Definable Topological Dynamics in Metastable Theories

Thomas Kirk

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Jeremiah Horrocks Institute for Maths, Physics and Astronomy
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Declaration

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I declare that while registered as a candidate for the research degree, I have not been a registered candidate or enrolled student for another award of the University or other academic or professional institution.

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No proof-reading service was used in the compilation of this thesis.

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Abstract

We initiate a study of the definable topological dynamics of groups definable in metastable theories. In stable theories, it is known that the quotient of a group G by its type-definable connected component G^{00} is isomorphic to the Ellis Group of the flow $(G(M), S_G(M))$; we consider whether these results could be extended to the broader metastable setting. Further, the definable topological dynamics of compactly dominated groups in the o-minimal setting is well understood. We investigate to what extent stable domination is a suitable analogue of compact domination in regards to describing the Ellis Group of metastable definable groups.

We first prove that when $G = SL_2(\mathbb{C}((t)))$, the Ellis Group of $(G(M), S_G(M))$ is not isomorphic to G/G^{00}. This counterexample provides a negative answer as to whether metastability is a suitable weakening of the Ellis Group conjecture of Newelski. We demonstrate that the Ellis Group is infact isomorphic to B/B^0, where B is the Borel subgroup of G of upper-triangular matrices. This is analogous to the definable topological dynamics of $SL_2(\mathbb{Q}_p)$ **[22]** in which they also found the Ellis Group was dependent on the Borel subgroup $B(\mathbb{Q}_p)$.

We see later in the thesis that $SL_2(\mathbb{C}[[t]])$ is definably extremely amenable; it admits a unique global left invariant type whose restriction to $SL_2(\mathbb{C})$ via the residue map is generic in $SL_2(\mathbb{C})$. This also provides positive evidence towards a further generalisation of a maximum modulus principal in $K \vDash ACVF$, which proves $SL_2(\mathcal{O}_K)$ admits a unique global left-invariant type.

We suspect that these results could be generalised to larger classes of metastable definable groups, and the description of Ellis Groups for groups which admit a stably-dominated / definably amenable group decomposition is a key focus of the final sections of the thesis. We suggest that for groups G with decompositions $G = HN$, where H is definably amenable and N is maximally stably dominated, with $H \cap N$ possibly infinite, we can provide complete descriptions for the minimal flows and Ellis Groups. We demonstrate that this is the case when we restrict H further to definable (extreme) amenability, and provide some preliminary work towards an explicit description of the Ellis Group for definable affine algebraic groups in a metastable theory.

Contents

Chapter 1

Introduction

1.1 Assumptions and Notation

We assume familiarity with the basics of model theory, topology and valued fields, though we recall key definitions and results where appropriate.

Notation will mostly be standard. We will use T to denote a complete theory in a language L. We will use M, N to denote models of a theory T, and will often make no distinction between M as a model and its universe. We denote by $\bar{M}$ an elementary, but not necessarily saturated, extension of a model M. We often use $\mathcal{U}$ to denote some global model which is homogeneous and sufficiently saturated.

We will use $a, b, x, y, ...$ for variables and parameters, making no distinction between points and tuples unless necessary. Our formulas will be denoted by $\phi(x), \psi(x), ...$ Types will be denoted by $p, q, ...$, using $p(x)$ to specify when p is a type in a variable x. We assume all types are complete.

In general when we say definable we mean with parameters unless otherwise stated.

Definable groups are denoted G, H, ... ; that is sets definable by some formula together with a definable group operation. We will denote by $S(M)$ the space of complete types with parameters from M. Further, $S_G(M)$ will denote the subset of

$S(M)$ consisting of types which contain the formula defining G. We say these types *concentrate* on G.

We will often simplify notation where the meaning is clear. For example, we make no distinction between a definable set X and the formula that defines it; similarly for definable groups $(G, \cdot)$. Further, we will often use concatenation or a comma rather than union; for example, we write A, b to mean $A \cup \{b\}$.

1.2 Motivations

Topological Dynamics is a pure abstraction of dynamical systems in the classical sense; systems used to describe the behaviour of points over time. In classical dynamical systems, one may apply the rectification theorem to describe the dynamics of a point in a small region. This region can be extended by combining similar small regions, and when this is possible for the entire space we say that the space is *integrable*. However, the study of these systems becomes difficult when dealing with periodic or near periodic behaviour as - in general - the rectification theorem cannot be used in the neighbourhood of a periodic orbit. Topological Dynamics can be used to study periodic or near periodic (see **Definition 1.3.2**) behaviour since such points are contained in (and indeed generate) minimal flows of the system, and such objects are well understood.

The focus of this thesis is to expand upon recent work in topological dynamics in the context of model theory. Early aims in the field have been focused upon demonstrating relationships between type definability and topological spaces.

Take some $R \succ \mathbb{R}$ a sufficiently saturated real closed field. Then we recall that the standard part map st associates, for every finite $r \in R$, the unique real number r_0 which is infinitely close to r. That is, $st(r) = r_0$. The standard part map induces a definable group homomorphism from a definably compact semialgebraic group $G(R)$ into $G(\mathbb{R})$. The kernel of this map is in fact a type-definable normal subgroup of bounded index ($< \kappa$ where R is κ-saturated), and hence $G(R)/ker(st)$ is a group. In this example, $ker(st)$ is what we now call G^{00}.

Studying this quotient becomes interesting once we consider topologies on G; namely, the standard Euclidean topology and the Logic Topology (Definition 1.4.5). Pillay conjectured that in the o-minimal context, a definable group G has a minimal type-definable normal subgroup of bounded index in G, denoted G^{00}, and that the quotient of G by G^{00} should be isomorphic as a topological group to some definably compact real Lie group. Pillay shows **[27]** that a subset of G/G^{00} is closed in the Euclidean topology precisely when it is closed in the Logic topology. The implication of this result is that every closed set in the Euclidean topology is type-definable.

The motivation for definable topological dynamics is to study analogues of this conjecture (now theorem) in the broader NIP setting. It is shown by Shelah **[30]** that G^{00} exists for all groups G definable in a NIP structure.

In **[11]** Pillay, Peterzil and Hrushovski introduce a notion of compact domination; definable subsets of a type-definable compactly dominated set X can be determined entirely via some surjective map into a compact space C. They relate this concept to both the existence of and uniqueness of Keisler measures and demonstrate that definably compact groups in o-minimal structures are compactly dominated (via G/G^{00}). The analogue of this notion for metastable structures is the notion of stable domination.

This leads us to the key motivation of this thesis. We investigate the relationship between the quotient G/G^{00} and the minimal objects of the dynamical system $(G(M), S_G(M))$, where G is definable in some metastable structure. The intentions are twofold. First, to investigate whether the Ellis Group conjecture of Newelski for the topological dynamics of stable groups could be generalised to the metastable setting. Secondly, in **[9]**, they consider the definable topological dynamics of a non-compactly dominated group definable in an o-minimal structure. We wish to investigate to what extent the results of [**9**] would extend to a non-stably dominated group definable in a metastable structure.

1.3 Topological Dynamics

Our aim in this section is to outline the basics of topological dynamics and describe the construction of the Ellis Semigroup. We will eventually provide model theoretic context to this construction by considering the action of a definable group G on the space of complete types $S_G(M)$. However, we first recall the key notions of topological dynamics and Ellis Groups. We assume full familiarity with the basic notions of topological spaces and their properties. References for general topology and topological dynamics are [**1**] and [**7**].

A topological group is a topological space (G, τ) such that G is itself a group where the group operation $\cdot : G \times G \to G$ is a continuous function with respect to the topology τ on G and the product topology on $G \times G$. When discussing actions of groups on topological spaces, we will say an action $\cdot : G \times X \to X$ is *continuous* if, for all $g \in G$, the induced action $\pi_g : X \to X$ is continuous.

Definition 1.3.1. *Let G be a topological group and X be a topological space. Then a flow is the pair (G, X) together with a continuous action $G \times X \to X$ such that;*

(i) $ex = x$ *for all* $x \in X$,

(ii) $g_1(g_2x) = (g_1g_2)x$,

where e is the identity of G, $g_1, g_2 \in G$ and $x \in X$.

When discussing compact topological spaces we will also assume those spaces are Hausdorff. In general this is not the case; there are topological spaces which are compact but not Hausdorff. However, in this thesis, we discuss no such topological spaces which have this property. Hence, we assume X is a Compact Hausdorff topological space unless otherwise stated.

We mention for later the notion of a *compactification* (H, f) of a topological group G, where $f : G \to H$ is continuous and H a compact (Hausdorff) topological group, with G dense in H.

Throughout the thesis we will be assuming that (G, τ) is a topological group with the discrete topology unless otherwise stated, and as such we will just denote $(G, \tau) = G$ for simplicity. The reason for this restriction is to ensure that the definable Bohr Compactification of G coincides with G^{00}.

If (G, X) is a flow with Y a closed subspace of X such that Y is closed under the action of G, then we say (G, Y) is a subflow of (G, X). If Y is a subset of X with $GY = Y$, we say that Y is *invariant*. A set is invariant if and only if it is the union of orbits.

If (G, X) is a flow, then a subset W of X is a *minimal set* if W is a closed, non-empty invariant set such that for any $V \subseteq W$ that is also closed and invariant, either $V = \emptyset$ or $V = W$. The flow (G, W) is called a *minimal subflow*. A subset W of X is minimal if and only if it is the orbit-closure of each of its points. As such we can consider a set minimal if every point of W can generate W via G-orbit.

Further, for any minimal flows W_1, W_2, either $W_1 = W_2$ or $W_1 \cap W_2 = \emptyset$. Minimal sets will be the key object of study for this thesis, namely finding them and explicitly describing them in various model theoretic contexts. We now state several definitions and results that have previously been used in the description of minimal sets.

Definition 1.3.2. *Let (G, X) be a flow.*

- *A point $x \in X$ is said to be a fixed point if $gx = x$ for all $g \in G$. In this case, the singleton $\{x\}$ is a minimal set of (G, X).*

- *A point $x \in X$ is said to be a periodic point if $g^n x = x$ for some $g \neq e \in G$*

and some $n \in \mathbb{N}$.

- *A point $x \in X$ is said to be an almost periodic point if, for every neighbourhood U of x, there is a subset H of G such that $Hx \subset U$, where H is such that there exists a compact subset K of G with $G = HK$.*

For some topological spaces, the almost periodic points precisely describe the minimal sets.

Definition 1.3.3. *Let (G, X) be a flow. Then the orbit-closure of x, denoted $cl(Gx)$ or $\bar{G}x$, is the closure of the set $gx : g \in G$ in X.*

Let (G, X) be a flow with X a locally compact Hausdorff space. Then $x \in X$ is almost periodic if and only if the orbit closure $\overline{Gx}$ is a compact minimal set. Further, if X is compact, there exists an almost periodic point $x \in X$ and again $\overline{Gx}$ is a compact minimal set.

Since minimal sets are our objects of study, we wish to restrict to cases where we know such sets exist. Namely, if X is a compact Hausdorff topological space and (G, X) a flow, then there exists a minimal set A of X. The proof of this result follows from Zorn's Lemma.

We now move to discuss semigroups in the context of topological spaces.

For a flow (G, X), it can be beneficial to instead consider the set of functions $\pi_g : X \to X$ induced from the action of G on X. These functions are then homeomorphisms on X , and so can be considered as a subset of the set X^X of functions from X to itself.

By Tychonoff's Theorem, we observe that if X is a compact topological space, then X^X is a compact space with respect to the product topology. We can consider the closure of the set $\{\pi_g : g \in G\}$ in X^X and obtain a semigroup as follows.

Fact 1.3.4. *[1] Let (G, X) be a flow with X a compact topological space. Then the closure of the set $\{\pi_g : g \in G\}$ in the space X^X forms a semigroup, called the*

Enveloping Semigroup and denoted $E(X)$, where the semigroup operation is given by composition of mappings.

Definition 1.3.5. *[6] A semigroup $(S, \cdot)$ is an Ellis Semigroup if it satisfies the following:*

1. *The set S is a compact Hausdorff topological space.*

2. *The mappings given by left translation by an element of S are continuous.*

By considering semigroups in place of flows, the question of invariant points or minimal subflows instead becomes a question of idempotents and ideals. For a semigroup $(S, \cdot)$, we define a closed left ideal I to be a closed subset of S such that, for all $s \in S$, $s \cdot I \subseteq I$.

However, minimal subflows are not necessarily unique even up to isomorphism. This becomes a problem in this particular area of study when we wish to provide explicit descriptions of these ideals. We instead work with the following theorem of Ellis which allows us to obtain a unique (up to isomorphism) object from a given Ellis Semigroup.

Theorem 1.3.6. *[7] Let $(S, \circ)$ be an Ellis Semigroup and let J be the set of idempotents of S. Then;*

(i) There exists a minimal closed left ideal I of S.

(ii) If I is such a closed left ideal, then $I \cap J \neq \emptyset$ and moreover, for any $u \in I \cap J$, the left translate $(u \circ I, \circ)$ forms a group, which we called the Ellis Group

(iii) Every such Ellis Group obtained by varying the choice of I and u are isomorphic.

Given a flow (G, X) we can construct the Enveloping (Ellis) Semigroup $(E(X), \circ)$. We now wish to find the Ellis Group, which is unique up to isomorphism, and contained within $(E(X), \circ)$. We can simplify this by instead considering a new G-flow on $E(X)$, denoted $(G, E(X))$, with action of $g \in G$ on a function $f \in E(X)$ given by $\pi_g \circ f$.

This flow and the Ellis Semigroup $(E(X), \circ)$ are very closely related, as shown in the following fact.

Fact 1.3.7. *Let (X, G) be a flow with $(E(X), \circ)$ and $(G, E(X))$ constructed as above. Then the minimal closed left ideals of $(E(X), \circ)$ coincide (set-wise) with the minimal subflows of $(G, E(X))$.*

The above fact is most useful when $E(X)$ is homeomorphic to X. In which case, we can ignore the $E(X)$ construction and instead focus on finding minimal subflows of (G, X). Applying Fact 1.3.7 in this case means we can find the minimal closed left ideals of $(E(X), \circ)$ by considering the minimal subflows of (G, X).

1.4 Model Theoretic Context

We now recall some model theoretic notions which will provide context for interpreting the topological dynamics in a definable way. The majority of the results here are well known, but form the cornerstone of interpreting topological dynamics in a meaningful and definable way. We refer to **[32]** for the majority of these definitions and for a general overview of NIP theories.

Fix a structure M. Recall a formula $\phi(x, y)$ has the *independence property* (IP) if and only if there is some indiscernible sequence $(a_i)_{i<\omega}$ and a tuple b, with $a_i, b \in M$, such that

$$\models \phi(a_i, b) \iff i \text{ is even.}$$

Otherwise, we say a formula has NIP. We say a theory T has NIP if every formula $\phi(x, y) \in L$ has NIP.

A type $p(x)$ over a set B is said to be *definable* if for every formula $\phi(x, y)$ without parameters, there is a formula $d_{\phi(y)}$ with parameters from B such that

$p \vdash \phi(x, b) \iff b \models d_{\phi(y)}$ for all $b \in B$.

Let L be language with M and L-structure and $\bar{M}$ some elementary extension of M. A set $X \subseteq M^n$ is said to be *externally definable* if there is a definable subset $Y \subseteq \bar{M}^n$ such that $X = Y \cap M^n$. By augmenting the language with a predicate for each externally definable subset, denoted L^{ext}, we can consider the L^{ext}-structure M^{ext}. We call this the Shelah Expansion of M.

Fact 1.4.1. *[**31**] Suppose T is a complete NIP theory and let M be a model of T. Let M^{ext} be the Shelah Expansion of M. Then;*

1. *$Th(M^{ext})$ is a complete NIP theory.*

2. *$Th(M^{ext})$ has quantifier elimination.*

3. *Complete types over M^{ext} are definable.*

The following is a well known definition, and we refer to [**26**] for details and surrounding results.

Definition 1.4.2. *[**26**] Let p be a type over some model M of a theory T, and $q \in S(B)$ an extension of p to $B \supset M$.*

- *We call q an heir of p if for every $L(M)$-formula $\phi(x, y)$ such that $\phi(x, b) \in q$ for some $b \in B$ there is some $m \in M$ with $\phi(x, m) \in p$.*

- *We call q a coheir of p if q is finitely satisfiable in M.*

We recall a well known result for complete types in NIP theories.

Fact 1.4.3. *[**26**] Let T have NIP and suppose that all complete types over M are definable. Then every complete type p over M has a unique heir and unique coheir in $S(\bar{M})$.*

We now define one of the main objects of consideration in this thesis. Recall that by bounded index, we mean of cardinality less than (or equal to) $2^{|M|+|L|}$.

Definition 1.4.4. *Let G be a definable group.*

- *The Connected Component (over A), denoted G^0_A, is the intersection of all definable subgroups of G of finite index (using parameters from A).*
- *The Type-Definable Connected Component (over A), denoted G^{00}_A is the intersection of all type-definable subgroups of G of bounded index (using parameters from A).*

If this subgroup is independent of the choice of parameters A, we drop the subscript and say G^0 (equivalently G^{00}) exists.

It is well known ([**30**]) that if G is definable in some model of a NIP theory T, then G^0 and G^{00} exist. We say a definable group G is connected if $G = G^0$. Further, if T is stable, then $G^0 = G^{00}$.

We remark that G^0 and G^{00} are both normal subgroups and so the quotient G/G^{00} is well-defined. This quotient is model invariant, in the sense that the number of cosets obtained by considering the M-points of G/G^{00} is consistent regardless of choice of M. In fact, even when taking a Shelah expansion, the description of G^{00} in L_M is equivalent to G^{00} in $L_{M,ext}$ ([**2**]).

Before we discuss topological dynamics in this context, we first recall some connections between model theory and classical topology.

The space of complete types $S(M)$ can be equipped with the Stone topology on definable sets, where the open sets are given by $[\phi] = \{p \in S(M) : \phi \in p\}$. It is easy to see that this space is a Hausdorff, compact and totally disconnected topological space. We also define another topology, though this is more specific to our purposes and especially G^{00}.

Definition 1.4.5. *[8] Let M be a model and let X be an M-definable set in some saturated elementary extension $\bar{M}$ of M. Suppose E is some type-definable (over M) equivalence relation with $\pi : X \to X/E$ the canonical projection.*

Then we define the logic topology on X/E via closed sets; where $Z \subseteq X/E$ is closed if $\pi^{-1}(Z) \subseteq X$ is type-definable (over M).

For more details on the logic topology we refer to **[11]**. Namely, X/E is known to be a compact topological space and moreover the projection π is definable (as a function over M). In connection with the above, we can easily replace E with a type-definable normal subgroup N, and see that G/N is a compact topological group under the logic topology.

Definition 1.4.6. *[8] Let G be a group definable in M. By a definable compactification of G we mean a compactification (C, f) where $f : G \to C$ is a definable homomorphism, C is a compact group, and G is dense in C.*

When considering the $G(\bar{M})$ points, the identity embedding of $G(M)$ into $G(\bar{M})$ induces a homomorphism from $G(M)$ into $G(\bar{M})/N$ with dense image. Further, this mapping is continuous in the topological sense, and hence by way of replacing N with G^{00} we obtain a universal (Bohr) compactification of G (**[8]**).

1.5 Definable Topological Dynamics

We now consider the dynamical system of a definable group G acting on its space of types $S_G(M)$. We assume from here on that T is a NIP theory and G is a definable group. The key references for this construction are **[18]** and **[19]**.

Recall that by $G(M)$, we mean the interpretation of G in M, sometimes called the M-points of G. The construction here uses several different actions, and though we are initially explicit for the sake of clarity, we will eventually return to denoting these actions via juxtaposition when the meaning is clear.

Consider the space of complete types $S_G(M)$ concentrating on G, with the induced Stone topology from $S(M)$.

We can obtain an action of $(G(M), \cdot)$ on $S_G(M)$ as follows;

$$G \times S_G(M) \to S_G(M)$$
$$(g, p) \mapsto tp(g \cdot a/M)$$

where $a \models p|_M$.

We note that this action is well defined since we insist p contains the formula defining G, and hence in particular $a \in G(\bar{M})$. Further, this action is independent from the choice of realisation of p. That is, if $a \models p|_M$ and $b \models p|_M$, then $tp(g \cdot a/M) = tp(g \cdot b/M)$.

Since we have a flow $(G(M), S_G(M))$ with $S_G(M)$ a compact Hausdorff topological space, we can construct the Enveloping Semigroup $(E(S_G(M)), \circ)$.

The application of the results of Ellis to the space $S_G(M)$ was developed by Newelski **[18]** in which he demonstrates a relationship between $(E(S_G(M)), \circ)$ and $S_G(M^{ext})$. Namely, that the compact spaces $E(S_G(M))$ and $S_G(M^{ext})$ are homeomorphic. This homeomorphism extends to an action $*$ on $S_G(M^{ext})$.

Again let $(G, \cdot)$ be a definable group, and define a semigroup action $*$ on $S_G(M)$ as follows;

$$* : S_G(M) \times S_G(M) \to S_G(M)$$
$$(p, q) \mapsto tp(a \cdot b/M)$$

where $a \models p|_M$ and $b \models q|_{Ma}$.

To be explicit here, as this action is the key focus of the thesis, $q|_{Ma}$ is the heir of q over $M \cup \{a\}$ where a is any realisation of $p|_M$. We will call this action *"type multiplication"*.

The choice of heirs here is just preference, and the action can instead be defined in terms of coheirs by choosing $b \models q|_M$ and a the realisation of a coheir of p (over Mb). We note that this action is only well-defined when the extension of a type to an heir (or coheir) is unique.

To summarise then, since complete types over M^{ext} are definable, we have that for any definable group G, the Enveloping (Ellis) Semigroup $(E(S_G(M)), \circ)$ is homeomorphic to the semigroup $(S_G(M^{ext}), *)$.

Of course, if types over M are already definable, $S_G(M^{ext})$ is homeomorphic to $S_G(M)$, and similarly the type multiplicaton $*$ behaves identically on $S_G(M)$. Hence in the study of minimal subflows of $(G, S_G(M))$ and of the properties of the Enveloping Semigroup $(E(S_G(M)), \circ)$, it suffices instead to consider the much easier question of closed left ideals of $(S_G(M), *)$. We will often talk about minimal subflows of the system and minimal ideals of the semigroup interchangeably, and it should be understood that there is no meaningful difference between either representation aside from preference or ease of understanding.

Work on groups and measures in the NIP setting was already being considered (see **[11]**, **[4]**). Their work was initially independent of topological dynamics, though the notions and results they developed were eventually found to have implications for the description of minimal subflows for the groups they studied. We now discuss special cases of these systems for which these groups have additional properties, and recall results that explicitly describe the minimal subflows.

Fact 1.5.1. *Let G be a definable group, let p be a type in $S_G(\bar{M})$, and let $G(\bar{M})$ be the interpretation of G in $\bar{M}$. Then;*

- ***[1]*** *$cl(G(\bar{M}) * p)$ is $G(\bar{M})$-invariant.*
- ***[1]*** *Let $X \subset S_G(\bar{M})$. X is minimal if and only if $cl(G(\bar{M}) * p) = X$ for all $p \in X$. That is, a set is minimal exactly when it is the orbit closure of each of its points.*
- *$cl(G(\bar{M}) * p) = S_G(\bar{M}) * p$.*
- ***[29]*** *If p is a global f-generic type, then p is almost periodic and further $cl(G(\bar{M}) * p) = G(\bar{M}) * p$.*

An important class of types in $S_G(M)$ which appear frequently in the study of $(E(S_G(M)), \circ)$are the generic types. A formula ϕ (in G) is *left (right) generic* if finitely many left (right) translates of ϕ cover G. A type $p \in S_G(M)$ is left (right) generic if every formula in p is left (right) generic. We say a type (formula) is generic if it is both left and right generic.

A definable group G has *finitely satisfiable generics* (fsg) if there is some global type $p(x)$ in G and some (small) model M such that every G-translate of $p(x)$ is finitely satisfiable in M.

To see that these types are indeed generic, we refer Proposition 4.2 in [**11**] though summarise here. Fix some (small) M and let p be as in the above definition. A definable set X is generic if and only if every left translate of X meet M (meaning every left translate of X contains some element of $G(M)$). Since any formula (definable set) in p has this property due to fsg, we see that every formula in p is generic.

Fact 1.5.2. *[11] Let G be a definable group with fsg and let M be any (small) model. Then there exists a generic global type $p \in S_G(\mathcal{U})$. Further, for such a type;*

(1) Every left and right translate of p is generic, and is also finitely satisfiable in M,

(2) G^{00} exists and is both the left and right stabilizer of p.

It is logical then to ask whether this property on the global generic types that concentrate on an fsg group is retained when restricting to smaller models. Given a global type p, we write $p^M = \{\phi(M) : \phi \in p\}$, where $\phi(M) = \phi(\mathcal{U}) \cap M^n$. This is a complete type in $S_G(M^{ext})$. Under this restriction, for G a definable group with fsg, it is shown in [**11**] that if p is a global generic type, then p^M is generic in any (small) model M. This fact allows us to identify the set of generic types in $S_G(M^{ext})$ with the set of global generics. We will denote the set of (left) generic types concentrating on G, with parameters from a model M, by $Gen_G(M)$. When

$S_G(M)$ contains generic types, we can quickly use the following result of Newelski to find the minimal flows.

Fact 1.5.3. *[18] Let G be a definable group and assume there exists some generic type $p \in S_G(M^{ext})$. Then the set of generic types $Gen_G(M^{ext})$ is the unique minimal subflow of $(G(M), S_G(M^{ext}))$.*

We remark that this result assumes the existence of a generic type, rather than claims one always exists. In fact, there are many examples of groups which do not admit any generic types ($(\mathbb{R}, +)$ for example). However, this result becomes especially powerful if we assume that translates of p are finitely satisfiable in M^{ext}. Explicitly, this is assuming G to have fsg.

We say that a generic type p is the *principal generic* (of G) if any realisation of p is contained in G^{00}. For G an M-definable fsg group in a NIP theory, the minimal flow of
$(G(M), Gen_G(M^{ext}))$ decomposes into Ellis Groups of the form $q * Gen_G(M^{ext})$, where q is a principal generic ([**18**], [**14**]). We can consider this in the context of the Ellis Semigroup $(S_G(M^{ext}), *)$, where the $*$ action is given by the type multiplication defined earlier. Firstly, for any type $p \in S_G(M^{ext})$, the translate of a type $q \in Gen_G(M^{ext})$ is itself generic. Moreover, $q * p$ belongs to the same Ellis semigroup as q. From this, the following important fact can be seen;

Fact 1.5.4. *[14] Let G be a definable group in some NIP theory with G admitting finitely satisfiable generics. Then the minimal subflow $(G(M), Gen_G(M^{ext}))$ of $(G(M), S_G(M^{ext}))$ is two-sided and corresponds to a two-sided ideal $Gen_G(M^{ext})$ of the semigroup $(S_G(M^{ext}), *)$.*

Later, work surrounding fsg groups, particularly in the o-minimal setting, was generalised further to an even larger class of groups. We now provide definitions and recall the preliminary results of the dynamical systems. We assume familiarity with the basics of measures, though refresh some simple definitions where appropriate.

A measure μ on a set X is said to be *finitely additive* if, for any disjoint subsets A, B of X, $\mu(A \cup B) = \mu(A) + \mu(B)$. Further, μ is said to be a *probability measure* if $\mu(X) = 1$ (and hence takes all values in the interval $[0, 1]$).

In general, for X a topological space, we can consider a Borel (probability) measure, where the measure is defined on the smallest σ-algebra containing the open sets of X as given by the topology. We say a topological group G is *amenable* if every flow (G, X) admits a G-invariant probability measure on the Borel sets of X. By a left invariant measure, we mean a measure μ such that $\mu(\phi(gx)) = \mu(\phi(x))$ for all $g \in G$.

We wish to provide amenability with some model theoretic restrictions to ensure that discussing such measures is well defined. We say a definable group G is *definably amenable* if every definable G-flow (G, X) there is a G-invariant Borel probability measure on X. In much of the literature, such a measure is called a Keisler measure, due to work in **[16]** which generalises the theory of forking from stability theory to using measures rather than complete types. Hence when we say a group is definably amenable, we mean there exists a Keisler measure on the boolean algebra of definable sets. These definitions extend to groups we call definably extremely amenable, with the added condition that the measure maps into $\{0, 1\}$ rather than the interval $[0, 1]$. Equivalently, a group G is definably extremely amenable if it admits a Keisler measure and there exists a type $p \in S_G(M^{ext})$ with p invariant under the action of G.

Several observations regarding definably amenable groups can be found in **[32]**.

When G is definably amenable, there exists some global left invariant Keisler measure μ. This measure is somewhat special as it also implies the existence of a right invariant measure; namely $v(\phi(x)) = \mu(\phi(x^{-1}))$. Note that this is not true in general for an arbitrary measure. It's easy to see that v will be right-invariant if μ is left invariant. Further, for any model M, if a Keisler measure μ_0 exists on

M-definable sets, then this extends to a measure μ on the $\bar{M}$-definable sets where $\bar{M}$ is a saturated elementary extension of M.

The existence of a Keisler measure has many implications for the minimal flows of the system $(G(M), S_G(M^{ext}))$ where G is definably amenable. The dynamical systems of these groups are well understood, though the techniques and results developed in their study will be useful to us as such groups often appear in the decompositions of non-definably amenable groups.

All groups that admit fsg are themselves definably amenable, though there are definably amenable groups which are not fsg. In particular, the measure on an fsg group is additionally generically stable; meaning the measure is both definable and finitely satisfiable in some (small) model M. Hence we obtain the following.

Fact 1.5.5. *[32] Let G be a definable group. Then G has fsg if and only if it is definably amenable and admits a generically stable left invariant measure.*

By a (left) f-generic type (over some set A) we mean a global type $p \in S_G(\mathcal{U})$ such that no (left) translate of p forks over A. The existence of these types is closely related to definable amenability.

Fact 1.5.6. *[32] Let G be a definable group over some model M. Then G admits a global f-generic type if and only if G is definably amenable.*

Fact 1.5.7. *[32] If a definable group G admits a global f-generic type p (over some A), then p is invariant under left translation by G^{00} and it follows that $Stab(p) = G^{00}$.*

Note every f-generic type is generic, but it is not necessarily true that every generic type is f-generic. Because of this, and the above fact, it becomes much easier to decide that a group is not definably amenable by simply demonstrating that no generic types exist in $S_G(M)$.

