

Relations and radicals in abstract lattices and in lattices of subspaces of Banach spaces and of ideals of Banach algebras. Amitsur's theory revisited.

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Abstract

We refine Amitsur's theory of radicals in complete lattices and apply the obtained results to the theory of radicals in the lattices of subspaces of Banach spaces and in the lattices of ideals of Banach and C^* -algebras and of Banach Lie algebras.

1 Introduction

In his research of the radical theory of algebras and rings, Amitsur [A-II, A-III] discovered that a significant part of the results of this theory can be formulated and proved in terms of the general theory of lattices. In [A-I] he developed the theory of radicals for relations in lattices that was used in various areas of algebra: group theory, non-associative rings, Lie algebras, universal algebras, etc.

Later Dixon [Di1] initiated the radical approach to some problems of functional analysis and laid the basis of the theory of topological radicals of Banach algebras. This theory was further developed and applied to the theory of invariant subspaces of operator algebras and to classification of Banach and operator Lie algebras in [ST], [KST3], [KST5].

In this paper we investigate both aspects of the theory of radicals. In part one: Sections 2 to 7 we revise and refine Amitsur's theory of radicals in complete abstract lattices. In part two: Sections 8 and 9 we apply the obtained results to the theory of radicals in the lattices of subspaces of Banach spaces and in the lattices of ideals of Banach and C^* -algebras and of Banach Lie algebras.

Recall that a partially ordered set (Q, \leq) with a reflexive, anti-symmetric, transitive relation \leq is a lattice if all $a, b \in Q$ have a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. It is *complete* if each $G \subseteq Q$ has a least upper bound $\vee G$ and a greatest lower bound $\wedge G$. Set $\mathbf{0} = \wedge Q$ and $\mathbf{1} = \vee Q$. A transitive relation in Q is an *order*; a relation \prec_1 in Q is *stronger* than a relation \prec if

$$a \prec_1 b \text{ implies } a \prec b \text{ for } a, b \in Q \text{ (we write } \prec_1 \subseteq \prec \text{)}. \quad (1.1)$$

We will only consider reflexive relations \ll in (Q, \leq) stronger than \leq and denote by $\text{Ref}(Q)$ the family of all such relations. For $x \leq y$ in Q , set $[x, y] = \{z \in Q: x \leq z \leq y\}$.

Amitsur [A-I] studied \mathbf{H} - and dual \mathbf{H} -relations \ll in complete lattices Q and, using a special procedure, constructed the \mathbf{R} -order \ll^{\triangleright} from an \mathbf{H} -relation and the dual \mathbf{R} -order \ll^{\triangleleft} from a dual \mathbf{H} -relation (Definition 3.2). He proved that \ll^{\triangleright} has a unique \ll^{\triangleright} -radical $\mathfrak{r}_{[a,b]}$ in each interval $[a, b]$ in Q and \ll^{\triangleleft} has a unique dual \ll^{\triangleleft} -radical $\mathfrak{p}_{[a,b]}$.

In our paper (Theorems 2.4 and 2.5) we establish that even weaker relations (**T**- and dual **T**-orders) have unique radicals in each $[a, b]$. Moreover, for a **T**-order, the map $\mathfrak{r}: a \in Q \rightarrow \mathfrak{r}_{[a,1]}$ is pre-radical map; for a dual **T**-order, the map $\mathfrak{p}: a \in Q \rightarrow \mathfrak{p}_{[0,a]}$ is a dual pre-radical maps ((2.11) and (2.12)) on Q . For **R**-orders and dual **R**-orders they are, respectively, radical and dual radical maps ((2.13) and (2.14)) on Q (Theorems 3.5 and 3.6). If a relation \ll is both a **T**- and a dual **T**-order, it coincides "locally" with \leq , i.e., Q is a union of mutually disjoint intervals $[a_\lambda, b_\lambda]$, $\lambda \in \Lambda$, such that $x \ll y$ if and only if $x, y \in [a_\lambda, b_\lambda]$ for some $\lambda \in \Lambda$, and $x \leq y$ (Theorem 2.12).

In Section 4 we investigate the relations \ll^{\triangleright} and \ll^{\triangleleft} constructed from arbitrary relations \ll . We compare them to other naturally constructed relations \ll^{lo} and \ll^{up} (Definition 4.2) and prove that each interval $[a, b]$ in Q has \ll^{\triangleright} - and dual \ll^{\triangleleft} -radicals which are not, however, unique.

For an **H**-relation \ll , we show in Section 5 that \ll^{\triangleright} is an **R**-order, it coincides with \ll^{up} and each $[a, b]$ has a unique \ll^{\triangleright} -radical. Similarly, for a dual **H**-relation \ll , \ll^{\triangleleft} is a dual **R**-order coinciding with \ll^{lo} and each $[a, b]$ has a unique dual \ll^{\triangleleft} -radical. We prove that \ll is an **R**-order (a dual **R**-order) if and only if $\ll = \ll^{\triangleright}$ ($\ll = \ll^{\triangleleft}$).

In Section 6 we investigate two incompatible relations: the gap ($<_{\mathfrak{g}}$) and continuous ($<_{\mathfrak{c}}$) relations in lattices Q . We show that if Q is modular then $<_{\mathfrak{g}}$ is an **H**- and a dual **H**-relation. If Q is modular and has (JID) and (MID) (see (6.5)) then $<_{\mathfrak{c}}$ is an **R**- and a dual **R**-order and Q is a union of disjoint sets without gaps.

Section 7 is devoted to the study of enveloping and inscribing sets in lattices Q . We prove that there is a bijection between all enveloping sets and all radical maps in Q , and between all inscribing sets and all dual radical maps in Q . We consider some enveloping sets in the lattice $\text{Ref}(Q)$. In particular, we show that the sets of all **T**-orders, of all dual **T**-orders, of all **R**-orders, of all dual **R**-orders are enveloping in $\text{Ref}(Q)$. We also study transfinite extensions of relations and, in particular, of the relations $<_{\mathfrak{g}}$ and $<_{\mathfrak{c}}$.

An important place in the theory of operator algebras is occupied by the study of lattices of subspaces of Banach spaces X and, especially, of the lattices $\text{Lat } \mathcal{A}$ of invariant subspaces of operator algebras \mathcal{A} on X . In Section 8 the above results are used to study such lattices.

Denote by $\text{Ln}(X)$ the lattice of all *linear* subspaces and by $\text{Cl}(X)$ the lattice of all *closed* subspaces of X . In Theorem 8.1 we describe ascending and descending \ll -series of subspaces in sublattices of $\text{Ln}(X)$ and $\text{Cl}(X)$ with respect to **H**- and dual **H**-relations \ll , and their \ll^{\triangleright} - and \ll^{\triangleleft} -radicals \mathfrak{r} and \mathfrak{p} , respectively. In the sublattices $\text{Lat } \mathcal{A}$ of $\text{Cl}(X)$ these radicals are superinvariant (Proposition 8.13), i.e., they are invariant for the operator Lie algebra

$$\text{Nor } \mathcal{A} = \{S \in B(X) : SA - AS \in \mathcal{A} \text{ for } A \in \mathcal{A}\}.$$

We concentrate on the study of the gap relation $<_{\mathfrak{g}}$ and the relations $\{\ll_n\}_{1 \leq n \leq \infty}$ defined in $\text{Ln}(X)$ and $\text{Cl}(X)$ by the condition $L \ll_n M$ if $\dim(M/L) < n$. As $\text{Ln}(X)$ is a modular lattice, $<_{\mathfrak{g}}$ and all \ll_n are both **H**- and dual **H**-relations (**HH**-relations) in it (Proposition 8.3 and Corollary 8.5).

The lattice $\text{Cl}(X)$ is not modular and, although all \ll_n are still **HH**-relations in it, there are sublattices where $<_{\mathfrak{g}}$ is neither **H**-, nor dual **H**-relation (Corollary 8.21). The main obstacle is the fact that the sum of subspaces from $\text{Cl}(X)$ is not necessarily closed. We construct in $\text{Cl}(X)$ an **H**-relation $\sqsubset_{\mathfrak{g}}$ and a dual **H**-relation $\prec_{\mathfrak{g}}$ stronger than $<_{\mathfrak{g}}$ that give us the $\sqsubset_{\mathfrak{g}}^{\triangleright}$ -radical and the dual $\prec_{\mathfrak{g}}^{\triangleleft}$ -radical in each sublattice of $\text{Cl}(X)$.

For a Hilbert space X , we define in (8.27) another class of the **HH**-relations \ll_n^{\perp} , $0 \leq n \leq \infty$, in $\text{Cl}(X)$ (Theorem 8.26). Moreover, each **HH**-relation in $\text{Cl}(X)$ is either \ll_n , or \ll_n^{\perp} for some n (see [K3] and Theorem 8.27). Each commutative subspace lattice (CSL) Q in $\text{Cl}(X)$ has properties

(JID) and (MID) (see (7.15)). So it is modular, $\langle_{\mathfrak{g}} = \sqsubset_{\mathfrak{g}} = \prec_{\mathfrak{g}}$ is an **HH**-relation in Q , and Q is a union of disjoint intervals each of which has no gaps (Theorem 8.23).

The study of ideals of Banach and Banach Lie algebras constitutes one of the main tools of research. In [KST3] the authors used the structure of chains of ideals in Banach Lie algebras to develop the radical theory of these algebras. In this paper we study chains of ideals generated by **H**- and dual **H**-relations in the lattices Id_A of all closed ideals of Banach algebras A .

As \ll_{∞} is **HH**-order in Id_A , each sublattice of Id_A has the unique $\ll_{\infty}^{\triangleright}$ -radical and the dual $\ll_{\infty}^{\triangleleft}$ -radical. Denoting by Σ_A the set of all subalgebras of finite codimension in a Banach algebra A , we show in Proposition 9.6 that $\bigcap_{S \in \Sigma_A} S = \bigcap \{I \in \text{Id}_A : \dim(A/I) < \infty\}$.

The relation $\langle_{\mathfrak{g}}$ in Id_A is not always an **H**-, or a dual **H**-relation, since the sum of closed ideals of A is not necessarily closed. If, however, each ideal of A has a bounded left or right approximate identity, then $I + J$ is closed for all $I, J \in \text{Id}_A$, so that $\langle_{\mathfrak{g}}$ is an **HH**-relation in Id_A (Corollary 9.2).

Let $LR(A)$ be the operator algebra generated by all operators of left and right multiplication by elements from A . Then $\text{Id}_A = \text{Lat } LR(A)$ is a sublattice of $\text{Cl}(A)$, the Lie algebra $\text{Nor } LR(A)$ contains all derivations of A and operators from $\text{Nor } LR(A)$ map I^2 in I , $I \in \text{Id}_A$ (Theorem 9.5).

If $\bigcap_{S \in \Sigma_A} S = \{0\}$, for a Banach Lie algebra A , then (Theorem 9.16) the dual $\ll_{\infty}^{\triangleleft}$ -radical \mathfrak{p} in the lattice of all characteristic Lie ideals of A (invariant for all derivations of A) is $\{0\}$. So there is a descending series $(I_{\lambda})_{1 \leq \lambda \leq \gamma}$ of characteristic Lie ideals of A such that $\dim(I_{\lambda}/I_{\lambda+1}) < \infty$ for $\lambda \neq \gamma$, $I_1 = A$ and $\mathfrak{p} = I_{\gamma} = \{0\}$.

Section 9.2 is devoted to the study of **H**- and dual **H**-relations in the lattices Id_A of all ideals of C^* -algebras A . In this case Id_A is a modular lattice and $\langle_{\mathfrak{g}}$ is an **HH**-relation in Id_A . The lattice Id_A has many **H**-relations that can be obtained in the following way. Let \mathfrak{A} be the set of all C^* -algebras. A subclass P of \mathfrak{A} is a *property*, if $\{0\} \in P$ and $A \in P$ implies $B \in P$ for all $B \approx A$. Each property P generates the relation \ll_P in Id_A by $I \ll_P J$ if $I \subseteq J$ in Id_A and $J/I \in P$.

A property P is lower stable if $A \in P$ implies $\text{Id}_A \subset P$; P is upper stable if $A \in P$ implies that the quotients $A/I \in P$ for all $I \in \text{Id}_A$. In Theorem 9.7 we prove that P is upper (lower) stable if and only if \ll_P is an **H**-relation (a dual **H**-relation). Some characteristics of the \ll_P^{\triangleright} -radicals \mathfrak{r}_P and the dual \ll_P^{\triangleleft} -radicals \mathfrak{p}_P in Id_A are discussed in Theorem 9.8. These radicals are invariant for all automorphisms of A (Corollary 9.9). Since many properties in \mathfrak{A} are upper, or lower stable, we have a large variety of **H**- and dual **H**-relations in Id_A . These relations were investigated in [KST4]; in this paper we briefly consider some of them.

For example, the classes CCR and GCR of all CCR - and GCR -algebras are lower and upper stable properties, while the class of all $NGCR$ -algebras is a lower, but not upper stable property (see [D]). So \ll_{CCR} and \ll_{GCR} are **HH**-relations. This gives a well-known result that the radical \mathfrak{r}_{CCR} of a C^* -algebra A is the largest GCR -ideal and A/\mathfrak{r}_{CCR} has no CCR -ideals. Moreover, if $\mathfrak{r}_{CCR} \not\subseteq I \neq A$ then J/I is a CCR -algebra for some $I \subsetneq J \in \text{Id}_A$.

We also consider the classes RZ of all real rank zero, AF of all approximately finite-dimensional and NU of all nuclear C^* -algebras. They are lower and upper stable properties, so that \ll_{RZ} , \ll_{AF} , \ll_{NU} are **HH**-relations in Id_A for all C^* -algebras A (Corollary 9.12). While the relations \ll_{CCR} and \ll_{RZ} are not transitive, the relations \ll_{AF} and \ll_{NU} are transitive. Moreover, they are **R**-orders in all Id_A . So the corresponding radicals are, respectively, the largest AF -algebra and the largest nuclear algebra in A (Corollary 9.14) (this result was initially obtained in [ST]).

2 Radicals and T-orders in complete lattices

Let (Q, \leq) be a complete lattice. Set $\mathbf{0} = \wedge Q$ and $\mathbf{1} = \vee Q$. For each subset $G \subseteq Q$, its \wedge -completion G^\wedge and \vee -completion G^\vee are defined by

$$G^\wedge = \{\wedge N : \emptyset \neq N \subseteq G\} \text{ and } G^\vee = \{\vee N : \emptyset \neq N \subseteq G\}. \quad (2.1)$$

A set G is \wedge -complete if $G = G^\wedge$, \vee -complete if $G = G^\vee$, and complete if $G^\wedge = G^\vee = G$.

Let (Q, \leq) be a complete lattice, let $x < y$ in Q and let \ll be a relation in $\text{Ref}(Q)$. Set

$$\begin{aligned} (x, y] &= [x, y] \setminus \{x\}, \quad [x, y) = [x, y] \setminus \{y\}, \\ [a, \ll] &= \{x \in Q : a \ll x\}, \quad [\ll, a] = \{x \in Q : x \ll a\} \text{ for each } a \in Q. \end{aligned} \quad (2.2)$$

Definition 2.1 A relation \ll from $\text{Ref}(Q)$ is called **up-contiguous** if $a \ll b$ implies $[a, b] \subseteq [\ll, b]$; **down-contiguous** if $a \ll b$ implies $[a, b] \subseteq [a, \ll]$; **up-expanded** if $[a, \ll]$ is \vee -complete, **down-expanded** if $[\ll, a]$ is \wedge -complete for all $a \in Q$; **contiguous, expanded** if \ll satisfies the corresponding up- and down-condition.

Note that if a relation has one of the properties defined above, its restriction to any interval also has it. We consider now some examples of these relations.

Example 2.2 1) Let $X = [0, 1] \subset \mathbb{R}$ and $Q = P(X)$ be its power set – the lattice of all subsets of X with $\leq = \subseteq$. For $A, B \in Q$, we write $A \ll B$ if $A \subseteq B \subseteq \overline{A}$. Then \ll is contiguous: if $A \ll B$ then $[A, B] \subseteq [A, \ll] \cap [\ll, B]$, since $\overline{C} = \overline{A}$ for each $C \in [A, B]$. It is also up-expanded, as $\vee[A, \ll] = \overline{A} \in [A, \ll]$. However, \ll is not down-expanded, as $\wedge[\ll, X] = \emptyset \notin [\ll, X]$.

2) Let X be the set of all Lebesgue measurable subsets of \mathbb{R} and Q be the set of all equivalence classes in X . For $A, B \in Q$, we write $A \leq B$ if $\mu(A \setminus B) = 0$, and we write $A \ll B$ if $A \leq B$ and $\mu(B \setminus A) < \infty$. Then \ll is contiguous, but neither up-, nor down-expanded.

3) Let $Q_N = \{1, \dots, N\}$ for $N \in \mathbb{N}$, the order \leq is defined as usual and $n \ll m$ if n is a divisor of m . Then \ll is expanded, but not contiguous. ■

Definition 2.3 (i) A transitive relation is a **T-order** if it is up-contiguous and up-expanded;

(ii) A transitive relation is a **dual T-order** if it is down-contiguous and down-expanded.

(iii) \ll is a **TT-order**, if it is a T- and a dual T-order, i.e., \ll is contiguous and expanded.

By the Duality Principle [Sk, Theorem 1.3'], the results for down-conditions follow from the corresponding results for up-conditions and vice versa. The results for dual T-orders follow from the corresponding results for T-orders and vice versa.

