi)_{\mathbf{G}^F} = \chi(w), \tag{2.5.1}$$

and so

$$R_w(1_{\mathbf{T}^{wF}}) = \sum_{\chi \in \mathrm{Irr}(W(\mathbf{T}))} \chi(w) A_\chi$$

since the unipotent characters form an orthonormal family. Since R_{id} coincides with the functor of parabolic induction from $\mathbf{T}^F$, we see that the unipotent characters are the characters of the irreducible representations with supercuspidal support $n \cdot 1_{\mathbf{k}^\times}$. Further, by [DM91, Proposition 12.13] we have that A_{triv} is the trivial character of $\mathbf{G}^F$. It follows from our discussion of Harish-Chandra series that $\sigma_{\mathfrak{P}_{\min}}(1) = \mathrm{St}(1, n)$ is the only other factor of $R_{\mathrm{id}}(\mathrm{triv})$ with multiplicity one. This is A_{sgn}, where $\mathrm{sgn} : W(\mathbf{T}) \to \overline{\mathbf{Q}}_l^\times$ is the sign character.

Lusztig series. Recall that two pairs $(\mathbf{T}_i, \theta_i)$ consisting of a rational maximal torus in $\mathbf{G}$ and a character of $\mathbf{T}_i^F$ are said to be *geometrically conjugate* if there exists $g \in \mathbf{G}(\overline{\mathbf{k}})$ such that $\mathbf{T}_2 = \mathrm{ad}(g)\mathbf{T}_1$ and, for all n such that $F^n(g) = g$, we have

$$\theta_1 \circ N_{\mathbf{k}_n/\mathbf{k}} = \theta_2 \circ N_{\mathbf{k}_n/\mathbf{k}} \circ \mathrm{ad}(g).$$

Here, the norm of an F-stable torus $\mathbf{S}$ is defined to be the morphism

$$N_{\mathbf{k}_n/\mathbf{k}} : \mathbf{S} \to \mathbf{S}, t \mapsto tF(t)\cdots F^{n-1}(t),$$

and we are asking for equality to hold on $\mathbf{T}_i^{F^n}$. For $w \in S_n$, we write N_w for the $\mathbf{k}_n/\mathbf{k}$-norm of the diagonal torus with Frobenius endomorphism wF.

By Langlands duality, we associate to each geometric conjugacy class $[\mathbf{S},\theta]$ a semisimple conjugacy class in $\mathbf{G}^F = \mathrm{GL}_n(\mathbf{k})$. To do so, fix norm-compatible generators ζ_n of every $\mathbf{k}_n^\times$, and fix an embedding $\overline{\mathbf{k}}^\times \to \overline{\mathbf{Q}}_l^\times$. Since all maximal tori of $\mathrm{GL}_{n,\mathbf{k}}$ split over $\mathbf{k}_n$, by [DM91, Proposition 13.7] there exists an exact sequence

$$0 \to Y(\mathbf{S}) \to Y(\mathbf{S}) \to \mathbf{S}^F \to 0$$

where $Y(\mathbf{S})$ is the cocharacter group of $\mathbf{S}$, the first arrow is $F-1$ and the second arrow is $y \mapsto N_{\mathbf{k}_n/\mathbf{k}}(y(\zeta_n))$. Inflating θ we obtain a character $\theta^+ : Y(\mathbf{S}) \to \overline{\mathbf{Q}}_l^\times$ trivial on $(F-1)Y(\mathbf{S})$. This identifies the geometric conjugacy classes with the $\mathbf{G}$-conjugacy classes of pairs $(\mathbf{S},\theta^+)$ consisting of a rational maximal torus and a character of $Y(\mathbf{S})$ trivial on $(F-1)Y(\mathbf{S})$ (see [DM91, Proposition 13.8]). These are in bijection with the $W(\mathbf{T})$-orbits on the set of characters $Y(\mathbf{T}) \to \overline{\mathbf{Q}}_l^\times$ that are trivial on $(wF-1)Y(\mathbf{T})$ for some $w \in W(\mathbf{T})$, hence descend to $\mathbf{T}^{wF}$ (recall that $\mathbf{T}$ is the diagonal torus, and that the action of $W(\mathbf{T})$ on the cocharacter group is $wy : z \mapsto wy(z)w^{-1}$ while that of F is $Fy = F \circ y$).

Fix dual bases (e_i) and (f_i) of $X(\mathbf{T})$ and $Y(\mathbf{T})$ over $\mathbf{Z}$, such that $f_j(x)$ is the diagonal matrix with x in the j-th place and 1 elsewhere on the diagonal. Via the inclusion $\overline{\mathbf{k}}^\times \to \overline{\mathbf{Q}}_l^\times$ we obtain from θ^+ an element $(\theta^+(f_i))$ of $X(\mathbf{T}) \otimes_{\mathbf{Z}} \overline{\mathbf{k}}^\times \cong (\overline{\mathbf{k}}^\times)^{\times n}$. If θ^+ is trivial on $(wF-1)Y(\mathbf{T})$, then F acts as w^{-1} on this element, and we find a bijection from the geometric conjugacy classes to the F-stable $W(\mathbf{T})$-conjugacy classes in $X(\mathbf{T}) \otimes_{\mathbf{Z}} \overline{\mathbf{k}}^\times$. There is a Langlands duality isomorphism $X(\mathbf{T}) \to Y(\mathbf{T})$ sending roots to coroots, and we obtain an F-stable $W(\mathbf{T})$-orbit in $Y(\mathbf{T}) \otimes_{\mathbf{Z}} \overline{\mathbf{k}}^\times \cong \mathbf{T}(\overline{\mathbf{k}})$, corresponding to an F-stable semisimple conjugacy class in $\mathrm{GL}_n(\overline{\mathbf{k}})$. Since an F-stable conjugacy class always contains rational points, this determines a unique geometric conjugacy class of $\mathrm{GL}_n(\mathbf{k})$, which for this group is the same as a conjugacy class. This constructs a bijection between geometric conjugacy classes of pairs $(\mathbf{S},\theta)$ in $\mathbf{G}$ and semisimple conjugacy classes in $\mathbf{G}^F \cong \mathrm{GL}_n(\mathbf{k})$.

Example 2.5.11. A pair $(\mathbf{S},\theta)$ such that $\mathbf{S}$ has type $w = (1,2,\dots,n)$ corresponds to $(\theta(\zeta_n), \theta(\zeta_n)^q, \dots, \theta(\zeta_n)^{q^{n-1}}) \in X(\mathbf{T}) \otimes_{\mathbf{Z}} \overline{\mathbf{k}}^\times$, where $\theta(\zeta_n) \in \mathbf{k}_n$. This corresponds to the conjugacy class in $\mathrm{GL}_n(\mathbf{k})$ whose characteristic polynomial is the appropriate power of the minimal polynomial of $\theta(\zeta_n)$ over $\mathbf{k}$.

By [DM91, Propositions 3.1, 3.3], two virtual characters $R_{w_i}(\theta_i)$ admit a common constituent if and only if the $(\mathbf{T}_{w_i},\theta_i)$ are geometrically conjugate, and every irreducible character of $\mathbf{G}^F$ is a constituent of some $R_w(\theta)$. It follows that the geometric conjugacy classes partition the set of irreducible characters of $\mathbf{G}^F$, and a class in this partition is the *Lusztig series* of the corresponding semisimple conjugacy class s in $\mathbf{G}^F$, denoted $\mathcal{E}(\mathbf{G}^F, s)$. The unipotent characters form the Lusztig series $\mathcal{E}(\mathbf{G}^F, [1])$. We record the following theorem, which implies that in certain cases Lusztig induction preserves irreducibility. We will apply it in the next paragraph.

Theorem 2.5.12. [DM91, Theorem 13.25] Let s be a semisimple element of $\mathrm{GL}_n(\mathbf{k})$, and let $\mathbf{L}$ be a rational Levi subgroup of $\mathbf{G}$ containing the centralizer $Z_{\mathbf{G}}(s)$. Then the map $\epsilon_{\mathbf{G}}\epsilon_{\mathbf{L}}R_{\mathbf{L}}^{\mathbf{G}}$ (taken with respect to any parabolic $\mathbf{P}$ with Levi factor $\mathbf{L}$) induces a bijection $\mathcal{E}(\mathbf{L}^F, [s]_{\mathbf{L}^F}) \to \mathcal{E}(\mathbf{G}^F, [s]_{\mathbf{G}^F})$.

Virtual representations. Let m be a positive divisor of n and let π_m be an irreducible supercuspidal representation of $\mathrm{GL}_m(\mathbf{k})$. Since the matrix of Kostka numbers is upper unitriangular, it follows from the structure of the $\pi_{\mathfrak{P}}(\pi_m)$ that they form a basis for the Grothendieck group of finite length representations of $\mathrm{GL}_n(\mathbf{k})$ all of whose factors have supercuspidal support $(n/m)\pi_m$. Then for any partition $\mathfrak{P}$ of n/m there exists an element $\sigma^+_{\mathfrak{P}}(\pi_m)$ of this Grothendieck group such that

$$\begin{aligned}(\sigma^+_{\mathfrak{P}}(\pi_m), \pi_{\mathfrak{P}'}(\pi_m))_{\mathrm{GL}_n(\mathbf{k})} &= 1 \text{ if } \mathfrak{P} = \mathfrak{P}' \\ &= 0 \text{ otherwise.}\end{aligned}$$

We now give an explicit construction of $\sigma^+_{\mathfrak{P}_{\min}}(\pi_m)$ in terms of Deligne–Lusztig theory.

Theorem 2.5.13. Let w be an n-cycle in S_n, and assume $\pi_m \cong \sigma[\theta_m]$ for a character $\theta_m : \mathbf{k}_m^\times \to \overline{\mathbf{Q}}_l^\times$. Let $\theta = N^*_{\mathbf{k}_n/\mathbf{k}_m}(\theta_m)$. Then

$$\sigma^+_{\mathfrak{P}_{\min}}(\pi_m) \cong \epsilon_{\mathbf{G}}\epsilon_w R_w(\theta) = (-1)^{n+1}R_w(\theta).$$

Proof. First observe that $R_w(\theta)$ is orthogonal to each of the $\pi_{\mathfrak{P}}(\pi_m)$ for $\mathfrak{P} \neq \mathfrak{P}_{\min}$, because these are full parabolic inductions, and the torus $\mathbf{T}_w$ has no conjugates in any proper split Levi subgroup of $\mathbf{G}$. So we have to prove that

$$(R_w(\theta), \mathrm{St}(\pi_m, n/m))_{\mathbf{G}^F} = (-1)^{n+1},$$

since $\pi_{\mathfrak{P}_{\min}}(\pi_m) = \sigma_{\mathfrak{P}_{\min}}(\pi_m) = \mathrm{St}(\pi_m, n/m)$.

Write $z = \theta_m(\zeta_m)$, so that the geometric conjugacy class of $(\mathbf{T}_w, \theta)$ corresponds to the minimal polynomial of z over $\mathbf{k}$ (a degree m polynomial) to the n/m-th power, as in example 2.5.11. The centralizer in $\mathrm{GL}_n(\mathbf{k})$ of any rational element in this conjugacy class is isomorphic to $\mathrm{GL}_{n/m}(\mathbf{k}_m)$, and it is the group of rational points of a Levi subgroup $\mathbf{L}_0$ of $\mathbf{G}_0$, isomorphic to $\mathrm{Res}_{\mathbf{k}_m/\mathbf{k}}\mathrm{GL}_{n/m,\mathbf{k}_m}$. By the discussion preceding lemma 2.5.5, the conjugacy classes of rational maximal tori in $\mathbf{L}_0 \times_{\mathbf{k}} \overline{\mathbf{k}}$ are in bijection with those in $\mathrm{GL}_{n/m,\mathbf{k}_m} \times_{\mathbf{k}_m} \overline{\mathbf{k}}$.

Under this bijection, the torus $\mathbf{T}_w$ has type corresponding to the n/m-cycles, which we write as $w_{n/m}$.

By lemma 2.5.5, the unipotent characters $\mathcal{E}(\mathbf{L}^F, [1])$ coincide with the unipotent characters of $\mathrm{GL}_{n/m}(\mathbf{k}_m)$ viewed as the group of $\mathbf{k}_m$-points of $\mathrm{GL}_{n/m,\mathbf{k}_m}$. Hence they are parametrized by $\chi \in \mathrm{Irr}(S_{n/m})$ as in our previous discussion: we write $\chi \mapsto A_\chi$ for this parametrization.

Lemma 2.5.14. The Lusztig series of the geometric conjugacy class of $[\mathbf{T}_w, \theta]$ is

$$\{(-1)^{n+n/m} R_{\mathbf{L}}^{\mathbf{G}}(\theta_m A_\chi) : \chi \in \mathrm{Irr}(S_{n/m})\}.$$

Proof. The character θ_m can be inflated to $\mathrm{GL}_{n/m}(\mathbf{k}_m)$ via the determinant, and its restriction to $\mathbf{k}_n^\times$ is θ. By [DM91, Proposition 13.30], the Lusztig series of $\mathbf{L}^F$ attached to the geometric conjugacy class of $(\mathbf{T}_w, \theta)$ consists of the twists by θ of the unipotent characters of $\mathbf{L}^F$. Then the lemma follows from theorem 2.5.12 and our discussion of unipotent characters. □

The theorem will follow from the lemma and the following two equations:

$$R_w(\theta) = \sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(w_{n/m}) R_{\mathbf{L}}^{\mathbf{G}}(\theta_m A_\chi) \tag{2.5.2}$$

$$\mathrm{St}(\pi_m, n/m) = (-1)^{n+n/m} R_{\mathbf{L}}^{\mathbf{G}}(\theta_m A_{\mathrm{sgn}}). \tag{2.5.3}$$

Indeed, we see that

$$\begin{aligned}(R_w(\theta), \sigma_{\mathfrak{P}_{\min}}(\pi_m))_{\mathbf{G}^F} &= \left(\sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(w_{n/m}) R_{\mathbf{L}}^{\mathbf{G}}(\theta_m A_\chi), (-1)^{n+n/m} R_{\mathbf{L}}^{\mathbf{G}}(\theta_m A_{\mathrm{sgn}}) \right)_{\mathbf{G}^F} \\ &= (-1)^{n+n/m} \mathrm{sgn}(w_{n/m}) = (-1)^{n+1}\end{aligned}$$

since $\mathrm{sgn}(w_{n/m}) = (-1)^{n/m+1}$.

Proof of equation (2.5.2). Transitivity of Lusztig induction (see [DM91, 11.5]) implies that $R_w(\theta) = R_{\mathbf{L}}^{\mathbf{G}}(R_{\mathbf{T}_w}^{\mathbf{L}}(\theta))$, where we have chosen an arbitrary parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ with Levi factor $\mathbf{L} = \mathbf{L}_0 \times_{\mathbf{k}} \overline{\mathbf{k}}$. By [DL76, Corollary 1.27], we have an equality $R_{\mathbf{T}_w}^{\mathbf{L}}(\theta) = \theta_m R_{\mathbf{T}_w}^{\mathbf{L}}(1_{\mathbf{T}_w})$. By lemma 2.5.5, the functor $R_{\mathbf{T}_w}^{\mathbf{L}}$ coincides with $R_{w_{n/m}}$ taken with respect to $\mathrm{GL}_{n/m,\mathbf{k}_m}$. But we have seen in (2.5.1) that

$$R_{w_{n/m}}(\mathrm{triv}) = \sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(w_{n/m}) A_\chi. \tag{2.5.4}$$

□

Lemma 2.5.15. The Lusztig series of $[\mathbf{T}_w, \theta]$ coincides with the Harish-Chandra series of $n/m \cdot \pi_m$.

Proof. Let $w_{m,n/m} \in S_n$ be the product of n/m disjoint m-cycles. Similarly to (2.5.4), we have

$$R^{\mathbf{L}}_{\mathbf{T}_{w_{m,n/m}}}(\theta_m^{\otimes n/m}) = \theta_m R_{\mathrm{id}}(\mathrm{triv}) = \sum_{\chi \in \mathrm{Irr}(S_{n/m})} \chi(\mathrm{id}) \theta_m A_\chi. \tag{2.5.5}$$

Let $\mathbf{M} = \mathrm{GL}_{m,\mathbf{k}}^{\times n/m}$ be a split Levi subgroup of $\mathbf{G}$. containing $\mathbf{T}_{w_{m,n/m}}$. By transitivity, $R^{\mathbf{G}}_{\mathbf{T}_{w_{m,n/m}}}(\theta_m^{\otimes n/m})$ is the character of the parabolic induction of $R^{\mathbf{M}}_{\mathbf{T}_{w_{m,n/m}}}(\theta_m)$, because Lusztig induction from a split Levi subgroup coincides with Harish-Chandra parabolic induction [DM91, 11.1]. Now we can apply lemma 2.5.6 to compute $R^{\mathbf{M}}_{\mathbf{T}_{w_{m,n/m}}}(\theta_m^{\otimes n/m}) = R^{\mathrm{GL}_{m,\mathbf{k}}}_{\mathbf{T}_{w_m}}(\theta_m)^{\otimes n/m}$. Finally, we deduce that

$$\begin{aligned}(-1)^{n+n/m} R^{\mathbf{G}}_{\mathbf{T}_{w_{m,n/m}}}(\theta_m^{\otimes n/m}) = \mathrm{Ind}_{\mathbf{M}^F}^{\mathbf{G}^F}\left((-1)^{m+1} R_{w_m}(\theta_m)\right)^{\otimes n/m} &= \mathrm{Ind}^{\mathrm{GL}_n(\mathbf{k})}_{\prod_{i=1}^{n/m} \mathrm{GL}_m(\mathbf{k})}(\pi_m^{\otimes n/m}) \\ &= \pi_{\mathfrak{P}_{\max}}(\pi_m).\end{aligned}$$

Then the lemma follows from (2.5.5), lemma 2.5.14, and the fact that the Harish-Chandra series of $n/m \cdot \pi_m$ coincides with the set of constituents of $\pi_{\mathfrak{P}_{\max}}(\pi_m)$. □

Proof of equation (2.5.3). The character A_{sgn} is the Steinberg character of $\mathrm{GL}_{n/m}(\mathbf{k}_m)$, and the Lusztig induction of a nondegenerate character is nondegenerate, by [DM91, Proposition 14.32] (the nondegenerate characters are the constituents of a Gelfand–Graev representation). Since nondegeneracy is preserved under twisting by abelian characters (because unipotent elements have determinant one), we see that $(-1)^{n+n/m} R^{\mathbf{G}}_{\mathbf{L}}(\theta_m A_{\mathrm{sgn}})$ is the character of a nondegenerate representation in the Lusztig series of $[\mathbf{T}_w, \theta]$. By lemma 2.5.15, this representation is $\mathrm{St}(\pi_m, n/m)$. □

This completes the proof of theorem 2.5.13. □

l-MODULAR REPRESENTATIONS. Now let $R = \overline{\mathbf{F}}_l$ for $l \neq p$. The irreducible $R[\mathrm{GL}_n(\mathbf{F}_q)]$-modules have been classified by Dipper and James: see [Vig96, III.2] for an exposition. We will need some results about the mod l reduction of cuspidal $\overline{\mathbf{Q}}_l$-representations, which we give in this paragraph.

Recall that a character $\chi : \mathbf{e}_n^\times \to \overline{\mathbf{Q}}_l^\times$ decomposes uniquely as the product of an l-singular part $\chi_{(l)}$ and an l-regular part $\chi^{(l)}$, whose orbits under $\mathrm{Gal}(\mathbf{e}_n/\mathbf{e})$ only depend on the orbit of χ. We use the mod l reduction map to identify the prime-to-l roots of unity in $\overline{\mathbf{Q}}_l$ and $\overline{\mathbf{F}}_l$. Then the reduction mod l of χ identifies with $\chi^{(l)}$.

Proposition 2.5.16. [Vig96, III.2.3 Théorème] The reduction $\mathbf{r}_l(\sigma[\chi])$ is irreducible and cuspidal, and only depends on $[\chi^{(l)}]$. We denote it by $\sigma_l[\chi^{(l)}]$.

The map $[\chi] \mapsto \sigma_l[\chi]$ defines a bijection, from the orbits of $\mathrm{Gal}(\mathbf{e}_n/\mathbf{e})$ on the characters of $(\mathbf{e}_n^\times)^{(l)}$ which have an $\mathbf{e}$-regular extension to $\mathbf{e}_n^\times$, to the set of cuspidal irreducible representations of $\mathrm{GL}_n(\mathbf{e})$ over $\overline{\mathbf{F}}_l$.

Proposition 2.5.17. [Vig96, III.2.8] and [MS14b, Théorème 2.36]. The representation $\sigma_l[\chi^{(l)}]$ is supercuspidal if and only if $[\chi^{(l)}]$ is $\mathbf{e}$-regular. If $\chi^{(l)}$ is norm-inflated from an $\mathbf{e}$-regular $\overline{\mathbf{F}}_l$-character $\chi^{(l),\mathrm{reg}}$ of $\mathbf{e}_{n/a}^\times$ for some positive divisor a of n, then the supercuspidal support of $\mathbf{r}_l(\sigma[\chi])$ is $a \cdot \sigma_l[\chi^{(l),\mathrm{reg}}]$.

Example 2.5.18. Since $\mathbf{F}_9^\times$ has eight elements, the only character $\mathbf{F}_9^\times \to \overline{\mathbf{F}}_2^\times$ is the trivial character. Hence $\mathrm{GL}_2(\mathbf{F}_3)$ has no supercuspidal representations over $\overline{\mathbf{F}}_2$, and precisely one cuspidal irreducible representation, with supercuspidal support $1 \otimes 1$.

The following proposition is a consequence of the classification of cuspidal representations.

Proposition 2.5.19. Let R be an algebraically closed field of characteristic $l \neq p$, let $\mathbf{e} = \mathbf{F}_q$, and let ψ be an R-character of $\mathbf{e}^\times$ such that $\psi\pi \cong \pi$ for all cuspidal R-representations of $\mathrm{GL}_n(\mathbf{e})$. Then $\psi = 1$.

Proof. Assume first that $R = \overline{\mathbf{Q}}_l$. By [SZ00, Theorem 1.1], the equality $\mathrm{tr}\sigma[\chi_1] = \mathrm{tr}\sigma[\chi_2]$ holds on primitive elements of $\mathbf{e}_n/\mathbf{e}$ if and only if $[\chi_1] = [\chi_2]$. This implies that $\psi \otimes \sigma[\chi] \cong \sigma[\psi\chi]$. Then the claim follows because if $\psi \neq 1$ there always exists an $\mathbf{e}$-regular character χ of $\mathbf{e}_n^\times$ with no $\mathrm{Gal}(\mathbf{e}_n/\mathbf{e})$-conjugate of the form $\psi\chi$. Indeed, if $\chi^{q^i} = \psi\chi$ then $\chi^{(q-1)(q^i-1)} = 1$, and taking χ to be a generator of the character group yields a contradiction if $0 < i \leq n-1$. Then the claim holds for any R of characteristic zero.

For $R = \overline{\mathbf{F}}_l$, such a ψ lifts to a character $\psi : (\mathbf{e}^\times)^{(l)} \to \overline{\mathbf{Q}}_l^\times$ such that for all $\mathbf{e}$-regular $\chi : \mathbf{e}_n^\times \to \overline{\mathbf{Q}}_l^\times$ we have $[\chi^{(l)}] = [\chi^{(l)}\psi]$. By duality, we get an element $x \in (\mathbf{e}^\times)^{(l)}$ such that whenever $z \in \mathbf{e}_n^\times$ is $\mathbf{e}$-regular we have $[z^{(l)}x] = [z^{(l)}]$.

Assume that $\mathbf{e}_n^\times$ contains an l-singular element—that is, some $\zeta \in \mu_{l^\infty}(\mathbf{e}_n)$—which is $\mathbf{e}$-regular. Then $\zeta^{(l)} = 1$ implies $[x] = [1]$, hence $x = 1$.

Otherwise, since $(\mathbf{e}_n^\times)_{(l)}$ is cyclic there exists a proper divisor $a|n$ such that $(\mathbf{e}_n^\times)_{(l)} = (\mathbf{e}_a^\times)_{(l)}$. Let τ be a generator of $(\mathbf{e}_n^\times)^{(l)}$: then τ is the l-regular part of some $\mathbf{e}$-regular element of $\mathbf{e}_n^\times$, which can be chosen to be a generator of $\mathbf{e}_n^\times$. There exists a proper divisor b of n such that $(\mathrm{Frob}_q)^b\tau = \xi\tau$ for some $\xi \in \mathbf{e}^\times$, because the set of $g \in \mathrm{Gal}(\mathbf{e}_n/\mathbf{e})$ with $(g\tau)\tau^{-1} \in \mathbf{e}^\times$ is a subgroup and by assumption it is not trivial if $x \neq 1$.

Let w be the order of $\xi \in \mathbf{e}^\times$, which is a divisor of $|\mathbf{e}^\times| = q-1$. Then $(\mathrm{Frob}_q)^b(\tau^w) = (\xi\tau)^w = \tau^w$, hence $(\mathrm{Frob}_q)^b$ fixes the subgroup $w \cdot (\mathbf{e}_n^\times)^{(l)}$, which has index at most $w^{(l)}$ in $(\mathbf{e}_n^\times)^{(l)}$. Since $\mathbf{e}_n^\times \cong (\mathbf{e}_n^\times)^{(l)} \times (\mathbf{e}_n^\times)_{(l)}$, we find a bound

$$q^n - 1 \leq w^{(l)}|(\mathbf{e}_b^\times)^{(l)}||(\mathbf{e}_a^\times)_{(l)}|.$$

Since $w|q-1$, we have that $w^{(l)}|(\mathbf{e}_a^\times)_{(l)}|$ divides $|\mathbf{e}_a^\times|$. Then the bound yields $q^n - 1 \leq (q^a-1)(q^b-1)$ for certain proper divisors $a, b|n$. This is impossible even if both a and b coincide with the largest proper divisor d of n, because

$$\frac{q^n-1}{q^d-1} = 1 + q^d + \cdots + q^{d(\frac{n}{d}-1)} > q^d - 1.$$

The claim then holds over $\overline{\mathbf{F}}_l$, and follows over arbitrary R of characteristic l because an irreducible $\overline{\mathbf{F}}_l$-representation of $\mathrm{GL}_n(\mathbf{e})$ is absolutely irreducible, and the number of irreducible representations over $\overline{\mathbf{F}}_l$ and R is the same (it is the number of l-regular conjugacy classes in $\mathrm{GL}_n(\mathbf{e})$). □

Remark 2.5.20. If R is an algebraically closed field of characteristic $l \neq 0, p$, we obtain a canonical bijection from the set of cuspidal irreducible representations of $R[\mathrm{GL}_n(\mathbf{e})]$ to the set of $\mathrm{Gal}(\mathbf{e}_n/\mathbf{e})$-orbits on certain characters $\mathbf{e}_n^\times \to R^\times$. More precisely, we consider characters that are dual to l-regular elements $x \in \mathbf{e}_n^\times$ with the following property: there exists an l-singular $y \in \mathbf{e}_n^\times$ such that xy is $\mathbf{e}$-regular.

We write the inverse of this bijection as $\chi \mapsto \sigma_R[\chi]$. It is characterized by being the mod l reduction of the Deligne–Lusztig induction (defined as in remark 2.5.10) over a large enough finite cyclotomic extension of $W(R)[1/l]$.

Algebraic and p-modular representations. The irreducible $\overline{\mathbf{F}}_p[\mathrm{GL}_n(k_F)]$-modules are well-known to be restriction to $\mathrm{GL}_n(k_F)$ of algebraic representations of restricted weight. In this paragraph, we recall this classification following the notation of [EG14], which relates them to Weyl modules in characteristic zero. We also construct some algebraic $\mathcal{O}_D^\times$-representations, which will intervene in our Breuil–Mézard statements for deformation rings with nontrivial regular weight. We let $E/\mathbf{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}_E$ and residue field k_E, which we assume to contain all $[F : \mathbf{Q}_p]$ embeddings of E.

Write $\mathbf{Z}_+^n$ for the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$ of integers such that $\lambda_1 \geq \ldots \geq \lambda_n$. This defines a dominant character $\mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod_{i=1}^n t_i^{\lambda_i}$ of the diagonal torus in $\mathrm{GL}_{n,F}$. There is an associated algebraic $\mathcal{O}_F$-representation of $\mathrm{GL}_{n,\mathcal{O}_F}$ with highest weight λ. We write M_λ for the $\mathcal{O}_F$-points of this representation, so that $M_\lambda \cong \mathrm{Ind}_{B_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(w_{\max}\lambda)$ for the upper-triangular Borel subgroup B_n and the longest element $w_{\max}$ of the Weyl group. Then fix $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}_{\mathbf{Q}_p}(F,E)}$ and define an $\mathcal{O}_E$-representation of $\mathrm{GL}_n(\mathcal{O}_F)$ by

$$L_\lambda = \otimes_{\tau: F \to E} \left(M_{\lambda_\tau} \otimes_{\mathcal{O}_F, \tau} \mathcal{O}_E \right).$$

Next we recall some mod p representations. Given $a \in \mathbf{Z}_+^n$ with $p - 1 \geq a_i - a_{i+1}$ for all

$1 \leq i \leq n-1$, define

$$P_a = \mathrm{Ind}_{B_n(\mathbf{f})}^{\mathrm{GL}_n(\mathbf{f})}(w_{\max} a)$$

and let N_a be the irreducible subrepresentation of P_a generated by a highest weight vector. The *Serre weights* of $\mathrm{GL}_n(\mathbf{f})$ are the elements $a \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(\mathbf{f},\mathbf{e})}$ such that for all $\sigma : \mathbf{f} \to \mathbf{e}$ we have $p-1 \geq a_{\sigma,i} - a_{\sigma,i+1}$ for $1 \leq i \leq p-1$, and $0 \leq a_{\sigma,n} \leq p-1$. We furthermore require that not all $a_{\sigma,n} = p-1$. To a Serre weight there corresponds an irreducible $\mathbf{e}$-representation of $\mathrm{GL}_n(\mathbf{f})$, defined by

$$F_a = \otimes_{\tau \in \mathrm{Hom}(\mathbf{f},\mathbf{e})} \left(N_{a_\tau} \otimes_{\mathbf{f},\tau} \mathbf{e} \right).$$

These are absolutely irreducible and pairwise non-isomorphic, and every irreducible $\mathbf{e}$-representation of $\mathrm{GL}_n(\mathbf{f})$ has this form.

Finally, we introduce analogues for $D^\times$. For every $\mathbf{Q}_p$-linear embedding $\tau : F \to E$, fix an embedding $\tau^+ : F_n \to E$ lifting τ and write M_λ^+ for the $\mathcal{O}_{F_n}$-points of the algebraic representation with highest weight λ (so that $M_\lambda^+|_{\mathrm{GL}_n(\mathcal{O}_F)}$ is isomorphic to $M_\lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F_n}$). Then we introduce

$$L_\lambda^+ = \otimes_{\tau^+} \left(M_{\lambda_\tau}^+ \otimes_{\mathcal{O}_{F_n},\tau^+} \mathcal{O}_E \right)$$

which has an action of $\mathcal{O}_D^\times$ via a choice of F_n-linear isomorphism $j : D \otimes_F F_n \to M_n(F_n)$ mapping the order $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_n}$ into $M_n(\mathcal{O}_{F_n})$, and the inclusion $D \to D \otimes_F F_n,\ d \mapsto d \otimes 1$. We have the following lemma.

Lemma 2.5.21. If $z_D \in \mathcal{O}_D^\times$ corresponds to $z \in \mathrm{GL}_n(\mathcal{O}_F)$, then $\mathrm{tr}_{L_\lambda^+}(z_D) = \mathrm{tr}_{L_\lambda}(z)$.

Proof. This is because L_λ^+ is a lattice in a $\mathrm{GL}_n(F_n)$-representation over E, $\mathrm{tr}_{L_\lambda}(z) = \mathrm{tr}_{L_\lambda^+}(z)$, and z and z_D are conjugate in $\mathrm{GL}_n(F_n)$ under any choice of j. □

3

The local Langlands correspondence for GL_n.

This section reviews the statement and basic properties of the local Langlands correspondence for the group $\mathrm{GL}_n(F)$, and the classical Jacquet–Langlands correspondence, both with complex and l-modular coefficients. Recall the exact sequence

$$0 \to I_F \to W_F \xrightarrow{\alpha} \mathbf{Z} \to 0$$

defining the Weil group of F as the subgroup of $g \in G_F$ that act on $\overline{k}_F$ through an integer power of the geometric Frobenius element $\mathrm{Frob}_q : x \mapsto x^{1/q}$. The map α is defined through the equality $g = \mathrm{Frob}_q^{\alpha(g)}$ on $\overline{k}_F$, and we give W_F the locally profinite topology in which I_F is an open subgroup with its defining topology.

Definition 3.0.1. A Langlands parameter for $\mathrm{GL}_{n,F}$ is a morphism

$$W_F \times \mathrm{SL}_2(\mathbf{C}) \to {}^L\mathrm{GL}_{n,F} = \mathrm{GL}_n(\mathbf{C}),$$

with the following properties:

1. $\varphi|_{I_F}$ has finite image,
2. $\varphi(\mathrm{Frob}_q)$ is semisimple, and
3. $\varphi|_{\mathrm{SL}_2(\mathbf{C})}$ is algebraic.

We regard two parameters as equivalent if they are conjugate under $\mathrm{GL}_n(\mathbf{C})$.

Because of the third condition, this definition could be formulated over any algebraically closed field of coefficients. In order to emphasize this fact, as well as the relationship with Galois representations, we introduce the following definition.

Definition 3.0.2. Let R be a field of characteristic different from p. A Weil–Deligne representation with coefficients in R is a pair (ρ, N) consisting of a smooth representation $\rho : W_F \to \mathrm{GL}_n(R)$ and a nilpotent element $N \in (\mathrm{Lie}\,\mathrm{GL}_n)(R)$ such that

$$(\mathrm{Ad}\,\rho(w))N = q^{-\alpha(w)}N$$

for all $w \in W_F$. We regard two Weil–Deligne representations (g_i, N_i) as equivalent if there exists $g \in \mathrm{GL}_n(R)$ such that $\rho_2 = g\rho_1 g^{-1}$ and $N_2 = \mathrm{Ad}(g)N_1$. We say that (ρ, N) is *Frobenius-semisimple* if ρ is a semisimple representation. We write $\Phi_{F,n}(R)$ for the set of isomorphism classes of n-dimensional Frobenius-semisimple Weil–Deligne representations over R.

Remark 3.0.3. We say that $\Phi \in W_F$ is a *Frobenius lift* if $\alpha(\Phi) = 1$. If Φ is a Frobenius lift and R has characteristic zero, then (ρ, N) is Frobenius-semisimple if and only if $\rho(\Phi)$ is a semisimple automorphism: this follows because the inertia group is acting on ρ through a finite quotient, hence $\rho(\Phi)^{\mathbf{Z}}$ has finite index in $\rho(W_F)$. But in characteristic zero a representation of a group is semisimple if and only if its restriction to a finite index subgroup is semisimple.

We recall the construction of a bijection from the equivalence classes of Frobenius-semisimple Weil–Deligne representations (ρ, N) over $\mathbf{C}$ to the equivalence classes of Langlands parameters, following [GR10, Section 2]. Since $\rho(I_F)$ is finite, the Lie algebra $\mathfrak{h} = \mathfrak{gl}_n(\mathbf{C})^{\rho(I_F)}$ is reductive, and it contains N. It is normalized by any fixed Frobenius lift Frob_q, and we write $\mathfrak{h}(\lambda)$ for the λ-eigenspace of the semisimple automorphism $\mathrm{Ad}\,\rho(\mathrm{Frob}_q)$ of $\mathfrak{h}$. By a refinement of the Jacobson–Morozov theorem, there exists an $\mathfrak{sl}_2$-triple in $\mathfrak{h}$ such that

$$e = N \in \mathfrak{h}(q^{-1}),\ h \in \mathfrak{h}(1),\ f \in \mathfrak{h}(q).$$

(Hence h is fixed by W_F.) Furthermore, any two such triples are conjugate by an element of $\mathrm{GL}_n(\mathbf{C})$ that centralizes both W_F and N.

Then the Langlands parameter corresponding to (ρ, N) is defined as follows. There exists a morphism $\varphi : \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$ such that $e = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and one checks that if we put $s = \varphi\mathrm{diag}(q^{1/2}, q^{-1/2})$ then $s\rho(\mathrm{Frob}_q)$ centralizes φ. So we obtain a morphism $\varphi : W_F \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$ which is ρ on I_F, φ on $\mathrm{SL}_2(\mathbf{C})$, and $s\rho(\mathrm{Frob}_q)$ at Frob_q.

Classification of Langlands parameters. Let E be a field of characteristic zero. Every element of $\Phi_{F,n}(E)$ can be written as a direct sum of indecomposable subobjects, whose isomorphism class is uniquely determined. Every indecomposable object is isomorphic to $\mathrm{Sp}_s(W)$ for a unique isomorphism class of irreducible W_F-representations and a unique positive integer s, which is defined as follows. The underlying W_F-representation of $\mathrm{Sp}_s(W)$ is

$$W \oplus W(1) \oplus \cdots \oplus W(s-1),$$

where $W(i) = W \otimes_{\mathbf{C}} |\mathrm{Art}_F^{-1}(-)|$. The monodromy operator is $N : W(i) \xrightarrow{\sim} W(i+1), x \otimes 1 \mapsto x \otimes 1$. Observe that $\mathrm{Sp}_s(W) \cong W \otimes \mathrm{Sp}_s(1)$ as a Weil–Deligne representation.

See [Del73, Proposition 3.1.3] for a proof of this classification. Here, $|-|_F$ is the normalized absolute value character of $F^\times$, which coincides with $\mathrm{nr}_{1/q}$. The Artin map is normalized so that uniformizers correspond to geometric Frobenius lifts.