Finally, a further method of showing definable amenability indirectly is to find definably amenable normal subgroups that satisfy the following result;

Fact 1.5.8. *[32] Let G be a definable group with H a definable normal subgroup of G. Then;*

- *If G is definably amenable, so is G/H,*
- *If both H and G/H are definable amenable, then so is G.*

1.6 Ellis Groups of Definable Dynamical Systems

In this section we take the results of minimal subflows above and recall the work so far in regards to the description of the Ellis Groups of these definable systems. We begin with the work of Newelski which motivated this area of study.

Newelski ([**18**], [**19**]) worked towards describing the Ellis Group of $(G(M), S_G(M))$ under the assumptions that M is a model of some stable theory T and G a definable group in M. He proves an explicit description of the Ellis Groups for groups definable in stable theories, and moreover he demonstrates a relationship to the quotient G/G^{00}.

Fact 1.6.1. *[20] Let T be a stable theory with M a model of T. Let G be a definable group. Then the set of generic types (over M) $Gen_G(M)$ is a minimal closed left ideal of $(S_G(M), *)$.*

Further, $(Gen_G(M), *)$ is a group and is isomorphic to G/G^0 $(= G/G^{00})$.

It was this that motivated the idea that, in the more general setting where T is NIP rather than stable, G/G^{00} could be described or retrieved by considering the definable topological dynamics on external types. Newelski conjectured that for a group G, definable in some model of a NIP theory, that G/G^{00} is definably isomorphic to the Ellis Group of $(G(M), S_G(M^{ext}))$. We remark here that this conjecture was eventually proven false by way of counterexample in [**9**], though we comment more on this later.

Work towards this conjecture naturally began in *o*-minimal theories; a NIP theory which is very much "not-stable", with Newelski himself proving a positive case for G a definably compact group. We recall from **[11]** that a topological group G is said to be definably compact if every definable map from the open unit interval into G has a limit in G as the interval tends to 0. A more commonly used approach is via a result in **[24]** that if G is affine it is definably compact if and only if it is closed and bounded; for example, $SO_2(\mathbb{R})$. When G is such a group, Newelski shows the following.

Fact 1.6.2. *[**19**] Let G be a definably compact group in an o-minimal theory. Then the Ellis Group of* $(G(M), S_G(M^{ext}))$ *is isomorphic to* G/G^{00}*.*

Attempts to generalise definable compactness, especially outside the *o*-minimal setting, is what gave rise to the application of the measures on definable groups in dynamical systems. We provide a comprehensive study of the research done so far, this time in the context of Ellis Groups and G/G^{00}. These groups commonly appear in group decompositions and as such the following results are useful tools in understanding type multiplication for larger classes of groups. Pillay made the first steps towards these generalisations. As before, the rest of this section assumes a NIP theory.

Fact 1.6.3. *[**28**] Let G be a group with* fsg*,* $\emptyset$*-definable in any model M of a NIP theory T. Then the Ellis Group of* $G(M)$*-flow* $(G(M), S_G(M))$*, denoted* $(u \cdot I, \cdot)$*, is isomorphic to the quotient* G/G^{00}*.*

As we mentioned before, this result is really a special case of definable amenability, though the proof that this isomorphism holds over this larger class of groups did not come until much later. In fact, a counterexample to the general conjecture of Newelski was found in the interim, and we recall this here.

Fact 1.6.4. *[9] Let* $G = SL_2$ *definable in a real closed field, the Ellis Group of* $(SL_2(\mathbb{R}), S_{SL_2(\mathbb{R})}(M))$ *is isomorphic to* $(\{\pm 1\}, \times)$ *; a group of 2 elements.*

Further, $SL_2(\mathbb{R})^{00} = SL_2(\mathbb{R})$ *and hence the quotient* G/G^{00} *is trivial and is not isomorphic to the Ellis Group of* $(SL_2(\mathbb{R}), S_{SL_2(\mathbb{R}}(M))$*.*

This counterexample motivated the study of definable topological dynamics to take two directions. The work of Yao [**33**] and Jagiella ([**14**], [**15**]) for example was primarily concerned with applying what was known about the dynamical systems to explicitly describe Ellis Groups for large classes of definable groups. The other direction, seen in work of Pillay, Simon, Chernikov [**2**] looked towards finding the largest class of NIP groups for which the conjecture of Newelski did hold.

We summarise both approaches here, as the results of both will be useful to us. We begin with the latter. Recall that fsg was itself a special case of definable amenability. A positive answer for any definably amenable group definable in an o-minimal theory was given ([**29**], [**2**]), and was later generalised to an arbitrary NIP theory in [**3**].

Fact 1.6.5. *[3] Let* G *be a definably amenable group definable in some model* M *of an* NIP *theory* T*. Then the Ellis Group of the* G*-flow* $(G(M), S_G(M))$ *is isomorphic to* G/G^{00}*.*

So far, this is the most general result for which the isomorphism between the Ellis Group and the quotient G/G^{00} is known to hold. As mentioned before, other work in the area has been pushing towards descriptions of Ellis Groups with little regard towards the relationship with G/G^{00}. Specifically, work in [**14**] focused on groups definable in an o-minimal expansion of the reals.

We make note here of some of the machinery and preliminary results developed in [**14**] as understanding their application and limitations is integral to our work in later chapters. Let G admit a group decomposition into the semidirect product of definable subgroups H and K; where H is torsion-free, K is definably compact and

$H \cap K$ is trivial. This is analagous to a model theoretic interpretation of an Iwasawa group decomposition (Expressing a square matrix as the product of an orthogonal and upper triangular matrix).

Further, by insisting on a trivial intersection, one can develop actions of H on K that acts by homomorphism. Namely, for each $h \in H$, construct a map $\phi_h : K \to K$ by $\phi_h(k) = k'$, where k' is the unique element of K such that there exists a $h' \in H$ with $hk = k'h'$. The trivial intersection of H and K force these maps to be well defined, and moreover when G is a topological group, H acts on K by homeomorphism.

Since H is torsion-free definable in an o-minimal theory, an application of results from **[23]** shows that H is definably connected. Hence the flow $(H(M), S_H(M))$ has a one-point minimal flow and is definable extremely amenable with a unique H-invariant type. Moreover, since K is definably compact and definable in an o-minimal theory, it is known to admit a generic type and hence has minimal subflow $Gen_K(M)$ with idempotents the principal generics. In full generality then, the Ellis Group for such decompositions can be explicitly described as follows.

Fact 1.6.6. *[14] Let G be an M-definable group, with $M = \mathbb{R}$. Let K, H be M-definable subgroups of G such that;*

(1) $G = KH$ and $K \cap H = \{1\}$

(2) $S_H(M)$ has a $H(M)$-invariant type p.

(3) $(K(M), S_K(M))$ has a minimal subflow I which is invariant under the action of H on $S_K(M)$.

Then $I * p$ is the minimal subflow of $(G(M), S_G(M))$.

Describing the Ellis Group from the minimal subflow here is not a trivial step, in the sense that understanding the idempotent elements here takes some work. It turns out that the idempotents of I are given by certain translates of G on the 1-point flow p of $(H(M), S_H(M))$, namely those in which the standard part map together with the action of H on K gives the identity of K. Though somewhat abstract, there is an isomorphism to a more well known subgroup of G.

Fact 1.6.7. *[14] Let G be a group as in 1.6.6. Then the Ellis Group of the flow $(G(M), S_G(M))$ is algebraically isomorphic to $N_G(H) \cap K(\mathbb{R})$.*

Of course, $SL_2(\mathbb{R})$ is a group which admits such a decomposition, and in fact in [**9**] they suggest the Ellis Group should be somehow related to the centre of the group though the above result of Jagiella is evidence against this.

Further work in [**33**] looked to generalise this result to arbitrary expansions of models of RCF. This is not a straightforward generalisation and several of their preliminary results are interesting in their own right. For example, in the case where $M = \mathbb{R}$, with $G = HK$ as above, the minimal flow $Gen_K(M)$ of $(K(\mathbb{R}), S_K(M))$ is $H(\mathbb{R})$-invarant, and hence is also $G(\mathbb{R})$-invariant. However, for an arbitrary model M this is not true, and they provide examples where $Gen_K(M)$ is in fact a proper subset of the minimal flow of $(K(M), S_K(M))$.

However, despite the differences in minimal flows, it turns out that 1.6.7 does generalise to an arbitrary model of RCF. Note that the results here are only true in general for the Shelah expansion M^{ext} as they make no assumptions about types over M being definable.

Fact 1.6.8. *Let G be a group definable over $\mathbb{R}$, with $G = KH$ a compact torsion-free decomposition. Let M be an arbitrary elementary extension of $\mathbb{R}$. The Ellis Group of $S_G(M^{ext})$ is algebraically isomorphic to $N_G(H) \cap K(\mathbb{R})$.*

Naturally, one might wonder whether the generalisation of definably compact to fsg allows you to move out of the *o*-minimal specific context, and indeed it does, albeit with some further restrictions on the decomposition. It is also required to consider to what extent torsion-free is a requisite, or whether that too could be replaced with another condition. From the above, the torsion-free part admits a unique 1-point minimal flow; $B(\mathbb{R})$ is definably extremely amenable. It would seem sensible then to consider instead the generalisation of torsion-free and o-minimal to definably extremely amenable and NIP.

To this extent, **[15]** defines the notion of a "good" decomposition. For M a model of an arbitrary NIP theory, and G an M-definable group, we say that $G = KH$ is a good decomposition if $K \cap H = \{1\}$, K has fsg and H is definably extremely amenable. Jagiella manages to reduce the problem as follows, though a precise general description of the Ellis Group seems too general a result for such a large class of groups.

Fact 1.6.9. *[15] Let G be an M-definable group in some NIP theory T. Suppose $G = KH$ is a "good" decomposition. Then the minimal subflow of the universal flow $(G(M), S_G(M^{ext}))$ is given by $I * p$, where I is the unique minimal flow of $(K(M), S_K(M^{ext}))$ and p is $H(M)$-invariant.*

Further, the Ellis Group of $(G(M), S_G(M^{ext}))$ is isomorphic to a subgroup of K/K^{00}.

Further work that attempts to describe Ellis Groups in cases where we do not insist on a rigid decomposition has been more specific than general, though perhaps similar generalisations will eventually be possible. A key example of such work is again the consideration of SL_2, though this time definable in the field of p-adic numbers. $SL_2(\mathbb{Q}_p)$ admits a decomposition into the product of $SL_2(\mathbb{Z}_p)$, which is maximally compact, and the Borel subgroup $B(\mathbb{Q}_p)$, which admits f-generic types and is hence definably amenable.

However, the main difference in this case study is the existence of a non-trivial intersection; namely $SL_2(\mathbb{Z}_p) \cap B(\mathbb{Q}_p) = B(\mathbb{Z}_p)$. This has difficult consequences for the computations. With a trivial intersection between the subgroups of a decomposition, there is a natural action by homomorphisms which can be used to simplify the action of G. However, $B(\mathbb{Z}_p)$ is an infinite subgroup of $B(\mathbb{Q}_p)$ and so a similar approach is not well defined.

Though $SL_2(\mathbb{Q}_p)$ is perfect, $SL_2(\mathbb{Z}_p)$ admits a proper connected component. By way of considering open neighbourhoods of the identity, they demonstrate a well-defined way of commuting the realisations of generic types in their setting. Further, they prove that an action of conjugation on finitely satisfiable generic types (via their realisations) remains finitely satisfiable in M. The combination of these facts allows them to compute the Ellis Group of the flow $(G(M), S_G(M))$, for $G = SL_2$ and $M = \mathbb{Q}_p$. which we include below.

Fact 1.6.10. *[22] Let $M = \mathbb{Q}_p$ and $G = SL_2$.*

The Ellis Group of the flow $(G(M), S_G(M))$ is isomorphic to $B(\mathbb{Q}_p)/B(\mathbb{Q}_p)^0$.

We now refer to work **[15]** in which the aim is to simplify definable systems by way of considering the existence of certain subgroups of G. Suppose that G admits some normal definable subgroup H and that there exists a $H(M)$-invariant type in $S_G(M)$. Precisely, H is definably extremely amenable. By a result from **[2]**, invariant types extend to the Shelah expansion, and so there exists some $H(M)$-invariant external type in $S_G(M^{ext})$. Since H is normal, there exists a canonical quotient $\pi_H : G \to G/H$ which extends to a projection from types in G over M to types in G/H over M. Jagiella shows the following;

Fact 1.6.11. *[15] Let G be a definable group with a normal definably extremely amenable subgroup H. Let p be the $H(M)$-invariant type in $S_G(M^{ext})$.*

Then the set $S_G(M^{ext}) * p$ is a subflow of $S_G(M^{ext})$ and is isomorphic to $S_{G/H}(M^{ext})$. Further, there is a minimal subflow of $S_G(M^{ext}) * p$ which projects to a minimal subflow of $S_{G/H}(M^{ext})$.

Essentially, if you wish to study the dynamics of a group which admits a normal definably extremely amenable subgroup, it suffices to consider only the quotient space instead, which will often reduce the computations significantly. Further, **[15]** shows that whenever we project minimal flows as above, idempotents map to idempotents and ideal subgroups also map isomorphically. The consequence of this then is the following corollary in which he demonstrates the Ellis Groups of the G-flow and G/H-flow are isomorphic.

Fact 1.6.12. *[15] Let G be a definable group with H a normal definably extremely amenable subgroup of G.*

Then the Ellis Groups of $(G(M), S_G(M^{ext}))$ and $(G(M)/H(M), S_{G/H}(M^{ext}))$ are isomorphic.

To conclude this section, we recall further conjectures of Newelski ([**18**], [**19**]) surrounding Ellis Groups in this model theoretic context. We paraphrase the summary of this conjecture, and its corollary, as given in [**15**].

Conjecture 1.6.13. *Let G be an M-definable group and let N be an elementary extension of M. Then the Ellis Group of $(G(N), S_G(N^{ext}))$ and $(G(M), S_G(M^{ext}))$ are isomorphic.*

We remark that this conjecture also suggests this isomorphism can be found in a definable way. Newelski ([**19**]) provides some partial solutions by constructing a so called $*$-elementary extension, obtained by taking an extension of M^{ext} rather than M and naturally restricting this extension to the original language. This provides an interpretation of externally definable sets in the extension and types in $S_G(M^{ext})$ extend properly to types in $S_G(N^{ext})$. However this is not true for an arbitrary elementary extension. One can then state the following;

Conjecture 1.6.14. *Let G be an M-definable group and let N be an elementary extension of M. Then there is an ideal subgroup in $S_G(N^{ext})$ whose restriction to M is an ideal subgroup of $S_G(M^{ext})$).*

In general, this restriction map is not a semigroup homomorphism, and so it is not known whether the image of an ideal subgroup under this map is itself an ideal subgroup.

These potential limitations of the ideal subgroups is something we need to be aware of. In contrast to results of **[2]**, where the description of G^{00} does not depend on the model, it is important we fix a base model for calculating the Ellis Groups and recognise that the choice of M is not arbitrary in our results.

1.7 Summary

We now summarise the content and key results of the thesis. In Chapter 2, we consider the group G of 2 by 2 special linear matrices with entries from $\mathbb{C}((t))$; the field of formal laurent series with complex coefficients. We recall results surrounding the model theory of this field, find a suitable group decomposition and prove that this group is not definably amenable. We provide explicit description of the minimal flows and Ellis Groups of the additive and multiplicative groups of the field, as well as for the Borel group $B(\mathbb{C}((t)))$ of upper triangular matrices. The main result of Chapter 2 is an explicit description of the Ellis Group of $(G(M), S_G(M))$, and demonstrating that this is not isomorphic to G/G^{00}. This provides a counterexample to the suggestion that the relationship between Ellis Groups and G/G^{00} in stable theories may extend to the metastable setting.

Theorem 1.7.1. *Let $M = \mathbb{C}((t))$, $G = SL_2$ and let B be the borel subgroup of G of upper triangular matrices. Let $\mathcal{I}$ be the minimal subflow of the additive group of $\mathbb{C}((t))$ that contains a single type whose realisations are negatively infinitely valued and lie in $\mathbb{K}^{*0}$. Let $\mathcal{J}$ be the minimal subflow of $(B(M), S_B(M))$.*

Then the Ellis Group of $(G(M), S_G(M))$ is $\mathcal{I} * \mathcal{J}$, and is isomorphic to B/B^0.

Though we use a nonstandard decomposition in Chapter 2, we use the fact that the Ellis Group is unique up to isomorphism to conjecture that $SL_2(\mathbb{C}[[t]])$ admits a unique left-invariant global type. In Chapter 3 we prove this and give an explicit description of this type.

Theorem 1.7.2. *Let $G = SL_2(\mathbb{C}[[t]])$. Then G is definably amenable, $(G, S_G(M))$ admits a 1-point minimal flow $(G, \{q\})$ and so the Ellis Group of $(G, S_G(M))$ is trivial.*

The restriction of q to the stable group $SL_2(\mathbb{C})$ via the residue map is the unique generic type of $SL_2(\mathbb{C})$.

Analogues of this result in the case of $K \models ACVF$ are already known. It is conjectured that their result should extend to metastable groups of algebraically closed residue but possibly non-divisible value group, and so we see that Chapter 3 is further evidence in support of that generalisation.

In Chapter 4 we turn our attention towards the description of Ellis Groups for larger classes of metastable definable groups. We begin by recalling recent results of metastable definable groups and interpreting those results in the context of definable topological dynamics. We demonstrate that for G stably dominated, the collection of stably dominated types forms a subflow of $(G(M), S_G(M))$, though this is not necessarily minimal. Our main results are explicit descriptions of minimal flows and Ellis Groups for metastable definable groups that admit a definably (extremely) amenable / maximally stably dominated group decomposition. The main results of Chapter 4 are as follows.

Theorem 1.7.3. *Suppose $G = NH$ with N normal and both N, $H = G/N$ stably dominated. Then the minimal subflow of $(G, S_G(M^{ext}))$ is precisely $cl(I * J)$, where I is the minimal subflow of $(N, S_N(M^{ext}))$ and J is the minimal subflow of $(H, S_H(M^{ext}))$.*

Theorem 1.7.4. *Let G be a definable group with subgroups H, J of G such that;*

- $G = HJ$
- *J is maximally stably dominated*
- *$(H, S_H(M))$ admits a unique 2-sided 1-point minimal flow, p.*

Then the minimal flow of $(G, S_G(M))$ is a subset $cl(G(M^{ext}) \cdot p * q)$ where q is the principal generic of $Gen_J(M^{ext})$ and the Ellis Group of $(G, S_G(M))$ is a subset of $p * Gen_J(M^{ext}))$.

Theorem 1.7.5. *Let G be a definable stably dominated group. Then there exists some stable group, $\mathfrak{G}$, such that the minimal subflow of $(\mathfrak{G}(M), S_{\mathfrak{G}}(M))$ is expressible as a section of the minimal subflow of $(G(M^{ext}), S_G(M^{ext}))$.*

In future work, we believe removing the assumption that H is definably extremely amenable may be possible.

In Chapter 5 we instead work towards generalisation of the group and metastable setting. Namely, we seek to provide an explicit description of the Ellis Group for G affine algbraic admitting a decomposition with a maximally stably dominated subgroup. We begin with an example of $SL_2(K)$, and compute the minimal flows, Ellis Groups and connected components for the additive, multiplicative and borel subgroups. We also give an explicit description for the left-invariant global type in $S_{SL_2(\mathcal{O}_K}(M)$, as well as consider the minimal flows of the infinite intersection $B(\mathcal{O}_K)$.

We summarise the difficulties in the area surrounding generalisations with infinite intersection, and suggest how we would progress in future work.

Chapter 2

Definable Topological Dynamics of $SL_2(\mathbb{C}((t)))$

We investigate definable, non-definably amenable groups in metastable theories. The motivation here is whether the results for stable groups - where G/G^{00} and the Ellis Group are always isomorphic - have analogues for groups in metastable theories. Work on metastability has been largely concerned with algebraically closed valued fields, though it is known that all valued fields with non-divisible value group and algebraically closed residue field are also metastable [**10**]. The field of formal Laurent series with coefficients from $\mathbb{C}$ is an example of such a field, and is the setting we will be focused on throughout this chapter. We denote this field by $\mathbb{C}((t))$ and note that elements of this field are of the form $\sum_{i=n}^{\infty} a_i t^i$ for some $n \in \mathbb{Z}$. We begin with some classical, preliminary results of $k((t))$, where k is some arbitrary algebraically closed field. Our key references here are [**5**] and [**10**].

2.1 Preliminaries

We consider $\mathbb{C}((t))$ as a valued field. For this chapter, unless otherwise stated, we will let $M = \mathbb{C}((t))$ and will use this notation interchangeably, making no formal distinction between M as a model with universe $\mathbb{C}((t))$ and simply considering $\mathbb{C}((t))$ itself as the model. Similarly, we will use $\overline{M} = \mathbb{K}$ for some elementary extension of $M = \mathbb{C}((t))$, not necessarily saturated, and again will use $\bar{M}$ and $\mathbb{K}$ interchangeably.

The valuation on $\mathbb{C}((t))$ is given by $v\left(\sum_{i=n}^{\infty} a_i t^i\right) = n$, where a_n is the first non-zero coefficient of the series. We call this the t-adic valuation. For $\mathbb{C}((t))$ equipped with the t-adic valuation, we have value group $\mathbb{Z}$ and residue field $\mathbb{C}$. We will denote the valuation ring by $\mathbb{C}[[t]]$ and the maximal ideal by $\mathfrak{M}$.

We now recall the definition of a Henselian valued field here.

Definition 2.1.1. *Let K be a valued field complete with respect to some valuation v. Let $\mathcal{O}_K$ be the valuation ring of K and k the residue field of K. Let $f(x) \in \mathcal{O}_K[X]$.*

Then K is Henselian if the reduction $\bar{f}(x) \in k[x]$ has a simple root (that is, a_0 such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$), then there exists a unique $a \in \mathcal{O}_K$ such that $f(a) = 0$ and the reduction (residue) $res(a) = a_0$.

It is well known that $\mathbb{C}((t))$ equipped with the t-adic valuation is a Henselian valued field.

Corollary 2.1.2. *Let K be a henselian valued field and suppose that an element $a \in K$ is in the coset $1 + \mathfrak{M}$. Then a has n^{th} roots for all $n \in \mathbb{N}$.*

One final thing to consider is the existence of (definably or otherwise) an angular component map.

Definition 2.1.3. *Let K be a valued field with valuation v. Let k be the a residue field of K and Γ the value group of K. Then an angular component map is a map $ac : K \to k$ such that;*

- $ac(0) = 0$
- *The restriction $ac* : K^* \to k^*$ is a group homomorphism (with multiplication from the fields K and k respectively).*

- *The restriction of ac to the group of units is the residue map.*

Not all valued fields may admit an angular component map, though power series field $k((t^\Gamma))$ do, in particular, if $f \in k((t^\Gamma))$ with $f = \sum_{\gamma > \gamma_0} a_\gamma t^\gamma$ with $\gamma_0 \neq 0$, then $ac(f) = a_{\gamma_0}$. That is, map an element f to it's first non-zero coefficient.

However, the angular component map is not (in general) definable, even in a 3-sorted language with sorts of the field, residue field and value group. It is known **[21]** that the theory of henselian valued fields of residue characteristic 0 with an angular component map ac eliminates field quantifiers in this 3-sorted language when equipped with a predicate for ac.

We will be considering $\mathbb{C}((t))$ as a valued field in the language of rings, together with the predicates $N(x), div(x, y)$ and $P_n(x)$ for all $n \in \mathbb{N}$, where;

- $P_n(x) \iff \exists y(y^n = x)$
- $N(x) \iff v(x) = 1$
- $div(x, y) \iff v(x) \leq v(y)$.

The angular component map is not definable in this language. We can however access the angular component for elements with valuation in $\mathbb{Z}$ since we allow parameters from our base model M. That is, if $f \in \mathbb{C}((t))$ with $v(f) = z \in \mathbb{Z}$, we can define the angular component of f as $res(t^{-z} f)$.

The main motivation for using the Delon language is that complete types over $\mathbb{C}((t))$ are all definable, and as such we can avoid working in a Shelah expansion M^{ext} in which the computational work is more difficult.

Fact 2.1.4. *[5]*

- $\mathbb{C}((t))$ *admits quantifier elimination in the language* $(0, 1, +, \times) \cup \{div, N(x)\} \cup \{P_n : n \in \mathbb{N}\}$.
- $Th(\mathbb{C}((t)))$ *is NIP.*

- *The complete* 1*-types over* $M = \mathbb{C}((t))$ *are definable.*

It was later proven in **[10]** that $Th(\mathbb{C}((t)))$ is metastable (over Γ). Of course, there are many groups definable in $\mathbb{C}((t))$ which are known to be definably amenable, the Ellis Groups of which are fully understood.

We remark here that the majority of the work that follows in this Chapter is submitted for publication. We confirm that the results of this chapter are my own and that the submitted publication is a single author paper with no collaboration.

In testing whether the relationship between the Ellis Group and G/G^{00} holds for an arbitrary metastable definable group, we need to look at a group which is not definably amenable. We show that $SL_2(\mathbb{C}((t)))$ is such a group.

Proposition 2.1.5. $SL_2(\mathbb{C}((t)))$ *is not definably amenable.*

Proof. First, it is easy to check that $SL_2(\mathbb{C}((t))) = SL_2(\mathbb{C}((t)))^{00}$, using the fact that $PSL_2(\mathbb{C}((t)))$ is simple and $SL_2(\mathbb{C}((t)))$ is perfect. Assume for contradiction that $SL_2(\mathbb{C}((t)))$ is definably amenable. Then there is a global left $SL_2(\mathbb{C}((t)))^{00}$-invariant, and hence $SL_2(\mathbb{C}((t)))$-invariant, type $p(x)$.

Let $x_1 \neq 0$ (we will handle $x_1 = 0$ later) be the top left entry of a 2×2 matrix. Then $p(x) \vdash x_1 \in C_i$ for some coset C_i of $\mathbb{K}^{*0} = \bigcap_n P_n(x)$.

Consider then the translation g of a realisation of $p(x)$, where $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ for some $a \in \mathbb{C}((t))$. Then $gp(x) \vdash ax_1 \in C_i$ if and only if a is in the identity coset $\mathbb{K}^{*0}$. Clearly since $\mathbb{C}((t))$ is not algebraically closed we can easily find some suitable $a \notin \mathbb{K}^{*0}$ and see $gp(x) \neq p(x)$.

For completeness sake, if the x_1 entry of a realisation of $p(x)$ is 0, since the determinant of the realisation is 1, we see that the top right entry $x_2 \neq 0$ and the above argument follows for the same g. Since p is not G^{00}-invariant then p is not an f-generic type and hence $SL_2(\mathbb{C}((t)))$ is not definably amenable by Fact 1.5.7. □

For the remainder of this chapter, we will use $G = SL_2$ with $G(M) = SL_2(\mathbb{C}((t)))$ unless otherwise stated.

To consider the action of G of $S_G(M)$ directly we would need to understand 3-types over $\mathbb{C}((t))$ which is itself a difficult problem. We instead attempt to build a subflow using complete 1-types and we do this by decomposing G into subgroups of a smaller dimension.

Let

$$H(M) = \{\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right) : \alpha \in M\}$$

and let

$$B(M) = \left\{ \left(\begin{smallmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{smallmatrix}\right) : \beta \in M^* \text{ and } \gamma \in M \right\}.$$

It is clear that the map $f : \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right) \mapsto \alpha$ is an isomorphism from $H(\bar{M})$ into $(\mathbb{K}, +)$. Similarly, one can see $B(\bar{M}) \cong \mathbb{K}^* \times \mathbb{K}$. Finally, we consider the subgroup $\mathbb{Z}/4\mathbb{Z} = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right\}$. We choose this notation as this subgroup is isomorphic to the cyclic group of 4 elements.

Proposition 2.1.6. *Every element of $G(M) = SL_2(\mathbb{C}((t)))$ can be expressed as a product of elements from $\mathbb{Z}/4\mathbb{Z}$, $H(M)$ and $B(M)$.*

Proof. Let $g = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$ be an arbitrary matrix in $G(M)$. Assume that $x_1 \neq 0$ and let $\beta = x_1 \neq 0$, $\gamma = x_2$, $\alpha = x_3(x_1^{-1})$. Then;

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} \beta & \gamma \\ \beta\alpha & \beta^{-1} + \alpha\gamma \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

It remains to show that we can obtain matrices where $x_1 = 0$. Choose $z \in \mathbb{Z}/4\mathbb{Z}$ to be the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and assume $x_1 = 0$. Let $\alpha = 0$, $\beta = x_3$ and $\gamma = x_4$. Then;

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} -\alpha\beta & -(\beta^{-1} + \alpha\gamma) \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Hence for any given arbitrary matrix we can solve for z, α, β and γ as above and thus proves the decomposition. □

We note that there are multiple ways (at most 4) to decompose an element of $G(M)$ in this way. We considered the decomposition used in **[22]**, namely that $SL_2(K) = B(K) \cdot SL_2(\mathcal{O})$ for a valued field K. However, an issue occurs in that $SL_2(\mathbb{C}[[t]])$ is not locally compact. We show later in this thesis that we could in fact deal with this subgroup in several ways, though for this chapter we recall the results as we originally proved them.

We now provide a classification of the 1-types over $\mathbb{C}((t))$ as given in **[5]**.