Following [A-I], for $\ll \in \text{Ref}(Q)$, define its *lower* and *upper complement* relations $\overleftarrow{\ll}$ and $\overrightarrow{\ll}$

$$a \overleftarrow{\ll} b \text{ if } [a, \ll] \cap [a, b] = \{a\}; \text{ and } a \overrightarrow{\ll} b \text{ if } [\ll, b] \cap [a, b] = \{b\} \text{ for } a \leq b \text{ in } Q. \quad (2.3)$$

An element $\mathfrak{r} \in Q$ is called a \ll -radical if $\mathbf{0} \ll \mathfrak{r} \overleftarrow{\ll} \mathbf{1}$; an element \mathfrak{p} is called a dual \ll -radical if $\mathbf{0} \overrightarrow{\ll} \mathfrak{p} \ll \mathbf{1}$. In particular, for an interval $[a, b] \subseteq Q$,

$$\mathfrak{r} \in [a, b] \text{ is a } \ll\text{-radical in } [a, b] \text{ if } a \ll \mathfrak{r} \overleftarrow{\ll} b, \quad (2.4)$$

$$\mathfrak{p} \in [a, b] \text{ is a dual } \ll\text{-radical in } [a, b] \text{ if } a \overrightarrow{\ll} \mathfrak{p} \ll b. \quad (2.5)$$

The set of radicals may be empty or have many elements. For example, let $Q = [0, 1] \subseteq \mathbb{R}$ and $\ll \in \text{Ref}(Q)$ be such that $x \ll y$ only if $0 \neq x \leq y \neq 1$. Then the set $[a, 1]$, $a \neq 0$, has no \ll -radicals and the dual \ll -radical $\mathfrak{p} = 1$; the set $[0, b]$, $b \neq 1$, has no dual \ll -radicals and the \ll -radical $\mathfrak{r} = 0$.

Theorem 2.4 (i) *If \ll is an up-expanded order, $\mathfrak{r} = \vee([a, b] \cap [a, \ll])$ is a \ll -radical in $[a, b] \subseteq Q$.*

(ii) *If \ll is a \mathbf{T} -order then \mathfrak{r} in (i) is a unique radical and $[a, \mathfrak{r}] = [a, b] \cap [\ll, \mathfrak{r}]$.*

(iii) *If each $[a, b] \subseteq Q$ has a unique \ll -radical then \ll is up-expanded.*

(iv) *An up-contiguous order \ll is a \mathbf{T} -order if and only if each $[a, b] \subseteq Q$ has a unique \ll -radical.*

Proof. (i) As \ll is up-expanded, $[a, \ll] \cap [a, b]$ is \vee -complete. So it contains \mathfrak{r} . Hence $a \ll \mathfrak{r}$. If $\mathfrak{r} \ll y$ for some $y \in (\mathfrak{r}, b]$, then $a \ll y$ by transitivity of \ll – a contradiction, since \mathfrak{r} is the largest element in $[a, \ll] \cap [a, b]$. Thus $\mathfrak{r} \not\ll b$. So \mathfrak{r} is a \ll -radical in $[a, b]$.

(ii) Let \ll be also up-contiguous and $a \ll z \ll b$ for some z . Then $z \leq \mathfrak{r}$, since \mathfrak{r} is the largest element in $[a, \ll] \cap [a, b]$. As $a \ll \mathfrak{r}$ and \ll is up-contiguous, $[a, \mathfrak{r}] \subseteq [\ll, \mathfrak{r}]$. So $z \ll \mathfrak{r}$. As $z \ll b$, we have $z = \mathfrak{r}$. Thus \mathfrak{r} is a unique \ll -radical.

(iii) Let $a \in Q$, $G \subseteq [a, \ll]$ and $b = \vee G$. Let \mathfrak{r} be the \ll -radical in $[a, b]$: $a \ll \mathfrak{r} \not\ll b$. If $x \in G$ then $a \ll x \leq b$. Let \mathfrak{r}_x be the \ll -radical in $[x, b]$: $x \ll \mathfrak{r}_x \not\ll b$. As \ll is an order, $a \ll \mathfrak{r}_x \not\ll b$. So \mathfrak{r}_x is a \ll -radical in $[a, b]$. As the \ll -radical in $[a, b]$ is unique, $\mathfrak{r} = \mathfrak{r}_x$. Thus $a \ll x \ll \mathfrak{r} \not\ll b$ for all $x \in G$. Hence $b = \vee G \leq \mathfrak{r}$. Thus $b = \mathfrak{r}$. So $a \ll b$. Therefore $b \in [a, \ll]$, so that \ll is up-expanded.

Part (iv) follows from (ii) and (iii). ■

By duality we get the following result.

Theorem 2.5 (i) *If \ll is a down-expanded order, $\mathfrak{p} = \wedge([a, b] \cap [\ll, b])$ is a dual \ll -radical in $[a, b]$.*

(ii) *If \ll is a dual \mathbf{T} -order then \mathfrak{p} in (i) is unique and $[\mathfrak{p}, b] = [a, b] \cap [\mathfrak{p}, \ll]$.*

(iii) *If each $[a, b] \subseteq Q$ has a unique dual \ll -radical then \ll is down-expanded.*

(iv) *A down-contiguous order \ll is a dual \mathbf{T} -order if and only if each $[a, b] \subseteq Q$ has a unique dual \ll -radical.*

As the following examples show, the results of Theorems 2.4 and 2.5 can not be strengthened.

Example 2.6 (i) *The existence of a unique \ll -radical in each $[a, b] \subseteq Q$ only guarantees that \ll is up-expanded, but not that it is up-contiguous.* Indeed, let \ll be the reflexive relation in $Q = [0, 1] \subseteq \mathbb{R}$ with only one non-trivial pair $0 \ll 1$. Each $[a, b] \subseteq Q$ has a unique \ll -radical: $\mathfrak{r}_{[a,b]} = a$ if $[a, b] \neq [0, 1]$, and $\mathfrak{r}_{[0,1]} = 1$. The relation \ll is up-expanded, but not up-contiguous (not a \mathbf{T} -order): $0 \ll 1$ but $Q = [0, 1] \not\subseteq [\ll, 1] = \{0, 1\}$.

(ii) *The existence of \ll -radicals in each $[a, b] \subseteq Q$ does not even guarantee that \ll is up-expanded.* Indeed, let $Q = \{\mathbf{0}, a, b, \mathbf{1}\}$, $\mathbf{0} < a < \mathbf{1}$ and $\mathbf{0} < b < \mathbf{1}$. Let \ll be the reflexive relation in Q such that $\mathbf{0} \ll a$ and $\mathbf{0} \ll b$. Then $[\mathbf{0}, \mathbf{1}]$ has two \ll -radicals: $\mathfrak{r}_1 = a$ and $\mathfrak{r}_2 = b$. The relation \ll is not up-expanded, since $[\mathbf{0}, \ll] = \{\mathbf{0}, a, b\}$ and $\vee[\mathbf{0}, \ll] = a \vee b = \mathbf{1} \notin [\mathbf{0}, \ll]$.

(iii) *If \ll is up-expanded but not up-contiguous then all $[a, b] \subseteq Q$ have radicals but not necessarily unique.* Indeed, let \ll be the reflexive relation in $Q = [0, 1] \subseteq \mathbb{R}$ such that $0 \ll 1$ and $0 \ll \frac{1}{2}$. It is up-expanded, but not up-contiguous (not a \mathbf{T} -order): $0 \ll 1$ but $Q = [0, 1] \not\subseteq [\ll, 1] = \{0, 1\}$. Each $[a, b] \neq [0, 1]$ has a unique \ll -radical, while $[0, 1]$ has two \ll -radicals: $\frac{1}{2}$ and 1. ■

For a dual \mathbf{T} -order \ll in Q , denote by $\mathfrak{p}(b)$ the unique dual \ll -radical in $[\mathbf{0}, b] \subseteq Q$:

$$\mathfrak{p}(b) = \mathfrak{p}_{\ll}(b) = \wedge[\ll, b], \quad \mathbf{0} \not\ll \mathfrak{p}(b) \ll b \text{ and } [\mathbf{0}, b] \cap [\mathfrak{p}, \ll] = [\mathfrak{p}, b]. \quad (2.6)$$

For a \mathbf{T} -order \ll in Q , denote by $\mathfrak{r}(a)$ the unique \ll -radical in $[a, \mathbf{1}]$:

$$\mathfrak{r}(a) = \mathfrak{r}_{\ll}(a) = \vee[a, \ll], \quad a \ll \mathfrak{r}(a) \not\ll \mathbf{1} \text{ and } [a, \mathbf{1}] \cap [\ll, \mathfrak{r}] = [a, \mathfrak{r}]. \quad (2.7)$$

Lemma 2.7 (i) *Let \ll be a dual \mathbf{T} -order in Q . Then*

$$\mathfrak{p}(\mathfrak{p}(b)) = \mathfrak{p}(b) \text{ for all } b \in Q; \quad a \ll b \text{ implies } \mathfrak{p}(b) = \mathfrak{p}(a) \ll a; \quad (2.8)$$

$$\mathfrak{p}(b) \leq a \leq b \text{ implies } \mathfrak{p}(b) = \mathfrak{p}(a) \ll a. \quad (2.9)$$

If \ll is also up-contiguous then $\mathfrak{p}(b) \leq a \leq b$ implies $a \ll b$.

(ii) *Let \ll be a \mathbf{T} -order in Q . Then*

$$\begin{aligned} \mathfrak{r}(\mathfrak{r}(a)) &= \mathfrak{r}(a) \text{ for all } a \in Q; \quad a \ll b \text{ implies } b \ll \mathfrak{r}(b) = \mathfrak{r}(a); \\ a \leq b \leq \mathfrak{r}(a) &\text{ implies } b \ll \mathfrak{r}(b) = \mathfrak{r}(a). \end{aligned} \quad (2.10)$$

If \ll is also down-contiguous then $a \leq b \leq \mathfrak{r}(a)$ implies $a \ll b$.

Proof. By (2.6), $\mathfrak{p}(\mathfrak{p}(b)) \ll \mathfrak{p}(b) \ll b$. As \ll is transitive, $\mathfrak{p}(\mathfrak{p}(b)) \ll b$. By (2.6), $\mathfrak{p}(b)$ is the smallest element in $[\ll, b]$. Hence $\mathfrak{p}(b) \leq \mathfrak{p}(\mathfrak{p}(b))$. Thus $\mathfrak{p}(\mathfrak{p}(b)) = \mathfrak{p}(b)$.

Let $a \ll b$. As $\mathfrak{p}(b)$ is the smallest element in $[\ll, b]$, we have $\mathfrak{p}(b) \leq a \leq b$. So, by (2.6), $\mathfrak{p}(b) \ll a$. Thus $a \ll b$ implies $\mathfrak{p}(b) \ll a$. Hence $\mathfrak{p}(b) \ll a$ implies $\mathfrak{p}(a) \ll \mathfrak{p}(b)$. So $\mathfrak{p}(a) \ll \mathfrak{p}(b) \ll b$. As \ll is transitive, $\mathfrak{p}(a) \ll b$. By (2.6), $\mathfrak{p}(b) \leq \mathfrak{p}(a)$. Thus $\mathfrak{p}(b) = \mathfrak{p}(a)$. As $\mathfrak{p}(a) \ll a$, (2.8) is proved.

Let $\mathfrak{p}(b) \leq a \leq b$. By (2.6), $\mathfrak{p}(b) \ll a$. By (2.8), $\mathfrak{p}(a) = \mathfrak{p}(\mathfrak{p}(b)) = \mathfrak{p}(b)$.

If, in addition, \ll is up-contiguous then $\mathfrak{p}(b) \ll b$ and $a \in [\mathfrak{p}(b), b]$ imply $a \ll b$.

Part (ii) can be proved similarly. ■

Let $g: Q \rightarrow Q$ be a map on Q . We say that g is

$$\text{a pre-radical map if } x \leq g(x) = g(g(x)) \text{ and } x < y < g(x) \text{ implies } g(y) = g(x); \quad (2.11)$$

$$\text{a dual pre-radical map if } g(g(x)) = g(x) \leq x \text{ and } g(x) < y < x \text{ implies } g(y) = g(x); \quad (2.12)$$

$$\text{a radical map if } x \leq g(x) = g(g(x)) \text{ and } x \leq y \text{ implies } g(x) \leq g(y); \quad (2.13)$$

$$\text{a dual radical map if } g(g(x)) = g(x) \leq x \text{ and } x \leq y \text{ implies } g(x) \leq g(y). \quad (2.14)$$

Radical maps were considered in [G, Definition I.3.26], where they are called closure operators.

Remark 2.8 *Radical maps are pre-radical, while pre-radical maps are not always radical. Indeed, $x < y < f(x)$ implies $f(x) \leq f(y) \leq f(f(x)) = f(x)$. So $f(y) = f(x)$ and f is pre-radical.*

On the other hand, let $Q = \{\mathbf{0}, a, b, c, d, \mathbf{1}\}$, $a < b$, $a < c < d$. Set $f(\mathbf{0}) = \mathbf{0}$, $f(a) = f(b) = b$, $f(c) = f(d) = d$, $f(\mathbf{1}) = \mathbf{1}$. Then f is a pre-radical map, but not a radical map, as $a < c$ does not imply $f(a) = b \leq f(c) = d$. Similar results hold for dual radical and dual pre-radical maps. ■

For a map $g: Q \rightarrow Q$, define the following relation \ll^g in Q :

$$x \ll^g y \text{ if } x \leq y \text{ and } g(x) = g(y) \text{ for } x, y \in Q, \quad (2.15)$$

Theorem 2.9 (i) For a dual \mathbf{T} -order \ll , the map

$$\mathfrak{p}_{\ll}: b \in Q \mapsto \mathfrak{p}_{\ll}(b) \text{ (see (2.6)) is a dual pre-radical map}$$

and $\ll \subseteq \ll^{\mathfrak{p}}$ (see (1.1)). If \ll is a contiguous dual \mathbf{T} -order then $\ll = \ll^{\mathfrak{p}}$.

(ii) The map $g \mapsto \ll^g$ is a bijection from the set of all dual pre-radical maps onto the set of all contiguous dual \mathbf{T} -orders. The map $\ll \mapsto \mathfrak{p}_{\ll}$ is its inverse, i.e., $\mathfrak{p}_{\ll^g} = g$.

Proof. (i) From (2.8), (2.9) and (2.12) it follows that \mathfrak{p} is a dual pre-radical map.

Let $a \ll b$. By (2.8), $\mathfrak{p}(a) = \mathfrak{p}(b)$. Thus $a \ll^{\mathfrak{p}} b$ by (2.15). So $\ll \subseteq \ll^{\mathfrak{p}}$.

Let $a \ll^{\mathfrak{p}} b$. Then, by (2.15), $\mathfrak{p}(b) = \mathfrak{p}(a) \leq a \leq b$. If \ll is also up-contiguous then, by Lemma 2.7(i), $a \ll b$, so that $\ll^{\mathfrak{p}}$ is stronger than \ll . Thus $\ll = \ll^{\mathfrak{p}}$.

(ii) If $x \ll^g y \ll^g z$ then $g(x) = g(y) = g(z)$ by (2.15). So $x \ll^g z$. Thus \ll^g is an order.

Let $x \ll^g y$ and $z \in [x, y]$. Then $g(y) = g(x) \leq x \leq z \leq y$ by (2.12) and $g(z) = g(y) = g(x)$. Thus $x \ll^g z \ll^g y$. So $[x, y] \subseteq [x, \ll^g]$ and $[x, y] \subseteq [\ll^g, y]$, i.e., \ll^g is contiguous.

Fix $y \in Q$. By (2.15) and (2.12), $[\ll^g, y] = \{x \in Q: g(y) = g(x) \leq x \leq y\} \subseteq [g(y), y]$. On the other hand, for each $g(y) \leq x \leq y$, it follows from (2.12) that $x \ll^g y$. Thus $[g(y), y] \subseteq [\ll^g, y]$. So $[\ll^g, y] = [g(y), y]$. Hence

$$\mathfrak{p}(y) = \mathfrak{p}_{\ll^g}(y) = \wedge[\ll^g, y] = g(y) \in [\ll^g, y].$$

Thus $[\ll^g, y]$ is \wedge -complete. Hence \ll^g is down-expanded. So \ll^g is a dual \mathbf{T} -order which is also up-contiguous and $\mathfrak{p}_{\ll^g} = g$. Applying (i), we complete the proof. ■

By duality we have the following result.

Theorem 2.10 (i) For a \mathbf{T} -order \ll , the map $\mathfrak{r}: a \mapsto \mathfrak{r}_{\ll}(a)$ (see (2.7)) is a pre-radical map, and $\ll \subseteq \ll^{\mathfrak{r}}$ (see (1.1)). If \ll is a contiguous \mathbf{T} -order then $\ll = \ll^{\mathfrak{r}}$.

(ii) The map $g \mapsto \ll^g$ is a bijection from the set of all pre-radical maps onto the set of all contiguous \mathbf{T} -orders. The map $\ll \mapsto \mathfrak{r}_{\ll}$ is its inverse, i.e., $\mathfrak{r}_{\ll^g} = g$.

Each \mathbf{TT} -order \ll defines the maps $\mathfrak{p}, \mathfrak{r}$ on Q by (2.6) and (2.7). We will show that Q decomposes into a union of disjoint intervals and the restriction of \ll to each of them coincides with \leq . Set

$$Q_{\mathfrak{r}} = \{x \in Q : x = \mathfrak{r}(x)\} \text{ and } Q_{\mathfrak{p}} = \{y \in Q : y = \mathfrak{p}(y)\}. \quad (2.16)$$

Proposition 2.11 Let \ll be a \mathbf{TT} -order from $\text{Ref}(Q)$. Then

(i) $\mathfrak{r}(a) = \mathfrak{r}(\mathfrak{p}(a))$ and $\mathfrak{p}(a) = \mathfrak{p}(\mathfrak{r}(a))$ for all $a \in Q$.