3.1 Statement of the correspondence.

The Langlands correspondence for $\mathrm{GL}_n(F)$ [HT01, Hen00, Sch13] is a series of bijections

$$\mathrm{rec}_{\mathbf{C}} : \mathrm{Irr}_{\mathbf{C}}(\mathrm{GL}_n(F)) \to \Phi_{F,n}(\mathbf{C})$$

compatible with central characters, unramified twists and contragredients, given by local class field theory when $n = 1$, and satisfying an equality of L-functions and ϵ-factors of pairs that we will not need in this thesis, although it is of fundamental theoretical importance. (See the introduction to [HT01] for a precise statement.) The normalized correspondence

$$r_\infty(\pi) = \mathrm{rec}_{\mathbf{C}}(\pi \otimes \mathrm{nr}_{q^{(1-n)/2}}),$$

for the positive real square root $q^{1/2} \in \mathbf{C}^\times$, commutes with all automorphisms of $\mathbf{C}$. We will sometimes drop the index $\mathbf{C}$ from the notation when the coefficients are clear from the context. (An automorphism $\tau \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ acts on the two sides of the correspondence by $\tau^*(\rho, N) = (\rho \otimes_\tau \mathbf{C}, N \otimes 1)$, and $\tau^*\pi = \pi \otimes_\tau \mathbf{C}$.)

However, in order to access Vignéras's results on the Langlands correspondence over more general coefficient fields, we will fix a square root of q in $\overline{\mathbf{Q}}_l$ for all prime numbers $l \neq p$, and we will work with rec. Any isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}}_l$ sending the positive real $q^{1/2}$ to our fixed choice defines the same bijection

$$\mathrm{rec}_{\overline{\mathbf{Q}}_l} : \mathrm{Irr}_{\overline{\mathbf{Q}}_l} \mathrm{GL}_n(F) \to \Phi_{F,n}(\overline{\mathbf{Q}}_l).$$

Remark 3.1.1. Most of our arguments will be concerned with the restriction to inertia of $\mathrm{rec}_{\overline{\mathbf{Q}}_l}(\pi)$, which is independent of the choice of $q^{1/2}$.

Now we recall the results of [Vig01a] on the reduction mod l of $\mathrm{rec}_{\overline{\mathbf{Q}}_l}$. The first step is to study the behaviour of the semisimple part of the Langlands parameter. Recall our notation and conventions for the supercuspidal support map

$$sc : \coprod_{n\geq 1} \operatorname{Irr}_R \operatorname{GL}_n(F) \to \operatorname{Div}^+(\operatorname{scusp}_R F).$$

Given $X = \pi_1 + \cdots + \pi_r \in \operatorname{Div}^+(\operatorname{scusp}_{\overline{\mathbf{Q}}_l} F)$, we define

$$\operatorname{rec}_{\overline{\mathbf{Q}}_l}(X) = \operatorname{rec}_{\overline{\mathbf{Q}}_l}(\pi_1) \oplus \cdots \oplus \operatorname{rec}_{\overline{\mathbf{Q}}_l}(\pi_r).$$

The semisimple Langlands correspondence is the map $\pi \mapsto \operatorname{rec}_{\overline{\mathbf{Q}}_l} sc(\pi)$.

Definition 3.1.2. Let G be a locally profinite group. A smooth $\overline{\mathbf{Q}}_l$-representation of G of finite length is integral if there exists a finite extension $E/\mathbf{Q}_l$ and a representation of G on a free $\mathcal{O}_E$-module L such that

1. $\overline{\mathbf{Q}}_l \otimes_{\mathcal{O}_E} L \cong \pi$, and
2. L is a finite $\mathcal{O}_E[G]$-module.

In the case that G is a p-adic reductive group, a profinite group, or the Weil group W_F, the semisimplification of $\overline{\mathbf{F}}_l \otimes_{\mathcal{O}_E} L$ is a finite length representation of $\overline{\mathbf{F}}_l[G]$ that is independent of L [Vig96, I.9.6]. It is denoted $\mathbf{r}_l(\pi)$. One checks that $\sigma \in \Phi_{F,n}(\overline{\mathbf{Q}}_l)$ is integral if and only if its determinant is integral, and $\pi \in \operatorname{scusp}_{\overline{\mathbf{Q}}_l}(\operatorname{GL}_n(F))$ is integral if and only if its central character is integral. At the same time, a semisimple $\overline{\mathbf{Q}}_l$-representation σ of W_F is integral if and only if its irreducible constituents are integral, and $\pi \in \operatorname{Irr}_{\overline{\mathbf{Q}}_l}(\operatorname{GL}_n(F))$ is integral if and only if the constituents of $sc(\pi)$ are integral. Since the central character of π corresponds to $\det(\operatorname{rec}_{\overline{\mathbf{Q}}_l}(\pi))$, we deduce the following statement.

Corollary 3.1.3. The semisimple Langlands correspondence over $\overline{\mathbf{Q}}_l$ preserves integrality of representations.

If $\pi \in \operatorname{Irr}_{\overline{\mathbf{Q}}_l} \operatorname{GL}_n(F)$ is integral and $sc(\pi) = \pi_1 + \cdots + \pi_k$, the factors of $\mathbf{r}_l(\pi)$ have the same supercuspidal support, which is equal to

$$sc(\mathbf{r}_l\pi_1) + \cdots + sc(\mathbf{r}_l\pi_k).$$

Theorem 3.1.4. [Vig01a, Théorème principal]

1. Let $\pi, \pi' \in \operatorname{Irr}_{\overline{\mathbf{Q}}_l} \operatorname{GL}_n(F)$ and write $\sigma = \operatorname{rec}_{\overline{\mathbf{Q}}_l} sc(\pi), \sigma' = \operatorname{rec}_{\overline{\mathbf{Q}}_l} sc(\pi')$. Then $sc(\mathbf{r}_l\pi) = sc(\mathbf{r}_l\pi')$ if and only if $\mathbf{r}_l\sigma = \mathbf{r}_l\sigma'$.
2. There exists a unique sequence of bijections

$$\operatorname{rec}_{\overline{\mathbf{F}}_l} : \operatorname{scusp}_{\overline{\mathbf{F}}_l} \operatorname{GL}_n(F) \to \operatorname{Irr}_{\overline{\mathbf{F}}_l, n}(W_F)$$

such that the induced semisimple correspondence over $\overline{\mathbf{F}}_l$ is compatible with mod l reduction, in the following sense: if $\pi \in \mathrm{Irr}_{\overline{\mathbf{Q}}_l} \mathrm{GL}_n(F)$ and $\sigma = \mathrm{rec}_{\overline{\mathbf{Q}}_l} sc(\pi)$, then $\mathbf{r}_l\sigma = \mathrm{rec}_{\overline{\mathbf{F}}_l} sc(\mathbf{r}_l\pi)$.

Remark 3.1.5. Let $\sigma \in \mathrm{Irr}_{\overline{\mathbf{Q}}_l} W_F$. Recall that $\mathbf{r}_l(\sigma)$ needs not be irreducible. This phenomenon corresponds to irreducible supercuspidal $\overline{\mathbf{Q}}_l$-representations whose reduction is cuspidal but not supercuspidal.

When R has positive characteristic, the semisimple correspondence is all we will need in this thesis. However, for completeness we indicate how it extends to a full Langlands correspondence over $\overline{\mathbf{F}}_l$. The irreducible $R[\mathrm{GL}_n(F)]$-representations with a fixed supercuspidal support can be classified via an analogue of Zelevinsky's theory.

Theorem 3.1.6. Let (σ, V) be a semisimple, finite-dimensional $R[W_F]$-module. The representations $\pi \in \mathrm{Irr}_R W_F$ such that $\mathrm{rec}_R sc(\pi) = \sigma$ are in bijection with the nilpotent conjugacy classes N in $\mathfrak{gl}(V)$ such that

$$\sigma(w)N\sigma(w)^{-1} = q^{-\alpha(w)}N = |w|N$$

for all $w \in W_F$.

Corollary 3.1.7. There exists a bijection

$$\mathrm{rec}_{\overline{\mathbf{F}}_l} : \mathrm{Irr}_{\overline{\mathbf{F}}_l} \mathrm{GL}_n(F) \to \Phi_{F,n}(\overline{\mathbf{F}}_l)$$

which refines the semisimple correspondence.

Remark 3.1.8. In the following lemma, we record explicitly the way that $\mathrm{rec}_{\mathbf{C}}$ refines the semisimple correspondence. Over $\overline{\mathbf{Q}}_l$, one can also ask whether the nilpotent part of the Langlands parameter is compatible with mod l reduction, in the same way as the semisimple correspondence. The answer is more complicated, since characters of $\mathrm{GL}_n(F)$ over $\overline{\mathbf{F}}_l$ need not have trivial nilpotent part (but $\overline{\mathbf{Q}}_l$-characters do). We will not need this finer information.

Lemma 3.1.9. Let $\pi \in \mathrm{Irr}_{\mathbf{C}} \mathrm{GL}_n(F)$ correspond to the multiset $\{\Delta_1, \dots, \Delta_r\}$, where $\Delta_i = (\rho_i, \dots, \rho_i(n_i - 1))$. Then $\mathrm{rec}_{\mathbf{C}}(\pi) = \bigoplus_i \mathrm{rec}(\rho_i) \otimes \mathrm{Sp}(n_i)$. It follows that if π has supercuspidal support

$$[\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F), \pi_1 \otimes \cdots \otimes \pi_r]$$

then the W_F-representation underlying $\mathrm{rec}_{\mathbf{C}}(\pi)$ is $\mathrm{rec}_{\mathbf{C}}(\pi_1) \oplus \cdots \oplus \mathrm{rec}_{\mathbf{C}}(\pi_r)$.

Finally, we study the behaviour of the Langlands correspondence with respect to restriction to inertia.

Lemma 3.1.10. Let $\pi_1, \pi_2 \in \operatorname{Irr}_R \mathrm{GL}_n(F)$, and assume R is $\mathbf{C}$ or $\overline{\mathbf{Q}}_l$. Then $\operatorname{rec}_R(\pi_1)|_{I_F} \cong \operatorname{rec}_R(\pi_2)|_{I_F}$ if and only if π_1 and π_2 are in the same Bernstein block.

Proof. If the π_i are inertially equivalent then the W_F representations underlying $\operatorname{rec}_R(\pi_i)$ have the same restriction to I_F by lemma 3.1.9 and the compatibility of rec_R with unramified twists.

Conversely, assume that $\operatorname{rec}_R(\pi_1)|_{I_F} \cong \operatorname{rec}_R(\pi_2)|_{I_F}$ and let τ_1 occur in the supercuspidal support of π_1. By lemma 3.1.9, $\operatorname{rec}_R(\tau_1)$ is a direct summand of $\operatorname{rec}_R(\pi_1)|_{W_F}$, hence $\operatorname{rec}_R(\tau_1)|I_F$ shares a constituent with $\operatorname{rec}_R(\tau_2)|_{I_F}$ for some τ_2 in the supercuspidal support of π_2. Since the restriction of an irreducible W_F-representation to I_F is multiplicity-free and consists of a single orbit of representations under the action of W_F/I_F, this implies that $\operatorname{rec}_R(\tau_1)|I_F \cong \operatorname{rec}_R(\tau_2)|I_F$. Hence $\operatorname{rec}_R(\tau_1)$ and $\operatorname{rec}_R(\tau_2)$ are unramified twists of each other, so that an unramified twist of τ_1 occurs in the supercuspidal support of π_2. The claim follows. □

Concerning the wild part of the Langlands parameter, we recall the following result of Bushnell and Henniart, which is for complex representations. Write $P_F^\vee$ for the set of irreducible smooth complex representations of the wild inertia group P_F, and $\mathcal{E}(F)$ for the set of endo-classes of simple characters over F. There is a left action of W_F on $P_F^\vee$ by conjugation. If σ is an irreducible representation of W_F, then let $r_F^1(\sigma) \in W_F \backslash P_F^\vee$ be the orbit contained in the restriction $\sigma|_{P_F}$ (which need not be multiplicity-free).

Theorem 3.1.11. [BH14, Ramification Theorem] The Langlands correspondence induces a bijection

$$\Phi_F : W_F \backslash P_F^\vee \to \mathcal{E}(F)$$

such that $\Phi_F(r_F^1(\sigma))$ is the endo-class of $\operatorname{rec}_{\mathbf{C}}^{-1}(\sigma)$, for any irreducible σ. If $\gamma : F \to F$ is a topological automorphism, extended in some way to an automorphism of W_F, then $\Phi_F(\gamma^*[\alpha]) = \gamma^*\Phi_F[\alpha]$ for all $[\alpha] \in W_F \backslash P_F^\vee$.

Remark 3.1.12. This result has been extended in the supercuspidal case to study the behaviour of the whole ramification filtration under the local Langlands correspondence, see [BH17]. However, we will not make use of this.

3.2 Relation with Galois representations.

Let l be a prime number (possibly equal to p) and $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$ is a continuous Galois representation. If ρ is potentially semistable, which is always the case if $l \neq p$, it is possible to attach to ρ an element of $\Phi_{F,n}(\overline{\mathbf{Q}}_l)$ and so a representation of $\overline{\mathbf{Q}}_l[\mathrm{GL}_n(F)]$. When $l \neq p$ the construction is based on Grothendieck's l-adic monodromy theorem, which also holds for more general coefficient rings. The proof is essentially the same as in the classical case of coefficients in a finite extension of $\mathbf{Q}_l$ [ST68, Appendix], but we give the details for lack of a suitable reference. Let

$$t_l : I_F \to \mathbf{Z}_l$$

be the map that is obtained by composing the Kummer theory map $I_F \to \mathbf{Z}_l(1)$ with an arbitrary isomorphism $i_l : \mathbf{Z}_l(1) \to \mathbf{Z}_l$. Then t_l is the projection to the maximal pro-l quotient of I_F. If L/F is a finite extension contained in $\overline{F}$, the restriction of t_l to I_L is a surjection to $e(L/F)\mathbf{Z}_l \cong \mathbf{Z}_l$, and it identifies with the projection to the maximal pro-l quotient of I_L. See [Ser72, 1.3, 1.4].

Theorem 3.2.1. Let $l \neq p$ and let A be a complete Noetherian local $\mathbf{Z}_l$-algebra with finite residue field. Assume $\rho : G_F \to \mathrm{GL}_n(A)$ is a continuous morphism. Then there exist an open subgroup $H = I_L \subset I_F$ and a unique element $N \in M_n(A)$ such that the equality

$$\rho(g) = \exp(e(L/F)^{-1} t_l(g) N) \text{ for all } g \in H$$

holds in $\mathrm{GL}_n(A[1/l])$. Furthermore, N becomes nilpotent in $M_n(A[1/l])$.

Proof. Let $H = \rho^{-1}(1+\mathfrak{m}^t M_n(A)) \cap I_F$, for t large enough. This is an open subgroup of I_F, hence it is the inertia group of a finite extension L/F. Since $t_l|_{I_L}$ maps onto $e(L/F)\mathbf{Z}_l$, we find that there exists $\alpha \in 1 + \mathfrak{m}^t M_n(A)$ such that

$$\rho(g) = \alpha^{e(L/F)^{-1} t_l(g)} \text{ for all } g \in H.$$

Now $\alpha - 1$ is topologically nilpotent, and $N = \log(\alpha)$ is well-defined. Furthermore, if t is large enough then $\exp(e(L/F)^{-1} t_l(g) N)$ converges in $M_n(A[1/l])$ and is equal to $\rho(g)$ for all $g \in H$. The uniqueness of N follows since $N = \log(\rho(g))$ for any $g \in H$ with $t_l(g) = e(L/F)$.

It remains to prove that the image of N in $M_n(A[1/l])$ is nilpotent. For this, recall that by Kummer theory we have

$$t_l(wgw^{-1}) = \chi_l(w) t_l(g)$$

for all $g \in I_F$ and $w \in G_F$, where $\chi_l : G_F \to \mathbf{Z}_l^\times$ is the cyclotomic character. It follows that

$$\rho(w) N \rho(w)^{-1} = \chi_l(w) N. \tag{3.2.1}$$

If $a_i(N)$ is the i-th coefficient in the characteristic polynomial of N, equation (3.2.1) implies that $a_i(N) = \chi_l(w)^i a_i(N)$ for all $w \in G_F$. Since χ_l has infinite image in $\mathbf{Z}_l^\times$, we can find $w \in G_F$ such that $\chi_l(w)$ is not a root of unity. But then the image of $a_i(N)$ in $M_n(A[1/l])$ is zero, and the claim follows. □

Let $\Phi \in G_F$ be a Frobenius lift. It follows from the theorem and equation (3.2.1) that we can attach to every continuous Galois representation $\rho : G_F \to \mathrm{GL}_n(A)$ a Weil–Deligne representation with coefficients in $A[1/l]$, defined on $w = \Phi^m i$ by

$$\rho'(w) = \rho(w) \exp(-e(L/F)^{-1} t_l(i) N).$$

One checks that the isomorphism class of this representation is independent of the choice of Φ and $i_l : \mathbf{Z}_l(1) \to \mathbf{Z}_l$.

When $l = p$, we only treat the case of coefficients in the ring of integers of a finite extension $E/\mathbf{Q}_p$. Fix a completed algebraic closure C/F. Let us recall that p-adic Hodge theory constructs a discretely valued field B_{dR}, with residue field C, and equipped with its valuation filtration and a "canonical topology" inducing the valuation topology on C. It is furthermore equipped with a continuous action of G_F such that $F = B_{\mathrm{dR}}^{G_F}$. Write F_0 for the maximal unramified extension of $\mathbf{Q}_p$ in F_0. We also have the crystalline period ring B_{cris}, which is an F_0-algebra equipped with an injective ring endomorphism $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ inducing $\sigma = [x \mapsto x^p]$ on F_0. There is a canonical ring extension $B_{\mathrm{cris}} \to B_{\mathrm{st}}$, non-canonically isomorphic to $B_{\mathrm{cris}}[X]$, to which φ extends. There is a monodromy operator $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$, which is a B_{cris}-linear derivation satisfying the equation $N\varphi = p\varphi N$, and we have $B_{\mathrm{cris}} = B_{\mathrm{st}}^{N=0}$.

A continuous representation $\rho : G_F \to \mathrm{GL}(V)$ on a finite-dimensional E-vector space V is *potentially semistable* if there exists a finite Galois extension L/F such that

$$D_{\mathrm{st}}^L(V) = (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}$$

is a free $L_0 \otimes_{\mathbf{Q}_p} E$-module of rank $\dim_E(V)$. By the p-adic monodromy theorem, this is equivalent to $\rho : G_F \to \mathrm{GL}(V)$ being a de Rham representation.

The module $D_{\mathrm{st}}^L(V)$ carries a Frobenius map φ induced by $\varphi \otimes 1$ and a residual action of $\mathrm{Gal}(L/F)$, which commute with each other and are $\sigma \otimes 1$-equivariant. It also has a monodromy operator N, induced by $N \otimes 1$ and satisfying $N\varphi = p\varphi N$. Following [Fon94], we introduce a "linearized" action of W_F on $D_{\mathrm{st}}^L(V)$, by letting

$$r(\sigma) = \sigma \varphi^{\alpha(\sigma)[k_F:\mathbf{F}_p]} \otimes \rho(\sigma).$$

This makes sense because $\varphi \otimes 1$ is bijective on $D_{\mathrm{st}}^L(V)$. Hence r is a smooth bounded representation of W_F on $D_{\mathrm{st}}^L(V)$, commuting with the endomorphism φ, and satisfying

$$(N \otimes 1) r(\sigma) = q^{\alpha(\sigma)} r(\sigma)(N \otimes 1)$$

Hence we have a Weil–Deligne representation with coefficients in $L_0 \otimes_{\mathbf{Q}_p} E$. We introduce the notation

$$\mathrm{WD}_{\tau_0}(V) = D^L_{\mathrm{st}}(V) \otimes_{L_0 \otimes_{\mathbf{Q}_p} E, \tau_0 \otimes 1} E$$

for the specialization through an embedding $\tau_0 : L_0 \to E$. When E contains all embeddings of L_0, we have the following lemma.

Lemma 3.2.2. The isomorphism class of $\mathrm{WD}_{\tau_0}(V)$ is independent of the choice of $\tau_0 : L_0 \to E$. We denote it by $\mathrm{WD}(V)$.

Proof. We have an identification of $\mathrm{WD}_{\tau_0}(V)$ with

$$D^{\tau_0}_{\mathrm{st}}(L) = \{x \in D^L_{\mathrm{st}}(V) : (l \otimes 1)x = (1 \otimes \tau_0(l))x \text{ for all } l \in L_0\}.$$

Then $\varphi : D^{\tau_0}_{\mathrm{st}}(V) \xrightarrow{\sim} D^{\tau_0\sigma^{-1}}_{\mathrm{st}}(V)$, because

$$(l \otimes 1)\varphi x = \varphi\big((\sigma^{-1} l \otimes 1)x\big) = \varphi\big((1 \otimes \tau_0\sigma^{-1} l)x\big) = (1 \otimes \tau_0\sigma^{-1} l)\varphi x$$

if $x \in D^{\tau_0}_{\mathrm{st}}(V)$. Furthermore, this map commutes with the W_F-action, and intertwines N and pN. Since σ acts transitively on the embeddings $\tau_0 : L_0 \to E$, the lemma follows. □

We define the inertial type of V as the isomorphism class of $\mathrm{WD}(\tau|_{I_F})$. Observe that we have an isomorphism

$$D^L_{\mathrm{st}}(V) \cong L_0 \otimes_{\mathbf{Q}_p} \mathrm{WD}(\tau)$$

of smooth W_F-representations, hence our definition agrees with that in [Kis08].

Remark 3.2.3. In [Kis08] the extension L is not specified, working instead with the functor

$$D_{\mathrm{pst}}(V) = \varinjlim_{L/F} D^L_{\mathrm{st}}(V).$$

The formalism of period rings implies that $\mathrm{WD}(V) \otimes_E \overline{\mathbf{Q}}_p$ is independent of the choice of L and E (provided that L_0 embeds in E). In addition, we will usually be working with a finite collection of Weil–Deligne representations, and we will assume that E is large enough that all their irreducible E-components are absolutely irreducible.

4

A parametrization of inertial classes.

In this chapter we give an explicit parametrization of inertial classes of irreducible representations of $\mathrm{GL}_m(D)$, in terms of type-theoretic invariants. Instances of this problem have already been studied, but never in a systematic way, and this has led to inaccuracies in the literature.

To see what might go wrong, let us consider a supercuspidal inertial class $\mathfrak{s}$ for $\mathrm{GL}_n(F)$. By theorem 2.4.10, it is uniquely determined by a $\mathrm{GL}_n(F)$-conjugacy class of maximal simple types. To construct these, we fix a maximal simple character θ and a β-extension κ, and we form a tensor product $(J_\theta, \kappa \otimes \sigma)$ with a cuspidal irreducible representation of J_θ/J^1_θ. By proposition 2.4.5, the conjugacy class of θ is uniquely determined by its endo-class $\Theta_F = \mathrm{cl}(\theta)$. It is therefore natural to regard the pair (Θ_F, σ) as an invariant of $\mathfrak{s}$.

However, suppose that we want to compare these invariants for $\mathfrak{s}$ and $\mathrm{JL}_{D^\times}(\mathfrak{s})$, as we will need to do in order to study our Jacquet–Langlands transfer of weights. Since the endo-classes are defined intrinsically to F, it makes sense to ask whether

$$\mathrm{cl}(\mathfrak{s}) = \mathrm{cl}(\mathrm{JL}_{D^\times}\mathfrak{s}),$$

and indeed this is a special case of [BSS12, Conjecture 9.5] and theorem A in the introduction. Hence one would like to express σ in a way that is independent of the group $\mathrm{GL}_n(F)$.

We know that J_θ/J^1_θ is isomorphic to $\mathrm{GL}_{n/\delta(\Theta_F)}(k_{F,f(\Theta_F)})$, and whenever an isomorphism is fixed we can apply 2.5.7 and write $\sigma = \sigma[\chi]$ for an appropriate choice of characters of $k^\times_{F,n/e(\Theta_F)}$. Furthermore, it will turn out that the group J_θ/J^1_θ on the side of $D^\times$ is isomorphic to $k^\times_{F,n/e(\Theta_F)}$. Hence our problem is equivalent to finding canonical isomorphisms between these groups.

However, we find that one needs to introduce an additional piece of data (a "rigidification") in order to obtain such a canonical isomorphism. Neglecting to do so leads to ambiguities in certain statements: see for example [SS16b, Section 9], which deals with the problem in a similar way, or section 4.1, in which we give a thorough exposition of another instance of this ambiguity, and we show how our method fixes it.

We conclude this introduction by giving a brief sketch of our construction. Let θ be a maximal simple character in $\mathrm{GL}_n(F)$, and fix a β-extension κ to J_θ. The space

$$\mathbf{K}_\kappa^+(\pi) = \mathrm{Hom}_{J_\theta^1}(\kappa, \pi)$$

carries an action of J_θ/J_θ^1, by $f \mapsto xfx^{-1}$. The resulting functor $\mathbf{K}_\kappa^+$ has been introduced in [SZ99] and further studied in [MS14b]. In section 4.2, we show that the choice of a lift Θ_E of $\Theta_F = \mathrm{cl}(\theta)$ to its unramified parameter field $E \subset \overline{F}$ induces a well-defined inner conjugacy class of isomorphisms

$$J_\theta/J_\theta^1 \xrightarrow{\sim} \mathrm{GL}_{\delta(\Theta_F)}(\mathbf{e}),$$

which is furthermore independent of the choice of θ in its G-conjugacy class. By means of this isomorphism, we identify $\mathbf{K}_\kappa^+(\pi)$ with a representation of $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$, written $\mathbf{K}_\kappa(\pi)$.

If π is supercuspidal, then so is $\mathbf{K}_\kappa(\pi)$, and we obtain a Galois conjugacy class of $\mathbf{e}$-regular characters

$$\chi : \mathbf{e}_{n/\delta(\Theta_F)}^\times \to R^\times.$$

The triple $(\Theta_F, \Theta_E, [\chi])$ is well-defined in terms of π, and only depends on the inertial class $\mathfrak{s}(\pi)$. We write

$$\mathfrak{s}(\pi) = \mathfrak{s}_G^\kappa(\Theta_F, \Theta_E, [\chi]).$$

At this point, it is straightforward to prove that the map $\mathfrak{s}_G$ induces a surjection from triples consisting of

1. an endo-class Θ_F of degree dividing n,
2. a lift of Θ_F to its unramified parameter field, and
3. a $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$-orbit on the set of $\mathbf{e}$-regular R-valued characters of $\mathbf{e}_{n/\delta(\Theta_F)}^\times$

to supercuspidal inertial classes of $\mathrm{GL}_n(F)$ over R. We extend this to simple inertial classes via the supercuspidal support and a notion of compatibility for β-extensions. In this thesis, we will only need this parametrization of simple inertial classes, but the results of [SS16a] make it straighforward to extend it to the whole of $\mathfrak{B}_R(\mathrm{GL}_n(F))$.

In this section we point out an explicit instance of the sort of ambiguities that our rigidification process is intended to fix. We are concerned with [BH05, Section 2.3], where $R = \mathbf{C}$. The authors start from an admissible pair $(E/F, \xi)$ of degree n and construct a cuspidal representation ${}_F\pi_\xi$ of $\mathrm{GL}_n(F)$. We claim that this construction is not well-defined: we will review it line by line, point out where it fails, and explain how to fix it.

Example 4.1.1. To fix ideas, one could let $F = \mathbf{Q}_p$, $E = \mathbf{Q}_{p^4}$, and take a character ξ of $E^\times$ such that $\xi | U_E^1$ is inflated from $U^1_{\mathbf{Q}_{p^2}}$ but not $U^1_{\mathbf{Q}_p}$. Admissible pairs $(E/F, \xi)$ with this property do exist.

Let E'/F be the minimal subextension of E/F such that $\xi | U_E^1$ factors through $N_{E/E'}$. In our example, $E' = \mathbf{Q}_{p^2}$. Write $\xi | U_E^1 = N^*_{E/E'}\phi$. Choose a simple stratum $[\mathfrak{A}, l, 0, \beta]$ in $A = M_n(F)$ and a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$ whose endo-class is $\mathrm{Res}_{E'/F}(\phi)$.

At this point of [BH05] we find the sentence "We then have $F[\beta] \cong E'$; we henceforward identify E' with $F[\beta] \subset A$." The point of our discussion is that the isomorphism class of the representation ${}_F\pi_\xi$ depends on the choice of this isomorphism $E' \xrightarrow{\sim} F[\beta]$ (which we take to be F-linear, even though [BH05] is not precise about this). In our example there are, of course, two such isomorphisms that we will denote ι_1, ι_2.

Let us assume that $\mathfrak{A}$ is maximal among $(E')^\times$-stable hereditary $\mathcal{O}_F$-orders in A, or equivalently that θ is a maximal simple character. The groups attached to θ are denoted $J^1 \subset J^0 \subset \mathbf{J} = E'^\times J^0 \subset \mathrm{GL}_n(F)$: they are the normalizers of θ in, respectively, $U^1_{\mathfrak{A}}, U_{\mathfrak{A}}$ and G. Let B be the centralizer of E' in A. Let η be the only irreducible representation of J^1 that contains θ.

Remark 4.1.2. Since endo-equivalent maximal simple characters in $\mathrm{GL}_n(F)$ are conjugate, the choice of a defining simple stratum $[\mathfrak{A}, l, 0, \beta]$ for θ can be ultimately proved not to introduce any ambiguities. Our aim in this section is just to show that the construction in [BH05] fails to associate a well-defined representation of $\mathrm{GL}_n(F)$ to an admissible pair $(E/F, \xi)$, hence we will work (as they do) with a fixed choice of $[\mathfrak{A}, l, 0, \beta]$.

The next lemmas in [BH05] assert the existence of β-extensions, as in our discussion in section 2.4. Fix a uniformizer π_F of F.

Lemma 4.1.3. There exists a unique irreducible representation κ of J^0 such that $\kappa | J^1 \cong \eta$, κ is intertwined by every element of $B^\times$, and $\det \kappa$ has finite order which is a power of p.

Lemma 4.1.4. There exists a unique representation $\widetilde{\kappa}$ of $\mathbf{J}$ such that $\widetilde{\kappa} | J \cong \kappa$, $\pi_F \in \ker \widetilde{\kappa}$, and $\det \widetilde{\kappa}$ has finite order which is a power of p.

Now we employ the uniformizer π_F to write down a factorization of ξ.

Lemma 4.1.5. There is a unique character ξ_w of $E^\times$ such that $\xi_w|U_E^1 = \xi|U_E^1$, $\xi_w(\pi_F) = 1$, and ξ_w has finite order which is a power of π.

Proof. These conditions determine $\xi_w|\pi_F^{\mathbf{Z}} \times U_E^1$. Since $E^\times/\pi_F^{\mathbf{Z}} \times U_E^1$ is a finite abelian group with prime-to-p order, uniqueness is true. For existence, first extend from $\pi_F^{\mathbf{Z}} \times U_E^1$ arbitrarily (using that $\mathbf{C}^\times$ is an injective abelian group) and then notice that the prime-to-p part of this extension factors through $E^\times/\pi_F^{\mathbf{Z}} \times U_E^1$. □

Set $\xi_t = \xi_w^{-1}\xi$. Then ξ_t is tamely ramified, and $(E/E', \xi_t)$ is admissible. Let $\mathfrak{B} = \mathfrak{A} \cap B$, which is a maximal $\mathcal{O}_{E'}$-order in B. At this point in [BH05] we find the sentence "We have $J = U_{\mathfrak{B}} J^1$ and $J/J^1 \cong U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 \cong \mathrm{GL}_d(\mathbf{k}_{E'})$, where $d = [E : E']$. We follow the procedure of 2.2 to define, from the admissible pair $(E/E', \xi_t)$, an irreducible representation Λ_t of $E^\times U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 \cong \mathbf{J}/J$."

The claimed isomorphism $J/J^1 \cong \mathrm{GL}_d(\mathbf{k}_{E'})$ is not well-defined. What we do have is an isomorphism

$$J/J^1 \xrightarrow{\sim} \mathrm{GL}_d(\mathbf{k}_{F[\beta]}),$$

and to go to $\mathrm{GL}_d(\mathbf{k}_{E'})$ one needs to choose an isomorphism $\mathbf{k}_{F[\beta]} \xrightarrow{\sim} \mathbf{k}_{E'}$. Of course this is provided by our choice of $E' \xrightarrow{\sim} F[\beta]$. The reason this matters is that changing this isomorphism induces, in general, an *outer* automorphism of J/J^1 (nontrivial on the centre), and this is going to act nontrivially on representations.

Remark 4.1.6. In our example, E' and $F[\beta]$ are unramified extensions of F, hence this is the same as choosing an isomorphism $E' \xrightarrow{\sim} F[\beta]$: this is the same ambiguity that was introduced at the start. In general, E' and $F[\beta]$ are isomorphic tame extensions, and the only ambiguity comes from the induced identification of their maximal unramified subextensions. So, for a totally tame endo-class, there is no ambiguity: this is the reason [BH05, Section 2.2] works, since it is concerned with the trivial endo-class, and the main results of [BH05] are not affected by this, since they work in the totally ramified case. The sequel [BH10] works with more general extensions, but the authors change the construction of ${}_F\pi_\xi$ in such a way that this is not an issue: however, this fix seems to only work in the essentially tame case, since it relies on identifying $\mathrm{GL}_n(F)$ with $\mathrm{GL}_F(E)$.

Let us inspect "the procedure of 2.2" to see that this really plays a role. (Of course, we will have to interpret what "the procedure" is, since this is not made precise: but it seems clear that the following is what is intended.) First let us review [BH05, Section 2.2]. This is concerned with admissible pairs $(E/F, \xi)$ such that $\xi|U_E^1$ is trivial. By definition, this implies that E/F is unramified and that ξ yields a $\mathrm{Gal}(\mathbf{k}_E/\mathbf{k}_F)$-regular character $\overline{\xi}$ of $\mathbf{k}_E^\times$. This yields an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbf{k}_F)$ that depends only on the $\mathrm{Gal}(\mathbf{k}_E/\mathbf{k}_F)$-conjugacy class of $\overline{\xi}$. This representation can be inflated to $\mathrm{GL}_n(\mathcal{O}_F)$, and then extended to a representation of $F^\times \mathrm{GL}_n(\mathcal{O}_F)$ be requiring that $\Lambda_\xi|F^\times$ be a multiple of ξ. Then $\text{c-Ind}_{\mathbf{J}}^G \Lambda_\xi$ is an irreducible cuspidal representation of $\mathrm{GL}_n(F)$.

In our previous setting, we have an admissible pair $(E/E', \xi_t)$. This yields a cuspidal representation of $\mathrm{GL}_d(\mathbf{k}_{E'})$, and our choice of isomorphism $\iota : \mathbf{k}_{E'} \xrightarrow{\sim} \mathbf{k}_{F[\beta]}$ yields a representation of $J/J^1 \cong U_{\mathfrak{B}}/U_{\mathfrak{B}}^1$ (this last isomorphism is canonical). If we extend it to a representation Λ_ξ of $\mathbf{J} = F[\beta]^\times U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 = E^\times U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 = E'^\times U_{\mathfrak{B}}/U_{\mathfrak{B}}^1$ by letting it be a multiple of ξ_t on E', we obtain an irreducible representation

$$ {}_F\pi_\xi = \text{c-Ind}_{\mathbf{J}}^{\mathrm{GL}_n(F)} \Lambda_\xi \otimes \widetilde{\kappa}. $$

To see that this need not be well-defined, we can assume that $\overline{\xi}_t$, a character of $\mathbf{k}_E$, is not just $\mathrm{Gal}(\mathbf{k}_E/\mathbf{k}_{E'})$-regular but also $\mathrm{Gal}(\mathbf{k}_E/\mathbf{k}_F)$-regular. In our example, this amounts to requiring the Galois conjugacy class $[\overline{\xi}_t]$ to have size four. But now $[\overline{\xi}_t]$ splits into two conjugacy classes for $\mathrm{Gal}(\mathbf{k}_E/\mathbf{k}_{E'})$, corresponding to two nonisomorphic cuspidal representations of $\mathrm{GL}_2(\mathbf{k}_{E'})$, namely $\{\sigma[\overline{\xi}_t], \sigma[\overline{\xi}_t^q]\}$, that are switched by the outer automorphism coming from $\mathrm{Gal}(\mathbf{k}_{E'}/\mathbf{k}_F)$. The upshot is that the representation Λ_ξ really depends on the choice of $\iota_i : E' \xrightarrow{\sim} F[\beta]$. To see this, notice that the construction in [BH05] requires us to always work with $\sigma[\overline{\xi}_t]$. It is immediate that the pushforwards of this representation under the two different isomorphisms

$$ \iota_1, \iota_2 : \mathrm{GL}_2(\mathbf{k}_{E'}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbf{k}_{F[\beta]}) \xleftarrow{\sim} J/J^1 $$

are nonisomorphic representations σ_1, σ_2 of J/J^1.

So the construction produces two nonisomorphic representations Λ_{ξ,ι_i} of $\mathbf{J}$, with the property that $\Lambda_{\xi,\iota_i}|J \cong \sigma_i$. But then there is the following observation.

Lemma 4.1.7. With the notation above, the representations $\pi_1 = \text{c-Ind}_{\mathbf{J}}^{\mathrm{GL}_n(F)}(\Lambda_{\xi,\iota_1} \otimes \widetilde{\kappa})$ and $\pi_2 = \text{c-Ind}_{\mathbf{J}}^{\mathrm{GL}_n(F)}(\Lambda_{\xi,\iota_2} \otimes \widetilde{\kappa})$ are nonisomorphic irreducible cuspidal representations of $\mathrm{GL}_n(F)$.

Proof. It is enough to prove that

$$ \mathbf{K}_\kappa(\pi_i) = \mathrm{Hom}_{J^1}(\kappa, \pi_i) $$

are not isomorphic as representations of J/J^1. One way of doing so is to apply [MS14b, Lemme 5.3], which says that

$$ \mathrm{Hom}_{J^1}(\kappa, \pi_i) \cong \sigma_i $$

as J/J^1-representations, since π_i contains the maximal simple type $\kappa \otimes \sigma_i$. □

This is an issue since both π_1, π_2 should be isomorphic to ${}_F\pi_\xi$, according to [BH05, Section 2.3].

Here is how to fix the problem in our example. The endo-class $\Theta_F = \mathrm{Res}_{E'/F}(\phi)$ has degree two, and unramified parameter field isomorphic to E'. By proposition 2.4.8 there is an internal lift of θ, denoted $\theta_{F[\beta]}$, to a simple character for the group $\mathrm{GL}_2(F[\beta]) \subset \mathrm{GL}_4(F)$, which is the centralizer of $F[\beta]$ in A. There are two lifts of Θ_F to E': both have degree 1 over E', hence they correspond to two characters of $U^1_{E'}$, which are conjugate under $\mathrm{Gal}(E'/F)$. But the datum of the admissible pair $(E/F, \xi)$ singles out one of these two lifts, namely the one corresponding to ϕ. Let us denote it $\Theta_{E'}$.

