Exact Simulation of Ornstein-Uhlenbeck Tempered Stable Processes

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Abstract

There are two types of tempered stable (TS) based Ornstein-Uhlenbeck (OU) processes: 1) *OU-TS process*, the OU process driven by TS subordinator, and 2) *TS-OU process*, the OU process with TS marginal law. They have various applications in financial engineering and econometrics. In the literature, only the second type under the stationary assumption has an exact simulation algorithm. In this paper, we develop a unified approach to exactly simulate both types without the stationary assumption. It is mainly based on the distributional decomposition of stochastic processes with an aid of acceptance-rejection scheme. As the inverse Gaussian distribution is an important special case of TS distribution, we also provide tailored algorithms for the corresponding OU processes. Numerical experiments and tests are reported to demonstrate the accuracy and effectiveness of our algorithms, and some further extensions are also discussed.

Keywords: Monte Carlo simulation; Exact simulation; Non-Gaussian Ornstein-Uhlenbeck process; Tempered stable subordinator; Tempered stable OU process; OU tempered stable process

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1 Introduction

A Lévy-driven Ornstein-Uhlenbeck (OU) process is the analogue of an *ordinary Gaussian OU process* (Uhlenbeck and Ornstein, 1930) with its Brownian motion part replaced by a Lévy process. This class of stochastic processes has been extensively studied in the literature, see Wolfe (1982), Sato and Yamazato (1984), Barndorff-Nielsen (1998) and Barndorff-Nielsen et al. (1998). Comparing with the Gaussian OU processes, the non-Gaussian counterparts offer greater flexibility that can additionally accommodate some crucial distributional features, such as jumps and volatility clustering, which are often observed in the real time series data[1]. Nowadays, these processes have been widely used as the continuous-time stochastic volatility models for the observed

[1] See empirical evidences in finance from Carr et al. (2002).

behaviour of price dynamics in finance and economics. The applicability has been enhanced substantially by Barndorff-Nielsen and Shephard (2001b, 2002). They proposed a variety of useful non-negative OU processes for modelling stochastic volatilities. This class of models not only possess mathematically tractable properties, but also has nice economic interpretations for which new information arrives in discrete packets and trades are made in blocks[2]. In addition, it has also been used in option pricing, see Nicolato and Venardos (2003), Kallsen et al. (2011) and Li and Linetsky (2014), and for describing high-frequency financial data in market microstructure, see Barndorff-Nielsen and Shephard (2003a,b) and Todorov and Tauchen (2006).

In fact, Barndorff-Nielsen and Shephard (2001a,b,c) proposed two general ways to construct non-Gaussian OU processes. One approach is to first specify the invariant marginal distribution of the underlying OU process and then study the implied behaviour of the driving non-negative Lévy process. The model building also involves an unusual change of time, in order to separate the marginal distribution and dynamic structure of the process. The alternative approach is the other way around but more natural: the process is constructed directly by specifying the driving non-negative Lévy process. Although the former approach appears to be more popular and is widely used in the current literature[3], the latter one is also very attractive as a natural alterative for describing financial data.

Due to numerous applications of these models, the availability of efficient and accurate simulation algorithms is particularly important in the context of model validation and statistical inference, as well as for risk analysis and derivative pricing. The most well-known simulation scheme is based on Rosiński's *infinite series representation* (Rosiński, 2001). One could alternatively use *Fourier inversion techniques* to numerically invert the underlying characteristic functions, see Glasserman and Liu (2010) and Chen et al. (2012). Both methods apply to a very general class of processes, however, they are not *exact* and would introduce truncation, discretisation or round-off errors. Our interest in this paper is the exact simulation rather than the approximation-based, and we consider two important types of non-Gaussian OU processes which are constructed from positive tempered stable (TS) distributions: (1) *OU-TS process*, i.e. the ordinary Gaussian OU process with its Brownian motion part replaced by a TS subordinator; (2) *TS-OU process*, i.e. the OU process with positive TS marginal law. One should be aware that these two types are very different although their names sound similar. The marginal distribution of TS-OU process is simply a

[2]See empirical evidences from the market microstructure in Easley and O'Hara (1987).

[3]The former approach has been widely used in the literature, for example, for modelling stochastic volatility, see Barndorff-Nielsen and Shephard (2001c, 2002, 2003a), Barndorff-Nielsen et al. (2002), Gander and Stephens (2007a,b) and Andrieu et al. (2010).

time-invariant TS distribution, whereas the marginal distribution of OU-TS process is time-variant and is not TS. As pointed in the concluding remarks of Barndorff-Nielsen and Shephard (2001c, p.19), these two types offer great flexibility and are mathematically and computationally tractable, which could lead to a variety of applications, e.g. Barndorff-Nielsen et al. (1998, 2002), Barndorff-Nielsen and Shephard (2001b, 2002, 2003a), Nicolato and Venardos (2003), Jongbloed et al. (2005), Gander and Stephens (2007a,b), Andrieu et al. (2010) and Todorov (2015) for modelling the stochastic volatility, and a series of time-changed models by Li and Linetsky (2013, 2014, 2015) and Mendoza-Arriaga and Linetsky (2014, 2016) for modelling the stochastic time clock.

The aim of this paper is to design a unified approach to exactly simulate OU-TS and TS-OU processes, with a particular focus on the first type. Our key methodology for simulation design is the *exact distributional decomposition*, which has also been recently used to simulate the classical Hawkes process (Dassios and Zhao, 2013), the point process with CIR intensity (Dassios and Zhao, 2017), tempered stable distributions (Dassios et al., 2018), Lévy-driven point processes (Qu et al., 2019a) and gamma-driven Ornstein-Uhlenbeck processes (Qu et al., 2020). We first develop an exact simulation scheme for the OU-TS process based on distributional decomposition by breaking the Lévy measure of the driving TS subordinator. More precisely, the transition distribution of OU-TS process can be decomposed into simple elements: one TS r.v. and one compound Poisson r.v., and each of them can be exactly simulated directly, or, via acceptance-rejection (A/R) scheme. Besides, this approach can also be easily extended to the TS-OU process. We find that the TS-OU process is the sum of one OU-TS process and one compound Poisson process, and this immediately reveals an associated exact simulation scheme for this process. In particular, the inverse Gaussian (IG) OU processes, i.e. OU-IG and IG-OU processes, are included as the important special cases, and their tailored algorithms are provided. In addition, some further extensions for Lévy-driven OU processes with the BDLPs beyond tempered stable processes are also discussed.

Recently, Zhang (2011) derived an exact simulation algorithm for the stationary TS-OU process, see also Zhang and Zhang (2009). His algorithm is developed mainly based on the well known *Lévy-Khintchine representation* for the (infinitely divisible) TS distribution. Apparently it is applicable, as the marginal distribution of a TS-OU process is a time-invariant TS distribution under the key assumption of stationarity. However, his approach is methodologically different from ours, and the key difference is that we do not use *Lévy-Khintchine representation* so the stationary assumption is not required. For example, his approach can not apply to the non-stationary OU-TS process whose marginal distribution is time-varying and is not necessarily TS or any identifiable distribution. We can deal with the non-stationary processes, each of which starts from a given time with a fixed initial value (rather than a stationary distribution for the initial value). The main contribution of this paper in the context of simulation is to provide the first method for exactly sampling the OU-TS process without the stationary assumption, which is also applicable to the TS-OU process.

This paper is organised as follows: Section 2 offers the preliminaries including formal mathematical definitions and introductions for the TS distribution, TS subordinator, non-Gaussian OU process, and OU-TS/TS-OU processes. In Section 3 and 4, we first derive some important distributional properties, and then present the associated algorithms of exact simulation for OU-TS/OU-IG and TS-OU/IG-OU processes, respectively. In Section 5, extensive numerical experiments have been carried out and reported in detail, which demonstrate the accuracy and effectiveness of our algorithms. Some further extensions for Lévy-driven OU processes with the BDLPs beyond tempered stable processes are discussed in Section 6. Finally, Section 7 draws a conclusion for this paper, and proposes some issues for possible further extensions and future research.

2 Preliminaries

This preliminary section offers a brief review for a number of well-known distributions and stochastic processes: tempered stable distribution, inverse Gaussian distribution, TS subordinator, non-Gaussian Ornstein-Uhlenbeck process, and two important types of non-Gaussian Ornstein-Uhlenbeck processes. They provide the foundations for developing simulation algorithms later in the next section.

2.1 Tempered Stable Distribution and Tempered Stable Subordinator

Positive tempered stable distribution can be obtained from a one-sided α*-stable law* by exponential tilting (Barndorff-Nielsen et al., 2002, p.14), see also Barndorff-Nielsen and Shephard (2001c, p.3). More precisely, it can be defined as below:

Definition 2.1 (Positive Tempered Stable Distribution)**. Positive tempered stable (TS) distribution**, *denoted by* $\mathcal{TS}(\alpha, \beta, \theta)$*, is an infinitely divisible distribution defined by its Lévy measure*

$$\nu(\mathrm{d}y) = \frac{\theta}{y^{\alpha+1}} e^{-\beta y} \mathrm{d}y, \qquad y \geq 0, \qquad \alpha \in (0,1), \quad \beta, \theta \in \mathbb{R}^+, \tag{2.1}$$

where α *is the stability index,* θ *is the intensity parameter and* β *is the tilting parameter.*

In particular, if $\alpha = \frac{1}{2}$, it reduces to a very important distribution, the inverse Gaussian (IG) distribution (which can be interpreted as the distribution of the first passage time of a Brownian motion to an absorbing barrier).

Tempered stable subordinator is a positive Lévy process whose one-dimensional distributions are positive TS distributions[4]. More precisely, it can be defined as below:

Definition 2.2 (Tempered Stable Subordinator)**. Tempered stable (TS) subordinator** *is a positive Lévy process* $\{Z_t : t \geq 0\}$ *such that* Z_1 *follows a positive TS distribution, i.e.* $Z_1 \sim \mathit{TS}(\alpha, \beta, \theta)$.

It is a Lévy *subordinator*[5] where the state space is restricted in the positive half real line, and $Z_t \sim \mathsf{TS}\big(\alpha, \beta, \theta t\big), \forall t > 0$. The stable index α determines the importance of small jumps for the process trajectories, the intensity parameter θ controls the intensity of jumps, and the tilting parameter β determines the decay rate of large jumps. TS subordinator (including IG subordinator as the special case) is one of the most general and widely used building blocks for further constructing many useful TS-based stochastic processes. Distinguishing examples include the TS-based non-Gaussian OU processes which are briefly reviewed as follows.