(ii) The map \mathfrak{p} maps Q onto $Q_{\mathfrak{p}}$ and it maps isomorphically $Q_{\mathfrak{r}}$ onto $Q_{\mathfrak{p}}$.

The map \mathfrak{r} maps Q onto $Q_{\mathfrak{r}}$ and it maps isomorphically $Q_{\mathfrak{p}}$ on $Q_{\mathfrak{r}}$ and $\mathfrak{r}|_{Q_{\mathfrak{p}}} = (\mathfrak{p}|_{Q_{\mathfrak{p}}})^{-1}$.

(iii) Let $\mathfrak{p}(c) \leq a \leq b \leq \mathfrak{r}(c)$ for some $c \in Q$. Then $a \ll b$, $\mathfrak{p}(a) = \mathfrak{p}(b) = \mathfrak{p}(c)$ and $\mathfrak{r}(a) = \mathfrak{r}(b) = \mathfrak{r}(c)$. Thus the relations \leq and \ll coincide on $[\mathfrak{p}(c), \mathfrak{r}(c)]$.

Proof. (i) By (2.6) and (2.7), $\mathfrak{p}(a) \ll a \ll \mathfrak{r}(a)$ for $a \in Q$. So (i) follows from (2.8) and (2.10).

(ii) By (2.11) and (2.12), $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{r}^2 = \mathfrak{r}$. Hence $y = \mathfrak{p}(x) \in Q_{\mathfrak{p}}$ for $x \in Q$, as $\mathfrak{p}(y) = \mathfrak{p}^2(x) = \mathfrak{p}(x) = y$. Similarly, $\mathfrak{r}(x) \in Q_{\mathfrak{r}}$ for $x \in Q$. For $y \in Q_{\mathfrak{p}}$, we have $\mathfrak{r}(y) \in Q_{\mathfrak{r}}$ and, by (i),

$\mathfrak{p}(\mathfrak{r}(y)) = \mathfrak{p}(y) = y$. Similarly, $\mathfrak{p}(x) \in Q_{\mathfrak{p}}$ and $\mathfrak{r}(\mathfrak{p}(x)) = x$ for $x \in Q_{\mathfrak{r}}$. From this (ii) follows immediately.

(iii) By (2.6) and (2.7), $\mathfrak{p}(c) \ll \mathfrak{r}(c)$. As \ll is contiguous, $\mathfrak{p}(c) \ll a \ll b \ll \mathfrak{r}(c)$. So, by (2.8) and (2.10), $\mathfrak{p}(a) = \mathfrak{p}(b) = \mathfrak{p}(c)$ and $\mathfrak{r}(a) = \mathfrak{r}(b) = \mathfrak{r}(c)$. Thus $\leq|_{[\mathfrak{p}(c), \mathfrak{r}(c)]}$ is stronger than $\ll|_{[\mathfrak{p}(c), \mathfrak{r}(c)]}$. As $\ll \subseteq \leq$ in Q , we have $\leq|_{[\mathfrak{p}(c), \mathfrak{r}(c)]} = \ll|_{[\mathfrak{p}(c), \mathfrak{r}(c)]}$. ■

Recall that subsets G_1 and G_2 of Q are disjoint if $G_1 \cap G_2 = \emptyset$.

Theorem 2.12 *Let \ll belong to $\text{Ref}(Q)$. The following conditions are equivalent.*

- (i) \ll is a **TT**-order.
- (ii) $Q = \cup_{\lambda \in Q_{\mathfrak{r}}} [\mathfrak{p}(\lambda), \lambda]$ and all intervals $[\mathfrak{p}(\lambda), \lambda]$, $\lambda \in Q_{\mathfrak{r}}$, are mutually disjoint. Moreover, $x \ll y$ if and only if $x, y \in [\mathfrak{p}(\lambda), \lambda]$ and $x \leq y$ for some $\lambda \in Q_{\mathfrak{r}}$.
- (iii) $Q = \cup_{\lambda \in Q_{\mathfrak{p}}} [\lambda, \mathfrak{r}(\lambda)]$ and all intervals $[\lambda, \mathfrak{r}(\lambda)]$, $\lambda \in Q_{\mathfrak{p}}$, are mutually disjoint. Moreover, $x \ll y$ if and only if $x, y \in [\lambda, \mathfrak{r}(\lambda)]$ and $x \leq y$ for some $\lambda \in Q_{\mathfrak{p}}$.

Proof. (ii) \Rightarrow (i). Let Q satisfy (ii). If $x \ll y$ then $[x, y] \subseteq [\mathfrak{p}(\lambda), \lambda]$ for some $\lambda \in Q_{\mathfrak{r}}$. As \ll on $[\mathfrak{p}(\lambda), \lambda]$ coincides with \leq , we have $[x, y] \subseteq [\ll, y]$ and $[x, y] \subseteq [x, \ll]$. Thus \ll is contiguous.

We also have $[x, \ll] = [x, \lambda]$ and $[\ll, y] = [\mathfrak{p}(\lambda), y]$. Hence $[x, \ll]$ is \vee -complete and $[\ll, y]$ is \wedge -complete. Thus \ll is expanded. Finally, if $x \ll y \ll z$ then $x, y, z \in [\mathfrak{p}(\lambda), \lambda]$ for some λ and $x \leq y \leq z$. Hence $x \leq z$, so that $x \ll z$. Thus \ll is transitive, so that \ll is a **TT**-order.

(i) \Rightarrow (ii). For a **TT**-order \ll , let $z \in [\mathfrak{p}(\lambda), \lambda] \cap [\mathfrak{p}(\nu), \nu]$, where $\lambda \neq \nu$ in $Q_{\mathfrak{r}}$. As $\mathfrak{p}(\lambda) \leq z \leq \lambda$, we have $\mathfrak{p}(z) = \mathfrak{p}(\lambda)$ by (2.12). Similarly, $\mathfrak{p}(z) = \mathfrak{p}(\nu)$, so $\mathfrak{p}(z) = \mathfrak{p}(\lambda) = \mathfrak{p}(\nu)$. By Proposition 2.11, \mathfrak{p} maps isomorphically $Q_{\mathfrak{r}}$ on $Q_{\mathfrak{p}}$. Thus $\lambda = \nu$, a contradiction. So all $[\mathfrak{p}(\lambda), \lambda]$ are disjoint.

Let $x \ll y$. Set $\lambda = \mathfrak{r}(y)$. By (2.10), $y \ll \mathfrak{r}(x) = \mathfrak{r}(y) = \lambda$. By (2.11) and (2.16), $\lambda \in Q_{\mathfrak{r}}$. So $\mathfrak{p}(\lambda) = \mathfrak{p}(\mathfrak{r}(x)) = \mathfrak{p}(x) \ll x \ll y \ll \lambda$ by Proposition 2.11 and (2.9). Thus $x, y \in [\mathfrak{p}(\lambda), \lambda]$.

Conversely, let $x \leq y$ in $[\mathfrak{p}(\lambda), \lambda]$. By Proposition 2.11, \ll and \leq coincide in $[\mathfrak{p}(\lambda), \lambda]$. So $x \ll y$.

Finally, let $x \in Q$. Set $\lambda = \mathfrak{r}(x)$. By (2.10) and Proposition 2.11, $x \ll \lambda \in Q_{\mathfrak{r}}$. Hence, by Proposition 2.11 and (2.9), $\mathfrak{p}(\lambda) = \mathfrak{p}(x) \ll x \ll \lambda$. So $x \in [\mathfrak{p}(\lambda), \lambda]$. Thus $Q = \cup_{\lambda \in Q_{\mathfrak{r}}} [\mathfrak{p}(\lambda), \lambda]$.

Similarly, one can prove (i) \Leftrightarrow (iii). ■

It should be noted that if $Q = \cup_{\lambda \in \Lambda} [a_{\lambda}, b_{\lambda}]$ and all intervals $[a_{\lambda}, b_{\lambda}]$, $\lambda \in \Lambda$, are mutually disjoint, then the relation in Q defined by $x \ll y$ if and only if $x, y \in [a_{\lambda}, b_{\lambda}]$ and $x \leq y$ for some $\lambda \in \Lambda$, is a **TT**-order.

3 Radicals, H-relations and R-orders in complete lattices

In this section we consider **H**-relations, and **R**-orders. We show that many results obtained for **T**-orders in the previous section can be strengthened for **R**-orders.

Lemma 3.1 (i) *For a relation \ll in Q , the following conditions are equivalent:*

- 1) \ll is up-contiguous and $a \wedge b \ll b$ implies $a \ll a \vee b$;
- 2) $a \ll b$ and $a \leq c$ imply $c \ll b \vee c$ for $a, b, c \in Q$;
- 3) $a \ll b$ implies $a \vee x \ll b \vee x$ for every $x \in Q$.

(ii) *For a relation \ll in Q , the following conditions are equivalent:*

- 1) \ll is down-contiguous and $a \ll a \vee b$ implies $a \wedge b \ll b$;

- 2) $a \ll b$ and $c \leq b$ in Q imply $a \wedge c \ll c$;
- 3) $a \ll b$ implies $a \wedge x \ll b \wedge x$ for every $x \in Q$.

Proof. (i) 1) \Rightarrow 2). Let $a \ll b$ and $a \leq c$. Then $b \wedge c \in [a, b]$. As \ll is up-contiguous, $b \wedge c \ll b$, so that $c \ll b \vee c$. 2) \iff 3) is proved in [A-I, Page 776].

3) \Rightarrow 1). For $c \in [a, b]$ and $a \ll b$, one has $c = a \vee c \ll b \vee c = b$. Thus \ll is up-contiguous. If $a \wedge b \ll b$ then $a = a \vee (a \wedge b) \ll a \vee b$. Part (ii) is proved similarly. ■

Definition 3.2 ([A-I]) (i) \ll is an **H-relation** if it satisfies conditions of Lemma 3.1(i).

(ii) \ll is a **dual H-relation** if it satisfies equivalent conditions of Lemma 3.1(ii).

(iii) \ll is an **HH-relation** if it is an **H-** and a **dual H-relation**.

(iv) \ll is an **R-order** if it is an up-expanded **H-order**.

(v) \ll is a **dual R-order** if it is a down-expanded **dual H-order**.

(vi) \ll is an **RR-order** if it is an **R-order** and a **dual R-order**.

By the Duality Principle [Sk, Theorem 1.3'], the results for **H-relations** and **R-orders** follow from the corresponding results for **dual H-relations** and **dual R-orders** and vice versa.

Each **H-order** \ll is *finitely up-expanded*: $b, c \in [a, \ll]$ implies $b \vee c \in [a, \ll]$ for all $a \in Q$. Indeed, by Lemma 3.1, $c = a \vee c \ll b \vee c$. By transitivity, $a \ll b \vee c$, as $a \ll c$. Similarly, each **dual H-order** \ll is *finitely down-expanded*, i.e., $a, c \in [\ll, b]$ implies $a \wedge c \in [\ll, b]$ for each $a \in Q$.

The notions of (dual) **R-orders** are stronger than the notions of (dual) **H-orders**, respectively, as they require up- and down-expandedness and not only its "finite" version. Amitsur [A-I] defined them in a different, but equivalent way (he called them **R-** and **dual R-relations**).

Comparing Definitions 2.3 and 3.2, we see that (dual) **R-orders** are (dual) **T-orders**. Thus the results for (dual) **T-orders** hold also for (dual) **R-orders**. In particular,

- 1) for a **dual R-order** \ll , $\mathfrak{p}(b) = \bigwedge[\ll, b]$ is the unique **dual** \ll -radical in $[0, b]$ and (2.6) holds;
- 2) for an **R-order** \ll , $\mathfrak{r}(a) = \bigvee[a, \ll]$ is the unique \ll -radical in $[a, 1]$ and (2.7) holds.

However, as the following example shows, (dual) **T-orders** are not necessarily (dual) **R-orders**.

Example 3.3 (i) Let $Q = \{0, a, b, 1\}$, $a \wedge b = 0$ and $a \vee b = 1$. Let \ll be a reflexive relation in Q .

- 1) If only $0 \ll a$ then \ll is a **TT-order** but not an **R-order** as $b = 0 \vee b \not\ll a \vee b = 1$.
- 2) If only $a \ll 1$ then \ll is a **TT-order** but not a **dual R-order**.

(ii) Let $Q = [0, 1] \subset \mathbb{R}$, $\ll|_{[0,1]} = \leq|_{[0,1]}$ and $1 \ll 1$. Then \ll is an **H-order** but not an **R-order**.

We consider now some additional properties of the maps $a \mapsto \mathfrak{p}(a)$, $a \mapsto \mathfrak{r}(a)$ (see Lemma 2.7).

Lemma 3.4 (i) Let \ll be a **dual R-order**. Then (2.8) and (2.9) hold and, for $a, b \in Q$,

$$\mathfrak{p}(a \wedge b) \ll \mathfrak{p}(a) \wedge \mathfrak{p}(b) \ll a \wedge b \text{ and } \mathfrak{p}(a) \leq \mathfrak{p}(b) \text{ if } a \leq b.$$

(ii) Let \ll be an **R-order**. Then (2.10) holds and, for $a, b \in Q$,

$$a \vee b \ll \mathfrak{r}(a) \vee \mathfrak{r}(b) \ll \mathfrak{r}(a \vee b) \text{ and } \mathfrak{r}(a) \leq \mathfrak{r}(b) \text{ if } a \leq b.$$

Proof. As a dual \mathbf{R} -order is a dual \mathbf{T} -order, the results of Lemma 2.7(i) hold for them.

As $\mathfrak{p}(b) \ll b$ and $\mathfrak{p}(a) \ll a$ by (2.6), we have $\mathfrak{p}(a) \wedge \mathfrak{p}(b) \ll \mathfrak{p}(a) \wedge b \ll a \wedge b$ from Lemma 3.1(ii). By transitivity, $\mathfrak{p}(a) \wedge \mathfrak{p}(b) \ll a \wedge b$. So, by (2.8), $\mathfrak{p}(a \wedge b) \ll \mathfrak{p}(a) \wedge \mathfrak{p}(b) \ll a \wedge b$.

If $a \leq b$ then $\mathfrak{p}(a) = \mathfrak{p}(a \wedge b) \leq \mathfrak{p}(a) \wedge \mathfrak{p}(b)$. So $\mathfrak{p}(a) \leq \mathfrak{p}(b)$. The proof of (ii) is similar. ■

Recall that, for a map $g: Q \rightarrow Q$, the relation \ll^g in Q is defined in (2.15).

Theorem 3.5 (i) For a dual \mathbf{R} -order \ll , the map $\mathfrak{p}_{\ll}: b \mapsto \mathfrak{p}_{\ll}(b)$ in (2.6) is a dual radical map and $\ll \subseteq \ll^{\mathfrak{p}_{\ll}}$ (see (1.1)). If \ll is a contiguous dual \mathbf{R} -order then $\ll = \ll^{\mathfrak{p}_{\ll}}$.

(ii) The map $g \mapsto \ll^g$ is a bijection from the set of all dual radical maps onto the set of all contiguous dual \mathbf{R} -orders. The map $\ll \mapsto \mathfrak{p}_{\ll}$ is its inverse, i.e., $\mathfrak{p}_{\ll^g} = g$.

Proof. (i) By (2.8) and (2.6), $\mathfrak{p}(\mathfrak{p}(b)) = \mathfrak{p}(b) \leq b$ for $b \in Q$. By Lemma 3.4, $\mathfrak{p}(a) \leq \mathfrak{p}(b)$ if $a \leq b$. Thus \mathfrak{p} satisfies (2.14), so that it is a dual radical map.

If \ll is a dual \mathbf{R} -order, it is a dual \mathbf{T} -order. So it follows from Theorem 2.9 that $\ll \subseteq \ll^{\mathfrak{p}}$ and that $\ll = \ll^{\mathfrak{p}}$ if \ll is also up-contiguous.

(ii) If g is a dual radical map then it is pre-dual by Remark 2.8. Hence, by Theorem 2.9, \ll^g is a contiguous dual \mathbf{T} -order and $\mathfrak{p}_{\ll^g} = g$.

If $a \ll^g b$ then $g(a) = g(b)$. Let $c \leq b$. By (2.14), $g(c) \leq g(b)$ and $g(x) \leq x$ for all $x \in Q$. Thus

$$g(c) = g(b) \wedge g(c) = g(a) \wedge g(c) \leq a \wedge c \leq c.$$

So $g(c) \leq a \wedge c \leq c$. As g is also pre-dual, $g(c) = g(a \wedge c)$ by (2.12). Hence $a \wedge c \ll^g c$. Thus we have from Lemma 3.1(ii) that \ll^g is a dual \mathbf{H} -relation. So \ll^g is a contiguous dual \mathbf{R} -order. Applying (i), we complete the proof. ■

By duality we get the following result.

Theorem 3.6 (i) For a \mathbf{R} -order \ll , the map $\mathfrak{r}: a \mapsto \mathfrak{r}(a)$ in (2.7) is a radical map and $\ll \subseteq \ll^{\mathfrak{r}}$ (see(1.1)). If \ll is a contiguous \mathbf{R} -order then $\ll = \ll^{\mathfrak{r}}$.

(ii) The map $g \mapsto \ll^g$ is a bijection from the set of all radical maps onto the set of all contiguous \mathbf{R} -orders. The map $\ll \mapsto \mathfrak{r}_{\ll}$ is its inverse, i.e., $\mathfrak{r}_{\ll^g} = g$.

We will now study properties of the maps \mathfrak{p} , \mathfrak{r} for relations satisfying combined conditions.

Lemma 3.7 (i) If \ll is an \mathbf{H} -relation and a dual \mathbf{T} -order then $\mathfrak{p}(a \vee b) \ll \mathfrak{p}(a) \vee \mathfrak{p}(b)$ for $a, b \in Q$.