Fact 2.1.7. *Let $K \prec L$ be valued fields with Γ_K and Γ_L the value groups of K and L respectively. Let $x \in L$. Then to describe the type of x over K, we introduce the following set;*

$$I_K(x) = \{g \in \Gamma_K : \exists k \in K \text{ such that } v(x-k) \geq g\}.$$

We distinguish the following cases;

- *If $\{v(x-k) : k \in K\} \subset \Gamma_K$ and $I_K(x)$ does not have a maximum element, we say that x is immediate over K.*
- *If $\{v(x-k) : k \in K\} \subset \Gamma_K$ and $I_K(x)$ has a maximum element, we say that x is residual over K.*
- *If there exists a $k_0 \in K$ such that $v(x-k_0) \notin \Gamma_K$, then we have that $\{v(x-k) : k \in K\} = I_K(x) \cup \{v(x-k_0)\}$, and we say that x is valuational over K.*

Remark 2.1.8. *We remark that $P_n(x)$ and $x \neq 0$ determine a finite index subgroup of $\mathbb{K}^*$. It is clear that the type $\bigcap_n P_n(x)$ determines the connected component $\mathbb{K}^{*0}$ of the multiplicative group $(\mathbb{K}^*, \times)$. We will use C_i to denote an arbitrary coset of the connected component $\mathbb{K}^{*0}$, with C_0 denoting the identity coset $\mathbb{K}^{*0}$ itself. We use $\mathbb{K}$ here for consistency with the rest of the thesis, though one can see via a countability argument (using the fact that in $ACVF$ an element a is an n^{th} power if and only if $n|v(a)$) that there are as many cosets of $\mathbb{K}^{*0}$ as there are of $\mathbb{C}((t))^{*0}$.*

We establish some final notation before progressing with this proof. By $aP_n(x)$ we mean the multiplicative coset of the set defined by $P_n(x)$. That is, $aP_n(x) = \{ap : p \in P_n(x)\}$. Similarly when using $a\mathbb{K}^{*0}$ we mean the multiplicative coset of $\mathbb{K}^{*0} = \bigcap_n P_n(x)$.

Lemma 2.1.9. *The complete 1-types over $M = (\mathbb{C}((t)), +, \times)$ are precisely the following;*

(a) The (realized) types $tp(a/M)$ for each $a \in \mathbb{C}((t))$.

(b) For each $a \in \mathbb{C}((t))$ and coset C of $\mathbb{K}^{*0}$, the type $p_{a,C}$ determined by the formulas $\{v(x-a) > n : \forall n \in \mathbb{Z}\}$ and $(x-a) \in C$.

(c) For each coset C of $\mathbb{K}^{*0}$, the type $p_{\infty,C}$ determined by the formulas $\{v(x) < n : \forall n \in \mathbb{Z}\}$ and $x \in C$.

(d) For each $a \in \mathbb{C}((t))$ with a of the form $\sum_{m=i}^{j}$, and for each $n \in \mathbb{Z}$ with $n > j$, the type $p_{a,n,trans}$ determined by the formulas $v(x-a) = n$, and $\{f(res((x-a)t^{-n})) \neq 0 : f \in \mathbb{C}[x]\}$.

Proof. It is well known that realised types are complete and since determined by a single formula are clearly consistent.

Claim: Types of Kind (b) are complete.

Proof of Claim. We first prove that p_{0,C_0} is a complete type, but see that the other types of kind (b) are just translations of p_{0,C_0} and completeness is preserved.

To see consistency, consider a finite subset A of the set of formulas $\{P_n(x) : n \in \mathbb{N}\} \cup \{v(x-0) > z : z \in \mathbb{Z}\}$. This finite subset A has a realisation in $\mathbb{C}((t))$ since we can take the product of all n (for each $P_n(x) \in A$) and the maximum value of z (of the $v(x-0) > z$ formulas in A). This obtains some finite integer for which any element of $\mathbb{C}((t))$ with this valuation is n-divisible (for each $P_n(x) \in A$), and hence an n^{th}-power by Hensel's Lemma. Further such an element has valuation strictly greater than the maximum z in the finite set of formulas of the form $v(x-0) > z$ in A. Hence the collection of formulas that we claim determine the types of kind (b) is finitely consistent in $\mathbb{C}((t))$, and hence consistent over $\mathbb{C}((t))$ by compactness.

Since we have quantifier elimination, we need only consider formulas of the form $P_n(f(x))$ and $N(f(x))$ where $f(x) \in \mathbb{C}((t))[x], n \in \mathbb{N}$. We begin with $P_n(f(x))$. Suppose that $f(x) = a_m x^m + ... + a_1 x + a_0$. Let $i \leq m$ be the least such that $a_i \neq 0$.

Let $x_0 \models p_{0,C_0}$. Then $x_0 \in C_0 = \mathbb{K}^{*0}$, and so is an n^{th} power for all $n \in \mathbb{N}$. Hence $P_n(f(x_0)) \iff P_n(x_0^{-i} f(x_0))$.

Since x_0 is infinitely valued, we claim that $P_n(x_0^{-i} f(x_0)) \iff P_n(a_i)$. Consider then $v(x_0^{-i} f(x_0))$. This is in infinitely valued as $x_0^{-i} f(x_0) = a_i + a_{i+1} x_0 + ... + a_m x_0^{m-i}$ and a_i is infinitely valued. As the value of each term in $x_0^{-i} f(x_0)$ is in a different coset of $\mathbb{Z}$, we can divide by a_i to translate the polynomial at x_0 into the coset $1+\mathfrak{M}$. Hence by Hensel's Lemma $a_i^{-1} x_0^{-i} f(x_0)$ has n^{th} roots, and so $P_n(x_0^{-i} f(x_0)) \iff P_n(a_i)$ as required.

We repeat this for $N(f(x))$. Again, let $x_0 \models p_{0,C}$. Recall that $N(f(x_0)) \iff v(f(x_0)) = 1$. Then, since $x_0 \models p_{0,C}$, $v(x_0) > z$ for all $z \in \mathbb{Z}$, and so $f(x_0)$ is the sum of objects from distinct cosets of $\mathbb{Z}$ (as an additive group). That is, $f(x_0) = a_0 + a_1 x_0 + ... + a_n x_0^n$, where $v(a_i) \in iv(x_0) + \mathbb{Z}$. This means that $v(f(x_0)) = 1$ exactly when $v(a_0) = 1$, meaning we have $N(f(x_0)) \iff N(a_0)$ for any $x_0 \models p_{0,C}$.

These arguments demonstrate that p_{0,C_0} is a complete type, and further that each can be determined by a single element proves definability (over $\emptyset$). □

We now prove that types of kind (c) are complete. We show that p_{∞,C_0} is complete, and observe that other types of kind (c) are translations of p_{∞,C_0} and completeness is preserved.

Claim Types of kind (c) are complete.

Proof of Claim. To first see consistency, consider a finite subset A of the set of formulas $\{P_n(x) : n \in \mathbb{N}\} \cup \{v(x) < z : z \in \mathbb{Z}\}$. This finite subset A has a realisation in $\mathbb{C}((t))$ since we can take the product of all n (for each $P_n(x) \in A$) and the modulus of the minimum value of z (of the $v(x < z)$ formulas in A). This obtains some finite integer, the additive inverse of which in $\mathbb{Z}$ has the propery that any element of $\mathbb{C}((t))$ with this valuation is n-divisible, and hence an n^{th}-power (for each $P_n(x) \in A$) by Hensel's Lemma. Further this is strictly less than all z where $v(x) < z$ is in A. Hence the set of formulas which we claim describes the types of kind (c) is finitely consistent, and hence consistent by compactness.

Since we have quantifier elimination, we need only consider formulas of the form $P_n(f(x))$ and $N(f(x))$ where $f(x) \in \mathbb{C}((t))[x], n \in \mathbb{N}$. We begin with $P_n(f(x))$. Suppose that $f(x) = a_m x^m + ... + a_1 x + a_0$. Let $i \leq m$ be the least such that $a_i \neq 0$.

Let $x_0 \models p_{\infty,C_0}$. Then $x_0 \in C_0 = \mathbb{K}^{*0}$, and so is an n^{th} power for all $n \in \mathbb{N}$. Hence $P_n(f(x_0)) \iff P_n(x_0^{-i} f(x_0))$.

We claim that $P_n(x_0^{-m} f(x_0)) \iff P_n(a_m)$. As $f(x_0) = a_m x_0^m + a_{m-1} x_0^{m-1} + ... + a_i x_0^i$, we have $x_0^{-m} f(x_0) = a_m + a_{m-1} x_0^{-1} + a_{m-2} x^{-2} + .. + a_i x_0^{i-m}$. Since $v(x_0) < z$ for all $\in \mathbb{Z}$, we see $v(x_0^{-1}) > z$ for all $z \in \mathbb{Z}$, and hence $x_0^{-m} f(x_0) \in a_m + \mathfrak{M}$. By factoring a_m and again applying Hensel's Lemma, we see $P_n(x_0^{-m} f(x_0)) \iff P_n(a_m)$ as required.

We repeat this for $N(f(x))$. Again, let $x_0 \models p_{\infty,C_0}$. We again see that $f(x_0) = x_0^m(a_m + a_{m-1} x_0^{-1} + ...)$. Then, if $m = 0$, f is a constant polynomial and $N(f(x_0)) \iff N(a_m)$. If $m \neq 0$, then since $x_0 \models p_{\infty,C_0}$, we have $f(x_0))$ negatively infinitely valued, and so $\models \neg N(f(x_0))$ for all non-constant $f \in \mathbb{K}[x]$. This determines every formula of the form $N(f(x))$.

Hence p_{∞,C_0} is a complete type as required. □

Claim: Types of kind (d) are complete.

Proof of Claim. Consistency is clear since for any element A of $\mathbb{C}$ there are infinitely many polynomials with $f(A) = 0$. Hence any finite subset of the formulas which determine types of kind (b) clearly has a realisation in $\mathbb{C}((t))$. That the infinite collection of formulas is consistent then follows from compactness.

Let $p_{b,n,trans}$ be as above. Let $x_0 \models p_{b,n,trans}$. Then $x_0 = b + x = b + \alpha t^n + ...$, where α is transcendental over the residue field of $\mathbb{K}$.

Then by QE, we consider polynomials f of $\mathbb{C}((t))[x]$. We may assume f is not a constant polynomial. Hence

$$\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_m x_0^m \\ &= a_0 + a_1(b+x) + a_2(b+x)^2 + ... + a_m(b+x)^m \\ &= (a_0 + a_1 b + a_2 b^2 + ... + a_m b^m) + (a_1 x + a_2(2bx + x^2) + ... + a_m x^m) \\ &= f(b) + c_1 x + c_2 x^2 + c_3 x^3 + ... c_m x^m \\ &= f(b) + g(x) \end{aligned}$$

Where the coefficients c_i are elements of $\mathbb{C}((t))^{*0}$. This is possible since $b \in \mathbb{C}((t))^{*0}$ and not a variable.

Remember that the angular component is not definable in this setting **[17]**, but since each $c_i x^i$ has some valuation $\beta \in \mathbb{Z}$, we can instead consider $res(c_i x^i t^{-\beta})$. Hence, since x is transcendental over $\mathbb{C}$, the angular component of each term $c_i x^i$ is transcendental over $\mathbb{C}((t))^{*0}$.

Also note that since $\mathbb{C}$ is algebraically closed, $x^i \notin \mathbb{C}$ for any i.

We can express $g(x) = d_0\delta_0 + d_1\delta_1 + d_2\delta_2 + ...$ with $d_i \in \mathbb{C}((t))$ and δ_i transcendental over $\mathbb{C}$. Using this, we know $P_n(g(x)) \iff P_n(d_0\delta_0) \iff P_n(d_0)$ since δ_0 is transcendental over the residue field, and so since $Th(\mathbb{C}((t)))$ knows it has algebraically closed residue, in some expansion $\mathbb{K}(x)$ with residue field $acl(res(\mathbb{K}(x_0)))$, δ_0 has n^{th} roots for all n, and so $\models P_n(\delta_0)$ for all n.

Hence $P_n(g(x)) \iff P_n(d_0)$. We then just need to account for $f(b)$, but since this is itself an element of $\mathbb{C}((t))^{*0}$, we just see $P_n(f(x_0)) \iff P_n(f(b)+g(x)) \iff P_n(f(b) + d_0)$. Hence $P_n(f(x_0))$ is determined.

We can determine $N(f(x_0))$ similarly by considering the valuation of $g(x)$. Since the transcendental coefficients have valuation 0, $N(g(x))$ is determined by the leading term, d say, which is an element of $\mathbb{C}((t))$. Hence $N(f(x_0)) \iff N(f(b)+d)$.

Hence $P_n(f(x_0))$ and $N(f(x_0))$ are determined and hence $p_{b,n,trans}$ is a complete type as required. □

Hence we have demonstrated that the types as above are complete 1-types over $\mathbb{C}((t))$.

Claim: There are no other complete 1-types over $\mathbb{C}((t))$.

Proof of Claim: To see that these are all the complete 1-types over $\mathbb{C}((t))$ we apply Fact 2.1.7 from [**5**]. As in Fact 2.1.7 we define the set $I_{\mathbb{C}((t))}(x) = \{g \in \mathbb{Z} : \exists a \in \mathbb{C}((t)) \text{ with } v(x-a) \geq g\}$.

Immediate: Clearly, the immediate types correspond to the realised types determined by the formula $x = a$.

Residual: To see that the residual types correspond to those of kind (d), assume that we have the type of x over $\mathbb{C}((t))$ for which $\{v(x-k) : k \in \mathbb{C}((t))\}$ is a subset of $\mathbb{Z}$ and $I_K(x)$ obtains a maximum value in $\mathbb{Z}$.

Since $I_{\mathbb{C}((t))}(x)$ obtains a maximum value in $\mathbb{Z}$ and $\mathbb{C}((t))$ can be considered as an additive group, we must have that $x \notin \mathbb{C}((t))$ (else the type of x over $\mathbb{C}((t))$ would be realised). Further, since $\{v(x-a) : a \in \mathbb{C}((t))\}$ is a subset of $\mathbb{Z}$, we must have that $v(x) \in \mathbb{Z}$. Hence x lies in some extension $\mathbb{K}$ of $\mathbb{C}((t))$ where $\Gamma_{\mathbb{K}} = \mathbb{Z}$, and so $res(x) \notin \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, $res(x)$ is transcendental over $\mathbb{C}$.

Moreover, for a fixed $a \in \mathbb{C}((t))$, by the above we have $v(x-a) \in \mathbb{Z}$. Since $I_{\mathbb{C}((t))}(x)$ obtains a maximum value n in $\mathbb{Z}$, the type of x over $\mathbb{C}((t))$ must contain the formula $v(x-a) = n$ for any $a \in \mathbb{C}((t))$ for which $v(a) \geq n$. What this means is, to describe the type of x over $\mathbb{C}((t))$ is suffices to consider only the terms in x which have value less than n. Specifically, these terms can be thought of as an element of $\mathbb{C}((t))$ with finite support.

Hence x is of the form $a+x'$ where $v(x')=n$, $a \in \mathbb{C}((t))$ and a has finite support. Hence, over $\mathbb{C}((t))$, we can determine all residual types via the formulas $v(x-a)=n$ and $\{f(res((x-a)t^{-n}) \neq 0 : f \in \mathbb{C}[x]\}$, where a is of the form $\sum_{m=i}^{j} a_m t^m$ and $j<n$ as required. This is precisely the description of (d). Note that the $P_n(x)$ predicates are not necessary here as the n^{th} powers are determined exclusively by the valuation (which is in $\mathbb{Z}$) since the residue field is algebraically closed.

Valuational: Let us consider the type of x over $\mathbb{C}((t))$ which is valuational over $\mathbb{C}((t))$ as given in Fact 2.1.7.

Then there exists an $a \in \mathbb{C}((t))$ for which $v(x-a) \notin \mathbb{Z}$. Since $Th(\mathbb{C}((t)))$ knows $\mathbb{Z}$ is discrete, we must have either $v(x-a)>z$ for all $z \in \mathbb{Z}$ or $v(x-a)<z$ for all $z \in \mathbb{Z}$. Clearly this means there is more than 1 possibility for the type of x over $\mathbb{C}((t))$ as these formulas are not finitely consistent. Moreover, if $v(x-a)<z$ for all $z \in \mathbb{Z}$, and $a \in \mathbb{C}((t))$, we must have $v(x)<z$ for all $z \in \mathbb{Z}$ since $v(x-a)=min\{v(x),v(a)\}$.

Since the angular component map is not definable in this language (see [**17**]), we can not define the properties x relative to the residue field since $v(x) \notin \mathbb{Z}$. It remains to consider the predicates of the language. Since it does not make sense to ask whether something not in $\mathbb{Z}$ is n-divisble, and since types must complete, for any $P_n(x)$ the type of x over $\mathbb{C}((t))$ must contain either $P_n(x)$ or $\neg P_n(x)$ for all n. Hence the type of x over $\mathbb{C}((t))$ must know which coset of $\bigcap_n P_n(x)$ it is contained in.

Hence the type of x over $\mathbb{C}((t))$ where x is valuational is determined by either $\{v(x-a)>z : z \in \mathbb{Z}\} \cup x \in C_i$, which is precisely the types of kind (b), or by $\{v(x)<z : z \in \mathbb{Z}\} \cup x \in C_i$, which is precisely the types of kind (c), as required. □

Hence we have shown that this is the complete list of complete 1-types over $\mathbb{C}((t))$ (in the Delon language) as required. □

Corollary 2.1.10. *Every (left) $\mathbb{K}^*$-translate of p_{0,C_0} is definable over M.*

Proof. Let $a \in \bar{M}^*$ and $x_0 \models p_{0,C}$ Suppose $P_n(f(x)) \in ap_{0,C}$. Then $ax_0 \models P_n(f(x)) \iff a^{-1}P_n(f(x)) \in p_{0,C}$. Since $P_n(\mathbb{K}^*)$ has finite index in $\mathbb{K}^*$, $\exists b \in M^*$ such that $a^{-1}P_n(\mathbb{K}^*) = bP_n(\mathbb{K}^*)$. Hence $P_n(f(x)) \in ap_{0,C} \iff bP_n(f(x)) \in p_{0,C}$. As p_{0,C_0} is $\emptyset$-definable, b is $\mathbb{C}((t))$-definable and so ap_{0,C_0} is definable over M as required.

□

2.2 Additive and Multiplicative Groups of $\mathbb{C}((t))$

We consider the additive and multiplicative groups of $\mathbb{C}((t))$.

We denote by $S_{\mathbb{G}_a}(M)$ the space of complete types concentrating on $\mathbb{G}_a$, where $\mathbb{G}_a(M) = (\mathbb{C}((t)), +)$, and so $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$ is a flow under the additive group action. We note that $\mathbb{G}_a(\overline{M}) = (\mathbb{K}, +)$. By global in this context we mean over some large sufficiently saturated model. Recall that by a (left) f-generic type (over some set A) we mean a global type $p \in S_G(\mathcal{U})$ such that no (left) translate of p forks over A.

Proposition 2.2.1.

(i) *The types $p(x) \in S_{\mathbb{G}_a}(M)$ of kind (c) are definable generic types of $(\mathbb{G}_a, +)$. Moreover, the global heir of $p_{\infty,C}$ is invariant under the action of $(\mathbb{K}, +)$ for any coset C of $\mathbb{K}^{*0}$.*

(ii) *The types $\{p_{\infty,C}\}$ are 1-point minimal subflows of $(\mathbb{G}_a(M), S_{\mathbb{G}_a}(M))$.*

(iii) *The global heirs of the types of kind (c) are precisely the global (strongly) f-generics of $(\mathbb{K}, +)$ and are all definable and invariant under $(\mathbb{K}, +)$, and hence $\mathbb{K}^{00} = \mathbb{K}^0 = \mathbb{K}$.*

Proof.

(i) Suppose that $a \in \mathbb{K}$ and $\beta \models p_{\infty,C}|_{Ma}$. Since $v(\beta) = \alpha < \Gamma$, and $v(a) = c \in \Gamma$, we have $v(\beta) < v(a)$ and hence $v(a + \beta) < \Gamma$. It remains to show that

$a+\beta \in P_n(\mathbb{K})$. Since $v(\beta) = -v(\beta^{-1})$, and $v(\beta^{-1}) > \Gamma$, we have $v(a\beta^{-1}) > \Gamma$. Hence $1 + a\beta^{-1} \in 1 + \mathfrak{M}$, and so by Hensel's Lemma $1 + a\beta^{-1}$ has n^{th} roots in $\mathbb{K}$ for all n. Hence $\beta(1+a\beta^{-1}) = a+\beta \in \beta\mathbb{K}^{*0}$, and so $a+\beta \models p_{\infty,C}$. Since this type is $\mathbb{K}$-invariant, in general, it must be $\mathbb{K}^{00}$-invariant and hence is f-generic.

(ii) This follows from (i). Let $q \in S_{(\mathbb{C}((t)),+)}(M)$ and consider $q * p_{\infty,C} = tp(a + \beta/M)$. Then from above we have $tp(a+\beta/M) = tp(\beta/M)$, and hence is a subflow of $S_{\mathbb{G}_a}(M)$ under the action of $(\mathbb{K},+)$. Minimality follows trivially since $\{p_{\infty,C}\}$ is a singleton, there can be no properly contained non-empty subflow.

(iii) Since $(\mathbb{K},+)$ is abelian, it is amenable as a discrete group and hence definably amenable **[3]**. Hence by Fact 1.5.6, $(\mathbb{K},+)$ admits some global f-generic type. Further, by Corollary 8.20 of **[3]**, as $(\mathbb{K},+)$ admits a global f-generic, it must admit an f-generic over any model; in particular, over M. Hence we need only check the complete 1-types over M.

We claim that the p_{∞,C_i} are all f-generic over M. Note that by Fact 1.5.7 if a type $q \in S_{\mathbb{G}_a}(M)$ is f-generic then $Stab(q) = (\mathbb{K}^{00},+)$.

Clearly, immediate types are not f-generic. The stabilisers of those types are trivial in all models and hence not of bounded index in $(\mathbb{K},+)$, meaning the stabilizer is not $(\mathbb{K}^{00},+)$ and so not f-generic.

To decide whether the transcendental types are f-generic, it suffices to consider the type $p_{0,0,trans}$. We see that $Stab(p_{0,0,trans})$ is precisely the valuation ring, which does not have bounded index in $(\mathbb{K},+)$ and further has an infinite chain of smaller subgroups, and so is not $(\mathbb{K}^{00},+)$.

This leaves the valuational types. It is clear that the infinitesimal types are not f-generic as their stabilizers in the additive group are also trivial, and hence not of bounded index in $(\mathbb{K}, +)$.

Hence the types p_{∞,C_i} must be f-generic, and we confirm this further by considering that $Stab(p_{\infty,C_i})$ is $(\mathbb{K}, +)$ itself, which is of course of bounded index and hence we conclude $(\mathbb{K}^{00}, +) = (\mathbb{K}^0, +) = (\mathbb{K}, +)$ as required.

□

Similarly, we denote by $S_{\mathbb{G}_m}(M)$ the space of complete types concentrating on $\mathbb{G}_m$, where $\mathbb{G}_m(M) = (\mathbb{C}((t))^*, \times)$, and so $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$ is a flow under multiplication. Again, we have $\mathbb{G}_m(\overline{M}) = (\mathbb{K}^*, \times)$ and will use both notations where appropriate.

Proposition 2.2.2.

(i) The types $P_0 = \{p_{0,C} : C$ some coset of $(\mathbb{K}^*)^0\}$ form a minimal subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$.

(ii) The types $P_\infty = \{p_{\infty,C} : C$ some coset of $(\mathbb{K}^*)^0\}$ form a minimal subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$.

(iii) The type-definable connected component $\mathbb{K}^{*00}$ coincides with the definable connected component $\mathbb{K}^{*0}$.

***Proof*.**

(i) To show P_0 is a minimal subflow, we show it is precisely the $S_{\mathbb{G}_m}(M)$-orbit (meaning under the semigroup action of type multiplication) of a type p_{0,C_0}.

Let $q \in S_{\mathbb{G}_m}(M)$ with a realising q and α realise the heir of p_{0,C_0} over (M, a). Then $q * p_{0,C_0} = tp(a\alpha/M)$. Since $v(a\alpha) = v(a) + v(\alpha) > \Gamma$, then $tp(a\alpha/M)$ must be a type of kind (b), with $a\alpha$ infinitesimally close to 0. Hence, $tp(a\alpha/M) = p_{0,C_i}$ for C_i some coset of $\mathbb{K}^{*0}$. Further, since α is an element of the identity coset C_0, we have $a\alpha \in C_i \iff a \in C_i$.

However, the choice of q (and a) was arbitrary. In particular, a could lie in any coset C_i, and so the $S_{\mathbb{G}_m}(M)$-orbit of p_{0,C_0} is $P_0 = \{p_{0,C_i} : C_i \text{ a coset of } \mathbb{K}^{*0}\}$ as required.

(ii) To show P_∞ is a minimal subflow, we show it is precisely the $S_{\mathbb{G}_m}(M)$-orbit of a type p_{∞,C_0}.

Let $q \in S_{\mathbb{G}_m}(M)$ with a realising q and α realise the heir of p_{∞,C_0} over (M, a). Then $q * p_{\infty,C_0} = tp(a\alpha/M)$. Since $v(a\alpha) = v(a) + v(\alpha) < \Gamma$, then $tp(a\alpha/M)$ must be of kind (c); that is, $tp(a\alpha/M) = p_{\infty,C_i}$ for C_i some coset of $\mathbb{K}^{*0}$. Again, since α is an element of the identity coset C_0, we have $a\alpha \in C_i \iff a \in C_i$.

However, the choice of q (and a) was arbitrary. In particular, a could lie in any coset C_i, and so the $S_{\mathbb{G}_m}(M)$-orbit of p_{∞,C_0} is $P_\infty = \{p_{\infty,C_i} : C_i \text{ a coset of } \mathbb{K}^{*0}\}$ as required.

(iii) Since $(\mathbb{K}^*, \times)$ is abelian, it is amenable as a discrete group and hence definably amenable **[3]**. Hence by Fact 1.5.6, $(\mathbb{K}^*, \times)$ admits some global f-generic type. Further, by Corollary 8.20 of **[3]**, as $(\mathbb{K}^*, \times)$ admits a global f-generic, it must admit an f-generic over any model; in particular, over M. Hence we need only check the complete 1-types over M.

We claim that the p_{∞,C_i} and p_{0,C_i} are all f-generic over M. Note that by Fact 1.5.7 if a type $q \in S_{\mathbb{G}_m}(M)$ is f-generic then $Stab(q) = (\mathbb{K}^*, \times)$

Clearly, immediate types are not f-generic. The stabilisers of those types are trivial in all models and hence not of bounded index in $(\mathbb{K}^*, \times)$, meaning they are not $(\mathbb{K}^{*00}, \times)$ and so not f-generic.

For the transcendental types, it is clear that the stabilizers of those types are the multiplicative group of the residue field, which are not of bounded index in $(\mathbb{K}^*, \times)$ and so the transcendental types are not f-generic.

We are left with the valuational types. The stabilizers of these types are equivalent, and are all equal to $\bigcap_n P_n(x) = (\mathbb{K}^{*0}, \times)$, which is of bounded index in $(\mathbb{K}^*, \times)$ and hence since at least one of these types must be f-generic, we conclude $(\mathbb{K}^{*0}, \times) = (\mathbb{K}^{*00}, \times)$ as required.

□

2.3 The Borel Subgroup of $SL_2(\mathbb{C}((t)))$

We now consider the Borel subgroup, $B(\bar{M})$, of upper triangular matrices. We will often associate the matrix $\left(\begin{smallmatrix} b & c \\ 0 & b^{-1} \end{smallmatrix}\right) \in B(\bar{M})$ with the pair (b, c) where $b \in \mathbb{K}^*$ and $c \in \mathbb{K}$. For this section, multiplication of these pairs is given by matrix multiplication and so $(b, c)(\beta, \gamma) = (b\beta, b\gamma + c\beta^{-1})$.

Lemma 2.3.1. $B(\bar{M})^{00} = B(\bar{M})^0 \cong \{(b, c) : b \in \mathbb{K}^{*0}, c \in \mathbb{K}\}$.

Proof. Consider the following mapping;

$$\pi : B(\bar{M}) \to \mathbb{K}^*$$
$$(b, c) \mapsto b$$

With $Ker(\pi) = (\mathbb{K}, +)$. Then it is clear that $B(\mathbb{K}^{00}) \to \mathbb{K}^{*00}$ has Kernel isomorphic to $(\mathbb{K}^{00}, +)$. Using the results of Propositions 2.2.1 and 2.2.2 that $\mathbb{K}^{*00} = \mathbb{K}^{*0}$ and $(\mathbb{K}^{00}, +) = (\mathbb{K}, +)$, we obtain $B(\bar{M})^{00} = B(\bar{M})^0 = \{(b, c) : b \in \mathbb{K}^{*0}, c \in \mathbb{K}\}$.

□

Recall that C_0 denotes $(\mathbb{K}^*)^0$, p_{0,C_0} is an f-generic of $S_{\mathbb{G}_a}(M)$ and that p_{∞,C_0} is an f-generic type in $S_{\mathbb{G}_m}(M)$. Let β realise $p_{0,C_0}|_{\bar{M}}$ and γ realize $p_{\infty,C_0}|_{\bar{M},\beta}$.

Consider then the pairs $(\beta, 0)$ and $(1, \gamma)$ and we identify these pairs with the types $tp((\beta, 0)/\bar{M})$ and $tp((1, \gamma)/\bar{M}, \beta)$ of the corresponding matrix. Then $p_{0,C_0} * p_{\infty,C_0} = tp((\beta, 0)/\bar{M}) * tp((1, \gamma)/\bar{M}, \beta) = tp((\beta, \gamma\beta)/\bar{M})$.

Further, since γ realises an heir over $\bar{M}\beta$, we have that p_{∞,C_0} is invariant under left multiplication by β^{-1}, and hence $tp((\beta, \gamma\beta^{-1})/\bar{M}) = tp((\beta, \gamma)/\bar{M})$.

Let $\bar{p_0} = tp((\beta, \gamma)/\bar{M}) \in S_B(\bar{M})$, and so by p_0 we mean the restriction of this type to M.

Lemma 2.3.2. *$\bar{p_0} \in S_B(\bar{M})$ is a $B^{00}(\bar{M})$-invariant global type, and moreover every left $B(\bar{M})$-translate is definable over M (i.e. definable over $\mathbb{C}((t))$).*

Proof. We first show that $\bar{p_0}$ is $B(\bar{M})^{00}$-invariant. Let $(b, c) \in B(\bar{M})^{00}$, which by Lemma 2.3.1, means $b \in (\mathbb{K}^*)^0$ and $c \in \mathbb{K}$. Since the operation here is matrix multiplication, we note that $(b, c)(\beta, \gamma) = (b\beta, b\gamma + c\beta^{-1})$.

We want to show that $tp((b\beta, b\gamma + c\beta^{-1})/\bar{M}) = tp((\beta, \gamma)/\bar{M})$.