2.2 Non-Gaussian Ornstein-Uhlenbeck Processes and Tempered Stable Ornstein-Uhlenbeck Processes

Non-Gaussian Ornstein-Uhlenbeck processes are well-equipped for capturing the mean-reverting dynamics as well as the skewness and leptokurtosis in the marginal distributions of the underlying financial time series. In the literature, there are two important types of non-Gaussian Ornstein-Uhlenbeck processes: the modified version with time change, and the original version without time change. Let us first review the original version[6] as proposed by Barndorff-Nielsen et al. (1998, p.995):

Definition 2.3 (Non-Gaussian Ornstein-Uhlenbeck Process)**.** X_t *is a* **non-Gaussian Ornstein-Uhlenbeck (OU) process** *that satisfies the stochastic differential equation (SDE)*

$$\mathrm{d}X_t = -\delta X_t \mathrm{d}t + \varrho \mathrm{d}Z_t, \qquad t \geq 0, \tag{2.2}$$

where

- $\varrho > 0$ *is a positive constant;*
- $\delta > 0$ *is the constant rate of exponential decay;*
- $Z_t \geq 0$ *with* $Z_0 = 0$ *is a pure-jump Lévy subordinator.*

[4]More generalised tempered stable distributions and hence the associated processes can be found in Rosiński (2007).
[5]A *subordinator* is a Lévy process with non-decreasing paths, see Bertoin (1998, Chapter 3) and Sato (1999).
[6]It is also named the *general Ornstein-Uhlenbeck process* in Norberg (2004).

Equivalently, given the initial level $X_0 > 0$ at time 0, the solution to this SDE (2.2) is given by

$$X_t = e^{-\delta t} X_0 + \varrho \int_0^t e^{-\delta(t-s)} \mathrm{d}Z_s. \tag{2.3}$$

Z_t, termed the *background driving Lévy process* (BDLP), is a homogenous Lévy process with positive increments almost surely. Hence, the resulting process X_t is non-negative, and it is the continuous-time analogue of a discrete-time *autoregression of order* 1 (AR(1)) (Barndorff-Nielsen et al., 1998, p.995). If Z_t is replaced by a standard Brownian motion, then, it returns to the *ordinary Gaussian OU process* (Uhlenbeck and Ornstein, 1930).

On the other hand, Barndorff-Nielsen and Shephard (2001a,b,c) proposed a more popular version based on change of time:

Definition 2.4 (Time-changed Non-Gaussian Ornstein-Uhlenbeck Process)**.** *Y_t is a* **time-changed non-Gaussian Ornstein-Uhlenbeck (OU) process** *that satisfies the SDE*

$$\mathrm{d}Y_t = -\delta Y_t \mathrm{d}t + \mathrm{d}R_t, \qquad t \geq 0, \tag{2.4}$$

where R_t is a time-changed Lévy subordinator such that the resulting marginal distribution of Y_t is independent of the decay rate δ.

This deliberately leads to a separation between the marginal distribution of OU process and its dynamic structure, which is the main attractiveness of this model.

Based on these two types of non-Gaussian OU processes defined above, the associated two types of TS-based OU processes can be naturally constructed, i.e. the so-called *OU-TS process* and *TS-OU process*[7], respectively:

Definition 2.5 (TS-Based Ornstein-Uhlenbeck Processes)**.** *Two types of TS-based Ornstein-Uhlenbeck processes:*

1. **OU-TS Process***: For the non-Gaussian OU process X_t of Definition 2.3, if Lévy subordinator Z_t is a TS process of Definition 2.2, then, X_t is an OU-TS process for any time t.*

2. **TS-OU Process***: For the non-Gaussian OU process Y_t of Definition 2.4, if the marginal distribution of Y_t is a positive TS distribution of Definition 2.1, then, Y_t is a TS-OU process for any time t.*

[7] We adopt the abbreviations of *OU-TS* and *TS-OU* from Barndorff-Nielsen et al. (2002, p.13), see also Barndorff-Nielsen et al. (1998), Barndorff-Nielsen and Shephard (2001c, 2003a) and Schoutens (2003, p.48).

In particular, if the stable index $\alpha = 1/2$, then they reduce to OU-IG and IG-OU processes. Simulated paths of OU-TS and OU-IG processes are plotted in Figure 1 and Figure 2, respectively.

OU-TS and TS-OU processes are very tractable which could facilitate many types of positive time series, such as stochastic volatilities, interest rates and default intensities. However, most of the literature concentrate on the second type, whereas in this paper, we focus on the first type. Meanwhile, we provide some important connections between the two.

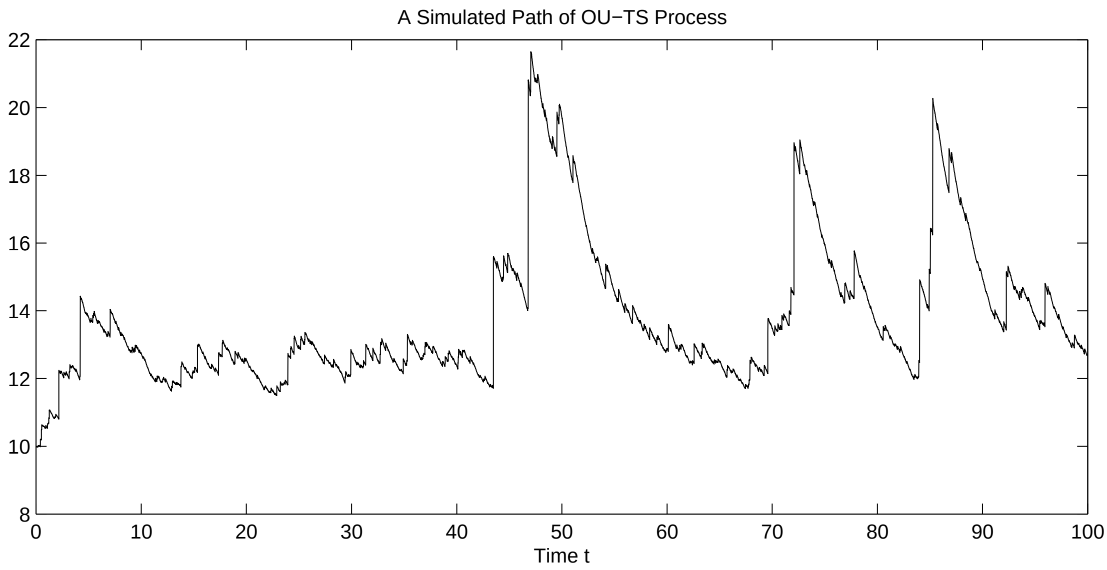

Figure 1: A simulated path of OU-TS process by Algorithm 3.1, with the parameter setting $(\delta, \varrho; \alpha, \beta, \theta; X_0) = (0.2, 1.0; 0.9, 0.2, 0.25; 10)$ within the time period of $[0, 100]$ and $10,000$ equally-spaced discretisation steps

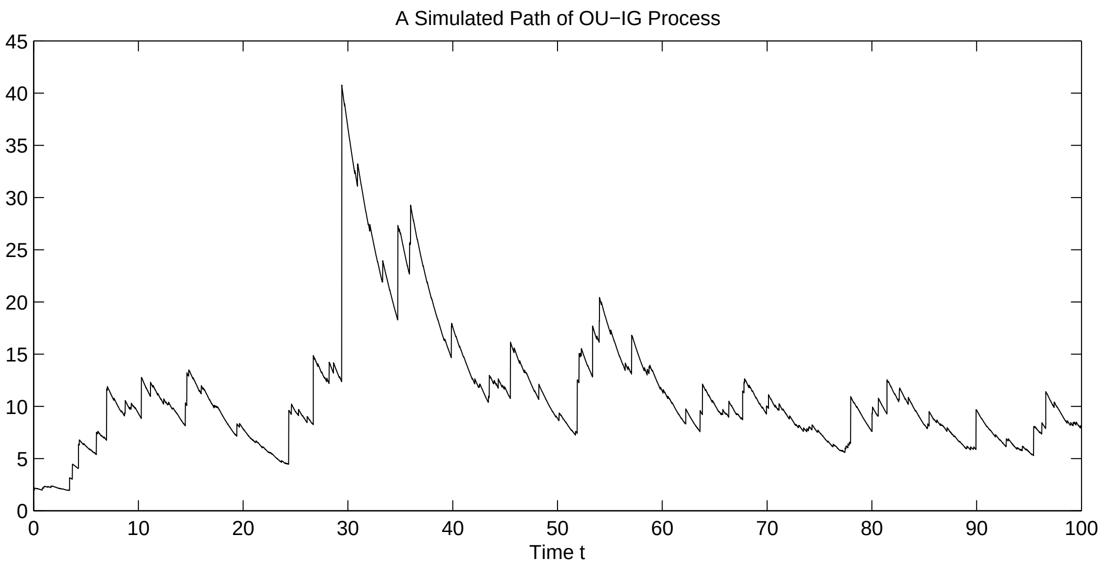

Figure 2: A simulated path of OU-IG process by Algorithm 3.3, with the parameter setting $(\delta, \varrho; c; X_0) = (0.2, 1.0; 0.5; 2.0)$ within the time period of $[0, 100]$ and $10,000$ equally-spaced discretisation steps

3 Exact Simulation of OU-TS Process

In this section, we develop the exact simulation scheme for OU-TS process based on the exact distributional decomposition, i.e. conditional on the value of OU-TS process at time $t \in \mathbb{R}^+$, the distribution of OU-TS process at time $t+\tau$ for any time lag $\tau \in \mathbb{R}^+$ can be broken into three simple elements: one constant, one TS r.v. and one compound Poisson r.v., and each of them can be exactly generated. This is achieved by the identification through the conditional Laplace transforms derived as below.

Proposition 3.1. *For a general non-Gaussian OU process X_t of Definition 2.3, the Laplace transform of $X_{t+\tau}$ conditional on X_t is given by*

$$\mathbb{E}\left[e^{-vX_{t+\tau}} \mid X_t\right] = e^{-vwX_t} \times \exp\left(-\frac{\varrho}{\delta}\int_{vw}^{v} \frac{\Phi(u)}{u}\mathrm{d}u\right), \qquad \tau \in \mathbb{R}^+, \tag{3.1}$$

where $w := e^{-\delta\tau}$ and $\Phi(u)$ is the Laplace exponent of Lévy subordinator Z_t.