From this, we get a natural choice of the isomorphism $\iota : E' \xrightarrow{\sim} F[\beta]$, by requesting that $\iota^* \,\mathrm{cl}(\theta_{F[\beta]}) = \Theta_{E'}$. This is *precisely* the outcome of the process of "rigidification via $\Theta_{E'} \to \Theta_F$" that will be described in the next section: in general, one does the same thing, except that there is no need to fix a choice of $E' \xrightarrow{\sim} F[\beta]$, but just of its restriction to the maximal unramified subextension, that we regard as a subfield of a fixed algebraic closure of F.

4.2 Lifts and rigidifications.

Let $A = M_m(D)$, a central simple algebra over F with $m \deg(D) = n$, and let $G = A^\times = \mathrm{GL}_m(D)$. Assume θ is a maximal simple character in G and write $\Theta_F = \mathrm{cl}(\theta)$.

Definition 4.2.1. A *parameter field* for θ in G is a subfield of A of the form $F[\beta]$ for a simple stratum $[\mathfrak{A}, \beta]$ defining θ. An *unramified parameter field* is a subfield of A of the form $F[\beta]^{\mathrm{ur}}$ for a parameter field $F[\beta]$ (the maximal unramified extension of F in $F[\beta]$).

Proposition 4.2.2. Let θ be a maximal simple character in G and let T_1, T_2 be unramified parameter fields for θ. Then

1. there exists $j \in J^1_\theta$ conjugating T_1 to T_2
2. if $j \in J^1_\theta$ normalizes an unramified parameter field T for θ, then it centralizes it.

It follows that there exists exactly one isomorphism $T_1 \to T_2$ which can be realized by conjugation by elements of J^1_θ.

Proof. This is very similar to [BH14, 2.6 Proposition]. Let $[\mathfrak{A}, \beta_i]$ be strata defining θ with $T_i = F[\beta_i]^{\mathrm{ur}}$. For the first part, given a generator ζ_1 of μ_{T_1}, there exists some generator $\zeta_2 \in \mu_{T_2}$ and some $j_1 \in J^1_\theta$ such that $\zeta_2 = \zeta_1 j_1$. This is because the inclusion yields isomorphisms $U(\mathfrak{B}_{\beta_i})/U^1(\mathfrak{B}_{\beta_i}) \to J_\theta/J^1_\theta$ embedding μ_{T_i} in the centre of J_θ/J^1_θ. The centre is given by the image of $\mathcal{O}^\times_{D'_i}$, hence might be larger than the image of μ_{T_i}, but it will still be a cyclic group. Since the μ_{T_i} have the same order, as the T_i have the same degree $f(\Theta_F)$ over F, they will have the same image under these maps.

By [BH14, 2.6 Conjugacy Lemma], $\zeta_2 = \zeta_1 j_1$ is J^1_θ-conjugate to some $\zeta_3 = \zeta_1 j_2$, where $j_2 \in J^1_\theta$ commutes with ζ_1. But then $j_2 = 1$ as its order has to be both a power of p (as $j_2 \in J^1_\theta$) and prime to p (as $j_2 = \zeta_1^{-1}\zeta_3$ and the factors at the right hand side commute). So the generator ζ_2 of μ_{T_2} is J^1_θ-conjugate to the generator ζ_1 of μ_{T_1}, and the claim follows.

The second part holds as μ_T generates T over F and embeds in J_θ/J^1_θ, on which the conjugation action of J^1_θ is trivial. □

The degree of an unramified parameter field of θ over F equals $f(\Theta_F)$, which is independent of the choice of $[\mathfrak{A}, \beta]$ defining θ, and even of the choice of a representative θ of Θ_F. Let $E = F_{f(\Theta_F)}$, the unramified extension of F in $\overline{F}$ of degree $f(\Theta_F)$. By proposition 4.2.2, between any two unramified parameter fields E_i for θ there is a distinguished F-linear isomorphism $\iota_{E_1,E_2} : E_1 \to E_2$. Choose F-linear isomorphisms

$$\iota_T : E \to T$$

for any parameter field T for θ, such that $\iota_{T_1,T_2}\iota_{T_1} = \iota_{T_2}$ throughout. Denote the system of the ι_T by ι.

Now fix a parameter field $F[\beta]$ for θ, and an $F[\beta]$-linear isomorphism $\Phi : B \to M_{m'}(D')$ for some central division algebra D' over $F[\beta]$, sending $\mathfrak{B} = B \cap \mathfrak{A}$ to $M_{m'}(\mathcal{O}_{D'})$. The choice of ι yields a distinguished embedding $\mathbf{e} \to \mathbf{d}'$, and the extension $\mathbf{d}'/\mathbf{e}$ has degree $d' = d/(d, \delta(\Theta_F))$, where $\delta(\Theta_F) = [F[\beta] : F]$, so we get a well-defined $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$-orbit of $\mathbf{e}$-linear isomorphisms $\mathbf{e}_{d'} \to \mathbf{d}'$ and $M_{m'}(\mathbf{d}') \to M_{m'}(\mathbf{e}_{d'})$. In all, the choice of ι specifies, for every maximal simple stratum $[\mathfrak{A}, \beta]$ defining θ and every $F[\beta]$-linear isomorphism $\Phi : B \to M_{m'}(D')$, an isomorphism

$$\Psi : J_\theta/J^1_\theta \to \mathrm{GL}_{m'}(\mathbf{e}_{d'}),$$

well-defined up to the action of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$ on matrix entries.

Proposition 4.2.3. The conjugacy class $\Psi(\iota)$ of this isomorphism under the natural action of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ on $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$, by inner automorphisms and Galois action on matrix entries, is independent of the choice of $[\mathfrak{A}, \beta]$ and Φ, and only depends on θ and ι.

Proof. Take two maximal simple strata $[\mathfrak{A}_i, \beta_i]$ defining θ, and fix $F[\beta_i]$-linear isomorphisms $\Phi_i : B_i \to M_{m'}(D'_i)$ to central division algebras D'_i over $F[\beta_i]$, sending $\mathfrak{B}_i$ to $M_{m'}(D'_i)$. We obtain isomorphisms

$$J_\theta/J^1_\theta \to U(\mathfrak{B}_i)/U^1(\mathfrak{B}_i) \to \mathrm{GL}_{m'}(\mathbf{d}'_i) \to \mathrm{GL}_{m'}(\mathbf{e}_{d'}) \tag{4.2.1}$$

well-defined up to Galois action on coefficients, where the first map is the inverse of the natural inclusion, the second is induced by Φ_i, and the third by an arbitrary choice of an isomorphism $\mathbf{d}'_i \to \mathbf{e}_{d'}$ that is $\mathbf{e}$-linear for the embedding $\mathbf{e} \to \mathbf{d}'_i$ induced by $\iota_{F[\beta_i]^{\mathrm{ur}}} : E \to F[\beta_i]^{\mathrm{ur}}$. The integers m'_i and d'_i coincide as they only depend on the endo-class of θ.

Observe that (4.2.1) arises from an analogous sequence

$$\mathfrak{j}(\beta_i, \mathfrak{A}_i)/\mathfrak{j}^1(\beta_i, \mathfrak{A}_i) \to \mathfrak{B}_i/\mathfrak{P}_1(\mathfrak{B}_i) \to M_{m'}(\mathbf{d}'_i) \to M_{m'}(\mathbf{e}_{d'})$$

of $\mathbf{e}$-linear ring isomorphisms between $\mathbf{e}$-algebras, on passing to the groups of units. The equality $\mathfrak{j}^1(\beta_1, \mathfrak{A}_i) = \mathfrak{j}^1(\beta_2, \mathfrak{A}_i)$ holds since $\mathfrak{j}^1(\beta_i, \mathfrak{A}_i) = J^1(\beta_i, \mathfrak{A}_i) - 1$. The orders $\mathfrak{j}(\beta_i, \mathfrak{A}_i)$ have the same group of units, since $\mathfrak{j}(\beta_i, \mathfrak{A}_i)^\times = J(\beta, \mathfrak{A}_i)$. The quotient $\mathfrak{j}(\beta_i, \mathfrak{A}_i)/\mathfrak{j}^1(\beta_i, \mathfrak{A}_i)$ is additively generated by its group of units (as for all matrix algebras over fields), hence $\mathfrak{j}(\beta_1, \mathfrak{A}_i) = \mathfrak{j}(\beta_2, \mathfrak{A}_i)$.

The $\mathbf{e}$-algebra structure on $\mathfrak{j}(\beta_i, \mathfrak{A}_i)/\mathfrak{j}^1(\beta_i, \mathfrak{A}_i)$ comes from the embedding $\iota_{F[\beta_i]^{\mathrm{ur}}}$ for $i \in \{1, 2\}$, and by construction these embeddings are conjugate by the action of J^1_θ. So these two $\mathbf{e}$-algebra structures coincide. The claim follows as we have two $\mathbf{e}$-linear ring isomorphisms $\mathfrak{j}(\beta_i, \mathfrak{A}_i)/\mathfrak{j}^1(\beta_i, \mathfrak{A}_i) \to M_{m'}(\mathbf{e}_{d'})$, which therefore differ by the action of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ by the Skolem–Noether theorem. □

Now we show how the choice of a lift $\Theta_E \to \Theta_F$ of Θ_F to E defines such a system ι of isomorphisms. Let $[\mathfrak{A}_i, \beta_i]$ for $i = 1, 2$ be a simple stratum in A defining θ, and consider the unramified parameter field $T_i = F[\beta_i]^{\mathrm{ur}}$ of F. Proposition 2.4.8 applies, as β_i commutes with T_i and $T_i[\beta_i] = F[\beta_i]$ is a field with $F[\beta_i]^\times \subseteq \mathfrak{K}(\mathfrak{A}_i)$, and we get an interior lift θ_{T_i}. Fix compatible isomorphisms $\iota_{T_i} : E \to T_i$ as in section 4.2. We get endo-classes

$$\Theta^i_E = \iota^*_{T_i} \operatorname{cl}(\theta_{T_i}).$$

Proposition 4.2.4. The endo-classes Θ^1_E and Θ^2_E are equal.

Proof. Because the ι_{T_i} are compatible, we have $\iota_{T_2} = \iota_{T_1,T_2}\iota_{T_1}$, for $\iota_{T_1,T_2} : T_1 \to T_2$ the only isomorphism induced by conjugation by elements of J^1_θ (see proposition 4.2.2). The relation

$$\Theta^2_E = \iota^*_{T_2} \operatorname{cl}(\theta_{T_2}) = \iota^*_{T_2}\iota^*_{T_1,T_2} \operatorname{cl}(\theta_{T_2})$$

holds. Assume ι_{T_1,T_2} is induced by conjugation by $j \in J^1_\theta$. Then

$$\iota^*_{T_1,T_2} \operatorname{cl}(\theta_{T_2}) = \operatorname{cl}(\operatorname{ad}(j)^*\theta_{T_2}).$$

However, $\operatorname{ad}(j)^*\theta_{T_2}$ is the T_1-lift of $\operatorname{ad}(j)^*\theta = \theta$, hence $\operatorname{ad}(j)^*\theta_{T_2} = \theta_{T_1}$, and the claim follows. □

Proposition 4.2.5. The group $\mathrm{Gal}(E/F)$ is simply transitive on the set $\mathrm{Res}_{E/F}^{-1}(\Theta_F)$ of E-lifts of Θ_F.

Proof. The action has been defined after (2.4.1). By [BH03, 1.5.1], $\mathrm{Gal}(E/F)$ is transitive on $\mathrm{Res}_{E/F}^{-1}(\Theta_F)$, which is in bijection with the set of simple components of $E \otimes_F F[\beta]$ for any parameter field $F[\beta]$ for θ. But E is F-isomorphic to the maximal unramified extension of F in $F[\beta]$, hence

$$E \otimes_F F[\beta] \cong \prod_{\sigma : E \to F[\beta]} F[\beta]$$

and so the fiber $\mathrm{Res}_{E/F}^{-1}(\Theta_F)$ has as many elements as $\mathrm{Gal}(E/F)$. □

It follows that for *any* unramified parameter field T for θ we can define $\iota_T : E \to T$ to be the only F-linear isomorphism such that $\iota_T^* \,\mathrm{cl}(\theta_T) = \Theta_E$; by proposition 4.2.5, ι_T is well-defined, and by proposition 4.2.4 this defines a compatible system of isomorphisms. So a choice of a lift $\Theta_E \to \Theta_F$ gives rise to a conjugacy class

$$\Psi(\Theta_E) : J_\theta / J_\theta^1 \to \mathrm{GL}_{m'}(\mathbf{e}_{d'})$$

for any maximal simple character θ in G with endo-class Θ_F, by setting $\Psi(\Theta_E) = \Psi(\iota)$ for the ι just constructed.

4.3 Level zero maps for supercuspidal inertial classes.

Using the construction in the previous section, we carry through the argument sketched at the beginning of this chapter and we attach to each supercuspidal inertial class $\mathfrak{s}$ of G a conjugacy class of characters, that we call the level zero part of $\mathfrak{s}$. We begin by recalling the definition and basic properties of the $\mathbf{K}$-functor associated to a β-extension κ of a maximal simple character θ in G, as in [MS14b, Section 5].

Definition 4.3.1. Define a functor from smooth representations of G to representations of J_θ / J_θ^1, by

$$\mathbf{K}_\kappa^+ : \pi \mapsto \mathrm{Hom}_{J_\theta^1}(\kappa|_{J_\theta^1}, \pi|_{J_\theta^1}),$$

with the action $x : f \mapsto x f x^{-1}$ for $x \in J_\theta$.

The functor $\mathbf{K}_\kappa^+$ is exact, and its behaviour on cuspidal representations of G is recorded in the following lemma.

Lemma 4.3.2. [MS14b, Lemme 5.3] Let ρ be a cuspidal irreducible representation of G. Then

1. if ρ does not contain θ, then $\mathbf{K}^+_\kappa(\rho) = 0$.

2. if ρ contains the maximal simple type $(J_\theta, \kappa \otimes \sigma)$ then

$$\mathbf{K}^+_\kappa(\rho) \cong \sigma \oplus \sigma^\phi \oplus \cdots \oplus \sigma^{\phi^{b(\rho)-1}},$$

where ϕ is a generator of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$ and $b(\rho)$ is the size of the orbit of σ under the action of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$.

If we fix a lift $\Theta_E \to \Theta_F$, we can attach to every cuspidal representation π of G containing θ an orbit of cuspidal representations of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ under $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$: it is enough to take the pushforward of $\mathbf{K}^+_\kappa(\pi)$ under any isomorphism in the conjugacy class $\Psi(\Theta_E)$. By propositions 2.5.7 and 2.5.16, and remark 2.5.20, we get an orbit of $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$ on the set of characters of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$. We introduce the notation $X_R(\Theta_F)$ for the set of $R^\times$-valued characters of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$, and $\Gamma(\Theta_F)$ for the group $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$. We will refer to the map just constructed

$$\Lambda_{\kappa,R} : (\text{cuspidal representations of endo-class } \Theta_F) \to \Gamma(\Theta_F)\backslash X_R(\Theta_F)$$

as the *level zero map* attached to $\Theta_E \to \Theta_F$ and κ.

Proposition 4.3.3. If $\theta_1 = \mathrm{ad}(g)^*\theta_2$ are conjugate maximal simple characters in G, and $\kappa_1 = \mathrm{ad}(g)^*\kappa_2$ are β-extensions of the θ_i, then $\Lambda_{\kappa_1,R} = \Lambda_{\kappa_2,R}$. Conversely, if κ_1, κ_2 are β-extensions of θ with $\Lambda_{\kappa_1,R} = \Lambda_{\kappa_2,R}$, then $\kappa_1 = \kappa_2$.

Proof. Let E_1 be an unramified parameter field for θ_1, and let $\iota_{E_1} : E \to E_1$ be the only F-linear isomorphism with $\iota_{E_1}^* \operatorname{cl}(\theta_{1,E_1}) = \Theta_E$. Then gE_1g^{-1} is an unramified parameter field for θ_2, and we have an isomorphism $\mathrm{ad}(g) \circ \iota_{E_1} : E \to gE_1g^{-1}$. Since $\theta_1 = \mathrm{ad}(g)^*\theta_2$, the relation $\theta_{1,E_1} = \mathrm{ad}(g)^*\theta_{2,gE_1g^{-1}}$ holds on the interior lifts. Hence $(\mathrm{ad}(g)\circ\iota_{E_1})^* \operatorname{cl}\theta_{2,gE_1g^{-1}} = \Theta_E$ and $\mathrm{ad}(g) \circ \iota_{E_1}$ is the isomorphism specified by Θ_E. So conjugation by g preserves the classes $\Psi(\Theta_E)$ of isomorphisms $J_{\theta_i}/J^1_{\theta_i} \to \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$, and since $\mathbf{K}_{\kappa_1} = \mathrm{ad}(g)^*\mathbf{K}_{\kappa_2}$ the first claim follows.

Now assume that the κ_i are β-extensions of θ and $\Lambda_{\kappa_1} = \Lambda_{\kappa_2}$. By the proof of proposition 2.4.9, the κ_i are twists of each other by a character χ of $\mathbf{e}^\times$. Then χ fixes all elements of $\Gamma(\Theta_F)\backslash X_R(\Theta_F)$ giving rise to cuspidal R-representations of $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$, because these also give rise to cuspidal representations of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ (recall that $m'd' = n/\delta(\Theta_F)$). By proposition 2.5.19, we have that $\chi = 1$. □

By [MS14a, Lemmas 6.1, 6.8], we see that $\Lambda_{\kappa,R}(\pi)$ is supercuspidal if and only if π is supercuspidal, and we know that an inertial class of supercuspidal representations consists of unramified twists of a single representation, and is determined by the corresponding maximal simple type. So $\Lambda_{\kappa,R}$ only depends on the inertial class, when restricted to supercuspidal representations. To extend the level zero map to simple inertial classes, we need to study the compatibility of $\mathbf{K}$-functors with parabolic induction, which we do in the next section, following [MS14b, Section 5.3].

4.4 Compatible β-extensions.

Because of proposition 4.3.3, we will work throughout this section with β-extensions up to inner conjugacy in $\mathrm{GL}_m(D)$. Let $(m_1, \dots, m_r)$ be a sequence of positive integers summing to m, defining a Levi subgroup M of $\mathrm{GL}_m(D)$. Let Θ_F be an endo-class whose degree divides all the integers $n_i = m_i d$. Fix a β-extension κ of a maximal simple character θ in $\mathrm{GL}_m(D)$ of endo-class Θ_F. Fix a lift $\Theta_E \to \Theta_F$ to the unramified parameter field.

Definition 4.4.1. For $1 \leq i \leq r$, let κ_i be a conjugacy class of maximal β-extension in $\mathrm{GL}_{m_i}(D)$ of endo-class Θ_F. We say that the sequence (κ_i) is compatible with (the conjugacy class of) κ if, for all irreducible representations of $\mathrm{GL}_{m_i}(D)$, there is an isomorphism

$$\mathbf{K}_\kappa(\pi_1 \times \cdots \times \pi_r) \to \mathbf{K}_{\kappa_1}(\pi_1) \times \cdots \times \mathbf{K}_{\kappa_r}(\pi_r), \tag{4.4.1}$$

where $\prod_i \mathrm{GL}_{m_i}(D)$ is block-diagonally embedded, and the parabolic induction at the right-hand side is for the Levi subgroup $\prod_i \mathrm{GL}_{m_i'}(\mathbf{e}_{d'})$ of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$.

Remark 4.4.2. We have $m_i'd' = m_i d/\delta(\Theta_F)$, and $d' = d/(d, \delta(\Theta_F))$, so that $m_i' = m_i(d, \delta(\Theta_F))/\delta(\Theta_F)$ and $\sum_i m_i' = m'$. So the parabolic induction makes sense.

Proposition 4.4.3. For any given choice of κ and (m_i), there exists a unique sequence (κ_i) that is compatible with κ. It is already determined by equation (4.4.1) for irreducible cuspidal representations π_i. If $m_i = m_j$, then $\kappa_i = \kappa_j$.

Proof. Let us first prove uniqueness. Assume we are given sequences (κ_j) and $(\varkappa_j)$, compatible with κ in the sense that equation (4.4.1) holds whenever the π_i are irreducible cuspidal representations of $\mathrm{GL}_{m_i}(D)$. There exist characters $\chi_i : \mathbf{e}^\times \to R^\times$ such that $\kappa_i = \chi_i \varkappa_i$. It suffices to prove that $\chi_i = 1$ for all i. To do so, let $[\xi]$ be a $\mathrm{Gal}(\mathbf{e}_{n_i/\delta(\Theta_F)}/\mathbf{e})$-orbit of R-characters of $\mathbf{e}^\times_{n_i/\delta(\Theta_F)}$, giving rise to a cuspidal representation of $\mathrm{GL}_{m_i'd'}(\mathbf{e})$. By proposition 2.5.19, it suffices to prove that $[\xi] = [\chi_i \xi]$.

Let σ be an irreducible cuspidal representation of $\mathrm{GL}_{m_i'}(\mathbf{e}_{d'})$ attached to $[\xi]$. For arbitrary j (possibly equal to i) let μ_j be an irreducible cuspidal representation of $\mathrm{GL}_{m_j'}(\mathbf{e}_{d'})$, such that $\mu_{j_1} = \mu_{j_2}$ if $m_{j_1}' = m_{j_2}'$, and $\mu_i = \sigma$. Let π_j be an irreducible cuspidal representation of $\mathrm{GL}_{m_j'}(D')$ containing the maximal simple type $\kappa_j \otimes \mu_j$.

Let X be a quotient of $\mu_1 \times \cdots \times \mu_i \times \cdots \times \mu_r$. By the compatibility assumption applied to $\mathbf{K}_\kappa(\pi_1 \times \cdots \times \pi_r)$, together with proposition 4.3.2, we deduce that X is also a quotient of

$$(\chi_1 \mu_1)^{g_1} \times \cdots \times (\chi_i \mu_i)^{g_i} \times \cdots \times (\chi_r \mu_r)^{g_r}$$

for certain $g_i \in \mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$. But then uniqueness of the cuspidal support for finite general linear groups implies that, whenever $m'_i = m'_j$, we have that σ and $\chi_j\sigma$ are in the same $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$-orbit. Applying this to $j = i$, it follows that $[\xi] = [\chi_i\xi]$.

The existence of compatible β-extensions is established during the construction of covers of maximal simple types, and the rest of proposition 4.4.3 is a consequence of [MS14b, Proposition 5.9], although this reference does not keep track of Θ_E. So we review the construction briefly. Fix a decomposition $D^m = D^{m_1} \oplus \cdots \oplus D^{m_r}$. Assume that θ is defined by a stratum $[\Lambda_{\max}, \beta]$ such that $F(\beta)$ preserves the decomposition and $\Lambda_{\max}$ conforms to the decomposition (see [MS14b, 5.3]). We obtain an embedded block-diagonal $\prod_i A_i \cong \prod_i M_{m_i}(D)$ in $A = M_m(D)$, for which $F(\beta)$ is diagonally embedded. The commutant $B = Z_A(E)$ contains the product $B_1 \times \cdots \times B_r$, where $B_i = Z_{A_i}(E)$. We have furthermore lattice chains $\Lambda_{\max,i} = \Lambda_{\max} \cap D^{m_i}$, and corresponding strata in the A_i.

Now transfer θ to maximal simple characters θ_i defined on $[\Lambda_{\max,i}, \beta_i]$, where β_i is the projection of β to B_i. We are going to see that κ determines a β-extension of each of these θ_i. For this, we need to fix an $F[\beta]$-linear isomorphism $\Phi : B \to M_m(F[\beta])$ identifying $U^0(\Lambda_{\max}) \cap B$ with $M_m(\mathcal{O}_{F[\beta]})$, and a simple stratum $[\Lambda, \beta]$ in A satisfying conditions (1) and (2) in [MS14b, Section 5.3].

Transfer θ to a simple character θ_Λ defined on $[\Lambda, \beta]$ (it won't be maximal) and let κ_Λ be the transfer of κ to a β-extension of θ_Λ. Then let N be the upper-triangular unipotent group defined by $(m_1, \ldots, m_r)$, and take the invariants of κ_Λ under $J(\beta, \Lambda) \cap N$: this is a representation of $J(\beta, \Lambda) \cap M$ which by [SS12, Proposition 6.6] decomposes as a tensor product of β-extensions κ_i of the θ_i. If $m_i = m_j$ then the same reference shows that $\kappa_i \cong \kappa_j$: this implies the second claim of the proposition once we prove that (κ_i) is compatible with κ.

Consider the functors $\mathbf{K}_{\kappa_i}$ that the maximal β-extensions κ_i define with the respect to the same lift $\Theta_E \to \Theta_F$. The lift Θ_E defines an F-linear embedding $\iota : E \to F[\beta]$, characterized by the equality $\mathrm{cl}(\iota^*\theta_{F[\beta]^{\mathrm{ur}}}) = \Theta_E$. By construction, the image of $F[\beta]$ under the projection to B_i is a parameter field for θ_i.

Lemma 4.4.4. The equality $\mathrm{cl}(\iota^*(\theta_{i,F[\beta]^{\mathrm{ur}}})) = \Theta_E$ holds.

Proof. Recall that θ_i is the transfer of θ to $\mathfrak{A}_{\max,i}$. We know by assumption that the interior lift $\theta_{F[\beta]^{\mathrm{ur}}}$ has endo-class Θ_E under ι, and the content of the lemma is that the same is true for these transfers. This follows from the compatibility between interior lifts and transfer maps, for which see for instance [BSS12, Theorem 6.7]. □

Now by [MS14b, Proposition 5.9], we have an isomorphism

$$\mathbf{K}^+_\kappa(\pi_1 \times \cdots \times \pi_r) \to \mathbf{K}^+_{\kappa_1}(\pi_1) \times \cdots \times \mathbf{K}^+_{\kappa_r}(\pi_r).$$

Here, both sides are representations of J_θ/J_θ^1 and the parabolic induction refers to $\prod_i J_{\theta_i}/J_{\theta_i}^1$, identified with a Levi subgroup of J_θ/J_θ^1. By lemma 4.4.4, any isomorphism $J_\theta/J_\theta^1 \to \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ in the class $\Psi(\Theta_E)$ restricts to an isomorphism $\prod_i J_{\theta_i}/J_{\theta_i}^1 \to \prod_i \mathrm{GL}_{m'_i}(\mathbf{e}_{d'})$ which is in the class $\Psi(\Theta_E)$ on each factor. The claim follows. □

Whenever we have $(m_1, \dots, m_r)$, and a sequence (κ_i) with the same endo-class as κ, it makes sense to ask whether they are compatible. As remarked above, a necessary condition is that $\kappa_i \cong \kappa_j$ if $i = j$. In the simple case, all the m_i are equal to m/r for some positive divisor r of m, and we can make the following definition.

Definition 4.4.5. We say that maximal β-extensions $\kappa_{m/r}$ in $\mathrm{GL}_{m/r}(D)$ and κ_m in $\mathrm{GL}_m(D)$, of endo-class Θ_F, are *compatible* if κ_m is compatible with $(\kappa_{m/r}, \dots, \kappa_{m/r})$.

Notice that every element of a compatible pair $(\kappa_{m/r}, \kappa_m)$ determines the other uniquely: this follows from proposition 4.4.3 and the fact that if $(\kappa_{m/r}, \kappa_m)$ are compatible then so are $(\chi\kappa_{m/r}, \chi\kappa_m)$ for all characters χ of $\mathbf{e}^\times$. We end this section by recording the following transitivity result with respect to refining the decomposition (m_i).

Proposition 4.4.6. Let $(m_i)_{i \in I}$ be a sequence summing to m. Assume that for all i we have a sequence $(m_{ij})_{j \in J_i}$ of positive integers summing to m_i, such that $\delta(\Theta_F)$ divides every $n_{ij} = m_{ij}d$. Let κ be a β-extension in $\mathrm{GL}_m(D)$ of endo-class Θ_F. Let $(\kappa_i)_{i \in I}$ be a sequence of β-extensions compatible with κ, and for all i let $(\kappa_{ij})_{j \in J_i}$ be a sequence compatible with κ_i. Then (κ_{ij}) is compatible with κ.

Proof. Fix irreducible cuspidal representations ρ_{ij} of $\mathrm{GL}_{m_{ij}}(D)$, and form $\mathbf{K}$-functors with respect to a fixed lift Θ_E. We can twist the ρ_{ij} by unramified characters, and assume that no two of them are linked. By [MS14a, Théorème 7.24], this implies that the parabolic inductions $\times_{j \in J_i} \rho_{ij}$ are irreducible. So by proposition 4.4.3 we deduce that there are isomorphisms

$$\mathbf{K}_\kappa(\times_{i,j}\rho_{ij}) \to \times_{i \in I}\mathbf{K}_{\kappa_i}(\times_{j \in J_i}\rho_{ij}) \to \times_{ij}\mathbf{K}_{\kappa_{ij}}(\rho_{ij}). \tag{4.4.2}$$

Since the $\mathbf{K}$-functors only depend on the restriction of a representation to a compact open subgroup, they are invariant under unramified twists. Hence (4.4.2) is actually true for all cuspidal irreducible representations ρ_{ij}. Then the claim follows from the uniqueness part of proposition 4.4.3. □

Remark 4.4.7. A compatibility of this kind is implicit in [MS14b, Remarque 5.17], so probably it can be proved directly (without the use of $\mathbf{K}$-functors).

Definition 4.5.1. Define the level zero map

$$\Lambda_{\kappa,R} : (\text{simple inertial classes of endo-class } \Theta_F) \to \Gamma(\Theta_F)\backslash X_R(\Theta_F)$$

by sending the inertial class of supercuspidal support $[\mathrm{GL}_{m/r}(D), \pi_r^{\times r}]$ to the inflation of $\Lambda_{\kappa_{m/r},R}(\pi_r)$ through the norm $\mathbf{e}^\times_{n/\delta(\Theta_F)} \to \mathbf{e}^\times_{n/r\delta(\Theta_F)}$. Here $\kappa_{m/r}$ is the maximal β-extension compatible with κ.

Proposition 4.5.2. Let π be a cuspidal, non-supercuspidal irreducible representation of $\mathrm{GL}_m(D)$. Then the two definitions we have given for the level zero part of π coincide.

Proof. Let l be the characteristic of R. Choose an element of $\Psi(\Theta_E)$ and then a factor σ of $\mathbf{K}_\kappa(\pi)$, viewed as a cuspidal representation of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$. Then σ corresponds to an orbit $[\chi]_{d'}$ of characters $\mathbf{e}^\times_{n/\delta(\Theta_F)} \to R^\times$ under $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e}_{d'})$. Our first definition yields $\Lambda_{\kappa,R}(\pi) = [\chi]$, the orbit under $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$.

Since π is not supercuspidal, neither is σ, and so χ is not itself $\mathbf{e}_{d'}$-regular. Hence it is norm-inflated from an $\mathbf{e}_{d'}$-regular character χ_s of some intermediate $\mathbf{e}^\times_{n/\delta(\Theta_F)s}$. The supercuspidal support of σ corresponds to the $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)s}/\mathbf{e}_{d'})$-orbit of χ_s.

The supercuspidal support of π is inertially equivalent to some $\left(\mathrm{GL}_{m/r}(D), \pi_r^{\times r}\right)$. The second definition of $\Lambda_{\kappa,R}(\pi)$ is the inflation of the character orbit $\Lambda_{\kappa_{m/r},R}(\pi_r)$. By construction, $\Lambda_{\kappa_{m/r_r},R}(\pi_r)$ is an orbit of $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)r}/\mathbf{e})$ on the $\mathbf{e}_{d'}$-regular characters of $\mathbf{e}^\times_{n/\delta(\Theta_F)r}$. By compatibility of κ and $\kappa_{m/r}$, the supercuspidal support of σ consists of representations attached to $\Lambda_{\kappa_{m/r},R}(\pi_0)$. It follows that $r = s$, and the two definitions agree. □

We have the following proposition concerning reduction modulo l, which uses the fact that a β-extension κ of a maximal simple $\overline{\mathbf{Q}}_l$-character θ is integral and the reduction $\mathbf{r}_l(\kappa)$ is a β-extension of $\mathbf{r}_l(\theta)$, a maximal simple $\overline{\mathbf{F}}_l$-character [MS14b, Proposition 2.37].

Lemma 4.5.3. Let π be an integral $\overline{\mathbf{Q}}_l$-representation of $\mathrm{GL}_m(D)$ which is simple of endo-class Θ_F. If τ is a Jordan–Hölder factor of $\mathbf{r}_l(\pi)$, then $\Lambda_{\mathbf{r}_l(\kappa),\overline{\mathbf{F}}_l}(\tau) = \Lambda_{\kappa,\overline{\mathbf{Q}}_l}(\pi)^{(l)}$.

Proof. In this proof, we write "support" for "supercuspidal support". There exists r such that the representation π is a subquotient of a parabolic induction $\chi_1\pi_r \times \cdots \times \chi_r\pi_r$ for an integral supercuspidal representation π_r of $\mathrm{GL}_{m/r}(D)$ and unramified characters χ_i valued in $\overline{\mathbf{Z}}_l^\times$. Then the Jordan–Hölder factors of $\mathbf{r}_l(\pi)$ form a subset of those of $\overline{\chi}_1\mathbf{r}_l(\pi_r) \times \cdots \times \overline{\chi}_n\mathbf{r}_l(\pi_r)$.

By construction, every representation appearing in the support of a factor of $\mathbf{K}_{\mathbf{r}_l(\kappa)}(\tau)$ is attached to $\Lambda_{\mathbf{r}_l(\kappa),\overline{\mathbf{F}}_l}(\tau)^{\mathrm{reg}}$. By [MS14b, Lemme 5.11], the equality $\mathbf{r}_l[\mathbf{K}_\kappa(\pi)] = [\mathbf{K}_{\mathbf{r}_l(\kappa)}(\mathbf{r}_l(\pi))]$ holds.

Let σ be a factor of $\mathbf{K}_\kappa(\pi)$. Then $sc(\sigma) \in [\mathbf{K}_{\kappa_{m/r}}(\pi_r)^{\times r}]$, where $\kappa_{m/r}$ is compatible with κ, by (4.4.1). Hence, the support of any factor of $\mathbf{r}_l[\mathbf{K}_\kappa(\pi)]$ coincides with the support of a factor of $\mathbf{r}_l[\mathbf{K}_{\kappa_{m/r}}(\pi_r)^{\times r}]$. By definition, $[\mathbf{K}_{\kappa_{m/r}}(\pi_r)]$ consists of the irreducible, cuspidal $\overline{\mathbf{Q}}_l$-representations attached to $\Lambda_{\kappa,\overline{\mathbf{Q}}_l}(\pi)$. The support of the reduction of any of these is a multiple of a representation attached to $\Lambda_{\kappa,\overline{\mathbf{Q}}_l}(\pi)^{(l),\mathrm{reg}}$. Hence, every representation appearing in the support a factor of $\mathbf{r}_l[\mathbf{K}_\kappa(\pi)]$ is attached to $\Lambda_{\kappa,\overline{\mathbf{Q}}_l}(\pi)^{(l),\mathrm{reg}}$. Since $[\mathbf{K}_{\mathbf{r}_l(\kappa)}(\tau)]$ is a subset of $\mathbf{r}_l[\mathbf{K}_\kappa(\pi)]$, the claim follows. $\square$

We can finally define our parametrization of simple inertial classes. Fix for every endo-class Θ_F a lift $\Theta_E \to \Theta_F$ and a conjugacy class of maximal β-extensions κ in $\mathrm{GL}_m(D)$ of endo-class Θ_F. In more detail, by proposition 2.4.5 any two maximal simple characters in $\mathrm{GL}_m(D)$ of endo-class Θ_F are conjugate, and it is possible to choose their β-extensions so that, whenever $\theta_1 = \mathrm{ad}(g)^*\theta_2$ for some $g \in \mathrm{GL}_m(D)$, one has $\kappa_1 = \mathrm{ad}(g)^*\kappa_2$. For this to be a well-defined conjugacy class we need to check that if $g \in G$ normalizes θ, then it normalizes κ; but the normalizer $\mathbf{J}(\theta)$ of θ in G normalizes J_θ, which is the unique maximal compact subgroup of $\mathbf{J}(\theta)$, and θ and κ have the same G-intertwining (this is a defining property of β-extensions), hence the claim follows.

This allows us to define two invariants of a simple inertial class $\mathfrak{s} \in \mathfrak{B}_R(G)$, namely the endo-class $\mathrm{cl}(\mathfrak{s})$ of any maximal simple character contained in the supercuspidal support of $\mathfrak{s}$, and the level zero part $\Lambda_{\kappa(\mathfrak{s}),R}(\mathfrak{s})$, where $\kappa(\mathfrak{s})$ is any β-extension in the class we have attached to $\mathrm{cl}(\mathfrak{s})$, computed with respect to the lift that we have fixed for $\mathrm{cl}(\mathfrak{s})$. This is well-defined by proposition 4.3.3.

Theorem 4.5.4. Fix lifts $\Theta_E \to \Theta_F$ for every endo-class Θ_F over F. The resulting map

$$\mathrm{inv} : \mathfrak{s} \mapsto (\mathrm{cl}(\mathfrak{s}), \Lambda_{\kappa(\mathfrak{s}),R}(\mathfrak{s}))$$

is a bijection from the set of simple inertial classes of R-representations of $\mathrm{GL}_m(D)$ to the set of pairs $(\Theta_F, [\chi])$ consisting of an endo-class Θ_F of degree dividing $n = md$ and a character orbit $[\chi] \in \Gamma(\Theta_F)\backslash X_R(\Theta_F)$.