It is equivalent to show that $tp(b\beta/\bar{M}) = tp(\beta)$ and that $tp(b\gamma + c\beta^{-1}/\bar{M}, b\beta)$ is an heir of $tp(\gamma/\bar{M})$.

As $b \in \mathbb{K}^{*0}$ we have that $tp(b\beta/\bar{M}) = tp(b/\bar{M})$. Then since $b\beta \equiv_{\bar{M}} \beta$, γ must also realise the heir of p_{∞,C_0} over $\bar{M}, b\beta$.

Since p_{∞,C_0} is invariant under multiplication by any element in $\mathbb{K}^{*0}$, we have $tp(b\gamma/\bar{M}, b\beta) = tp(\gamma/\bar{M}, b\beta)$.

Moreover, as $v(\gamma) < \Gamma \cap dcl(M, \beta)$, $tp(b\gamma + c\beta^{-1}/\bar{M}, b\beta) = tp(b\gamma/\bar{M}, b\beta)$ and so $tp(b\gamma + c\beta^{-1}/\bar{M}, b\beta)$ is an heir of $tp(\gamma/\bar{M})$.

Again, since p_{∞,C_0} is invariant under multiplication by elements of $\mathbb{K}^{*0}$, we have that $tp((b\beta)^{-1}(b\gamma + c\beta^{-1})/\bar{M}, b\beta)$ is an heir of $tp(\gamma/\bar{M})$.

Since $b\beta$ realises p_{0,C_0} and $(b\beta)^{-1}(b\gamma + c\beta^{-1})$ realises the unique heir of p_{∞,C_0} over $(M, b\beta)$, we have that $p_{0,C_0} * p_{\infty,C_0} = tp((b\beta, b\gamma + c\beta^{-1})/\bar{M}) = tp((\beta, \gamma)/\bar{M})$.

Hence $\bar{p_0}$ is a $B(\bar{M})^{00}$-invariant type of $S_B(\bar{M})$.

Finally, since $tp(\beta/\bar{M})$ is definable over M, and $tp(\gamma/\bar{M},\beta)$ is the heir of p_{∞,C_0}, which is also definable over M, we have that $tp((\beta,\gamma)/\bar{M})$ is definable over M. It is clear using the above argument that every left $B(\bar{M})$-translate of $\bar{p_0}$ is definable over M.

□

Proposition 2.3.3.

(i) The $B(\bar{M})$-orbit of $\bar{p_0}$ is closed and is a minimal $B(\bar{M})$-subflow of $S_B(\bar{M})$. We call this subflow $\bar{\mathcal{J}}$.

(ii) The restriction of $\bar{\mathcal{J}}$ to M, denoted $\mathcal{J}$, is a subgroup of $(S_B(M),*)$, is isomorphic to $B(\bar{M})/B(\bar{M})^0$ and hence is the Ellis Group of the flow $(B(M),S_B(M))$.

Proof.

(i) The fact that the orbit is closed follows from Lemma 1.15 of **[29]**, and it is well known that a non-empty set is a minimal flow if and only if it is the orbit closure of each of its points, a proof of which can be found in **[1]**.

(ii) First, we note that p_0 is itself contained in $S_{B^0}(M)$, and since $\bar{p_0}$ is $B(\bar{M})^0$-invariant by Lemma 2.3.2, we have that the restriction to p_0 is $B(M)^0$-invariant and hence idempotent.

That subflows are preserved under restrictions is a consequence of Proposition 5.4 of **[29]**. Since $\bar{\mathcal{J}}$ is the minimal subflow of $(B(\bar{M}),S_B(\bar{M}))$, the restriction $\mathcal{J}$ is a minimal subflow of $(B(M),S_B(M))$. We can then form the Ellis group of $(B(M),S_B(M))$, which is $(p_0*\mathcal{J},*)$.

We now show that $p_0*\mathcal{J}=\mathcal{J}$. Clearly, $p_0*\mathcal{J}\subseteq\mathcal{J}$, as $\mathcal{J}$ is a minimal subflow of $S_B(M)$. Let $p_i\in\mathcal{J}$. We claim that $p_0*p_i=p_i$.

Claim: $p_0*p_i=p_i$.

Proof of Claim. Let (b, c) realise p_0 and (β, γ) realise the heir of p_i over $(M, (b, c))$. Then we want to show that $tp((b\beta, b\gamma + c\beta^{-1})/M) = tp((\beta, \gamma)/M)$.

The valuational arguments in Lemma 2.3.2 carry over, namely that if β is negatively infinitely valued then so is $b\beta$, and that $v(\gamma) < dcl(M, \beta) \cap \Gamma$. It remains to prove that $b\beta$ lies in the same coset as β, and that $b\gamma + c\beta^{-1}$ lies in the same coset as γ.

Since $\mathbb{K}^{*0}$ acts as the identity on $\mathbb{K}^*/\mathbb{K}^{*0}$, and $b \in \mathbb{K}^{*0}$, we know that $b\beta$ lies in the same coset as β.

For $b\gamma + c\beta^{-1}$ this is not as clear. Instead observe that $\gamma + b^{-1}c\beta^{-1}$ is in the same coset as γ if and only if $\gamma^{-1}(\gamma + b^{-1}c\beta^{-1}) = 1 + \gamma^{-1}b^{-1}c\beta^{-1} \in \mathbb{K}^{*0}$. Since $v(\gamma^{-1})$ is infinite over (M, b, c, β), we see that $1 + \gamma^{-1}b^{-1}c\beta^{-1} \in 1 + \mathfrak{M}$.

Hence by the corollary to Hensel's Lemma (2.1.2), $1+\gamma^{-1}b^{-1}c\beta^{-1}$ has n^{th} roots for all n and so lies in $\mathbb{K}^{*0}$. Hence $\gamma + b^{-1}c\beta^{-1}$ lies in the same coset as γ. Finally, since $b \in \mathbb{K}^{*0}$, $b(\gamma + b^{-1}c\beta^{-1}) = b\gamma + c\beta^{-1}$ also lies in the same coset as γ as required. Hence $p_0 * p_i = p_i$ for all $p_i \in \mathcal{J}$. □

So $p_0 * \mathcal{J} = \mathcal{J}$, with p_0 acting as identity we see $(p_0 * \mathcal{J}, *) = (\mathcal{J}, *)$.

From this, we obtain the following map;

$$\pi : \mathcal{J} \to B(\bar{M})/B(\bar{M})^0$$
$$tp(t/M) \mapsto tB(\bar{M})^0$$

It is clear that π is well defined, and we now show that π is an isomorphism.

First, we show π is a group homomorphism. Note that it is not longer necessary to consider entries in the realisations of these types, and so for simplicities sake we make a small change in notation here.

Let t_0 realise p_0. Then t_0 is a matrix in $B(\bar{M})^0$, and so clearly $t_0B(\bar{M})^0 = B(\bar{M})^0$, the identity element of the quotient group B/B^0.

Since $\mathcal{J}$ is a group, let $p_i \in \mathcal{J}$ with inverse p_i^{-1}. Then p_i is realised by some t_i, and so $\pi(p_i) = t_i(B(\bar{M})^0)$.

The heir of p_i^{-1} over (M, t_i), and so in particular p_i^{-1} itself, is realised by some s_i, and so $\pi(p_i^{-1}) = s_i(B(\bar{M})^0)$. Note we are not claiming s_i is the inverse of t_i in G, just that s_i is a realisation of the inverse of p_i in $\mathcal{J}$.

We claim that $s_i(B(\bar{M})^0) \cdot t_i(B(\bar{M})^0) = B(\bar{M})^0$.

From coset multiplication we have $s_i(B(\bar{M})^0) \cdot t_i(B(\bar{M})^0) = (t_i s_i)B(\bar{M})^0$. Then as $p_i^{-1} * p_i = p_0$, we see $s_i t_i \in B(\bar{M})^0$, and so $s_i(B(\bar{M})^0) \cdot t_i(B(\bar{M})^0) = B(\bar{M})^0$.

Hence $\pi((p_i)^{-1}) = ((t_i)(B(\bar{M})^0))^{-1}$ as required, and so π is a group homomorphism.

It is easy to see that π is bijective. Note $\mathcal{J}$ is a section of $B(\bar{M})/B(\bar{M})^0$. That is to say π is surjective since, for every coset $t(B(\bar{M})^0)$, we can associate a type $p_i \in \mathcal{J}$ with $t' \in t(B(\bar{M})^0)$ and $t' \models p_i$ such that $\pi(p_i) = t(B(\bar{M})^0)$. Injectivity follows from the definition of $\mathcal{J}$, observing that each type in $\mathcal{J}$ is determined uniquely by a coset of $\mathbb{K}^{*0}$.

□

We make no claims that the above minimal subflow is unique. In fact, taking $v(\beta) < \mathbb{Z}$ and $v(\gamma) < \Gamma \cap dcl(M, \beta)$, both in C_0, describes another left-invariant idempotent element p_0' of $(S_B(M), *)$. It is easy to check that $p_0 * p_0' = p_0'$ and $p_0' * p_0 = p_0$. This dichotomy is important as it confirms that the types p_0 and p_0' cannot share a minimal flow using a result of topological dynamics found in [**1**]. As such, $\mathcal{J}$ is definitely not the unique minimal flow of $(B(M), S_B(M))$. This is simply noted for interest, since the isomorphism of Ellis Groups means our choice of minimal flow is unimportant beyond personal preference or perhaps ease of computation.

2.4 The Minimal Subflow of $(G(M), S_G(M))$.

Recall $\mathcal{J}$ is the minimal subflow of $(B(M), S_B(M))$. As before, p_i will denote a type in $\mathcal{J}$, where p_i specifies in which coset C_i of $\mathbb{K}^*/\mathbb{K}^{*0}$ the first coordinate of the realisation of p_i lies. Later in this section we may need to be specific about the coset, and we clarify this change in notation when it used.

We will again use the notation in 2.1.9 for the valuational types, with p_{∞,C_0} the minimal subflow of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$.

We will often associate some $h \vDash p_{\infty,C_0}$ with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$, and likewise some $t \vDash p_i$ with the matrix $\left(\begin{smallmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{smallmatrix}\right)$. We will not distinguish between $z \in \mathbb{Z}/4\mathbb{Z}$ and the type determined by the formula $x = z$. As before, $\mathbb{K}$ will denote some elementary extension of $\mathbb{C}((t))$ with C_0 denoting the identity coset $\mathbb{K}^{*0}$ itself.

We approach the construction of the minimal flow differently to how it has been done in the past. In the literature, they take the closure of the $*$-product of the minimal flows of groups in the decomposition for G and prove that it is indeed minimal. In the following work, we instead begin with an idempotent element and build our minimal flow around it by taking the closure of the $G(M)$-orbit. In general, this will not be a minimal flow; for example, the realised type of the identity element will have $G(M)$-orbit equal to $G(M)$ itself which is rarely minimal.

Proposition 2.4.1. *Let $p_0 \in \mathcal{J}$ as in Lemma 2.3.2 and p_{∞,C_0} a minimal subflow of $S_{\mathbb{G}_a}(M)$. Then the type $p_{\infty,C_0} * p_0$ is an idempotent element of $(S_G(M), *)$.*

Proof. To show this is an idempotent, we need to show $(p_{\infty,C_0} * p_0) * (p_{\infty,C_0} * p_0) = p_{\infty,C_0} * p_0$.

Let h_0 realise p_{∞,C_0}, let t_0 realise the heir of p_0 over (M, h_0), let h realise the heir of p_{∞,C_0} over (M, h_0, t_0) and let t realise the heir of p_0 over (M, h_0, t_0, h). Then $(p_{\infty,C_0} * p_0)^2 = tp(h_0 t_0 h t/M)$. Then;

$$
\begin{aligned}
h_0t_0ht &= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b + c\alpha & c \\ b^{-1}\alpha & b^{-1} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{b^{-1}\alpha}{b+c\alpha} & 1 \end{pmatrix} \begin{pmatrix} b + c\alpha & c \\ 0 & (b+c\alpha)^{-1} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ a + \frac{b^{-1}\alpha}{b+c\alpha} & 1 \end{pmatrix} \begin{pmatrix} \beta(b+c\alpha) & \gamma(b+c\alpha) + c\beta^{-1} \\ 0 & \beta^{-1}(b+c\alpha)^{-1} \end{pmatrix}
\end{aligned}
$$

Then, since $(\beta, \gamma) \models p_0|_{M,h_0,t_0,h}$, we see that $(\beta(b + c\alpha), \gamma(b + c\alpha) + c\beta^{-1})$ also realises $p_0|_{M,h_0,t_0,h}$ since $p_0 \in \mathcal{J}$ (the minimal subflow) and all $b, c, \beta, \gamma, \alpha \in \mathbb{K}^{*0}$.

We prove that $a + \frac{b^{-1}\alpha}{b+c\alpha}$ realises p_{∞,C_0}. Write $\frac{b^{-1}\alpha}{b+c\alpha} = (b^2\alpha^{-1} + bc)^{-1}$. Then, since $v(c) < dcl(M, b, \alpha) \cap \Gamma$, we see $v(b^2\alpha^{-1} + bc) < dcl(M, b, \alpha) \cap \Gamma$.

Hence $v((b^2\alpha^{-1} + bc)^{-1}) > dcl(M, b, \alpha) \cap \Gamma$. Then $v(a + \frac{b^{-1}\alpha}{b+c\alpha}) = v(a) < \mathbb{Z}$, and hence $a + \frac{b^{-1}\alpha}{b+c\alpha} \models p_{\infty,C_0}$.

Hence $p_{\infty,C_0} * p_0$ is idempotent in $(S_G(M), *)$. □

Consider a type q in $S_G(M)$. Then by using the group decomposition from Proposition 2.1.6, we see that we can express any realisation g of q in the form $g = zht$ for $z \in \mathbb{Z}/4\mathbb{Z}$, $h \in H(\bar{M})$ and $t \in B(\bar{M})$. The same can be done for any $g \in G(M)$, this time with $z \in \mathbb{Z}/4\mathbb{Z}$, $h \in H(M)$ and $t \in B(M)$.

We first compute the action of $G(M)$ on $p_{\infty,C_0} * p_0$, and do so by considering the action of $H(M)$, $B(M)$ and $\mathbb{Z}/4\mathbb{Z}$ separately.

Proposition 2.4.2. *The $H(M)$-orbit of $p_{\infty,C_0} * p_0$ is $p_{\infty,C_0} * p_0$.*

Proof. Clearly, since p_{∞,C_0} is a minimal flow of the additive group, $H(M)$ acts trivially, and we see $H(M) * p_{\infty,C_0} * p_0 = p_{\infty,C_0} * p_0$. □

The following computations require us to be more precise about the cosets of $\mathbb{K}^{*0}$ than previously in the thesis. In the following work, $p_0 \in \mathcal{J}$ will still mean the identity coset, and p_i, p_j,... will denote arbitrary elements of $\mathcal{J}$. However, by $p_{k\mathbb{K}^{*0}}$, we mean a type in $\mathcal{J}$ realised by some (β, γ) with $\beta, \gamma \in k\mathbb{K}^{*0}$.

Similarly, for one types, C_0 will still denote the identity coset, but when necessary we will be explicit and use $k\mathbb{K}^{*0}$ in place of C_i to denote the specific coset of $\mathbb{K}^{*0}$ which contains the realisations.

We claim that we can express the $B(M)$ orbit of $p_{\infty,C_0} * p_0$ as a subset V of $S_1(M) * \mathcal{J}$. We first show left inclusion.

Proposition 2.4.3. *The $B(M)$-orbit of $p_{\infty,C_0} * p_0$ is a proper subset V of $S_1(M) * \mathcal{J}$, where $V = \{p_{\infty,k^{-2}\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}} : k \in \mathbb{C}((t))^*\} \cup \{p_{a,k^{-2}\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}} : a \neq 0, k \in \mathbb{C}((t))^*\}$.*

Proof. Again, let p_0 be as in Lemma 2.3.2. We compute $B(M) \cdot p_{\infty,C_0} * p_0$. Let $t_0 = \left(\begin{smallmatrix} b & c \\ 0 & b^{-1} \end{smallmatrix}\right)$ be an element of $B(M)$.

Let $h = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ realise $p_{\infty,C_0}|_{M,t_0}$.

Let $t = \left(\begin{smallmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{smallmatrix}\right)$ realise $p_0|_{M,t_0,h}$.

Then $t_0 * p_{\infty,C_0} * p_0 = tp(t_0ht/M)$. We split into two cases; where $c = 0$ and $c \neq 0$.

Case 1: Let $c = 0$.

Then;

$$\begin{aligned} t_0ht &= \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\ &= \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ b^{-2}\alpha & 1 \end{pmatrix} \begin{pmatrix} b\beta & b\gamma \\ 0 & \beta^{-1}b^{-1} \end{pmatrix}. \end{aligned}$$

Since $v(\alpha) < \mathbb{Z}$, and $v(b) \in \mathbb{Z}$, we see $v(b^{-2}\alpha) < \mathbb{Z}$. Further, since $\alpha \in \mathbb{K}^{*0}$, $b^{-2}\alpha \in b^{-2}\mathbb{K}^{*0}$. Hence $b^{-2}\alpha \vDash p_{\infty,b^{-2}\mathbb{K}^{*0}}$.

Next, since $(\beta, \gamma) \vDash p_0$, which has $B(M)$-orbit $\mathcal{J}$ as $\mathcal{J}$ is minimal, we see that $(b\beta, b\gamma)$ realises $p_{b\mathbb{K}^{*0}} \in \mathcal{J}$.

Hence, when $t_0 = (b, c) \in B(M)$ with $c = 0$, $t_0 * p_{\infty, C_0} * p_0 = p_{\infty, b^{-2}\mathbb{K}^{*0}} * p_b\mathbb{K}^{*0}$.

Case 2: Let $c \neq 0$.

Then;

$$t_0 h t = \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{b^{-1}\alpha}{b+c\alpha} & 1 \end{pmatrix} \begin{pmatrix} \beta(b + c\alpha) & \gamma(b + c\alpha) + c\beta^{-1} \\ 0 & \beta^{-1}(b + c\alpha)^{-1} \end{pmatrix}$$

As $b\alpha^{-1} \neq 0$, we can write $\frac{b^{-1}\alpha}{b+c\alpha} = (b^2\alpha^{-1} + cb)^{-1} = (bc)^{-1}(1 + bc^{-1}\alpha^{-1})^{-1}$.

Since $(1 + bc^{-1}\alpha^{-1})$ is in the infinitesimal neighbourhood of 1, which is itself a multiplicative group, we know that $(1 + bc^{-1}\alpha^{-1})^{-1}$ is of the form $1 + x$ where $v(x) \geq \mathbb{Z}$. Then $(1 + bc^{-1}\alpha^{-1})(1 + x) = 1 + bc^{-1}\alpha^{-1} + x + xbc^{-1}\alpha^{-1}$.

Since $(1 + bc^{-1}\alpha^{-1})(1 + x) = 1$, we see $bc^{-1}\alpha^{-1} + x + xbc^{-1}\alpha^{-1} = 0$. Hence $x = -bc^{-1}\alpha^{-1} - xbc^{-1}\alpha^{-1}$, and since $v(x) \geq \mathbb{Z}$, and $(b, c) \in B(M)$, we see $v(x) = v(-bc^{-1}\alpha^{-1})$, and the coset of $\mathbb{K}^{*0}$ which contains x is determined by $-bc^{-1}$.

Hence $\frac{b^{-1}\alpha}{b+c\alpha} = (bc)^{-1}(1 + x) = (bc)^{-1} + (bc^{-1})x$, and $x \in -bc^{-1}\mathbb{K}^{*0}$, and hence $((bc)^{-1})x \in c^{-2}\mathbb{K}^{*0}$.

Hence, $\frac{b^{-1}\alpha}{b+c\alpha} \vDash p_{(bc)^{-1}, c^{-2}\mathbb{K}^{*0}}$, where $(bc)^{-1} \neq 0 \in \mathbb{C}((t))$.

Next, since $(\beta, \gamma) \vDash p_0$, which has $B(M)$-orbit $\mathcal{J}$ since $\mathcal{J}$ is a minimal flow of $(B, S_B(M))$, we see that $(\beta(b + c\alpha), \gamma(b + c\alpha) + c\beta^{-1})$ realises some $p_j \in \mathcal{J}$, where the coset C_j of $\mathbb{K}^{*0}$ in which $\beta(b + c\alpha)$ and $\gamma(b + c\alpha) + c\beta^{-1}$ lie is determined by $(b + c\alpha)$.

Since $\alpha \vDash p_{\infty, C_0}$, we see that this coset is determined by c.

Hence when $t_0 = (b, c)$ with $c \neq 0$, we see that $(b, c) \cdot p_{\infty, C_0} * p_0 = p_{(bc)^{-1}, c^{-2}\mathbb{K}^{*0}} * p_{c\mathbb{K}^{*0}}$.

Note that b has no bearing on the cosets here and $c \neq 0$. As such, for any

$a \in \mathbb{C}((t))$ and coset $c^{-2}\mathbb{K}^{*0}$ we can find an element $t_0 \in B(M)$ such that $t_0 \cdot p_{\infty,C_0} * p_0 = p_{a,c^{-2}\mathbb{K}^{*0}} * p_{c\mathbb{K}^{*0}}$; namely, where $t_0 = ((ac)^{-1}, c)$.

Hence, the $B(M)$-orbit of $p_{\infty,C_0} * p_0$ is a set of the form $\{p_{\infty,k^{-2}\mathbb{K}^{*0}} * p_k\} \cup \{p_{a,k^{-2}\mathbb{K}^{*0}} * p_k : a \neq 0\}$, where p_k here means entries in the realisations of $p \in \mathcal{J}$ are elements of the coset $k\mathbb{K}^{*0}$. □

We now demonstrate that the right hand side is a subset of the left; that any type in V can be factorised into an element of the $B(M)$-orbit of $p_{\infty,C_0} * p_0$.

Proposition 2.4.4. *Let V be as in the above proposition. Then V is a subset of $B(M) \cdot p_{\infty,C_0} * p_0$ and hence the two sets are equal.*

Proof. We show that for any $q \in V$, there exists some $(b, c) \in B(M)$ such that $q = (b, c) * p_{\infty,C_0} * p_0$.

We first consider the case where $q = p_{\infty,k^{-2}\mathbb{K}^{*0}} * p_k$. Let $h \models p_{\infty,k^{-2}\mathbb{K}^{*0}}$ and let $t = (\beta, \gamma) \models p_k|_{M,h}$. Then $ht \models q$.

Since $\beta, \gamma \in k\mathbb{K}^{*0}$, we can write $\beta = k\beta'$ and $\gamma = k\gamma'$, where β' and γ' lie in the coset $C_0 = \mathbb{K}^{*0}$. Since $k \in \mathbb{C}((t))$, we see $(\beta', \gamma') \models p_0|_{M,h}$.

Hence $(k, 0) \cdot p_0 = p_k$.

$$p_{\infty,k^2\mathbb{K}^{*0}} \cdot (k, 0) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} k & 0 \\ k^{-1}\alpha & k^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k^2\alpha & 1 \end{pmatrix}$$

Finally, since α lies in $k^{-2}\mathbb{K}^{*0}$ we can write $\alpha = k^{-2}\alpha'$, for $\alpha' \in p_{\infty,C_0}$. Hence $k^2\alpha = k^2k^{-2}\alpha' = \alpha' \in \mathbb{K}^{*0}$.

Hence, for $q \in V$ of the form $p_{\infty,k^{-2}\mathbb{K}^{*0}} * p_k$, we can find an element t_0 of $B(M)$ such that $q = t_0 \cdot p_{\infty,C_0} * p_0$.

We now show that we can do the same when $q \in V$ is of the form $p_{a,k^{-2}\mathbb{K}^{*0}} * p_k$. Let $\alpha_0 \models p_{a,k^{-2}\mathbb{K}^{*0}}$, and let $(\beta, \gamma) \models p_k|_{M,\alpha_0}$. We show there exists some element $t_0 \in B(M)$ such that $q = t_0 * p_{\infty,C_0} * p_0$.

Since $\alpha_0 \models p_{a,k^{-2}\mathbb{K}^{*0}}$, we can write $\alpha_0 = a + \epsilon$, where $\epsilon \in k^{-2}\mathbb{K}^{*0}$ and $v(\epsilon) > \mathbb{Z}$. Further, we can write $\epsilon = -k^{-2}\epsilon'$ for some $\epsilon' \in \mathbb{K}^{*0}$ with $v(\epsilon') > \mathbb{Z}$.

Hence $\alpha_0 = a(1 + a^{-1}\epsilon) = a(1 - a^{-1}k^{-2}\epsilon')$.

Let b be such that $a = (bk)^{-1} \in \mathbb{C}((t))$.

Then $\alpha_0 = a(1 = bk^{-1}\epsilon')$. Now, $bk^{-1}\epsilon' \in -bk^{-1}\mathbb{K}^{*0}$, and $v(bk^{-1}\epsilon') > \mathbb{Z}$. Hence $1 - bk^{-1}\epsilon'$ is in the infinitesimal neighbourhood of 1, and hence so is $(1 - bk^{-1}\epsilon')^{-1}$ since this neighbourhood forms a multiplicative group.

Then $(1 - bk^{-1}\epsilon')(1 + x) = 1$, for some $x \in dcl(\mathbb{C}((t)), \alpha_0)$, and one can see that $x \in bk^{-1}\mathbb{K}^{*0}$ with $v(x) > \mathbb{Z}$.

Hence we can write $\alpha_0 = a(1 + bk^{-1}\alpha)^{-1}$ where $\alpha \in \mathbb{K}^{*0}$ and $v(\alpha) > \mathbb{Z}$. Hence $\alpha \models p_{0,C_0}$ and we can write $\alpha_0 = a(1+bk^{-1}\alpha)^{-1} = \frac{a}{1+bk^{-1}\alpha} = \frac{b^{-1}\alpha^{-1}}{b+k\alpha^{-1}}$, using $a = (bk)^{-1}$ from above.

Since $\mathcal{J}$ is a minimal flow of $(B(M), S_B(M))$, we can find some element $y \in B(\bar{M}) \cap dcl(M, \alpha_0)$ such that $y \cdot p_0 = p_k$.

We claim that this $y = (b + k\alpha^{-1}, k)$ where $\alpha^{-1} \models p_{\infty,C_0}$, since $\alpha \models p_{0,C_0}$ and $\alpha \in dcl(M, \alpha_0)$. Further, we see that this does indeed preserve the required cosets, since $\alpha^{-1} \in \mathbb{K}^{*0}$ and $v(\alpha^{-1}) < \mathbb{Z}$ and so $b + k\alpha^{-1} \in k\mathbb{K}^{*0}$.

$$\begin{pmatrix} 1 & 0 \\ \alpha_0 & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha_0 & 1 \end{pmatrix} \begin{pmatrix} b+k_{\alpha}^{-1} & k \\ 0 & (b+k\alpha^{-1})^{-1} \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} b+k\alpha^{-1} & k \\ \alpha_0(b+k\alpha^{-1}) & \alpha_0 k + (b+k\alpha^{-1})^{-1} \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} b+k\alpha^{-1} & k-1 \\ \frac{b^{-1}\alpha^{-1}}{b+k\alpha^{-1}}(b+k\alpha^{-1}) & \frac{b^{-1}\alpha^{-1}}{b+k\alpha^{-1}}k + (b+k\alpha^{-1})^{-1} \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} b+k\alpha^{-1} & k \\ b^{-1}\alpha^{-1} & \frac{b^{-1}\alpha^{-1}}{b+k\alpha^{-1}}k + (b+k\alpha^{-1})^{-1} \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} b+k\alpha^{-1} & k \\ b^{-1}\alpha^{-1} & b^{-1} \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} b & k \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta^{-1} \end{pmatrix}.$$

Where (β', γ') realise $p_0|_{M,\alpha_0}$.

Then $(b, k) = t_0 \in B(M)$, $\alpha^{-1} \models p_{\infty,C_0}$. Further, $(\beta', \gamma') \models p_0|_{M,\alpha_0}$ and since $\alpha^{-1} \in dcl(M, \alpha_0)$, we have $(\beta', \gamma') \models p_0|_{M,\alpha^{-1}}$ as required.

Hence for any $q \in V$ of the form $p_{a,k^{-2}\mathbb{K}^{*0}} * p_k$, we can find some $t_0 \in B(M)$ such that $q = t_0 \cdot p_{\infty,C_0} * p_0$.

Hence $V \subseteq B(M) \cdot p_{\infty,C_0} * p_0$ and by Proposition 2.4.3, we see $V = B(M) \cdot p_{\infty,C_0} * p_0$. □

Finally, we must consider how $\mathbb{Z}/4\mathbb{Z}$ acts on V. We consider the union over V of $\mathbb{Z}/4\mathbb{Z}$ orbits. As before, we first demonstrate left inclusion of the union of orbits.

Proposition 2.4.5. *The union* $\bigcup_{v \in V} \mathbb{Z}/4\mathbb{Z} \cdot v$ *is a subset of* $V \cup \{(p_{0,k^{-2}\mathbb{K}^{*0}} * p_{k^3\mathbb{K}^{*0}}) : k \in \mathbb{C}((t))^*\}$

Proof. Let $z \in \mathbb{Z}/4\mathbb{Z}$. Since $\mathbb{Z}/4\mathbb{Z}$ is small, we consider each element case by case. Clearly if $z = I_2$, the identity of $\mathbb{Z}/4\mathbb{Z}$, then $z \cdot v = v$ for all $v \in V$.

First let $h \vDash p_{\infty,k^2\mathbb{K}^{*0}}$ and $t \vDash p_{k\mathbb{K}^{*0}}|_{M,h}$ for some $k \in \mathbb{C}((t))$. Then $ht \vDash v$ for some $v \in V$.

Suppose $z = -I_2$. Since $-I_2$ is in the centre of SL_2, we can write $z \cdot p_{\infty,k^2\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}} = p_{\infty,k^2\mathbb{K}^{*0}} * z \cdot p_{k\mathbb{K}^{*0}}$.