Proof. Note that, in general, the Laplace exponent for Z_t is

$$\Phi(u) = \int_0^\infty \left(1 - e^{-uy}\right)\nu(\mathrm{d}y),$$

where ν is the Lévy measure of Z_t. The infinitesimal generator[8] $\mathcal{A}$ of process (X_t, t) acting on any function $f(x,t)$ within its domain $\Omega(\mathcal{A})$ is given by

$$\mathcal{A}f(x,t) = \frac{\partial f}{\partial t} - \delta x\frac{\partial f}{\partial x} + \varrho\left(\int_0^\infty \left[f(x+y,t) - f(x,t)\right]\nu(\mathrm{d}y)\right), \tag{3.2}$$

where $\Omega(\mathcal{A})$ is the domain for the generator $\mathcal{A}$ such that $f(x,t)$ is differentiable with respect to x and t for all x and t, and

$$\left|\int_0^\infty \left[f(x+y,t) - f(x,t)\right]\nu(\mathrm{d}y)\right| < \infty.$$

By applying the *piecewise-deterministic Markov processes theory* (Davis, 1984) and martingale approach (Dassios and Embrechts, 1989), we can derive the conditional Laplace transform for X_t.

[8] Infinitesimal generator for (X_t, t) is defined as

$$\mathcal{A}f(x,t) := \lim_{\Delta t \to 0} \frac{\mathbb{E}[f(X_{t+\Delta t}, t+\Delta t) \mid X_t = x] - f(x,t)}{\Delta t},$$

see Øksendal (2010). The associated infinitesimal generator for Lévy subordinators can be easily found in the existing literature, for example, Sato (1999, Ch.6).

More precisely, set $\mathcal{A}f(x,t)=0$, we find a martingale

$$\exp\left(-X_t k e^{\delta t}\right)\exp\left(\varrho\int_0^t \Phi\left(k e^{\delta s}\right)\mathrm{d}s\right), \qquad \forall\, k\in\mathbb{R}^+, \tag{3.3}$$

with the proof in Appendix A. By martingale property and setting $k=ve^{-\delta(t+\tau)}$, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-vX_{t+\tau}}\mid X_t\right] &= \exp\left(-ve^{-\delta\tau}X_t\right)\exp\left(-\varrho\int_t^{t+\tau}\Phi\left(ve^{-\delta(t+\tau-s)}\right)\mathrm{d}s\right)\\ &= \exp\left(-ve^{-\delta\tau}X_t\right)\exp\left(-\frac{\varrho}{\delta}\int_{ve^{-\delta\tau}}^{v}\frac{\Phi(u)}{u}\mathrm{d}u\right).\end{aligned}$$

□

The conditional Laplace transform (3.1)[9] in Proposition 3.1 is the key tool to develop our exact simulation scheme later in this section. It can also be used to obtain an analytical formula for the associated conditional expectation in Proposition 3.2.

Proposition 3.2. *The expectation of $X_{t+\tau}$ conditional on X_t is given by*

$$\mathbb{E}\left[X_{t+\tau}\mid X_t\right] = wX_t+\frac{\varrho}{\delta}(1-w)\mathbb{E}\left[Z_1\right], \qquad \tau\in\mathbb{R}^+, \tag{3.4}$$

where $\mathbb{E}[Z_1]=\int_0^\infty s\nu(\mathrm{d}s)$.

Proof. Based on Proposition 3.1, we have

$$\begin{aligned}\mathbb{E}\left[X_{t+\tau}\mid X_t\right] &= -\frac{\partial}{\partial v}\mathbb{E}\left[e^{-vX_{t+\tau}}\mid X_t\right]\Big|_{v=0}\\ &= e^{-\delta\tau}X_t+\frac{\varrho}{\delta}\lim_{v\to 0}\left(\Phi'(v)-\Phi'\left(e^{-\delta\tau}v\right)\right)\\ &= e^{-\delta\tau}X_t+\frac{\varrho}{\delta}\lim_{v\to 0}\left(\int_0^\infty se^{-vs}\nu(\mathrm{d}s)-w\int_0^\infty se^{-wvs}\nu(\mathrm{d}s)\right)\\ &= e^{-\delta\tau}X_t+\frac{\varrho}{\delta}\left(1-e^{-\delta\tau}\right)\int_0^\infty s\nu(\mathrm{d}s).\end{aligned}$$

□

In particular, for the OU-TS process, we have $\mathbb{E}\left[Z_1\right]=\theta\beta^{1-\alpha}\Gamma(1-\alpha)$, where $\Gamma(\cdot)$ is *gamma*

[9] An alternative proof of this result via the characteristic function of stochastic integral for a continuous function proposed by Lukacs (1969) can also be found in Wolfe (1982).

function, i.e. $\Gamma(u) := \int_0^\infty s^{u-1}e^{-s}\mathrm{d}s$. Set $t=0$ and $\tau = T > 0$ in (3.4), we have

$$\mathbb{E}\left[X_T \mid X_0\right] = X_0 e^{-\delta T} + \frac{\varrho}{\delta}\left(1-e^{-\delta T}\right)\theta\beta^{\alpha-1}\Gamma(1-\alpha), \tag{3.5}$$

which will be used later for numerically validating our simulation scheme in Section 5.

3.1 Simulation Algorithm for OU-TS Process

The conditional distribution of OU-TS process is decomposable, due to the *infinite divisibility* property of TS distribution. We choose a "cutting" value to break the OU-TS process into several simple elements such that each one can be exactly simulated. Theorem 3.1 illustrates the exact distributional decomposition of OU-TS process via integral transforms.

Theorem 3.1. *For the OU-TS process X_t of Definition 2.5, the Laplace transform of $X_{t+\tau}$ conditional on X_t can be expressed by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-vX_{t+\tau}} \mid X_t\right] \qquad (3.6)\\
= \;& e^{-vwX_t} \times \exp\left(-\frac{\varrho\theta(1-w^\alpha)}{\alpha\delta}\int_0^\infty (1-e^{-vs})\frac{e^{-\frac{\beta}{w}s}}{s^{\alpha+1}}\mathrm{d}s\right)\\
&\times \exp\left(-\frac{\varrho\theta\beta^\alpha\Gamma(1-\alpha)D_w}{\alpha\delta}\int_0^\infty (1-e^{-vs})\int_1^{\frac{1}{w}}\frac{(\beta u)^{1-\alpha}}{\Gamma(1-\alpha)}s^{(1-\alpha)-1}e^{-\beta us}\frac{u^{\alpha-1}-u^{-1}}{D_w}\mathrm{d}u\mathrm{d}s\right),
\end{aligned}$$

where $w := e^{-\delta\tau}$ and

$$D_w := \frac{1}{\alpha}\left(w^{-\alpha}-1\right) + \ln w. \tag{3.7}$$

Proof. Since the Lévy measure of TS is (2.1), the Laplace exponent is specified by

$$\Phi(u) = \int_0^\infty (1-e^{-uy})\frac{\theta}{y^{\alpha+1}}e^{-\beta y}\mathrm{d}y = \frac{\theta\Gamma(1-\alpha)}{\alpha}\left[(\beta+u)^\alpha - \beta^\alpha\right]. \tag{3.8}$$

Based on Proposition 3.1, we have

$$\mathbb{E}\left[e^{-vX_{t+\tau}} \mid X_t\right] = e^{-vwX_t}\exp\left(-\frac{\varrho}{\delta}\int_{vw}^{v}\frac{1}{u}\int_0^\infty\left(1-e^{-uy}\right)\frac{\theta}{y^{\alpha+1}}e^{-\beta y}\mathrm{d}y\mathrm{d}u\right),$$

where

$$\int_{vw}^{v}\frac{1}{u}\int_0^\infty\left(1-e^{-uy}\right)\theta y^{-\alpha-1}e^{-\beta y}\mathrm{d}y\mathrm{d}u$$

$$= \int_0^\infty \frac{1-e^{-vs}}{s} \int_s^{\frac{s}{w}} \theta y^{-\alpha-1} e^{-\beta y} \mathrm{d}y \mathrm{d}s$$

$$= \int_0^\infty \frac{1-e^{-vs}}{s} \int_s^{\frac{s}{w}} \frac{\theta}{y^{\alpha+1}} \left(e^{-\beta \frac{s}{w}} + e^{-\beta y} - e^{-\beta \frac{s}{w}}\right) \mathrm{d}y \mathrm{d}s$$

$$= \int_0^\infty \frac{1-e^{-vs}}{s} \int_s^{\frac{s}{w}} \frac{\theta}{y^{\alpha+1}} e^{-\beta \frac{s}{w}} \mathrm{d}y \mathrm{d}s \;+\; \int_0^\infty \frac{1-e^{-vs}}{s} \int_s^{\frac{s}{w}} \frac{\theta}{y^{\alpha+1}} \left(e^{-\beta y} - e^{-\beta \frac{s}{w}}\right) \mathrm{d}y \mathrm{d}s. \quad (3.9)$$

Since $y < \frac{s}{w}$, the two terms in (3.9) are both positive for any $y \in [s, \frac{s}{w}]$. In particular, for the first term of (3.9), we have

$$\int_0^\infty \frac{1-e^{-vs}}{s} \int_s^{\frac{s}{w}} \frac{\theta}{y^{\alpha+1}} e^{-\beta \frac{s}{w}} \mathrm{d}y \mathrm{d}s = \frac{\theta(1-w^\alpha)}{\alpha} \int_0^\infty \left(1-e^{-vs}\right) \frac{e^{-\frac{\beta}{w}s}}{s^{\alpha+1}} \mathrm{d}s; \quad (3.10)$$

for the second term of (3.9), we have

$$\begin{aligned}
&\int_0^\infty \left(1-e^{-vs}\right) \frac{1}{s} \int_s^{\frac{s}{w}} \frac{\theta}{y^{\alpha+1}} \left(e^{-\beta y} - e^{-\beta \frac{s}{w}}\right) \mathrm{d}y \mathrm{d}s \\
&= \theta \int_0^\infty \left(1-e^{-vs}\right) \int_1^{\frac{1}{w}} s^{-\alpha} x^{-\alpha-1} \frac{e^{-\beta s x} - e^{-\beta \frac{s}{w}}}{s} \mathrm{d}x \mathrm{d}s \\
&= \theta \int_0^\infty \left(1-e^{-vs}\right) \int_1^{\frac{1}{w}} x^{-\alpha-1} s^{-\alpha} \int_x^{\frac{1}{w}} \beta e^{-\beta s u} \mathrm{d}u \mathrm{d}x \mathrm{d}s \\
&= \theta \int_0^\infty \left(1-e^{-vs}\right) \int_1^{\frac{1}{w}} s^{-\alpha} \beta e^{-\beta s u} \int_1^{u} x^{-\alpha-1} \mathrm{d}x \mathrm{d}u \mathrm{d}s \\
&= \frac{\theta\beta^\alpha}{\alpha} \Gamma(1-\alpha) D_w \int_0^\infty \left(1-e^{-vs}\right) \int_1^{\frac{1}{w}} \frac{(\beta u)^{1-\alpha}}{\Gamma(1-\alpha)} s^{(1-\alpha)-1} e^{-\beta u s} \frac{1}{D_w} \left(u^{\alpha-1} - u^{-1}\right) \mathrm{d}u \mathrm{d}s,
\end{aligned} \quad (3.11)$$

where

$$D_w = \int_1^{\frac{1}{w}} \left(u^{\alpha-1} - u^{-1}\right) \mathrm{d}u = \frac{1}{\alpha}\left(w^{-\alpha} - 1\right) + \ln w.$$

□

The exact distributional decomposition of $X_{t+\tau}$ conditional on X_t can be immediately identified from the representation of Laplace transforms in Theorem 3.1, and hence implies an exact simulation scheme summarised in Algorithm 3.1.