Proof. A supercuspidal inertial class $\mathfrak{s}$ is determined by the conjugacy class of maximal simple types it contains, which can be recovered from the image of this map. To see this, assume given $(\Theta_F, [\chi])$ such that $[\chi]$ is $\mathbf{e}_{d'}$-regular, and let θ be a maximal simple character in $\mathrm{GL}_m(D)$ with endo-class Θ_F and β-extension κ. Then the maximal simple types $(J_\theta, \kappa \otimes \sigma_i)$, where the σ_i are any two supercuspidal representations of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ in the orbit corresponding to $[\chi]$, inflated to J_θ/J^1_θ under any element of $\Psi(\Theta_E)$, are conjugate in $\mathrm{GL}_m(D)$ and correspond to a supercuspidal inertial class. Indeed, if $[\mathfrak{A}, \beta]$ is a simple stratum defining θ, and we fix an $F[\beta]$-linear isomorphism $B \to M_{m'}(D')$, then the normalizer $\mathbf{J}(\theta) = \pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$, and conjugation by $\pi_{D'}$ acts as the Frobenius element of $\mathrm{Gal}(\mathbf{d}'/\mathbf{e})$ on $\mathbf{d}'$. Hence our map is injective when restricted to supercuspidal inertial classes, and its image consists of those $(\Theta_F, [\chi])$ such that $[\chi]$ is $\mathbf{e}_{d'}$-regular.

The result then follows because, if $\mathfrak{s} = [\mathrm{GL}_{m/r}(D), \pi_r^{\times r}]$, then

$$\mathrm{inv}_{\mathrm{GL}_m(D)}(\mathfrak{s}) = \mathrm{inv}_{\mathrm{GL}_{m/r}(D)}[\mathrm{GL}_{m/r}(D), \pi_r],$$

and we can use the result for supercuspidal inertial classes of $\mathrm{GL}_{m/r}(D)$, where we define $\mathrm{inv}_{\mathrm{GL}_{m/r}(D)}$ using the compatible β-extensions. □

Example 4.5.5. Fix Θ_F and notice that the simple inertial classes in $\mathrm{GL}_m(D)$ of endo-class Θ_F are in bijection with the union of the supercuspidal inertial classes in $\mathrm{GL}_t(D)$ of endo-class Θ_F, where t is a divisor of m such that $\delta(\Theta_F)$ divides n/t. We are claiming that these are also in bijection with $\Gamma(\Theta_F)\backslash X_R(\Theta_F)$. Every element of $\Gamma(\Theta_F)\backslash X_R(\Theta_F)$ is regular for precisely one subfield of $\mathbf{e}_{n/\delta(\Theta_F)}$ containing $\mathbf{e}_{d'}$, and $\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e}_{d'}$ has degree $m' = \frac{n}{\delta(\Theta_F)d'}$. It is possible to see that the two sets have the same number of elements, independently of the previous discussion. To do this, it suffices to prove that for a divisor $t|m$, we have that $\delta(\Theta_F)|n/t$ and $t|m'$ are equivalent conditions.

Assume first that $\delta(\Theta_F)|n/t$. Then $\mathrm{GL}_t(D)$ has a maximal simple character of endo-class Θ_F, with a parameter field whose commutant is isomorphic to some $M_{m'_t}(D'_t)$ with $m'_t = \frac{n}{t\delta(\Theta_F)d'}$, whereas $m' = \frac{n}{\delta(\Theta_F)d'}$. Since m'_t is an integer and $m' = m'_t \cdot t$, we have $t|m'$. Conversely, assume that $t|m'$ and write $m' = m''t$ for an integer m''. Then $n/\delta(\Theta_F) = m'd' = m''td'$, so $n/t = \delta(\Theta_F)m''d'$ and $\delta(\Theta_F)|n/t$.

Definition 4.5.6. To emphasize the dependence of $\Lambda_{\kappa(\mathfrak{s})}$ on the choice of a lift of $\mathrm{cl}(\mathfrak{s})$ to its unramified parameter field, we will write the inverse to inv as a map $(\Theta_F, \Theta_E, [\chi]) \mapsto \mathfrak{s}_G(\Theta_F, \Theta_E, [\chi])$, with finite fibers consisting of orbits of $\mathrm{Gal}(\mathbf{e}/\mathbf{f})$ acting diagonally by $g \cdot (\Theta_F, \Theta_E, [\chi]) = (\Theta_F, g^*\Theta_E, (g^{-1})^*[\chi])$.

Example 4.5.7. The triple $(\Theta_F, \Theta_E, [\chi])$ corresponds to a supercuspidal inertial class of $\mathrm{GL}_n(F)$ if and only if $[\chi]$ consists of $\mathbf{e}$-regular characters of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$. If this happens, then $\mathfrak{s}_G(\Theta_F, \Theta_E, [\chi])$ is supercuspidal for all inner forms G of $\mathrm{GL}_n(F)$. When the inner form is $D^\times$ for a division algebra D, one has $\mathbf{e}_{n/\delta(\Theta_F)} = \mathbf{e}_{d'}$, so every triple is supercuspidal for $D^\times$—of course, this is as expected because $D^\times$ has no nontrivial rational parabolic subgroups and so every irreducible smooth representation is supercuspidal.

Remark 4.5.8. In [BSS12] there is assigned an endo-class $\Theta(\mathfrak{s})$ to every simple inertial class $\mathfrak{s}$ of complex representations of $\mathrm{GL}_m(D)$, defined to be the endo-class of any simple character contained in representations in $\mathfrak{s}$. Since $\mathfrak{s}$ needs not be supercuspidal, these characters need not be maximal simple characters, but if $\mathfrak{s}_0$ is supercuspidal then $\Theta(\mathfrak{s}_0) = \mathrm{cl}(\mathfrak{s}_0)$ by definition. By [SS16b, Remark 6.8], we have that $\Theta(\mathfrak{s}) = \Theta(\mathfrak{s}_0)$ if $\mathfrak{s}$ is inertially equivalent to a multiple of a supercuspidal inertial class $\mathfrak{s}_0$ (this is implicit in the construction of compatible β-extensions). Then by construction we see that $\mathrm{cl}(\mathfrak{s}) = \Theta(\mathfrak{s})$ for every simple inertial class $\mathfrak{s}$.

Remark 4.5.9. In [MS14b, Section 3.4] there is defined a number of invariants attached to a cuspidal representation ρ of $\mathrm{GL}_m(D)$. If $(J, \kappa \otimes \sigma)$ is a maximal simple type in ρ, these are

1. $n(\rho)$, the *torsion number*, which is the number of unramified characters χ of G such that $\rho \otimes \chi \cong \chi$
2. $b(\rho)$, the size of the orbit of σ under the action of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$
3. $s(\rho)$, the order of the stabilizer of σ in $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$
4. $f(\rho) = n/e(\Theta_F)$.

These only depend on the inertial class of ρ and can be read off from our parametrization. We make this explicit over the complex numbers. Write $\mathrm{inv}(\rho) = (\Theta_F, [\chi])$. We have the equality $f(\rho) = n(\rho)s(\rho)$, by an explicit computation using [BH11, 2.6.2(4)(b)] (or see [MS14b, Equation (3.6)]). We also note that [BH11, Section 2] defines a *parametric degree* for all simple representations, and for a supercuspidal ρ this equals $n/s(\rho)$ (see [SS16b, Section 3.1]).

The stabilizer S_1 of any representative χ of $[\chi]$ under $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$ is isomorphic to the stabilizer S_2 of the corresponding cuspidal representation of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ under $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$. Indeed, S_1 surjects onto S_2 by restriction and $S_1 \cap \mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e}_{d'}) = 1$, because $[\chi]$ consists of $\mathbf{e}_{d'}$-regular characters. The quantity $s(\rho)$ therefore also equals $s[\chi]$, the order of the stabilizer of any element of $[\chi]$ under $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$. We will also denote by $b[\chi]$ the size of the orbit under $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$ of any representation of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ of the form $\sigma[\chi]$ for $\chi \in [\chi]$.

5

The inertial Jacquet–Langlands correspondence.

We now proceed to the proof of theorems A and B from the introduction. As in the previous chapter, we let $G = \mathrm{GL}_m(D)$ and we fix lifts $\Theta_E \to \Theta_F$ of all endo-classes of degree dividing n. For any given Θ_F, we will work with the conjugacy class of p-primary maximal β-extensions in $\mathrm{GL}_m(D)$ of endo-class Θ_F. By compatibility, this determines a conjugacy class of maximal β-extensions of any endo-class in any group $\mathrm{GL}_m(D)$, as m varies. These compatible conjugacy classes, however, need not be p-primary. We write Λ_κ and inv for the corresponding level zero and invariant maps. We will work over the complex numbers unless stated otherwise.

5.1 A character formula.

Consider a supercuspidal irreducible representation π of G. Let $\mathfrak{s} = \mathfrak{s}_G(\Theta_F, \Theta_E, [\chi])$ be the inertial class of π, and let (J, λ) be a maximal simple type for $\mathfrak{s}$, so that π is the compact induction $\text{c-Ind}_{\mathbf{J}(\lambda)}^G \Pi$ of an extension Π of λ to its normalizer $\mathbf{J}(\lambda)$. The type (J, λ) is constructed from a maximal simple character θ with endo-class Θ_F, and we fix a simple stratum $[\mathfrak{A}, \beta]$ defining θ. Let $T = F[\beta]^{\mathrm{ur}}$, an unramified parameter field for θ.

Write $B = Z_A(F[\beta]) \cong M_{m'}(D')$ for the commutant of $F[\beta]$ in A. Fix an extension $L/F[\beta]$ in B that has maximal degree, is unramified and normalizes the order $\mathfrak{A}$. Such an L exists by the arguments in the proof of proposition 2.4.5. Consider the maximal unramified extension $K = L^{\mathrm{ur}}$ of F in L, write A_K for the commutant $Z_A(K)$, and let $G_K = A_K^\times$. In this context, the normalizer $N_G(K)$ acts on G_K, and there is an isomorphism

$$N_G(K)/G_K \to \mathrm{Gal}(K/F)$$

by the conjugation action on K. It follows that $\mathrm{Gal}(K/F)$ has a right action on isomorphism classes of representations of G_K: if τ is a representation and $t_\alpha \in N_G(K)$ maps to $\alpha \in \mathrm{Gal}(K/F)$, denote by τ^α the representation $g \mapsto \tau(t_\alpha g t_\alpha^{-1})$. The isomorphism class of τ^α is independent of the choice of preimage t_α of α. If τ has endo-class Θ_K, then τ^α has endo-class $\alpha^*\Theta_K$.

Since β commutes with K and generates a field $L = K[\beta]$ over K, and $L^\times \subseteq \mathfrak{K}(\mathfrak{A})$, proposition 2.4.8 applies and θ has an interior K-lift θ_K. This is a character of $H^1_K = H^1_\theta \cap B$, and it is defined by the simple stratum $[\mathfrak{A}_K, \beta]$ for $\mathfrak{A}_K = \mathfrak{A} \cap A_K$. It is a maximal simple character, because β generates L over K, L is self-centralizing in A_K, and L has a unique hereditary $\mathcal{O}_L$-order, namely $\mathcal{O}_L$ itself, which is a maximal order. (See lemma 2.4.17.)

Take the Heisenberg representation η_K of J^1_K attached to θ_K. Let κ_K be its p-primary β-extension to J_K. Then, if we set $\lambda_K = \kappa_K$ we obtain a maximal simple type in G_K: the representation κ_K is the extension of θ_K to J_K in which μ_L acts trivially, and λ_K thus corresponds to the trivial character of $J_K/J^1_K \cong \mu_L$.

By the discussion after theorem 2.4.10, the normalizer $\mathbf{J}(\lambda_K)$ equals $L^\times J_K$. The representation λ_K extends to $K^\times J_K = \pi_K^{\mathbf{Z}} \times J_K$ by letting π_K act trivially, and since $\mathbf{J}(\lambda_K)/K^\times J^1_K$ is cyclic of order $e(F[\beta]/F)$ it also extends to $\mathbf{J}(\lambda_K)$. However, as $F[\beta]/F$ might be wildly ramified, we cannot normalize the extension via the order of the determinant as we did in proposition 2.4.9. Instead, we will refer to any representation obtained by inducing one of these extensions from $\mathbf{J}(\lambda_K)$ to G_K as a *K-lift* of π. These are all supercuspidal irreducible representations of G_K. This ambiguity in the definition will not affect arguments concerning the inertial class.

Let τ be a K-lift of π. The aim of this section if to prove the following theorem, that relates the characters of π and τ In the context of essentially tame endo-classes, it is due to Bushnell and Henniart (see [BH11, Section 6]). Recall that π contains a maximal simple type (J_θ, λ) constructed from $[\mathfrak{A}, \beta]$, and that $T = F[\beta]^{\mathrm{ur}}$.

Theorem 5.1.1. Let $\zeta \in \mu_K$ generate the field K over F, and let u be an elliptic, regular and pro-unipotent element of G_K. Then

$$\mathrm{tr}\,\pi(\zeta u) = (-1)^{m'+1} s[\chi]^{-1} \epsilon_\theta(\zeta) \sum_{\alpha \in \mathrm{Gal}(\mathbf{k}/\mathbf{f})} \chi(\zeta^\alpha) \mathrm{tr}\,\tau^\alpha(u)$$

where χ is evaluated at ζ via any $\mathbf{e}$-linear isomorphism $\iota : \mathbf{k} \to \mathbf{e}_{n/\delta(\Theta_F)}$, where $\mathbf{k}$ is an $\mathbf{e}$-algebra via $\iota(\Theta_E)_T : \mathbf{e} \to \mathbf{t}$.

Remark 5.1.2. It is not immediate that the formula makes sense as written, but we will see while proving the theorem that the characters of τ and τ^α coincide on u whenever $\alpha \in \mathrm{Gal}(\mathbf{k}/\mathbf{t})$, hence the right hand side is independent of the choice of representatives of $[\chi]$ and of the choice of ι. Recall that $V_\theta = J^1_\theta/H^1_\theta$ is a symplectic representation of μ_K over $\mathbf{F}_p$, defining a sign $\epsilon_\theta(\zeta)$ as in proposition 2.4.19, and that m' is defined by $B = Z_A(F[\beta]) \cong M_{m'}(D')$. A *pro-unipotent* element u of G is one for which $u^{p^n} \to 1$ as $n \to +\infty$. See remark 4.5.9 for the definition of $s[\chi]$.

Proof. By the following lemma, the Harish-Chandra character of π at ζu can be computed by the Mackey formula for an induced representation

$$\mathrm{tr}\pi(\zeta u) = \sum_{y \in \mathbf{J}(\lambda)\backslash G} \mathrm{tr}\Pi(y\zeta u y^{-1}),$$

as in [BH11, Section 1.2] and [BH96, Appendix].

Lemma 5.1.3. The element $\zeta u \in G$ is elliptic and regular over F.

Proof. Since $F[\zeta u]$ is a finite-dimensional F-vector space, it is complete, and so it contains ζ, which is the limit of a sequence of element of the form $(u\zeta)^{p^n}$. But then it contains u, hence $F[\zeta u] = K[u]$ is a maximal field extension of F in A. □

The theorem will follow from a careful study of this formula.

Lemma 5.1.4. If $y \in G$ and $y\zeta u y^{-1} \in \mathbf{J}(\lambda)$, then $y\zeta u y^{-1} \in J_\theta$ and there exists $\widetilde{y}$ in the normalizer $N_G(K)$ such that $\mathbf{J}(\lambda)y = \mathbf{J}(\lambda)\widetilde{y}$. For any such $\widetilde{y}$, one has $\widetilde{y}u\widetilde{y}^{-1} \in J^1_K$.

Proof. Since the valuation of the determinant of ζu is zero, and $\mathbf{J}(\lambda)/J_\theta$ is infinite cyclic generated by some power of a uniformizer of D', necessarily $y\zeta u y^{-1} \in J_\theta$ if $y\zeta u y^{-1} \in \mathbf{J}(\lambda)$. The quotient J_θ/J^1_θ is isomorphic to a general linear group $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$, and the degree $[K : F]$ equals $n/e(\Theta_F)$, so $\mathbf{k}^\times$ embeds in $\mathrm{GL}_{m'}(\mathbf{d}')$ as a maximal elliptic torus. Now the claim follows as in the proof of [BH10, Lemma 13]: first prove that $y\zeta y^{-1} \in J_\theta$ by raising to a suitable power of p, and then notice that there exists some other $\zeta' \in \mu_K$ generating K over F with $y\zeta y^{-1}$ conjugate in J_θ to $\zeta' u'$ for some $u' \in J^1_\theta$. By [BH14, 2.6 Conjugacy Lemma] or [BH10, Lemma 14], we can further change y in its J_θ-coset and assume that u' and ζ' commute, and this implies that $u' = 1$. But then $y\zeta u y^{-1} = \zeta' y u y^{-1}$ with yuy^{-1} commuting with ζ' and contained in J_θ. As the image of ζ' in J_θ/J^1_θ is a regular elliptic element, it commutes with no unipotent elements except the identity, so $yuy^{-1} \in J^1_\theta$. □

Lemma 5.1.5. The group $\mathbf{J}(\lambda) \cap G_K$ equals $\mathbf{J}(\lambda_K)$, and the order of the image of $\mathbf{J}(\lambda) \cap N_G(K)$ under the isomorphism $N_G(K)/G_K \to \mathrm{Gal}(K/F)$ equals $n/\delta(\Theta_F)b[\chi] = m'd'/b[\chi]$, where $b[\chi]$ equals the index of $\mathbf{J}(\lambda)$ in $\pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$.

Proof. (Compare [BH10, Proposition 9].) We can determine an element in $\mathrm{Gal}(K/F)$ by its action on μ_K, and $\mu_K = \mu_L$. Any choice of isomorphism $\psi : J_\theta/J_\theta^1 \to \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ in $\Psi(\Theta_E)$ induces a surjective group homomorphism

$$\widetilde{\psi} : \pi_{D'}^{\mathbf{Z}} \ltimes J_\theta \to \mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$$

which sends $\pi_{D'}$ to a generator of $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$ and maps μ_L isomorphically onto its image, which is an elliptic maximal torus in $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$, hence self-centralizing in $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ (to see this, embed this group in $\mathrm{GL}_{m'd'}(\mathbf{e})$, where the image of μ_L is still an elliptic maximal torus). So, if $x \in \pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$ centralizes μ_K then it is contained in $\pi_{D'}^{d'\mathbf{Z}} \ltimes J_\theta$, which equals $\pi_{F[\beta]}^{\mathbf{Z}} \times J_\theta$ as $\mathcal{O}_{D'}^\times \subseteq J_\theta$. This implies that

$$\mathbf{J}(\lambda) \cap Z_G(K) = (\pi_{F[\beta]}^{\mathbf{Z}} \times J_\theta) \cap Z_G(K) = \pi_{F[\beta]}^{\mathbf{Z}} \times J_K = \mathbf{J}(\lambda_K).$$

Every automorphism of μ_K induced by a conjugation in $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ is also induced by a conjugation in $\pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$; to see this, observe that if $x \in \pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$ and $x\zeta x^{-1} = \zeta' u$ for some $u \in J_\theta^1$ then we can change x in its J_θ-coset and assume that ζ' and u commute, by [BH10, Lemma 14]. Then since the order of ζ and ζ' is prime to p, and J_θ^1 is a pro-p group, we conclude that $u = 1$, and the claim follows.

The group of automorphisms of an elliptic maximal torus T in $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ induced by $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ is cyclic of order $m'd'$: this holds because up to conjugacy T arises from restricting scalars of the $\mathbf{e}_{m'd'}$-vector space $\mathbf{e}_{m'd'}$ to $\mathbf{e}_{d'}$. Restricting scalars further to $\mathbf{e}$, we see that the normalizer of $\mathbf{e}_{d'}^\times$ in $\mathrm{GL}_{m'd'}(\mathbf{e})$ is $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e}) \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$, and it contains the normalizer of $\mathbf{e}_{m'd'}^\times$ in $\mathrm{GL}_{m'd'}(\mathbf{e})$, which induces $\mathrm{Gal}(\mathbf{e}_{m'd'}/\mathbf{e})$ on $\mathbf{e}_{m'd'}^\times$.

The lemma now follows since the group $\mathbf{J}(\lambda)$ has index $b[\chi]$ in $\pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$ and contains J_θ, hence maps under $\widetilde{\psi}$ to $\Delta \ltimes \mathrm{GL}_{m'}(\mathbf{e}_{d'})$, for $\Delta \subset \mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$ the only subgroup of index $b[\chi]$. □

The space $\mathbf{J}(\lambda)N_G(K)$ decomposes into double cosets

$$\mathbf{J}(\lambda)N_G(K) = \bigcup_{\sigma \in \mathrm{Gal}(K/F)} \mathbf{J}(\lambda) t_\sigma G_K$$

where $t_\sigma \in N_G(K)$ induces σ on K upon conjugation, and $\mathbf{J}(\lambda)t_\sigma G_K = \mathbf{J}(\lambda)t_\tau G_K$ if and only if $\tau\sigma^{-1}$ is induced by $\mathbf{J}(\lambda)$. Then by lemma 5.1.4 and lemma 5.1.5 we may rewrite the sum as

$$\mathrm{tr}\pi(\zeta u) = \sum_{y \in \mathbf{J}(\lambda)\backslash \mathbf{J}(\lambda)N_G(K)} \mathrm{tr}\Pi(y\zeta u y^{-1}) = (\delta(\Theta_F)b[\chi]/n) \sum_{\alpha \in \mathrm{Gal}(K/F)} \sum_{y \in \mathbf{J}(\lambda_K)\backslash G_K} \mathrm{tr}\Pi(y\alpha(\zeta u)y^{-1}).$$

Here, $\alpha(\zeta u) = t_\alpha \zeta u t_\alpha^{-1}$.

These y commute with all the $\alpha(\zeta)$. We are now going to fix an isomorphism $\psi : J_\theta/J_\theta^1 \to \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ in the conjugacy class $\Psi(\Theta_E)$, and a representation σ of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$ so that $\lambda = \kappa \otimes \psi^*\sigma$. Furthermore, we will choose a representative χ for $[\chi]$ such that $\sigma = \sigma[\chi]$ under the Green parametrization*. Then

$$\mathrm{tr}\Pi(y\alpha(\zeta u)y^{-1}) = \mathrm{tr}\Pi(\alpha(\zeta)y\alpha(u)y^{-1}) = \mathrm{tr}\sigma(\zeta^\alpha)\mathrm{tr}\kappa(\zeta^\alpha y\alpha(u)y^{-1})$$

since Π extends λ, where $\zeta^\alpha = \alpha(\zeta)$ and σ is evaluated at ζ^α via ψ.

Lemma 5.1.6. The equality

$$\mathrm{tr}\kappa(\zeta^\alpha y\alpha(u)y^{-1}) = \epsilon_\theta(\zeta^\alpha)\mathrm{tr}\lambda_K(y\alpha(u)y^{-1})$$

holds whenever $y\alpha(u)y^{-1} \in J_K^1$.

Proof. Compare [BH14, Section 5.2]. We use the Glauberman correspondence (proposition 2.4.19 for the cyclic group $\Gamma \subseteq \mu_K$ generated by ζ, acting on $J_\theta^1/\ker(\theta)$ and normalizing η. This implies that there exist a unique irreducible representation η^Γ of $(J^1/\ker(\theta))^\Gamma$ and sign $\epsilon = \pm 1$ such that

$$\mathrm{tr}\eta^\Gamma(x) = \epsilon\mathrm{tr}\widetilde{\eta}(\zeta x) \tag{5.1.1}$$

for all $x \in (J^1/\ker(\theta))^\Gamma$ and every generator ζ of Γ. Recall that $\widetilde{\eta}$ is the only irreducible representation of $\Gamma \ltimes J_\theta^1$ with trivial determinant on Γ.

By construction, $\widetilde{\eta}$ is isomorphic to the restriction of the p-primary β-extension κ to $\Gamma \ltimes J_\theta^1$, since $\det\kappa$ has order a power of p and Γ has order prime to p. Since ζ generates K over F, the fixed point space $(J_\theta^1)^\Gamma = J_K^1$, and since $\ker(\theta)$ is a pro-p group and Γ has order prime to p, a cohomological vanishing argument as in [BH10, Proposition 6] implies that $(J_\theta^1/\ker(\theta))^\Gamma = J_K^1/\ker(\theta_K)$. We claim that actually $\eta^\Gamma = \eta_K$, the Heisenberg representation associated to θ_K. To see this, it is enough to prove that η^Γ contains θ_K, by the uniqueness statement in proposition 2.4.11. Replacing x by xh for $h \in H_K^1$ in(5.1.1), we find

$$\mathrm{tr}\eta^\Gamma(xh) = \epsilon\mathrm{tr}\widetilde{\eta}(\zeta xh) = \epsilon\mathrm{tr}\widetilde{\eta}(\zeta x)\theta(h) = \theta_K(h)\mathrm{tr}\eta^\Gamma(x).$$

Setting $x = 1$ and letting h vary through H_K^1 yields the claim.

Finally, we compute ϵ by letting $x = 1$ in $\mathrm{tr}\eta_K(x) = \epsilon\mathrm{tr}\widetilde{\eta}(\zeta x)$. The resulting equality $\epsilon\mathrm{tr}\widetilde{\eta}(\zeta) = \dim\eta_K$ implies that ϵ equals the sign of the trace of ζ on $\widetilde{\eta}$, which by definition is $\epsilon_\theta(\zeta)$. □

*Recall that $[\chi]$ only determines a $\mathrm{Gal}(\mathbf{e}_{d'}/\mathbf{e})$-orbit of representations of $\mathrm{GL}_{m'}(\mathbf{e}_{d'})$. Via the choice of ψ and σ, we are fixing an element of this orbit.

If $y\alpha(u)y^{-1} \notin J^1_K$, then $y\alpha(\zeta u)y^{-1} \notin \mathbf{J}(\lambda)$ by lemma 5.1.4. So we have

$$\mathrm{tr}\Pi(y\alpha(\zeta u)y^{-1}) = \epsilon_\theta(\zeta^\alpha)\mathrm{tr}\sigma(\zeta^\alpha)\mathrm{tr}\lambda_K(y\alpha(u)y^{-1})$$

where the traces of Π and λ_K are extended by zero to G and G_K respectively. Since τ is induced from an extension of λ_K to $\mathbf{J}(\lambda_K)$, we deduce that

$$\sum_{y \in \mathbf{J}(\lambda_K)\backslash G_K} \mathrm{tr}\lambda_K(y\alpha(u)y^{-1}) = \mathrm{tr}\tau(\alpha(u))$$

and so

$$\mathrm{tr}\pi(\zeta u) = (\delta(\Theta_F)b[\chi]/n) \sum_{\alpha \in \mathrm{Gal}(\mathbf{k}/\mathbf{f})} \epsilon_\theta(\zeta^\alpha)\mathrm{tr}\sigma(\zeta^\alpha)\mathrm{tr}\tau(\alpha(u)).$$

Now the Galois twists $\tau^\alpha = \tau \circ \mathrm{ad}(t_\alpha)$ have character $x \mapsto \mathrm{tr}\tau(\alpha(x))$, and the endo-class $\mathrm{cl}(\tau^\alpha)$ of τ^α satisfies $\mathrm{cl}(\tau^\alpha) = \alpha^*(\mathrm{cl}\,\tau)$. By [BH03, 1.5.1], the group $\mathrm{Gal}(\mathbf{k}/\mathbf{f})$ is transitive over the K-lifts of Θ_F, and there is the same number of these as simple components of $K \otimes_F F[\beta]$. So the stabilizer of $\mathrm{cl}(\tau)$ in $\mathrm{Gal}(\mathbf{k}/\mathbf{f})$ is $\mathrm{Gal}(\mathbf{k}/\mathbf{t})$, and it follows that for $\alpha \in \mathrm{Gal}(\mathbf{k}/\mathbf{t})$ the supercuspidal representations τ and τ^α of G_K have the same endo-class. By proposition 2.4.5, they both contain the simple character θ_K, so their restriction to J^1_K contains η_K. Since $J_K/J^1_K \cong \mu_L = \mu_K$, which by construction acts trivially on τ and τ^α, these representations contain the same maximal simple type (J_K, κ_K). So they are inertially equivalent, and their characters therefore agree on elements whose reduced norm has valuation 0, such as u.

We can now rearrange the sum further to

$$\mathrm{tr}\pi(\zeta u) = (\delta(\Theta_F)b[\chi]/n)\epsilon_\theta(\zeta) \sum_{\gamma \in \mathrm{Gal}(\mathbf{t}/\mathbf{f})} \left(\mathrm{tr}\tau^\gamma(u) \sum_{\delta \in \gamma \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \mathrm{tr}\sigma(\zeta^\delta) \right)$$

since $\epsilon_\theta(\zeta)$ only depends on the subgroup of μ_K generated by ζ.

The trace $\mathrm{tr}\sigma(\zeta^\delta)$ can be computed as follows. We are evaluating σ at ζ^δ using a fixed choice of isomorphism $\psi : J_\theta/J^1_\theta \to \mathrm{GL}_{m'}(\mathbf{e}_{d'})$ in $\Psi(\Theta_E)$. Recall that any such ψ comes from an $\mathbf{e}$-linear isomorphism $\psi : \mathfrak{j}_\theta/\mathfrak{j}^1_\theta \to M_{m'}(\mathbf{e}_{d'})$ by passing to groups of units. The elliptic maximal torus $\psi(\mu_K)$ is conjugate to $\mathbf{e}^\times_{m'd'}$, where the trace of $\sigma = \sigma(\chi)$ is given by explicit formulas, and this isomorphism $\mu_K \to \mathbf{e}^\times_{m'd'}$ (ψ followed by conjugation) comes from an $\mathbf{e}$-linear isomorphism $\mathbf{k} \to \mathbf{e}_{m'd'}$ by passing to groups of units. Then one has the character formula

$$\begin{aligned}\sum_{\delta\in\mathrm{Gal}(\mathbf{k}/\mathbf{t})}\mathrm{tr}\sigma(\zeta^{\delta}) &= (-1)^{m'+1}\sum_{\delta\in\mathrm{Gal}(\mathbf{k}/\mathbf{t})}\sum_{\nu\in\mathrm{Gal}(\mathbf{e}_{m'd'}/\mathbf{e}_{d'})}\chi(\zeta^{\delta\nu})\\ &= (-1)^{m'+1}\sum_{\nu\in\mathrm{Gal}(\mathbf{e}_{m'd'}/\mathbf{e}_{d'})}\sum_{\chi_0\in[\chi]}\chi_0(\zeta^{\nu})\\ &= (-1)^{m'+1}m'\sum_{\delta\in\mathrm{Gal}(\mathbf{k}/\mathbf{t})}\chi(\zeta^{\delta})\end{aligned}$$

where σ is evaluated on ζ^{δ} via ψ and χ is evaluated on ζ^{δ} via any $\mathbf{e}$-linear isomorphism $\iota:\mathbf{k}\to\mathbf{e}_{m'd'}$. Because the sums are taken over $\mathrm{Gal}(\mathbf{k}/\mathbf{t})$, the answer is independent of the choice of ψ and ι, and the second line shows that the answer does not depend on the choice of χ in $[\chi]$.

Now since $n/\delta(\Theta_F)=m'd'$ we have

$$(\delta(\Theta_F)b[\chi]/n)m' = b[\chi]/d' = s[\chi]^{-1},$$

and rearranging further we obtain

$$\begin{aligned}\mathrm{tr}\pi(\zeta u) &= (-1)^{m'+1}s[\chi]^{-1}\epsilon_{\theta}(\zeta)\sum_{\gamma\in\mathrm{Gal}(\mathbf{t}/\mathbf{f})}\left(\mathrm{tr}\tau^{\gamma}(u)\sum_{\delta\in\gamma\mathrm{Gal}(\mathbf{k}/\mathbf{t})}\chi(\zeta^{\delta})\right)\\ &= (-1)^{m'+1}s[\chi]^{-1}\epsilon_{\theta}(\zeta)\sum_{\gamma\in\mathrm{Gal}(\mathbf{k}/\mathbf{f})}\mathrm{tr}\tau^{\gamma}(u)\chi(\zeta^{\gamma}).\end{aligned}$$

□

5.2 Results from l-modular representation theory.

Let $l\neq p$ be a prime number, and fix an isomorphism $\iota:\mathbf{C}\to\overline{\mathbf{Q}}_l$. In [SS16b, Section 4.1] there is defined a notion of *mod l inertial supercuspidal support* for irreducible smooth $\overline{\mathbf{Q}}_l$-representations of $G=\mathrm{GL}_m(D)$. It is an inertial class of supercuspidal supports for $\mathrm{GL}_m(D)$ over $\overline{\mathbf{F}}_l$, and it only depends on the inertial class of the representation. Write $\mathbf{i}_l(\mathfrak{s})$ for the mod l inertial supercuspidal support of the inertial class $\mathfrak{s}$, and say that two classes $\mathfrak{s}_1$ and $\mathfrak{s}_2$ for the category of $\overline{\mathbf{Q}}_l$-representations of G are in the same *l-block* if $\mathbf{i}_l(\mathfrak{s}_1)=\mathbf{i}_l(\mathfrak{s}_2)$.

Given simple inertial classes $\mathfrak{s}_1,\mathfrak{s}_2$ of complex representations of G, we say that they are *l-linked* if the $\overline{\mathbf{Q}}_l$-components corresponding to them under ι are in the same l-block. By [SS16b, Lemma 5.2] this is independent of the choice of ι. We say that the $\mathfrak{s}_i$ are *linked* if there exist prime numbers $l_1,\dots,l_r$ all distinct from p and inertial classes $\mathfrak{s}^0,\dots,\mathfrak{s}^r$ such that $\mathfrak{s}^0=\mathfrak{s}_0$, $\mathfrak{s}^r=\mathfrak{s}_1$, and $\mathfrak{s}^{i-1}$ and $\mathfrak{s}^i$ are l_i-linked for all i.

By [SS16b, Lemma 4.3], the mod l inertial supercuspidal support of an integral representation π coincides with the supercuspidal support of every factor of $\mathbf{r}_l(\pi)$. From this and proposition 4.5.3 it follows that

$$\mathrm{cl}(\mathbf{i}_l(\mathfrak{s})) = \mathrm{cl}(\mathfrak{s}) \text{ and } \Lambda_{\mathbf{r}_l(\kappa)}(\mathbf{i}_l(\mathfrak{s})) = \Lambda_\kappa(\mathfrak{s})^{(l)}.$$

So, two simple inertial classes $\mathfrak{s}_i$ are l-linked if and only if $\mathrm{cl}(\mathfrak{s}_1) = \mathrm{cl}(\mathfrak{s}_2)$ and $\Lambda_\kappa(\mathfrak{s}_1)^{(l)} = \Lambda_\kappa(\mathfrak{s}_2)^{(l)}$. Letting l vary, we see that the $\mathfrak{s}_i$ are linked if and only if they have the same endo-class (compare [SS16b, Propositions 5.5, 5.8]). The compatibility of the Jacquet–Langlands correspondence with respect to mod l reduction [MS17] then implies the following result.

Theorem 5.2.1. [SS16b, Corollary 6.3, Theorem 6.4] Let $H = \mathrm{GL}_n(F)$ and consider the Jacquet–Langlands transfer of simple inertial classes of complex representations

$$\mathrm{JL}_G : \mathfrak{B}_{\mathrm{ds}}(G) \to \mathfrak{B}_{\mathrm{ds}}(H)$$

Let $l \neq p$ be a prime number. Then two simple inertial classes $\mathfrak{s}_1$ and $\mathfrak{s}_2$ for G are l-linked if and only if $\mathrm{JL}_G(\mathfrak{s}_1)$ and $\mathrm{JL}_G(\mathfrak{s}_2)$ are l-linked.

5.3 Proof of theorems A and B.

Consider central simple algebras $A_1 = M_n(F)$ and $A_2 = M_m(D)$ over F. Write $\mathrm{JL}_{G_2} : \mathbf{D}(G_2) \to \mathbf{D}(G_1)$ for the Jacquet–Langlands correspondence between their unit groups, as well as for the map it induces on simple inertial classes. Let $\mathfrak{s}_i$ be simple inertial classes of the G_i with

$$\mathfrak{s}_1 = \mathrm{JL}_{G_2}(\mathfrak{s}_2).$$

Theorem 5.3.1. We have the equality $\mathrm{cl}(\mathfrak{s}_1) = \mathrm{cl}(\mathfrak{s}_2)$.

Proof. Let d be the reduced degree of D over F. For all integers $a \geq 1$ there exist simple inertial classes $\mathfrak{s}_i^*$ in $\mathrm{GL}_{an}(F)$ and $\mathrm{GL}_{am}(D)$ which correspond under the Jacquet–Langlands transfer on these groups and have endo-class $\mathrm{cl}(\mathfrak{s}_i^*) = \mathrm{cl}(\mathfrak{s}_i)$: it suffices to let their supercuspidal support be a multiple of the supercuspidal support of the $\mathfrak{s}_i$. Letting $a = d$, we can assume that d divides $\frac{n}{\delta(\mathrm{cl}\,\mathfrak{s}_i)}$ for all $i = 1, 2$.

We can assume that both $\mathfrak{s}_i$ are supercuspidal: to see this, recall that the parametric degree of a simple inertial class is preserved under the Jacquet–Langlands correspondence [BH11, 2.8 Corollary 1]. Since a simple inertial class of $\mathrm{GL}_n(F)$ is supercuspidal if and only if it has maximal parametric degree, we see that the transfer of a supercuspidal representation of $\mathrm{GL}_n(F)$ is supercuspidal. Let $\mathfrak{s}_1^+$ be a supercuspidal inertial class for $\mathrm{GL}_n(F)$, with $\mathrm{cl}(\mathfrak{s}_1) = \mathrm{cl}(\mathfrak{s}_1^+)$. Then $\mathfrak{s}_1$ and $\mathfrak{s}_1^+$ are linked, hence by theorem 5.2.1 also $\mathfrak{s}_2$ and $\mathrm{JL}_{G_2}^{-1}(\mathfrak{s}_1^+)$ are linked, and so they have the same endo-class. So if $\mathrm{cl}(\mathrm{JL}_{G_2}^{-1}(\mathfrak{s}_1^+)) = \mathrm{cl}(\mathfrak{s}_1^+)$ then the theorem follows.

When the $\mathfrak{s}_i$ are supercuspidal, their parametric degree is maximal and by the formulas in remark 4.5.9 the invariants $s[\chi_i]$ are equal to 1, where $[\chi_i] = \Lambda_\kappa(\mathfrak{s}_i)$. Fix maximal simple characters θ_i in G_i of endo-class $\mathrm{cl}(\mathfrak{s}_i)$, with underlying simple strata $[\mathfrak{A}_i, \beta_i]$. Let $T_i = F[\beta_i]^{\mathrm{ur}}$. Let L_i be an extension of $F[\beta_i]$ contained in $Z_{A_i}(F[\beta_i])$ which has maximal degree, is unramified, and normalizes $\mathfrak{A}_i$, as in the proof of proposition 2.4.5. Let K_i be the maximal unramified extension of F contained in L_i. The quantity $t = \frac{n}{\delta(\Theta_F^i)} f(F[\beta_i]/F)$ is preserved under the Jacquet–Langlands correspondence (it is the torsion number of the inertial class, by the formulas in remark 4.5.9, which is preserved since the Jacquet–Langlands correspondence commutes with twists by unramified characters), hence the K_i have the same degree t over F.