Then $\mathcal{J}$ is a minimal subflow of $(B(M), S_B(M))$ and so $z \cdot p_{k\mathbb{K}^{*0}} = p_j$ for some $p_j \in \mathcal{J}$. However, since $-1 \in \mathbb{K}^{*0}$, every entry of $z \cdot p_{k\mathbb{K}^{*0}}$ is in the same coset as the corresponding entry in $p_{k\mathbb{K}^{*0}}$, and so $z \cdot p_{k\mathbb{K}^{*0}} = p_{k\mathbb{K}^{*0}}$ and hence $z \cdot p_{\infty,k^2\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}} = p_{\infty,k^2\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}}$.

Finally, suppose $z = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Then $z \cdot p_{\infty,k^2\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}} = tp(zht/M)$ and we see;

$$\begin{aligned} zht &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -\alpha & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} -\alpha & -1 \\ 0 & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} -\alpha\beta & -\alpha\gamma - \beta^{-1} \\ 0 & -\alpha^{-1}\beta^{-1} \end{pmatrix} \end{aligned}$$

Observe that $-\alpha^{-1} \in k^{-2}\mathbb{K}^{*0}$ and $v(-\alpha^{-1}) > \mathbb{Z}$. Further, $-\alpha\beta \in k^3\mathbb{K}^{*0}$, $-\alpha\gamma \in k^3\mathbb{K}^{*0}$.

Hence $-\alpha^{-1} \vDash p_{0,k^{-2}\mathbb{K}^{*0}}$ and $(-\alpha\beta, -\alpha\gamma - \beta^{-1}) \vDash p_{k^3\mathbb{K}^{*0}}|_{M,-\alpha^{-1}}$.

The case where $z = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ follows similarly. Taking the union over elements of V we see that $\bigcup_{v \in V} \mathbb{Z}/4\mathbb{Z} \cdot v \subseteq V \cup \{(p_{0,k^{-2}\mathbb{K}^{*0}} * p_{k^3\mathbb{K}^{*0}}) : k \in \mathbb{C}((t))^*\}$. □

Finally, we show that V' is included in the union of $\mathbb{Z}/4\mathbb{Z}$-orbits, and hence demonstrate equality between the sets.

Proposition 2.4.6. *$V' = V \cup \{(p_{0,k^{-2}\mathbb{K}^{*0}} * p_{k^3\mathbb{K}^{*0}}) : k \in \mathbb{C}((t))^*\}$ is a subset of $\bigcup_{v \in V} \mathbb{Z}/4\mathbb{Z} \cdot v$.*

Proof. Clearly any element $v \in V$ lies in the set $\bigcup_{v \in V} \mathbb{Z}/4\mathbb{Z} \cdot v$ since we could choose $z \in \mathbb{Z}/4\mathbb{Z}$ to be the identity element.

Hence we just need to show that $(p_{0,k^{-2}\mathbb{K}^{*0}} * p_{k^3\mathbb{K}^{*0}})$ can be expressed in the form $z \cdot v$ for some $z \in \mathbb{Z}/4\mathbb{Z}$ and some $v \in V$.

Fix some arbitrary non-zero $k \in \mathbb{C}((t))$. Let $h = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$, where $\alpha \vDash p_{0,k^{-2}\mathbb{K}^{*0}}$. Let $t = (\beta, \gamma) \vDash p_{k^3\mathbb{K}^{*0}}|_{M,h}$.

Since $\mathcal{J}$ is a minimal subflow of $(B(M), S_B(M))$, and $t \vDash p_{k^3\mathbb{K}^{*0}}|_{M,h}$, we can find some $(b, c) \in B(\bar{M}) \cap dcl(M, h)$ such that $(b, c) * p_{k\mathbb{K}^{*0}}|_{M,h} = p_{k^3\mathbb{K}^{*0}}$.

In particular, we know that b uniquely determines the coset in this factorisation. Note that since $\alpha \vDash p_{0,k^{-2}\mathbb{K}^{*0}}$, we see $-\alpha^{-1} \vDash p_{\infty,k^2\mathbb{K}^{*0}}$. As such, we can choose $(b, c) = (-\alpha^{-1}, -1)$, and hence;

$$
\begin{aligned}
ht &= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} -\alpha^{-1} & -1 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -\alpha^{-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \beta' & \gamma' \\ 0 & \beta'^{-1} \end{pmatrix} \\
&= zh't'
\end{aligned}
$$

Where $z \in \mathbb{Z}/4\mathbb{Z}$, $h' \vDash p_{\infty,k^2\mathbb{K}^{*0}}$ and $t' \vDash p_{k\mathbb{K}^{*0}}|_{M,h}$.

Since $h' \in dcl(M, h)$, we also see $t' \vDash p_{k\mathbb{K}^{*0}|_{M,h'}}$, and so $zh't' \vDash z \cdot p_{\infty,k^2\mathbb{K}^{*0}} * p_{k\mathbb{K}^{*0}}$, which is an element of $\bigcup_{v \in V} \mathbb{Z}/4\mathbb{Z} \cdot v$ as required. □

Hence, we have demonstrated that the $G(M)$-orbit of $p_{\infty,C_0} * p_0$ is precisely $V' = V \cup \{(p_{0,k^{-2}\mathbb{K}^{*0}} * p_{k^3\mathbb{K}^{*0}}) : k \in \mathbb{C}((t))^{*0}\}$. This set is not closed, and hence not a subflow; minimal or otherwise. However, we can take the closure of V' and we claim that this is indeed minimal, as we now show.

Taking closures of both sides, we see that $cl(G(M) \cdot p_{\infty,C_0} * p_0) = cl(V')$. By Fact 1.5.1, for any type $p \in S_G(M)$, $cl(G(M) * p) = S_G(M) * p$. Hence the closure of the $G(M)$-orbit of $p_{\infty,C_0} * p_0$ is the $S_G(M)$-orbit of $p_{\infty,C_0} * p_0$. Hence $S_G(M) \cdot p_{\infty,C_0} * p_0 = cl(V')$.

Lemma 2.4.7. *$S_G(M) * p_{\infty,C_0} * p_0 \subseteq S_1(M) * \mathcal{J}$.*

Moreover, every element $s * p_{\infty,C_0} * p_0$ is of the form $r * p$ with $r \in S_1(M)$, $p \in \mathcal{J}$.

Proof. We can see from Propositions 2.4.2, 2.4.4 and 2.4.6 and see that the $G(M)$-orbit of $p_{\infty,C_0} * p_0 = V' \subset S_1(M) * \mathcal{J}$. The $S_G(M)$-orbit behaves in a similar way, though it will be a proper superset of V', however still a subset of $S_1(M) * \mathcal{J}$. □

Lemma 2.4.8. *Let $p, p' \in \mathcal{J}$. Let $q \in S_1(M)$. Then $p * q * p'$ can be expressed as an element of $S_1(M) \setminus \{p_{\infty,C_k} : k \in \mathbb{C}((t))\} * \mathcal{J}$.*

Proof. This is easy to see using the same method as in Proposition 2.4.4, and note that insisting $p \in \mathcal{J}$ removes the case where $c = 0$. □

Proposition 2.4.9. *$cl(V')$ is a minimal subflow of $(G(M), S_G(M))$.*

Proof. Any point in $cl(V')$ is of the form $s * p_{\infty,C_0} * p_0$, and by the above lemma we can show any type of the form $s * p_{\infty,C_0} * p_0$ can be expressed as an element $q * p$ of $S_1(M) * \mathcal{J}$.

We claim that for any r in $S_1(M) * \mathcal{J}$, we can demonstrate that $p_{\infty,C_0} * p_0$ is in the orbit-closure $cl(G(M) * r) = S_G(M) * r$.

Let $r = q * p \in S_1(M) * \mathcal{J}$.

Then we can find some type $p_{\infty,C_0} \cdot p_j$, such that $p_{\infty,C_0} * p_j * q * p = p_{\infty,C_0} * q' * p_0$, using a similar argument to Proposition 2.4.4. However, here we see that the realisation (b, c) of the heir of p_j ensures that $q' \notin \{p_{\infty,C_k} : k \in \mathbb{C}((t))^*\}$. As such, $p_{\infty,C_0} * q' = p_{\infty,C_0}$.

Hence we can find some $s \in S_G(M)$ such that $s * r = p_{\infty,C_0} * p_0$, and so $p_{\infty,C_0} * p_0$ is in the orbit-closure of r for any $r \in cl(V')$.

Since $p_{\infty,C_0} * p_0$ is in the $S_G(M)$-orbit of any element of $cl(V')$, and $cl(V') = S_G(M) * p_{\infty,C_0} * p_\infty$, we see that the $cl(V')$ is the orbit-closure of any type in $cl(V')$, and hence minimal. □

2.5 The Ellis Group of $(G(M), S_G(M))$.

To obtain the Ellis Group of $(G(M), S_G(M))$, from Theorem 1.3.6, we act on the minimal subflow of $(G(M), S_G(M))$ (equivalently, the minimal closed left ideal of $(S_G(M), *)$) by an idempotent, namely $p_{\infty,C_0} * p_0$.

Theorem 2.5.1. *The Ellis Group of* $(G(M), S_G(M))$ *is* $p_{\infty,C_0} * p_0 * cl(V') = p_{\infty,C_0} * \mathcal{J}$*, and is isomorphic to* B/B^0.

Proof. This is clear to see. Take any element $r = q * p \in cl(V')$. We compute $p_{\infty,C_0} * p_0 * r$.

We note that $p_0 * r$ is of the form $q' * p'$ for some $q' \in S_1(M)/\{p_{\infty,C_k} : k \in \mathbb{C}((t))^*\}$ and $p' \in \mathcal{J}$.

Then $p_{\infty,C_0} * p_0 * q * p = p_{\infty,C_0} * p'$ for some $p' \in \mathcal{J}$, and hence $p_{\infty,C_0} * p_0 * cl(V') \subseteq p_{\infty,C_0} * \mathcal{J}$.

To demonstrate equality, we must show that p' can range over all cosets of $\mathbb{K}^{*0}$. That is, for any $p_{\infty,C_0} * p_j$, we can find some $r \in cl(V')$ with $p_{\infty,C_0} * p_0 * r = p_{\infty,C_0} * p_j$.

This is clear to see from Proposition 2.4.4. Since $V \subset cl(V')$, we see that types of the form $p_{\infty,k^2\mathbb{K}^{*0}} * p_k$ for all $k \in \mathbb{C}((t))$ are contained $cl(V')$. One can simply choose $r = p_{\infty,j^2\mathbb{K}^{*0}} * p_j$ and show $p_{\infty,C_0} * p_0 * r = p_{\infty,C_0} * p_j$. Hence $p_{\infty,C_0} * \mathcal{J} \subseteq p_{\infty,C_0} * p_0 * cl(V')$.

Hence $p_{\infty,C_0} * p_0 * cl(V') = p_{\infty,C_0} * \mathcal{J}$, and so the Ellis Group of $(G(M), S_G(M))$ is precisely $p_{\infty,C_0} * \mathcal{J}$.

That $(q_0 * \mathcal{J}, *)$ is isomorphic to $(\mathcal{J}, *)$ follows almost identically to Theorem 3.7 of **[22]**. $\square$

Hence, we have demonstrated that the Ellis Group of $(G(M), S_G(M))$ is not isomorphic to G/G^{00}; namely the Ellis Group has infinite elements whereas G/G^{00} is trivial.

To summarise then, we have demonstrated that the Ellis Group of $SL_2(\mathbb{C}((t)))$ is determined by the Borel Subgroup. This is similar to what we see in **[22]** where the Ellis Group of $SL_2(\mathbb{Q}_p)$ is isomorphic to a subgroup of $B(\mathbb{Q}_p)$.

The main result of this chapter provides a negative answer to the hypothesis that the restriction of NIP to Metastable Theories is not a suitable weakening of Newelski's conjecture.

Since the Ellis Groups of $(G(M), S_G(M))$ are all isomorphic for any choice of minimal flow and idempotent, we can guess here that if we had chosen $SL_2(\mathbb{C}((t)) = SL_2(\mathbb{C}[[t]]) \times B(\mathbb{C}((t)))$ as our group decomposition we would have found that $SL_2(\mathbb{C}[[t]])$ has an invariant type. We show in the following chapter that this is indeed the case.

Chapter 3

Definable Topological Dynamics of $SL_2(\mathbb{C}[[t]])$

From the Ellis Group of $SL_2(\mathbb{C}((t)))$ in Chapter 2, we expect to find that $SL_2(\mathbb{C}[[t]])$ admits a unique left-invariant generic type. In this chapter we prove that this is indeed the case. We recall a result from **[10]** at the end of this chapter which proves a similar result for $SL_2(\mathcal{O}_K)$ where $K \vDash ACVF$. They conjecture that this result should extend to metastable groups with algebraically closed residue, and we note here that the results of this chapter support this by providing an explicit description of a unique left-invariant type in $SL_2(\mathbb{C}[[t]])$.

In this chapter we give a group decomposition for $SL_2(\mathbb{C}[[t]])$ into subgroups of smaller dimension and find their minimal subflows. We build a minimal subflow for $SL_2(\mathbb{C}[[t]])$ and demonstrate that this is a 1-point minimal subflow, and hence that $SL_2(\mathbb{C}[[t]])$ is a definably (extremely) amenable group.

3.1 Group Decomposition of $SL_2(\mathbb{C}[[t]])$

We first demonstrate a group decomposition of $SL_2(\mathbb{C}[[t]])$. Recall that a sequence $G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \dots. \xrightarrow{f_n} G_n$ of groups G_i and group homomorphisms f_j is called exact if $Im(f_k) = ker(f_{k+1})$ for all k. A short exact sequence is of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ where f is an injective homomorphism and g a surjective homomorphism. Recall that for a short exact sequence, we have $C \cong B/Im(f)$.

Lemma 3.1.1 (**Splitting Lemma**). *If a short exact sequence admits a morphism $t : B \to A$ such that $t \circ f$ is the identity on A, or, a morphism $u : C \to B$ such that $g \circ u$ is identity on C, then B is the semidirect product of A and C.*

We define a mapping $\pi : SL_2(\mathbb{C}[[t]]) \to SL_2(\mathbb{C})$ that maps entrywise from elements of $\mathbb{C}[[t]]$ to their residue in $\mathbb{C}$.

Proposition 3.1.2. *$SL_2(\mathbb{C}[[t]]) \xrightarrow{\pi} SL_2(\mathbb{C}) \to 0$ is exact*

Proof. We show that π is a surjective homomorphism. Firstly, since π acts as identity on any element of $SL_2(\mathbb{C})$, which itself is a subset of $SL_2(\mathbb{C}[[t]])$, res is surjective.

That π is a group homomorphism follows easily from the properties of the residue map. □

We see then that

$$Ker(\pi) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, d \in 1 + t\mathbb{C}[[t]]; c, b \in t\mathbb{C}[[t]] \text{ and } ad - bc = 1\}$$

and moreover this is a normal subgroup of $SL_2(\mathbb{C}[[t]])$.

Let $Ker(\pi) = H \triangleleft SL_2(\mathbb{C}[[t]])$.

Proposition 3.1.3. *$0 \to H \xrightarrow{id} SL_2(\mathbb{C}[[t]])$ is exact.*

Proof. This is trivial as the identity map is clearly an injective homomorphism. □

Proposition 3.1.4. *$0 \to H \xrightarrow{id} SL_2(\mathbb{C}[[t]]) \xrightarrow{\pi} SL_2(\mathbb{C}) \to 0$ is a short exact sequence.*

Proof. From Propositions 3.1.2 and 3.1.3 the functions id and π are injective and surjective respectively. It is clear by construction that $Im(id) = Ker(\pi)$. □

Proposition 3.1.5. *There exists a morphism $u : SL_2(\mathbb{C}) \to SL_2(\mathbb{C}[[t]])$ such that $\pi \circ u$ acts as identity on $SL_2(\mathbb{C})$.*

Proof. Clearly choosing u to be the identity map satisfies the above.

□

Proposition 3.1.6. $SL_2(\mathbb{C}[[t]]) = SL_2(\mathbb{C}) \ltimes H$

Proof. From Proposition 3.1.4 we have a short exact sequence

$$0 \to H \xrightarrow{id} SL_2(\mathbb{C}[[t]]) \xrightarrow{\pi} SL_2(\mathbb{C}) \to 0.$$

Further, there exists a morphism $u : SL_2(\mathbb{C}) \to SL_2(\mathbb{C}[[t]])$ such that $\pi \circ u$ acts as identity on $SL_2(\mathbb{C})$ by Proposition 3.1.5.

Hence we can apply the Splitting Lemma 3.1.1 and this proves the result. □

We look to factor H into subgroups of smaller dimension. We remark that $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} b & c \\ \alpha b & \alpha c + b^{-1} \end{pmatrix}$.

Then $\begin{pmatrix} b & c \\ \alpha b & \alpha c + b^{-1} \end{pmatrix}$ is an element of H exactly when;

- $b \in 1 + t\mathbb{C}[[t]]$
- $c \in t\mathbb{C}[[t]]$
- $\alpha \in t\mathbb{C}[[t]]$

It is clear that any element of $H(M)$ can be written uniquely as a product of matrices from $\{\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} : \alpha \in t\mathbb{C}[[t]]\}$ and $\{\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} : b \in 1 + t\mathbb{C}[[t]]$ and $c \in t\mathbb{C}[[t]]\}$.

Likewise, any element in the product of
$\{\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} : \alpha \in t\mathbb{C}[[t]]\}$ and $\{\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} : b \in 1 + t\mathbb{C}[[t]]$ and $c \in t\mathbb{C}[[t]]\}$ is an element of H.

We see that $(\{\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} : \alpha \in t\mathbb{C}[[t]]\}, \cdot)$ is isomorphic to $(t\mathbb{C}[[t]], +)$ and that $(\{\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} : b \in 1 + t\mathbb{C}[[t]]$ and $c \in t\mathbb{C}[[t]]\}, \times)$ is isomorphic to $(1 + t\mathbb{C}[[t]], \times) \cdot (t\mathbb{C}[[t]], +)$.

Hence $H = \{\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) : x_1, x_4 \in 1 + t\mathbb{C}[[t]]; x_2, x_3 \in t\mathbb{C}[[t]]$ and $x_1x_4 - x_2x_3 = 1\}$ and $SL_2(\mathbb{C}[[t]])$ is isomorphic to $((t\mathbb{C}[[t]], +) \cdot ((1+t\mathbb{C}[[t]], \times) \cdot (t\mathbb{C}[[t]], +))) \rtimes SL_2(\mathbb{C})$.

Finally, we remark that H has infinite torsion elements. In particular, for every diagonal (scalar) matrix A in H and every $n \in \mathbb{N}$, there exists some matrix B in H such that $B^n = A$. This is a consequence of Hensel's Lemma 2.1.2. We mention torsion elements as to draw parallels to the work on $SL_2(\mathbb{R})$ in **[9]** in which they make use of a torsion-free group decomposition. In this setting however, all parts of our group decomposition have infinitely many torsion elements.

3.2 Additive and Multiplicative groups in $\mathbb{C}[[t]]$

As before, we wish to build a minimal flow by considering the components of a group decomposition individually. We once again start by considering additive and multiplicative groups contained in the ring. We will use the same notation as in Lemma 2.1.9, and for convenience recall the result here.

Lemma 3.2.1. *The complete 1-types over $M = (\mathbb{C}((t)), +, \times)$ are precisely the following;*

(a) The (realized) types $tp(a/M)$ for each $a \in \mathbb{C}((t))$.

(b) For each $a \in \mathbb{C}((t))$ and coset C of $\mathbb{K}^{*0}$, the type $p_{a,C}$ determined by the formulas $\{v(x-a) > n : \forall n \in \mathbb{Z}\}$ and $(x-a) \in C$.

(c) For each coset C of $\mathbb{K}^{*0}$, the type $p_{\infty,C}$ determined by the formulas $\{v(x) < n : \forall n \in \mathbb{Z}\}$ and $x \in C$.

(d) For each $a \in \mathbb{C}[t]$ and for some $n \in \mathbb{Z}$, the type $p_{a,n,trans}$ determined by the formulas
$v(x-a) = n$, $deg(a) < n$ and $\{f(res((x-a)t^{-n})) \neq 0 : f \in \mathbb{C}[x]\}$.
If $a = 0$ then, we can drop the $deg(a) < n$ from the description.

We first consider the additive group $(t\mathbb{C}[[t]], +)$.

Proposition 3.2.2. *The type* $p_{0,1,trans}$ *is an idempotent element of* $(S_{(t\mathbb{C}[[t]],+)}(M), *)$.

Proof. Let $a \models p_{0,1,trans}$ and let $\alpha \models p_{0,1,trans}|_{M,a}$.

Then $p^2_{0,1,trans} = tp(a + \alpha/M)$. We check that $a + \alpha \models p_{0,1,trans}|_M$.

We know $a = a_1t^1 + ...$ and $\alpha = \alpha_1t^1 + ...$ with a_1 transcendental over $\mathbb{C}$ and α_1 transcendental over $res(\mathbb{C}((t))(a))$.

Then $a + \alpha = (a_1 + \alpha_1)t^1 + ...$, and since α is transcendental over $res(\mathbb{C}((t))(a))$ we see $a_1 + \alpha_1 \notin res(\mathbb{C}((t))(a))$. In particular, this shows $a_1 + \alpha_1 \neq 0$ and so $v(a + \alpha) = 1$.

Further, this shows that $a_1 + \alpha_1$ is transcendental over $res(\mathbb{C}((t))(a))$, and so in particular transcendental over $\mathbb{C}$.

Hence $a + \alpha \models p_{0,1,trans}|_M$ as required. □

Proposition 3.2.3. *The* $t\mathbb{C}[[t]]$*-orbit of* $p_{0,1,trans}$ *is precisely* $p_{0,1,trans}$ *itself, and hence* $\{p_{0,1,trans}\}$ *is a minimal subflow of* $((t\mathbb{C}[[t]], +), S_{(t\mathbb{C}[[t]],+)}(M))$.

Proof. Let $a \in t\mathbb{C}[[t]]$ and let α realise the heir of $p_{0,1,trans}$ over (M, a). Then

$$tp(a/M) * p_{0,1,trans} = tp((a + \alpha)/M)$$

Now, $v(\alpha + a) \geq min\{v(\alpha), v(a)\} = 1$. As $ac(\alpha)$ is transcendental over $\mathbb{C}$, it is clear that $v(\alpha + a) = min\{v(\alpha), v(a)\} = 1$. Further, as $v(\alpha) = 1$ and $v(a + \alpha) = 1$, we see that $res(a + \alpha)$ is transcendental over $\mathbb{C}$ because $res(\alpha)$ is transcendental over $\mathbb{C}$.

Hence $a + \alpha$ realies $p_{0,1,trans}$, and so $tp((a + \alpha)/M) = p_{0,1,trans}$. This is a subflow by consturction, and is clearly minimal as it is a singleton there can be no properly contained non-empty subsets, and hence $p_{0,1,trans}$ is a minimal subflow of $(t\mathbb{C}[[t]], +)$ as required. □

Corollary 3.2.4. *The single element set* $\{p_{0,n,trans}\}$ *is a minimal subflow of* $((t^n\mathbb{C}[[t]], +), S_{(t^n\mathbb{C}[[t]],+)}(M))$ *for all* $1 \leq n \in \mathbb{N}$.

Proof. This follows easily using the same proof as in 3.2.3 by replacing $v(\alpha) = 1$ with $v(\alpha) = n$ as necessary. □

We now consider the multiplicative group $(1 + t\mathbb{C}[[t]], \times)$.

Proposition 3.2.5. *The type* $p_{1,1,trans}$ *is an idempotent element of* $(S_{1+t\mathbb{C}[[t]]}(M), *)$.

Proof. Let $a \vDash p_{1,1,trans}$ and $\alpha \vdash p_{1,1,trans}|_{M,a}$. Then $p^2_{1,1,trans} = tp(a\alpha/M)$.

We see that $a = 1 + a_1 t^1 + ...$ and $\alpha = 1 + \alpha_1 t^1 + ...$ Then $a\alpha = 1 + (a_1 + \alpha_1)t^1 + a_1\alpha_1 t^2 + ...$

Clearly $a\alpha - 1$ has valuation 1, and the proof that $a_1 + \alpha_1$ is transcendental over $res(\mathbb{C}((t))) = \mathbb{C}$ follows using the same argument as in Proposition 3.2.2. □

Proposition 3.2.6. *The* $1 + t\mathbb{C}[[t]]$*-orbit of* $p_{1,1,trans}$ *is precisely* $\{p_{1,1,trans}\}$ *itself, and is a minimal subflow of* $(1 + t\mathbb{C}[[t]], S_{1+t\mathbb{C}[[t]]}(M))$.

Proof. Let $1+a \in 1+t\mathbb{C}[[t]]$ and let $1+\alpha$ realise the heir of $p_{1,1,trans}$ over $(M, 1+a)$. Then;

$$\begin{aligned} tp((1+a)/M) * p_{1,1,trans} &= tp((1+a)(1+\alpha)/M) \\ &= tp((1+a+\alpha+a\alpha)/M) \\ &= tp((1+\alpha)+a(1+\alpha))/M). \end{aligned}$$

Write $(1 + \alpha) + a(1 + \alpha) = (1 + \alpha)(1 + a)$. We show that $(1 + \alpha)(1 + a)$ realises $p_{1,1,trans}$.

Since $v(\alpha) = 1$, and $a \in t\mathbb{C}[[t]]$, we see that $v((1 + a)(1 + \alpha) - 1) = 1$. Further, $(1 + a)(1 + \alpha) - 1$ has angular component transcendental over $\mathbb{C}$, and so clearly $res(t^{-1}((1 + a)(1 + \alpha) - 1))$ is transcendental over $\mathbb{C}$. Finally observe that $v(1) = 0 < v((1 + a)(1 + \alpha) - 1) = 1$.

Hence $(1 + \alpha) + a(1 + \alpha)$ realises $p_{1,1,trans}$, and so $tp((1 + \alpha) + a(1 + \alpha))/M) = p_{1,1,trans}$. The set $\{p_{1,1,trans}\}$ is a subflow of $(1+t\mathbb{C}[[t]], S_{1+t\mathbb{C}[[t]]}(M))$ by construction, and is clearly minimal as there can be no properly contained non-empty subsets.

Hence $\{p_{1,1,trans}\}$ is a minimal subflow as required.

□

3.3 Minimal Flows of $H = Ker(\pi)$

Having established minimal flows for the additive groups $(t\mathbb{C}[[t]], +)$ and $(1+t\mathbb{C}[[t]], \times)$, we can begin to construct a minimal subflow for H.

Recall that $H = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, d \in 1 + t\mathbb{C}[[t]] : b, c \in t\mathbb{C}[[t]]$ and $ad - bc = 1\}$, which we can decompose further into the product of $\{\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right) : \alpha \in t\mathbb{C}[[t]]\}$ and $\{\left(\begin{smallmatrix} b & c \\ 0 & b^{-1} \end{smallmatrix}\right) : b \in 1 + t\mathbb{C}[[t]]$ and $c \in t\mathbb{C}[[t]]\}$.

Define the Borel subgroup of H to be $B_H(M) = \{\left(\begin{smallmatrix} b & c \\ 0 & b^{-1} \end{smallmatrix}\right) : b \in 1+t\mathbb{C}[[t]]$ and $c \in t\mathbb{C}[[t]]\}$.

As in Chapter 2 we once again associate a matrix in $B_H(M)$ with pairs (b, c), this time with $b \in 1 + t\mathbb{C}[[t]]$ and $c \in t\mathbb{C}[[t]]$. Remember that the group operation here is matrix multiplication, and so given pairs (b, c) and (β, γ), we have that $(b, c)(\beta, \gamma) = (b\beta, b\gamma + c\beta^{-1})$.

Define a type $p = tp((b, c)/M)$ where $b \models p_{1,1,trans}$ and $c \models p_{0,1,trans}|_{M,b}$. Then $p \in S_{B_H}(M)$ and we claim that $\{p\}$ is a 1-point minimal flow of $B_H(M)$.

Proposition 3.3.1. *The $B_H(M)$-orbit of p is $\{p\}$. This orbit is closed and clearly minimal, and hence is a minimal subflow of $(B_H, S_{B_H}(M))$.*

Proof. Let $(b, c) \in B_H(M)$.

Let β realise the heir of $p_{1,1,trans}$ over $(M, (b, c)) = M$, and let γ realise the heir of $p_{0,1,trans}$ over (M, β). Then $(\beta, \gamma) \vdash p$.

Then;

$$\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} b\beta & b\gamma + c\beta^{-1} \\ 0 & b^{-1}\beta^{-1} \end{pmatrix}$$

Since β realises $p_{1,1,trans}$, a minimal subflow of the multiplicative group, we see that $b\beta$ also realises $p_{1,1,trans}$ over M.

We now consider $b\gamma + c\beta^{-1}$. It is clear that this has valuation 1 with angular component $(\gamma_1 + c_1)$. Since γ was chosen such that γ_1 is transcendental over the residue field of (M, β), we know that $(\gamma_1 + c_1)$ is also transcendental over the residue field of (M, β), and hence $res(t^{-1}b\gamma + c\beta^{-1})$ is transcendental over $res(\mathbb{K}) \cap dcl(M, \beta)$.

Therefore, $b\gamma + c\beta^{-1}$ realises the heir of $p_{0,1,trans}$ over (M, β), hence $(b\beta, b\gamma + c\beta^{-1})$ realises $p|_M$ and so the B_H-orbit of p is $\{p\}$. □

Corollary 3.3.2. *The $S_{B_H}(M)$-orbit of p is $\{p\}$ and hence p is an idempotent of $(S_{B_H}(M), *)$ and the Ellis Group of $(B_H, S_{B_H}(M))$ is trivial. Further, B_H is definably (extremely) amenable and $B_H^{00} = B_H$.*

Proof. The fact that the $S_{B_H}(M)$-orbit of p is trivial follows similarly to the proof of Proposition 3.3.1, and it follows that $p^2 = p$ since $p \in S_{B_H}(M)$.

This shows that the Ellis Group of $(B_H, S_{B_H}(M))$ is trivial.

Finally, this means B_H is definably extremely amenable as B_H admits a 1-point minimal flow, and as such $B_H^{00} = B_H$ as $B_H^{00} = Stab_{B_H}(p) = B_H$ by Fact 1.5.2. □

We are now in a position to construct a minimal subflow for $(H, S_H(M))$. Recall that $H \cong A \times B_H$, where $A = \{\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) : a \in t\mathbb{C}[[t]]\}$.