Algorithm 3.1 (Exact Simulation for OU-TS Process)**.** *The distribution of $X_{t+\tau}$ conditional on X_t can be exactly decomposed by*

$$X_{t+\tau} \mid X_t \overset{\mathcal{D}}{=} wX_t + \widetilde{TS} + \sum_{i=1}^{N} S_i, \qquad \tau \in \mathbb{R}^+,$$

where $w := e^{-\delta\tau}$,

- *$\widetilde{TS}$ is a TS r.v. of*

$$\widetilde{TS} \sim \mathsf{TS}\Big(\alpha,\ \frac{\beta}{w},\ \frac{\varrho\theta}{\alpha\delta}(1-w^{\alpha})\Big), \tag{3.12}$$

  which can be exactly simulated by Algorithm D.1;

- *N is a Poisson r.v. of rate $\frac{\varrho\theta}{\alpha\delta}\beta^{\alpha}\Gamma(1-\alpha)D_w$;*

- *$\{S_i\}_{i=1,2,\dots}$ are conditionally independent and conditionally gamma r.v.s of*

$$S_i \mid V \sim \mathsf{Gamma}\big(1-\alpha,\ \beta V\big)^{10}, \tag{3.13}$$

  given that V can be exactly simulated via Algorithm 3.2;

- *$\widetilde{TS}$, N and $\{S_i\}_{i=1,2,\dots}$ are independent of each other.*

Proof. From Theorem 3.1, we can see that, the original Laplace transform has been broken into three parts, and each part is a well-defined Laplace transform. In particular, (3.10) is the Laplace transform of a TS r.v. with Lévy measure

$$\nu(\mathrm{d}s) = \frac{\theta(1-w^{\alpha})}{\alpha} s^{-\alpha-1} e^{-\frac{\beta}{w}s}\mathrm{d}s.$$

(3.11) is the Laplace transform of a compound Poisson r.v. with the jump sizes following a Gamma distribution of shape parameter $(1-\alpha)$ and rate parameter βV. Here, V is a well-defined r.v. with density function

$$f_V(u) = \frac{1}{D_w}\left(u^{\alpha-1} - u^{-1}\right), \qquad u \in \left[1, \frac{1}{w}\right]. \tag{3.14}$$

□

Note that, there are several different algorithms for generating TS r.v.s in the literature, such as *simple stable rejection* (SSR) (Algorithm D.1), *double rejection* (Devroye, 2009), *fast rejection* (Hofert, 2011), *backward recursive* (Dassios et al., 2018) and *two-dimensional single rejection* (Qu et al., 2019b). The choice for the fundamental TS generator is indeed not our main focus of this

[10] $\mathsf{Gamma}(1-\alpha, \beta V)$ means a Gamma distribution with *shape parameter* $(1-\alpha)$ and *rate parameter* βV.

paper. Here, we directly adopt the SSR scheme just for the purpose of illustration. It is the simplest and most widely-used algorithm for exact simulation, and it works more efficiently when α is larger and β, θ are smaller.

Algorithm 3.2 (A/R Scheme for V)**.** *The r.v. V, defined by its density (3.14), can be exactly simulated via the following A/R procedure:*

1. *Generate a candidate r.v.*

$$E_e \stackrel{\mathcal{D}}{=} \left(1 + \sqrt{\alpha C_w U^{(1)}}\right)^{\frac{2}{\alpha}}, \qquad U^{(1)} \sim \mathcal{U}[0,1], \tag{3.15}$$

where

$$C_w := \frac{1}{\alpha}\left(w^{-\frac{\alpha}{2}} - 1\right)^2.$$

2. *Generate a standard uniform r.v. $U^{(2)} \sim \mathcal{U}[0,1]$.*

3. *If*

$$U^{(2)} \le \frac{1}{2}\frac{E_e^{\alpha} - 1}{E_e^{\alpha} - E_e^{\frac{\alpha}{2}}},$$

then, accept this candidate by setting $V = E_e$; otherwise, reject this candidate and go back to Step 1.

Proof. Based on the density function (3.14), it is easy to derive the CDF of V by

$$F_V(u) := \Pr\{V \le u\} = \frac{1}{D_w}\left[\frac{1}{\alpha}\left(u^{\alpha} - 1\right) - \ln u\right], \qquad u \in \left[1, \frac{1}{w}\right].$$

However, its inverse function has no explicit form, and the explicit inverse transform is not available. Then, it is natural to consider the A/R scheme for exact simulation. For a detailed introduction to A/R scheme, see Glasserman (2003) and Asmussen and Glynn (2007). We choose an envelop r.v. E_e defined by its density function

$$g_e(u) = \frac{1}{C_w}\left(u^{\alpha-1} - u^{-\frac{\alpha}{2}-1}\right), \qquad u \in \left[1, \frac{1}{w}\right].$$

We can derive its CDF

$$G_e(u) = \frac{1}{\alpha C_w}\left(u^{\frac{\alpha}{2}} - 1\right)^2, \qquad u \in \left[1, \frac{1}{w}\right],$$

which can be inverted explicitly by

$$G_e^{-1}(x) = \left(1 + \sqrt{\alpha C_w x}\right)^{\frac{2}{\alpha}}, \qquad x \in [0,1].$$

Hence, E_e can be exactly simulated by explicit inverse transform (3.15). Obviously, $\frac{u^\alpha-1}{u^\alpha-u^{\frac{\alpha}{2}}}$ is a strictly decreasing function of $u \in \left[1, \frac{1}{w}\right]$. By L'Hôpital's rule, we can find its upper bound

$$\lim_{u\downarrow 1} \frac{u^\alpha - 1}{u^\alpha - u^{\frac{\alpha}{2}}} = 2.$$

Then, we have

$$\frac{f_V(u)}{g_e(u)} = \frac{C_w}{D_w}\frac{u^\alpha - 1}{u^\alpha - u^{\frac{\alpha}{2}}} \leq \frac{C_w}{D_w}\lim_{u\downarrow 1}\frac{u^\alpha - 1}{u^\alpha - u^{\frac{\alpha}{2}}} = 2\frac{C_w}{D_w} := \bar{c}_w, \qquad \forall u \in \left[1, \frac{1}{w}\right]. \tag{3.16}$$

□

Remark 3.1. Note that, $\bar{c}_w$ of (3.16) is the expected number of candidates generated until one is accepted, hence, $1/\bar{c}_w$ is the *acceptance probability*, i.e. the probability of acceptance on each attempt. Obviously, it is preferable for us to have $\bar{c}_w$ close to 1. In fact, our Algorithm 3.2 is pretty efficient, as we can prove in Appendix B that, the acceptance probability is guaranteed to be above 50%. More precisely, we have $\bar{c}_w \in (1, 2)$ and

$$\begin{cases} \bar{c}_w \to 1, & \text{when} \quad w \to 1, \\ \bar{c}_w \to 2, & \text{when} \quad w \to 0. \end{cases} \tag{3.17}$$

3.2 Algorithm for OU-IG Process

We provide a tailored algorithm for the special case of OU-IG process. The enhancement is mainly achieved by replacing the TS r.v. of (3.12) in Algorithm 3.1 by an IG r.v.[11], and it is well known that IG r.v.s can be very efficiently simulated without A/R using the classical algorithm developed by Michael et al. (1976).

Algorithm 3.3 (Algorithm for OU-IG Process)**.** *For the OU process X_t with Lévy subordinator $Z_t \sim \mathit{IG}\left(\frac{t}{c}, t^2\right)$, $c \in \mathbb{R}^+$, we can exactly simulate $X_{T+\tau}$ conditional on X_t via modifying Algorithm 3.1 by*

1. *setting $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}c^2$ and $\theta = \frac{1}{\sqrt{2\pi}}$;*
2. *replacing the general TS r.v. (3.12) by the IG r.v.*

$$\widetilde{IG} \sim \mathit{IG}\left(\mu_{IG} = \frac{2\varrho}{\delta c}\left(\sqrt{w} - w\right), \ \lambda_{IG} = \left[\frac{2\varrho}{\delta}\left(1 - \sqrt{w}\right)\right]^2\right),$$

where μ_{IG} is the mean parameter and λ_{IG} is the rate parameter.

[11]Mathematical properties of IG distributions are well documented in Chhikara and Folks (1989).