Because d divides $\frac{n}{\delta(\Theta_F^i)}$, it divides $[K_i : F]$, hence the commutant $Z_{A_2}(K_2)$ is a split central simple algebra over K_2: indeed, we have $Z_{A_2}(K_2) \cong M_{m'}(D')$ for a central division algebra D' of reduced degree $d/(d, [K_2 : F]) = 1$ over K_2. We can therefore fix an F-linear isomorphism $\alpha_0 : K_2 \to K_1$, and then an α_0-linear isomorphism $\alpha : Z_{A_2}(K_2) \to Z_{A_1}(K_1)$. We emphasize that these isomorphisms are chosen arbitrarily, and that this ambiguity will not affect our results. Using α and α_0 to identify K_1 and K_2, and $Z_{A_1}(K_1)$ and $Z_{A_2}(K_2)$, we can write K for any of the K_i and A_K for any of the $Z_{A_i}(K_i)^\dagger$.

Choose supercuspidal irreducible representations π_i in the inertial classes $\mathfrak{s}_i$, corresponding to each other under the Jacquet–Langlands correspondence. Let τ_i be some K-lift of π_i. Choose $\zeta \in \mu_K$ generating K over F, and an elliptic, regular and pro-unipotent element u of $G_K = A_K^\times$. The ζu are matching elements of A_1 and A_2, and by proposition 5.1.1 and its proof, we have equalities

$$\mathrm{tr}\pi_i(\zeta u) = (-1)^{m_i'+1}\epsilon_{\theta_i}(\zeta) \sum_{\gamma \in \mathrm{Gal}(\mathbf{t}_i/\mathbf{f})} \left(\mathrm{tr}\tau_i^\gamma(u) \sum_{\delta \in \gamma \mathrm{Gal}(\mathbf{k}/\mathbf{t}_i)} \chi_i(\zeta^\delta) \right).$$

By [BH03, 1.5.1], the group $\mathrm{Gal}(\mathbf{k}/\mathbf{f})$ is transitive on the set of K-lifts of $\mathrm{cl}(\mathfrak{s}_i)$, which has $f(\mathrm{cl}\ \mathfrak{s}_i)$ elements. Since $\mathrm{cl}(\tau_i^\gamma) = \gamma^* \mathrm{cl}(\tau_i)$, the representations τ_i^γ as γ runs through $\mathrm{Gal}(\mathbf{t}_i/\mathbf{f})$ are pairwise inertially inequivalent (as they have different endo-classes). They are furthermore *totally ramified* representations of G_K, in the sense that their unramified parameter fields all coincide with K.

Lemma 5.3.2 (Linear independence lemma)**.** Let $\pi_1, \ldots, \pi_r$ be irreducible, supercuspidal, totally ramified representations of $\mathrm{GL}_m(D)$ for a central division algebra D over F,

†Formally, K is really an inverse limit of the diagram $\alpha_0 : K_2 \to K_1$, and similarly for A_K and $\alpha : Z_{A_2}(K_2) \to Z_{A_1}(K_1)$.

whose central characters agree on μ_F. Assume that they are pairwise inertially inequivalent. Then the characters $\mathrm{tr}\pi_i$ are linearly independent on the set of elliptic, regular, pro-unipotent elements of $\mathrm{GL}_m(D)$.

Proof. This follows from [BH11, Lemma 6.6], as we can twist the π_i by unramified characters of $\mathrm{GL}_m(D)$ until the central characters also agree on a uniformizer of F. This does not change the inertial classes of the π_i, nor the character values on elliptic, regular, pro-unipotent elements of $\mathrm{GL}_m(D)$ as these have reduced norms of valuation 0. □

The central characters of the τ_i are trivial on μ_K by construction. Then by the linear independence lemma either there exists $\gamma \in \mathrm{Gal}(K/F)$ such that τ_1^γ and τ_2 are inertially equivalent, or

$$\sum_{\delta \in \gamma \mathrm{Gal}(\mathbf{k}/\mathbf{t}_i)} \chi_i(\zeta^\delta) = 0$$

for all values of i, γ and ζ. That this does not happen follows when $i = 1$ by [SZ00, Theorem 1.1], stating that there exists no character χ of $\mathbf{k}^\times$ such that $\sum_{\gamma \in \mathrm{Gal}(\mathbf{k}/\mathbf{f})} \chi(\zeta^\gamma) = 0$ for all $\mathbf{f}$-regular elements of $\mathbf{k}$.

So we have proved that τ_1 and τ_2^γ are inertially equivalent for some $\gamma \in \mathrm{Gal}(K/F)$. But then they have the same endo-class, and since their endo-classes are K-lifts of $\mathrm{cl}(\mathfrak{s}_1)$ and $\mathrm{cl}(\mathfrak{s}_2)$ respectively, the theorem follows. □

We pass now to the study of the level zero part of the $\mathfrak{s}_i$. Let us first assume that the $\mathfrak{s}_i$ are supercuspidal. Choose maximal simple characters θ_i contained in the $\mathfrak{s}_i$, defined by strata $[\mathfrak{A}_i, \beta_i]$, and let $T_i = F[\beta_i]^{\mathrm{ur}}$. As in the proof of proposition 2.4.5, we take a maximal unramified extension L_i of $F[\beta_i]$ in $Z_{A_i}(F[\beta_i])$ normalizing $\mathfrak{A}_i$, and identify the maximal unramified extensions K_i of F in L_i. Now we know that the θ_i have the same endo-class, and we take the only F-linear ring isomorphism $\alpha_0 : K_2 \to K_1$ such that

$$\alpha_0^* \,\mathrm{cl}(\theta_{1,K_1}) = \mathrm{cl}(\theta_{2,K_2})$$

for the interior lifts θ_{i,K_i}. Notice, however, that the commutants $Z_{A_i}(K_i)$ need not be isomorphic. As in the proof of theorem 5.3.1, we write K for any of K_2 and K_1 and T for any of T_2 and T_1, using the isomorphism α_0.

Theorem 5.3.3. The equality $\epsilon^1_{\theta_1}\Lambda_\kappa(\mathfrak{s}_1) = \epsilon^1_{\theta_2}\Lambda_\kappa(\mathfrak{s}_2)$ holds, where the $\epsilon^1_{\theta_i}$ are the symplectic sign characters (definition 2.4.23), and κ is the conjugacy class of p-primary β-extensions.

Proof. We choose supercuspidal representations π_i of G_i in $\mathfrak{s}_i$, Jacquet–Langlands transfers of each other, and we let τ_i be some K-lift of π_i; then, because of our choice of α_0,

τ_1 and τ_2 have the same endo-class.

Fix a root of unity $\zeta \in \mu_K$ generating K over F, and let u_1 be an elliptic, regular, pro-unipotent element of $G_{K,1} = Z_{G_1}(K)$. The matching conjugacy class in $G_{K,2}$ then consists of pro-unipotent elements, as in the case of an elliptic regular element this is a condition which can be checked on the eigenvalues of the characteristic polynomial. Let u_2 be an element of this conjugacy class. We apply proposition 5.1.1 and obtain an equality

$$\mathrm{tr}\pi_i(\zeta u_i) = (-1)^{m_i'+1}\epsilon_{\theta_i}(\zeta) \sum_{\gamma \in \mathrm{Gal}(\mathbf{t}/\mathbf{f})} \left(\mathrm{tr}\tau_i^\gamma(u_i) \sum_{\delta \in \gamma \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \chi_i(\zeta^\delta) \right)$$

where $\Lambda_\kappa(\mathfrak{s}_i) = [\chi_i]$. By the linear independence lemma (lemma 5.3.2), we have that τ_1^γ and $\mathrm{JL}_{G_{K,2}}(\tau_2)$ are inertially equivalent for some $\gamma \in \mathrm{Gal}(\mathbf{t}/\mathbf{f})$ (this is the Jacquet–Langlands correspondence for the groups $G_{K,i}$, which are inner forms of each other). This γ is unique, as the τ_i^γ have pairwise different endo-classes for $\gamma \in \mathrm{Gal}(\mathbf{t}/\mathbf{f})$. By theorem 5.3.1, the endo-class of $\mathrm{JL}_{G_{K,2}}(\tau_2)$ is $\mathrm{cl}(\theta_{2,K_2})$. By our choice of α_0, this implies $\gamma = 1$.

Fix u_i so that the characters of the τ_i are nonzero at u_i; this is possible by the linear independence lemma, because the τ_i are totally ramified. Then the Jacquet–Langlands character relation

$$(-1)^{n_K}\mathrm{tr}\tau_1(u_1) = (-1)^{m_K}\mathrm{tr}\tau_2(u_2)$$

holds, where $Z_{A_1}(K_1) \cong M_{n_K}(K)$ and $Z_{A_2}(K_2) \cong M_{m_K}(D_K)$ for some central division algebra D_K over K.

We now have an equality

$$(-1)^{m+m'+m_K+1}\epsilon_{\theta_2}(\zeta) \sum_{\delta \in \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \chi_2(\zeta^\delta) = (-1)^{n+n'+n_K+1}\epsilon_{\theta_1}(\zeta) \sum_{\delta \in \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \chi_1(\zeta^\delta) \qquad (5.3.1)$$

on comparing $\mathrm{tr}\pi_1(\zeta u_1)$ and $\mathrm{tr}\pi_2(\zeta u_2)$ by the Jacquet–Langlands correspondences over F and over K. This equality holds for all $\zeta \in \mu_K$ generating K over F—equivalently, for all $\zeta \in \mathbf{k}^\times$ generating $\mathbf{k}$ over $\mathbf{f}$. To be more precise‡, we are evaluating χ_i at $\zeta^\delta \in \mu_{K_i}$ via a choice of $\mathbf{e}$-linear isomorphism $\iota_i : \mathbf{k}_i \to \mathbf{e}_{n/\delta(\Theta_F)}$, as in theorem 5.1.1. Since $\alpha_0^* \,\mathrm{cl}(\theta_{1,K_1}) = \theta_{2,K_2}$, we have $\alpha_0 \iota_{T_2} = \iota_{T_1}$, hence the ι_i can be chosen compatibly with $\alpha_0 : \mathbf{k}_2 \to \mathbf{k}_1$, allowing us to evaluate both characters χ_i at $\zeta^\delta \in \mu_K$.

Recall that there exist a sign $\epsilon^0_{\theta_i}$ and a quadratic character $\epsilon^1_{\theta_i}(-)$ of μ_K such that, whenever $z \in \mu_K$ generates a subgroup Δ of μ_K with $V_{\theta_i}^{\mu_K} = V_{\theta_i}^{\Delta}$, one has

$$\epsilon(z, V_i) = \epsilon^0_{\theta_i}\epsilon^1_{\theta_i}(z).$$

‡This becomes clearer if we consider K to be an inverse limit of the diagram $\alpha_0 : K_2 \to K_1$.

In our case, every ζ generating K over F satisfies $V_i^{\zeta} = V_i^{\mu_K}$, even if ζ does not generate μ_K, by a cohomological vanishing argument as in [BH10, Proposition 6]. Comparing coefficients, one gets an equality

$$\begin{aligned}(-1)^{n+n'+n_K+1}\epsilon^0_{\theta_1} &\sum_{\delta\in \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \epsilon^1_{\theta_1}(\zeta^\delta)\chi_1(\zeta^\delta) \\ &= (-1)^{m+m'+m_K+1}\epsilon^0_{\theta_2} \sum_{\delta\in \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \epsilon^1_{\theta_2}(\zeta^\delta)\chi_2(\zeta^\delta)\end{aligned}$$

which we rewrite as

$$\begin{aligned}(-1)^{n'+1} &\sum_{\delta\in \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \epsilon^1_{\theta_1}(\zeta^\delta)\chi_1(\zeta^\delta) \\ &= (-1)^{n+n_K+m+m'+m_K+1}\epsilon^0_{\theta_1}\epsilon^0_{\theta_2} \sum_{\delta\in \mathrm{Gal}(\mathbf{k}/\mathbf{t})} \epsilon^1_{\theta_2}(\zeta^\delta)\chi_2(\zeta^\delta). \qquad (5.3.2)\end{aligned}$$

This equation stays true if ζ varies over all generators of the extension $\mathbf{k}/\mathbf{f}$. Since $\mathfrak{s}_1$ is a supercuspidal inertial class for $\mathrm{GL}_n(F)$, the character χ_1 is $\mathbf{e}$-regular.

For any $\mathbf{e}$-regular character χ_1 of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$, there exists a $\mathbf{e}$-regular character χ_2 of the same group such that

$$\mathfrak{s}_{G_1}(\Theta_F, \Theta_E, [\chi_1]) = \mathrm{JL}(\mathfrak{s}_{G_2}(\Theta_F, \Theta_E, [\chi_2])).$$

This follows from theorem 5.3.1 and the invariance of parametric degrees under the Jacquet–Langlands correspondence. Then equation (5.3.2) continues to stay true.

At the left hand side of (5.3.2), one has the trace of the supercuspidal irreducible representation $\sigma[\epsilon^1_{\theta_1}\chi_1]$ of $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{t})$. By [BH10, 2.3 Corollary] we deduce that[§]

$$\begin{gathered}(-1)^{n+n'+n_K+1}\epsilon^0_{\theta_1} = (-1)^{m+m'+m_K+1}\epsilon^0_{\theta_2} \text{ and} \\ \sigma[\epsilon^1_{\theta_1}\chi_1] \cong \sigma[\epsilon^1_{\theta_2}\chi_2],\end{gathered}$$

and the claim follows. □

It follows from theorem 5.3.3 that, twisting the p-primary β-extension by the symplectic sign character (a quadratic character), we obtain conjugacy classes κ_i of β-extensions in G_i, of endo-class Θ_F, such that $\Lambda_{\kappa_1}(\mathfrak{s}_1) = \Lambda_{\kappa_2}(\mathfrak{s}_2)$ whenever the $\mathfrak{s}_i$ are supercuspidal

[§]We couldn't apply this directly to equation 5.3.1 to deduce

$$\begin{gathered}[\chi_1] = [\chi_2] \\ (-1)^{n+n'+n_K+1}\epsilon_{\theta_1}(\zeta) = (-1)^{m+m'+m_K+1}\epsilon_{\theta_2}(\zeta)\end{gathered}$$

because the sign $\epsilon_{\theta_i}(\zeta)$ may not be constant on the ζ which generate K over F, since these may generate proper subgroups of μ_K.

inertial classes and Jacquet–Langlands transfers of each other. Notice also that by [BH11, 6.9], the sign ϵ^0 and the character ϵ^1 determine each other: ϵ^1 is the nontrivial quadratic character if and only if p is odd and $\epsilon^0 = -1$. It follows that the quadratic character $\epsilon^1_{\theta_1}\epsilon^1_{\theta_2}$ is nontrivial if and only if p is odd and $n + n' + n_K + m + m' + m_K$ is odd.

Theorem 5.3.4. With the notation of the previous paragraph, the equality $\Lambda_{\kappa_1}(\mathfrak{s}_1) = \Lambda_{\kappa_2}(\mathfrak{s}_2)$ also holds for non-cuspidal $\mathfrak{s}_i$.

Proof. This follows from theorem 5.3.3 by the method of [SS16b, Lemma 9.1]. Namely, write $[\chi_i]$ for $\Lambda_{\kappa_i}(\mathfrak{s}_i)$ and assue that χ_1 is not $\mathbf{e}$-regular, as the $\mathbf{e}$-regular case has already been treated in theorem 5.3.3. Write $\xi(\kappa_1, \kappa_2)$ for the permutation of $\Gamma(\Theta_F)\backslash X_{\mathbf{C}}(\Theta_F)$ such that

$$\xi(\kappa_1, \kappa_2)\Lambda_{\kappa_1}(x_1) = \Lambda_{\kappa_2}(x_2)$$

for all simple inertial classes $x_1 = \mathrm{JL}_{G_2}(x_2)$ of endo-class Θ_F. By the results in section 5.2 we see that for any prime number $l \neq p$ this permutation preserves the equivalence relation of having the same l-regular parts on $\Gamma(\Theta_F)\backslash X_{\mathbf{C}}(\Theta_F)$. We will prove that $\xi[\chi_1] = [\chi_1]$.

Because the parametric degree of simple inertial classes, as defined in [BH11], is preserved under the Jacquet–Langlands correspondence, one finds that $\mathbf{e}[\chi_1] = \mathbf{e}[\chi_2]$ (since by the formulas in remark 4.5.9 the parametric degree of $\mathfrak{s}_{G_i}(\Theta_F, \Theta_E, [\chi_i])$ equals $n/s[\chi_i]$). Hence ξ preserves the size of a Frobenius orbit.

Let a be some large integer ($a \geq 7$ will suffice) and write κ_i^* for the maximal β-extension in $\mathrm{GL}_{an}(F)$ or $\mathrm{GL}_{am}(D)$ compatible with κ_i. Let $\mathfrak{s}_i$ correspond to the supercuspidal support $\pi_i^{\otimes r_i}$, and let $\mathfrak{s}_{i,a}$ be the simple inertial class with supercuspidal support $\pi_i^{\otimes a r_i}$. Then the $\mathfrak{s}_i^*$ are Jacquet–Langlands transfers of each other, and we claim that that $\Lambda_{\kappa_i^*}(\mathfrak{s}_i^*)$ is the inflation $[\chi_i^*]$ of $\Lambda_{\kappa_i}(\mathfrak{s}_i)$. To see this, observe that π_i is a supercuspidal representation of some $\mathrm{GL}_{n_0}(F)$ or $\mathrm{GL}_{m_0}(D)$ and write $\kappa_{i,*}$ for the β-extension in this group compatible with κ_i. Then by construction $\Lambda_{\kappa_i}(\mathfrak{s}_i)$ is the inflation of $\Lambda_{\kappa_{i,*}}(\pi_i)$. By transitivity, $\kappa_{i,*}$ and κ_i^* are compatible, hence $\Lambda_{\kappa_i^*}(\mathfrak{s}_i^*)$ is the inflation of $\Lambda_{\kappa_{i,*}}(\pi_i)$, and the claim follows.

So $\xi(\kappa_1^*, \kappa_2^*)[\chi_1]^* = [\chi_2]^*$, and since the norm is surjective in finite extensions of finite fields it suffices to prove that $\xi(\kappa_1^*, \kappa_2^*)[\chi_1]^* = [\chi_1]^*$. By [SS16b, Lemma 8.5] we can find a prime number $l \neq p$ not dividing the order of $\mathbf{e}[\chi_i]^\times$, an integer $a \geq 1$ and an $\mathbf{e}$-regular character β of $\mathbf{e}^\times_{an/\delta(\Theta_F)}$ with the same l-regular part of χ_1^*. Then $(\xi[\chi_1^*])^{(l)} = (\xi[\beta]])^{(l)}$, so it suffices to prove that $\xi[\beta] = [\beta]$ and $\xi[\chi_1^*]$ is l-regular. That $\xi[\chi_1^*]$ is l-regular follows as ξ preserves parametric degrees and l does not divide the order of $\mathbf{e}[\chi_i]^\times$.

By theorem 5.3.3 we know that there exists some β-extension $\varkappa$ in $\mathrm{GL}_{an}(F)$ such that $\xi(\varkappa, \kappa_2^*)[\beta] = [\beta]$, hence there exists some character δ of $\mathbf{e}^\times$ such that $\xi(\kappa_1^*, \kappa_2^*)[\beta] = [\delta\beta]$ for *every* $\mathbf{e}$-regular character β of $\mathbf{e}^\times_{an/\delta(\Theta_F)}$, because κ_1^* and $\varkappa$ are unramified twists of each other. An argument analogous to the one used in the proof of theorem 6.2.2 to follow then proves that $\delta = 1$, and the theorem follows. □

6

Canonical β-extensions.

In this chapter we show that the methods of section 5.3, when applied to the local Langlands correspondence, yield a canonical choice of β-extensions for $\mathrm{GL}_m(D)$ that is characterized by a compatibility with the semisimple tame part of Langlands parameters. As a byproduct, we generalize some of the results of [BH14] to arbitrary representations.

We let R be an algebraically closed field of characteristic $l \neq p$.

6.1 Clifford theory on the Weil group.

Our definition of canonical β-extension for $\mathrm{GL}_n(F)$ will go through a comparison of the map $\Lambda_{\kappa^{\mathrm{can}}}$ with $\Lambda \circ \mathrm{rec}$, where Λ is a map that we construct in this section, by an application of Clifford theory as in [Vig01b] and [BH14, Section 1]. We keep the same choice of lifts $\Theta_E \to \Theta_F$ for all endo-classes of degree dividing n that we have fixed when defining our parametrization. The definition of Λ will depend on this choice (in the same way as Λ_κ).

Definition 6.1.1. A *supercuspidal inertial type* for W_F over R is the restriction to inertia of an irreducible R-representation σ of W_F.

Let σ be an irreducible R-representation of W_F of dimension n. Since P_F is a normal subgroup of W_F, the restriction $\sigma|_{P_F}$ is semisimple and consists of a single W_F-orbit of irreducible representations (possibly with multiplicity). Let α be a representative of this W_F-orbit. Let $T = T_\alpha = Z_F(\alpha)$ be the tamely ramified extension of F corresponding to the stabilizer of α in W_F. It is a subfield of $\overline{F}$.

By [BH14, 1.3], there exists a unique extension ρ_α of α to I_T with p-primary determinant, and ρ_α extends to W_T. We denote by $\rho(\alpha)$ an arbitrary choice of extension of ρ_α to W_T.

As in [Vig01b, Section 2.6], there exists a unique tamely ramified representation $\sigma^{\mathrm{tr}}(\alpha)$ of W_T, denoted τ in [BH14], such that $\sigma \cong \mathrm{Ind}_T^F(\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha))$.

Lemma 6.1.2. The α-isotypic component of σ, denoted σ_α, is isomorphic to $\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha)$ as a representation of W_T.

Proof. Notice that $\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha)$ is an irreducible W_T-subspace of σ_α. Let $\{g_i\}$ be a set of representatives of W_F/W_T. Then $g_i(\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha)) \subset \sigma_{g_i\alpha}$, hence

$$R[W_F](\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha)) = \bigoplus_i g_i(\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha))$$

and $R[W_F](\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha))$ would be a proper W_F-subspace of σ if $\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha)$ were properly contained in σ_α. □

The representation $\sigma^{\mathrm{tr}}(\alpha)$ can be written uniquely as an induced representation $\mathrm{Ind}_{T_d}^T(\chi_1(\alpha))$ for some unramified extension T_d/T of degree $d > 0$ and some $\mathrm{Gal}(T_d/T)$-orbit of T-regular characters $[\chi_1(\alpha)]$ of $T_d^\times$, where $\chi_1(\alpha)$ is trivial on $U^1(T_d)$ and is inflated to a character of W_{T_d} via the Artin reciprocity map

$$\mathrm{Art}_{T_d}^{-1} : W_{T_d} \to T_d^\times.$$

One finds that $\sigma \cong \mathrm{Ind}_{T_d}^F(\rho_d(\alpha) \otimes \chi_1(\alpha))$ for the restriction $\rho_d(\alpha)$ of $\rho(\alpha)$ to W_{T_d}.

Remark 6.1.3. Write $\chi(\alpha) = \chi_1(\alpha)|\mu_{T_d}$. Let $[\chi(\alpha)]$ be its orbit under $\mathrm{Gal}(T_d/T)$. The restriction of σ_α to $I_{T_d} = I_T$ is a direct sum of the twists $\rho_\alpha \otimes \chi$ for $\chi \in [\chi(\alpha)]$, hence we can recover $[\chi(\alpha)]$ from σ as follows. Take the α-isotypic component σ_α and restrict it to I_{T_α}. The restriction will decompose as a direct sum of twists of ρ_α, which is the only irreducible extension of α to I_{T_α} with p-primary determinant character, by characters of μ_{T_d}. Since μ_{T_d} has order coprime to p, the map $\chi \mapsto \rho_\alpha \otimes \chi$ is injective, and this determines $[\chi(\alpha)]$ as the set of characters such that $\rho_\alpha \otimes \chi$ is a constituent of $\sigma_\alpha|I_{T_\alpha}$.

By theorem 3.1.11, the W_F-orbit $[\alpha]_F$ defines an endo-class Θ_F (denoted $\Phi_F[\alpha]_F$ in that reference).

Lemma 6.1.4. We have the equality $d = n/\delta(\Theta_F)$.

Proof. By [BH14, Tame Parameter Theorem], the field T is isomorphic over F to a tame parameter field for Θ_F, and the degree $\delta(\Theta_F)$ equals $[T : F]\dim\alpha$. Since σ decomposes as the direct sum of its α-isotypic components for $\alpha \in [\alpha]_F$, and the orbit $[\alpha]_F$ has $[T : F]$ elements, and $\rho(\alpha)$ extends α, we have the equality

$$n = [T : F](\dim\alpha)(\dim\sigma^{\mathrm{tr}}(\alpha)).$$

Hence $d = \dim\sigma^{\mathrm{tr}}(\alpha) = n/\delta(\Theta_F)$. □

Let's introduce the maximal unramified extension $E = T^{\mathrm{ur}}$ of F in T. This is independent of the choice of α, and it is the unramified parameter field of Θ_F in $\overline{F}$. At this stage, we have attached to σ an endo-class Θ_F of degree dividing $n = \dim(\sigma)$, and whenever we choose a representative α of the orbit $[\alpha]_F$ attached to Θ_F, we obtain a $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$-orbit $[\chi(\alpha)]$ of $\mathbf{e}$-regular characters of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$, since $\mu_T = \mu_E$, $\mu_{T_d} = \mu_{E_d}$ and $d = n/\delta(\Theta_F)$. This is, of course, very similar to the data of a triple $(\Theta_F, \Theta_E, [\chi])$. To push the resemblance further, we now consider how $[\chi(\alpha)]$ changes when we change representative $\alpha \in [\alpha]_F$.

Lemma 6.1.5. Let $g \in W_F$. Then $[\chi(\mathrm{ad}(g)^*\alpha)]$ only depends on the image of g in $W_F/W_E \cong \mathrm{Gal}(E/F)$.

Proof. By our explicit description of $[\chi(\alpha)]$ in terms of the α-isotypic component of σ (see remark 6.1.3) it follows that $g^*[\chi(\mathrm{ad}(g)^*\alpha)] = [\chi(\alpha)]$. However, by definition, the group W_E fixes the $\mathrm{Gal}(E_d/E)$-conjugacy classes of characters of μ_{E_d}, for every d. □

A choice of lift Θ_E of Θ_F to E defines an orbit of W_E on $[\alpha]_F$, and we are now in a similar situation as for $\mathrm{GL}_n(F)$, except that we have no ambiguity coming from the β-extension: we can define

$$\Lambda^+_{\Theta_E} : \sigma \mapsto [\chi(\alpha)],$$

for any α such that $\Theta_E = \Phi_E[\alpha]_E$. Notice that $[\chi(\alpha)] \in \Gamma(\Theta_F)\backslash X_R(\Theta_F)$. The behaviour of level zero maps under change of lifts is also the same as for $\mathrm{GL}_n(F)$.

Lemma 6.1.6. Let $\gamma \in \mathrm{Gal}(E/F)$. Then $\gamma^* \circ \Lambda^+_{\gamma^*\Theta_E} = \Lambda^+_{\Theta_E}$.

Proof. By theorem 3.1.11, if $\Theta_E = \Phi_E[\alpha]$ then $\gamma^*\Theta_E = \Phi_E(\mathrm{ad}(g)^*[\alpha])$ for any lift $g \in W_F$ of γ. We have seen that $g^*[\chi(\mathrm{ad}(g)^*\alpha)] = [\chi(\alpha)]$, which implies the lemma. □

From now we fix a lift $\Theta_E \to \Theta_F$ and write Λ^+ for $\Lambda^+_{\Theta_E}$.

Proposition 6.1.7. Two irreducible W_F-representations σ_1 and σ_2 containing $\alpha \in \mathrm{Irr}_R\, P_F$ have the same image under Λ^+ if and only if they have isomorphic restriction to I_F.

Proof. By remark 6.1.3, the restriction $\sigma_i|_{I_F}$ determines $\Lambda^+_{\Theta_E}(\sigma_i)$. Conversely, one can construct $\sigma|_{I_F}$ if one knows that $\Lambda^+_{\Theta_E}(\sigma) = [\chi]$. Indeed, choose a representative α of the W_{T_E}-orbit of representations of P_F attached to Θ_E. Then the isotypic component σ_α is isomorphic to

$$\rho(\alpha) \otimes \mathrm{Ind}_{T_{d,\alpha}}^{T_\alpha} \chi_1(\alpha)$$

for some $\rho(\alpha)$ and some extension $\chi_1(\alpha)$ of χ to a character of $T_d^\times$ trivial on $U^1(T_d)$. These are not determined by the level zero part. However, we see that

$$\sigma_\alpha|I_{T_\alpha} \cong \rho_\alpha \otimes \bigoplus_{\xi \in [\chi]} \xi \tag{6.1.1}$$

and so the restriction $\sigma_\alpha | I_{T_\alpha}$ is determined by $\Lambda^+_{\Theta_E}(\sigma)$. Similarly one computes all the $\sigma_\gamma|_{I_{T_\gamma}}$ for $\gamma \in [\alpha]_F$, applying lemma 6.1.6, and then the restriction $\sigma | I_F$ is determined by the Mackey formula for induction and restriction

$$\mathrm{Res}^{W_F}_{I_F} \sigma = \mathrm{Res}^{W_F}_{I_F} \mathrm{Ind}^{W_F}_{W_{T_\alpha}} \sigma_\alpha = \bigoplus_{\gamma \in W_{T_\alpha} \backslash W_F / I_F} \mathrm{Ind}^{I_F}_{\gamma^{-1} I_{T_\alpha} \gamma} \mathrm{Res}^{\gamma^{-1} W_{T_\alpha} \gamma}_{\gamma^{-1} I_{T_\alpha} \gamma} \mathrm{ad}(\gamma)^* \sigma_\alpha. \qquad (6.1.2)$$

□

By the proposition, we have a well-defined injection

$$\Lambda^+ : (\text{supercuspidal inertial types of dimension } n \text{ over } R \text{ containing } \Theta_F) \to \Gamma(\Theta_F) \backslash X_R(\Theta_F)$$

with image the $\mathbf{e}$-regular orbits. The left-hand side consists, of course, of those representations whose restriction to P_F corresponds to Θ_F.

Now we extend this to non-regular orbits. The Langlands parameter of a simple $R[\mathrm{GL}_n(F)]$-representation π has restriction to inertia isomorphic to $\sigma^{\oplus m}$, for some $m | n$ and some supercuspidal inertial type σ of dimension n/m. This motivates the following definition.

Definition 6.1.8. A simple inertial type of endo-class Θ_F is a representation of $R[W_F]$ isomorphic to $\sigma^{\oplus m}$, for some supercuspidal inertial type σ of endo-class Θ_F.

We extend the map Λ^+ to simple inertial types of endo-class Θ_F and dimension n by putting

$$\Lambda^+(\sigma^{\oplus m}) = N^* \Lambda^+(\sigma)$$

where $N : \mathbf{e}^\times_{n/\delta(\Theta_F)} \to \mathbf{e}^\times_{n/m\delta_{\Theta_F}}$ is the norm.

Remark 6.1.9. To see that $n/(m\delta(\Theta_F))$ is an integer, one could notice that it equals $\dim \sigma^{\mathrm{tr}}$ by the proof of lemma 6.1.4, since σ is an n/m-dimensional irreducible representation of W_F of endo-class Θ_F.

It will be convenient (because of the statement of theorem 6.2.3 to follow) to twist Λ^+ by a certain automorphism of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$. Let p^r be the degree of any parameter field P of Θ_F over the maximal tamely ramified extension of F it contains (this is the degree of the "wildly ramified part" of the endo-class Θ_F).

Definition 6.1.10. Let τ be a simple inertial type of endo-class Θ_F. Define the level zero map as

$$\Lambda^{\Theta_E}(\tau) = \Lambda^+_{\Theta_E}(\tau)^{p^{-r}}.$$

Now every maximal β-extension κ in $\mathrm{GL}_n(F)$ defines a map Λ_κ on simple inertial classes with endo-class Θ_F. The local Langlands correspondence over $\mathbf{C}$ puts the simple $\mathbf{C}$-inertial classes with endo-class Θ_F in bijection with the simple $\mathbf{C}$-inertial types with endo-class Θ_F. We write rec for this bijection. We define a permutation $\xi(\kappa)$ of the set $\Gamma(\Theta_F)\backslash X_{\mathbf{C}}(\Theta_F)$, depending on κ, via

$$\xi(\kappa)(\Lambda_\kappa(\pi)) = \Lambda(\operatorname{rec}\pi)$$

for any simple irreducible representation π of $\mathrm{GL}_n(F)$ with endo-class Θ_F. Any isomorphism $\iota_l : \mathbf{C} \to \overline{\mathbf{Q}}_l$ defines a bijection $\operatorname{rec}_{\overline{\mathbf{Q}}_l}$ analogous to rec, identifying both simple inertial classes and simple inertial types over $\mathbf{C}$ with their analogues over $\overline{\mathbf{Q}}_l$. These identifications commute with the level zero maps through ι_l. The map $\operatorname{rec}_{\overline{\mathbf{Q}}_l}$ defines a permutation $\xi_l(\kappa)$ of $\Gamma(\Theta_F)\backslash X_{\overline{\mathbf{Q}}_l}(\Theta_F)$ in the same way, and $\xi_l(\kappa)$ is intertwined with $\xi(\kappa)$ by ι_l.

Lemma 6.1.11. Define the parametric degree of $[\chi] \in \Gamma(\Theta_F)\backslash X_{\mathbf{C}}(\Theta_F)$ as the size of the orbit $[\chi]$. Then the map $\xi(\kappa)$ preserves parametric degrees.

Proof. This is an immediate consequence of the definition of Λ_κ and Λ, together with lemma 3.1.9. □

Theorem 6.1.12. Two elements of $\Gamma(\Theta_F)\backslash X_{\mathbf{C}}(\Theta_F)$ have the same l-regular part if and only if their images under $\xi(\kappa)$ have the same l-regular part.

Proof. By the discussion above, it suffices to prove the theorem for $\xi_l(\kappa)$ instead of $\xi(\kappa)$. Since $\xi_l(\kappa)$ is a bijection, it suffices to prove that it preserves equality of l-regular parts. Consider two simple irreducible integral representations π_i with endo-class Θ_F and such that $\Lambda_{\kappa,\overline{\mathbf{Q}}_l}(\pi_i) = [\psi_i]$ and $[\psi_1]^{(l)} = [\psi_2]^{(l)}$.

It suffices to prove that $\Lambda(\operatorname{rec}_{\overline{\mathbf{Q}}_l}\pi_1) = \Lambda(\operatorname{rec}_{\overline{\mathbf{Q}}_l}\pi_2)$. Assume that ψ_i is norm-inflated from an $\mathbf{e}$-regular character μ_i of $\mathbf{e}^\times_{n/a_i\delta(\Theta_F)}$. By proposition 4.5.3, the equality $\Lambda_{\mathbf{r}_l\kappa,\overline{\mathbf{F}}_l}(\mathbf{r}_l\pi_i) = [\psi_i]^{(l)}$ holds.

Lemma 6.1.13. We can choose the π_i in their inertial class so that the $\mathbf{r}_l(\pi_i)$ have the same supercuspidal support.

Proof. First choose the π_i so that they have supercuspidal support of the form $a_i \cdot (\pi_i^0)$ for integral representations π_i^0. By the classification of cuspidal representations in [MS14a, Section 6], the supercuspidal support of $\mathbf{r}_l(\pi_i^0)$ has the form $\tau_i \otimes \cdots \otimes \tau_i(m_i - 1)$ for some τ_i. It follows from our assumption on $\Lambda_{\kappa,\overline{\mathbf{Q}}_l}(\pi_i)$ that $a_1 m_1 = a_2 m_2$ and τ_1 is an unramified twist of τ_2. So there exists unramified $\overline{\mathbf{Q}}_l$-characters χ_i such that any two $\overline{\mathbf{Q}}_l$-representations π_i with supercuspidal support $\chi_i\pi_i^0 \otimes \chi_i\pi_i^0(m_i) \otimes \cdots \otimes \chi_i\pi_i^0((a_i-1)m_i)$ satisfy the conclusion of the lemma. □

Write $\tau_i = \operatorname{rec}_{\overline{\mathbf{Q}}_l}(sc(\pi_i))$ for the semisimple W_F-representation underlying $\operatorname{rec}_{\overline{\mathbf{Q}}_l}(\pi_i)$. It is a direct sum of a_i copies of some irreducible representation σ_i. By theorem 3.1.4, we know that $\mathbf{r}_l(\tau_1) = \mathbf{r}_l(\tau_2)$.

By theorem 3.1.11, there exists an irreducible representation α of P_F that is contained in σ_1 and σ_2. Let W_T be the stabilizer of α in W_F. Then σ_i can be written as the induction of its α-isotypic component:

$$\sigma_i \cong \mathrm{Ind}_T^F(\rho(\alpha) \otimes \sigma_i^{\mathrm{tr}}(\alpha)).$$

There exist integers d_i and characters $\chi_i = \chi_i(\alpha)$ of $T_{d_i}^\times$ such that $\sigma_i^{\mathrm{tr}}(\alpha) = \mathrm{Ind}_{T_{d_i}}^T \chi_i(\alpha)$ and $\sigma_i = \mathrm{Ind}_{T_{d_i}}^F \rho(\alpha) \otimes \chi_i(\alpha)$. (Notice that the χ_i may be characters of different groups, and at this stage we don't attempt to compare them with the ψ_i.) It suffices to prove that $(\chi_1|\mu_{T_{d_1}})^{(l)}$ and $(\chi_2|\mu_{T_{d_2}})^{(l)}$ are both norm-inflated from μ_T-regular characters of the same μ_{T_r} for some $r > 0$, and that these characters of μ_{T_r} are conjugate over T.