(ii) If \ll is an \mathbf{H} -relation and a dual \mathbf{R} -order, then $\mathfrak{p}(a \vee b) = \mathfrak{p}(a) \vee \mathfrak{p}(b)$ for $a, b \in Q$.

(iii) If \ll is a dual \mathbf{H} -relation and a \mathbf{T} -order then $\mathfrak{r}(a) \wedge \mathfrak{r}(b) \ll \mathfrak{r}(a \wedge b)$ for $a, b \in Q$.

(iv) If \ll is a dual \mathbf{H} -relation and an \mathbf{R} -order then $\mathfrak{r}(a) \wedge \mathfrak{r}(b) = \mathfrak{r}(a \wedge b)$ for $a, b \in Q$.

Proof. (i) By (2.6), $\mathfrak{p}(a) \ll a$. As \ll is an \mathbf{H} -relation, $\mathfrak{p}(a) \vee \mathfrak{p}(b) \ll a \vee \mathfrak{p}(b)$ by Lemma 3.1. Similarly, $\mathfrak{p}(b) \ll b$ implies $a \vee \mathfrak{p}(b) \ll a \vee b$. By transitivity of \ll , $\mathfrak{p}(a) \vee \mathfrak{p}(b) \ll a \vee b$. Since \ll is also a dual \mathbf{T} -order, $\mathfrak{p}(a \vee b) \ll \mathfrak{p}(a) \vee \mathfrak{p}(b)$ by (2.8).

(ii) If \ll is a dual \mathbf{R} -order, $\mathfrak{p}(a) \vee \mathfrak{p}(b) \leq \mathfrak{p}(a \vee b)$ by Lemma 3.4. So the proof follows from (i). Parts (iii) and (iv) follow from duality. ■

Recall that \mathbf{TT} -orders are \mathbf{T} - and dual \mathbf{T} -orders, and \mathbf{RR} -orders are \mathbf{R} - and dual \mathbf{R} -orders. Each \mathbf{TT} -order defines both maps $\mathfrak{r}: a \mapsto \mathfrak{r}(a)$ and $\mathfrak{p}: a \mapsto \mathfrak{p}(a)$ on Q . In terms of them we will find necessary and sufficient conditions for a \mathbf{TT} -order to be an \mathbf{R} -, or a dual \mathbf{R} -, or an \mathbf{RR} -order.

Corollary 3.8 (i) Let \ll be a **TT**-order in Q . The following conditions are equivalent.

- 1) \ll is a dual **R**-order;
- 2) $\mathfrak{p}(a) \leq \mathfrak{p}(b)$ if $a \leq b$ in Q ;
- 3) $\mathfrak{p}(a \wedge b) \leq \mathfrak{p}(a) \wedge \mathfrak{p}(b)$ for $a, b \in Q$;
- 4) $\mathfrak{p}(a \wedge b) = \mathfrak{p}(\mathfrak{p}(a) \wedge \mathfrak{p}(b))$ for $a, b \in Q$.

(ii) Let \ll be a **TT**-order in Q . The following conditions are equivalent:

- 1) \ll is an **R**-order;
- 2) $\mathfrak{r}(a) \leq \mathfrak{r}(b)$ if $a \leq b$ in Q ;
- 3) $\mathfrak{r}(a) \vee \mathfrak{r}(b) \leq \mathfrak{r}(a \vee b)$ for $a, b \in Q$;
- 4) $\mathfrak{r}(\mathfrak{r}(a) \vee \mathfrak{r}(b)) = \mathfrak{r}(a \vee b)$ for $a, b \in Q$.

(iii) A **TT**-order \ll is an **RR**-order if and only if $\mathfrak{p}(a) \leq \mathfrak{p}(b)$ and $\mathfrak{r}(a) \leq \mathfrak{r}(b)$ for all $a \leq b$.

Proof. (i) 1) \Rightarrow 2) follows from Lemma 3.4(i).

2) \Rightarrow 3) As $a \wedge b \leq a$, we have $\mathfrak{p}(a \wedge b) \leq \mathfrak{p}(a)$ from 2). Similarly, $\mathfrak{p}(a \wedge b) \leq \mathfrak{p}(b)$ and 3) follows.

3) \Rightarrow 4) We have $\mathfrak{p}(a \wedge b) \leq \mathfrak{p}(a) \wedge \mathfrak{p}(b) \leq a \wedge b$. By (2.9), $\mathfrak{p}(a \wedge b) = \mathfrak{p}(\mathfrak{p}(a) \wedge \mathfrak{p}(b))$.

4) \Rightarrow 3). As $\mathfrak{p}(x) \leq x$ for $x \in Q$, we have $\mathfrak{p}(\mathfrak{p}(a) \wedge \mathfrak{p}(b)) \leq \mathfrak{p}(a) \wedge \mathfrak{p}(b)$.

3) \Rightarrow 1). Let $x \ll y$ and $z \in Q$. By Theorem 2.12, $\mathfrak{p}(\lambda) \leq x \leq y \leq \lambda$ and $\mathfrak{p}(\mu) \leq z \leq \mu$ for some $\lambda, \mu \in Q_{\mathfrak{r}}$. Hence, by 3),

$$\mathfrak{p}(\lambda \wedge \mu) \leq \mathfrak{p}(\lambda) \wedge \mathfrak{p}(\mu) \leq x \wedge z \leq y \wedge z \leq \lambda \wedge \mu \leq \mathfrak{r}(\lambda \wedge \mu).$$

By Proposition 2.11(iii), $x \wedge z \ll y \wedge z$. Thus \ll is a dual **H**-relation. So it is a dual **R**-order.

Part (ii) can be proved similarly. Part (iii) follows from (i) and (ii). ■

Maps f, g on Q are *conjugate* if $g(x) = g(f(x))$ and $f(x) = f(g(x))$ for all $x \in Q$.

Corollary 3.9 (i) For an **RR**-order \ll , the maps \mathfrak{r}_{\ll} and \mathfrak{p}_{\ll} are conjugate.

(ii) Let maps f, g on Q be conjugate and let

$$g(x) \leq x \leq f(x) \text{ for } x \in Q, \text{ and } f(x) \leq f(y) \text{ and } g(x) \leq g(y) \text{ for } x \leq y \text{ in } Q. \quad (3.1)$$

Then there is an **RR**-order \ll such that $f = \mathfrak{r}_{\ll}$ and $g = \mathfrak{p}_{\ll}$. Moreover, $\ll = \ll^f = \ll^g$.

Proof. (i) Let \ll be an **RR**-order. By Theorems 3.5 and 3.6, \mathfrak{r}_{\ll} is a radical map and \mathfrak{p}_{\ll} is a dual radical map. By (2.6) and (2.7), $\mathfrak{p}_{\ll}(x) \ll x \ll \mathfrak{r}_{\ll}(x)$ for all $x \in Q$, and

$$\mathfrak{p}_{\ll}(x) \stackrel{(2.8)}{=} \mathfrak{p}_{\ll}(\mathfrak{r}_{\ll}(x)) \text{ and } \mathfrak{r}_{\ll}(x) \stackrel{(2.10)}{=} \mathfrak{r}_{\ll}(\mathfrak{p}_{\ll}(x)).$$

(ii) As f and g are conjugate, $f(f(x)) = f(g(f(x))) = f(g(x)) = f(x)$. Similarly, $g(g(x)) = g(x)$. From this and from (3.1) we get that f is a radical and g is a dual radical map ((2.13), (2.14)).

Let $x \ll^f y$. By (2.15), $f(x) = f(y)$. As f and g are conjugate, $g(x) = g(f(x)) = g(f(y)) = g(y)$. So $x \ll^g y$. Similarly, $x \ll^g y \Rightarrow x \ll^f y$. Thus $\ll^f = \ll^g$. By Theorems 3.5 and 3.6, $\ll = \ll^f = \ll^g$ is an **RR**-order, $f = \mathfrak{r}_{\ll}$ and $g = \mathfrak{p}_{\ll}$. ■

4 Construction of contiguous and expanded relations

We proved in the previous section that \mathbf{T} -orders have unique radicals. However, majority of relations in applications are neither contiguous, nor expanded. In this section we consider two ways to construct from a relation some new relations that are contiguous, or expanded, or both, so that the new relations have radicals. We start with the complements of \ll : $\overleftarrow{\ll}$ and $\overrightarrow{\ll}$ defined in (2.3).

Proposition 4.1 (i) $\overleftarrow{\ll}$ is a down-contiguous relation; the set $[a, \overleftarrow{\ll}]$ is \wedge -complete for each $a \in Q$.
(ii) $\overrightarrow{\ll}$ is an up-contiguous relation; the set $[\overrightarrow{\ll}, b]$ is \vee -complete for each $b \in Q$.
(iii) If \ll is stronger than \prec , then $\overleftarrow{\ll}$ is stronger than $\overleftarrow{\prec}$ and $\overrightarrow{\ll}$ is stronger than $\overrightarrow{\prec}$.

Proof. (i) Let $a \overleftarrow{\ll} b$. By (2.3), $[a, \ll] \cap [a, b] = \{a\}$. Then $[a, \ll] \cap [a, x] = \{a\}$ for each $x \in [a, b]$. Thus $a \overleftarrow{\ll} x$, so that $\overleftarrow{\ll}$ is down-contiguous (see Definition 2.1).

Let $G \subseteq [a, \overleftarrow{\ll}]$. By (2.3), $[a, \ll] \cap [a, b] = \{a\}$ for all $b \in G$. Thus $[a, \ll] \cap [a, \wedge G] = \{a\}$. Hence $a \overleftarrow{\ll} \wedge G$, so that $[a, \overleftarrow{\ll}]$ is \wedge -complete. Part (ii) is proved similarly.

(iii) Let $a \overleftarrow{\prec} b$. By (2.3), $[a, \prec] \cap [a, b] = \{a\}$. As $\ll \subseteq \prec$ (see (1.1)), we have $[a, \ll] \subseteq [a, \prec]$. Hence $[a, \ll] \cap [a, b] = \{a\}$. By (2.3), $a \overleftarrow{\ll} b$. The proof of the second statement is similar. ■

We introduce now two constructions that play important part in this paper.

Definition 4.2 Let \ll be a relation from $\text{Ref}(Q)$. We say that $G \subseteq Q$ is

- 1) a **lower \ll -set** if, for each $x \in G \setminus \{\wedge G\}$, there is $y \in G$ such that $x \neq y$ and $y \ll x$;
- 2) an **upper \ll -set** if, for each $x \in G \setminus \{\vee G\}$, there is $y \in G$ such that $x \neq y$ and $x \ll y$.
- 3) We write $a \ll^{\text{lo}} b$ if $[a, b]$ is a lower \ll -set; and $a \ll^{\text{up}} b$ if $[a, b]$ is an upper \ll -set.

Proposition 4.3 (i) $\ll^{\text{lo}} = \overleftarrow{(\overrightarrow{\ll})}$ is a down-contiguous order. If \ll is down-contiguous, $\ll \subseteq \ll^{\text{lo}}$.
(ii) $\overrightarrow{\ll} \subseteq \overleftarrow{\ll^{\text{lo}}}$ and $(\ll^{\text{lo}})^{\text{lo}} = \ll^{\text{lo}}$.
(iii) $\ll^{\text{up}} = \overrightarrow{(\overleftarrow{\ll})}$ is an up-contiguous order. If \ll is up-contiguous, $\ll \subseteq \ll^{\text{up}}$.
(iv) $\overleftarrow{\ll} \subseteq \overleftarrow{\ll^{\text{up}}}$ and $(\ll^{\text{up}})^{\text{up}} = \ll^{\text{up}}$.

Proof. (i) If $a \overleftarrow{(\overrightarrow{\ll})} b$ then a is not $\overrightarrow{\ll}$ -related to all $x \in (a, b]$ by (2.3). So, by (2.3),

$$a \overleftarrow{(\overrightarrow{\ll})} b \text{ if and only if, for each } x \in (a, b], \text{ there is } y \in [a, x) \text{ such that } y \ll x. \quad (4.1)$$

That is, if and only if $[a, b]$ is a lower \ll -set: $a \ll^{\text{lo}} b$. Hence $\ll^{\text{lo}} = \overleftarrow{(\overrightarrow{\ll})}$. So, by Proposition 4.1(i), \ll^{lo} is down-contiguous. Clearly, it is an order.

If \ll is down-contiguous and $a \ll b$ then $[a, b] \subseteq [a, \ll]$. So $a \ll x$ for all $x \in (a, b]$. Thus $[a, b]$ is a lower \ll -set: $a \ll^{\text{lo}} b$. Thus $\ll \subseteq \ll^{\text{lo}}$. The proof of (iii) is similar.

(ii) By Proposition 4.1(ii), $\overrightarrow{\ll}$ is up-contiguous. So, by (iii) and (i), $\overrightarrow{\ll} \subseteq (\overrightarrow{\ll})^{\text{up}} = \overrightarrow{(\overleftarrow{\ll})} = \ll^{\text{lo}}$. Hence, by (i) and by Proposition 4.1(iii), $(\ll^{\text{lo}})^{\text{lo}} = \overleftarrow{\ll^{\text{lo}}} \subseteq \overleftarrow{(\overrightarrow{\ll})} = \ll^{\text{lo}}$.

Let a be not $\overset{\leftarrow}{\ll}^{\text{lo}}$ -related to b . By (4.1), there is $x \in (a, b]$ such that y is not \ll^{lo} -related to x for each $y \in [a, x)$. In particular, a is not \ll^{lo} -related to x . Then there is $z \in (a, x]$ such that $u \not\ll z$ for each $u \in [a, z)$. Hence a is not \ll^{lo} -related to b . Thus $\ll^{\text{lo}} \subseteq \overset{\leftarrow}{\ll}^{\text{lo}} = (\ll^{\text{lo}})^{\text{lo}}$. So $\ll^{\text{lo}} = (\ll^{\text{lo}})^{\text{lo}}$.

The proof of part (iv) is similar. ■

To describe the second way of constructing new relations from \ll , we study chains in lattices. Recall that a chain C in Q is a linearly ordered subset: either $x \leq y$ or $y \leq x$ for $x, y \in C$. A chain C is *from a to b* if $\wedge C = a$ and $\vee C = b$. A chain C is *maximal* in $G \subseteq Q$ if $C \subseteq G$ and G has no larger chains.

Definition 4.4 *Let \ll be a relation in Q and C be a chain from a to b . Then*

- 1) C is a **lower (upper) \ll -chain**, if it is a lower (upper) \ll -set.
- 2) C is a **lower \ll -gap chain** if each $x \in C \setminus \{a\}$ has an immediate \ll -predecessor $p_x \in C$:
 $p_x \neq x$, $p_x \ll x$ and $[p_x, x] \cap C = \{p_x, x\}$.
- 3) C is an **upper \ll -gap chain** if each $x \in C \setminus \{b\}$ has an immediate \ll -successor $s_x \in C$:
 $x \neq s_x$, $x \ll s_x$ and $[x, s_x] \cap C = \{x, s_x\}$.

Lemma 4.5 *Let G be a \wedge -complete set in Q , C be a chain in G and $b := \vee C \in G$. Set $a = \wedge C$.*

- (i) *There is a maximal chain in $G \cap [a, b]$ containing C .*
- (ii) *If C is a maximal chain in $G \cap [a, b]$, it is \wedge -complete. If G is complete, C is complete.*
- (iii) (cf. Lemma I.34 [G]) *If C is a \wedge -complete lower \ll -gap chain and $b \in C$, then C is complete.*

Proof. (i) Apply Zorn's Lemma to the set of all chains in $G \cap [a, b]$ that contain C .

(ii) As C^\wedge is a chain in $G \cap [a, b]$ containing C and C is maximal, $C = C^\wedge$. If G is complete then $C^\vee \subseteq G \cap [a, b]$. As C^\vee is a chain containing C and C is maximal, $C = C^\vee$. So C is complete.

(iii) For $\Gamma \subseteq C$, $\Gamma \neq \{a\}$, let $K = \{x \in C : y \leq x \text{ for all } y \in \Gamma\}$. As $b \in K$, $K \neq \emptyset$. As C is \wedge -complete, $d := \wedge K \in C$. Then $a < d$ and $y \leq d$ for all $y \in \Gamma$. So $d \in K$. As C is a lower \ll -gap chain, there is an immediate \ll -predecessor p of d in C : $C \cap [p, d] = \{p, d\}$. If $y \leq p$ for all $y \in \Gamma$, then $p \in K$ and $d = \wedge K \leq p$, a contradiction. Hence there is $y \in \Gamma$ such that $p < y \leq d$. So $y = d$, i.e., $\vee \Gamma = d \in C$. Thus C is \vee -complete. ■

We shall now consider a particular type of extension of lower \ll -gap chains.

Definition 4.6 *Let D be a chain, $a = \wedge D$ and $b = \vee D \in D$. A chain C **down-extends** D if*

$$C \subseteq [0, b] \text{ and } D = C \cap [a, b], \text{ i.e., } C = D \cup E, \text{ where } E \text{ is a chain in } [0, a]. \quad (4.2)$$

*A chain D is a **maximal down-extended** lower \ll -gap chain (resp. lower \ll -chain), if there does not exist another lower \ll -gap chain (resp. lower \ll -chain) that down-extends D .*

Let $Q = [0, 1] \subset \mathbb{R}$. Then (Q, \leq) is a maximal chain, a maximal down-extended lower \leq -chain, but not a lower \leq -gap chain. The set $\mathfrak{Q}_{[0,1]}$ of all rational numbers in $[0, 1]$ is a maximal down-extended lower \leq -chain, but is neither a maximal lower \leq -chain, nor a lower \leq -gap chain. The set $\{0\} \cup \mathfrak{Q}_{[\frac{1}{2}, 1]}$ is a maximal down-extended lower \leq -chain but not a lower \leq -gap chain; $\mathfrak{Q}_{[\frac{1}{2}, 1]}$ is a lower \leq -chain but not maximal down-extended. The sets $\{0, 1\}$ and $\{0, \frac{1}{2}, 1\}$ are maximal down-extended lower \leq -gap chains and $\{\frac{1}{2}, 1\}$ is a lower \ll -gap chain but not maximal down-extended.