Proof. For an IG r.v. $IG \sim \mathsf{IG}\left(\frac{1}{c}, 1\right)$, the Lévy measure is given by

$$\nu(\mathrm{d}s) = \frac{1}{\sqrt{2\pi s^3}} e^{-\frac{c^2}{2}s} \mathrm{d}s,$$

then, we have

$$IG \;\sim\; \mathsf{TS}\left(\frac{1}{2},\, \frac{c^2}{2},\, \frac{1}{\sqrt{2\pi}}\right).$$

If we set $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}c^2$ and $\theta = \frac{1}{\sqrt{2\pi}}$, then, it recovers the special case of OU-IG process. In particular, (3.12) turns to be

$$\mathsf{TS}\left(\frac{1}{2},\, \frac{c^2}{2w},\, \frac{2\varrho}{\delta\sqrt{2\pi}}\left(1 - \sqrt{w}\right)\right),$$

with the associated Laplace exponent

$$\int_0^\infty \left(1 - e^{-vs}\right) \frac{\frac{2\varrho\left(1-\sqrt{w}\right)}{\delta}}{\sqrt{2\pi s^3}} e^{-\frac{\left(\frac{c}{\sqrt{w}}\right)^2}{2}s} \mathrm{d}s.$$

Note that, in general, the Laplace exponent of $\mathsf{IG}\left(\mu_{\mathsf{IG}}, \lambda_{\mathsf{IG}}\right)$ is given by

$$\int_0^\infty \left(1 - e^{-vs}\right) \frac{\sqrt{\lambda_{\mathsf{IG}}}}{\sqrt{2\pi s^3}} e^{-\frac{\left(\frac{\sqrt{\lambda_{\mathsf{IG}}}}{\mu_{\mathsf{IG}}}\right)^2}{2}s} \mathrm{d}s, \qquad \mu_{\mathsf{IG}}, \lambda_{\mathsf{IG}} \in \mathbb{R}^+. \tag{3.18}$$

Under the parameter setting of $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}c^2$ and $\theta = \frac{1}{\sqrt{2\pi}}$, the general TS r.v. in (3.12) can be replaced by an IG r.v. as

$$\mathsf{TS}\left(\frac{1}{2},\, \frac{c^2}{2w},\, \frac{2\varrho}{\delta\sqrt{2\pi}}(1 - \sqrt{w})\right) \;\stackrel{\mathcal{D}}{=}\; \mathsf{IG}\left(\frac{2\varrho}{\delta c}\left(\sqrt{w} - w\right),\, \left[\frac{2\varrho}{\delta}\left(1 - \sqrt{w}\right)\right]^2\right).$$

□

4 Exact Simulation of TS-OU Process

In this section, we extend our approach developed in Section 3 to the TS-OU process Y_t of Definition 2.5. Analogue to Proposition 3.1 for the OU-TS process X_t, the conditional Laplace transform of TS-OU process Y_t is given by

$$\mathbb{E}\left[e^{-vY_{t+\tau}} \mid Y_t\right] = e^{-vwY_t} \times \exp\left(-\int_{vw}^{v} \frac{\Phi(u)}{u} \mathrm{d}u\right). \tag{4.1}$$

According to the general theory of OU processes (Barndorff-Nielsen and Shephard, 2001b, p.173), the stationary TS-OU process Y_t has a TS marginal law with Lévy measure $\nu_{\mathsf{TS}}(\mathrm{d}y)$ specified in (2.1). The one-dimensional distributions of the process is self-decomposable, and the Laplace exponent $\Phi(u)$ in (4.1) is of the form

$$\Phi(u) = \int_0^\infty \left(1 - e^{-ur}\right) \nu_{\mathsf{BDLP}}(\mathrm{d}r),$$

with

$$\nu_{\mathsf{BDLP}}(\mathrm{d}r) = -\nu_{\mathsf{TS}}(\mathrm{d}r) - r\frac{\partial}{\partial r}\nu_{\mathsf{TS}}(\mathrm{d}r) = \left(\alpha r^{-1} + \beta\right)\theta r^{-\alpha} e^{-\beta r}\mathrm{d}r, \tag{4.2}$$

where $\nu_{\mathsf{BDLP}}(\mathrm{d}r)$ is the Lévy measure of BDLP R_t in (2.4).

Given the Lévy measure of R_t, we provide the conditional expectation of Y_t as below.

Proposition 4.1. *The expectation of $Y_{t+\tau}$ conditional on Y_t is given by*

$$\mathbb{E}\left[Y_{t+\tau} \mid Y_t\right] = wY_t + (1-w)\theta\beta^{\alpha-1}\Gamma(1-\alpha), \tag{4.3}$$

$$\lim_{\tau\to\infty} \mathbb{E}\left[Y_{t+\tau} \mid Y_t\right] = \theta\beta^{1-\alpha}\Gamma(1-\alpha). \tag{4.4}$$

Proof. Based on (4.2), (4.3) can be immediately derived from (3.4) by

$$\int_0^\infty s\nu_{\mathsf{BDLP}}(\mathrm{d}s) = \alpha\theta\beta^{\alpha-1}\Gamma(1-\alpha) + \theta\beta^{\alpha-1}\Gamma(2-\alpha).$$

Take the limit of $\tau \to \infty$ for (4.3), then, $w \to 0$ and we have (4.4). □

In fact, we can see from (4.2) that, the Lévy measure of the BDLP of TS-OU process is the sum of Lévy measures of a TS process and a compound Poisson process. Hence, based on the exact decomposition of OU-TS process in Algorithm 3.1, the distribution of TS-OU process at a given time is equivalent to the sum of a TS r.v. and two compound Poisson r.v.s. as specified by Algorithm 4.1, with the proof outlined in Appendix C.

Algorithm 4.1 (Exact Simulation for TS-OU Process)**.** *The distribution of $Y_{T+\tau}$ conditional on Y_t can be exactly decomposed by*

$$Y_{t+\tau} \mid Y_t \overset{\mathcal{D}}{=} wY_t + \breve{\widetilde{TS}} + \sum_{i=1}^{\tilde{N}} S_i + \sum_{j=1}^{\breve{N}} \breve{S}_j, \qquad \tau \in \mathbb{R}^+, \tag{4.5}$$

where $w := e^{-\delta\tau}$,

- $\widetilde{\breve{TS}}$ *is a TS r.v. of*

$$\widetilde{\breve{TS}} \sim \mathsf{TS}\Big(\alpha,\ \frac{\beta}{w},\ \theta\big(1-w^{\alpha}\big)\Big); \tag{4.6}$$

- $\tilde{N}$ *is a Poisson r.v. of rate* $\theta\beta^{\alpha}\Gamma(1-\alpha)D_w$*, and the jump sizes* $\{S_i\}_{i=1,2,\dots}$ *are i.i.d. and exactly the same as (3.13);*

- $\breve{N}$ *is another Poisson r.v. of rate* $\theta\beta^{\alpha}\Gamma(1-\alpha)\ln\left(\frac{1}{w}\right)$*, and the jump sizes* $\{\breve{S}_j\}_{j=1,2,\dots}$ *are conditionally independent and conditionally gamma r.v.s of*

$$\breve{S}_j \mid V^* \sim \mathit{Gamma}\big(1-\alpha,\ \beta V^*\big),$$

given that

$$V^* \stackrel{\mathcal{D}}{=} \exp\left(\delta\tau U^{(3)}\right), \qquad U^{(3)} \sim \mathit{U}[0,1]; \tag{4.7}$$

- $\widetilde{\breve{TS}}, \tilde{N}, \breve{N}$*,* $\{S_i\}_{i=1,2,\dots}$ *and* $\{\breve{S}_j\}_{j=1,2,\dots}$ *are independent of each other.*

Accordingly, we also offer a tailored scheme for the IG-OU process:

Algorithm 4.2 (Algorithm for IG-OU Process)**.** *For the OU process* Y_t *with an* $\mathit{IG}\left(\frac{1}{c},1\right)$ *marginal law, we can exactly simulate* $Y_{T+\tau}$ *conditional on* Y_t *via modifying Algorithm 4.1 by*

1. *setting* $\alpha=\frac{1}{2}$*,* $\beta=\frac{1}{2}c^2$ *and* $\theta=\frac{1}{\sqrt{2\pi}}$ *in Algorithm 4.1;*

2. *replacing the general TS r.v. (4.6) by the IG r.v.*

$$\widetilde{\breve{IG}} \sim \mathit{IG}\Big(\mu_{\mathit{IG}}=\frac{1}{c}\left(\sqrt{w}-w\right),\ \lambda_{\mathit{IG}}=\left(1-\sqrt{w}\right)^2\Big).$$

Proof. When $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}c^2$ and $\theta=\frac{1}{\sqrt{2\pi}}$, the Laplace exponent of (4.6) is

$$\int_0^{\infty}\left(1-e^{-vs}\right)\frac{1-\sqrt{w}}{\sqrt{2\pi s^3}}e^{-\frac{\left(\frac{c}{\sqrt{w}}\right)^2}{2}s}\mathrm{d}s.$$

Comparing with (3.18), we have

$$\mathsf{TS}\left(\frac{1}{2},\ \frac{c^2}{2w},\ \theta(1-\sqrt{w})\right) \stackrel{\mathcal{D}}{=} \mathsf{IG}\left(\frac{1}{c}\left(\sqrt{w}-w\right),\ \left(1-\sqrt{w}\right)^2\right).$$

□

5 Numerical Examples

In this section, we illustrate the performance and effectiveness of our exact simulation schemes through extensive numerical experiments. We have implemented the exact simulation scheme for four cases, OU-TS/OU-IG and TS-OU/IG-OU processes within the fixed time period $[0, T]$, respectively. They are mainly implemented on a desktop with Intel Core i7-6700 CPU@3.40GHz processor, 24.00GB RAM, Windows 10 Professional, 64-bit Operating System. The algorithms are coded and performed in MatLab (R2012a), and the computing time is measured by the elapsed CPU time in seconds. We use our algorithms to simulate paths of X_t and Y_t from time 0 to T, and numerically validate and test them based on the true values of means (3.5) and (4.3) at the terminal time T for OU-TS/OU-IG processes and TS-OU/IG-OU processes, respectively. The associated errors from the true values are reported by three standard measures:

1. *difference* = estimated value − true value;
2. *relative error (error %)* = $\frac{\text{estimated value} - \text{true value}}{\text{true value}}$;
3. *root mean square error* RMSE = $\sqrt{\text{bias}^2 + \text{SE}^2}$, where the SE is the standard error of the simulation output, and the bias is the difference between the expectation of the estimator and the associated true (theoretical) value. For our exact simulation schemes, the bias is zero.

We set the parameters $(\delta, \varrho; \alpha, \beta, \theta; X_0 = Y_0; T) = (0.2, 1.0; 0.25, 0.5, 0.25; 10.0; 5.0)$ for OU-TS/TS-OU processes and $(\delta, \varrho; c; X_0 = Y_0; T) = (0.2, 1.0; 1.0; 10.0; 5.0)$ for OU-IG/IG-OU processes, and experiment with different numbers of equally-spaced discretisation steps within the period $[0, T]$, i.e. $n_\tau := T/\tau$. Of course, all of our algorithms can be directly applied to the irregularly-spaced time points which may be more useful in practice[12], and the equally-spaced cases here just serve for illustration purpose.

Simulated paths of OU-TS/OU-IG processes have been presented earlier in Figure 1 and Figure 2, respectively. Numerical verification for the four cases, OU-TS/TS-OU, OU-IG/IG-OU, are reported in Table 1. The efficiency enhancement for simulating OU-IG/IG-OU processes using the tailored schemes (Algorithm 3.3, 4.2) against the associated general schemes (Algorithm 3.1, 4.1) can be clearly observed through numerical results reported in Table 2. Overall, from these numerical results reported in this section, it is evident that each algorithm developed in this paper can achieve a very high level of accuracy as well as efficiency.