Now we proceed as in [Vig01b, Section 6.2.1]. Since the wild inertia group P_F is a pro-p group, we can identify its representations over $\overline{\mathbf{Q}}_l$ and $\overline{\mathbf{F}}_l$. Then, we use that $\mathbf{r}_l(\sigma_i)$ is the semisimplification of $\mathrm{Ind}_{T_{d_i}}^F(\rho(\alpha) \otimes \mathbf{r}_l(\chi_i))$. The character $\xi_i = \mathbf{r}_l(\chi_i)$ needs not be l-regular, and it extends to its stabilizer in W_T, the Weil group of some intermediate unramified extension T_{r_i} of T. Since $\rho(\alpha)$ extends to W_T, hence to $W_{T_{r_i}}$, the induction $\mathrm{Ind}_{T_{d_i}}^{T_{r_i}} \rho(\alpha) \otimes \xi_i$ semisimplifies to a direct sum (possibly with multiplicity) of representations of the form $\rho(\alpha) \otimes \widetilde{\xi}_i$, where $\widetilde{\xi}_i$ ranges over extensions of ξ_i to T_{r_i}. All these extensions are unramified twists of each other.

By [Vig01b, Proposition 4.2] and its proof, the induction functor from T_{r_i} to G gives rise to an equivalence between the categories of $\rho(\alpha) \otimes \xi_i$-isotypic modules for $\overline{\mathbf{F}}_l[T_{r_i}]$ and $\overline{\mathbf{F}}_l[G]$. It follows that each induced representation $\mathrm{Ind}_{T_{r_i}}^F(\rho(\alpha) \otimes \widetilde{\xi}_i)$ is irreducible. So $\mathbf{r}_l(\sigma_i)$ is a direct sum of unramified twists of a single irreducible representation, which can be taken to be any of the $\mathrm{Ind}_{T_{r_i}}^F(\rho(\alpha) \otimes \widetilde{\xi}_i)$.

Since $\mathbf{r}_l(\tau_1) = \mathbf{r}_l(\tau_2)$ and $\mathbf{r}_l(\tau_i)$ is a multiple of $\mathbf{r}_l(\sigma_i)$ in the Grothendieck group, we see that $\mathrm{Ind}_{T_{r_1}}^F \rho(\alpha) \otimes \widetilde{\xi}_1$ and $\mathrm{Ind}_{T_{r_2}}^F \rho(\alpha) \otimes \widetilde{\xi}_2$ are unramified twists of each other. This implies that $r_1 = r_2$ and the restriction to $\mathcal{O}_{T_{r_i}}^\times$ of the $\widetilde{\xi}_i$ are conjugate over T, by (6.1.1). But since $\xi_i = \mathbf{r}_l(\chi_i)$ this implies that $(\chi_1|\mu_{T_{d_1}})^{(l)}$ and $(\chi_2|\mu_{T_{d_2}})^{(l)}$ are conjugate over T, after descending to μ_{T_r} via the norm (here $r = r_1 = r_2$). □

6.2 Canonical β-extensions.

By requesting that the level zero maps for the same $\Theta_E \to \Theta_F$ on $\mathrm{GL}_n(F)$ and on W_F coincide, we obtain a canonical normalization for maximal β-extensions. In this section we work over the complex numbers.

Definition 6.2.1. Fix a lift $\Theta_E \to \Theta_F$. We say that a maximal β-extension κ^{can} in $\mathrm{GL}_n(F)$ of endo-class Θ_F is *canonical* if $\Lambda_\kappa^{\mathrm{can}}(\pi) = \Lambda(\mathrm{rec}(\pi))$ for all simple irreducible representations of $\mathrm{GL}_n(F)$ of endo-class Θ_F. Equivalently, $\xi(\kappa^{\mathrm{can}})$ is the identity.

Theorem 6.2.2. Let κ^{can} be a maximal β-extension of endo-class Θ_F such that $\xi(\kappa^{\mathrm{can}})$ fixes the $\mathbf{e}$-regular elements of $\Gamma(\Theta_F)\backslash X(\Theta_F)$. Then $\xi(\kappa^{\mathrm{can}}) = 1$.

Proof. This is proved in the same way as [SS16b, Lemma 9.11]. Assume that α is a character of $\mathbf{e}^{\times}_{n/\delta(\Theta_F)}$ which is not $\mathbf{e}$-regular: we will prove that $\xi(\kappa^{\mathrm{can}})[\alpha] = [\alpha]$. Consider a simple representation π of $\mathrm{GL}_n(F)$ with supercuspidal support $\pi_0^{\otimes r}$ and $\Lambda_{\kappa^{\mathrm{can}}}(\pi) = [\alpha]$.

Let $a \geq 1$ be some large integer ($a \geq 7$ will suffice) and write $\kappa^{\mathrm{can}*}$ for the maximal β-extension in $\mathrm{GL}_{an}(F)$ compatible with κ, and let π_a be a representation of $\mathrm{GL}_{an}(F)$ with supercuspidal support $\pi_0^{\otimes ar}$. Then it follows from proposition 4.4.6 that $\Lambda_{\kappa^{\mathrm{can}*}}\pi_a$ is the inflation $[\alpha^*]$ of α to $\mathbf{e}^{\times}_{an/\delta(\Theta_F)}$.

By lemma 3.1.9 we have $\mathrm{rec}(\pi_a)|_{I_F} = \mathrm{rec}(\pi_0)|_{I_F}^{\oplus ar}$, so that if $\Lambda(\mathrm{rec}\,\pi) = [\mu]$ then $\Lambda(\mathrm{rec}\,\pi_a) = [\mu^*]$. Hence by definition we have $[\mu] = \xi(\kappa^{\mathrm{can}})[\alpha]$ and $[\mu^*] = \xi(\kappa^{\mathrm{can}*})[\alpha^*]$, although at this stage we do not know whether $[\alpha] = [\mu]$. It follows that $\xi(\kappa^{\mathrm{can}*})[\alpha^*] = (\xi(\kappa^{\mathrm{can}})[\alpha])^*$ and so it suffices to prove that $\xi(\kappa^{\mathrm{can}*})[\alpha^*] = [\alpha^*]$, because the norm is surjective in finite extensions of finite fields.

Write $\mathbf{e}[\alpha]^{\times}$ for the fixed field of the stabilizer of α in $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$. By lemma 8.5 and [SS16b, Remark 8.7], there exist an $\mathbf{e}$-regular character β of $\mathbf{e}^{\times}_{an/\delta(\Theta_F)}$ and a prime number $l \neq p$ not dividing the order of $\mathbf{e}[\alpha]^{\times}$ such that α^* is the l-regular part of β. By proposition 6.1.12 we have that $(\xi(\kappa^{\mathrm{can}*})[\alpha^*])^{(l)} = (\xi(\kappa^{\mathrm{can}*})[\beta])^{(l)}$, and it suffices now to prove that $\xi(\kappa^{\mathrm{can}*})[\beta] = [\beta]$ and that $\xi(\kappa^{\mathrm{can}*})[\alpha^*]$ is l-regular. That $\xi(\kappa^{\mathrm{can}*})[\alpha^*]$ is l-regular follows by proposition 6.1.11, because it has the same parametric degree as $[\alpha^*]$ and l does not divide the order of $\mathbf{e}[\alpha]^{\times}$.

Now, we know by theorem 6.2.3 that there exists some β-extension $\varkappa$ in $\mathrm{GL}_{an}(F)$ such that $\xi(\varkappa)[\beta] = [\beta]$. So there exists some character δ of $\mathbf{e}^{\times}$ such that $\xi(\kappa^{\mathrm{can}*})[\beta] = [\delta\beta]$ for *every* $\mathbf{e}$-regular character β of $\mathbf{e}^{\times}_{an/\delta(\Theta_F)}$, because $\varkappa$ and $\kappa^{\mathrm{can}*}$ are $\mathbf{e}^{\times}$-twists of each other. We will prove that δ is trivial: this implies the theorem.

Fix some $\mathbf{e}$-regular character α_+ of $\mathbf{e}^{\times}_{n/\delta(\Theta_F)}$. Because a is large enough, there exists some prime number $l \neq p$ not dividing the order of $\mathbf{e}^{\times}_{n/\delta(\Theta_F)} = \mathbf{e}[\alpha_+]$ (maybe not the same l as before) and some $\mathbf{e}$-regular character β_+ of $\mathbf{e}^{\times}_{an/\delta(\Theta_F)}$ such that α_+^* is the l-regular part of β_+.

We know that $\xi(\kappa^{\mathrm{can}})[\alpha_+] = [\alpha_+]$ by regularity of α_+ and by definition of κ^{can}. At the same time, $\xi(\kappa^{\mathrm{can}*})[\alpha_+^*] = [\delta\beta_+]^{(l)} = [\delta^{(l)}\alpha_+^*]$ (and this δ is the same δ as before), and since $\xi(\kappa^{\mathrm{can}*})[\alpha_+^*] = (\xi(\kappa^{\mathrm{can}})[\alpha_+])^*$ we find that $[\alpha_+^*] = [\delta^{(l)}\alpha_+^*]$.

It follows that we can write $\delta = \delta_{(l)}(\alpha_+^*)^{|\mathbf{e}|^i - 1}$ for some l-primary character $\delta_{(l)}$ and some integer $i \in \{0, \dots, \frac{n}{\delta(\Theta_F)} - 1\}$. The order of δ divides $|\mathbf{e}| - 1$, as it is a character of $\mathbf{e}^{\times}$, and so $(\delta_{(l)})^{1-|\mathbf{e}|} = (\alpha_+^*)^{(|\mathbf{e}|^i - 1)(|\mathbf{e}|-1)}$. But the order of $\delta_{(l)}$ is a power of l, and l is coprime to $|\mathbf{e}^{\times}_{n/\delta(\Theta_F)}| = |\mathbf{e}|^{n/\delta(\Theta_F)} - 1$, hence to $|\mathbf{e}| - 1$. So $\delta_{(lR)} = 1$.

Finally, we can take α_+ to be a generator of the character group of $\mathbf{e}^\times_{n/\delta(\Theta_F)}$, hence we can assume that α_+ has order $|\mathbf{e}|^{n/\delta(\Theta_F)}-1$. But the order of α_+^* divides $(|\mathbf{e}|^i-1)(|\mathbf{e}|-1)$ by the above; and since $|\mathbf{e}| \geq 2$ we have $|\mathbf{e}|^{n/\delta(\Theta_F)} - 1 > (|\mathbf{e}|^i - 1)(|\mathbf{e}| - 1)$, hence $i = 0$ and δ is trivial. □

The existence of κ^{can} satisfying the assumptions of theorem 6.2.2 can be deduced from the Types Theorem of Bushnell and Henniart, see [BH14]. We will give an explicit description of κ^{can} as a twist of a p-primary β-extension.

Theorem 6.2.3. Let θ be a maximal character in $\mathrm{GL}_n(F)$ of endo-class Θ_F. Let κ be the p-primary β-extension of θ and let ϵ^1_θ be the symplectic sign character of θ (as defined in 2.4.23, or equivalently in [BH14, 5.4]). Write ϵ_{Gal} for the quadratic character of $\mathbf{e}^\times$ which is nontrivial if and only if $p \neq 2$ and the degree of a tame parameter field of Θ_F over F is even. Then $\kappa^{\mathrm{can}} = \epsilon_{\mathrm{Gal}}\epsilon^1_\theta\kappa$ has the property that

$$\Lambda_{\kappa^{\mathrm{can}}}(\pi) = \Lambda(\mathrm{rec}(\pi))$$

for all supercuspidal irreducible representations π of $\mathrm{GL}_n(F)$ with endo-class Θ_F.

Proof. Let σ be an irreducible representation of W_F with $\Lambda(\sigma) = [\chi]$, so that if we fix $\alpha \in [\alpha]_E$ corresponding to Θ_E then the isotypic component σ_α of σ is isomorphic to $\rho(\alpha) \otimes \sigma^{\mathrm{tr}}(\alpha)$ for some choice of $\rho(\alpha)$ and $\sigma(\alpha)$. The Types Theorem in [BH14] then says that $\mathrm{rec}^{-1}(\sigma)$ contains an extended maximal simple type of the form

$$\psi \odot \lambda_{\sigma^{\mathrm{tr}}(\alpha)} \ltimes \nu.$$

Let us give definitions for these objects. First, one fixes a simple stratum $[\mathfrak{A}, \beta]$ defining θ, with tame parameter field $T_\theta \subseteq F[\beta]$, and identifies $T = T_\alpha$ with T_θ. This is done via an isomorphism $\iota : T_\alpha \to T_\theta$ in such a way that the pullback to T of the endo-class over T_θ of the interior lift θ_{T_θ} coincides with the endo-class $\Theta_T = \Phi_T(\alpha)$ corresponding to the W_T-orbit $[\alpha]_T$ of representations of the wild inertia group $P_F = P_T$.

By construction, $\Phi_E(\alpha) = \Theta_E$, and by [BH14, 6.2 Proposition] we have that Θ_T is a lift of Θ_E to T. But $\mathrm{cl}(\theta_{T_\theta})$ is a lift to T_θ of the endo-class $\mathrm{cl}(\theta_{T_\theta^{\mathrm{ur}}})$ of the interior lift of θ to the unramified parameter field T_θ^{ur} contained in $F[\beta]$. It follows that this isomorphism $\iota : T \to T_\theta$ induces the isomorphism $\iota_{T_\theta^{\mathrm{ur}}} : E \to T_\theta^{\mathrm{ur}}$ associated to Θ_E, because

$$(\iota_{T_\theta^{\mathrm{ur}}})^* \,\mathrm{cl}(\theta_{T_\theta^{\mathrm{ur}}}) = (\iota|_E)^* \mathrm{Res}_{T_\theta/T_\theta^{\mathrm{ur}}} \mathrm{cl}(\theta_{T_\theta}) = \mathrm{Res}_{T/E}(\iota^* \,\mathrm{cl}(\theta_{T_\theta})) = \mathrm{Res}_{T/E}\Theta_T = \Theta_E.$$

Then write $d = n/\delta(\Theta_F) = \dim \sigma^{\mathrm{tr}}(\alpha)$. Take an unramified extension $F[\beta]_d$ of $F[\beta]$ of degree d, contained in the centralizer of $F[\beta]$ in $M_n(F)$, such that $F[\beta]_d^\times$ normalizes θ. Let K_d be the maximal unramified extension of F in $F[\beta]_d$. The representation ν is a full Heisenberg representation of $\mathbf{J}_\theta$ over θ in the sense of [BH14, 3.2 Definition], such that the trace of ν is constant over K_d/F-regular elements of μ_{K_d}. Since this condition determines $\nu|J_\theta$ uniquely, we find that $\nu|J_\theta \cong \epsilon_\theta^1\kappa$.

The representation $\lambda_{\sigma^{\mathrm{tr}}(\alpha)}$ is constructed as follows (see [BH14, 3.6]). Consider the characters $[\chi_1(\alpha)]$ of T_d attached to $\sigma^{\mathrm{tr}}(\alpha)$, and the restrictions $\chi(\alpha) = \chi_1(\alpha)|\mathcal{O}_{T_d}^\times$. Observe that the isomorphism $\iota : T \to T_\theta$ extends to an isomorphism $\iota : T_d \to T_{\theta,d}$ to the maximal tamely ramified extension $T_{\theta,d}$ of F in $F[\beta]_d$, which is a degree d unramified extension of T_θ. We get via ι a well-defined orbit $[\chi_1(\alpha)]$ of T_θ-regular characters of $T_{\theta,d}$ under $\mathrm{Gal}(T_{\theta,d}/T_\theta)$.

Inflate χ_1 to a character χ_1^* of $F[\beta]_d^\times$ via the norm $N_{F[\beta]_d/T_{\theta,d}} : F[\beta]_d^\times \to T_{\theta,d}^\times$, and let $\mathfrak{B}$ be the intersection of $\mathfrak{A}$ with the centralizer of $F[\beta]$ in $M_n(F)$. The restriction of χ_1^* to $\mu_{F[\beta]_d} = \mu_{T_d} \cong \mathbf{t}_d^\times = \mathbf{e}_d^\times = \mathbf{e}_{n/\delta(\Theta_F)}^\times$ is an $\mathbf{e}$-regular character χ^*. Embedding $\mu_{T_d} \cong \mathbf{e}_d^\times$ in $\mathrm{GL}_d(\mathbf{e})$ as a maximal elliptic torus, we see that there exists a unique supercuspidal irreducible representation $\widetilde{\sigma}[\chi^*]$ of $U(\mathfrak{B})/U^1(\mathfrak{B})$ whose trace on $\mu_{F[\beta]_d} = \mu_{T_{\theta,d}}$ is given in terms of χ^* under the Green parametrization and the isomorphism $\iota : \mu_{T_d} \to \mu_{T_{\theta,d}}$.

The representation $\widetilde{\sigma}[\chi^*]$ is extended to $\mathbf{J}_\theta = F[\beta]^\times U(\mathfrak{B}) J_\theta^1$ by letting J_θ^1 act trivially, and the extension is denoted $\lambda_{\chi_1^*}^{\mathbf{J}}$. Then, by definition,

$$\lambda_{\sigma^{\mathrm{tr}}(\alpha)} \ltimes \nu = \lambda_{\chi_1^*}^{\mathbf{J}} \otimes \nu.$$

Since $\iota : T_d \to T_{\theta,d}$ induces $\iota_{T_\theta^{\mathrm{ur}}} : E \to T_\theta^{\mathrm{ur}}$, any isomorphism in the conjugacy class $\Psi(\Theta_E) : U(\mathfrak{B})/U^1(\mathfrak{B}) \to \mathrm{GL}_d(\mathbf{e})$ induces the isomorphism $\iota^{-1} : \mu_{T_{\theta,d}} \to \mu_{T_d} \cong \mathbf{e}_{n/\delta(\Theta_F)}^\times$, up to $\mathrm{GL}_d(\mathbf{e})$ conjugacy and the action of $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$. Then, $\widetilde{\sigma}[\chi^*]$ is isomorphic to the inflation of the representation $\sigma[\chi^*]$ of $\mathrm{GL}_d(\mathbf{e})$ through any isomorphism in the conjugacy class $\Psi(\Theta_E)$. It follows that the restriction of $\lambda_{\sigma^{\mathrm{tr}}(\alpha)} \ltimes \nu$ to J_θ is a maximal simple type corresponding to the unique Bernstein component $\mathfrak{s}$ with endo-class Θ_F and $\Lambda_{\epsilon_\theta^1\kappa}(\mathfrak{s}) = [\chi^{p^r}]$, for $p^r = [F[\beta] : T_\theta] = [F[\beta]_d : T_d]$. Indeed, the norm $N : F[\beta]_d^\times \to T_d^\times$ induces on the residue field the automorphism of raising to the p^r-th power, and the trace of $\lambda_{\sigma^{\mathrm{tr}}(\alpha)}^{\mathbf{J}}$ on $\mu_{F[\beta]_d}$ is given in terms of $\chi^* = \chi^{p^r}$.

By definition, ψ is a character of $T^\times$ trivial on $U^1(T)$ and corresponding to ϵ_{Gal} on μ_T. By part (1) of [BH14, 3.6 Proposition], one has

$$\psi \odot \lambda_{\sigma^{\mathrm{tr}}(\alpha)} \ltimes \nu = \lambda_{\sigma^{\mathrm{tr}}(\alpha)} \ltimes (\psi \odot \nu)$$

where the operation $\psi \odot \nu$ is defined in [BH14, 3.2.1] as given by $\psi^{\mathbf{J}} \otimes \nu$ for the θ-flat character $\psi^{\mathbf{J}}$ of $\mathbf{J}_\theta$ attached to ψ. This character is defined in [BH14, 3.1 Definition], and by part (1) of [BH14, 3.1 Proposition] we have $\psi^{\mathbf{J}}(x) = \psi(\det_T(x))$ for all $x \in \mathbf{J}_\theta \cap Z_G(T)$. But then the restriction of $\psi \odot \lambda_{\sigma^{\mathrm{tr}}(\alpha)} \ltimes \nu$ to J_θ is a maximal simple type for the unique Bernstein component $\mathfrak{s}'$ with endo-class Θ_F and $\Lambda_{\kappa^{\mathrm{can}}}(\mathfrak{s}') = [\chi^{p^r}]$. □

This together with theorem 6.2.2 completes the construction of canonical β-extensions, and now we prove that they behave well under transfer.

Proposition 6.2.4. Let κ^{can} be the canonical β-extension in $\mathrm{GL}_n(F)$ of endo-class Θ_F. Consider a sequence (n_i) of positive integers summing to n, such that $\delta(\Theta_F)$ divides each n_i. Let κ_i be the corresponding sequence of compatible β-extensions in the $\mathrm{GL}_{n_i}(F)$. Then each κ_i is canonical.

Proof. Fix a lift $\Theta_E \to \Theta_F$. By proposition 4.4.6, it suffices to prove that κ^{can} is compatible with $\kappa^{\mathrm{can}}_{\delta(\Theta_F)}$, the canonical β-extension in $\mathrm{GL}_{\delta(\Theta_F)}(F)$. Write κ_+ for the β-extension in $\mathrm{GL}_n(F)$ compatible with $\kappa^{\mathrm{can}}_{\delta(\Theta_F)}$. Let $\pi_{\delta(\Theta_F)}$ be a supercuspidal representation of $\mathrm{GL}_{\delta(\Theta_F)}(F)$ with endo-class Θ_F and $\Lambda_{\kappa^{\mathrm{can}}_{\delta(\Theta_F)}}(\pi_{\delta(\Theta_F)}) = [1]$. There exists a character χ of $\mathbf{e}^\times$ such that $\chi\kappa^{\mathrm{can}} \cong \kappa_+$, and then $\Lambda_{\kappa^{\mathrm{can}}}(\pi) = \chi\Lambda_{\kappa_+}(\pi)$ for all simple representations π of endo-class Θ_F.

Let π be a simple representation of $\mathrm{GL}_n(F)$ with supercuspidal support inertially equivalent to $\pi_{\delta(\Theta_F)}^{\otimes n/\delta(\Theta_F)}$. Then $\Lambda_{\kappa_+}(\pi)$ is inflated from $[1] = \Lambda_{\kappa^{\mathrm{can}}_{\delta(\Theta_F)}}(\pi_{\delta(\Theta_F)}) = \Lambda(\mathrm{rec}\,\pi_{\delta(\Theta_F)})$, by compatibility and the fact that $\kappa^{\mathrm{can}}_{\delta(\Theta_F)}$ is canonical. But we also know that $\Lambda_{\kappa^{\mathrm{can}}}(\pi) = \Lambda(\mathrm{rec}\,\pi)$, since κ^{can} is canonical. By construction, we have that $\Lambda(\mathrm{rec}\,\pi)$ is inflated from $\Lambda(\mathrm{rec}\,\pi_{\delta(\Theta_F)})$, hence $\Lambda_{\kappa_+}(\pi) = \Lambda_{\kappa^{\mathrm{can}}}(\pi) = [1]$. It follows that $\chi = 1$, hence κ^{can} is compatible with $\kappa^{\mathrm{can}}_{\delta(\Theta_F)}$. $\square$

Finally, we mention that the connection between **K**-functors and level zero parts of Langlands parameters carries over to arbitrary Bernstein components of $\mathrm{GL}_n(F)$. We briefly sketch how to see this. Given an inertial class of supercuspidal supports in $\mathrm{GL}_n(F)$

$$\mathfrak{s} = \left[\prod_{i=1}^{r} \mathrm{GL}_{m_i}(F), \times_{i=1}^{r}\pi_i\right]$$

we can assume that the π_i are ordered according to their endo-class, so that there is a partition $I_1, \ldots, I_t$ of $\{1, \ldots, r\}$ such that $i \in I_j$ if and only if π_i has endo-class Θ_j. For $1 \leq j \leq t$ write $n_j = \sum_{i \in I_j} m_j$. In [SS16a, Section 6] there is constructed a functor

$$\mathbf{K} : (\text{smooth representations of } \mathrm{GL}_n(F)) \to \left(\text{representations of } \prod_{j=1}^{t} \mathrm{GL}_{n_j}(\mathbf{e}(\Theta_j))\right)$$

with the following two properties:

1. **K** only depends on the choice of a maximal β-extension κ_j in $\mathrm{GL}_{n_j}(F)$ of endo-class Θ_j.

2. (see [SS16a, Theorem 6.2]) taking the β-extensions κ_i in $\mathrm{GL}_{m_i}(F)$ compatible

with κ_j, for $i \in I_j$, there is an isomorphism

$$\mathbf{K}(\mathrm{Ind}_P^G(\otimes_{i=1}^r \pi_i)) \to \times_{i=1}^r \mathbf{K}_i(\pi_i).$$

The induction at the left-hand side is unnormalized, but this does not affect conclusions regarding inertial classes. If π is a representation with supercuspidal support in $\mathfrak{s}$, we see that $\mathrm{rec}(\pi)|_{I_F} \cong \oplus_{i=1}^r \mathrm{rec}(\pi_i)|_{I_F}$ by lemma 3.1.9. It follows that, if all the κ_i are canonical and we compute with the corresponding $\mathbf{K}$-functor, then the supercuspidal support of $\mathbf{K}(\pi)$ encodes the level zero part of the Langlands parameter of π.

THE CASE OF $\mathrm{GL}_m(D)$. The canonical β-extension of endo-class Θ_F for the group $\mathrm{GL}_n(F)$ is

$$\kappa_{\mathrm{GL}_n(F)}^{\mathrm{can}} = \epsilon_{\mathrm{Gal}} \epsilon_\theta^1 \kappa_p,$$

where κ_p is p-primary and ϵ_{Gal} is nontrivial if and only if $p \neq 2$ and the degree of a tame parameter field of Θ_F over F is even. We make the same definition for $\mathrm{GL}_m(D)$.

Definition 6.2.5. Let θ be a maximal simple character in $\mathrm{GL}_m(D)$. The canonical β-extension of θ is $\kappa_{\mathrm{GL}_m(D)}^{\mathrm{can}} = \epsilon_{\mathrm{Gal}} \epsilon_\theta^1 \kappa_p$, where we used the same notation as in the previous paragraph.

Because of theorems A and B, $\kappa_{\mathrm{GL}_m(D)}^{\mathrm{can}}$ has the property that if π is an essentially square-integrable representation of $\mathrm{GL}_m(D)$, then $\Lambda_{\kappa_{\mathrm{GL}_m(D)}^{\mathrm{can}}}(\pi)$ coincides with the level zero part of the Langlands parameter $\mathrm{rec}(\mathrm{JL}(\pi))$, as defined in 6.1.10. Furthermore, since the canonical β-extensions of endo-class Θ_F in $\mathrm{GL}_n(F)$ are compatible with each other for varying n, we see (by theorem 5.3.3 and an argument similar to the proof of proposition 4.4.3) that the same is true for $\mathrm{GL}_m(D)$ and varying m.

7

$\mathbf{K}$-types for $\mathrm{GL}_n(F)$ and $D^\times$.

In this chapter we recall the results of Schneider and Zink about $\mathrm{GL}_n(F)$-typical representations of $\mathbf{K} = \mathrm{GL}_n(\mathcal{O}_F)$, which we also refer to as $\mathbf{K}$-*types*, although they are not types in the sense of section 2.3 (see the discussion after proposition 7.0.8). Then we establish analogues for the maximal compact subgroup of $D^\times$, and we prove some formulas for the trace of a maximal $\mathbf{K}$-type in terms of its level zero part and its base change to unramified extensions, similar to the ones in chapter 5.

We will also fix the maximal compact subgroup $\mathbf{K} = \mathrm{GL}_n(\mathcal{O}_F)$ of $\mathrm{GL}_n(F)$ for the rest of this thesis. Unless otherwise stated, in this chapter we work over an algebraically closed field R of characteristic zero.

$\mathbf{K}$-TYPES FOR $\mathrm{GL}_n(F)$. Let $A = M_n(F)$ and $G = A^\times = \mathrm{GL}_n(F)$. We recall some results from [SZ99] and translate them in the form we will need later on. Let $F[\beta]$ be a field extension of F in $A = M_n(F)$, and let $B = Z_A(F[\beta])$. Choose a pair $\mathfrak{B}_{\min} \subseteq \mathfrak{B}_{\max}$ of hereditary $\mathcal{O}_{F[\beta]}$-orders in B, such that $\mathfrak{B}_{\min}$ is minimal and $\mathfrak{B}_{\max}$ is maximal. Recall that hereditary $\mathcal{O}_{F[\beta]}$-orders $\mathfrak{B}$ in B are in bijection with $\mathcal{O}_{F[\beta]}$-lattice chains in $V = F^n$ viewed as an $F[\beta]$-vector space via the inclusion $F[\beta] \subset A$. Since these are also $\mathcal{O}_F$-lattice chains, there corresponds to $\mathfrak{B}$ a unique hereditary $\mathcal{O}_F$-order $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$ of A, which is actually the continuation of $\mathfrak{B}$ to A in the sense of theorem 2.4.1. It satisfies $\mathfrak{A}(\mathfrak{B}) \cap B = \mathfrak{B}$. Following [SZ99, Section 5], we associate to $\mathfrak{B}$ a subgroup $J = J(\mathfrak{B}) = J(\beta, \mathfrak{A}(\mathfrak{B}))$ of the unit group $\mathfrak{A}(\mathfrak{B})^\times$ such that $J = J^1\mathfrak{B}^\times$ for $J^1 = J^1(\mathfrak{B}) = J \cap U^1(\mathfrak{A}(\mathfrak{B}))$. We write $J_{\max}$ and $J^1_{\max}$ for the groups corresponding to $\mathfrak{B}_{\max}$.

Remark 7.0.1. From now on, we make the assumption that the group $J_{\max}$ is contained in our fixed maximal compact subgroup $\mathbf{K}$. This can always be achieved after possibly replacing $F[\beta]$ with a conjugate.

We let θ be a simple character of the stratum $[\mathfrak{A}_{\max}, \beta]$, so that $J_{\max} = J_\theta$ and $J^1_{\max} = J^1_\theta$ by definition, and we write $\kappa_{\max} = \kappa(\mathfrak{B}_{\max})$ for the corresponding canonical β-extension (the paper [SZ99] works with an arbitrary β-extension). Notice that θ is maximal. There is a corresponding family of representations $\kappa(\mathfrak{B})$ of $J(\beta)$, one for each for any hereditary $\mathcal{O}_{F[\beta]}$-order $\mathfrak{B}_{\min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{\max}$, satisfying a coherence property as in [SZ99, Lemma 5.1].

Writing Θ_F for the endo-class of θ, and E/F for the unramified parameter field of Θ_F in $\overline{F}$, we have attached an inner conjugacy class of isomorphisms

$$J_{\max}/J^1_{\max} \to \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$$

to every lift of Θ_F to an endo-class Θ_E over E. Fix such a lift, and let ψ be a representative of the corresponding conjugacy class, such that ψ identifies $\mathfrak{B}^\times_{\min} J^1_{\max}/J^1_{\max}$ with the upper-triangular Borel subgroup (compare the discussion after [SZ99, Lemma 5.5]).

The functor $\mathbf{K}^+_{\kappa_{\max}}$ is denoted $V \mapsto V(\kappa_{\max})$ in [SZ99], and we will sometimes adopt this notation. Also, we will compose it with our isomorphism ψ, and denote the resulting functor still by $V \mapsto V(\kappa_{\max})$. It sends admissible representations to finite-dimensional representations.

For any positive divisor r of $n/\delta(\Theta_F)$ we have a standard parabolic subgroup of $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$, with Levi factor isomorphic to $\prod_{i=1}^r \mathrm{GL}_{n/r\delta(\Theta_F)}(\mathbf{e})$, and consisting of block upper triangular matrices. It coincides with the image under ψ of $\mathfrak{B}^\times J^1_{\max}/J^1_{\max}$ for some principal order $\mathfrak{B}_{\min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{\max}$ that we fix. If $\sigma_0 = \sigma[\chi]$ is a cuspidal representation of $\mathrm{GL}_{n/r\delta(\Theta_F)}(\mathbf{e})$, and $\sigma = \sigma_0^{\otimes r}$ is inflated to $J(\mathfrak{B})/J^1(\mathfrak{B})$, then the pair $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \sigma)$ is a simple type in $\mathrm{GL}_n(F)$. It is a maximal simple type precisely when $r = 1$. The next lemma connects this construction with our parametrization of simple inertial classes.

Lemma 7.0.2. With the notation of the previous paragraph, the pair $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \sigma)$ is a type for the Bernstein component $\mathfrak{s}_G(\Theta_F, \Theta_E, [\chi])$.

Proof. Let V be an irreducible representation of $\mathrm{GL}_n(F)$ contained in the inertial class of the simple type $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \sigma)$. It suffices to prove that the supercuspidal support of V is inertially equivalent to $r \cdot V_0$, for some $V_0 \in \mathrm{Irr}(\mathrm{GL}_n(F))$ with $\mathrm{inv}(V_0) = (\Theta_F, \Theta_E, [\chi])$.

By [SZ99, Proposition 5.3] the supercuspidal support of $V(\kappa_{\max})$ is

$$\left[\prod_{i=1}^r \mathrm{GL}_{n/r\delta(\Theta_F)}(\mathbf{e}), \sigma_0^{\otimes r}\right].$$

Let θ_r be a maximal simple character in $\mathrm{GL}_{n/r}(F)$ of endo-class Θ_F. Let $\kappa_{\max,r}$ be its canonical β-extension to J_{θ_r}. The representations $\kappa_{\max}$ and $\kappa_{\max,r}$ are compatible, by proposition 6.2.4. Then the claim follows from (4.4.1), which implies that the supercuspidal support of V is inertially equivalent to $\left[\prod_{i=1}^r \mathrm{GL}_{n/r}(F), V_0^{\otimes r}\right]$, where V_0 is any irreducible representation with $\mathbf{K}_{\kappa_{\max},r}(V_0) = \sigma_0$. $\square$

Write $\mathfrak{s} = \mathfrak{s}_G(\Theta_F, \Theta_E, [\chi])$ and $\sigma_0 = \sigma[\chi]$. We define two classes of virtual representations of $\mathbf{K}$ attached to $\mathfrak{s}$, depending only on the maximal simple character θ. Assume that $J_\theta \subseteq \mathbf{K}$. If $\mathfrak{P}$ is a partition of r, the constructions of section 2.5 provide us with a representation $\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0)$ of $J_\theta / J_\theta^1 \cong \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$ (via the lift Θ_E).

Definition 7.0.3. Write $\sigma_{\mathfrak{P}}(\mathfrak{s}) = \mathrm{Ind}_{J_\theta}^{\mathbf{K}}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0))$, which by the discussion at the end of [SZ99, Section 5] is an irreducible smooth representation of $\mathbf{K}$. Write $\sigma_{\mathfrak{P}}^+(\mathfrak{s})$ for the virtual representation $\mathrm{Ind}_{J_\theta}^{\mathbf{K}}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}^+(\sigma_0))$ of $\mathbf{K}$. We will refer to these representations as **K**-*types* for $\mathfrak{s}$.

Next we show how the **K**-types provide a refinement of Bushnell–Kutzko type theory for generic representations. Via the Bernstein–Zelevinsky classification, we can attach to each irreducible representation $V \in \mathrm{Irr}(\mathfrak{s})$ a partition $\mathfrak{P}(V)$ of r, in the following way.

Definition 7.0.4. Let $V \in \mathrm{Irr}(\mathfrak{s})$. We define $\mathfrak{P}(V)(i)$ to be the number of times a segment of length i appears in the multiset corresponding to V. We will sometimes shorten notation to $\mathfrak{P} = \mathfrak{P}(V)$.

Proposition 7.0.5. Let $V \in \mathrm{Irr}\,\mathrm{GL}_n(F)$ be generic. Then $\mathrm{Hom}_{J_\theta}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0), V) \neq 0$ if and only if $V \in \mathfrak{s}$ and its partition $\mathfrak{P}(V)$ satisfies $\mathfrak{P} \leq \mathfrak{P}(V)$.

Proof. By [SZ99, Lemma 5.2], the nonvanishing implies that $V \in \mathrm{Irr}(\mathfrak{s})$. By [SZ99, Proposition 5.9] we have that $V(\kappa_{\max}) \cong \pi_{\mathfrak{P}(V)}(\sigma_0)$ whenever $V \in \mathrm{Irr}(\mathfrak{s})$ is generic. Then the claim follows from the existence of an isomorphism

$$\mathrm{Hom}_{J_\theta}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0), V) \to \mathrm{Hom}_{\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})}(\sigma_{\mathfrak{P}}(\sigma_0), V(\kappa_{\max})).$$

$\square$

Remark 7.0.6. By [SZ99, Lemma 5.2], the fact that $V \in \mathrm{Irr}(\mathfrak{s})$ if $\mathrm{Hom}_{J_\theta}(\kappa_{\max} \otimes \sigma_{\mathfrak{P}}(\sigma_0), V) \neq 0$ holds for any $V \in \mathrm{Irr}\,\mathrm{GL}_n(F)$, with no genericity assumptions.

Example 7.0.7. Let V be irreducible and generic. We have $(\sigma_{\mathfrak{P}_{\min}}^+(\mathfrak{s}), V)_{\mathbf{K}} \neq 0$ if and only if $V \in \mathrm{Irr}(\mathfrak{s})$ and $V(\kappa_{\max}) \cong \pi_{\mathfrak{P}_{\min}}(\sigma_0)$, in which case it equals one. This happens if and only if $\mathfrak{P}(V) = \mathfrak{P}_{\min}$, that is the multiset of V has only one segment (because $\mathfrak{P}_{\min}$ is the partition with only one summand). Equivalently, V is an essentially square-integrable representation in $\mathfrak{s}$.

Finally, we remove the dependence of the **K**-types on θ.

Proposition 7.0.8. The representations $\sigma_{\mathfrak{P}}(\mathfrak{s})$ and $\sigma_{\mathfrak{P}}^+(\mathfrak{s})$ are independent of the choice of θ.