Proposition 4.7 *Let G be a \wedge -complete set in Q , let S be a complete, lower \ll -gap chain in G and $b := \vee G \in S$. Then there is a maximal down-extended complete, lower \ll -gap chain C in G that*

$$1) \text{ down-extends } G \text{ and } G \cap [\ll, \mathfrak{p}] = \{\mathfrak{p}\}, \text{ where } \mathfrak{p} = \wedge C \in C. \quad (4.3)$$

If G is a lower \ll -set, there is a complete lower \ll -gap chain C in G such that $\wedge G = \wedge C = \mathfrak{p}$ and $\vee C = b$.

Proof. Let Γ_S be the set of all complete, lower \ll -gap chains in G that down-extend S . It is partially ordered by the inclusion "down-extends". Let Ω be a linearly ordered subset of Γ_S . Set $K = \cup_{C \in \Omega} C$ and $T = K \cup \{\wedge K\}$. Then T is a chain in G and it down-extends each $D \in \Omega$, as

$$T \cap [\wedge D, b] = (\cup_{C \in \Omega} C) \cap [\wedge D, b] = \cup_{C \in \Omega} (C \cap [\wedge D, b]) \stackrel{(4.2)}{=} D.$$

Let $x \in K$ and $\wedge K < x$. Then $x \in D$ and $\wedge D < x$ for some $D \in \Omega$. As D is a lower \ll -gap chain, there is $p \in D$ such that $p \ll x \neq p$ and $D \cap [p, x] = \{p, x\}$. By the above,

$$T \cap [p, x] = T \cap ([\wedge D, b] \cap [p, x]) = D \cap [p, x] = \{p, x\},$$

so that T is a lower \ll -gap chain in G that down-extends S .

To prove that T is \wedge -complete, let $E \subseteq T$. If $\wedge K \in E$ then $\wedge E = \wedge K \in T$. If $\wedge K \notin E$, set $E_C = C \cap E$ for $C \in \Omega$. As each C is complete, $e_C := \wedge E_C \in C$. If all e_C lie in a chain $D \in \Omega$ then, as D is complete,

$$\wedge E \stackrel{(5.4)}{=} \wedge_{C \in \Omega} (\wedge E_C) = \wedge_{C \in \Omega} e_C \in D \subseteq T.$$

If not then, for each $D \in \Omega$, there is $C \in \Omega$ such that $e_C \notin D$. Thus $\wedge C \leq e_C < \wedge D$. Hence

$$\wedge K \stackrel{(5.4)}{=} \wedge_{C \in \Omega} (\wedge C) \leq \wedge_{C \in \Omega} e_C \leq \wedge_{D \in \Omega} \wedge D = \wedge K.$$

Therefore $\wedge E = \wedge_{C \in \Omega} e_C = \wedge K \in T$, so that T is \wedge -complete. By Lemma 4.5(iii), T is complete. Thus $T \in \Gamma_S$ and T is a supremum of Ω . By Zorn's lemma, Γ_S has a maximal element – a complete, lower \ll -gap chain C in G that down-extends S .

Set $\mathfrak{p} = \wedge C$. As C is complete, $\mathfrak{p} \in C \subseteq G$. If $x \ll \mathfrak{p}$ for some $x \in G$, $\mathfrak{p} \neq x$, then $C \cup \{x\} \in \Gamma_S$ and larger than C . As C is maximal, such x does not exist. So $G \cap [\ll, \mathfrak{p}] = \{\mathfrak{p}\}$.

Let G be a lower \ll -set. If $\mathfrak{p} \neq \wedge G$ then there is $x \in G$ such that $x \neq \mathfrak{p}$ and $x \ll \mathfrak{p}$ which contradicts (4.3). Hence $\mathfrak{p} = \wedge G$. The chain $S = \{b\}$ is a complete, lower \ll -gap chain in G containing b . By (i) and the above argument, there is a complete, lower \ll -gap chain C in G that down-extends S and $\mathfrak{p} = \wedge C = \wedge G$. ■

Let γ be an ordinal number. A set $(x_\lambda)_{1 \leq \lambda \leq \gamma}$ in Q is a *descending* (respectively, an *ascending*) \ll -series from a to b if

$$\begin{aligned} & x_{\lambda+1} \ll x_\lambda \text{ for } \lambda < \gamma, \quad x_\beta = \wedge_{\lambda < \beta} (x_\lambda) \text{ for limit ordinals } \beta, \quad x_1 = b, \quad x_\gamma = a; \\ & \text{respectively, } x_\lambda \ll x_{\lambda+1} \text{ for } \lambda < \gamma, \quad x_\beta = \vee_{\lambda < \beta} (x_\lambda) \text{ for limit ordinals } \beta, \quad x_1 = a, \quad x_\gamma = b. \end{aligned} \quad (4.4)$$

Proposition 4.8 *Let $\ll \in \text{Rel}(Q, \leq)$ and C be a complete chain in Q from a to b . Then*

- (i) *C is an upper \ll -gap chain if and only if it is an ascending \ll -series.*
- (ii) *C is a lower \ll -gap chain if and only if it is a descending \ll -series.*

Proof. (i) Any ascending \ll -series $(x_\lambda)_{1 \leq \lambda \leq \gamma}$ is, clearly, a \vee -complete, upper \ll -gap chain. By dual to Lemma 4.5(iii), it is complete. Conversely, let C be a complete, upper \ll -gap chain from a to b . Let $\emptyset \neq G \neq \{b\}$ be any subset of C . Then $\wedge G \in C$, as C is complete, and $\wedge G \neq b$. Hence there is $d \in C$ such that $[\wedge G, d] \cap C$ is a gap. So $\wedge G \in G$.

Suppose that, for some ordinal β ,

$$C = C_\beta \cup G_\beta, \quad G_\beta \cap C_\beta = \emptyset \text{ and } \vee C_\beta \leq \wedge G_\beta, \quad (4.5)$$

where $C_\beta = (c_\alpha)_{\alpha < \beta}$ is an ascending \ll -series. Let $G_\beta \neq \emptyset$. If there is α_0 such that $\beta = \alpha_0 + 1$ then c_{α_0} is the largest element in C_β . As C is an upper \ll -gap chain, take $c_\beta = c_{\alpha_0+1}$ equal to an immediate \ll -successor of c_{α_0} . Clearly, c_β belongs to G_β . If such α_0 does not exist then β is a limit ordinal. Set $c_\beta = \wedge G_\beta$. By the above, $c_\beta \in G_\beta$ and $c_\alpha < c_\beta$ for all $\alpha < \beta$. In both cases, extend C_β to the ascending \ll -series $C_{\beta+1} = (c_\alpha)_{\alpha < \beta+1}$ and restrict G_β to $G_{\beta+1} = G_\beta \setminus \{c_\beta\}$. Then $C = C_{\beta+1} \cup G_{\beta+1}$.

Let γ be a limit ordinal. Suppose that there are $\{C_\beta, G_\beta\}$, $\beta < \gamma$, satisfying (4.5). Set $C_\gamma = \cup_{\beta < \gamma} C_\beta$ and $G_\gamma = \cap_{\beta < \gamma} G_\beta$. It is easy to see that $C = C_\gamma \cup G_\gamma$, $C_\gamma \cap G_\gamma = \emptyset$ and $c \leq g$ for all $c \in C_\gamma$ and $g \in G_\gamma$, so that $C_\gamma \leq \wedge G_\gamma$. Thus $\{C_\gamma, G_\gamma\}$ satisfies (4.5). Moreover, C_γ is an ascending \ll -series. Thus, by transfinite induction, (4.5) holds for all ordinals. As C is a set, find an ordinal γ such that $G_\gamma = \emptyset$. Then $C = C_\gamma$ is an ascending \ll -series and $b = \vee C = a_\gamma$. The proof of (ii) is similar. ■

For each $\ll \in \text{Ref}(Q)$, define the reflexive relations \ll^\triangleleft and \ll^\triangleright from $\text{Ref}(Q)$:

$$\begin{aligned} a \ll^\triangleleft b & \text{ if } a < b \text{ and there is a complete lower } \ll \text{-gap chain from } a \text{ to } b; \\ a \ll^\triangleright b & \text{ if } a < b \text{ and there is a complete upper } \ll \text{-gap chain from } a \text{ to } b. \end{aligned} \quad (4.6)$$

If $a \ll b$ then $\{a, b\}$ is a complete, lower and upper \ll -gap chain, i.e., $a \ll^\triangleleft b$ and $a \ll^\triangleright b$. Thus

$$\ll \subseteq \ll^\triangleleft \text{ and } \ll \subseteq \ll^\triangleright \text{ (see (1.1)).} \quad (4.7)$$

Theorem 4.9 (i) *Let \ll be a relation from $\text{Ref}(Q)$. Then \ll^\triangleleft and \ll^\triangleright are orders,*

$$\ll^{\text{lo}} \subseteq \ll^\triangleleft = (\ll^\triangleleft)^\triangleleft \text{ and } \overrightarrow{\ll} = \overrightarrow{\ll^\triangleleft}; \quad \ll^{\text{up}} \subseteq \ll^\triangleright = (\ll^\triangleright)^\triangleright \text{ and } \overleftarrow{\ll} = \overleftarrow{\ll^\triangleright}. \quad (4.8)$$

(ii) *If \ll is a down-expanded order, $\ll = \ll^\triangleleft$. If \ll is a dual \mathbf{T} -order then $\ll = \ll^\triangleleft = \ll^{\text{lo}}$.*

(iii) *If \ll is an up-expanded order then $\ll = \ll^\triangleright$. If \ll is a \mathbf{T} -order then $\ll = \ll^\triangleright = \ll^{\text{up}}$.*

Proof. (i) Clearly, \ll^\triangleleft and \ll^\triangleright are transitive, so they are orders.

If $a \ll^{\text{lo}} b$ then $[a, b]$ is a complete lower \ll -set. Hence, by Proposition 4.7, there is a complete, lower \ll -gap chain from a to b . Thus $a \ll^\triangleleft b$, so that $\ll^{\text{lo}} \subseteq \ll^\triangleleft$.

By (4.7), $\ll^\triangleleft \subseteq (\ll^\triangleleft)^\triangleleft$. Conversely, let $a (\ll^\triangleleft)^\triangleleft b$, $a \neq b$. Then there is a descending \ll^\triangleleft -series $(x_\alpha)_{1 \leq \alpha \leq \gamma}$ such that $x_1 = b$ and $x_\gamma = a$. For each ordinal α , $x_{\alpha+1} \ll^\triangleleft x_\alpha$, so that there is a descending \ll -series $(u_{\alpha\beta})_{1 \leq \beta \leq \gamma(\alpha)}$ such that $u_{\alpha 1} = x_\alpha$ and $u_{\alpha \gamma(\alpha)} = x_{\alpha+1}$. Renumbering $\cup_{1 \leq \alpha \leq \gamma} (u_{\alpha\beta})_{1 \leq \beta \leq \gamma(\alpha)}$, we get a descending \ll -series $(v_\lambda)_{1 \leq \lambda \leq \delta}$ such that $v_1 = b$ and $v_\delta = a$. This shows that $(\ll^\triangleleft)^\triangleleft \subseteq \ll^\triangleleft$. So $(\ll^\triangleleft)^\triangleleft = \ll^\triangleleft$.

As $\ll \subseteq \ll^\triangleleft$ by (4.7), we have $\overrightarrow{\ll} \subseteq \overrightarrow{\ll^\triangleleft}$ by Proposition 4.1(iii).

Let $a \overrightarrow{\ll} b$. Suppose that $x \ll^{\triangleleft} b$ for some $x \in [a, b)$. By (4.6), there is a complete lower \ll -gap chain C from x to b . Hence, by Definition 4.4, b has an immediate \ll -predecessor $p \in C$: $a \leq x \leq p < b$ and $p \ll b$. This contradicts $a \overrightarrow{\ll} b$. So there is no $x \in [a, b)$ such that $x \ll^{\triangleleft} b$, i.e., $a \overrightarrow{\ll} b$. Hence $\overrightarrow{\ll} \subseteq \overrightarrow{\ll}^{\triangleleft}$ whence $\overrightarrow{\ll} = \overrightarrow{\ll}^{\triangleleft}$ which completes the proof of the first part of (4.8).

The proof of the first part of (4.8) is similar.

(ii) By (4.7), $\ll \subseteq \ll^{\triangleleft}$. Let $a \ll^{\triangleleft} b$. By (4.6) and Proposition 4.8, there is a descending \ll -series $(x_\lambda)_{1 \leq \lambda \leq \gamma}$ from a to b : $x_{\lambda+1} < x_\lambda$, $x_{\lambda+1} \ll x_\lambda$ for $\lambda < \gamma$, $x_\beta = \bigwedge_{\lambda < \beta} (x_\lambda)$ for limit ordinals β , $x_1 = b$, $x_\gamma = a$. Suppose that $x_\lambda \ll b$ for some λ . Then $x_{\lambda+1} \ll x_\lambda \ll b$. As \ll is an order, $x_{\lambda+1} \ll b$. Let β be a limit ordinal and $x_\lambda \ll b$ for all $\lambda < \beta$. As \ll is down-expanded, $x_\beta = \bigwedge_{\lambda < \beta} (x_\lambda) \ll b$. By transfinite induction, $a = x_\gamma \ll b$. Thus $\ll^{\triangleleft} \subseteq \ll$ whence $\ll = \ll^{\triangleleft}$.

If \ll is a dual \mathbf{T} -order, it is down-contiguous. Hence $\ll \subseteq \ll^{\text{lo}}$ by Proposition 4.3. By (4.8), $\ll^{\text{lo}} \subseteq \ll^{\triangleleft}$. As $\ll = \ll^{\triangleleft}$, we have $\ll = \ll^{\text{lo}} = \ll^{\triangleleft}$.

Part (iii) is proved by duality. ■

Even if \ll is contiguous, the relations \ll^{\triangleright} and \ll^{\triangleleft} are not necessarily contiguous.

Example 4.10 Let $Q = \{0, a, b, 1\}$, $0 < a < 1$, $0 < b < 1$. Let \ll be a reflexive relation in Q , $0 \ll a \ll 1$ and $0 \not\ll 1$. Then \ll is contiguous, while \ll^{\triangleright} and \ll^{\triangleleft} are not contiguous.

Indeed, $0 \ll^{\triangleright} 1$. However, $[0, 1] \not\subseteq [\ll^{\triangleright}, 1]$ and $[0, 1] \not\subseteq [0, \ll^{\triangleright}]$, since $b \in [0, 1]$, $0 \not\ll^{\triangleright} b$ and $b \not\ll^{\triangleright} 1$. Thus \ll^{\triangleright} is not contiguous. Similarly, \ll^{\triangleleft} is not contiguous. ■

If, however, Q is a complete chain and \ll is contiguous then \ll^{\triangleright} and \ll^{\triangleleft} are contiguous.

Lemma 4.11 Let Q be a complete chain. If \ll from $\text{Ref}(Q)$ is up-contiguous then the relations \ll^{\triangleright} , \ll^{\triangleleft} are up-contiguous. If \ll is down-contiguous, \ll^{\triangleright} , \ll^{\triangleleft} are down-contiguous.

Proof. Let $a \ll^{\triangleright} b$ and let T be a complete upper \ll -gap chain from a to b . Then $a \in [\ll^{\triangleright}, b]$. Let $x \in (a, b)$. Set $c = \bigvee \{t \in T: t \leq x\}$ and $c' = \bigwedge \{t \in T: x \leq t\}$. As T is complete, $c, c' \in T$ and $c \leq x \leq c'$. If there is $r \in T$ such that $c < r < c'$, then either $x \leq r$, or $r < x$, as Q is a chain. If $x \leq r$ then $c' \leq r$, a contradiction. Similarly, the condition $r < x$ gives a contradiction. Thus $[c, c']_T$ is a gap, so that $c' = c_s$ is the immediate \ll -successor of c in T .

Let \ll be up-contiguous. Then $c \ll c_s$ and $c \leq x \leq c_s$ imply $x \ll c_s$. Hence $\{x\} \cup \{t \in T: c_s \leq t\}$ is a complete upper \ll -gap chain from x to b . Thus $x \ll^{\triangleright} b$ and \ll^{\triangleright} is up-contiguous.

If \ll is down-contiguous then $c \ll c_s$ and $c \leq x \leq c_s$ imply $c \ll x$. Hence $\{t \in T: t \leq c\} \cup \{x\}$ is a complete upper \ll -gap chain from a to x . Thus $a \ll^{\triangleright} x$ and \ll^{\triangleright} is down-contiguous.

Similarly, if \ll be down-contiguous then \ll^{\triangleright} , \ll^{\triangleleft} are down-contiguous. ■

By Theorem 4.9, if \ll is a down-expanded order then $\ll^{\triangleleft} = \ll$ is down-expanded. If, however, \ll is **not** down-expanded, \ll^{\triangleleft} is not necessarily down-expanded even if Q is a complete chain.

Example 4.12 Let Q be a complete chain. If \ll is **not** down-expanded then \ll^{\triangleleft} is not necessarily down-expanded. If \ll is **not** up-expanded then \ll^{\triangleright} is not necessarily up-expanded.