From Proposition 3.2.3 we see that $(A, S_A(M))$ has a 1-point minimal subflow, $\{q\}$ where q is realised by a matrix $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$, with $a \models p_{0,1,trans}$. We will often associate the matrix $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$ with the bottom left entry a, and so by $a \models q$ we mean $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \models q$.

We claim that $q * p$ is a minimal subflow of $(H, S_H(M))$, and prove this by calculating $cl(q * p) = S_H(M) * q * p$. Note that in the following proof we make mention to the angular component. We discussed earlier in the thesis that this map is not definable in this context, though since the valuations here are in $\mathbb{Z}$, we can access the angular component of some element a with $v(a) = z$ with the expression $res(f^{-z}a)$.

Proposition 3.3.3. *The B_H-orbit of q, $B_H(M) * q = q * (p_{1+at,2,trans} * q)$, where $(p_{1+at,2,trans} * q)$ denotes a type $tp((x,y)/M)$ with $x \in X \subset (1 + t\mathbb{C}[[t]](\alpha), \times)$ and $y \in t\mathbb{C}[[t]]$.*

Proof. Let $(b,c) \in B_H(M)$ with α realising the heir of q over $(M, (b,c))$.

Write $(b,c) = \left(\begin{smallmatrix} b & c \\ 0 & b^{-1} \end{smallmatrix}\right)$ and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$. Then;

$$
\begin{aligned}
(b,c) \cdot p_{0,1,trans} &= \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \\
&= \begin{pmatrix} b + c\alpha & c \\ b^{-1}\alpha & b^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \frac{b^{-1}\alpha}{b+c\alpha} & 1 \end{pmatrix} \begin{pmatrix} b + c\alpha & c \\ 0 & (b+c\alpha)^{-1} \end{pmatrix}
\end{aligned}
$$

We first show that $\frac{b^{-1}\alpha}{b+c\alpha}$ realises q. Since $b \in 1 + t\mathbb{C}[[t]]$, b^{-1} is also, and since $c \in t\mathbb{C}[[t]]$ and $v(\alpha) = 1$, it is easy to see that $(b + c\alpha)^{-1}$ has valuation 0 with angular component 1. Further, $v(b^{-1}\alpha) = v(\alpha) = 1$, and $ac(b^{-1}\alpha) = ac(\alpha) = \alpha_1$. So $(b^{-1}\alpha)(b + c\alpha)^{-1}$ has valuation $0 + 1 = 1$. Moreover, we know that $ac(x)ac(y) = ac(xy)$ (since angular components are non-zero), and hence the angular component of $(b^{-1}\alpha)(b + c\alpha)^{-1}$ is α_1, which is transcendental over $\mathbb{C}$ by assumption.

Hence $\frac{b^{-1}\alpha}{b+c\alpha}$ has valuation 1 with $res\left(t^{-1}(\frac{b^{-1}\alpha}{b+c\alpha})\right)$ transcendental (over $\mathbb{C}$). So $\frac{b^{-1}\alpha}{b+c\alpha}$ realises $p_{0,1,trans}$.

It remains to decide which types the pair $(b + c\alpha, c)$ could realise. Given that the $*$ mapping would add $\frac{b^{-1}\alpha}{b+c\alpha}$ in as parameters, we lose information about the transcendental nature of α.

Hence $(b + c\alpha, c)$ realises a 2-type where $b + c\alpha$ is of the form $1 + t^2\mathbb{C}[[t]](\alpha)$ and $c \in t\mathbb{C}[[t]]$. □

Proposition 3.3.4. *The H-orbit of $q*p$ is $q*p$ and moreover $\{q*p\}$ is a minimal subflow of $(H, S_H(M))$.*

Proof. We can consider the H-orbit as being an action of $A \times B_H$.

Let $a \in A$, and $(b, c) \in B_H$. Suppose that α realises q, and that (β, γ) realise the heir of p over (M, α).

Then;

$$
\begin{aligned}
H * q * p &= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \gamma^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{b^{-1}\alpha}{b+c\alpha} & 1 \end{pmatrix} \begin{pmatrix} b + c\alpha & c \\ 0(b+c\alpha)^{-1} & \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ a + \frac{b^{-1}\alpha}{b+c\alpha} & 1 \end{pmatrix} \begin{pmatrix} \beta(b+c\alpha) & \gamma(b+c\alpha) + c\beta^{-1} \\ 0 & (b+c\alpha)^{-1}\beta^{-1} \end{pmatrix}.
\end{aligned}
$$

Then from 3.3.3, we see that $\frac{b^{-1}\alpha}{b+c\alpha}$ realises q, which is a 1-point minimal subflow of A, and since α is transcendental over M, we see that $a + \frac{b^{-1}\alpha}{b+c\alpha}$ realises q.

Further, since (β, γ) realises the heir of p over (M, α), and that p is a minimal subflow of B_H, we see that $(\beta(b + c\alpha), \gamma(b + c\alpha) + c\beta^{-1})$ realises the heir of p over (M, α).

Hence $H * q * p = q * p$.

Since the $H(M)$-orbit is $\{q * p\}$, which is closed, a subflow by construction, and can clearly contains no proper non-empty subsets, we must have that $q * p$ is a 1-point minimal subflow of $(H, S_H(M))$ as required. □

Corollary 3.3.5. *The Ellis Group of $(H, S_H(M))$ is trivial, H is definably (extremely) amenable and $H^{00} = H$.*

Proof. The Minimal Subflow of $(H, S_H(M))$ contains a single element, and hence it follows that the Ellis Group is isomorphic to the trivial group. Further, by 1.5.2

we see $H^{00} = Stab_H(q * p) = H$. Since $q * p$ is a left-invariant type, H is definably (extremely) amenable by definition. □

From Proposition 3.3.4 it is easy to see that that $q * p$ extends uniquely to a $S_H(M)$-invariant global type.

3.4 The Ellis Group of $SL_2(\mathbb{C}[[t]])$

We now apply well known results of Newelski for stable groups, as well as the group decomposition from Proposition 3.1.6 to construct a minimal flow of $SL_2(\mathbb{C}[[t]])$.

Fact 3.4.1. *As $G = SL_2(\mathbb{C})$ is a stable group, the Ellis Group is isomorphic to $SL_2(\mathbb{C})/SL_2(\mathbb{C})^{00}$ by Fact 1.6.1. Further, $SL_2(\mathbb{C})/SL_2(\mathbb{C})^{00}$ is trivial.*

We need not give an explicit description of any minimal flow in $SL_2(\mathbb{C}[[t]])$. Instead, we can simply demonstrate this the group is definably amenable and construct the Ellis Group using the connected component as follows.

Proposition 3.4.2. *$SL_2(\mathbb{C}[[t]])$ is definably extremely amenable and hence the flow of $SL_2(\mathbb{C}[[t]])$ acting on the its space of types has trivial Ellis Group.*

Proof. We know H is definably amenable, with H normal in $SL_2(\mathbb{C}[[t]])$. Further, $SL_2(\mathbb{C}) = SL_2(\mathbb{C}[[t]])/H$ is definably amenable by 3.4.1. Hence by Fact 1.5.8, $SL_2(\mathbb{C}[[t]])$ is definably amenable.

Further, $SL_2(\mathbb{C}[[t]])$ contains no normal subgroups of bounded index, and so clearly $SL_2(\mathbb{C}[[t]])^{00} = SL_2(\mathbb{C}[[t]])$.

Hence since $SL_2(\mathbb{C}[[t]])$ is definably amenable and $SL_2(\mathbb{C}[[t]])^{00} = SL_2(\mathbb{C}[[t]])$, the Ellis Group is isomorphic to $SL_2(\mathbb{C}[[t]])/SL_2(\mathbb{C}[[t]])^{00}$, which is trivial. □

We first comment briefly on how this, and Chapter 2, relate to the $SL_2(\mathbb{Q}_p)$ case. In Fact 1.6.10 we see that the Ellis Group of $(SL_2(\mathbb{Q}_p), S_G(M))$ is $B(\mathbb{Q}_p)/B(\mathbb{Q}_p)^0$.

In Theorem 2.5.1 we demonstrate that the Ellis Group of $(SL_2(\mathbb{C}((t))), S_G(M))$ is isomorphic to $B(\mathbb{C}((t)))/B(\mathbb{C}((t)))^0$.

We suggest that the Ellis Group of $G = SL_2(K)$, where K is a characteristic 0 Henselian valued fields, could have a generalisation of the form $B(K)/B(K)^0$. In $ACVF$, we suspect that $B(K)/B(K)^0$ is trivial. In this setting, we would see Ellis Group is isomorphic to G/G^{00} despite G not having any generic types (and hence G here is not definably amenable).

This would be a new example of a setting in which, for G not definably amenable, G/G^{00} is isomorphic to the Ellis Group of $(G(M), S_G(M))$.

To conclude this chapter, we recall a result for $ACVF$ from **[10]**. Note that in this context, invariant type refers to a complete global type $p \in S(\mathcal{U})$ which is $Aut(\mathcal{U}/C)$-invariant for some (small) C. This is equivalent to saying p does not split over C. In our context we will take $C = M$, and so p is some global type left fixed by $Aut(\mathcal{U}/M)$. In this sense, the restriction of p M-invariant will be M-invariant and hence form a 1-point minimal flow of $(G, S_G(M))$.

Fact 3.4.3. *Let G be an affine algebraic group over an algebraically closed valued field K, with H a C-definable subgroup of $G(K)$. Let p be a C-definable global type of elements of H. Then the following are equivalent;*

- *p is invariant and stably dominated.*
- *For any $f \in C[G], b \vDash p$ and $a \in H$, then $|f(a)| \leq |f(b)|$.*

It is suggested in **[10]** that this result should extend to valued fields of algebraically closed residue but with non-divisible value group, though it is not explicitly proven.

By taking $G = SL_2(\mathbb{C}((t)))$ with $H = SL_2(\mathbb{C}[[t]])$ here, we apply the results of Proposition 3.4.2 to see that H admits a translation invariant type p. Using our construction, we see that p is described via a generic of $SL_2(\mathbb{C})$ together with a generic of $Ker(res)$ which is trivial. As such, we can map p into a type over $\mathbb{C}$ and see that this restriction is a generic of $SL_2(\mathbb{C})$.

It is observed in [**10**] that the same is true for $SL_n(K)$ and $SL_n(\mathcal{O}_K)$ for $K \vDash ACVF$.

It is easy to see that the realisations of p also obtain a maximum modulus here. Observe that $|f(b)|$ maximal is equivalent to considering $v(f(b))$ minimal. This is clear since $v(f(b)) = 0$ for all f by way of b being transcendental when $b \vDash p$.

Chapter 4

Dynamical Systems in Metastable Theories

In this chapter we investigate the definable topological dynamics of groups definable in metastable theories. Our approach to finding the Ellis Group of $SL_2(\mathbb{C}((t)))$ was computational, though here we take a more general approach and aim to provide descriptions of minimal flows and Ellis Groups for larger classes of metastable definable groups.

We begin by recalling the definitions of metastability as well as preliminary results of definable metastable groups. The key aims in this chapter are to investigate to what extent stably dominated groups in metastable theories admit analogues to compactly dominated groups in o-minimality. We also seek to interpret the results of [12] into the context of definable topological dynamics.

4.1 Preliminaries

The notation in this chapter is standard, and follows the conventions outlined previously in the thesis unless otherwise stated. We add that for all a, b, C, we write $a \downarrow_C b$ if there exists an $acl(C)$-definable type p such that $a \models p|_{acl(Cb)}$ **[12]**.

CHAPTER 4

For the following recap of stability theory, we refer to **[10]**, **[25]** and **[26]**.

A formula $\phi(x, y)$ is *stable* if it does not have an order property. That is, there does not exist any a_i, b_i for $i < \omega$ such that $\models \phi(a_i, b_j) \iff i \leq j$. We will say that a theory T is *stable* if every formula $\phi(x, y)$ is stable.

If $D \subseteq \mathcal{U}$ is some C-definable set, we say D is *stable* if the structure with domain D, equipped with all C-definable relations, is stable.

We recall some equivalences of stability. First, T is stable if and only if, for all models M of T, every type over M is definable. Equivalently, T is stable if there exists some cardinal κ such that for any M, the type space $S(M)$ has size at most $|M|^\kappa$.

Suppose that T is a complete theory with $M \models T$ and D a $\emptyset$-definable set in M^{eq}. We say D is *stably embedded* if, for any n, any definable subset of D^n is definable with parameters from D. If a C-definable set D in $\mathcal{U}$ is stable, then D is also stably embedded.

A *projective system* is family of objects X_i and morphisms $f_{ij} : X_i \to X_j$ such that f_{ii} is the identity on X_i, and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

A *pro-definable set* X is a projective filtered system $(X_i)_{i \in I}$ of definable sets and definable maps. We think of X as the inverse limit $\varprojlim_i X_i$. We can think of types as pro-definable sets by considering systems of definable sets ordered by inclusion. In this context it can be useful to view types this way as we will want to construct mappings from types into stable, stably embedded sets. A *pro-definable map* **[12]** $f : X \to Y$, where Y is definable, is a definable map from some X_i into Y. If Y is itself a pro-definable set, then a pro-definable map is a compatible collection of maps $f_j : X \to Y_j$.

For some set of parameters C, let $(D_i)_i$ be all C-definable stable sets. Define the *stable part* St_C to be the pro-definable set $\varprojlim_i D_i$. We can also choose to view St_C as a structure whose sorts are the D_i together with all C-definable relations R_j on the sets D_i. As a structure, St_C is stable.

For a tuple b, define $St_C(b)$ to be $dcl(Cb) \cap St_C$. By $A \mathop{\smile\!\!\!\!\!\!|}_C b$ we mean $a \mathop{\smile\!\!\!\!\!\!|}_C b$ for all $a \in A$.

Definition 4.1.1. *[12] Let $p = tp(a/C)$ and $\alpha : p \to St_C$ be a pro-C-definable map. Then we say p is stably dominated (via α) if, for any tuple b, whenever $St_C(b) \mathop{\smile\!\!\!\!\!\!|}_C \alpha(a)$, then*

$$tp(b/C\alpha(a)) \vdash tp(b/Ca).$$

A type p over C is stably dominated if it is stably dominated via some α.

Put simply, a type is stably dominated if, whenever the stable part of its realisations are sufficiently independent from the stable part of some tuple b, then the type of b over Ca can be determined using only the stable part $C\alpha(a)$ of the parameters.

Let $\theta_C(a)$ be a pro-definable map that enumerates $St_C(a)$ where $a \vDash p$. Then from [12] we see p is stably dominated if and only if it is stably dominated via θ_C.

We take opportunity here to draw a closer parallel between this property and that of compact domination. Recall the following definitions.

Definition 4.1.2. *[11] Let X be an A-definable set in M, C a compact Hausdorff space of bounded size, and f a map from X to C. We will say that f is definable over A if, for any closed subset C_1 of C, $f^{-1}(C_1) \subseteq X$ is type-definable over A in M.*

Definition 4.1.3. *[11] Let X be some type-definable set and $\pi : X \to C$ a definable surjective map from X into some compact space C. Let μ be a probability measure on C.*

Then we say X is compactly dominated by (C, μ, π) if, for any definable subset Y of X, and for every $c \in C$ outside a set of μ measure zero, either $\pi^{-1}(c) \subseteq Y$ or $\pi^{-1}(c) \subseteq X \setminus Y$.

One can observe that these notions are similar. The key difference is the replacement of topological properties on C with sufficient notions of stability and independence. Our hope then is that the results from o-minimality surrounding compact domination and Ellis Groups can be adapted to the metastable case by considering stable domination instead.

One says a type-definable group G is *compactly dominated (as a group)* if G is compactly dominated by some (H, m, π) where m here is specifically the unique normalised Haar measure on H (some compact group) and π is a group homomorphism. We say that a definable group G is *Stably Dominated* if $S_G(M)$ contains a stably dominated generic type. Hence a group will be stably dominated if it is stably dominated via some group homomorphism into a stable group, as the existence of a stably dominated generic implies G/G^{00} is a stable group. If H is a stably dominated subgroup of G, we say that H is maximal if there exists no proper stably dominated subgroup. We will call such H the *maximally stably dominated* subgroup of G.

It is seen in **[28]** and **[8]** that for all type-definable (over M) groups G, with G compactly dominated, the Ellis Group of $(G(M), S_G(M))$ is isomorphic to G/G^{00}. In this chapter we investigate to what extent these results have analogues in the metastable case for G some definable stably dominated group.

Let Γ be an $\emptyset$-definable stably embedded set. We assume Γ is orthogonal to the stable part; no infinite definable subset of Γ^{eq} is stable. For any C, a, let $\Gamma_C(a)$ denote $(C \cup \Gamma^{eq}) \cap dcl(Ca)$.

By a global C-invariant type p we mean that p does not split over C. That is, p is $Aut(\mathcal{U}/C)$-invariant.

Definition 4.1.4. *[12] The theory T is metastable (over Γ) if, for any C:*

(1) There exists $D \supseteq C$ such that, for any tuple a, $tp(a/\Gamma_D(a))$ is stably dominated.

(2) If $C = acl(C)$, then for all tuples a there exists a global C-invariant type p with

$a \models p|_C$.

A set D as in (1) *is called a metastability basis.*

The motivating example of a metastable theory is $ACVF$, and it is shown in [10] that $Th(\mathbb{C}((t)))$ is also metastable.

For this chapter, unless otherwise stated, we use T to denote an arbitrary metastable theory. We will use $M \models T$ with the universe of M denoted by K. We use $\bar{M}$ to denote an elementary extension of M, with the universe of $\bar{M}$ denoted $\mathbb{K}$.

4.2 Topological Dynamics of Stably Dominated Groups

In metastable structures many types can be determined via reduction to parameters in a stable structure. We ask if we can apply the results of stability to definable topological dynamics to study minimal flows. In this section we recall several results concerning stably dominated types and draw some preliminary conclusions for definable topological dynamics in the metastable setting.

Fact 4.2.1. *[10] For all a, C;*

(1) $tp(a/C)$ is stably dominated if and only if $tp(a/acl(C))$ is.

(2) If $C = acl(C)$ and $tp(a/C)$ is stably dominated via α, then $tp(a/C)$ has a unique C-definable extension p. Moreover, for all $B \supseteq C$, $a \models p|_B$ if and only if $St_C(B) \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$\mid$}}_C \alpha(a)$.

Hence, when working with stably dominated types, there is no loss of generality in considering C algebraically closed. By doing so, we can assume stably dominated types will admit unique heirs. Given that for an arbitrary metastable structure

M, types over M may not be definable, this fact allows our action of G on stably dominated types to be well defined in all metastable structures.

Fact 4.2.2. *[10] Assume $tp(a/C)$ is stably dominated.*

(1) Let q be a global $acl(C)$-definable type and $b \vDash q|_{acl(C)}$, $a \mathop{\smile\kern-0.75em\vert}_C b$ implies $b \mathop{\smile\kern-0.75em\vert}_C a$. In particular, if $tp(b/C)$ is stably dominated, $a \mathop{\smile\kern-0.75em\vert}_C b$ if and only if $b \mathop{\smile\kern-0.75em\vert}_C a$.

(2) $a \mathop{\smile\kern-0.75em\vert}_C bd$ if and only if $a \mathop{\smile\kern-0.75em\vert}_C b$ and $a \mathop{\smile\kern-0.75em\vert}_{Cb} d$.

(3) If $tp(b/Ca)$ is stably dominated, then so is $tp(ab/C)$

(4) If $b \in acl(Ca)$, then $tp(b/C)$ is stably dominated.

It follows immediately that we can say something meaningful about the definable topological dynamics of maximally stably dominated groups.

Proposition 4.2.3. *Let M be a metastable structure with the universe of M algebraically closed. Let G be an M-definable maximal stably dominated group, and $X \subset S_G(M)$ the set of stably dominated types in G (over M). Then the flow (G, X) is closed.*

***Proof*.** This follows from (4) of Fact 4.2.2. Take any type $p \in X$ and let $a \vDash p$. Then clearly any $g \in G$ is contained in M algebraically closed, and so $tp(g/M)$ is stably dominated.

The action of G on X can be considered as an action of types of $g \in G$ over M on X wherein the action is given by $tp(g/M) * p = tp(ga/M)$ where $a \vDash p$.

As $tp(g/M)$ stably dominated, and since $p = tp(a/M) = tp(a/Mg)$ (since $g \in M$) is stably dominated, we have $tp(ga/M)$ stably dominated by (3) of Fact 4.2.2.

Hence X is closed under action of G and (G, X) is a subflow of $(G, S_G(M))$. □

We can refine this further by extending the action of G on X to an action of X on itself.

Proposition 4.2.4. *Let M be a metastable structure with the universe of M algebraically closed. Let G be an M-definable maximal stably dominated group, and $X \subset S_G(M)$ the set of stably dominated types in G (over M).*

Let $p, q \in X$ with $a \vDash p$ and $b \vDash q|_{M,a}$. Then the action

$$* : X \times X \to X$$
$$(p, q) \mapsto tp(ab/M),$$

is well-defined and moreover $(X, *)$ is a semigroup.

Proof. The fact that the action is well defined follows immediately from (2) of Fact 4.2.1. Namely, that extensions to heirs over supersets of M are unique.

We now show this is a semigroup. Closure follows from (3) of Fact 4.2.2, by showing that $p * q \in X$. Associativity is a consequence of (2) of 4.2.2, though also from the general topological dynamics of type multiplication as shown in Chapter 1.

Hence $(X, *)$ is a semigroup as required. □

What we are showing here is simply that the collection of stably dominated types form a subflow of $(G, S_G(M))$ where G is maximally stably dominated. We make no claims about minimality of such a subflow; in fact in general this subflow will not be minimal. However, it will contain a minimal subflow and as such this preliminary result can be used to narrow down our search for idempotent elements.

We now recall a notion of symmetry for stably dominated types, and extend this once again to the action on the type space to demonstrate 2-sidedness of type multiplication in stably dominated groups. We say a (generic) type $p \in S_G(M)$ is 2-sided if, for all $q \in S_G(M)$, we have $q * p = p * q$.

Definition 4.2.5. *[12] Let p be a definable type. We say that p is symmetric if for all definable types q and parameters C such that p and q are defined over C, $a \vDash p|_{acl(C)}$ and $b \vDash q|_{acl(Ca)}$, then $a \vDash p|_{acl(Cb)}$.*

It is remarked in [12] that stably dominated types are symmetric. Note that for $p, q \in S_G(M)$ stably dominated types, this is not claiming $p * q = q * p$. However, it does imply that a type multiplication $*'$ given by taking heirs on the left rather than the right would be a similarly well-defined semigroup action on the set $X \subseteq S_G(M)$ of stably dominated types.

Fact 4.2.6. *[12] Assume G admits a symmetric right generic type. Then left and right generics coincide, they are all symmetric, and there is a unique left (right) orbit of generics. In particular, there are only boundedly many generics in G.*

Since stably dominated types are symmetric, it follows that all generics of G are both left and right generics and are all stably dominated. The boundedness condition is especially interesting here, and we should note the comparison to G/G^{00} having boundedly many cosets when asking if G/G^{00} and Ellis Groups of maximally stably dominated groups coincide.

The existence of unique symmetric types has consequences for the connectedness of G, as shown by the following result of Hrushovski and Rideau.

Fact 4.2.7. *[12] Assume G admits a symmetric generic type p. Then the following are equivalent;*

- *p is the unique generic of G.*
- *For all $g \in G$, $pg = p$.*
- *G is connected.*

What this shows is that whenever G admits a symmetric generic type, then there is a unique principal generic. By a principal generic type we mean that the realisations are contained in G^{00}. Note that some literature uses G^0 here, as this setting is close to stability where $G^0 = G^{00}$, but we will use G^{00} to be consistent with the majority of the NIP literature on generics.

An easy consequence of these equivalences is as follows.

Proposition 4.2.8. *If G is a stably dominated connected group, then G is definably extremely amenable.*

Proof. As G is stably dominated, by definition it admits a symmetric generic type p.

By Lemma 4.2.7, if G is connected, then p is the unique such type.

The existence of a unique generic is sufficient for G to be definably extremely amenable. □

We note that this is more of a specific case of the following result.

Proposition 4.2.9. *Let M be a metastable structure with the universe of M algebraically closed. Let G be an M-definable maximal stably dominated group, and $Gen_G(M) \subset S_G(M)$ the set of stably dominated generic types in G (over M).*

Then $(G, Gen_G(M))$ is a subflow of $(G, S_G(M))$.

Proof. Since G is a maximal stably dominated group, there exists at least one stably dominated generic type in $S_G(M)$. It follows from 4.2.6 that in fact every generic type is stably dominated and by the above $Gen_G(M)$ is nonempty. It is easy to see then that if p is generic, p is stably dominated and hence has unique extensions by (2) of Fact 4.2.1, as are all of its left G-translates. As p is generic, $gp = tp(ga/M)$ is also generic and hence also stably dominated. Hence $Gen_G(M)$ is closed under left-action by G and hence $(G, S_G(M))$ is a subflow of $(G, S_G(M))$. □

I make no claims currently that such a subflow is minimal, though a remark of Hrushovski in [11] suggests that maximally stably dominated groups admit finitely satisfiable generics, and as such we can apply Fact 1.5.4 and conclude that $(G, Gen_G(M^{ext}))$ is minimal and two-sided.

This would mean that that stably dominated groups are definably amenable, and as such questions as to whether Ellis Groups and G/G^{00} coincide are already known. In general, we want to study the dynamics of groups which are not themselves stably dominated, but instead admit a group decomposition containing definable maximally stably dominated subgroups. In doing so, we wish to assess to what extent the description of Ellis Groups can be simplified using results of stable domination and stability.

The following results are useful for such study, and we see some preliminary consequences for groups that admit slightly simpler decompositions than we plan on working with.

Fact 4.2.10. *[12] The class of stably dominated pro-definable groups is closed under Cartesian products and image under definable group homomorphisms. If G and H are stably dominated pro-definable groups with p and q stably dominated generics of G and H (resp.), then $p \otimes q$ is a stably dominated generic of $G \times H$.*

Note here that $\otimes$ is a map $S_G(M) \times S_H(M) \to S_{G\times H}(M)$, and is distinct from the type multiplication $*$. Specifically, $p \otimes q = tp((a, b)/M)$, where $a \models p$ and $b \models q|_{M,a}$; mapping pairs of complete n and m-types (depending on the dimension of G, H) to complete $n + m$-types.

However, if G and H have the same group operation, and $G \cap H$ is trivial, we could construct the group GH and observe that there exists a homeomorphism between the minimal flows of $(G \times H, S_{G\times H}(M))$ and $(GH, S_{GH}(M))$.

Fact 4.2.11. *[12] Let G be a pro-definable group and let $N \triangleleft G$ be a stably dominated pro-definable subgroup. Assume that there exists a stably dominated type concentrating on G/N whose orbit under G-translations is bounded. Then G is stably dominated.*

In particular, if $N \triangleleft G$ and $N, G/N$ are stably dominated, then so is G.

We remark the similarities to Fact 1.5.8 and the results of [15] in which groups admitting similar decompositions, though insisting on definable amenability rather than stable domination, are shown to have the same property. Of course this is somewhat trivial knowing stably dominated groups are themselves definably amenable, though this context-specific result may be beneficial in describing minimal flows in terms of stable groups.

Fact 4.2.12. *[12] Let G be an algebraic group with $N \leq G$ an algebraic subgroup. Let $H \leq G$ be definable in $ACVF$ and stably dominated. Then $H \cap N$ is stably dominated.*

This result is especially beneficial when it comes to manipulating types in group decompositions with a large intersection. For example, let K be an algebraically closed valued field and consider that every element of $SL_2(K)$ can be expressed as the product of elements from $B(K)$ and $SL_2(\mathcal{O})$. This decomposition has an infinite intersection of elements in the decomposition; namely $B(\mathcal{O})$. Using the above with the Borel subgroup algebraic (so $G = B$) and $H = SL_2(\mathcal{O})$ maximally stably dominated, we see $B(\mathcal{O})$ is itself stably dominated and admits stably dominated generics in $S_{B(\mathcal{O})}(M)$.

We now include several results of [12] which demonstrate the existence of a group homomorphism from stably dominated group into a stable group and describes a relationship between the generics of these groups. We use these later to demonstrate how the minimal subflows of a stably dominated group can be described via reduction to the stable group.

Fact 4.2.13. *[12] Let G be a stably dominated pro-definable group. There exists a pro-definable stable group $\mathfrak{G}$ and a pro-definable homomorphism $\pi : G \to \mathfrak{G}$ such that the generics of G are stably dominated via π.*

We remark some consequences of this fact for G maximally stably dominated. Consider the above π and observe that π extends to a function $S_G(M) \to S_{\mathfrak{G}}(M)$ that acts on realisations. First, since the action of G on $S_G(M)$ acts by homomorphism, we can see that $\pi(g * p) = \pi(g) * \pi(p)$. This means that we reduce closed left ideals of $(S_G(M), *)$ to some closed set in $(S_{\mathfrak{G}}(M), *)$, and we claim later that this reduction coincides with the closed left ideals of $(S_{\mathfrak{G}}(M), *)$.

This would mean that, given some metastable definable group that admits a decomposition with maximally stably dominated subgroups, we could describe the dynamics of that subgroup quickly via reduction to a stable group. To do this, we would like π to have some maximal property, or perhaps some notion of uniqueness, to ensure that the reduction of G to $\mathfrak{G}$ is well determined. We see using the following result that such a notion is known to exist for these homomorphisms.

Fact 4.2.14. *[12] There exists a pro-C-definable stable group $\mathfrak{G}$ and a pro-C-definable homomorphism $\pi : G \to \mathfrak{G}$, maximal in the sense that any pro-C-definable homomorphism $\pi' : G \to \mathfrak{G}'$ into a pro-C-definable stable group factors through π.*

The kernel of this maximal π is uniquely determined. If G is stably dominated it will be stably dominated via this maximal homomorphism.

Further, as minimal flows of stable groups are known to contain the generic types, we would want that the projection to generic types in $\mathfrak{G}$ should come from types generic in G. Again, we see that there is a duality between the generics in a stably dominated group G and the stable group $\mathfrak{G}$.