Proof. Let θ_1 and θ_2 be conjugate maximal simple characters in $\mathrm{GL}_n(F)$, with $J_{\theta_i} \subseteq \mathbf{K}$. The orders $\mathfrak{A}_i$ attached to the θ_i are principal orders with the same ramification, corresponding to lattice chains containing the lattice chain defined by $\mathbf{K}$ (because $\mathfrak{A}$ is the continuation of $\mathfrak{B}$ and $\mathfrak{B}^\times \subseteq J_\theta \subseteq \mathbf{K}$). Hence the $\mathfrak{A}_i$ are $\mathbf{K}$-conjugate. Since intertwining maximal simple characters defined on the same order are conjugate under the group of units of that order (see theorem [BK93, Theorem 3.5.11]), we see that the θ_i are conjugate under $\mathbf{K}$, hence so are the J_{θ_i}. Write $J_{\theta_2} = \mathrm{ad}(g)J_{\theta_1}$. Since the lift $\Theta_E \to \Theta_F$ is fixed, by the proof of proposition 4.3.3 the inner conjugacy classes $[\psi_i] : J_{\theta_i}/J^1_{\theta_i} \to \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$ satisfy $[\psi_1] = \mathrm{ad}(g)^*[\psi_2]$. It follows that we get isomorphic representations when inducing. □

So there are well-defined representations $\sigma_{\mathfrak{P}}(\mathfrak{s})$ and $\sigma^+_{\mathfrak{P}}(\mathfrak{s})$ of $\mathbf{K}$ for every simple Bernstein component $\mathfrak{s}$ of $\mathrm{GL}_n(F)$. By remark 7.0.6, these are *typical* representations: each of them determines the Bernstein component of an irreducible representations of $\mathrm{GL}_n(F)$ that contains it. We do not claim that these are the only typical representations of $\mathbf{K}$, although some variant of this statement (perhaps assuming $p > n$) is expected to hold, as in [EG14, Conjecture 4.1.3]. This is closely related to the problem of "uniqueness of types", for which see [Paš05] and [BM02] appendix.

K-types for $D^\times$. The group $D^\times$ has a unique maximal compact subgroup $\mathcal{O}_D^\times$. Let $(J_\theta, \lambda = \kappa_\theta \otimes \chi)$ be a maximal simple type in $D^\times$. Then $J_\theta \subseteq \mathcal{O}_D^\times$. Fix a simple stratum $[\mathcal{O}_D^\times, \beta]$ for θ and a uniformizer $\pi_{D'}$ of the central division algebra $D' = Z_D(F[\beta])$ over $F[\beta]$. Then the normalizer $\mathbf{J}(\theta)$ of θ in $D^\times$ is $\pi_{D'}^{\mathbf{Z}} \ltimes J_\theta = (D')^\times J^1_\theta$, and the normalizer $\mathbf{J}(\lambda)$ of λ in $D^\times$ has index in $\mathbf{J}(\theta)$ equal to the size $b(\chi)$ of the orbit of χ under $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$ (see remark 4.5.9).

By [BH11, Proposition 2.6.1], the $D^\times$-intertwining set of (J_θ, λ) coincides with $\mathbf{J}(\lambda)$, which intersects $\mathcal{O}_D^\times$ in J_θ. It follows that the intertwining set of λ in $\mathcal{O}_D^\times$ is J_θ and that $\mathrm{Ind}_{J_\theta}^{\mathcal{O}_D^\times} \lambda$ is irreducible. By Frobenius reciprocity, it is a type for the Bernstein component corresponding to (J_θ, λ). We will refer to these representations as **K**-*types* for $D^\times$.

Another construction of **K**-types in this context can be given through Clifford theory, as follows. (See [Roc09, Remark 1.6.1.3, Lemma 1.6.3.1] for details.) A smooth irreducible representation π of D restricts to a semisimple representation of $\mathcal{O}_D^\times$, whose irreducible constituents form a unique orbit under conjugation by a uniformizer Π_D of $D^\times$. Each constituent occurs with multiplicity one. If τ is another smooth irreducible representation, it follows that $\mathrm{Hom}_{\mathcal{O}_D^\times}(\pi, \tau)$ is nonzero if and only if $\pi|_{\mathcal{O}_{D^\times}}$ and $\tau|_{\mathcal{O}_{D^\times}}$ are isomorphic, and this is equivalent to π and τ being unramified twists of each other. It follows that any irreducible constituent of $\pi|_{\mathcal{O}_{D^\times}}$ is a **K**-type for the inertial class of π. This is the construction used in [GG15].

In contrast with the case of $\mathrm{GL}_n(F)$ (see [Paš05]), the **K**-type of a supercuspidal representation needs not be unique: the number of constituents of $\pi_{\mathcal{O}_{D^\times}}$ equals the *torsion number* of π, namely, the number of unramified characters χ of $D^\times$ such that $\chi\pi \cong \pi$.

Remark 7.0.9. This numerical coincidence can also be explained as follows. The maximal simple types giving rise to the same Bernstein component as $(J_\theta, \kappa_\theta \otimes \chi)$ are all conjugate in $D^\times$, and the normalizer $\mathbf{J}(\lambda)$ has index in $\pi_{D'}^{\mathbf{Z}} \ltimes J_\theta$ equal to the size $b(\chi)$ of the orbit of χ. Since the valuation of $\pi_{D'}$ in $D^\times$ is $[\mathbf{e} : \mathbf{f}]$, we see that there are $b(\chi)[\mathbf{e} : \mathbf{f}]$ orbits of $\mathcal{O}_D^\times$ on the set of $D^\times$-conjugates of $(J_\theta, \kappa_\theta \otimes \chi)$ (the same as the cardinality of $D^\times/\mathcal{O}_D^\times \mathbf{J}(\lambda)$). On the other hand, a computation as in [Séc09, Proposition 4.1] implies that the torsion number of the type equals $b(\chi)[\mathbf{e} : \mathbf{f}]$.

Locally algebraic types. For applications for the Breuil–Mézard conjecture, we will need the following class of locally algebraic representations of maximal compact subgroups.

Definition 7.0.10. Let τ be a discrete series inertial type for F, defining simple inertial classes $\mathfrak{s} \in \mathfrak{B}(\mathrm{GL}_n(F))$ and $\mathfrak{s}_D \in \mathfrak{B}(D^\times)$. Fix a choice of a $\mathcal{O}_D^\times$-type $\sigma(\mathfrak{s}_D)$ for $\mathfrak{s}_D$. (Our results will be independent of this choice.) Let $E/\mathbf{Q}_p$ be a finite extension such that τ, $\sigma_{\mathfrak{P}}(\mathfrak{s})$ and $\sigma(\mathfrak{s}_D)$ are defined over E, for all partitions $\mathfrak{P}$. Finally, let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}_{\mathbf{Q}_p}(F,E)}$ be a weight.

Then we define

$$\sigma_{\mathfrak{P}}(\tau, \lambda) = \sigma_{\mathfrak{P}}(\mathfrak{s}) \otimes_E L_\lambda,\ \sigma_{\mathfrak{P}}^+(\tau, \lambda) = \sigma_{\mathfrak{P}}^+(\mathfrak{s}) \otimes_E L_\lambda,\ \sigma_D(\tau, \lambda) = \sigma(\mathfrak{s}_D) \otimes_E L_\lambda^+.$$

Trace formulas for $\mathbf{K}$-types. A conjugacy class in a profinite group is *pro-p-regular* if its elements are p-regular in all finite quotients of G (that is, its order is coprime to p). We have the following lemma.

Lemma 7.0.11. If G is a profinite group, H is a finite group, and $\pi : G \to H$ is a continuous surjection with pro-p kernel, then π induces a bijection from the pro-p-regular classes of G to the p-regular classes of H.

Proof. Since a filtered inverse limit of nonempty finite sets is nonempty, it suffices to prove the claim for G a finite group. In this case, the surjectivity of π on p-regular classes follows because every p-regular element $\pi(x)$ of H admits a p-regular lift, since if $x = x_{(p)}x^{(p)}$ then the images of these under π commute, hence $\pi(x_{(p)}) = \pi(x)_{(p)} = 1$ and $\pi(x^{(p)}) = \pi(x)$. Then the claim follows because G and H have the same number of p-regular classes, since every irreducible $\overline{\mathbf{F}}_p$-representation of G is trivial on $\ker(\pi)$, and the number of irreducible $\overline{\mathbf{F}}_p$-representations of a finite group equals the number of its p-regular conjugacy classes [Ser77, Section 18.2, Corollary 3]. □

Consider a pair $(J_\theta, \lambda = \kappa_\theta \otimes \sigma[\chi])$ in $G = \mathrm{GL}_n(F)$ for a maximal simple character θ of endo-class Θ_F and a character $\chi : \mathbf{e}^\times_{n/\delta(\Theta_F)} \to R^\times$, and let $[\mathfrak{A}, \beta]$ be a simple stratum for θ. As in the above, we assume that $J_\theta \subseteq \mathbf{K}$. Fix a maximal unramified extension K^+ of $F[\beta]$ in $Z_A(F[\beta])$ normalizing $\mathfrak{A}$, which exists by the argument in the proof of proposition 2.4.5. Let K be the maximal unramified extension of F in K^+. We remark that the unit group $K^\times$ normalizes $\mathbf{K}$: this is because $\mathcal{O}_K \subseteq \mathcal{O}_{K^+} \subseteq \mathfrak{B}$, and $K^\times = p^{\mathbf{Z}} \times \mathcal{O}_K^\times$, and p is central.

By theorem 2.5.13 we have

$$\sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s}_G(\Theta_F, \Theta_E, [\chi])) \cong \mathrm{Ind}_{J_\theta}^{\mathbf{K}}(\lambda),$$

which is a virtual representation of $\mathbf{K}$ if χ is not $\mathbf{e}$-regular. When χ is $\mathbf{e}$-regular, this is a maximal simple type. We shorten notation to $\sigma^+ = \sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s}_G(\Theta_F, \Theta_E, [\chi]))$.

Proposition 7.0.12. If $x \in \mathbf{K}$ is a pro-p-regular element that is not $\mathbf{K}$-conjugate to an element of μ_K then $\mathrm{tr}\sigma^+(x) = 0$.

Proof. By the Frobenius formula for an induced character we have

$$\mathrm{tr}\sigma^+(x) = \sum_{y \in J_\theta \backslash \mathbf{K}} \mathrm{tr}\lambda(yxy^{-1}).$$

By lemma 7.0.11, the pro-p-regular conjugacy classes of J_θ are in bijection with the semisimple conjugacy classes of $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$, via our isomorphism $J_\theta / J_\theta^1 \to \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$. Now the claim follows by proposition 2.5.9, as $\sigma[\chi]$ vanishes on semisimple conjugacy classes that are not represented in a maximal elliptic torus (equivalently, their characteristic polynomial has more than one irreducible factor up to multiplicity), and $\mu_K = \mu_{K^+}$ maps isomorphically to such a torus. □

We now give a formula for $\mathrm{tr}\sigma^+(x)$ when $x \in \mu_K$ generates an unramified extension L/F. For this, we take the interior lift θ_L, and notice the decomposition

$$J_{\theta_L} = \mathrm{GL}_{n/[L[\beta]:F]}(\mathcal{O}_{L[\beta]}) J^1_{\theta_L}.$$

We write $G_L = Z_G(L)$ and notice the equality $Z_{\mathbf{K}}(L) = \mathbf{K} \cap G_L$. Since $K^\times$ normalizes $\mathbf{K}$, this is a maximal compact subgroup of G_L that we denote $\mathbf{K}_L$.

Recalling the representation $\widetilde{\eta}(\theta)$ of $\mu_K \ltimes J^1_\theta$ from our discussion of symplectic invariants, we see that it is isomorphic to the restriction of $\epsilon_{\mathrm{Gal}} \epsilon^1_\theta \kappa_\theta$ to $\mu_K \ltimes J^1_\theta$, since this is the p-primary β-extension of θ. Also, inflating η^{μ_L} through $J_{\theta_L} \to Q_{\theta_L} \to Q_\theta^{\mu_L}$ (the first map is the canonical surjection) yields $\eta(\theta_L)$ by the uniqueness property of Heisenberg representations. It follows from this and proposition 2.4.22 that if $x \in \mu_K$ generates L/F, then

$$\mathrm{tr}\kappa_\theta(x) = \epsilon_{\mathrm{Gal}}(x) \epsilon^1_\theta(x) \epsilon(x, V_\theta) \dim \eta(\theta_L) = \epsilon_{\mathrm{Gal}}(x) \epsilon^0_{\theta_L} \epsilon^0_\theta \dim \eta(\theta_L) \qquad (7.0.1)$$

Now consider the pair $(J_{\theta_L}, \lambda_L = \kappa_{\theta_L}\sigma[\chi])$ where the Deligne–Lusztig induction is taken from $\mathbf{e}^\times_{n/\delta(\Theta_F)}$ to the centralizer of the image of x in $\mathrm{GL}_{n/[F[\beta]:F]}(\mathbf{e})$, which is the group $\mathrm{GL}_{n/[L[\beta]:F]}(\mathbf{e}[x])$. When χ is $\mathbf{e}[x]$-regular, this is a maximal simple type in G_L. The corresponding $\mathbf{K}_L$-type $\sigma_L^+ = \mathrm{Ind}_{J_{\theta_L}}^{\mathbf{K}_L} \lambda_L$ has dimension equal to

$$\dim(\sigma_L^+) = \dim \eta(\theta_L)|J_{\theta_L}\backslash \mathbf{K}_L|(\mathrm{GL}_{n/[L[\beta]:F]}(\mathbf{e}[x]) : \mathbf{e}^\times_{n/\delta(\Theta_F)})_{p'}. \tag{7.0.2}$$

Remark 7.0.13. The dimension of a virtual representation is the value of its character at the identity. In this case, it is independent of χ.

Proposition 7.0.14. Let $x \in \mu_K$ generate an unramified extension L/F. Then

$$\mathrm{tr}\sigma^+(x) = (-1)^{n/[F[\beta]:F]+n/[L[\beta]:F]}\epsilon_{\mathrm{Gal}}(x)\epsilon^0_{\theta_L}\epsilon^0_\theta \dim(\sigma_L^+) \sum_{\gamma\in\mathrm{Gal}(L/F)} \chi(\gamma x).$$

Remark 7.0.15. As in theorem 5.1.1, we are evaluating χ at x through any isomorphism $\mu_K \to \mathbf{e}^\times_{n/\delta(\Theta_F)}$ that is induced by a $\mathbf{e}$-linear isomorphism $\mathbf{k} \to \mathbf{e}_{n/\delta(\Theta_F)}$, where $\mathbf{k}$ is an $\mathbf{e}$-algebra through $\iota(\Theta_E)$.

Proof. We begin with the Frobenius formula

$$\mathrm{tr}\sigma^+(x) = \sum_{J_\theta\backslash\mathbf{K}} \mathrm{tr}\lambda(yxy^{-1}) = \sum_{J_\theta\backslash\mathbf{K}} \sigma[\chi](yxy^{-1})\mathrm{tr}\kappa_\theta(yxy^{-1})$$

and the remark that if $y \in \mathbf{K}$ and $\mathrm{tr}\lambda(yxy^{-1}) \neq 0$ then there exists an element of J_θ conjugating yxy^{-1} to an element of μ_K. Indeed, by lemma 7.0.11 the pro-p-regular classes of J_θ are in bijection with those of $\mathrm{GL}_{n/[F[\beta]:F]}(\mathbf{e})$, and by proposition 7.0.12 the only ones on which $\mathrm{tr}\sigma[\chi]$ is nonzero are those represented in μ_K. It follows that $J_\theta y = J_\theta \widetilde{y}$ for some $\widetilde{y} \in N_{\mathbf{K}}(L)$ (this is the normalizer of L for the conjugation action of $\mathbf{K}$). We have an isomorphism $N_{\mathbf{K}}(L)/\mathbf{K}_L \to \mathrm{Gal}(L/F)$, and the intersection $N_{\mathbf{K}}(L) \cap J_\theta$ maps onto $\mathrm{Gal}(L/L \cap F[\beta])$. To see this, notice that if $x \in J_\theta = \mathfrak{B}^\times_{\max} J^1_\theta$ normalizes μ_L then its image $\overline{x}$ in $\mathrm{GL}_{n/[F[\beta]:F]}(\mathbf{e})$ normalizes the image of μ_L, and the automorphism of μ_L induced by $\mathrm{ad}\,\overline{x}$ determines that induced by $\mathrm{ad}\,x$, which therefore is also induced by an element of $\mathfrak{B}^\times_{\max} \cong \mathrm{GL}_{n/[F[\beta]:F]}(\mathcal{O}_{F[\beta]})$.

It follows that the space $J_\theta N_{\mathbf{K}}(L)$ decomposes into double cosets

$$J_\theta N_{\mathbf{K}}(L) = \bigcup_{\sigma\in\mathrm{Gal}(L/F)} J_\theta t_\sigma \mathbf{K}_L$$

where $t_\sigma \in \mathbf{K}$ induces σ on L by conjugation, and $J_\theta t_\sigma \mathbf{K}_L = J_\theta t_\tau \mathbf{K}_L$ if and only if $\tau\sigma^{-1} \in \mathrm{Gal}(L/L \cap F[\beta])$. Since $J_\theta t_\sigma \mathbf{K}_L = J_\theta \mathbf{K}_L t_\sigma$, we deduce that

$$\mathrm{tr}\sigma^+(x) = [L : F[\beta] \cap L]^{-1} \sum_{\gamma\in\mathrm{Gal}(L/F)} |J_{\theta_L}\backslash\mathbf{K}_L|\mathrm{tr}\lambda(\gamma x).$$

Recalling formula (7.0.1), this implies that

$$\mathrm{tr}\sigma^+(x) = [L : F[\beta] \cap L]^{-1} \epsilon_{\mathrm{Gal}}(x) \epsilon^0_{\theta_L} \epsilon^0_\theta \dim \eta(\theta_L) \sum_{\gamma \in \mathrm{Gal}(L/F)} |J_{\theta_L} \backslash \mathbf{K}_L| \sigma[\gamma x].$$

By proposition 2.5.9, we have

$$\sigma[\chi] = (-1)^{n/[F[\beta]:F] + n/[L[\beta]:F]} (\mathrm{GL}_{n/[L[\beta]:F]}(\mathbf{e}[x])) : \mathbf{e}^\times_{n/\delta(\Theta_F)})_{p'} \sum_{\alpha \in \mathrm{Gal}(\mathbf{e}[x]/\mathbf{e})} \chi(\alpha x).$$

Recall that $\mathbf{e}$ is isomorphic to the residue field of $F[\beta]$, and since $L[\beta]/F[\beta]$ is an unramified extension generated by x, $\mathbf{e}[x]$ is isomorphic to the residue field of $L[\beta]$. The restriction map is an isomorphism

$$\mathrm{res} : \mathrm{Gal}(L[\beta]/F[\beta]) \to \mathrm{Gal}(L/F[\beta] \cap L)$$

and it follows that

$$[L : F[\beta] \cap L]^{-1} \sum_{\gamma \in \mathrm{Gal}(L/F)} \sum_{\alpha \in \mathrm{Gal}(\mathbf{e}[x]/\mathbf{e})} \chi(\alpha \gamma x) = \sum_{\gamma \in \mathrm{Gal}(L/F)} \chi(\gamma x).$$

The claim now follows from (7.0.2). □

We end this section by proving an analogous result for $D^\times$. Let $(J_\theta, \lambda = \kappa_\theta \otimes \chi)$ be a maximal simple type in $D^\times$ and let $\sigma^+_D = \mathrm{Ind}^{\mathcal{O}^\times_D}_{J_\theta} \lambda$ be the associated $\mathbf{K}$-type. Fix a simple stratum $[\mathcal{O}^\times_D, \beta]$ defining θ and fix a maximal unramified extension K^+ of $F[\beta]$ in $D' = Z_D(F[\beta])$. Let $x \in \mu_{K^+}$ generate an extension L/F. Let θ_L be the interior lift of θ to L, and write λ_L for any maximal simple type in $Z_{D^\times}(L)$ with maximal simple character θ_L. Write $\sigma^+_L = \mathrm{Ind}^{Z_{\mathcal{O}^\times_D}(L)}_{J_{\theta_L}} \lambda_L$ for the corresponding $\mathbf{K}_L$-type.

Proposition 7.0.16. We have an equality

$$\mathrm{tr}\sigma^+_D(x) = \epsilon_{\mathrm{Gal}}(x) \epsilon^0_\theta \epsilon^0_{\theta_L} \dim(\sigma^+_L) \chi(x).$$

Remark 7.0.17. Here we use the same conventions as in remark 7.0.15.

Proof. If $y \in \mathcal{O}^\times_D$ and $yxy^{-1} \in J_\theta = \mathcal{O}^\times_{D'} J^1_\theta$, then yxy^{-1} is J_θ-conjugate to an element of μ_{K^+}, because μ_{K^+} represents the pro-p-regular conjugacy classes in J_θ by lemma 7.0.11. Again by lemma 7.0.11, elements of μ_{K^+} are pairwise nonconjugate in $\mathcal{O}^\times_D$. So $J_\theta y = J_\theta \widetilde{y}$ for some $\widetilde{y} \in Z_{\mathcal{O}^\times_D}(L)$, and we deduce that

$$\mathrm{tr}\sigma^+_D(x) = |J_{\theta_L} \backslash Z_{\mathcal{O}^\times_D}(L)| \mathrm{tr}\lambda(x).$$

By the discussion of symplectic invariants, we know that

$$\epsilon_{\mathrm{Gal}}(x)\epsilon_\theta^1(x)\mathrm{tr}\kappa_\theta(x) = \epsilon_\theta(x)\dim\eta(\theta_L) = \epsilon_\theta(x)\dim(\lambda_L),$$

hence the claim follows from proposition 2.4.22. □

The formal degree formula. Let $\mathfrak{s}$ be a supercuspidal inertial class for $\mathrm{GL}_n(F)$, and let $\mathfrak{s}_D = \mathrm{JL}^{-1}(\mathfrak{s})$ be its Jacquet–Langlands transfer to $D^\times$. We give a relation between the dimension of a $\mathbf{K}$-type σ_D^+ for $\mathfrak{s}_D$ and the dimension of a $\mathbf{K}$-type σ^+ for $\mathfrak{s}$. We assume that σ_D^+ and σ^+ have been constructed as in the above. Write $q = |\mathbf{f}|$ and $t(\mathfrak{s}) = t(\mathfrak{s}_D)$ for the torsion numbers of the inertial classes. Normalize the formal degrees for $\mathrm{GL}_n(F)$ so that the Steinberg representation has formal degree one, and let $\mathrm{Iw} \subseteq \mathbf{K}$ be an Iwahori subgroup.

Theorem 7.0.18. [BH04, 1.4.1] The formal degree of any irreducible representation containing a maximal simple type (J_θ, λ) corresponding to $\mathfrak{s}$ is

$$d(\pi) = t(\mathfrak{s})\dim(\lambda)\frac{q^n-1}{(q-1)^n}\frac{\mu_G(\mathrm{Iw})}{\mu_G(J_\theta)}$$

for any Haar measure μ_G on G.

Proposition 7.0.19. We have an equality $\dim(\sigma^+) = (\mathrm{GL}_n(\mathbf{f}) : \mathbf{f}_n^\times)_{p'} \dim(\sigma_D^+)$.

Proof. Multiplying numerator and denominator of the equation in theorem 7.0.18 by $\mu_G(\mathbf{K})^{-1}$ yields

$$d(\pi) = t(\mathfrak{s})\dim(\sigma^+)(\mathrm{GL}_n(\mathbf{f}) : \mathbf{f}_n^\times)_{p'}^{-1}$$

because $\dim(\sigma^+) = \dim(\lambda)(\mathbf{K} : J_\theta)$ and $(\mathbf{K} : \mathrm{Iw}) = \frac{(q^n-1)\cdots(q^n-(q^{n-1}))}{q^{n(n-1)/2}(q-1)^n}$. We have seen that any irreducible representation in $\mathfrak{s}_D$ restricts to $\mathcal{O}_D^\times$ to a sum of $t(\mathfrak{s}_D)$ representations each appearing with multiplicity one and all conjugate under $D^\times$, which are precisely the $\mathbf{K}$-types for $\mathfrak{s}_D$. Since $d(\pi) = \dim(\mathrm{JL}^{-1}\pi)$, we deduce that

$$t(\mathfrak{s}_D)\dim(\sigma_D^+) = t(\mathfrak{s})\dim(\sigma^+)(\mathrm{GL}_n(\mathbf{f}) : \mathbf{f}_n^\times)_{p'}^{-1}$$

and the claim follows since $t(\mathfrak{s}_D) = t(\mathfrak{s})$, as the Jacquet–Langlands correspondence commutes with character twists. □

8

Jacquet–Langlands transfer of weights and types.

In this chapter we construct a Jacquet–Langlands transfer of Serre weights from $D^\times$ to $\mathrm{GL}_n(F)$, and prove its compatibility with the inertial Jacquet–Langlands correspondence. We also consider analogues for l-adic coefficients when $l \neq p$, so we begin by fixing a prime number l (allowing, of course, the case $l = p$). We will mostly be interested in proving that our transfer preserves congruences of locally algebraic types, in the sense of theorem 8.0.4. However, we point out that in the case of trivial regular weight we can interpret our result as describing a Jacquet–Langlands correspondence for representations of maximal compact subgroups of $D^\times$ and $\mathrm{GL}_n(F)$. This is because of the following lemma.

Lemma 8.0.1. Let R be an algebraically closed field of any characteristic (including char $R = p$) and let τ be an irreducible smooth R-linear representation of $\mathcal{O}_D^\times$. Then τ occurs in the restriction to $\mathcal{O}_D^\times$ of an irreducible smooth representation of $D^\times$.

Proof. We regard τ as a representation of $F^\times\mathcal{O}_D^\times$ with π_F acting trivially. As in [Vig01b, Section 4], τ extends to a representation τ' of its normalizer $N = N_{D^\times}(\tau)$, and the induction $\mathrm{Ind}_N^{D^\times}(\tau')$ is an irreducible representation of $D^\times$ containing τ. □

Choosing an isomorphism $\iota_l : \overline{\mathbf{Q}}_l \to \mathbf{C}$, one gets a Jacquet–Langlands transfer from inertial classes of $\overline{\mathbf{Q}}_l$-representations of $D^\times$ to inertial classes of $\overline{\mathbf{Q}}_l$-representations of $\mathrm{GL}_n(F)$. Because the Harish–Chandra character is compatible with automorphisms of the coefficient field, this transfer is independent of the choice of ι_l [MS17, 10.1].

Definition 8.0.2. Define a map $\mathrm{JL}_{\mathbf{K}} : R_{\overline{\mathbf{Q}}_l}(\mathcal{O}_D^\times) \to R_{\overline{\mathbf{Q}}_l}(\mathrm{GL}_n(\mathcal{O}_F))$ as follows. Let σ_D be an irreducible representation of $\mathcal{O}_D^\times$. Then σ_D is a type for some Bernstein component $\mathfrak{s}_D$ of $D^\times$, by lemma 8.0.1, and we let $\mathfrak{s} = \mathrm{JL}(\mathfrak{s}_D)$. We define $\mathrm{JL}_{\mathbf{K}}(\sigma_D) = \sigma^+_{\mathfrak{P}_{\min}}(\mathfrak{s})$.

Mod p reduction. Set $l = p$. We construct a map

$$\mathrm{JL}_p : R_{\overline{\mathbf{F}}_p}(\mathcal{O}_D^\times) \to R_{\overline{\mathbf{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F))$$

and prove our first main result of this chapter, the equality

$$\mathrm{JL}_p(\overline{\sigma}_D(\tau, \lambda)) = \overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau, \lambda).$$

Since every irreducible smooth $\overline{\mathbf{F}}_p$-representation of a pro-p group is trivial, it is enough to define a map

$$\mathrm{JL}_p : R_{\overline{\mathbf{F}}_p}(\mathbf{d}^\times) \to R_{\overline{\mathbf{F}}_p}(\mathrm{GL}_n(\mathbf{f})).$$

We choose any $\mathbf{f}$-linear isomorphism $\iota : \mathbf{d} \to \mathbf{f}_n$ and we define this to be the semisimplified mod p reduction of $\chi \mapsto \sigma[\chi]$, composed with the isomorphism $R_{\overline{\mathbf{F}}_p}(\mathbf{f}_n^\times) \to R_{\overline{\mathbf{Q}}_p}(\mathbf{f}_n^\times)$. (Compare remark 2.5.8.) Since $\chi \mapsto \sigma[\chi]$ is constant on $\mathrm{Gal}(\mathbf{f}_n/\mathbf{f})$-orbits, this is independent of the choice of ι. Recall the explicit formula 2.5.9, and observe that this is a direct generalization of the construction in [GG15, Section 2].

For a general profinite group G, one defines the Brauer character of a finite-dimensional representation V of G over a finite field $\mathbf{F}_q$ as in the finite group case, obtaining a function $\chi(V)$ on the set of pro-p-regular conjugacy classes of G valued in $\overline{\mathbf{Q}}_p$. From lemma 7.0.11, and the corresponding assertion for finite groups, we find that whenever G has an open normal pro-p subgroup the Brauer character induces an isomorphism $R_{\overline{\mathbf{F}}_p}(G) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p \to \mathcal{C}^{(p)}(G, \overline{\mathbf{Q}}_p)$, where $R_{\overline{\mathbf{F}}_p}(G)$ is the Grothendieck group of finite length smooth representations of G over $\overline{\mathbf{F}}_p$, and the target denotes the space of functions from the set of pro-p-regular classes of G to $\overline{\mathbf{Q}}_p$. We get an induced map

$$\mathrm{JL}_p : \mathcal{C}^{(p)}(\mathbf{f}_n^\times, \overline{\mathbf{Q}}_p) \to \mathcal{C}^{(p)}(\mathrm{GL}_n(\mathbf{f}), \overline{\mathbf{Q}}_p)$$

such that if $x \in \mathrm{GL}_n(\mathbf{f})$ has a conjugate in $\mathbf{f}_n^\times$ with degree $\deg(x)$ over $\mathbf{f}$ then

$$\mathrm{JL}_p(f)(x) = (-1)^{n+n/\deg(x)}(\mathrm{GL}_{n/\deg(x)}(\mathbf{f}_{\deg(x)}) : \mathbf{f}_n^\times)_{p'} \sum_{\gamma \in \mathrm{Gal}(\mathbf{f}_{\deg(x)}/\mathbf{f})} f(\gamma x). \qquad (8.0.1)$$

Remark 8.0.3. Since JL_p is defined by an explicit formula, we need not worry about subtleties in the definition of Deligne–Lusztig induction over $\overline{\mathbf{Q}}_p$. See remark 2.5.8.

Theorem 8.0.4. Let τ be a discrete series inertial type for I_F and $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}_{\mathbf{Q}_p}(F, \overline{\mathbf{Q}}_p)}$.

Then we have the equality

$$\mathrm{JL}_p(\overline{\sigma}_D(\tau,\lambda)) = \overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau,\lambda)$$

Proof. We have an equality of Brauer characters

$$\chi(\overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau,\lambda)) = \chi(\overline{L}_\lambda)\chi(\overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau)),$$

and similarly

$$\chi(\overline{\sigma}_D(\tau,\lambda)) = \chi(\overline{L}^+_\lambda)\chi(\overline{\sigma}_D(\tau)).$$

The representation $\sigma^+_{\mathfrak{P}_{\min}}(\tau)$ is smooth and defined over a finite extension $E/\mathbf{Q}_p$, so we can compute $\chi(\overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau))$ as the restriction of the trace of $\sigma^+_{\mathfrak{P}_{\min}}(\tau)$ to p-regular conjugacy classes: this follows from the corresponding statement in the finite group case, via lemma 7.0.11. By proposition 7.0.12, both $\chi(\overline{\sigma}_D(\tau))$ and $\chi(\overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau))$ vanish away from certain conjugacy classes represented by roots of unity. If z and z_D are matching p-regular roots of unity, then lemma 2.5.21 actually implies that $\chi(\overline{L}_\lambda)(z) = \chi(\overline{L}^+_\lambda)(z_D)$, because the Brauer character of a representation of the finite groups generated by z and z_D can be computed on a lift to characteristic zero. Hence it is enough to prove that $\mathrm{JL}_p\left(\chi(\overline{\sigma}_D(\tau))\right) = \chi(\overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau))$.

Fix a pair $(J_\theta, \kappa_\theta \otimes \sigma[\chi])$ in $\mathrm{GL}_n(F)$, a simple stratum $[\mathfrak{A}, \beta]$ for θ, and a maximal unramified extension $K^+/F[\beta]$ in $Z_A(F[\beta])$ such that $J_\theta \subseteq \mathbf{K}$ and the maximal unramified extension K of F in K^+ normalizes the group $\mathbf{K}$. By proposition 7.0.8, $\sigma^+_{\mathfrak{P}_{\min}}(\tau) \cong \mathrm{Ind}_{J_\theta}^{\mathbf{K}}(\kappa_\theta \otimes \sigma[\chi])$. where we have fixed a lift $\Theta_E \to \Theta_F$ of $\Theta_F = \mathrm{cl}(\theta)$ in order to get the character orbit $[\chi]$. By theorems A and B, we have the equality $\mathfrak{s}_D(\tau) = \mathrm{JL}^{-1}(\mathfrak{s}(\tau)) = \mathfrak{s}_{D^\times}(\Theta_F, \Theta_E, [\chi])$.

It follows that we can find a maximal simple character θ_D in $D^\times$ with $\mathrm{cl}(\theta) = \mathrm{cl}(\theta_D)$ and a simple stratum $[\mathcal{O}_D, \beta_D]$ for θ_D such that $\sigma_D(\tau)$ is isomorphic to the induction $\mathrm{Ind}_{J_{\theta_D}}^{\mathcal{O}_D^\times}(\kappa_\theta \otimes \chi_D)$, for a character $\chi_D : \mathbf{e}^\times_{n/\delta(\Theta_F)} \to \overline{\mathbf{Q}}_p^\times$ such that the $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$-orbits $[\chi]$ and $[\chi_D]$ coincide. We fix a maximal unramified extension $K_D^+/F[\beta_D]$ in $Z_D(F[\beta_D])$ and write K_D for the maximal unramified extension of F in K_D^+. Since the Jacquet–Langlands correspondence preserves torsion numbers, we have $[K : F] = [K_D : F]$, and there exists a unique isomorphism $\iota : K_D \to K$ such that the equality of endo-classes $\mathrm{cl}(\theta_{D,K}) = \iota^* \mathrm{cl}(\theta_K)$ holds.

Let $z \in \mu_K$ and $z_D \in \mu_{K_D}$ generate isomorphic extensions of F, which we identify via ι with an unramified extension L/F. By propositions 7.0.14 and 7.0.16 we have equalities

$$\mathrm{tr}\sigma^+_{\mathfrak{P}_{\min}}(\tau)(z) = (-1)^{n/[F[\beta]:F]+n/[L[\beta]:F]}\epsilon_{\mathrm{Gal}}(z)\epsilon^0_{\theta_L}\epsilon^0_\theta \dim(\sigma^+_L) \sum_{\gamma \in \mathrm{Gal}(L/F)} \chi(\gamma z) \tag{8.0.2}$$

and

$$\mathrm{tr}\sigma_D(\tau)(z_D) = \epsilon_{\mathrm{Gal}}(z_D)\epsilon^0_{\theta_{D,L}}\epsilon^0_{\theta_D}\dim(\sigma^+_{D,L})\chi_D(x). \tag{8.0.3}$$

These compute the Brauer characters of the mod p reductions $\overline{\sigma}^+_{\mathfrak{P}_{\min}}(\tau)$ and $\overline{\sigma}_D(\tau)$ at z and z_D. It follows that

$$\mathrm{JL}_p\left(\overline{\sigma}_D(\tau)(z)\right) = (-1)^{n+n/[L:F]}(\mathrm{GL}_{n_L}(\mathbf{f}_{[L:F]}) : \mathbf{f}_n^\times)_{p'}\epsilon_{\mathrm{Gal}}(z_D)\epsilon^0_{\theta_{D,L}}\epsilon^0_{\theta_D}\dim(\sigma^+_{D,L})\sum_{\gamma\in\mathrm{Gal}(L/F)}\chi_D(\gamma z_D)$$

by formula (8.0.1), and we have to compare this to (8.0.2).

The two sums are equal because $[\chi] = [\chi_D]$. Now recall from remark 7.0.13 that $\dim(\sigma_L^+)$ and $\dim(\sigma^+_{D,L})$ are equal to the dimensions of the $\mathbf{K}$-types corresponding to an arbitrary choice of *maximal* simple types with maximal simple character θ_L, respectively $\theta_{D,L}$. By our choice of $\iota : F[z_D] \to F[z]$, these characters have the same endo-class, hence we can choose σ_L and $\sigma_{D,L}$ so that they determine inertial classes corresponding to each other under the Jacquet–Langlands correspondence between $Z_{D^\times}(F[z_D])$ and $Z_{\mathrm{GL}_n(F)}(F[z])$ (identified with groups over L via ι). By proposition 7.0.19, we see that

$$\dim(\sigma^+_{D,L})(\mathrm{GL}_{n_L}(\mathbf{f}_{[L:F]}) : \mathbf{f}_n^\times)_{p'} = \dim(\sigma_L^+)$$

since $\mathbf{f}_{[L:F]}$ is isomorphic to the residue field of L and $\mathbf{f}_n^\times \cong ((\mathbf{f}_{[L:F]})_{n_L})^\times$. The characters ϵ_{Gal} coincide as they only depend on the endo-classes $\mathrm{cl}(\theta) = \mathrm{cl}(\theta_D)$. Finally, the computations at the end of the proof of 5.3.3 show that*

$$(-1)^{n+n/[K:F]+n/[F[\beta]:F]}\epsilon^0_\theta = -\epsilon^0_{\theta_D}$$

and

$$(-1)^{n/[L:F]+n/[K:F]+n/[L[\beta]:F]}\epsilon^0_{\theta_L} = -\epsilon^0_{\theta_{D,L}}.$$

□

We remark that when the weight $\lambda = 0$, theorem 8.0.4 implies that the following diagram

$$\begin{array}{ccc} R_{\overline{\mathbf{Q}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\mathbf{K}}} & R_{\overline{\mathbf{Q}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \downarrow \mathbf{r}_p & & \downarrow \mathbf{r}_p \\ R_{\overline{\mathbf{F}}_p}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_p} & R_{\overline{\mathbf{F}}_p}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{8.0.4}$$

commutes.

*the integers there denoted m are all equal to one, since D is a division algebra.

Mod l reduction. Now assume $l \neq p$. By our discussion of $\mathbf{K}$-types for $D^\times$, lemma 8.0.1 implies that every irreducible smooth $\overline{\mathbf{Q}}_l$-representation τ of $\mathcal{O}_D^\times$ is a $\mathbf{K}$-type for a Bernstein component of $D^\times$. As such, there exists a simple character θ such that $\tau \cong \mathrm{Ind}_{J_\theta}^{\mathcal{O}_D^\times}(\kappa \otimes \chi)$ for some character χ of J_θ/J_θ^1, and the $\mathcal{O}_D^\times$-conjugacy class of the maximal simple type $(J_\theta, \kappa \otimes \chi)$ is uniquely determined by τ.