Indeed, let $Q = 0 \cup \{\frac{1}{n}\}_{n=1}^\infty$ be a subset of $[0, 1]$ with usual order \leq . Then (Q, \leq) is a complete chain. Let \ll be a reflexive relation in Q such that only $\frac{1}{n} \ll 1$ for all $n = 1, 2, \dots$. Then

$$\ll \text{ is an order, } \ll = \ll^{\triangleleft} = \ll^{\triangleright} \text{ and } [\ll^{\triangleleft}, 1] = [\ll, 1] = \{1/n\}_{n=1}^\infty.$$

Hence $\bigwedge [\ll^{\triangleleft}, 1] = 0 \notin [\ll^{\triangleleft}, 1]$. Thus $[\ll^{\triangleleft}, 1]$ is not \wedge -complete. So \ll, \ll^{\triangleleft} are not down-expanded.

Similar example shows that if \ll is not up-expanded, \ll^{\triangleleft} is not necessarily up-expanded. ■

While a relation \ll may have neither \ll -radicals, nor dual \ll -radicals, the relation \ll^{\triangleright} always has \ll^{\triangleright} -radicals and the relation \ll^{\triangleleft} always has dual \ll^{\triangleleft} -radicals in all $[a, b] \subseteq Q$.

Corollary 4.13 *For $\ll \in \text{Ref}(Q)$, each interval $[a, b] \subseteq Q$ has dual \ll^{\triangleleft} -radicals and \ll^{\triangleright} -radicals.*

Proof. The chain $S = \{b\}$ is a complete, lower \ll -gap chain in $[a, b]$ containing b . It follows from Proposition 4.7 that there is a complete, lower \ll -gap chain C in $[a, b]$ that down-extends S and $[a, b] \cap [\ll, \mathfrak{p}] = \{\mathfrak{p}\}$, where $\mathfrak{p} = \wedge C$. Hence $a \ll \mathfrak{p} \ll^{\triangleleft} b$ and $[a, \mathfrak{p}] \cap [\ll, \mathfrak{p}] = \{\mathfrak{p}\}$ by (2.3). By (4.8), $\ll \ll^{\triangleleft}$. So $a \ll \mathfrak{p}$ and \mathfrak{p} is a dual \ll^{\triangleleft} -radical in $[a, b]$. Similarly, \ll^{\triangleright} -radicals exist. ■

For the uniqueness of the radicals in Corollary 4.13 we need some extra conditions.

5 H-relations \ll and the corresponding \ll^{\triangleleft} and \ll^{\triangleright} relations.

We constructed above the relations \ll^{lo} , \ll^{up} , \ll^{\triangleleft} and \ll^{\triangleright} . In this section we show that, if \ll is an **H**-relation then $\ll^{\text{up}} = \ll^{\triangleright}$ is an **R**-order; if \ll is a dual **H**-relation, $\ll^{\text{lo}} = \ll^{\triangleleft}$ is a dual **R**-order.

The presence of upper (lower) \ll -chains effects the structure of lattices.

Proposition 5.1 *Let \ll be a dual **H**-relation in Q . Let C be a \wedge -complete, lower \ll -chain in a \wedge -complete set $G \subseteq Q$. Let $b = \vee C \in C$ and $a = \wedge C$. Then*

- (i) *For each $z \in [\mathbf{0}, b] \setminus [\mathbf{0}, a]$, there is $c \in C$ such that $z \neq c \wedge z$ and $c \wedge z \ll z$. In particular, the sets $[\mathbf{0}, b] \setminus [\mathbf{0}, a]$ and $[a, b]$ are lower \ll -sets.*
- (ii) *Each chain in $[a, b]$ larger than C is a lower \ll -chain.*
- (iii) *There is a maximal chain S in $G \cap [a, b]$ containing C ; it is a \wedge -complete, lower \ll -chain. If G is complete, S is complete.*

Proof. (i) Let $M = \{y \in C : z \leq y\}$. Then $M \neq \emptyset$, as $b \in M$. Let $d = \wedge M$. Then $z \leq d \in C$, since C is \wedge -complete, and $a < d$, as $z \notin [\mathbf{0}, a]$. Hence there is $c \in C$ such that $d \neq c \ll d$. Then $c \notin M$. By Lemma 3.1, $c \wedge z \ll d \wedge z = z$. Moreover, $c \wedge z \neq z$, since otherwise $z \leq c$ and $c \in M$. Thus $[\mathbf{0}, b] \setminus [\mathbf{0}, a]$ is a lower \ll -set.

If $z \in [a, b]$ then $c \wedge z \in [a, b]$. So $[a, b]$ is a lower \ll -set.

(ii) Let D be a chain in $[a, b]$, $C \subset D$ and $a \neq z \in D \setminus C$. By (i), there is $c \in C$ such that $z \neq c \wedge z \ll z$. As D is a chain, $c = c \wedge z$. Thus $z \neq c \ll z$, so that D is a lower \ll -chain.

(iii) By Lemma 4.5, S exists and it is \wedge -complete. If G is complete, S is complete. By (ii), S is a lower \ll -chain. ■

Let G be a \wedge -complete set in Q and $b := \vee G \in G$. It follows from Proposition 4.7 that, for different complete, lower \ll -gap chains S in G containing b (for example, $S = \{b\}$), there are maximal down-extended complete, lower \ll -gap chains C_S in G that down-extend S and the elements $\mathfrak{p}_{C_S} = \wedge C_S$ satisfy (4.3). We will show now that if \ll is a dual **H**-relation then all \mathfrak{p}_{C_S} coincide.

Theorem 5.2 *Let G be a \wedge -complete set in Q , $b := \vee G \in G$ and \ll be a dual **H**-relation. Then*

- (i) *There is a unique $\mathfrak{p} \in G$ satisfying $G \cap [\ll, \mathfrak{p}] = \{\mathfrak{p}\}$.*
- (ii) *If S is a complete, lower \ll -gap chain in G and $b \in S$ then $\mathfrak{p} \leq \wedge S$. If $\mathfrak{p} < \wedge S$, there is a*

maximal down-extended complete lower \ll -gap chain $C \subseteq G$ down-extending S , $\mathfrak{p} = \wedge C$.

(iii) The following conditions are equivalent.

- 1) G is a lower \ll -set;
- 2) There is a \wedge -complete, lower \ll -chain C in G from a to b and $b \in C$;
- 3) There is a complete, lower \ll -gap chain S in G from a to b and $b \in S$.

Proof. (i) and (ii). Let S and T be complete, lower \ll -gap chains in G containing b (for example, $S = \{b\}$). By Proposition 4.7, there is a maximal down-extended complete, lower \ll -gap chain C in G that down-extends S and $\mathfrak{p}_C = \wedge C$ satisfies

$$G \cap [\ll, \mathfrak{p}_C] = \{\mathfrak{p}_C\}. \quad (5.1)$$

Let D be another maximal down-extended complete, lower \ll -gap chain in G that either down-extends S , or T . Set $\mathfrak{p}_D = \wedge D$. If $\mathfrak{p}_C \notin [\mathbf{0}, \mathfrak{p}_D]$ then, as \ll is a dual \mathbf{H} -relation, it follows from Proposition 5.1(i) that there is $z \in D \subseteq G$ such that $\mathfrak{p}_C \neq z \wedge \mathfrak{p}_C \ll \mathfrak{p}_C$ which contradicts (5.1). Thus $\mathfrak{p}_C \in [\mathbf{0}, \mathfrak{p}_D]$. Similarly, $\mathfrak{p}_D \in [\mathbf{0}, \mathfrak{p}_C]$ whence $\mathfrak{p}_D = \mathfrak{p}_C$. Thus \mathfrak{p}_C is uniquely defined.

(iii) As \ll is a dual \mathbf{H} -relation, 2) \Rightarrow 1) follows from Proposition 5.1(i).

3) \Rightarrow 2) is evident. As G is a lower \ll -set, 1) \Rightarrow 3) follows from Proposition 4.7. ■

The following corollary gives an analogue of Theorem 5.2(i) and (ii) for lower \ll -chains.

Corollary 5.3 *Let G, b, \ll and \mathfrak{p} be as in Theorem 5.2. If $T \subseteq G$ is a \wedge -complete, lower \ll -chain containing b then $\mathfrak{p} \leq \wedge T$. If $\mathfrak{p} < \wedge T$, there is a maximal down-extended \wedge -complete lower \ll -chain \mathcal{T} in G from \mathfrak{p} to b that down-extends T and $\wedge \mathcal{T} = \mathfrak{p}$.*

Proof. Set $t = \wedge T \in T \subseteq G$. As \ll is a dual \mathbf{H} -relation, it follows from Theorem 5.2 (iii) that there is a complete, lower \ll -gap chain S in G from t to b . By Theorem 5.2(ii), $\mathfrak{p} \leq t$. If $\mathfrak{p} < t$ then there is a complete, lower \ll -gap chain C in G from \mathfrak{p} to b that down-extends S . Then $T' = C \cap [\mathfrak{p}, t]$ is a complete, lower \ll -gap chain from \mathfrak{p} to t . Hence $\mathcal{T} = T' \cup T$ is a \wedge -complete, lower \ll -chain in G from \mathfrak{p} to b that down-extends T . As $\wedge \mathcal{T} = \mathfrak{p}$, \mathcal{T} can not be down-extended. So \mathcal{T} is maximal down-extended. ■

The results analogous to Theorem 5.2 and Corollary 5.3 hold by duality for upper \ll -chains and upper \ll -gap chains if \ll is an \mathbf{H} -relation.

To refine the results of Theorem 4.9 for \mathbf{H} -relations, we define for each $a \in Q$, the maps Υ_a, λ_a on Q by setting

$$\Upsilon_a(x) = a \vee x \text{ and } \lambda_a(x) = a \wedge x \text{ for } x \in Q. \quad (5.2)$$

For $G \subseteq Q$, let $\Upsilon_a(G) = \{\Upsilon_a(x) : x \in G\}$ and $\lambda_a(G) = \{\lambda_a(x) : x \in G\}$. Then

$$\lambda_a(\wedge G) = \wedge \lambda_a(G) \text{ and } \Upsilon_a(\vee G) = \vee \Upsilon_a(G). \quad (5.3)$$

For example, $\lambda_a(\wedge G) \leq \lambda_a(g)$ for $g \in G$. So $\lambda_a(\wedge G) \leq \wedge \lambda_a(G)$. Conversely, $\wedge(\lambda_a(G)) \leq g$ for $g \in G$, so that $\wedge(\lambda_a(G)) \leq \wedge G$. As $\wedge(\lambda_a(G)) \leq a$, we have $\wedge(\lambda_a(G)) \leq \lambda_a(\wedge G)$. Thus $\lambda_a(\wedge G) = \wedge \lambda_a(G)$.

We also have (see [Sk, Theorem 1.4]) that

$$\text{if } G = \cup_{\alpha \in A} G_\alpha \text{ then } \wedge G = \wedge \{\wedge G_\alpha : \alpha \in A\} \text{ and } \vee G = \vee \{\vee G_\alpha : \alpha \in A\}. \quad (5.4)$$

Proposition 5.4 *Let \ll be a dual \mathbf{H} -relation and let $G, \{G_\alpha\}_{\alpha \in A}$ be \wedge -complete lower \ll -sets.*

(i) *For each $a \in Q$, the set $\lambda_a(G)$ is a \wedge -complete, lower \ll -set.*

(ii) *Let $b = \vee G_\alpha \in G_\alpha$ for all $\alpha \in A$. Then the \wedge -completion $(\cup_{\alpha \in A} G_\alpha)^\wedge$ of $\cup_{\alpha \in A} G_\alpha$ (see (2.1)) is a lower \ll -set.*

Proof. (i) Let $\wedge \lambda_a(G) < \lambda_a(g)$ for some $g \in G$. Set $E_g = \{z \in G: \lambda_a(g) \leq \lambda_a(z)\}$. Then $g \in E_g$ and $e_g := \wedge E_g \in G$, as G is \wedge -complete. Moreover, $e_g \in E_g$ and $\lambda_a(g) = \lambda_a(e_g)$, since

$$\lambda_a(e_g) = \lambda_a(\wedge E_g) \stackrel{(5.3)}{=} \wedge(\lambda_a(E_g)) = \lambda_a(g), \text{ as } g \in E_g \text{ and } \lambda_a(g) \leq \lambda_a(z) \text{ for } z \in E_g.$$

We have $\lambda_a(\wedge G) \stackrel{(5.3)}{=} \wedge \lambda_a(G) < \lambda_a(g) = \lambda_a(e_g)$ by above. As $\wedge G \leq e_g$, it follows that $\wedge G < e_g$. As G is a lower \ll -set, there is $h \in G$ such that $h \ll e_g$ and $h < e_g$. As e_g is minimal in E_g , $\lambda_a(h) \neq \lambda_a(g)$. So $\lambda_a(h) < \lambda_a(e_g) = \lambda_a(g)$. As \ll is a dual \mathbf{H} -relation, $\lambda_a(h) = a \wedge h \ll a \wedge e_g = \lambda_a(e_g) = \lambda_a(g)$, i.e., $\lambda_a(h)$ is a \ll -predecessor of $\lambda_a(g)$. Thus $\lambda_a(G)$ is a lower \ll -set.

For $N \subseteq \lambda_a(G)$, there is $M \subseteq G$ with $N = \lambda_a(M)$. As G is \wedge -complete, $\wedge M \in G$. Hence $\wedge N = \wedge \lambda_a(M) \stackrel{(5.3)}{=} \lambda_a(\wedge M) \in \lambda_a(G)$. Thus $\lambda_a(G)$ is \wedge -complete.

(ii) Set $K = \cup_{\alpha \in A} G_\alpha$. Let $x \in K^\wedge$ be such that $\wedge K < x$. Then $x = \wedge M$ for some $M \subseteq K$. Let $F_\alpha = M \cap G_\alpha$, $N_\alpha = G_\alpha \setminus F_\alpha$ and $n_\alpha = \wedge N_\alpha$ for all $\alpha \in A$. Then $K = (\cup_\alpha N_\alpha) \cup M$ and

$$\wedge K \stackrel{(5.4)}{=} (\wedge \{n_\alpha: \alpha \in A\}) \wedge x = \wedge \{n_\alpha \wedge x: \alpha \in A\}.$$

If $n_\alpha \wedge x = x$ for all α , then $\wedge K = x$ – a contradiction. Thus $n_\beta \wedge x < x$ for some β . As $N_\beta \subseteq G_\beta$, we have $\wedge \lambda_x(G_\beta) = \lambda_x(\wedge G_\beta) \leq n_\beta \wedge x < x$. As $x \leq b \in G_\beta$, we have $x = \lambda_x(b) \in \lambda_x(G_\beta)$. By (i), $\lambda_x(G_\beta)$ is a \wedge -complete, lower \ll -set. Hence there is $u \in G_\beta$ such that $\lambda_x(u) \ll \lambda_x(b)$ and $\lambda_x(u) \neq \lambda_x(b)$, i.e., $x \neq x \wedge u \ll x$. As $x \wedge u \in K^\wedge$, K^\wedge is a lower \ll -set. ■

We will use Proposition 5.4 to prove the main results of this section.

Theorem 5.5 (i) *Let \ll be a dual \mathbf{H} -relation. Then*

- 1) $\ll^\triangleleft = \ll^{\text{lo}}$ is a dual \mathbf{R} -order and $\overrightarrow{\ll} = \overleftarrow{\ll}^\triangleleft = (\overrightarrow{\ll})^\triangleright = (\overleftarrow{\ll})^{\text{up}}$ is an \mathbf{R} -order;
- 2) for each $[a, b] \subseteq Q$, there is a unique dual \ll^\triangleleft -radical $\mathfrak{p} = \mathfrak{p}_{[a, b]}$ such that $a \overrightarrow{\ll} \mathfrak{p} \ll^\triangleleft b$.
Moreover, $a \leq x \ll^\triangleleft b$ implies $x \in [\mathfrak{p}, b]$.

(ii) *Let \ll be an \mathbf{H} -relation and $[a, b] \subseteq Q$. Then*

- 1) $\ll^\triangleright = \ll^{\text{up}}$ is an \mathbf{R} -order, and $\overleftarrow{\ll} = \overrightarrow{\ll}^\triangleright = (\overleftarrow{\ll})^\triangleleft = (\overrightarrow{\ll})^{\text{lo}}$ is a dual \mathbf{R} -order;
- 2) for each $[a, b] \subseteq Q$, there is a unique \ll^\triangleright -radical $\mathfrak{r} = \mathfrak{r}_{[a, b]}$ such that $a \overleftarrow{\ll} \mathfrak{r} \ll^\triangleright b$.
Moreover, $a \ll^\triangleright x \leq b$ implies $x \in [a, \mathfrak{r}]$.

Proof. (i) 1) Let \ll be a dual \mathbf{H} -relation. Firstly, let us show that \ll^\triangleleft is down-expanded. Let $G = [\ll^\triangleleft, c]$ for some $c \in Q$. For each $g \in G \setminus \{c\}$, $G_g = \{g, c\}$ is a complete, lower \ll^\triangleleft -set and $G = \cup_{g \in G \setminus \{c\}} G_g$. Hence G^\wedge is a \wedge -complete, lower \ll^\triangleleft -set by Proposition 5.4. As $\wedge(G^\wedge) = \wedge G$, it follows from Theorem 5.2(iii) that there is a complete lower \ll^\triangleleft -gap chain C in G^\wedge such that $\wedge C = \wedge G$ and $\vee C = c$. Hence $\wedge G (\ll^\triangleleft)^\triangleleft c$. As $\ll^\triangleleft = (\ll^\triangleleft)^\triangleleft$ by Theorem 4.9, we have $\wedge G \ll^\triangleleft c$. Thus $[\ll^\triangleleft, c]$ is \wedge -complete whence \ll^\triangleleft is down-expanded.

It follows from Theorem 5.2(iii) that $[a, b]$ is a lower \ll -set ($a \ll^{\text{lo}} b$ (see Definition 4.2)) if and only if $a \ll^{\heartsuit} b$. Thus $\ll^{\heartsuit} = \ll^{\text{lo}}$ is down-expanded.