[12] The data in practice, such as trade transactions from market microstructure, are often observed at irregularly-spaced time points, see Engle and Russell (1998).

Table 1: Comparison between the true means and the associated simulation results of our exact simulation schemes for $1,024,000$ replications, based on the parameter setting $(\delta, \varrho; \alpha, \beta, \theta; X_0 = Y_0; T) = (0.2, 1.0; 0.25, 0.5, 0.25; 10.0; 5.0)$ for OU-TS/TS-OU processes and $(\delta, \varrho; c; X_0 = Y_0; T) = (0.2, 1.0; 1.0; 10.0; 5.0)$ for OU-IG/IG-OU processes with $n_\tau = 1, 2, 5, 10$, respectively.

n_τ	True	Estimation	Difference	Error%	Time	True	Estimation	Difference	Error%	Time
			OU-TS					**TS-OU**		
1	5.3072	5.3090	0.0018	0.03%	2,564.31	4.0045	4.0059	0.0015	0.04%	105.42
2	5.3072	5.3100	0.0027	0.05%	332.94	4.0045	4.0055	0.0010	0.03%	202.75
5	5.3072	5.3070	-0.0002	-0.00%	383.53	4.0045	4.0043	-0.0002	-0.00%	492.27
10	5.3072	5.3098	0.0026	0.05%	662.92	4.0045	4.0026	-0.0019	-0.05%	995.53
			OU-IG					**IG-OU**		
1	6.8394	6.8344	-0.0017	-0.02%	54.58	4.3109	4.3110	0.0000	0.00%	90.56
2	6.8394	6.8413	0.0019	0.03%	102.08	4.3109	4.3100	-0.0009	-0.02%	179.05
5	6.8394	6.8385	-0.0009	-0.01%	240.36	4.3109	4.3101	-0.0008	-0.02%	430.52
10	6.8394	6.8401	0.0007	0.01%	474.41	4.3109	4.3120	0.0011	0.03%	857.55

Table 2: Comparison between the true means and the associated simulation results of our exact simulation schemes for $1,024,000$ replications, based on the parameter setting $(\delta, \varrho; c; X_0 = Y_0; T) = (0.2, 1.0; 1.0; 10.0; 5.0)$ for OU-IG/IG-OU processes with $n_\tau = 1, 2, 5, 10$, respectively.

n_τ	True	Estimation	Difference	Error%	Time	True	Estimation	Difference	Error%	Time
			OU-IG	**Algo.** 3.1				**OU-IG**	**Algo.** 3.3	
1	6.8394	6.8402	0.0008	0.01%	4,487.67	6.8394	6.8417	0.0022	0.03%	58.25
2	6.8394	6.8451	0.0057	0.08%	353.39	6.8394	6.8374	-0.0020	-0.03%	100.39
5	6.8394	6.8371	-0.0023	-0.03%	374.17	6.8394	6.8400	0.0006	0.01%	242.95
10	6.8394	6.8401	0.0007	0.01%	660.09	6.8394	6.8369	-0.0025	-0.04%	478.70
			IG-OU	**Algo.** 4.1				**IG-OU**	**Algo.** 4.2	
1	4.3109	4.3103	0.0006	0.01%	120.68	4.3109	4.3123	0.0014	0.03%	93.81
2	4.3109	4.3109	0.0000	0.00%	216.94	4.3109	4.3106	-0.0030	-0.01%	188.57
5	4.3109	4.3107	-0.0002	-0.00%	501.08	4.3109	4.3098	-0.0011	-0.03%	434.11
10	4.3109	4.3108	-0.0001	-0.00%	1009.34	4.3109	4.3107	-0.0002	-0.00%	862.09

Conditional mean provides us the easiest way to test and verify newly-developed algorithms as the its true value can be much easier to be derived in a simple analytic form in most circumstances as given by (3.5) and (4.3). In fact, our tests and validations based on the means have been carried out using a vast number of various different parameter sets. The results based on other parameter sets show very similar levels of accuracy and efficiency, so we do not present all of them here in order to make our presentation more concise. Of course, other higher moments, values of probabilities or density functions can be also convenient to be used for testing as long as they have analytic forms so we have already known the true values precisely. For example, the conditional Laplace transforms that we have derived in Proposition 3.1 and Theorem 3.1 could be used for testing as well. But we have to first discretize and truncate the infinite integrals in the Laplace transforms, which would introduce estimation errors. Basically, means can be tested by a sufficient number of different parameter choices, the aim of testing and verifying our algorithms numerically can be achieved very similarly based on the simple mean and more complicated moments, so we choose means for simplicity and it also avoid additional estimation errors.

Alternatively, a widely used and simpler approach for simulating stochastic processes is the Euler time-discretisation scheme. However, it is well known that this scheme is not exact and it introduces biases for the estimators. For example, the continuous-time OU-TS process X_t following the SDE (2.2) can be approximated by $\hat{X}_t$ via the Euler discretisation scheme,

$$\hat{X}_{t+h} - \hat{X}_t = -\delta \hat{X}_t h + \varrho(Z_{t+h} - Z_t),$$

or,

$$\hat{X}_{t+h} = (1 - \delta h)\hat{X}_t + \varrho(Z_{t+h} - Z_t), \qquad h = T/n_g,$$

where $n_g \in \mathbb{N}^+$ is the total number of grids within the time interval $[0, T]$, and $(Z_{t+h} - Z_t) \sim \mathsf{TS}\big(\alpha, \beta, \theta h\big)$. According to the principle of optimal allocation of computation budget proposed by Duffie and Glynn (1995), the number of time-discretisation grids is set equal to the square root of the number of sample paths, i.e., $n_g = \sqrt{n_p}$ where n_p is the total number of sample paths. The comparison results between Algorithm 3.1 and Euler discretisation scheme for the OU-TS process, based on the parameter setting $(\delta, \varrho; \alpha, \beta, \theta; X_0) = (0.2, 1.0; 0.25, 0.5, 0.25; 10.0)$ and $T = 1, 5$ are reported in Table 3 with convergence comparison in Figure 3. Obviously, our algorithm outperforms the discretisation scheme in terms of RMSE and CPU time. In conclusion, our exact simulation scheme is far more efficient and accurate than the Euler discretisation scheme.

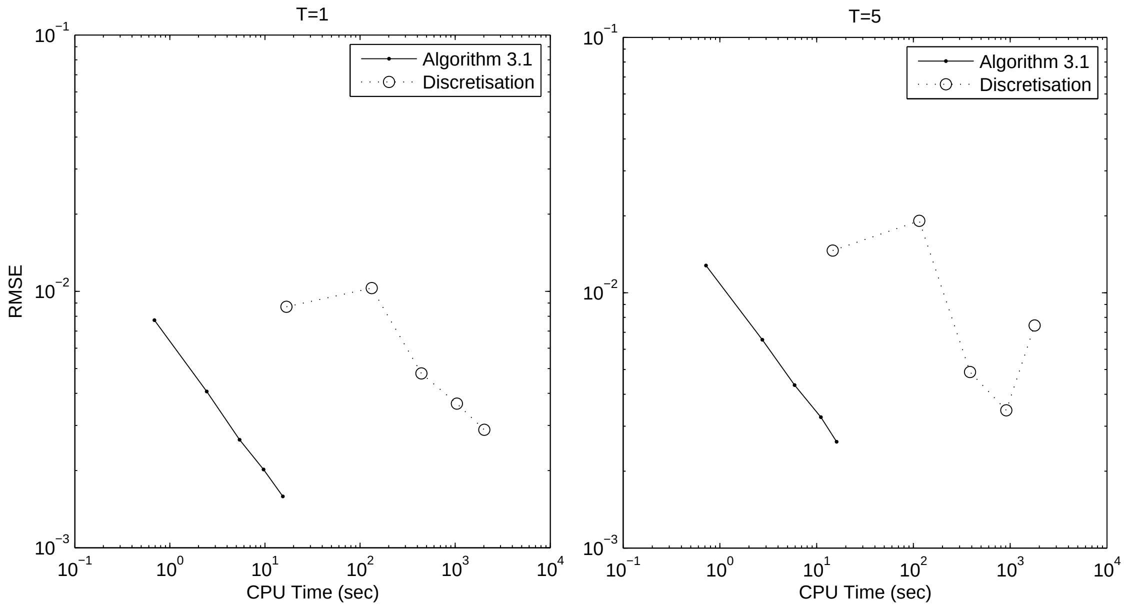

Figure 3: Convergence comparison between Algorithm 3.1 and discretisation scheme for OU-TS process, based on the parameter setting $(\delta, \varrho; \alpha, \beta, \theta; X_0) = (0.2, 1.0; 0.25, 0.5, 0.25; 10.0)$ and $T = 1, 5$, respectively, with the associated detailed numerical results reported in Table 3

Table 3: Comparison between the true means and the associated simulation results for Algorithm 3.1 and discretisation scheme for OU-TS process, based on the parameter setting $(\delta, \varrho; \alpha, \beta, \theta; X_0) = (0.2, 1.0; 0.25, 0.5, 0.25; 10.0)$ and $T = 1, 5$, respectively, with the associated plots provided in Figure 3

Paths n_p	True	Estimation	Difference	Error%	RMSE	Time	Grids n_g	Estimation	Difference	Error%	RMSE	Time
		Algo. 3.1	$T=1$						**Discretisation**	$T=1$		
10,000	8.6543	8.6443	-0.0099	-0.11%	0.0077	1	100	8.6574	0.0031	0.04%	0.009	17
40,000	8.6543	8.6600	0.0058	0.07%	0.0041	2	200	8.6447	-0.0096	-0.11%	0.010	133
90,000	8.6543	8.6531	-0.0011	-0.01%	0.0026	5	300	8.6503	-0.0040	-0.05%	0.005	442
160,000	8.6543	8.6560	0.0017	0.02%	0.0020	10	400	8.6573	0.0031	0.04%	0.004	1,045
250,000	8.6543	8.6554	0.0011	0.01%	0.0016	15	500	8.6567	0.0024	0.03%	0.003	2,026
		Algo. 3.1	$T=5$						**Discretisation**	$T=5$		
10,000	5.3072	5.3112	0.00	0.08%	0.0128	1	100	5.3003	-0.0069	-0.13%	0.015	15
40,000	5.3072	5.3126	0.01	0.10%	0.0065	3	200	5.2892	-0.0180	-0.34%	0.019	115
90,000	5.3072	5.3124	0.01	0.10%	0.0043	6	300	5.3050	-0.0023	-0.04%	0.005	384
160,000	5.3072	5.3104	0.00	0.06%	0.0033	11	400	5.3084	0.0012	0.02%	0.003	912
250,000	5.3072	5.3099	0.00	0.05%	0.0026	16	500	5.3002	-0.0070	-0.13%	0.007	1,793

6 Extensions

Based on the results proposed in Section 3, one could further decompose certain types of Lévy-driven OU processes with the BDLPs beyond tempered stable processes. The details are provided in Proposition 6.1.