Fact 4.2.15. *[12] Let G be a pro-C-definable group stably dominated via some pro-C-definable group homomorphism $\pi : G \to \mathfrak{G}$. Then $tp(a/C)$ is generic in G if and only if $tp(\pi(a)/C)$ is generic in $\mathfrak{G}$.*

The existence of a homomorphism into a stable group will be useful to us in describing minimal flows via the reduction to stable groups, and is especially useful at quickly finding explicit descriptions of generic types in stably dominated groups.

For example, let K be an algebraically closed valued field with residue field $res(K)$. Then $SL_2(\mathcal{O}_K)$ is maximally stably dominated via the homomorphism that maps matrices entrywise to their residue. We see an explicit description of the unique stably dominated generic of $SL_2(\mathcal{O}_K)$ later in the thesis, and note that this homomorphism maps that generic into the unique stably dominated generic of $SL_2(res(\mathbb{K}))$.

We first show that we can use this homomorphism to demonstrate minimality of the subflow of generics as given in Proposition 4.2.9.

Proposition 4.2.16. *Let G be a Stably Dominated Group. Then the Minimal Subflow of $(G, S_G(M^{ext}))$ is precisely the set of stably dominated (left) generic types, denoted $Gen_G(M^{ext})$.*

Proof. We first prove that this is a subflow. By definition, G contains at least one stably dominated generic type, and so $Gen_G(M^{ext})$ is non-empty. Further, it is clear that the set of generic types forms a subflow as any translate of a generic type is itself generic.

It remains to show that this is minimal.

By Fact 4.2.13, there exists a definable stable group $\mathfrak{G}$ and pro-definable map $\pi : G \to \mathfrak{G}$ a group homomorphism. Further, by Fact 4.2.15, this extends to a projection $\pi' : Gen_G(M^{ext}) \to Gen_{\mathfrak{G}}(M^{ext})$ that sends $tp(a/M^{ext}) \in Gen_G(M^{ext})$ to $tp(\pi(a)/M^{ext}) \in Gen_{\mathfrak{G}}(M^{ext})$.

Suppose for contradiction that there exists some subflow $(H, S_G(M^{ext}))$ with $H \subset Gen_G(M^{ext})$.

Then $\pi'(H) = \mathfrak{H}$ is clearly a minimal subflow of $(\mathfrak{G}, S_{\mathfrak{G}}(M^{ext}))$ since the action of G on H projects via π to an action of $\mathfrak{G}$ on $\mathfrak{H}$ which remains closed since π is a homomorphism.

However, we know from results on the topological dynamics of definable stable groups, that $Gen_{\mathfrak{G}}(M^{ext})$ is the unique minimal subflow of $(\mathfrak{G}, S_{\mathfrak{G}}(M^{ext}))$, and so we have a contradiction.

Hence $(Gen_G(M^{ext}), S_G(M^{ext}))$ is a minimal subflow of $(G, S_G(M))$ as required.

□

An easy Corollary of this is as follows.

Proposition 4.2.17. *For G a Stably Dominated Group, the flow $(G, S_G(M^{ext}))$ has Ellis Group $p * Gen_G(M^{ext})$, where p is a stably dominated principal generic.*

Proof. It is clear that p is idempotent, since the principal generic knows its realisations lie in G^0, a subgroup of G. Then by definition $p * Gen_G(M^{ext})$ is the Ellis Group of $(G, S_G(M^{ext}))$. □

We now move to more general results where we study larger classes of groups which admit decompositions with stably dominated components, and observe how we can describe their Ellis Groups using the results of this section.

4.3 Metastable Definable, Non-Stably Dominated Groups

We demonstrated in Chapter 2 that a non-definably amenable group definable in the metastable structure $\mathbb{C}((t))$ provided another negative answer to the Ellis group conjecture of Newelski. However, the follow up work in Chapter 3 shows something interesting for the flow of $SL_2(\mathbb{C}[[t]])$; that $S_{SL_2(\mathbb{C}[[t]])}(M)$ admits an invariant type. We see now that such a type is actually stably dominated via the residue map.

Given that we can write $SL_2(\mathbb{C}((t)))$ as the product of $SL_2(\mathbb{C}[[t]]) \times B(\mathbb{C}((t)))$ and all Ellis Groups are isomorphic, we can conlude the following; the maximally stably dominated part of $SL_2(\mathbb{C}((t)))$ collapses completely when finding the Ellis Group. We hypothesize here that this is a consequence of a trivial minimal flow in the stably dominated part, which is not necessarily true for all stably dominated groups.

Proposition 4.3.1. *Suppose $G \cong NH$ with N normal, both N, H stably dominated, and H definably isomorphic to G/N. Then the minimal subflow of $(G, S_G(M^{ext}))$ is precisely $cl(I * J)$, where I is the minimal subflow of $(N, S_N(M^{ext}))$ and J is the minimal subflow of $(G/N, S_{G/N}(M^{ext}))$.*

Proof. The assumptions here imply that G itself is stably dominated by Fact 4.2.11.

Further, by Fact 4.2.11, we know that the product of any types p, q in I, J (resp.) will be a stably dominated generic type in G.

Next, we see that $I * J$ is left $G(M^{ext})$-invariant. Consider some $g \in G(M^{ext})$. Then we can express $g = nh$ for some $n \in N(M^{ext}))$ and $h \in G/N(M^{ext})$. Let $n'h' \models q * p \in I * J$ (over M^{ext}, nh). Using the normality of N in G, we can find some $\bar{h} \in G/N(\overline{M^{ext}})$ such that $nhn'h' = nn'\bar{h}h'$. Then since $n' \models q|_{M^{ext},n}$, and q is in the minimal flow I of $(N, S_N(M^{ext}))$, $nn' \models q' \in I$. Similarly, since $h' \models p|_{M,n,h,n'}$, and $p \in J$ the minimal flow of $(G/N, S_{G/N}(M^{ext}))$, we see $\bar{h}h' \models p'|_{M,nn'}$ as required, and hence $g \cdot q * p = q' * p' \in I * J$ as required.

It remains to prove that $cl(I * J)$ is minimal. Note that by Fact 1.5.1, we can express every element of $cl(I * J)$ as $s * q * p$ for some $s \in S_G(M^{ext})$, $q \in I$ and $p \in J$. We prove that the $S_G(M^{ext})$-orbit of $s * q * p$ generates all of $cl(I * J)$, for which it is sufficient to show $I * J \subseteq S_G(M^{ext}) * s * q * p$.

Let $q' * p' \in I * J$. We need to find some $r \in S_G(M^{ext})$ such that $q' * p' = r * s * q * p$. The normality assumption makes this somewhat easy. Let r be realised by some nh. Let the heir of s (over nh) be realised by some n_1h_1. Let the heir of q (over nh, n_1h_1) be realised by some n_2. Let the heir of p (over nh, n_1h_1, n_2) be realised by some h_2. In this description, $n_i \in N$ and $h_i \in G/N$.

Then $r * s * q * p = tp(nhn_1h_1n_2h_2/M)$. Using normality of N we can find some $\bar{h} \in G/N(\overline{M^{ext}})$ such that $nhn_1h_1n_2h_2 = nn_1n_2\bar{h}h_2$.

Since $n_2 \models q \in I$ we can choose n such that $nn_1n_2 \models q' \in I$. Similarly, since $h_2 \models p \in J$, we can choose h such that the resulting $\bar{h}h_2 \models p'|_{M^{ext},nn_1n_2}$.

Let $r \in S_G(M^{ext})$ be the type realised by nh chosen as above. Hence $I * J \subseteq S_G(M^{ext}) * s * q * p$. Then $cl(I * J)$ is the orbit-closure of every element in $cl(I * J)$ and hence a minimal flow of $(G(M^{ext}), S_G(M^{ext}))$ as required. □

Proposition 4.3.2. *Let G be a definable group with subgroups H, J of G such that;*

- $G = HJ$

- *J is maximally stably dominated*
- *$(H, S_H(M))$ admits a unique 2-sided 1-point minimal flow, p.*

Then the minimal flow of $(G, S_G(M))$ is $cl(G(M^{ext}) \cdot p * q)$ where q is the principal generic of $Gen_J(M^{ext})$. The Ellis Group of $(G, S_G(M))$ is a subset of $p * Gen_J(M^{ext}))$.

Proof. First, since J is maximally stably dominated, $S_J(M)$ contains stably dominated generic types and hence the set $Gen_J(M^{ext})$ is non-empty. Let q be the principal generic (i.e. centred on J^{00}). Further, for any $s \in S_J(M^{ext})$, $s * Gen_J(M^{ext}) = Gen_J(M^{ext})$ by the properties of generic types as $Gen_J(M^{ext})$ the minimal flow of $(J(M^{ext}), S_J(M^{ext}))$.

The fact that $cl(G(M^{ext}) \cdot p * q)$ is minimal follows similarly to the proof of Proposition 4.3.1. Note that we do not assume normality in this case, but that the arguments in Proposition 4.3.1 can be sufficiently adapted using the 2-sided 1-point minimal flow of $(H(M^{ext}), S_H(M^{ext}))$.

To see then that the Ellis Group is a subgroup of $p * Gen_J(M^{ext})$, consider that $cl(G(M^{ext}) \cdot p * q)$ contains some idempotent element $s * q * p$.

Claim: There exists an idempotent in $cl(G(M^{ext}) \cdot p * q)$ of the form $s * q * p$, where $s = q * p'$ for some $p' \in Gen_J(M^{ext})$.

Proof of Claim. We show there exists a p' such that $q * p' * q * p$ is idempotent. First, note we can write $p' * q$ as $q_0 \cdot p_0$ for some $q_0 \in S_H(M^{ext})$ and $p_0 \in S_J(M^{ext})$. Hence $q * p' * q * p = q * q_0 * p_0 * p$.

Then, since q is a 2-sided 1-point minimal flow of $(H(M^{ext}), S_H(M^{ext}))$ and $p \in Gen_J(M^{ext})$, we can write $q * q_0 = q$ and $p_0 * p = p_1 \in Gen_J(M^{ext})$.

Hence $q * p' * q * p = q * p_1$. Then $(q * p' * q * p)^2 = (q * p_1)^2$, and we can use the same argument to write $(q * p_1)^2 = q * p_2$.

We claim now that p' can be chosen such that $p_2 = p_1$.

Let $h \vDash q$ and let $j \vDash p_1|_{M^{ext},h}$. Let $h' \vDash q|_{M^{ext},h,j}$ and let $j' \vDash p_1|_{M^{ext},h,j,h'}$.

Then $(q * p_1)^2 = tp(hjh'j'/M^{ext})$. Further, there exists some $h_0 \in H(\bar{M}^{ext})$ and $j_0 \in J(\bar{M}^{ext})$ such that $jh = h_0 j_0$. hence $tp(hjh'j'/M^{ext}) = tp(hh_0 j_0 j'/M^{ext})$.

Then as $Gen_J(M^{ext})$ is a section of J/J^{00} is a minimal subflow of $(J(M^{ext}), S_J(M^{ext}))$, there exists some j_0 such that $j_0 \cdot p_1 = p_1$. Hence we can choose p' appropriately such that $j_0 j' \vDash p_1$.

Hence, as required, we can find some $p' \in S_J(M^{ext})$ such that $q * p' * q * p$ is idempotent as required. □

Finally then, as we have demonstrated the idempotent is of this form, we can act on $cl(G(M^{ext}) * q * p)$ on the left by $q * p' * q * p$. Hence the Ellis Group of $(G(M^{ext}), S_G(M^{ext}))$ is precisely $q*p'*q*p*cl(G(M^{ext})*q*p)$. Using similar arguments to the above, since q is a 2-sided 1 point minimal flow of $(H(M^{ext}), S_H(M^{ext}))$, and $p \in Gen_J(M^{ext})$), we can rearrange the realisations of elements of $q * p' * q * p * cl(G(M^{ext}) * q * p)$ such that it realises some $q * p' \in q * Gen_J(M^{ext})$.

We make no claims that this covers all of $Gen_J(M^{ext})$, simply that the Ellis Group is a subgroup contained in the set $q * Gen_J(M^{ext})$. □

Proposition 4.3.3. *Let G be a definable stably dominated group. Then there exists some stable group, $\mathfrak{G}$, such that the minimal subflow of $(\mathfrak{G}(M), S_{\mathfrak{G}}(M))$ is expressible as a section of the minimal subflow of $(G(M^{ext}), S_G(M^{ext}))$.*

Proof. Since G is stably dominated, there exists some stable group $\mathfrak{G}$ and group homomorphism $\pi : G \to \mathfrak{G}$ by Fact 4.2.13 such that G is stably dominated via π. Moreover, by Fact 4.2.14, we may assume π is maximal in the sense that any other such homomorphism cycles through π.

By definition, since G is a stably dominated group, $S_G(M^{ext})$ contains stably dominated generic types. Hence the minimal flow of $(G(M^{ext}), S_G(M^{ext}))$ is the set $Gen_G(M^{ext})$ of generic types in $S_G(M^{ext})$, by Fact 1.5.3.

Finally, by Fact 4.2.15, $\{\pi(p) : p \in Gen_G(M^{ext})\}$ is precisely the set $Gen_{\mathfrak{G}}(M^{ext})$ of generic types in $S_{\mathfrak{G}}(M^{ext})$, which is itself a minimal subflow by Fact 1.6.1. Further, since $\mathfrak{G}$ is a group definable in a stable structure, we can take $M^{ext} = M$.

Consider the binary relation $\bar{\pi}$ on $Gen_G(M^{ext})$ defined by $\bar{\pi}(p, q)$ if $\pi(p) = \pi(q)$. It is easy to see using the fact that π is a group homomorphism that this is an equivalence relation, and moreover that $Gen_G(M^{ext})/\bar{\pi} = Gen_{\mathfrak{G}}(M)$ as required. □

Of course, this result would be especially useful when $|Gen_G(M^{ext})| = |Gen_{\mathfrak{G}}(M)|$, so that the quotient by $\bar{\pi}$ is trivial. In general however, $S_G(M^{ext})$ may contain more generic types. We leave this as an open question.

Question 4.3.4. *Consider the result of Proposition 4.3.3. When can the generic types of $S_G(M^{ext})$ be recovered using the generic types of $S_{\mathfrak{G}}(M)$?*

Clearly there exists some preliminary but somewhat trivial answers to this; namely when π^{-1} is itself a group homomorphism (and hence π is a group isomorphism). In general however, this will not be the case. We ask whether there are any properties of the topologies or of the model theory that allow us to recover the generic types in the opposite direction. We would not necessarily require a result that provides a description; an argument which counts the generic types in $S_G(M^{ext})$ using the generic types of $S_{\mathfrak{G}}(M)$ would be useful in itself. Such a result would allow us to know quickly if G were definably extremely amenable for example. Similarly, due to the isomorphism between the Ellis Group and G/G^{00}, we could quickly find the type-definable connected component of more complicated groups if we could count the generics using the - likely simpler - reduction of G to a stable group.

Chapter 5

Definable Topological Dynamics of $SL_2(K)$

We look to generalise the work of Chapter 2 to a larger class of groups; namely definable, non-definably amenable groups in a metastable structure. The intention is that the work here should generalise to affine algebraic groups over algebraically closed valued fields, providing those groups admit similar group decompositions.

Unlike $\mathbb{C}((t))$, there is no known language in which all models of $ACVF$ admit definable types over some small (base) model M. This property is required for type multiplication to be well defined. We have two options in solving this issue; work in the Shelah expansion M^{ext}, or restrict to a smaller class of algebraically closed valued fields for which types over M are definable. We choose the latter, and restrict to models K of $ACVF$ which are maximally complete and have value group isomorphic to $(\mathbb{R}, +)$; it is known that all complete types over K are definable in this setting.

Further, it is well known that $ACVF$ is not a complete theory. Since we look to generalise the results of Chapter 2, we restrict to $ACVF_{0,0}$, and hence K will be of equicharacteristic 0.

Where the work in Chapter 2 was focused on demonstrating non-equivalence of

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Ellis Groups and 00-components in unstable metastable settings, the work in this chapter is aimed at providing a description for Ellis Groups in this setting, and determining to what extent maximally stably dominated subgroups determine Ellis Groups.

5.1 Preliminaries

The notation in this chapter should be consistent with the rest of this thesis unless otherwise stated. We will work in the language of rings, augmented with a predicate div for division where $div(x,y) \iff v(x) \leq v(y)$. Let L_{div} the language $L_{ring} \cup \{div\}$. We use M to denote an L_{div}-structure with domain K, where K is an algebraically closed value field. As before, $\bar{M}$ will denote some elementary extension of M, and we use L to denote the domain of $\bar{M}$. We use $\mathcal{U}$ to denote some global (monster) model. For a given field K, we use Γ_K, $\mathcal{O}_K$ and $res(K)$ for the value group, valuation ring and residue field of K respectively. Models of $ACVF$ in this language do not necessarily admit definable types over M.

Fact 5.1.1. *[10] Let K be an algebraically closed valued field.*

(i) The theory of K has quantifier elimination in L_{div}.

(ii) The theory of K has quantifier elmimination in a 2-sorted language with a sort K for the field in L_{ring}, a sort Γ for the value group (written multiplcatively in the language $(<, .., 0)$ and an absolute value map $|-| : K \to \Gamma$ with $|0| = 0$).

(iii) The theory of K has quantifier elimination in a 3-sorted language $L_{\Gamma k}$ with the sorts and language of (ii) together with a sort k for the residue field in L_{ring} and a map $Res : K^2 \to k$ given by $Res(xy^{-1})$ equal to the residue of xy^{-1} and taking value $0 \in k$ if $|x| > |y|$.

This fact has a partial converse; namely that any non-trivially valued field whose theory has QE in L_{div} must be algebraically closed.

We now recall a well known fact about L_{div} definable sets in models of $ACVF$, with the intention of giving a description of complete 1-types over K. Let K be a valued field with value group Γ. Then any definable subset of K (in 1-variable) is a Boolean combination of open balls $B_\gamma(a) = \{x \in K : |x - a| < \gamma\}$. In the topology induced by the valuation, all open balls of non-zero radius are clopen.

Recall that a residual extension of K an algebraically closed valued field is a field $L \supset K$ such that $res(L) \supset res(K)$. Likewise a ramified (or valuational) extension is an extension L of K such that $\Gamma_L \supset \Gamma_K$. An Immediate extension of K is such that $res(L) = res(K)$ and $\Gamma_L = \Gamma_K$.

Fact 5.1.2. *Let $K \vDash ACVF$, with $K \preceq L$, $t \in L \setminus K$ and consider the type $p = tp(t/K)$. Then;*

- *If $K(t)/K$ is a residual extension, then p is definable.*
- *If $K(t)/K$ is a ramified (valuational) extension with $val(t) = \gamma \notin \Gamma_K$, then p is definable if and only if the cut definable by $val(t)$ in $\Gamma(K)$ is rational.*
- *If $K(t)/K$ is an immediate extension, then p is not definable.*

This result has a useful consequence; by taking a complete valued field such that no proper immediate extension exists, and insisting that Γ_K is isomorphic to $\mathbb{R}$, one can see the following result.

Fact 5.1.3. *Suppose $K \vDash ACVF$. Then the following are equivalent;*

- *K is maximally complete, and the value group $\Gamma_K \cong (\mathbb{R}, +, <)$.*
- *Every type $p \in S_1(K)$ is definable.*
- *Every type $p \in S_n(K)$ is definable for all $n \in \mathbb{N}$.*

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I recall several well-known results of algebraically closed valued fields that will be useful to us in this section. Firstly, every valued field has an immediate maximally complete extension. As such, for $K \models ACVF_{0,0}$, we will assume K to be maximally complete. Further, any maximally complete field is Henselian.

The fields $K = k((t^{\Gamma})) = \{f = \sum_{\gamma \in \Gamma} a_{\gamma} t^{\gamma} : supp(f)$ is well ordered $\}$ are all maximally complete. As such, generalising to G affine algebraic over some maximally complete $K \models ACVF_{0,0}$, seems reasonable; taking $G = SL_2$ and $K = \mathbb{C}((t^{\mathbb{R}}))$ as a canonical example.

For this chapter, I assume that K is maximally complete, with Γ_K isomorphic to $(\mathbb{R}, +, <)$.

We first provide an analogue for Lemma 2.1.9 where K is a maximally complete algebraically closed valued field in $L_{ring} \cup \{div\}$.

Fact 5.1.4. *[13] Let $K \models ACVF$ with L an extension of K. Let v be the valuation on K with value group Γ. Suppose that for some $t \in L$, $L = K(t)$. Then we can determine the valuation on L via the set $I = \{v(t-a) : a \in K\}$ as follows;*

- *If $I \supset \Gamma$ - that is, there exists some $v(t-a) = \gamma \notin \Gamma$ - then $\Gamma_L = \Gamma(\gamma)$ and the residue field of L is the residue field of K.*
- *If I takes a maximal value $v(b)$ in Γ at some $a \in K$, then $k_L = k_K(e)$ where $e = res((t-a)/b)$.*
- *If $I = \Gamma$ with no maximal value for all $a \in K$, then $K(t)$ is an immediate extension.*

The first case is an extension of the value group in which we can determine the possibilities for the complete types via the o-minimal theory. The non-immediate types here are of elements with infinite or infinitesimal value with respect to Γ_K. The second case is what we referred to as transcendental types or residual types. These are types whose realisations extend the residue field. In the case of $ACVF$

these are elements whose residue is transcendental over $res(K)$. These are uniquely determined by being "transcendentally close" to an element of K. Finally, we have immediate extensions and these coincide with the immediate types of elements already in K.

Following from Fact 5.1.4 we see that the complete descriptions of 1-types in this setting are as follows. Types of kind (a) are immediate, the valuational types correspond to 1-types in an o-minimal setting and the transcendental types are determined as above as the ac map is not definable in this language by **[21]**.

Proposition 5.1.5. *[13] Let $K \vDash ACVF$ be a henselian, maximally complete field of equicharacteristic 0.*

Then the complete 1-types over K are precisely the following;

(a) $tp(a/M)$ for $a \in K$ determined by the formula $x = a$.

(b) Types p_∞ determined by the formulas $v(x) < \gamma$ for all $\gamma \in \Gamma$.

(c) Types p_a determined by the formulas $v(x-a) > \gamma$ for all $\gamma \in \Gamma$.

(d) Types p_{a^-} determined by $v(x) < v(a)$ for some $a \in K$ and $v(x) > \gamma$ for all $\gamma < v(a) \in \Gamma$.

(e) Types p_{a^+} determined by $v(x) > v(a)$ for some $a \in K$ and $v(x) < \gamma$ for all $\gamma > v(a) \in \Gamma$.

(f) Types $p_{a,\gamma}$ determined by $a \in K$, $v(a) < v(x-a) = \gamma$, and $res((x-a)/\gamma)$ is transcendental over k. (If $a = 0$, we simply use $v(x) = \gamma$ instead.)

We compute the Ellis Group for $SL_2(K)$ by hand as to better inform the general case where G is affine algebraic over K. Although $\mathbb{C}((t))$ does not elementarily embed into K, we expect that the Ellis Groups should be similar. As before, it can be shown $SL_n(K)$ is not definably amenable.

Proposition 5.1.6. *For $K \vDash ACVF$ and any $n \in \mathbb{N}$ with $n > 1$, the group $SL_n(K)$ has no definable generic types, and hence is not definably amenable.*

Proof. This can be seen directly as in Proposition 2.1.5, or found in Example 1.14 of [12]. □

We know now that $\mathbb{C}((t))$ does indeed admit a decomposition into a product of $SL_2(\mathbb{C}[[t]])$ and $B(\mathbb{C}((t)))$, though until the work in Chapter 3, we were unaware that $SL_2(\mathbb{C}[[t]])$ was definably amenable with a 1-point minimal subflow. We show later in this chapter that $\mathcal{O}_K$ similarly admits a unique invariant type, and as such we can consider $SL_2(K) = B(K) \cdot SL_2(\mathcal{O}_K)$, and note that this decomposition is a definably amenable / maximally stably dominated decomposition. Note that $B(K) \cap SL_2(\mathcal{O}_K) = B(\mathcal{O}_K)$ is infinite. This is similar to the $SL_2(\mathbb{Q}_p)$ case in [22], however we remark later that this case is much more difficult to work with.

As before, we first compute the minimal flows, idempotents and Ellis Groups of the additive and multiplicative groups of K. The motivation here is to construct minimal flows of $B(K)$ and $SL_2(\mathcal{O}_K)$.

5.2 Additive and Multiplicative Flows

We now use the types in Proposition 5.1.5 to explicitly describe the minimal subflows of the additive and multiplicative groups of some $K \vDash ACVF$ in L_{div} with equicharacteristic 0 and $\Gamma \cong (\mathbb{R}, +)$. This should carry through to an arbitrary value group providing 1-types over M are definable. We denote these groups by $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively with corresponding flows $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$ and $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$. Recall we also assume that complete types over K are definable.

Proposition 5.2.1. *Let $K \vDash ACVF$ and consider the additive group $\mathbb{G}_a$ of K. Then the type p_∞ is the unique 1-point minimal subflow of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$.*

***Proof*.** Let $a \in \mathbb{G}_a$ and let $\alpha \models p_\infty|_{M,a}$. Then $ap_\infty = tp(a + \alpha/M)$ and we show that $a + \alpha \models p_\infty$.

Clearly, $v(a + \alpha) = v(\alpha) < \gamma$ for all $\gamma \in \Gamma_K$. Hence $\{p_\infty\}$ is a subflow of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$. Further, as $\{p_\infty\}$ is closed and cannot contain any proper non-trivial subflows it must be a minimal subflow of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$ as required.

To see that this is the unique minimal subflow of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$, recall that minimal subflows coincide with minimal closed left ideals of $(S_{\mathbb{G}_a}(M), *)$. For any two minimal flows W_1, W_2, either $W_1 = W_2$ or $W_1 \cap W_2 = \emptyset$.
Suppose for contradiction there exists some other closed left ideal $W \in (S_{\mathbb{G}_a}(M), *)$. Then we can assume since p_∞ is minimal that $p_\infty \notin W$. Since W is a closed left ideal we have $p * W \subset W$.

However, let $q \in W$ and suppose $\alpha \models p_\infty$ and $\beta \models q|_{M,\alpha}$. Then $q*p = tp(\alpha+\beta/M)$. However, it is easy to see that since β does not realise $p_\infty|_{M,\alpha}$ that $v(\alpha + \beta) = v(\alpha)$. Hence $q * p = p \in W$, a contradiction.

Hence $\{p_\infty\}$ is the unique such minimal flow. □

Proposition 5.2.2. *Let $K \models ACVF$. Then;*

(1) The type p_∞ is an idempotent of $(S_{\mathbb{G}_a}(M), *)$.

(2) The Ellis Group of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$ is trivial.

(3) $\mathbb{G}_a^{00} = \mathbb{G}_a^0 = \mathbb{G}_a$.

***Proof*.** (1) Let $\alpha \models p_\infty$ and $\beta \models p_\infty|_{M,\alpha}$. Then $p_\infty^2 = tp(\alpha + \beta/M)$. Since $v(\beta) < \Gamma \cap dcl(M, \alpha)$, we see $v(\alpha + \beta) = v(\beta) < \gamma$ for all $\gamma \in \Gamma_K$.

Hence $\alpha + \beta \models p_\infty|_M$ and hence $p_\infty^2 = p_\infty$ as required.

(2) Since $\{p_\infty\}$ is a minimal subflow of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$, and p_∞ is an idempotent of $(S_{\mathbb{G}_a}(M), *)$, we see the Ellis Group is $p_\infty * p_\infty = p_\infty$ by definition. Hence the trivial group $(\{p_\infty\}, *)$ is the Ellis Group of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$

(3) Finally, the above demonstrates that p_∞ is an f-generic type and as such $\mathbb{G}_a^{00} = Stab_{\mathbb{G}_a}(p_\infty) = \mathbb{G}_a$. Hence $\mathbb{G}_a^{00} = \mathbb{G}_a^0 = \mathbb{G}_a$ as required.

□

We now repeat the above for the multiplicative group of K.

Proposition 5.2.3. *Let $K \models ACVF$ and consider the multiplicative group $\mathbb{G}_m$ of K. Then the minimal subflows of $(\mathbb{G}_m, S_{\mathbb{G}_m})$ are precisely the following;*

(1) The type p_∞ is a 1-point minimal subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$.

(2) The type p_0 is a 1-point minimal subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$.

Proof. We first demonstrate that the types p_∞ and p_0 are invariant under left action of $\mathbb{G}_a$.

(1) Let $a \in \mathbb{G}_m$ and let $\alpha \models p_\infty|_{M,a}$. Then $ap_\infty = tp(a\alpha/M)$. Note $v(a\alpha) = v(a) + v(\alpha)$, and so $v(a\alpha) < \gamma$ for all $\gamma \in \Gamma_K$ since $v(\alpha) < \Gamma \cap dcl(M, a) = \Gamma_K$. Hence $ap_\infty = p_\infty$ and so $\{p_\infty\}$ is a 1-point subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$. Clearly this is minimal as $\{p_\infty\}$ cannot contain any proper non-trivial subflows.

(2) Let $a \in \mathbb{G}_m$ and let $\alpha \models p_0|_{M,a}$. Then $ap_0 = tp(a\alpha/M)$. Note $v(a\alpha) = v(a) + v(\alpha)$, and so $v(a\alpha) > \gamma$ for all $\gamma \in \Gamma_K$ since $v(\alpha > \Gamma \cap dcl(M, a) = \Gamma_K$. Hence $ap_0 = p_0$ and so $\{p_0\}$ is a 1-point subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$. Clearly this is minimal as $\{p_0\}$ cannot contain any proper non-trivial subflows.

We now prove the claim that these are the only 2 minimal subflows of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$. Recall for any minimal subflows W_1 and W_2, either $W_1 = W_2$ or $W_1 \cap W_2 = \emptyset$.

Suppose there exists some minimal subflow $W \subseteq S_{\mathbb{G}_m}(M)$. Assume first that W does not contain the type p_∞. Then $p_0 * W \subseteq W$. But clearly for any type q in $S_{\mathbb{G}_m}(M) \setminus \{p_\infty\}$ we have $p * q = p$. Hence $p_0 \in W$. But $\{p_0\}$ is minimal and hence $W = \{p_0\}$ since a minimal subflow cannot contain another minimal subflow.

There is a duality in this argument by switching p_∞ and p_0, and so we see that for any subflow W of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$, W contains either p_0 or p_∞, and so these are the only 2 subflows of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$. □

Proposition 5.2.4. *Let $K \models ACVF$. Then;*

(1) The type p_∞ is an idempotent element of $(S_{\mathbb{G}_m}(M), *)$.