Let us now prove that $\overrightarrow{\ll}$ is an \mathbf{H} -order. Suppose that $a \overrightarrow{\ll} b$ and $a \leq c$. Assume that $c \leq x \ll b \vee c$ for some x . As \ll is a dual \mathbf{H} -relation, $a \leq x \wedge b \ll (b \vee c) \wedge b = b$. As $a \overrightarrow{\ll} b$, we have $x \wedge b = b$ by (2.3). So $b \leq x$. Thus $x = b \vee c$. From this we conclude that $c \overrightarrow{\ll} b \vee c$. It follows from Lemma 3.1 that $\overrightarrow{\ll}$ is an \mathbf{H} -relation.

To show that $\overrightarrow{\ll}$ is an order, let $a \overrightarrow{\ll} b \overrightarrow{\ll} c$ and $a \leq x \ll c$. By Lemma 3.1, $a \leq x \wedge b \ll c \wedge b = b$. As $a \overrightarrow{\ll} b$, we have from (2.3) that $x \wedge b = b$. Hence $b \leq x \ll c$. As $b \overrightarrow{\ll} c$, we have from (2.3) that $x = c$. Thus $a \overrightarrow{\ll} c$ by (2.3). So $\overrightarrow{\ll}$ is transitive. Thus $\overrightarrow{\ll}$ is an \mathbf{H} -order.

Similarly, if \ll is an \mathbf{H} -relation, $\overleftarrow{\ll}$ is a dual \mathbf{H} -order. Hence if \ll is a dual \mathbf{H} -relation then $\overleftarrow{\overleftarrow{\ll}}$ is a dual \mathbf{H} -order. By Proposition 4.3, $\ll^{\text{lo}} = \overleftarrow{\overleftarrow{\ll}}$ whence \ll^{lo} is a dual \mathbf{H} -order. Combining this with the fact that $\ll^{\heartsuit} = \ll^{\text{lo}}$ is down-expanded, we get that $\ll^{\heartsuit} = \ll^{\text{lo}}$ is a dual \mathbf{R} -order.

By duality, $\ll^{\spadesuit} = \ll^{\text{up}}$ is an \mathbf{R} -order, if \ll is an \mathbf{H} -relation. Hence, as $\overrightarrow{\ll}$ is an \mathbf{H} -order by above, $(\overrightarrow{\ll})^{\spadesuit} = (\overrightarrow{\ll})^{\text{up}}$. Therefore, since $\ll^{\text{lo}} = \overleftarrow{\overleftarrow{\ll}}$ and $\ll^{\text{up}} = \overleftarrow{\overleftarrow{\ll}}$ by Proposition 4.3,

$$(\overrightarrow{\ll})^{\spadesuit} = (\overrightarrow{\ll})^{\text{up}} = \overrightarrow{\overleftarrow{\overleftarrow{\ll}}} = \overleftarrow{\ll^{\text{lo}}} = \overleftarrow{\ll^{\heartsuit}}. \quad (5.5)$$

By (4.7) and Proposition 4.1, $\overleftarrow{\ll^{\heartsuit}} \subseteq \overrightarrow{\ll}$. By Proposition 4.3, $\overrightarrow{\ll} \subseteq \overleftarrow{\ll^{\text{lo}}}$. As $\ll^{\heartsuit} = \ll^{\text{lo}}$, we have $\overleftarrow{\ll^{\heartsuit}} \subseteq \overrightarrow{\ll} \subseteq \overleftarrow{\ll^{\heartsuit}}$. So $\overrightarrow{\ll} = \overleftarrow{\ll^{\heartsuit}}$. By (5.5), $\overrightarrow{\ll} = (\overrightarrow{\ll})^{\spadesuit}$. So, as $\overrightarrow{\ll}$ is an \mathbf{H} -order, $(\overrightarrow{\ll})^{\spadesuit}$ is an \mathbf{R} -order by above. Thus $\overrightarrow{\ll}$ is an \mathbf{R} -order. The proof of 1) is complete.

2) As \ll^{\heartsuit} is a dual \mathbf{R} -order, it is a dual \mathbf{T} -order. By Theorem 2.4 and (2.5), for each $[a, b] \subseteq Q$ there is a unique dual \ll^{\heartsuit} -radical $\mathfrak{p} \in [a, b]$ such that $a \overleftarrow{\ll^{\heartsuit}} \mathfrak{p} \ll^{\heartsuit} b$. As $\overleftarrow{\ll^{\heartsuit}} = \overrightarrow{\ll}$ by 1), $a \overrightarrow{\ll} \mathfrak{p} \ll^{\heartsuit} b$.

Let $a \leq x \ll^{\heartsuit} b$. As \ll^{\heartsuit} is a dual \mathbf{R} -order, $a \leq x \wedge \mathfrak{p} \ll^{\heartsuit} b \wedge \mathfrak{p} = \mathfrak{p}$ by Lemma 3.1(ii). As $a \overrightarrow{\ll} \mathfrak{p}$ by above, we have from (2.3) that $x \wedge \mathfrak{p} = \mathfrak{p}$. Hence $\mathfrak{p} \leq x$.

The proof of part (ii) is similar. ■

The following corollary strengthens Theorem 2.5

Corollary 5.6 *Let \ll be a dual \mathbf{H} -relation. The following conditions are equivalent.*

- (i) \ll is a dual \mathbf{R} -order. (ii) $\ll = \ll^{\heartsuit}$.
- (iii) \ll is an order and $\bigwedge_{n \in \mathbb{N}} (x_n) \ll x_1$ for each descending \ll -series $(x_n)_{n \in \mathbb{N}}$.
- (iv) For each $[a, b] \subseteq Q$, there is $c \in [a, b]$ such that $a \overrightarrow{\ll} c \leq b$.
- (v) Each $[a, b] \subseteq Q$ has a unique dual \ll -radical.

Proof. (ii) \Rightarrow (i) As \ll is a dual \mathbf{H} -relation, \ll^{\heartsuit} is a dual \mathbf{R} -order by Theorem 5.5. So \ll is a dual \mathbf{R} -order.

(i) \Rightarrow (iii). As $x_{n+1} \ll x_n$ for all n , and \ll is transitive, $x_n \ll x_1$. As \ll is down-expanded by Definition 3.2, $\bigwedge (x_n)_{n \in \mathbb{N}} \ll x_1$.

(iii) \Rightarrow (ii). If $a \ll^{\heartsuit} b$, there is a descending \ll -series $(x_\alpha)_{1 \leq \alpha \leq \gamma}$ with $a = x_\gamma$ and $b = x_1$. If $x_\alpha \ll b$ for some α , then $x_{\alpha+1} \ll x_\alpha \ll b$ implies $x_{\alpha+1} \ll b$ by transitivity of \ll . Hence all $x_{\alpha+n} \ll b$. If β is a limit ordinal and α is the previous limit ordinal then, $x_\beta = \bigwedge_{n \in \mathbb{N}} (x_{\alpha+n}) \ll x_\alpha \ll b$. So $x_\beta \ll b$. Thus, by transfinite induction, $a \ll b$. So $\ll = \ll^{\heartsuit}$.

(i) \Rightarrow (iv). As dual \mathbf{R} -orders are dual \mathbf{T} -orders, it follows from Theorem 2.4 (ii) that each $[a, b] \subseteq Q$ has a unique dual \ll -radical \mathfrak{p} : $a \xrightarrow{\ll} \mathfrak{p} \ll b$ (see (2.5)). If $\mathfrak{p} = a$ then $a \ll b$. If $\mathfrak{p} \neq a$, set $c = \mathfrak{p}$. As $\mathfrak{p} \ll b$ implies $\mathfrak{p} \leq b$, (iv) holds.

(iv) \Rightarrow (ii). Let $a \ll^{\triangleleft} b$ for $a < b$. If $a \not\ll b$ then, by (iv), $a \xrightarrow{\ll} c \leq b$ for some $c \neq a$. On the other hand, by Theorem 5.5, $\ll^{\triangleleft} = \ll^{\text{lo}}$. Thus, by Definition 4.2, $[a, b]$ is a lower \ll -set, so that there is $d \in [a, b]$ such that $c \neq d \ll c$ which contradicts $a \xrightarrow{\ll} c$. Thus $a \ll b$. So $\ll^{\triangleleft} \subseteq \ll$. Hence (4.7) implies $\ll = \ll^{\triangleleft}$.

(i) \Rightarrow (v). As dual \mathbf{R} -orders are dual \mathbf{T} -orders, it follows from Theorem 2.5 that each $[a, b] \subseteq Q$ has a unique dual \ll -radical.

(v) \Rightarrow (iv). If $[a, b] \subseteq Q$ has a dual \ll -radical \mathfrak{p} then $a \xrightarrow{\ll} \mathfrak{p} \ll b$ by (2.5). So (iv) holds. \blacksquare

It should be noted that Amitsur [A-I] got some results equivalent to the implication (ii) \Leftrightarrow (iv).

By duality, the following results are equivalent for an \mathbf{H} -relation \ll :

(i) \ll is an \mathbf{R} -order; (ii) $\ll = \ll^{\triangleright}$; (iii) Each $[a, b] \subseteq Q$ has a unique \ll -radical.

If \ll is an \mathbf{H} -relation in Q , it follows from Theorem 5.5 that \ll^{\triangleright} is an \mathbf{R} -order and $\xleftarrow{\ll}$ is a dual \mathbf{R} -order in Q . Hence each $[a, b] \subseteq Q$ has a unique \ll^{\triangleright} -radical and a unique dual $\xleftarrow{\ll}$ -radical.

Similarly, if \ll is a dual \mathbf{H} -relation in Q , then \ll^{\triangleleft} is a dual \mathbf{R} -order and $\xrightarrow{\ll}$ is an \mathbf{R} -order in Q . So each $[a, b] \subseteq Q$ has a unique dual \ll^{\triangleleft} -radical and a unique $\xrightarrow{\ll}$ -radical.

Proposition 5.7 (i) Let \ll be an \mathbf{H} -relation. Then, for each $[a, b] \subseteq Q$,

- 1) the \ll^{\triangleright} -radical and the dual $\xleftarrow{\ll}$ -radical in $[a, b]$ coincide;
- 2) if there exists a \ll -radical in $[a, b]$, it coincides with the \ll^{\triangleright} -radical.

(ii) Let \ll be a dual \mathbf{H} -relation. Then, for each $[a, b] \subseteq Q$,

- 1) the dual \ll^{\triangleleft} -radical and the $\xrightarrow{\ll}$ -radical in $[a, b]$ coincide;
- 2) if there exists a dual \ll -radical in $[a, b]$, it coincides with the dual \ll^{\triangleleft} -radical.

Proof. (i) 1) By Theorem 5.5(ii), the \ll^{\triangleright} -radical \mathfrak{r} satisfies $a \ll^{\triangleright} \mathfrak{r} \xleftarrow{\ll} b$. By (2.5), the dual $\xleftarrow{\ll}$ -radical \mathfrak{p} in $[a, b]$ satisfy $a \xrightarrow{\ll} \mathfrak{p} \xleftarrow{\ll} b$. By Proposition 4.3, $\xleftarrow{\ll} = \ll^{\text{up}}$. By Theorem 5.5, $\ll^{\text{up}} = \ll^{\triangleright}$, as \ll is an \mathbf{H} -relation, So $\xleftarrow{\ll} = \ll^{\triangleright}$. Hence $a \xrightarrow{\ll} \mathfrak{p} \xleftarrow{\ll} b$ turns into $a \ll^{\triangleright} \mathfrak{p} \xleftarrow{\ll} b$. Comparing it to $a \ll^{\triangleright} \mathfrak{r} \xleftarrow{\ll} b$ and taking into account the uniqueness of the radicals, we get $\mathfrak{r} = \mathfrak{p}$.

2) If \mathfrak{r}_1 is the \ll -radical in $[a, b]$ then $a \ll \mathfrak{r}_1 \xleftarrow{\ll} b$. As $\ll \subseteq \ll^{\triangleleft}$ by (4.7), $a \ll^{\triangleleft} \mathfrak{r}_1 \xleftarrow{\ll} b$. Since (see (i)) $a \ll^{\triangleright} \mathfrak{r} \xleftarrow{\ll} b$, it follows from the uniqueness of the radical that $\mathfrak{r} = \mathfrak{r}_1$. \blacksquare

For a subset $F \subseteq \text{Ref}(Q)$, define the relations $\ll_{\wedge F} := \bigcap_{\ll \in F} \ll$ and $\ll_{\vee F} := \bigcup_{\ll \in F} \ll$ by

$$a \ll_{\wedge F} b \text{ if } a \ll b \text{ for all } \ll \text{ in } F; \quad a \ll_{\vee F} b \text{ if } a \ll b \text{ for some } \ll \text{ in } F. \quad (5.6)$$

They belong to $\text{Ref}(Q)$. In particular, for \prec and \ll , $a (\prec \cap \ll) b$ if and only if $a \prec b$ and $a \ll b$.

If \ll is an \mathbf{H} -relation and \prec is not an \mathbf{H} -relation, then $\prec \cap \ll$ is not necessarily an \mathbf{H} -relation.

Lemma 5.8 Let \ll be an \mathbf{H} -relation, let \prec, \sqsubset be relations in $\text{Ref}(Q)$ and \ll be stronger than \sqsubset (see (1.1)). If $\prec \cap \sqsubset$ is an \mathbf{H} -relation then $\prec \cap \ll$ is also an \mathbf{H} -relation.

The same is true for dual \mathbf{H} -relations.

Proof. Let $a (\prec \cap \ll) b$ for $a, b \in Q$. Then $a \prec b$ and $a \ll b$. As \ll is stronger than \sqsubset , we have $a \sqsubset b$. By (5.6), $a (\prec \cap \sqsubset) b$. Since $\prec \cap \sqsubset$ is an **H**-relation, $a \vee c (\prec \cap \sqsubset) b \vee c$ for all $c \in Q$, by Definition 3.2. Hence $a \vee c \prec b \vee c$ by (5.6). As \ll is an **H**-relation, we also have $a \vee c \ll b \vee c$. So $a \vee c (\prec \cap \ll) b \vee c$ by (5.6). Thus $\prec \cap \ll$ is a **H**-relation (see Definition 3.2). ■

For convenience, we summarize below some results about **H**-relations which will be used later.

Theorem 5.9 (i) *Let \ll be an **H**-relation and $a \in Q$. Then \ll^\triangleright is an **R**-order and*

- 1) $\mathfrak{r}(a) = \vee[\ll^\triangleright, a]$ is the \ll^\triangleright -radical in $[a, \mathbf{1}]$, i.e., $a \ll^\triangleright \mathfrak{r}(a) \lll \mathbf{1}$;
- 2) $\mathfrak{r}(b) = \mathfrak{r}(a)$ for all $b \in [a, \mathfrak{r}(a)]$ – there is an ascending \ll -series from b to $\mathfrak{r}(a)$;
- 3) each $b \in [a, \mathbf{1}] \setminus [\mathfrak{r}(a), \mathbf{1}]$ has a \ll -successor s_b : $b \ll s_b \neq b$; $\mathfrak{r}(a)$ has no \ll -successor.

(ii) *Let \ll be a dual **H**-relation and $b \in Q$. Then \ll^\triangleleft is a dual **R**-order and*

- 1) $\mathfrak{p}(b) = \wedge[\ll^\triangleleft, b]$ is the dual \ll^\triangleleft -radical in $[\mathbf{0}, b] \subseteq Q$, i.e., $\mathbf{0} \ggg \mathfrak{p}(b) \ll^\triangleleft b$;
- 2) $\mathfrak{p}(a) = \mathfrak{p}(b)$ for all $a \in [\mathfrak{p}(b), b]$ – there is a descending \ll -series from a to $\mathfrak{p}(b)$;
- 3) each $a \in [\mathbf{0}, b] \setminus [\mathbf{0}, \mathfrak{p}(b)]$ has a \ll -predecessor $p_a \ll a \neq p_a$; $\mathfrak{p}(b)$ has no \ll -predecessor.

Let Q be a complete lattice. For $a \in Q$, set

$$\sigma(a) = \wedge[\ll, a] \text{ and } s(a) = \vee[a, \ll]. \quad (5.7)$$

Then if \ll is an **H**-relation, we have

$$[a, \ll] \stackrel{(4.7)}{\subseteq} [a, \ll^\triangleright], \text{ so that } s(a) \leq \vee[a, \ll^\triangleright] \stackrel{(2.7)}{=} \mathfrak{r}_{\ll^\triangleright}(a). \quad (5.8)$$

If \ll is a dual **H**-relation, we have

$$[\ll, a] \stackrel{(4.7)}{\subseteq} [\ll^\triangleleft, a], \text{ so that } \sigma(a) \geq \wedge[\ll^\triangleleft, a] \stackrel{(2.6)}{=} \mathfrak{p}_{\ll^\triangleleft}(a). \quad (5.9)$$

A bijection $\theta: Q \rightarrow Q'$ is an isomorphism if $x \leq y \Leftrightarrow \theta(x) \leq \theta(y)$ for all $x, y \in Q$. Then

$$\theta(\vee G) = \vee \theta(G) \text{ and } \theta(\wedge G) = \wedge \theta(G) \text{ for each } G \subseteq Q. \quad (5.10)$$

Indeed, $\theta(x) \leq \theta(\vee G)$, as $x \leq \vee G$ for $x \in G$. Thus $\vee \theta(G) \leq \theta(\vee G)$. As θ^{-1} is also an isomorphism, $\vee G = \vee \theta^{-1}(\theta(G)) \leq \theta^{-1}(\vee \theta(G))$. So $\theta(\vee G) \leq \vee \theta(G)$. Hence $\theta(\vee G) = \vee \theta(G)$. Similarly, $\theta(\wedge G) = \wedge \theta(G)$.