Proposition 6.1. *Let X_t being a non-Gaussian OU process of Definition 2.3 and the Lévy measure of the BDLP Z_t is of the form*

$$\nu(\mathrm{d}y) = \frac{\theta h(y)}{y^{\alpha+1}}\mathrm{d}y, \tag{6.1}$$

with ν satisfies the following condition

$$\int_0^\infty \min\{1, y\}\nu(\mathrm{d}y) < \infty.$$

For the following two cases, the distribution of $X_{t+\tau}$ conditional on X_t can be exactly decomposed.

CASE I: If

$$\hat{D}_w := \int_0^\infty \int_1^{\frac{1}{w}} \frac{h(su) - h(\frac{s}{w})}{s^{\alpha+1}u^{\alpha+1}}\mathrm{d}u\mathrm{d}s < \infty, \tag{6.2}$$

and $w := e^{-\delta\tau}$, then, $X_{t+\tau}|X_t$ can be expressed as

$$X_{t+\tau}|X_t \stackrel{\mathcal{D}}{=} wX_t + w\hat{Z} + \sum_{i=1}^{\hat{N}} \hat{S}_i, \quad \tau \in \mathbb{R}^+,$$

where

- *$\hat{Z}$ is a Lévy subordinator with Lévy measure*

$$\nu(\mathrm{d}s) = \frac{\varrho\theta(1/w^\alpha - 1)}{\alpha\delta}\frac{h(s)}{s^{\alpha+1}}\mathrm{d}s;$$

– *$\hat{N}$ is a Poisson r.v. of rate $\frac{\varrho\theta\hat{D}_w}{\delta}$;*

– *$\{\hat{S}_i\}_{i=1,2,...}$ are i.i.d r.v.s with density*

$$f_{\hat{S}}(s) = \frac{1}{\hat{D}_w}\int_1^{\frac{1}{w}} \frac{h(su) - h(\frac{s}{w})}{s^{\alpha+1}u^{\alpha+1}}\mathrm{d}u, \qquad s \in (0,\infty);$$

– *$\hat{Z}$, $\hat{N}$, and $\{\hat{S}_i\}_{i=1,2,...}$ are independent of each other.*

CASE II: If $h(\cdot)$ in (6.1) satisfies the conditions

$$h(y) \geq e^{-\beta y}, \qquad \forall y \in (0,\infty), \tag{6.3}$$

and

$$\bar{D}_w \;:=\; \int_0^\infty \int_1^{\frac{1}{w}} \frac{h(su) - e^{-\beta su}}{s^{\alpha+1}u^{\alpha+1}}\mathrm{d}u\mathrm{d}s \;<\; \infty, \tag{6.4}$$

and $w := e^{-\delta\tau}$, then, $X_{t+\tau}|X_t$ can be expressed as

$$X_{t+\tau} \mid X_t \overset{\mathcal{D}}{=} wX_t + \widetilde{TS} + \sum_{i=1}^{N} S_i + \sum_{i=1}^{\bar{N}} \bar{S}_i, \qquad \tau \in \mathbb{R}^+,$$

where

– *$\widetilde{TS}$, N, $\{S_i\}_{i=1,2,...}$ are suggested in Algorithm 3.1;*

– *$\bar{N}$ is a Poisson r.v. of rate $\frac{\varrho\theta\bar{D}_w}{\delta}$;*

– *$\{\bar{S}_i\}_{i=1,2,...}$ are i.i.d r.v.s with density*

$$f_{\bar{S}}(s) = \frac{1}{\bar{D}_w}\int_1^{\frac{1}{w}} \frac{h(su) - e^{-\beta su}}{s^{\alpha+1}u^{\alpha+1}}\mathrm{d}u, \qquad s \in (0,\infty);$$

– *$\widetilde{TS}$, N, $\bar{N}$, $\{S_i\}_{i=1,2,...}$ and $\{\bar{S}_i\}_{i=1,2,...}$ are independent of each other.*

Proof. According to Proposition 3.1, the Laplace transform of $X_{t+\tau}$ conditional on X_t is given by

$$\mathbb{E}\left[e^{-vX_{t+\tau}}|X_t\right] \;=\; e^{-vwX_t}\exp\left(-\frac{\varrho\theta}{\delta}\int_0^\infty \left(1-e^{-vs}\right)\frac{1}{s}\int_s^{\frac{s}{w}}\frac{h(y)}{y^{\alpha+1}}\mathrm{d}y\mathrm{d}s\right).$$

For *CASE I*, we have

$$\mathbb{E}\left[e^{-vX_{t+\tau}}|X_t\right]$$

$$
\begin{aligned}
&= e^{-vwX_t}\exp\left(-\frac{\varrho\theta}{\delta}\int_0^\infty\left(1-e^{-vs}\right)\frac{1}{s}\int_s^{\frac{s}{w}}\frac{h(\frac{s}{w})}{y^{\alpha+1}}\mathrm{d}y\mathrm{d}s\right)\exp\left(-\frac{\varrho\theta}{\delta}\int_0^\infty\left(1-e^{-vs}\right)\frac{1}{s}\int_s^{\frac{s}{w}}\frac{h(y)-h(\frac{s}{w})}{y^{\alpha+1}}\mathrm{d}y\mathrm{d}s\right)\\
&= e^{-vwX_t}\exp\left(-\frac{\varrho\theta(1/w^\alpha-1)}{\alpha\delta}\int_0^\infty\left(1-e^{-vws}\right)\frac{h(s)}{s^{\alpha+1}}\mathrm{d}s\right)\\
&\quad\times\exp\left(-\frac{\varrho\theta\hat{D}_w}{\delta}\int_0^\infty\left(1-e^{-vs}\right)\int_1^{\frac{1}{w}}\frac{h(su)-h(\frac{s}{w})}{\hat{D}_w s^{\alpha+1}u^{\alpha+1}}\mathrm{d}u\mathrm{d}s\right),
\end{aligned}
\tag{6.5}
$$

where $\hat{D}_w$ is specified in (6.2). We can see that, $X_{t+\tau}|X_t$ can be expressed as the sum of a deterministic constant, a Lévy subordinator with measure proportional to (6.1), and a compound Poisson r.v. under the condition $\hat{D}_w < \infty$.

For *CASE II*, we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-vX_{t+\tau}}|X_t\right]\\
&= e^{-vwX_t}\exp\left(-\frac{\varrho\theta}{\delta}\int_0^\infty\left(1-e^{-vs}\right)\frac{1}{s}\int_s^{\frac{s}{w}}\frac{e^{-\beta y}}{y^{\alpha+1}}\mathrm{d}y\mathrm{d}s\right)\exp\left(-\frac{\varrho\theta}{\delta}\int_0^\infty\left(1-e^{-vs}\right)\frac{1}{s}\int_s^{\frac{s}{w}}\frac{h(y)-e^{-\beta y}}{y^{\alpha+1}}\mathrm{d}y\mathrm{d}s\right)\\
&= e^{-vwX_t}\exp\left(-\frac{\varrho\theta(1-w^\alpha)}{\alpha\delta}\int_0^\infty(1-e^{-vs})\frac{e^{-\frac{\beta}{w}s}}{s^{\alpha+1}}\mathrm{d}s\right)\\
&\quad\times\exp\left(-\frac{\varrho\theta\beta^\alpha\Gamma(1-\alpha)D_w}{\alpha\delta}\int_0^\infty(1-e^{-vs})\int_1^{\frac{1}{w}}\frac{(\beta u)^{1-\alpha}}{\Gamma(1-\alpha)}s^{(1-\alpha)-1}e^{-\beta us}\frac{u^{\alpha-1}-u^{-1}}{D_w}\mathrm{d}u\mathrm{d}s\right)\\
&\quad\times\exp\left(-\frac{\varrho\theta\bar{D}_w}{\delta}\int_0^\infty\left(1-e^{-vs}\right)\int_1^{\frac{1}{w}}\frac{h(su)-e^{-\beta su}}{\bar{D}_w s^{\alpha+1}u^{\alpha+1}}\mathrm{d}u\mathrm{d}s\right),
\end{aligned}
\tag{6.6}
$$

where D_w is given by (3.7) and $\bar{D}_w$ is given in (6.4). We can see that, based on the exact decomposition of OU-TS process in Algorithm 3.1, the distribution of this new Lévy-driven OU process at a given time is equivalent to the sum of a TS r.v. and two compound Poisson r.v.s under the conditions $h(y) \geq e^{-\beta y}$ for all $y \in (0,\infty)$ and $\bar{D}_w < \infty$.

□

Remark 6.1. The availability to exact simulate the Lévy-driven OU process X_t suggested in Proposition 6.1 depends on the ability to sample $\hat{S}_i$ from density $f_{\hat{S}}$ and $\bar{S}_i$ from density $f_{\bar{S}}$. Since we do not have a general scheme to exactly sample $\hat{S}_i$ and $\bar{S}_i$, therefore the first task is to develop simulation schemes to sample these random variables when the function $h(\cdot)$ is specified. After that, given the specified $h(\cdot)$, if there exists an available simulation algorithm to generate Z_t, then one could follow *CASE I* in Proposition 6.1 to simulate X_t by sampling the corresponding Lévy suboridinator $\hat{Z}$ and the compound Poisson r.v. $\sum_{i=1}^{\hat{N}} \hat{S}_i$, respectively. If the simulation scheme for Z_t is not available but the specified $h(\cdot)$ satisfies the condition in (6.3), then one could follow *CASE II* in Proposition 6.1 to simulate X_t by sampling a tempered stable r.v. $\widetilde{TS}$ and two compound Poisson r.v.s $\sum_{i=1}^{N} S_i$ and $\sum_{i=1}^{\bar{N}} \bar{S}_i$, respectively.