(2) The type p_0 is an idempotent element of $(S_{\mathbb{G}_m}(M), *)$.

(3) The Ellis Group of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$ is trivial.

(4) $\mathbb{G}_m^{00} = \mathbb{G}_m^0 = \mathbb{G}_m$.

Proof. (1) Let $\alpha \models p_\infty$ and $\beta \models p_\infty|_{M,\alpha}$. Then $p_\infty^2 = tp(\alpha\beta/M)$. Since $v(\beta) < \Gamma \cap dcl(M, \alpha)$, we see $v(\alpha\beta) = v(\alpha) + v(\beta) < \gamma$ for all $\gamma \in \Gamma_K$.

Hence $\alpha\beta \models p_\infty|_M$ and hence $p_\infty^2 = p_\infty$ as required.

(2) Let $\alpha \models p_0$ and $\beta \models p_0|_{M,\alpha}$. Then $p_0^2 = tp(\alpha\beta/M)$. Since $v(\beta) < \Gamma \cap dcl(M, \alpha)$, we see $v(\alpha\beta) = v(\alpha) + v(\beta) < \gamma$ for all $\gamma \in \Gamma_K$. Hence $\alpha\beta \models p_0|_M$ and hence $p_0^2 = p_0$ as required.

(3) Since $\{p_0\}$ is a minimal subflow of $(\mathbb{G}_m, S_{\mathbb{G}_m}(M))$, and p_0 is an idempotent of $(S_{\mathbb{G}_m}(M), *)$, we see the Ellis Group is $p_0 * p_0 = p_0$. Hence the trivial group $(\{p_0\}, *)$ is the Ellis Group of $(\mathbb{G}_a, S_{\mathbb{G}_a}(M))$. It is clear to see that choosing the subflow $\{p_\infty\}$ would similarly provide a trivial Ellis Group, as we would expect since all such groups should be isomorphic.

(4) The global extensions of both p_0 and p_∞ are f-generic over M. Hence $\mathbb{G}_m^{00} = Stab_{\mathbb{G}_m}(p_0) = \mathbb{G}_m$. Hence $\mathbb{G}_m^{00} = \mathbb{G}_m^0 = \mathbb{G}_m$.

□

5.3 Borel Subgroup

As in the $\mathbb{C}((t))$ case, we can consider the Borel subgroup as the direct product of the multiplicative and additive groups of K, obtaining a pair $(b,c) \in \mathbb{G}_m \times \mathbb{G}_a$ where the operation is given by matrix multiplication.

Similarly, we once again construct the projection $\pi : B(K) \to K^*$ that sends (b,c) to b. This has kernel isomorphic to K. Given that $(K^{00},+) = (K,+)$ and $(K^{*00},\times) = (K^*,\times)$, we conclude $B^{00}(K) = B(K)$.

We consider then the 2-type $p = tp((b,c)/M^{ext})$ where $v(b) \leq \Gamma_K$ and $v(c) < \Gamma \cap dcl(M,b)$. We claim that p is idempotent, and further that p is left-invariant under action of $B(K)$ (and more generally, of $S_B(M)$) as follows.

Proposition 5.3.1. *The type $p = tp((b,c)/M^{ext})$ where $v(b) \leq \Gamma_K$ and $v(c) < \Gamma \cap dcl(M,b)$ is an idempotent of $(S_B(M),*)$.*

Proof. Let $b \models p_\infty|_{M^{ext}}$ and $c \models p_\infty|_{(M^{ext},b)}$. Then $(b,c) \models p$. Define β and γ similarly such that $(\beta,\gamma) \models p|_{(M^{ext},(b,c))}$.

Then $p^2 = tp((b,c)(\beta,\gamma)/M^{ext}) = tp((b\beta, b\gamma + c\beta^{-1})/M^{ext})$.

We prove that $(b\beta, b\gamma + c\beta^{-1}) \equiv_{M^{ext}} (b,c)$ and hence $(b\beta, b\gamma + c\beta^{-1}) \models p$.

We see $v(b\beta) = v(b) + v(\beta) \approx v(\beta) < \Gamma_K$ as $\beta \models p_\infty|_{(M^{ext},(b,c))}$ and so in particular $\beta \models p_\infty|_{M^{ext}}$. Hence $b\beta \models p_\infty|_{M^{ext}}$.

Next, we consider $b\gamma + c\beta^{-1}$. Clearly, $v(b\gamma + c\beta^{-1}) = min\{v(b\gamma), v(c\beta^{-1})\} = v(b\gamma)$ as $\gamma \models p_\infty|_{(M^{ext},(b,c),\beta)}$. Since $v(\gamma) < \Gamma \cap dcl(M,b)$, $v(\gamma) + v(b) \approx v(\gamma)$ and so we see $b\gamma \models p_\infty|_{(M^{ext},(b,c),\beta)}$. In particular, it is clear that $b\gamma \models p_\infty|_{(M^{ext},b\beta)}$.

Hence $(b\beta, b\gamma + c\beta^{-1}) \models p|_{M^{ext}}$ and so $p^2 = p$ as required. □

Proposition 5.3.2. *The type p as above is a 1-point minimal subflow of $(B(M), S_B(M))$.*

Proof. Let (b,c) represent a matrix in $B(M)$. Let $\beta \models p_\infty|_M$ and $\gamma \models p_\infty|_{(M,\beta)}$. Then $(\beta,\gamma) \models p|_{M,(b,c)}$ and so $tp((b,c)/M) * p = tp((b,c)(\beta,\gamma)/M) = tp((b\beta, b\gamma + c\beta^{-1})/M)$.

We prove that $(b\beta, b\gamma + c\beta^{-1}) \equiv_M (\beta, \gamma)$ and so $(b\beta, b\gamma + c\beta^{-1}) \models p$.

First notice that $v(b\beta) = v(b) + v(\beta) \approx v(\beta) < \Gamma_K$ as $\beta \models p_\infty|_{(M,(b,c))}$. Hence $b\beta \models p_\infty|_M$.

Next, we consider $b\gamma + c\beta^{-1}$. Clearly, $v(b\gamma + c\beta^{-1}) = min\{v(b\gamma), v(c\beta^{-1})\} = v(b\gamma)$ as $\gamma \models p_\infty|_{(M,(b,c),\beta)}$. Since $v(\gamma) < \Gamma \cap dcl(M, b)$, $v(\gamma) + v(b) \approx v(\gamma)$ and so we see $b\gamma + c\beta{-}1 \models p_\infty|_{(M,(b,c),\beta)}$. In particular, it is clear that $b\gamma \models p_\infty|_{(M^{ext},b\beta)}$.

Hence $(b\beta, b\gamma + c\beta^{-1}) \models p$ and so $(b, c) * p = p$ and so $\{p\}$ is a 1-point minimal subflow of $(B(M), S_B(M))$ as required. □

Proposition 5.3.3. *Let $K \models ACVF$ with $G = SL_2$ a definable group. Let $B \subset G$ be the Borel Subgroup of upper triangular matrices. Let $p \in S_B(M)$ be an invariant type as above. Then;*

- *$B(K)$ is definably extremely amenable and hence the Ellis Group of $(B(K), S_B(M))$ is trivial.*
- *$B^{00}(K) = B^0(K) = B(K)$ and $B/B^{00} \cong (\{p\}, *)$*

Proof.
- Since $(S_B(M), *)$ has a left-invariant type, $B(K)$ is definably (extremely) amenable by definition. It follows that the Ellis Group of $(B(K), S_B(M))$ is trivial as the minimal flow of $(B(K), S_B(M))$ is a singleton.
- Since B is definably (extremely) amenable, we know that the Ellis Group of $(B(K), S_B(M))$ is isomorphic to B/B^{00}. Hence $B/B^{00} \cong (\{p\}, *)$ and hence $B = B^0 = B^{00}$.

□

This is not the only minimal subflow of $(B, S_B(M))$. The type realised by the pair (β, γ) where $\beta \models p_0|_M$ and $\gamma \models p_\infty|_{M,\beta}$ is another 1-point minimal flow of $(B, S_B(M))$.

We take the opportunity here to consider the minimal flows and idempotents of $SL_2(\mathcal{O}_K) \cap B(K) = B(\mathcal{O}_K)$. We can use Fact 4.2.12 and immediately see that

$B(\mathcal{O}_K)$ is stably dominated, and is in fact maximally stably dominated in $B(K)$. As a subgroup of $SL_2(\mathcal{O}_K)$, we expect that $B(\mathcal{O}_K)$ admits an invariant type satisfying the conditions in Fact 3.4.3. We expect a type in $S_{B(\mathcal{O}_K)}(M)$ invariant under action of $B(\mathcal{O}_K)(M)$. We give an explicit description of this type below and note that this forms a 1-point minimal subflow of $B(\mathcal{O}_K)$.

Proposition 5.3.4. *The type $p_{\mathcal{O}} \in S_{B(\mathcal{O})}(M)$ realised by (β, γ) - with $\beta \models p_{0,0,trans}$ and $\gamma \models p_{0,0,trans}|_{M,\beta}$ - is an idempotent.*

Proof. Let $(b, c) \models p_{\mathcal{O}}$ and let $(\beta, \gamma) \models p_{\mathcal{O}}|_{M,(b,c)}$. Then;

$$\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} b\beta & b\gamma + c\beta^{-1} \\ 0 & b^{-1}\beta^{-1} \end{pmatrix}$$

Then $b\beta$ clearly has valuation $v(b) + v(\beta) = 0$, and since $\beta \models p_{0,0,trans}|_{M,(b,c)}$, we have that $f(res(b\beta)) \neq 0$ for all $f \in res(K(b, c))$, and so in particular $f(res(b\beta)) \neq 0$ for all $f \in res(K)$.

Hence $b\beta \models p_{0,0,trans}$.

We now consider $b\gamma + c\beta^{-1}$. First, we observe that b, γ, c and β^{-1} all have valuation 0, and so $v(b\gamma + c\beta^{-1}) \geq 0$. However, since $\gamma \models p_{0,0,trans}|_{M,(b,c),\beta}$, clearly $b\gamma$ and $c\beta^{-1}$ are linearly independent.

Hence $v(b\gamma + c\beta^{-1}) = 0$.

Further, since $\gamma \models p_{0,0,trans}|_{M,(b,c),\beta}$, we see that $f(res(b\gamma + c\beta^{-1})) \neq 0$ for all $f \in res(K(b, c, \beta))$, and so in particular for all $f \in res(K(b, \beta)) \supset res(K(b\beta))$.

Hence $b\gamma + c\beta^{-1} \models p_{0,0,trans}|_{M,b\beta}$ and thus $(b\beta, b\gamma + c\beta^{-1}) \models p_{0,0,trans}$ as required. □

Proposition 5.3.5. *The type $p_{\mathcal{O}} \in S_{B(\mathcal{O})}(M)$ realised by (β, γ) - with $\beta \models p_{0,0,trans}$ and $\gamma \models p_{0,0,trans}|_{M,\beta}$ - is a 1-point minimal subflow of $B(\mathcal{O}), S_{B(\mathcal{O})}(M)$.*

Proof. Let $(b, c) \models r \in S_{B(\mathcal{O})}(M)$ and let $(\beta, \gamma) \models p_{\mathcal{O}}|_{M,(b,c)}$. Then;

$$\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} b\beta & b\gamma + c\beta^{-1} \\ 0 & b^{-1}\beta^{-1} \end{pmatrix}$$

We prove that $(b\beta, b\gamma + c\beta^{-1})$ realises $p_{0,0,trans}$.

We first observe that $v(b) = v(b^{-1}) = 0$ regardless of type r. Hence $v(b\beta) = v(b) + v(\beta) = 0$. The fact that this is transcendental over $res(K)$ follows easily since we chose $\beta \models p_{0,0,trans}|_{M,(b,c)}$.

We now prove that $b\gamma + c\beta^{-1}$ realises $p_{0,0,trans}|_{M,b\beta}$. By the same argument above, and that $b\gamma$ is linearly independent from $c\beta^{-1}$, we see $v(b\gamma + c\beta^{-1}) = 0$.

Proving that $res(b\gamma + c\beta^{-1})$ is transcendental over $res(K(b\beta))$ follows again from the fact that $\gamma \models p_{0,0,trans}|_{M,(b,c),\beta}$, we see that $f(res(b\gamma + c\beta^{-1})) \neq 0$ for all $f \in res(K(b, c, \beta))$, and so in particular for all $f \in res(K(b, \beta)) \supset res(K(b\beta))$. □

Proposition 5.3.6. *Let $K \models ACVF$ with $G = SL_2$ a definable group. Let $B \subset G$ be the Borel Subgroup of upper triangular matrices. Let $p_{\mathcal{O}} \in S_{B(\mathcal{O}_K)}(M)$ be an invariant type as above. Then;*

- *$B(\mathcal{O}_K)$ is definably extremely amenable, stably dominated in $SL_2(\mathcal{O}_K)$ and maximally stably dominated in $B(K)$.*
- *The Ellis Group of $(B(\mathcal{O}_K), S_{B(\mathcal{O}_K)}(M))$ is trivial.*
- *$B^{00}(\mathcal{O}_K) = B^0(\mathcal{O}_K) = B(\mathcal{O}_K)$ and $B(\mathcal{O}_K)/B(\mathcal{O}_K)^{00} \cong (\{p_{\mathcal{O}}\}, *)$*

Proof.

The fact that $B(\mathcal{O}_K)$ is stably dominated, and maximally so in $B(\mathbb{K})$, follows from Fact 4.2.12. That $B(\mathcal{O}_K)$ is definably extremely amenable follows from Proposition 5.3.5; a 1-point minimal subflow is sufficient to demonstrate definable extreme amenability.

Since $(B(\mathcal{O}_K), S_{B(\mathcal{O}_K)}(M))$ admits a 1-point minimal subflow, the Ellis Group is trivial.

Since $B(\mathcal{O}_K)$ is definably amenable,$B^{00}(\mathcal{O}_K) = Stab_{B(\mathcal{O}_K)}(p_{\mathcal{O}}) = B(\mathcal{O}_K)$. Hence $B^{00}(\mathcal{O}_K) = B^0(\mathcal{O}_K) = B(\mathcal{O}_K)$ and $B(\mathcal{O}_K)/B(\mathcal{O}_K)^{00} \cong (\{p_{\mathcal{O}}\}, *)$ is trivial. □

5.4 $SL_2(\mathcal{O})$

We now calculate the minimal flow of the maximally stably dominated part of the decomposition of $SL_2(K)$.

Let q be a type in $S_{G(\mathcal{O})}(M^{ext})$ realised by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $ad - bc = 1$ and each of a, b, c and d each realise the type $p_{0,0}$ of kind (f) in Proposition 5.1.5 in such a way that $c \vDash p_{0,0}|_{M^{ext},a}$ and $b \vDash p_{0,0}|_{M^{ext},a,c}$.

Proposition 5.4.1. *The type q is a 1-point minimal subflow of $(G(\mathcal{O}), S_{G(\mathcal{O})}(M))$.*

Proof. Let $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ realise some $r \in G(\mathcal{O})$ and let $y = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ realise $q|_{M^{ext},x}$.

Then $r * q = tp(xy/M^{ext})$ and further;

$$\begin{aligned} xy &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} \end{aligned}$$

We first show that each entry of xy has valuation 0. We note that the realisations of r here may have entries with valuations greater than 0.

However, as $ad - bc = 1$, we have $v(ad - bc) = 0 \geq min\{v(ad), v(-bc)\}$. If $v(ad) < v(-bc)$, then $v(a) = v(d) = 0$. Likewise if $v(-bc) < v(ad)$.

If $v(ad) = v(-bc)$, then $v(a) + v(d) = v(-b) + v(c) \leq 0$. Clearly all a, b, c and d have valuation 0 in this case since we consider only $\mathcal{O}$-points.

Hence for each entry of xy, at least one of the terms has valuation 0. Since addition takes the minimum value, every entry of xy has valuation 0.

It remains to show that the translation does not affect the transcendental properties of the entries. Let $\alpha' = a\alpha + b\gamma$ and $\gamma' = c\alpha + d\gamma$.

Clearly, as both α and γ are transcendental over K, we have that α' is transcendental over K and hence $\alpha' \vDash p_{0,0}$.

Showing that $\gamma' \vDash p_{0,0}|_{M^{ext},\alpha'}$ requires some additional steps. We also need to be careful of when x contains 0-entries, and so we split case-by-case.

Case 1: Let $b = 0$. Then $a, d \neq 0$ and we have $\alpha' = a\alpha$, $\gamma' = c\alpha + d\gamma$. Then clearly as $\gamma \vDash p_{0,0}|_{M^{ext},\alpha}$, $\gamma' \vDash p_{0,0}|_{M^{ext},\alpha'}$.

We note c could be 0 also, but this does not affect the above argument.

Case 2: Let $b \neq 0$. Suppose for contradiction that $K(\alpha') = K(\gamma')$.

Then there exists some $r, s \in K$ such that $r + s\alpha' = \gamma'$. Clearly any solution to this would require $r = 0$, as both α and γ are transcendental over K.

Hence we claim $s(a\alpha + b\gamma) = c\alpha + d\gamma$.

Comparing coeffients, we obtain $sa = c$ and $sb = d$. We seek to show that this system has no solutions for $s \in K$ for these a, b, c, d.

Since $b \neq 0$, we write $s = \frac{d}{b}$, and substituting into $sa = c$ we obtain $\frac{da}{b} = c$.

But this gives $da = bc$, which is a contradiction since $det(x) = 1$.

Hence $trdeg(K(\alpha', \gamma')) \geq 2$. Since there are only 2 elements in this extension, we have $trdeg(K(\alpha', \gamma')) = 2$, and further $\gamma' \vDash p_{0,0}|_{M^{ext},\alpha'}$.

We finally remark that even in the case where $a = c = 0$ this argument follows through. As γ is transcendental over $K(\alpha)$ and α is transcendental over K, there is a duality in the sense that α is transcendental over $K(\gamma)$ and γ is transcendental over K.

Hence $\gamma' \vDash p_{0,0}|_{M^{ext},\alpha'}$ as required.

The final step is observing that $\beta' = a\beta + b\delta$ realises $p_{0,0}|_{M^{ext},\alpha',\gamma'}$. This is clear since β is transcendental over both α and γ, and hence transcendental over α' and γ' as necessary. Hence q is left-invariant under $G(\mathcal{O})$ and is a minimal subflow of

$(G(\mathcal{O}), S_G(\mathcal{O}))$. □

Corollary 5.4.2. *The type $q \in S_{SL_2(\mathcal{O})}(M)$ as above is an idempotent element of $(SL_2(\mathcal{O}), *)$ and hence $(\{q\}, *)$ is the Ellis Group of $(SL_2(\mathcal{O}), S_{SL_2(\mathcal{O})}(M))$.*

The result that $SL_2(\mathcal{O}_K)$ admits a 1-point minimal subflow and is definably extremely amenable follows quickly from 3.4.3, though we wished to give an explicit description of the subflow as we will require it to compute the minimal flow of $SL_2(K)$.

Corollary 5.4.3. *The type $q \in S_{SL_2(\mathcal{O})}(M)$ as above is 2-sided, in the sense that for any $r \in S_{SL_2(\mathcal{O})}(M)$, $r * q = q * r = q$.*

Proof. This can be seen directly using a similar proof to Proposition 5.4.1, but can also be seen as follows.

Since $SL_2(\mathcal{O})$ is maximally stably dominated, then a type in $S_{SL_2(\mathcal{O})}(M)$ is generic if and only if its restriction to $SL_2(k)$, where k is algebraically closed, is a generic in $S_{SL_2(k)}(M)$; by Fact 4.2.13 and Fact 4.2.15. Since $SL_2(k)$ admits a unique generic p whose entries are transcendental over k and algebraically independent from one another, the only generic types in $S_{SL_2(\mathcal{O})}(M)$ are those who restrict to p via the entry-wise residue map on matrices in $SL_2(\mathcal{O})$. Hence q is the unique generic of $S_{SL_2(\mathcal{O})}(M)$.

Further $SL_2(\mathcal{O})$ is definably extremely amenable and hence admits fsg, and so left and right generics coincide by Fact 1.5.2. Hence the global heir of q is unique and has fsg, and so it is invariant under right-translation by $G(\bar{M})$. Extending this action to the $*$ action on types, we see $q * r = q$. □

5.5 The Minimal Subflow of $SL_2(K)$

What we have demonstrated now is that $SL_2(K)$ admits a group decomposition with a maximally stably dominated / definably extremely amenable group decomposition, with possibly infinite $B(\mathcal{O})$. We demonstrated in Proposition 4.3.2 that for groups with a similar decomposition we can explicitly describe their minimal flow and Ellis Group.

$SL_2(K)$ admits a similar group decomposition in the sense that $SL_2(\mathcal{O})$ admits a 2-sided 1-point minimal flow. $B(K)$ is definably extremely amenable, however it is not a 2-sided minimal flow. Indeed there is a second generic type p' in $B(K)$ which is also a 1-point minimal flow, but for which $p * p' = p'$ and $p' * p = p$. That is; the $S_G(M)$-orbits of p and p' are disjoint.

We claim that the minimal flow of $(G(M), S_G(M))$ - where $G = SL_2$ and K is some maximally complete algebraically closed valued field with value group isomorphic to $(\mathbb{R}, +)$ - can be obtained as in Proposition 4.3.2. We now demonstrate this explicitly.

Proposition 5.5.1. *Let q be the 2-sided 1-point minimal flow of $SL_2(\mathcal{O})$ as in Proposition 5.4.1. Let p be the 1-point minimal flow of $B(K)$ as in 5.3.1. Then $q*p$ is an idempotent element of $(S_G(M), *)$.*

Proof. Let $h_0 \vDash q|_M$, $t_0 \vDash p|_{M,h_0}$, $h_1 \vDash q|_{M,h_0,t_0}$ and $t_1 \vDash p|_{M,h_0,t_0,h_1}$. Then $(q * p)^2 = tp(h_0 t_0 h_1 t_1/M)$. We show that $h_0 t_0 h_1 t_1 \vDash q * p$.

We write $t_0 h_1$ as $h't'$ for some $h', t' \in dcl(M, t_0, h_1, h_0)$. We note that the choice of h', t' is not unique here since $B(K) \cap SL_2(\mathcal{O})$ is infinite. However, this should not be an issue.

Since h' realises the heir of some $r \in S_{SL_2(\mathcal{O})}(M)$ over (M, h_0), we can see from a duality argument that h_0 realises the coheir of q over M, h'. Hence $h_0 h' \vDash q|_M$ since q is 2-sided by Corollary 5.4.3.

Further, since $t_1 \vDash p|_{M,h_0,t_0,h_1}$, and $t' \in dcl(M, h_0, t_0, h_1)$, we see that $t' t_1 \vDash p|_{M,h_0,t_0,h_1}$.

Hence $(q * p)^2 = tp(h_0 t_0 h_1 t_1/M) = tp(h_0 h' t' t_1/M) = q * p$ and hence $q * p$ is idempotent as required. □

Proposition 5.5.2. *Let $q * p$ be the idempotent element of $(S_G(M), *)$ as in Proposition 5.5.1. Then $cl(G(M) \cdot q * p)$ is a minimal subflow of $(G(M), S_G(M))$.*

Proof. From Fact 1.5.1, we see that $cl(G(M) \cdot q * p) = S_G(M) * q * p$. This is a subflow by construction but not necessarily minimal. To demonstrate minimality, we show that for any type r in $cl(G(M) \cdot q * p)$. The orbit-closure of r is precisely $cl(G(M) \cdot q * p)$.

This follows similary to the proof of Proposition 2.4.9. Since $cl(G(M) \cdot q * p)$ is, by construction, the orbit-closure of $q * p$, it suffices to prove that $q * p$ is in the orbit-closure of any $r \in cl(G(M) \cdot q * p)$.

Let $r = s * q * p$, and consider that any element in the closure of the $G(M)$-orbit of r is an element of the set $S_G(M) * s * q * p$. We show that $q * p \in S_G(M) * s * q * p$, which is to show there exists some $s' \in S_G(M)$ such that $s' * r = s' * s * q * p = q * p$.

Choose $s' = q * p'$ for any $p' \in S_B(M)$. Then since q is two-sided, and p is left-invariant under the $*$-operation, it is clear that $q * p' * s * q * p = q * p$ using a similar proof to that of Corollary 5.4.3. Hence $cl(G(M) \cdot q * p)$ is minimal. □

Corollary 5.5.3. *The Ellis Group of $(G(M), S_G(M))$ is precisiely the trivial group $(\{q * p\}, *)$. This is isomorphic to G/G^{00}.*

Proof. Since $cl(G(M) \cdot q * p)$ is minimal, and $q * p$ is an idempotent element in $cl(G(M) \cdot q * p)$, we can compute the Ellis Group of $(G(M), S_G(M))$ by constructing the set $q * p * S_G(M) * q * p$.

Again, since q is a 2-sided 1-point minimal flow, and p is left-invariant, we can use identical arguments as in Propositions 5.5.2 and 5.5.1 and see $q*p*r*q*p = q*p$ for any $r \in S_G(M)$.

Hence $(\{q*p\}, *)$ is the Ellis Group of $(G(M), S_G(M))$. To see this is isomorphic to G/G^{00} is easy, since $G^{00} = G$ for $G = SL_2$. □

This is one of the only examples we are aware of where G/G^{00} and the Ellis Group of $(G(M), S_G(M))$ are isomorphic despite G not being definably amenable. However, I do not believe this result generalises very far. Insisting on a decomposition where one of the subgroups has a 2-sided 1-point minimal flow is an exceptionally strong property, and in general will not be the case for an arbitrary affine algebraic group over $K \models ACVF$. It does however provide insight into how we may generalise using slightly weaker group decompositions.

5.6 Generalisations and Future Work

We first note that the work above is not specific to the case where $G = SL_2$, and that the proof should follow identically under similar assumptions. We remark that generalising in this way does require a lot of assumptions, though we suggest afterwards how to relax many of these assumptions to apply to a much more general setting.

Theorem 5.6.1. *Let G be a non-definably amenable group, definable in a metastable structure M for which $M = M^{ext}$. Suppose G is expressible as the product of groups H and J such that;*

- *H is maximally stably dominated.*
- *J is definably extremely amenable and $(J(M), S_J(M))$ contains a 2-sided 1-point minimal subflow $(J(M), p)$.*

Then the minimal subflow of $(G(M), S_G(M))$ is a subflow of $cl(G(M) \cdot p * q)$, where q is the principal generic in the set $Gen_H(M)$ of stably dominated generic types in $S_H(M)$.

Hence the Ellis Group is a subgroup of $(p * Gen_H(M))$.

Proof. The proof of this statement follows by generalising Proposition 5.5.2, Proposition 5.5.1 and Corollary 5.5.3, and further observing that H maximally stably dominated ensures $Gen_H(M)$ is non-empty by definition. □

A special case of this, an example of which is demonstrated by the construction of the Ellis Group of $(SL_2(K), S_{SL_2}(M))$, is as follows.

Theorem 5.6.2. *Let G be an M-definable group as in Theorem 5.6.1. Suppose further that G is an affine algebraic group over M and let $q \in S_H(\bar{M})$ be the unique global stably dominated generic type of H.*

Then the minimal subflow of $(G(M), S_G(M))$ is precisely $cl(G(M) \cdot p * q)$, and the Ellis Group of $(G(M), S_G(M))$ is $(\{p * q\}, *)$ and is trivial.

Proof. This is a special case of Theorem 5.6.1 which makes use of the fact that both subgroups in the decomposition admit 1-point minimal flows. □

We remark that the above example of $G(M) = SL_2(K)$, where K is maximally complete with value group isomorphic to $(\mathbb{R}, +)$, is an example of a group for which Theorem 5.6.2 applies. In this example we also note that $G = G^{00}$ and hence there is an isomorphism between the Ellis Group of $(G(M), S_G(M))$ and G/G^{00}. This is especially interesting as $S_{SL_2}(K)$ contains no global generics, and as such $G = SL_2$ is not definably amenable. We expect that this is more a coincidence rather than evidence towards a non-definably amenable setting for which the Ellis Group conjecture of Newelski holds.

Comparing this example to the work of Chapters 2 and 3 where $G(M) = SL_2(\mathbb{C}((t))$, we remark some similarities in that the interpretation of G over the valuation ring both admit a unique invariant type. In that sense, both Ellis Groups can be described via the non-stably dominated part of the group decomposition. That is, where $G = SL_2$ and M is either $\mathbb{C}((t))$ or $K \models ACVF$ (maximally complete with additive real value group), the Ellis Group of $(G(M), S_G(M))$ is precisely descibed by B/B^0. This is also the case in **[22]** where $M = \mathbb{Q}_p$.

In the metastable setting, we believe that this is likely a consequence of Fact 3.4.3, in that SL_2 here is affine algebraic over K and admits maximum modulus on regular functions, and hence has a unique global stably dominated generic type.

In settings where this is not the case, we suspect the Ellis Group will have some dependance on the stably dominated subgroup in the decomposition.

In future work, we would look towards providing a description for the Ellis Groups of the $G(M)$-flow of metastable definable groups which admit decomposition containing a maximally stably dominated subgroup. We would like this decomposition to hold for any affine algebraic G over some K metastable field, with no dependency on the existence of a unique global generic type in the stably dominated subgroup.

Further, we would like to generalise the result towards the Ellis Groups of $G(M^{ext})$-flows. This would allow a more complete description that does not restrict to metastable structures with additive real value group. We would also like to remove the condition that K be maximally complete, though the space of types here becomes more complex as we have immediate extensions of K which do not extend the residue field or value group. Ideally, we would like to find a general description of the Ellis Group for a large class of metastable definable groups which holds over expansions of any metastable M^{ext}. This would be a metastable analogue of the work of Yao [**33**] that described the Ellis Group for a large class of groups definable in arbitrary expansions of models of RCF.

We demonstrated in Chapter 4 that the Ellis Group of stably dominated groups can be described via their reduction to a stable group. We would also like to investigate under what conditions the converse is true. That is to ask under what conditions we could recover the Ellis Group of a stably dominated group by instead considering its reduction to a stable group. Such a result would be useful in describing the Ellis Groups of more complex stably dominated groups as the definable topological dynamics for stable groups is well understood.

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