Lemma 8.0.5. Every irreducible $\overline{\mathbf{F}}_l$-representation of $\mathcal{O}_D^\times$ is the mod l reduction of a $\overline{\mathbf{Q}}_l$-representation of $\mathcal{O}_D^\times$. The mod l reduction of an irreducible $\overline{\mathbf{Q}}_l$-representation τ of $\mathcal{O}_D^\times$ is irreducible.

Proof. The first claim is a consequence of the Fong–Swan theorem, since $\mathcal{O}_D^\times$ is a solvable group. For the second claim, observe first that $\tau|_{1+\mathfrak{p}_D}$ is a direct sum with multiplicity one of representations forming a unique $\mathcal{O}_D^\times$-orbit. This follows from Clifford theory: by [Vig01b, Proposition 4.1] the group $\mathrm{Hom}_{1+\mathfrak{p}_D}(\sigma, \tau)$ is a simple module for the Hecke algebra $\mathcal{H}(\mathcal{O}_D^\times, \sigma)$, for every irreducible representation σ of $1 + \mathfrak{p}_D$. Since the quotient $\mathcal{O}_D^\times/1 + \mathfrak{p}_D$ is cyclic, by [Vig01b, Proposition 4.2] this Hecke algebra is commutative, hence its simple $\overline{\mathbf{Q}}_l$-modules are one-dimensional, proving the claim of multiplicity one.

Now, if τ^0 is any $\overline{\mathbf{Z}}_l$-lattice in τ then the reduction $\overline{\tau}^0|_{1+\mathfrak{p}_D}$ will again be a direct sum with multiplicity one of irreducible $\overline{\mathbf{F}}_l$-representations of $1 + \mathfrak{p}_D$, because $1 + \mathfrak{p}_D$ is a pro-p group, and $\mathcal{O}_D^\times$ will act transitively on the summands. Hence every irreducible $\mathcal{O}_D^\times$-subrepresentation of $\overline{\tau}^0$ has to coincide with $\overline{\tau}^0$. ☐

Theorem 8.0.6. There exists a unique map JL_l making the following diagram

$$\begin{array}{ccc} R_{\overline{\mathbf{Q}}_l}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_{\mathbf{K}}} & R_{\overline{\mathbf{Q}}_l}(\mathrm{GL}_n(\mathcal{O}_F)) \\ \downarrow{\mathbf{r}_l} & & \downarrow{\mathbf{r}_l} \\ R_{\overline{\mathbf{F}}_l}(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{JL}_l} & R_{\overline{\mathbf{F}}_l}(\mathrm{GL}_n(\mathcal{O}_F)) \end{array} \tag{8.0.5}$$

commute.

Proof. The mod l reduction map for $\overline{\mathbf{Q}}_l$-representations is defined as the direct limit of the reduction maps over finite extensions of $\mathbf{Q}_l$. That JL_l is unique follows from the first claim in lemma 8.0.5, since the left vertical arrow is surjective. For the existence, by lemma 8.0.5 it suffices to prove that if τ_1 and τ_2 are irreducible representations of $\mathcal{O}_D^\times$ with the same mod l reduction, then $\mathbf{r}_l(\mathrm{JL}_{\mathbf{K}}(\tau_1)) = \mathbf{r}_l(\mathrm{JL}_{\mathbf{K}}(\tau_2))$. Indeed, this allows us to define $\mathrm{JL}_l(\overline{\sigma})$ as $\mathbf{r}_l\,\mathrm{JL}_{\mathbf{K}}(\sigma)$ for any irreducible lift σ of $\overline{\sigma}$, and then commutativity of the diagram holds by definition and the second part of lemma 8.0.5.

Since $\mathbf{r}_l(\tau_1) = \mathbf{r}_l(\tau_2)$, we have $\tau_1 \cong \tau_2 \otimes \psi$ for some character $\psi : \mathcal{O}_D^\times/1+\mathfrak{p}_D \to \overline{\mathbf{Q}}_l^\times$, because the restrictions $\tau_i|_{1+\mathfrak{p}_D}$ are isomorphic modulo l, hence they are isomorphic over $\overline{\mathbf{Q}}_l$ as $1+\mathfrak{p}_D$ is a pro-p group. Hence there exists a simple character θ_D with endo-class Θ_F such that $\tau_i = \mathrm{Ind}_{J_{\theta_D}}^{\mathcal{O}_D^\times}(\kappa \otimes \chi_i)$ (where the χ_i are computed with respect to a lift $\Theta_E \to \Theta_F$). By assumption, the representations $\mathbf{r}_l(\kappa \otimes \chi_i)$ intertwine in $\mathcal{O}_D^\times$, as they have isomorphic inductions to $\mathcal{O}_D^\times$. Since κ is a β-extension, the intertwining set of κ in $D^\times$ coincides with that of θ_D, which is also equal to its normalizer $\pi_{D'}^{\mathbf{Z}} \ltimes J_{\theta_D}$ (where we have fixed a parameter field $F[\beta]$ for θ_D, and $D' = Z_D(F[\beta])$). Hence we see that $\mathbf{r}_l[\chi_1] = \mathbf{r}_l[\chi_2]$, where $[\chi_i]$ denotes the orbit under $\mathrm{Gal}(\mathbf{e}_{n/\delta(\Theta_F)}/\mathbf{e})$.

There exists a maximal simple character θ in $\mathrm{GL}_n(F)$ with the same endo-class as θ_D, together with a conjugacy class of isomorphisms $J_\theta/J_\theta^1 \to \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$ induced by $\Theta_E \to \Theta_F$. We assume that the subgroup J_θ is contained in $\mathbf{K}$, so that the virtual representation $\mathrm{JL}_{\mathbf{K}}(\tau_i)$ is the induction $\mathrm{Ind}_{J_\theta}^{\mathbf{K}}(\kappa_\theta \otimes \sigma[\chi_i])$.

To conclude, it suffices to prove that $\mathbf{r}_l\sigma[\chi_1] = \mathbf{r}_l\sigma[\chi_2]$, or equivalently that the l-Brauer characters of the $\sigma[\chi_i]$ coincide. These are the restrictions to l-regular classes in $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$ of the characters of the $\sigma[\chi_i]$. An element of $\mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$ is l-regular if and only if its semisimple part is l-regular, because the unipotent elements of this group have order a power of p. The character formula of Deligne and Lusztig [DL76, Theorem 4.2] expresses the value of $\sigma[\chi_i]$ at $g \in \mathrm{GL}_{n/\delta(\Theta_F)}(\mathbf{e})$ with Jordan decomposition $g = su$ in terms of a Green function evaluated at u (this is independent of χ_i) and the value of χ_i at those conjugates of s contained in the inducing torus. Since s is an l-regular element, and we have seen that $[\chi_1^{(l)}] = [\chi_2^{(l)}]$, the claim follows. $\square$

9

Breuil–Mézard conjectures.

9.1 Galois deformation theory.

Working in the framework of [EG14, Section 4], we recall the definition and some properties of potentially semistable deformation rings of fixed Hodge type and discrete series Galois type, and introduce the monodromy stratification on these rings. Then we state a form of the geometric Breuil–Mézard conjecture for the mod p fibers of these rings, and deduce a description of the cycle corresponding to discrete series lifts. In this section, we fix p-adic coefficients consisting of a finite extension $E/\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$, uniformizer π_E, and residue field $\mathbf{e}$. We let $\overline{\rho} : G_F \to \mathrm{GL}_n(\mathbf{e})$ be a continuous representation, and we assume that E is sufficiently large (so that, for instance, it contains all $[F : \mathbf{Q}_p]$ embeddings of F).

Let τ be a discrete series inertial type. Our results will relate $\sigma^{+}_{\mathfrak{P}_{\min}}(\tau)$ to the locus in the deformation space of $\overline{\rho}$ consisting of discrete series lifts of inertial type τ: that is, Galois representations lifting $\overline{\rho}$ whose associated Weil–Deligne representation is the Langlands parameter of an essentially square integrable representation in $\mathfrak{s} = \mathrm{rec}^{-1}(\tau)$. Making this precise requires an account of the monodromy operator on the universal deformation ring.

To start with, we recall some commutative algebra. Let A be a commutative ring with 1 and let M be a finite projective A module of rank n with a nilpotent endomorphism $N : A \to A$. To each prime ideal $x \in \mathrm{Spec}(A)$ we attach a partition $\mathfrak{P}_x$ of n by considering the Jordan canonical form of the nilpotent endomorphism $N(x)$ on $M \otimes_A k(x)$, where $k(x)$ is the residue field at x.

Lemma 9.1.1. Each partition $\mathfrak{P}$ of n defines a closed subset of $\mathrm{Spec}(A)$

$$\mathrm{Spec}(A)_{\geq \mathfrak{P}} = \{x \in \mathrm{Spec}(A) : \mathfrak{P}_x \geq \mathfrak{P}\}.$$

Proof. See for instance [Pyv, Section 4]. By our definition of $\mathfrak{P}_x \geq \mathfrak{P}$ as the reverse of the dominance partial order on partitions, we find that $\mathfrak{P}_x \geq \mathfrak{P}$ if and only if $\dim(\ker N(x)^i) \geq \dim(\ker N(\mathfrak{P})^i)$ for all i, where $N(\mathfrak{P})$ has Jordan canonical form given by $\mathfrak{P}$. Since $\dim(\ker N(x)^i) = \dim(\mathrm{coker} N(x)^i)$ and $\mathrm{coker} N(x)^i \cong (\mathrm{coker} N^i) \otimes_A k(x)$, the claim follows since the set

$$\{x \in \mathrm{Spec}(A) : \dim_{k(x)}((\mathrm{coker} N^i) \otimes_A k(x)) \geq m\}.$$

is closed for all $m \in \mathbf{Z}$. □

Remark 9.1.2. One gets a canonical subscheme structure on this set by the vanishing, on open subsets where M is free, of the $(n-m+1) \times (n-m+1)$ minors of the matrix of N with respect to a basis of M. We will not need this.

Remark 9.1.3. It follows that if $\mathrm{Spec}(A)$ is irreducible then the function $x \mapsto \mathfrak{P}_x$ is constant on a dense open subset of $\mathrm{Spec}(A)$, where it attains its minimal value. So we can define subsets $\mathrm{Spec}(A)_{\mathfrak{P}}$ as the union of irreducible components of $\mathrm{Spec}(A)$ where the minimal value of $\mathfrak{P}_x$ is $\mathfrak{P}$—equivalently, where the monodromy is generically $\mathfrak{P}$.

Potentially semistable deformation rings. Let $\tau : I_F \to \mathrm{GL}_n(E)$ be a discrete series inertial type and $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}_{\mathbf{Q}_p}(F,E)}$. To be more precise, we assume that τ is a multiple of a representation τ_0 of $E[I_F]$ that extends to an absolutely irreducible representation of $E[W_F]$. Let L/F be a finite Galois extension such that τ is trivial on I_L. By [Kis08, Theorem 2.7.6] there is a quotient $(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)$ of the generic fibre of the universal lifting $\mathcal{O}_E$-algebra $R^{\square}_{\overline{\rho}}$, whose points in a finite extension E'/E correspond to potentially semistable lifts of $\overline{\rho}$ with Hodge type λ and inertial type τ. By [Kis08, Theorem 2.5.5], there is a finite projective $L_0 \otimes_{\mathbf{Q}_p} (R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)$-module $D_{\overline{\rho}}(\tau, \lambda)[1/p]$ with an automorphism φ, semilinear with respect to $\sigma \otimes 1$, and a $L_0 \otimes_{\mathbf{Q}_p} (R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)$-linear nilpotent endomorphism N, specializing to $D^*_{\mathrm{st}}(r^{\mathrm{univ}}_x|_{G_L})$ for any $\mathcal{O}_E$-linear ring homomorphism $x : (R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda) \to E'$. Since $D_{\overline{\rho}}(\tau, \lambda)[1/p]$ is a direct factor of a free $L_0 \otimes_{\mathbf{Q}_p} (R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)$-module, it is also projective over $(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)$. By lemma 9.1.1 we have a stratification $\mathrm{Spec}(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)_{\geq \mathfrak{P}}$, and $\mathrm{Spec}(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)_{\geq \mathfrak{P}_{\max}}$ corresponds to the vanishing of the monodromy operator, hence to potentially crystalline deformations of $\overline{\rho}$ (recall that $\mathfrak{P}_{\max}$ is the partition $n = 1 + \cdots + 1$).

We write $(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$ for the reduced quotient corresponding to the set $\mathrm{Spec}(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$, and we let $R_{\overline{\rho}}(\tau, \lambda)_{\mathfrak{P}}$ be the image of $R^{\square}_{\overline{\rho}} \to (R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$. This is a reduced π_E-torsion free $\mathcal{O}_E$-algebra whose generic fibre is isomorphic to $(R^{\square}_{\overline{\rho}}[1/p])(\tau, \lambda)_{\mathfrak{P}}$, and its minimal primes have characteristic zero (by $\mathcal{O}_E$-flatness), hence they are in bijection with those of the generic fibre, which are the components where the monodromy is generically $\mathfrak{P}$. (By definition $R_{\overline{\rho}}(\tau, \lambda)_{\mathfrak{P}}$ is the Zariski closure in $R^{\square}_{\overline{\rho}}$ of the set of these components of the generic fibre.) We define $R_{\overline{\rho}}(\tau, \lambda)_{\geq \mathfrak{P}}$ similarly. By [Kis08, Theorem 3.3.4], these rings are equidimensional of the same dimension, which we denote by d.

Cycles. Since the rings $R_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}}$ are equidimensional and π_E-torsion free, their special fibres $R_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}}/\pi_E$ are also equidimensional, and define a $(d-1)$-cycle on $R_{\overline{\rho}}^{\square}$ [BM14, Lemma 2.1]. The geometric conjecture in [EG14, Section 4.2] states that for each Serre weight a for $\mathrm{GL}_n(\mathbf{f})$ there exists a cycle $\mathcal{C}_a$ on $R_{\overline{\rho}}^{\square}$ such that

$$Z(R_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}_{\max}}/\pi_E) = \sum_a n_a\mathcal{C}_a$$

where the multiplicity n_a is equal to the multiplicity of the representation F_a in $\overline{\sigma}_{\mathfrak{P}_{\max}}(\tau,\lambda)$, the semisimplified mod π_E reduction of $\sigma_{\mathfrak{P}_{\max}}(\tau,\lambda)$. Notice that $R_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}_{\max}}$ is a potentially crystalline deformation ring of $\overline{\rho}$. This can be reformulated by defining a group homomorphism

$$\overline{\mathrm{cyc}} : R_{\mathbf{e}}(\mathrm{GL}_n(\mathbf{f})) \to Z^{d-1}(R_{\overline{\rho}}^{\square}),\ F_a \mapsto \mathcal{C}_a$$

and one can generalize the statement of the conjecture, and ask whether

$$Z(R_{\overline{\rho}}(\tau,\lambda)_{\geq\mathfrak{P}}/\pi_E) = \overline{\mathrm{cyc}}(\overline{\sigma}_{\mathfrak{P}}(\tau,\lambda)).$$

This is motivated by the fact that $\sigma(\tau)_{\mathfrak{P}}$ is contained in a generic absolutely irreducible representation π of $E[\mathrm{GL}_n(F)]$ if and only if the inertial class of $\pi\otimes_E\overline{\mathbf{Q}}_p$ corresponds to τ and the partition $\mathfrak{P}(\pi)$ attached to π satisfies $\mathfrak{P}(\pi)\geq\mathfrak{P}$, that is

$$\mathrm{Hom}_{\mathbf{K}}(\sigma_{\mathfrak{P}}(\tau),\pi)\neq 0 \text{ if and only if } \mathrm{rec}_{\overline{\mathbf{Q}}_p}(\pi\otimes_E\overline{\mathbf{Q}}_p)|_{I_K}\cong\tau\otimes_E\overline{\mathbf{Q}}_p \text{ and } \mathfrak{P}(\pi)\geq\mathfrak{P}.$$

Under some assumptions on $\overline{\rho}$, this is true when $F=\mathbf{Q}_p$ and $n=2$ by [Kis09] or when $n=2$ and $\lambda=0$ by [GK14]. However, we expect that this statement has to be modified for $n\geq 3$ to account for multiplicities: it is not true in general that $\mathrm{Hom}_{\mathbf{K}}(\sigma_{\mathfrak{P}}(\tau),\pi)$ is one-dimensional when it is nonzero. The general statement should be

$$Z(R_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}}/\pi_E) = \overline{\mathrm{cyc}}(\overline{\sigma}^{+}_{\mathfrak{P}}(\tau,\lambda)),$$

because of the multiplicities

$$\dim\mathrm{Hom}_{\mathbf{K}}(\sigma^{+}_{\mathfrak{P}}(\tau),\pi) = \begin{cases} 1 \text{ if } \mathrm{rec}_{\overline{\mathbf{Q}}_p}(\pi\otimes_E\overline{\mathbf{Q}}_p)|_{I_K}\cong\tau \text{ and } \mathfrak{P}(\pi)=\mathfrak{P} \\ 0 \text{ otherwise.}\end{cases}$$

We offer two pieces of evidence towards this. The first is our theorem C, concerning the case $\mathfrak{P}_{\min}$, which gives a compatibility with the analogous statement on central division algebras. Second, observe that $\dim_E\mathrm{Hom}_{\mathbf{K}}(\sigma_{\mathfrak{P}}(\tau),\pi_{\mathfrak{P}'}(\tau))$ equals the Kostka number $K_{\mathfrak{P},\mathfrak{P}'}$, and so we have an equality in the Grothendieck group

$$\overline{\sigma}_{\mathfrak{P}}(\tau,\lambda) = \sum_{\deg \mathfrak{P}' = \deg \mathfrak{P}} K_{\mathfrak{P},\mathfrak{P}'} \overline{\sigma}^{+}_{\mathfrak{P}'}(\tau,\lambda)$$

and

$$\overline{\sigma}^{+}_{\mathfrak{P}}(\tau,\lambda) = \sum_{\deg \mathfrak{P}' = \deg \mathfrak{P}} K^{+}_{\mathfrak{P},\mathfrak{P}'} \overline{\sigma}_{\mathfrak{P}'}(\tau,\lambda)$$

where $(K^{+}_{\mathfrak{P},\mathfrak{P}'})$ is the inverse of the matrix $(K_{\mathfrak{P},\mathfrak{P}'})$ of Kostka numbers. Now [Sho18, Corollary 4.9] says that the direct analogues of our formulas give the right answer for deformation rings with l-adic coefficients, where $l \neq p$ is a prime number. This is also consistent with the work of Yao described in the introduction.

Remark 9.1.4. A computation of Kostka numbers implies that the multiplicities

$$\dim_E \operatorname{Hom}_{\mathbf{K}}(\sigma_{\mathfrak{P}}(\tau), \pi) \leq 1 \text{ if } \mathfrak{P} = \mathfrak{P}_{\min},$$

so we do expect that $Z(R_{\overline{\rho}}(\tau,\lambda)/\pi_E) = \overline{\operatorname{cyc}}(\overline{\sigma}_{\mathfrak{P}_{\min}}(\tau,\lambda))$.

9.2 Cycles on discrete series deformation rings.

Fix a prime number l, possibly equal to p, and a finite extension $E/\mathbf{Q}_l$ of coefficients. Let $\overline{\rho} : \operatorname{Gal}(\overline{F}/F) \to \operatorname{GL}_n(k_E)$ be a continuous representation. Let $R^{\square}_{\overline{\rho}}$ be the framed deformation ring of $\overline{\rho}$ over $\mathcal{O}_E$.

Case $l = p$. We prove theorem C in the introduction. Assume that E is large enough that all irreducible components of $R^{\square}_{\overline{\rho}}$ are geometrically irreducible, and all irreducible $\overline{\mathbf{F}}_p$-representations of $\operatorname{GL}_n(\mathcal{O}_F)$ and $\mathcal{O}_D^{\times}$ are defined over $\mathbf{e}$. Assuming the existence of a homomorphism

$$\overline{\operatorname{cyc}} : R_{\mathbf{e}}(\operatorname{GL}_n(\mathcal{O}_F)) \to Z^{d-1}(R^{\square}_{\overline{\rho}}/\pi_E)$$

such that $\overline{\operatorname{cyc}}(\overline{\sigma}^{+}_{\mathfrak{P}_{\min}}(\tau,\lambda)) = Z(R^{\square}_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}_{\min}}/\pi_E)$, we see that if we set

$$\overline{\operatorname{cyc}}_{D^{\times}} = \overline{\operatorname{cyc}} \circ \operatorname{JL}_p : R_{\mathbf{e}}(\mathcal{O}_D^{\times}) \to Z^{d-1}(R^{\square}_{\overline{\rho}}/\pi_E)$$

then $\overline{\operatorname{cyc}}_{D^{\times}}(\overline{\sigma}_D(\tau,\lambda)) = Z(R^{\square}_{\overline{\rho}}(\tau,\lambda)_{\mathfrak{P}_{\min}}/\pi_E)$ by theorem 8.0.4, providing a description of the mod p fibres of discrete series lifting rings in terms of the representation theory of $\mathcal{O}_D^{\times}$ and the type theory of $D^{\times}$.

Case $l \neq p$. In this case, we cannot find a finite extension $E/\mathbf{Q}_l$ such that all irreducible $\mathbf{e}$-representations of $\mathcal{O}_D^{\times}$ are absolutely irreducible. We assume that E is large enough that whenever $\overline{\rho}$ has a lift of inertial type τ to some finite extension of $\mathbf{Q}_l$, then τ and all the corresponding $\mathbf{K}$-types for $\mathrm{GL}_n(F)$ and $D^\times$ are defined over E. We also assume that E and k_E are large enough that all irreducible components of $\mathrm{Spec}(R^\square_{\overline{\rho}}[1/p])$ and $\mathrm{Spec}(R^\square_{\overline{\rho}}/\pi_E)$ are geometrically irreducible. For any pair (τ, N) consisting of an inertial type and a monodromy operator, write $R^\square_{\overline{\rho}}(\tau, N)$ for the corresponding quotient of the $\mathcal{O}_E$-deformation ring $R^\square_{\overline{\rho}}$, as in [Sho18]. The characteristic zero points of $R^\square_{\overline{\rho}}(\tau, N)$ correspond to lifts of $\overline{\rho}$ whose attached Weil–Deligne representation has inertial type τ, N. Define a map

$$\mathrm{cyc} : R_E(\mathrm{GL}_n(\mathcal{O}_F)) \to Z^d(R^\square_{\overline{\rho}}), \quad \sigma \mapsto \sum_{\tau,N} \dim_{\overline{\mathbf{Q}}_l} \mathrm{Hom}_{\overline{\mathbf{Q}}_l[\mathrm{GL}_n(\mathcal{O}_F)]}(\sigma^\vee \otimes_E \overline{\mathbf{Q}}_l, \pi_{\tau,N})[R^\square_{\overline{\rho}}(\tau, N)]$$

where $\pi_{\tau,N}$ is any irreducible generic $\overline{\mathbf{Q}}_l$-representation of $\mathrm{GL}_n(F)$ such that $\mathrm{rec}_{\overline{\mathbf{Q}}_l}(\pi_{\tau,N})$ has inertial type τ, N. (This is compatible with definition 7.0.4.) The map $\mathrm{rec}_{\overline{\mathbf{Q}}_l}$ is only well-defined up to the choice of a square root of q in $\overline{\mathbf{Q}}_l$, but this plays no role when considering the inertial type. Similarly, we introduce a map

$$\mathrm{cyc}_{D^\times} : R_E(\mathcal{O}_D^\times) \to Z^d(R^\square_{\overline{\rho}}), \quad \sigma \mapsto \sum_{\tau,N} \dim \mathrm{Hom}_{\overline{\mathbf{Q}}_l[\mathrm{GL}_n(\mathcal{O}_F)]}(\sigma^\vee \otimes_E \overline{\mathbf{Q}}_l, \mathrm{JL}^{-1}(\pi_{\tau,N}))[R^\square_{\overline{\rho}}(\tau, N)].$$

In this formula we set $\mathrm{JL}^{-1}(\pi) = 0$ when π is a generic representation that is not essentially square-integrable (this is consistent with the fact that the Langlands–Jacquet transfer is nonzero on elliptic representations only, and the only generic elliptic representations are the essentially square-integrable representations. See [Dat07].)

Theorem 9.2.1 (Breuil–Mézard conjecture for $D^\times$, case $l \neq p$)**.** Assume $p \neq 2$. There exists a unique map $\overline{\mathrm{cyc}}_{D^\times,l}$ making the following diagram commute.

$$\begin{array}{ccc} R_E(\mathcal{O}_D^\times) & \xrightarrow{\mathrm{cyc}_{D^\times}} & Z^d(R^\square_{\overline{\rho}}) \\ \downarrow \mathbf{r}_l & & \downarrow \mathrm{red} \\ R_{k_E}(\mathcal{O}_D^\times) & \xrightarrow{\overline{\mathrm{cyc}}_{D^\times,l}} & Z^{d-1}(R^\square_{\overline{\rho}}/\pi_E) \end{array} \tag{9.2.1}$$

Proof. Since the map $\mathbf{r}_l$ is surjective for $\mathcal{O}_D^\times$, it suffices to prove that if $x \in \ker(\mathbf{r}_l)$ then $x \in \ker(\mathrm{red} \circ \mathrm{cyc}_{D^\times})$. This says that every congruence between $\mathbf{K}$-types gives rise to a congruence between deformation rings: it is not a formal statement.

By [Sho18, Theorem 4.6], there exists a commutative diagram

$$\begin{array}{ccc} R_E(\mathrm{GL}_n(\mathcal{O}_F)) & \xrightarrow{\mathrm{cyc}} & Z^d(R^\square_{\overline{\rho}}) \\ \downarrow \mathbf{r}_l & & \downarrow \mathrm{red} \\ R_{k_E}(\mathrm{GL}_n(\mathcal{O}_F)) & \xrightarrow{\overline{\mathrm{cyc}}_l} & Z^{d-1}(R^\square_{\overline{\rho}}/\pi_E). \end{array} \tag{9.2.2}$$

Let $x_{\overline{\mathbf{Q}}_l}$ be the image of x in $R_{\overline{\mathbf{Q}}_l}(\mathcal{O}_D^\times)$. Fix a finite extension L/E large enough that all irreducible summands in $x_{\overline{\mathbf{Q}}_l}$ and $\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l})$ are defined over L. Then $\mathrm{cyc}_{D^\times}(x_L) = \mathrm{cyc}(\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l}))$, where we regard $\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l})$ as an element of $R_L(\mathrm{GL}_n(\mathcal{O}_F))$ and the two sides as cycles on the deformation ring with $\mathcal{O}_L$-coefficients. Indeed, if σ is an L-representation of $\mathcal{O}_D^\times$ then we have by construction the equality

$$\dim \mathrm{Hom}_{\overline{\mathbf{Q}}_l[\mathcal{O}_D^\times]}(\sigma_{\overline{\mathbf{Q}}_l}, \mathrm{JL}^{-1}(\pi_{\tau,N})) = \dim \mathrm{Hom}_{\overline{\mathbf{Q}}_l[\mathrm{GL}_n(\mathcal{O}_F)]}(\mathrm{JL}_{\mathbf{K}}(\sigma_{\overline{\mathbf{Q}}_l}), \pi_{\tau,N})$$

because this equality holds on $\mathbf{K}$-types, and by lemma 8.0.1 the $\mathbf{K}$-types span $R_{\overline{\mathbf{Q}}_l}(\mathcal{O}_D^\times)$. Because of our assumptions on E, the natural maps $Z^d(R_{\overline{\rho}}^\square) \to Z^d(R_{\overline{\rho}}^\square \otimes_{\mathcal{O}_E} \mathcal{O}_L)$ and $Z^{d-1}(R_{\overline{\rho}}^\square/\pi_E) \to Z^{d-1}(R_{\overline{\rho}}^\square \otimes_{\mathcal{O}_E} k_L)$ are isomorphisms, hence it suffices to prove that

$$\mathrm{red}\,\mathrm{cyc}\,\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l}) = 0.$$

Since diagram (9.2.2) commutes (working with L-coefficients in the diagram), we have that $\mathrm{red}\,\mathrm{cyc}\,\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l}) = \overline{\mathrm{cyc}}_l\,\mathbf{r}_l\,\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l})$. By theorem 8.0.6, we have $\mathbf{r}_l\,\mathrm{JL}_{\mathbf{K}}(x_{\overline{\mathbf{Q}}_l}) = \mathrm{JL}_l\,\mathbf{r}_l(x_{\overline{\mathbf{Q}}_l}) = 0$, and the claim follows. $\square$

References

[Ber84] J. N. Bernstein, *Le "centre" de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32. MR 771671

[BF83] Colin J. Bushnell and Albrecht Fröhlich, *Gauss sums and p-adic division algebras*, Lecture Notes in Mathematics, vol. 987, Springer-Verlag, Berlin-New York, 1983. MR 701540

[BH96] Colin J. Bushnell and Guy Henniart, *Local tame lifting for* $\mathrm{GL}(N)$*. I. Simple characters*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 105–233. MR 1423022

[BH03] ———, *Local tame lifting for* $\mathrm{GL}(n)$*. IV. Simple characters and base change*, Proc. London Math. Soc. (3) **87** (2003), no. 2, 337–362. MR 1990931

[BH04] ———, *Local Jacquet–Langlands correspondence and parametric degrees*, Manuscripta Math. **114** (2004), no. 1, 1–7. MR 2136524

[BH05] ———, *The essentially tame local Langlands correspondence. I*, J. Amer. Math. Soc. **18** (2005), no. 3, 685–710. MR 2138141

[BH10] ———, *The essentially tame local Langlands correspondence, III: the general case*, Proc. Lond. Math. Soc. (3) **101** (2010), no. 2, 497–553. MR 2679700

[BH11] ———, *The essentially tame Jacquet–Langlands correspondence for inner forms of* $\mathrm{GL}(n)$, Pure Appl. Math. Q. **7** (2011), no. 3, Special Issue: In honor of Jacques Tits, 469–538. MR 2848585

[BH14] ———, *To an effective local Langlands correspondence*, Mem. Amer. Math. Soc. **231** (2014), no. 1087, v+88. MR 3236840

[BH17] ———, *Higher ramification and the local Langlands correspondence*, Ann. of Math. (2) **185** (2017), no. 3, 919–955. MR 3664814

[BHS] Corinne Blondel, Guy Henniart, and Shaun Stevens, *Jordan blocks of cuspidal representations of symplectic groups*, arXiv:1704.03545.

[BK93] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR 1204652

[BK98] ———, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417

[BL02] P. Broussous and B. Lemaire, *Building of* $\mathrm{GL}(m, D)$ *and centralizers*, Transform. Groups **7** (2002), no. 1, 15–50. MR 1888474

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572

[BM14] ———, *Multiplicités modulaires raffinées*, Bull. Soc. Math. France **142** (2014), no. 1, 127–175. MR 3248725

[BSS12] P. Broussous, V. Sécherre, and S. Stevens, *Smooth representations of* $\mathrm{GL}_m(D)$ *V: Endo-classes*, Doc. Math. **17** (2012), 23–77. MR 2889743

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR 0579172

[Car85] Roger W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication. MR 794307

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Camb. J. Math. **4** (2016), no. 2, 197–287. MR 3529394

[Dat07] Jean-François Dat, *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Invent. Math. **169** (2007), no. 1, 75–152. MR 2308851

[Dat17] ———, *A functoriality principle for blocks of p-adic linear groups*, Around Langlands correspondences, Contemp. Math., vol. 691, Amer. Math. Soc., Providence, RI, 2017, pp. 103–131. MR 3666052

[Dat18] ———, *Simple subquotients of big parabolically induced representations of p-adic groups*, J. Algebra **510** (2018), 499–507. MR 3828793

[Del73] P. Deligne, *Formes modulaires et représentations de* $\mathrm{GL}(2)$, 55–105. Lecture Notes in Math., Vol. 349. MR 0347738

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161. MR 0393266

[DM91] François Digne and Jean Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21, Cambridge University Press, Cambridge, 1991. MR 1118841

[Dud] Olivier Dudas, *Lectures on modular Deligne–Lusztig theory*, arXiv:1705.08234.

[EG14] Matthew Emerton and Toby Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13** (2014), no. 1, 183–223. MR 3134019

[Fon94] Jean-Marc Fontaine, *Représentations l-adiques potentiellement semi-stables*, Astérisque (1994), no. 223, 321–347, Périodes p-adiques (Bures-sur-Yvette, 1988). MR 1293977

[Frö87] A. Fröhlich, *Principal orders and embedding of local fields in algebras*, Proc. London Math. Soc. (3) **54** (1987), no. 2, 247–266. MR 872807

[GG15] Toby Gee and David Geraghty, *The Breuil–Mézard conjecture for quaternion algebras*, Annales de l'Institut Fourier (2015), 1557–1575.

[GK14] Toby Gee and Mark Kisin, *The Breuil–Mézard conjecture for potentially Barsotti-Tate representations*, Forum Math. Pi **2** (2014), e1, 56. MR 3292675

[GR10] Benedict H. Gross and Mark Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. **154** (2010), no. 3, 431–508. MR 2730575

[Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

[Kal11] Tasho Kaletha, *Endoscopic character identities for depth-zero supercuspidal L-packets*, Duke Math. J. **158** (2011), no. 2, 161–224. MR 2805068

[Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546. MR 2373358

[Kis09] ______, *The Fontaine–Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22** (2009), no. 3, 641–690. MR 2505297

[KV12] David Kazhdan and Yakov Varshavsky, *On endoscopic transfer of Deligne-Lusztig functions*, Duke Math. J. **161** (2012), no. 4, 675–732. MR 2891532

[LLHL] Daniel Le, Bao V. Le Hung, and Brandon Levin, *Weight elimination in serre-type conjectures*, https://arxiv.org/abs/1610.04819.

[Lus95] George Lusztig, *Classification of unipotent representations of simple p-adic groups*, Internat. Math. Res. Notices (1995), no. 11, 517–589. MR 1369407

[Lus77] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, Invent. Math. **38** (1976/77), no. 2, 101–159. MR 0453885

[MS14a] Alberto Mínguez and Vincent Sécherre, *Représentations lisses modulo ℓ de* $\mathrm{GL}_m(D)$, Duke Math. J. **163** (2014), no. 4, 795–887. MR 3178433

[MS14b] ______, *Types modulo ℓ pour les formes intérieures de* GL_n *sur un corps local non archimédien*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 4, 823–891, With an appendix by Vincent Sécherre et Shaun Stevens. MR 3273486

[MS17] ______, *Correspondance de Jacquet–Langlands locale et congruences modulo ℓ*, Invent. Math. **208** (2017), no. 2, 553–631. MR 3639599

[Paš05] Vytautas Paškūnas, *Unicity of types for supercuspidal representations of* GL_N, Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654. MR 2180458

[Paš15] ______, *On the Breuil-Mézard conjecture*, Duke Math. J. **164** (2015), no. 2, 297–359. MR 3306557

[Pyv] Alexandre Pyvovarov, *On the Breuil–Schneider conjecture: Generic case*, arXiv:1803.01610.

[Ren10] David Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR 2567785

[Roc09] Alan Roche, *The Bernstein decomposition and the Bernstein centre*, Ottawa lectures on admissible representations of reductive p-adic groups, Fields Inst. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 2009, pp. 3–52. MR 2508719

[Sch13] Peter Scholze, *The local Langlands correspondence for* GL_n *over p-adic fields*, Invent. Math. **192** (2013), no. 3, 663–715. MR 3049932

[Séc04] Vincent Sécherre, *Représentations lisses de* $\mathrm{GL}(m, D)$*,* I*: caractères simples*, Bulletin de la Société Mathématique de France **132** (2004), no. 3, 327–396.

[Séc05] ______, *Représentations lisses de* $\mathrm{GL}(m, D)$*. II. β-extensions*, Compos. Math. **141** (2005), no. 6, 1531–1550. MR 2188448

[Séc09] ______, *Proof of the Tadić conjecture (U0) on the unitary dual of* $\mathrm{GL}_m(D)$, J. Reine Angew. Math. **626** (2009), 187–203. MR 2492994

[Ser72] Jean-Pierre Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. MR 0387283

[Ser77] ______, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. MR 0450380

[Sho18] Jack Shotton, *The Breuil–Mézard conjecture when $l \neq p$*, Duke Math. J. **167** (2018), no. 4, 603–678. MR 3769675

[SS08] V. Sécherre and S. Stevens, *Représentations lisses de* $\mathrm{GL}_m(D)$*. IV. Représentations supercuspidales*, J. Inst. Math. Jussieu **7** (2008), no. 3, 527–574. MR 2427423

[SS12] ______, *Smooth representations of $GL_m(D)$ VI: semisimple types*, Int. Math. Res. Not. IMRN (2012), no. 13, 2994–3039. MR 2946230

[SS16a] ______, *Block decomposition of the category of ℓ-modular smooth representations of* $\mathrm{GL}_n(\mathrm{F})$ *and its inner forms*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 3, 669–709. MR 3503829

[SS16b] ______, *Towards an explicit Jacquet–Langlands correspondence beyond the cuspidal case*, https://arxiv.org/abs/1611.04317, 2016.

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517. MR 0236190

[SZ99] P. Schneider and E.-W. Zink, *K-types for the tempered components of a p-adic general linear group*, J. Reine Angew. Math. **517** (1999), 161–208, With an appendix by Schneider and U. Stuhler. MR 1728541

[SZ00] Allan J. Silberger and Ernst-Wilhelm Zink, *The characters of the generalized Steinberg representations of finite general linear groups on the regular elliptic set*, Trans. Amer. Math. Soc. **352** (2000), no. 7, 3339–3356. MR 1650042

[Vig96] Marie-France Vignéras, *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$*, Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1395151

[Vig01a] ______, *Correspondance de Langlands semi-simple pour* $\mathrm{GL}(n, F)$ *modulo* $\ell \neq p$, Invent. Math. **144** (2001), no. 1, 177–223. MR 1821157

[Vig01b] ______, *La conjecture de Langlands locale pour* $\mathrm{GL}(n, F)$ *modulo ℓ quand $\ell \neq p, \ell > n$*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 6, 789–816. MR 1872421

[Zel80] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR 584084