In general, when the function $h(\cdot)$ is specified, it is highly likely that the simulation scheme for the corresponding Lévy subordinator Z_t is not available, therefore, one has to consider *CASE II*. However, there are some rare cases when the simulation schemes for Z_t with Lévy measure in (6.1) are indeed available. One typical example is an indicator function $h(y) = \mathbf{1}_{\{0<y<b\}}$ with b being a positive constant. The associated Lévy subordinator Z_t, namely *truncated stable process*, can be simulated via the exact simulation scheme proposed by Dassios et al. (2020). Hence, by ensuring the corresponding $\hat{D}_w < \infty$, one could use the decomposition scheme above to sample this truncated stable-driven OU process. The details of the simulation procedures for this truncated stable-driven OU process are provided in Dassios et al. (2020, p.17:22).

7 Conclusion

The main contribution of this paper is providing the first exact simulation algorithm to generate OU-TS processes. This approach can be extended to generate TS-OU processes and beyond. Besides, it can also be used to exact simulation certain types of two-sided Lévy-driven OU processes by taking a difference of two Lévy-driven OU processes. Our algorithms are accurate and efficient which have been numerically verified and tested by our extensive experiments. They could be easily adopted for generating sample paths for modelling the dynamics of stochastic volatilities and interest rates to name a few. They would be especially useful for simulation-based statistical inference, derivative pricing and risk management in practice. Model extensions to the processes with time-varying parameters as well as multi-dimensional versions may be also possible, and we propose them for future research.

Appendices

A Proof for the Martingale of (3.3)

Proof. We adopt a similar approach as Dassios and Jang (2003) and Dassios and Zhao (2011) to find the martingale solution to $\mathcal{A}f = 0$ for the generator (3.2). We try a solution of exponential form $e^{-xA(t)}e^{B(t)}$ where $A(t)$ and $B(t)$ are deterministic and differentiable functions of time t.

Then, we get

$$-xA'(t) + B'(t) + \delta x A(t) - \varrho \int_0^\infty \left[1 - e^{-yA(t)}\right] \nu(\mathrm{d}y) = 0,$$

which is rewritten as

$$x\Big(\delta A(t) - A'(t)\Big) + B'(t) - \varrho \int_0^\infty \left[1 - e^{-yA(t)}\right] \nu(\mathrm{d}y) = 0,$$

holding for any x. It implies two equations

$$\begin{aligned} A'(t) &= \delta A(t), \\ B'(t) &= \varrho \Phi(A(t)), \end{aligned}$$

which can be easily solved as

$$A(t) = ke^{\delta t}, \qquad B(t) = \varrho \int_0^t \Phi\left(ke^{\delta s}\right) \mathrm{d}s, \qquad \forall\, k \in \mathbb{R}^+,$$

where $\Phi(u)$ is the Laplace exponent for Z_t, i.e.,

$$\Phi(u) = \int_0^\infty \left(1 - e^{-uy}\right) \nu(\mathrm{d}y).$$

□

B Proof of the Acceptance Rate $\bar{c}_w \in (1, 2)$ for A/R Algorithm 3.2

Proof. To further investigate how the acceptance rate $\bar{c}_w$ of (3.16) depends on w, i.e. the range of $\bar{c}_w$, we let $x = \frac{1}{w}$, and then, we have

$$\frac{C_w}{D_w} = \frac{C_{\frac{1}{x}}}{D_{\frac{1}{x}}} = \frac{\frac{1}{\alpha}\left(x^{\frac{\alpha}{2}} - 1\right)^2}{\frac{1}{\alpha}\left(x^\alpha - 1\right) - \ln x}, \qquad x > 1. \tag{B.1}$$

Obviously,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{C_{\frac{1}{x}}}{D_{\frac{1}{x}}}\right) = \left(x^{\frac{\alpha}{2}} - 1\right) x^{\frac{\alpha}{2}-1} \frac{\frac{1}{\alpha}\left(x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}}\right) - \ln x}{\left[\frac{1}{\alpha}\left(x^\alpha - 1\right) - \ln x\right]^2} > 0, \qquad \forall x > 1,$$

so, $\frac{C_{\frac{1}{x}}}{D_{\frac{1}{x}}}$ in (B.1) is a strictly increasing function of $x > 1$. When $w \to 1$ or $x \to 1$, by L'Hôpital's rule, we obtain the lower bound

$$\lim_{x \downarrow 1} \frac{C_{\frac{1}{x}}}{D_{\frac{1}{x}}} = \lim_{x \downarrow 1} \frac{x^{\alpha-1} - x^{\frac{\alpha-1}{2}}}{x^{\alpha-1} - x^{-1}} = \lim_{x \downarrow 1} \frac{(\alpha-1)x^{\alpha-2} - \left(\frac{\alpha}{2} - 1\right) x^{\frac{\alpha}{2}-2}}{(\alpha-1)x^{\alpha-2} + x^{-2}} = \frac{1}{2};$$

when $w \to 0$ or $x \to \infty$, we obtain the upper bound

$$\lim_{x \to \infty} \frac{C_{\frac{1}{x}}}{D_{\frac{1}{x}}} = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha}} = 1.$$

Therefore, $\frac{C_w}{D_w} \in \left(\frac{1}{2}, 1\right)$ for $w \in (0,1)$, or, $\bar{c}_w \in (1,2)$ for $w \in (0,1)$, and we have (3.17). $\square$

C Proof for Algorithm 4.1

Proof. According to (4.1), we have

$$\begin{aligned}
& \mathbb{E}\left[e^{-vY_{t+\tau}} \mid Y_t\right] \\
= \; & e^{-vwY_t} \exp\left(-\int_{vw}^{v} \frac{1}{u} \int_0^{\infty} \left(1 - e^{-uy}\right)\left(\theta\alpha y^{-1-\alpha} + \theta\beta y^{-\alpha}\right) e^{-\beta y} \mathrm{d}y \mathrm{d}u\right) \\
= \; & e^{-vwY_t} \exp\left(-\int_{vw}^{v} \frac{1}{u} \int_0^{\infty} \left(1 - e^{-uy}\right) \theta\alpha y^{-1-\alpha} e^{-\beta y} \mathrm{d}y \mathrm{d}u\right) \exp\left(-\theta\beta \int_{vw}^{v} \frac{1}{u} \int_0^{\infty} \left(1 - e^{-uy}\right) y^{-\alpha} e^{-\beta y} \mathrm{d}y \mathrm{d}u\right) \\
= \; & e^{-vwY_t} \times \mathbb{E}\left[\exp\left(-v\left(\breve{S} + \sum_{i=1}^{\tilde{N}} S_i\right)\right)\right] \times \exp\left(-\theta\beta \int_{vw}^{v} \frac{1}{u} \int_0^{\infty} \left(1 - e^{-uy}\right) y^{-\alpha} e^{-\beta y} \mathrm{d}y \mathrm{d}u\right).
\end{aligned} \tag{C.1}$$

From (C.1), we can identify that:

1. The first term of (C.1) is the Laplace transform of constant wY_t.
2. The second term of (C.1) is the Laplace transform of an OU-TS process such that $TS \sim \mathsf{TS}(\alpha, \beta, \alpha\theta)$ with initial value equal to 0, and it can be broken into a TS r.v. and a compound Poisson r.v. by Theorem 3.1.
3. Within the third term of (C.1), we have

$$\begin{aligned}
& \theta\beta \int_{vw}^{v} \frac{1}{u} \int_0^{\infty} (1 - e^{-uy}) y^{-\alpha} e^{-\beta y} \mathrm{d}y \mathrm{d}u \\
& = \theta\beta\Gamma(1-\alpha) \int_0^{\infty} (1 - e^{-vs}) \int_1^{\frac{1}{w}} \frac{1}{\Gamma(1-\alpha)} s^{(1-\alpha)-1} e^{-\beta us} u^{-\alpha} \mathrm{d}u \mathrm{d}s
\end{aligned}$$

$$= \theta\beta^{\alpha}\Gamma(1-\alpha)\int_{0}^{\infty}(1-e^{-vs})\int_{1}^{\frac{1}{w}}\frac{(\beta u)^{1-\alpha}}{\Gamma(1-\alpha)}s^{(1-\alpha)-1}e^{-\beta us}\frac{1}{u}\mathrm{d}u\mathrm{d}s$$

$$= \theta\beta^{\alpha}\Gamma(1-\alpha)\ln\left(\frac{1}{w}\right)\int_{0}^{\infty}(1-e^{-vs})\int_{1}^{\frac{1}{w}}\frac{(\beta u)^{1-\alpha}}{\Gamma(1-\alpha)}s^{(1-\alpha)-1}e^{-\beta us}\frac{1}{\ln\left(\frac{1}{w}\right)u}\mathrm{d}u\mathrm{d}s. \quad \text{(C.2)}$$

In fact, (C.2) is the Laplace exponent of a compound Poisson r.v. $\sum_{j=1}^{\breve{N}}\breve{S}_j$. The intermediate r.v. V^* has a simple density function

$$f_{V^*}(u) = \frac{1}{\ln\left(\frac{1}{w}\right)}\frac{1}{u}, \qquad u \in \left[1, \frac{1}{w}\right],$$

and the CDF can be inverted explicitly. Hence, V^* can be exactly simulated via the explicit inverse transform (4.7).

□

D Simple Stable Rejection (SSR) Scheme

Algorithm D.1 (Simple Stable Rejection (SSR) Scheme)**.** *For simulating one r.v.* $TS \sim \mathsf{TS}(\alpha,\beta,\theta)$*:*

1. *Generate a stable r.v.* $\mathsf{S}(\alpha,\theta)$ *via Zolotarev's integral representation (Zolotarev, 1966) of*

$$\mathsf{S}(\alpha,\theta) \overset{\mathcal{D}}{=} \Big(-\theta\Gamma(-\alpha)\Big)^{\frac{1}{\alpha}}\frac{\sin\left(\alpha U_s+\frac{1}{2}\pi\alpha\right)}{\left(\cos(U_s)\right)^{\frac{1}{\alpha}}}\left[\frac{\cos\left((1-\alpha)U_s-\frac{1}{2}\pi\alpha\right)}{E_s}\right]^{\frac{1-\alpha}{\alpha}}, \quad \text{(D.1)}$$

   where $U_s \sim \mathsf{U}\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$, $E_s \sim \mathsf{Exp}(1)$*, and they are independent;*

2. *Generate a uniformly distributed r.v.* $U \sim \mathsf{U}[0,1]$*;*

3. *If* $U \le e^{-\beta\mathsf{S}(\alpha,\theta)}$*, then, accept and set* $TS = \mathsf{S}(\alpha,\theta)$*; otherwise, reject and go back to Step 1.*

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