Let θ be an automorphism of Q . It preserves a relation \ll from $\text{Ref}(Q)$ if

$$\theta(x) \ll \theta(y) \Leftrightarrow x \ll y \text{ for all } x, y \in Q. \quad (5.11)$$

For $x < y$ in Q , θ is an isomorphism from $[x, y]$ onto $[\theta(x), \theta(y)]$. Clearly, $[x, y]$ is a lower (upper) \ll -set if and only if $[\theta(x), \theta(y)]$ is a lower (upper) \ll -set. So $x \ll^{\text{lo}} y$ if and only if $\theta(x) \ll^{\text{lo}} \theta(y)$, and $x \ll^{\text{up}} y$ if and only if $\theta(x) \ll^{\text{up}} \theta(y)$. Thus θ preserves the relations \ll^{lo} and \ll^{up} .

Proposition 5.10 *Let an automorphism θ of Q preserve a relation \ll from $\text{Ref}(Q)$.*

- (i) *If \ll is an **H**-relation, θ preserves \ll^\triangleright ; if \ll is a dual **H**-relation, θ preserves \ll^\triangleleft .*
- (ii) *Let $\theta(a) = a$ for some $a \in Q$. Then $\theta(\sigma(a)) = \sigma(a)$ and $\theta(s(a)) = s(a)$. Moreover,*
 - 1) *if \ll is a **T**-order then $\theta(\mathfrak{r}(a)) = \mathfrak{r}(a)$; if it is a dual **T**-order then $\theta(\mathfrak{p}(a)) = \mathfrak{p}(a)$;*
 - 2) *if \ll is an **H**-relation, $\theta(\mathfrak{r}_{\ll^\triangleright}(a)) = \mathfrak{r}_{\ll^\triangleright}(a)$; if it is a dual **H**-relation, $\theta(\mathfrak{p}_{\ll^\triangleleft}(a)) = \mathfrak{p}_{\ll^\triangleleft}(a)$.*

Proof. (i) follows, as θ preserves \ll^{lo} and \ll^{up} , and as $\ll^{\text{lo}} = \ll^{\triangleleft}$, $\ll^{\text{up}} = \ll^{\triangleright}$ by Theorem 5.5.
(ii) As θ and θ^{-1} are automorphisms, $\theta([\ll, a]) = [\ll, a]$ and $\theta([a, \ll]) = [a, \ll]$. By (5.10),

$$\theta(\sigma(a)) = \theta(\wedge[\ll, a]) = \wedge[\ll, a] = \sigma(a) \text{ and } \theta(s(a)) = \theta(\vee[a, \ll]) = \vee[a, \ll] = s(a). \quad (5.12)$$

By (2.7), $s(a) = \mathfrak{r}(a)$ if \ll is a \mathbf{T} -order. By (2.6), $\sigma(a) = \mathfrak{p}(a)$ if \ll is a dual \mathbf{T} -order. So 1) follows from (5.12). Part 2) follows from (i) and 1). ■

6 Gap $<_{\mathfrak{g}}$ and continuity $<_{\mathfrak{c}}$ relations in complete lattices.

Definition 6.1 Let $a \leq b$ in Q . We write $a <_{\mathfrak{g}} b$ if either $a = b$, or $[a, b]$ is a gap.

We write $a <_{\mathfrak{c}} b$ if either $a = b$, or there is a complete **continuous** chain C from a to b : For all $x < y$ in C , there is $z \in C$ such that $x < z < y$, i.e., C has no gaps.

The relations $<_{\mathfrak{g}}$ and $<_{\mathfrak{c}}$ belong to $\text{Ref}(Q)$ and $<_{\mathfrak{c}}$ is an order.

Proposition 6.2 A complete lattice Q has no gaps if and only if $<_{\mathfrak{c}} = \leq$.

Proof. Clearly, if $<_{\mathfrak{c}} = \leq$ then Q has no gaps. Conversely, let $a < b$. By Lemma 4.5, there is a maximal complete chain C from a to b . If C is not continuous, there is a gap $[x, y]_C$ in C . As Q has no gaps, there is $z \in Q$, $z \notin C$, with $x < z < y$. Hence the chain $C \cup \{z\}$ is larger than C , a contradiction. Thus C is continuous. So $a <_{\mathfrak{c}} b$. Hence $\leq \subseteq <_{\mathfrak{c}}$. As $<_{\mathfrak{c}} \subseteq \leq$, we have $<_{\mathfrak{c}} = \leq$. ■

Example 6.3 In general, the order $<_{\mathfrak{c}}$ is neither contiguous, nor expanded.

(i) Let $Q = [0, 1] \cup \{a\}$, where $[0, 1] \subset \mathbb{R}$, $\mathbf{0} = 0$, $\mathbf{1} = 1$, $a \wedge t = \mathbf{0}$ and $a \vee t = \mathbf{1}$ for $t \in (0, 1)$. Then Q is a complete lattice and $\mathbf{0} <_{\mathfrak{c}} \mathbf{1}$. As $a \not<_{\mathfrak{c}} \mathbf{1}$, we have $[0, 1] \not\subseteq [a, \mathbf{1}]$. So $<_{\mathfrak{c}}$ is *not contiguous*.

(ii) Let Q_n , $2 \leq n$, be the interval in \mathbb{R}^2 from $(\frac{1}{n}, \frac{1}{n})$ to $(1, 0)$. Let $L = \cup_{n=2}^{\infty} Q_n$ and $(x, y) \leq (u, v)$ in L if they lie in the same Q_n and $x \leq u$. Then $\mathbf{1} = (1, 0)$. Set $Q = (0, 0) \cup L$ and $\mathbf{0} = (0, 0) < (x, y)$ for all $(x, y) \in L$. Then Q is a complete lattice, $[a, \mathbf{1}] = L$, $\mathbf{0} = \wedge[a, \mathbf{1}]$ and $\mathbf{0} \not<_{\mathfrak{c}} \mathbf{1}$. So $<_{\mathfrak{c}}$ is *not expanded*. ■

We need now the following lemma.

Lemma 6.4 Let T be a complete chain from a to b . Let each $t \in T \setminus \{b\}$ have an immediate successor t_s in T , i.e., $[t, t_s]_T = \{t, t_s\}$ is a gap, and let C_t be some complete chain from t to t_s . Set $C_b = \{b\}$. Then the chain $C = \cup_{t \in T} C_t$ is complete.

Proof. Let $G \subseteq C$. For $t \in T$, set $G_t = G \cap C_t$ and $Y = \{t \in T: G_t \neq \emptyset\}$.

Let $y = \vee Y$ and $c_t = \vee G_t$ for $t \in Y$. As T and all C_t are complete, $y \in T$ and $c_t \in C_t$. As $G = \cup_{t \in Y} G_t$, we have from (5.4) that $\vee G = \vee\{c_t: t \in Y\}$. If $y \in Y$ then $\vee G = c_y \in C_y \subseteq C$. If $y \notin Y$ then $t < t_s \leq y$ for all $t \in Y$, as $[t, t_s]_T$ is a gap. So $y = \vee\{t_s: t \in Y\}$. As $t \leq c_t \leq t_s$, we have $y = \vee Y \leq \vee G = \vee\{c_t: t \in Y\} \leq \vee\{t_s: t \in Y\} = y$. Hence $\vee G = y \in T \subseteq C$. Thus C is \vee -complete.

Let $z = \wedge Y$ and $g_t = \wedge G_t$ for $t \in Y$. As T and all C_t are complete, $z \in T$ and $g_t \in C_t$. As $G = \cup_{t \in Y} G_t$, we have from (5.4) that $\wedge G = \wedge\{g_t: t \in Y\}$. If $z \notin Y$ then $z < t$ for all $t \in Y$, and $z = \wedge Y$. As $[z, z_s]_T$ is a gap, we get a contradiction. Hence $z \in Y$. Then $G_z \neq \emptyset$ and $\wedge G = \wedge G_z = g_z \in C_z \subseteq C$. Thus C is \wedge -complete. So C is complete. ■

Lemma 6.4 also holds if each $t \in T \setminus \{a\}$ has an immediate predecessor t_p in T : $[t_p, t]_T$ is a gap.

Proposition 6.5 (i) $\prec_c = \prec_c^\triangleleft = \prec_c^\triangleright$.

(ii) Each complete continuous chain from a to b in Q is maximal in $[a, b]$.

Proof. (i) Let $a \prec_c^\triangleright b$. Then there is a complete upper \prec_c -gap chain T from a to b , i.e., each $t \in T \setminus \{b\}$ has an immediate \prec_c -successor t_s in T , $t \prec_c t_s$. Then there is a complete continuous chain C^t from t to t_s . By Lemma 6.4, $C = \{b\} \cup (\cup_{t \in T} C^t)$ is a complete chain from a to b .

If C is not continuous, there are $x < y$ in C such that $[x, y]_C$ is a gap. Let $t \in T$ be such that $x \in C^t \setminus \{t_s\}$. If $y \in C^t$ then $[x, y]_C = [x, y]_{C^t}$ is not a gap, as C^t is continuous. If $y \notin C^t$ then $y \neq t_s$ and $y \in C^u$, $t < u \in T$. Hence $x < t_s < y$. So $[x, y]_C$ is not a gap. Thus C is continuous and $a \prec_c b$. So $\prec_c^\triangleright \subseteq \prec_c$. As also $\prec_c \subseteq \prec_c^\triangleright$ (see (4.7)), $\prec_c = \prec_c^\triangleright$. Similarly, $\prec_c^\triangleleft = \prec_c$.

(ii) Let C be a complete continuous chain from a to b . If C is not maximal, there is $u \in Q$, $u \notin C$, such that either $z < u$ or $u < z$ for each $z \in C$. Let $c = \vee \{z \in C: z < u\}$ and $d = \wedge \{z \in C: u < z\}$. As C is complete, $c, d \in C$ and $c < u < d$. As C is continuous, there is $x \in C$ such that $c < x < d$. Hence either $x < u$, or $u < x$, a contradiction. Thus C is maximal. ■

For $a \in Q$, we defined in (5.2) the maps $\Upsilon_a(x) = a \vee x$ and $\lambda_a(x) = a \wedge x$ for $x \in Q$. Then

$$\Upsilon_a(x) \leq \Upsilon_a(y) \text{ and } \lambda_a(x) \leq \lambda_a(y) \text{ if } x \leq y. \quad (6.1)$$

For each $G \subseteq Q$, its images $\Upsilon_a(G)$ and $\lambda_a(G)$ satisfy (5.3).

A complete lattice (Q, \leq) is *modular* if

$$\Upsilon_a \lambda_b = \lambda_b \Upsilon_a \text{ for all } a \leq b \text{ in } Q. \quad (6.2)$$

This is equivalent to the condition that Q has no sublattice P order-isomorphic to a pentagon:

$$P = \{a, b, c, d, e\}, \quad a < b < c < d, \quad a < e < d, \quad b \vee e = d \text{ and } c \wedge e = a. \quad (6.3)$$

The lattice Q in Example 6.3(i) is not modular, as it contains pentagons $\{\mathbf{0}, a, t, t', \mathbf{1}\}$ for $\mathbf{0} < t < t' < \mathbf{1}$.

Proposition 6.6 Let C be a complete chain from a to b in Q and $z \in Q$.

(i) $\lambda_z(C)$ is a \wedge -complete chain and $\Upsilon_z(C)$ is a \vee -complete chain.

(ii) Let Q be a modular complete lattice.

1) If $[x, y]$ is a gap then a) either $\lambda_z(x) = \lambda_z(y)$, or $[\lambda_z(x), \lambda_z(y)]$ is a gap;

b) either $\Upsilon_z(x) = \Upsilon_z(y)$, or $[\Upsilon_z(x), \Upsilon_z(y)]$ is a gap.

2) If C is continuous then the chains $\lambda_z(C)$ and $\Upsilon_z(C)$ are continuous.

Proof. (i) By (6.1), $\lambda_z(C)$ is a chain from $z \wedge a$ to $z \wedge b$. Let $\Gamma = \lambda_z(G)$ for some $G \subseteq C$. As $\wedge G \in C$, we have $\wedge \lambda_z(G) = \lambda_z(\wedge G) \in \lambda_z(C)$ by (5.3). Thus $\lambda_z(C)$ is \wedge -complete. Similarly, $\Upsilon_z(C)$ is \vee -complete.

(ii) 1) a) Let $[x, y]$ be a gap and $\lambda_z(x) \neq \lambda_z(y)$. Assume that $\lambda_z(x) < u < \lambda_z(y)$ for some $u \in Q$. Then $u < z$ and $u < y$. If $u \leq x$, then $u \leq \lambda_z(x)$, a contradiction. Thus $x < u \vee x \leq y$. As $[x, y]$ is a gap, $y = u \vee x$. As $u < z$, it follows from (6.2) that

$$\lambda_z(y) = \lambda_z \Upsilon_u(x) = \Upsilon_u \lambda_z(x) = u \vee \lambda_z(x) = u < \lambda_z(y),$$

a contradiction. Thus $[\lambda_z(x), \lambda_z(y)]$ is a gap. Similarly, we can prove b).

2) Let C be continuous. If $\lambda_z(C)$ is not continuous, $[\lambda_z(x), \lambda_z(y)]_{\lambda_z(C)}$ is a gap for some $x < y$ in C . Then, for all $\xi \in [x, y]_C$, either $\lambda_z(\xi) = \lambda_z(x)$, or $\lambda_z(\xi) = \lambda_z(y)$. Set $K = \{\xi \in [x, y]_C: \lambda_z(\xi) = \lambda_z(x)\}$, $u = \vee K$, $L = \{\xi \in [x, y]_C: \lambda_z(\xi) = \lambda_z(y)\}$ and $v = \wedge L$. As C is complete, $u, v \in [x, y]_C$. As C is continuous, $u = v$.

Let $u \in K$, so that $\lambda_z(u) = \lambda_z(x)$ and $u < \xi$ for all $\xi \in L$. As $\lambda_z(\xi) = \lambda_\xi(z)$,

$$\Upsilon_u \lambda_z(y) = \Upsilon_u \lambda_z(\xi) = \Upsilon_u \lambda_\xi(z) \stackrel{(6.2)}{=} \lambda_\xi \Upsilon_u(z) = \xi \wedge \Upsilon_u(z) \leq \xi.$$

Thus $\Upsilon_u \lambda_z(y)$ is a lower bound of L . Hence $u \vee \lambda_z(y) = \Upsilon_u \lambda_z(y) \leq u$. So $\lambda_z(y) \leq u$. As $\lambda_z(y) \leq z$, we have $\lambda_z(y) \leq z \wedge u = \lambda_z(u) = \lambda_z(x)$, a contradiction. Similarly, we get a contradiction, if we assume that $u \in L$. Thus $\lambda_z(C)$ is continuous. Similarly, $\Upsilon_z(C)$ is continuous. ■

Corollary 6.7 *If Q is modular then $<_{\mathfrak{g}}$ is an **HH**-relation.*

Proof. Let Q be modular. If $x <_{\mathfrak{g}} y$, $x \neq y$, then $[x, y]$ is a gap. By Proposition 6.6(ii), for each $z \in Q$, either $x \wedge z = y \wedge z$, or $[x \wedge z, y \wedge z]$ is a gap. So $x \wedge z <_{\mathfrak{g}} y \wedge z$.

Similarly, $x \vee z <_{\mathfrak{g}} y \vee z$. By Definition 3.2, $<_{\mathfrak{g}}$ is an **HH**-relation. ■

For each $G \subseteq Q$ and any $z \in Q$,

$$\vee\{z \wedge x : x \in G\} \leq z \wedge (\vee G) \text{ and } z \vee (\wedge G) \leq \wedge\{z \vee x : x \in G\}. \quad (6.4)$$

Consider the Join and Meet Infinite Distributive Identities (JID) and (MID)

$$z \vee (\wedge G) = \wedge\{z \vee x : x \in G\} \text{ and } z \wedge (\vee G) = \vee\{z \wedge x : x \in G\} \text{ for all } z \in Q \text{ and } G \subseteq Q. \quad (6.5)$$

For example, the lattice of *all subsets* of a set X with $\leq = \subseteq$, $\wedge = \cap$ and $\vee = \cup$ has properties (6.5), while the lattice of *all closed subsets* of a topological space X does not always have them.

If Q is a chain then conditions (6.5) hold. Note that a lattice Q satisfies (6.5) if and only if Q has a complete embedding into a complete boolean lattice (see [G, Theorem 166]).

Definition 6.8 A lattice Q has properties (JIDC) and (MIDC) if, respectively, the first and second part of (6.5) holds for each chain G in Q .

The properties (JIDC) and (MIDC) are weaker than (JID) and (MID). In particular, they do not imply that Q is distributive.

Lemma 6.9 *Let Q have properties (JIDC) and (MIDC). If \ll is an **HH**-relation then*

- (i) \ll^{\triangleleft} is an **H**-order and a dual **R**-order; (ii) \ll^{\triangleright} is an **R**-order and a dual **H**-order.

Proof. (i) By Proposition 5.5, \ll^{\triangleleft} is a dual **R**-order. Let us prove that it is an **H**-order.

Let $z \in Q$. If $x \ll^{\triangleleft} y$ then there is a complete lower \ll -gap chain T from y to x : each $t \in T \setminus \{x\}$ has the immediate \ll -predecessor t_p in T : $[t_p, t]_T$ is a gap. Fix $t \in T \setminus \{x\}$ and set $S_t = \{s \in T: z \vee s = z \vee t\}$. As T is complete, $u^t := \wedge S_t \in T$. By (6.5),

$$z \vee u^t = z \vee (\wedge S_t) = \wedge\{z \vee s : s \in S_t\} = z \vee t, \text{ so that } u^t \in S_t. \quad (6.6)$$

Let $u^t \neq x$. Then $z \vee x \neq z \vee u^t = z \vee t$, otherwise $u^t = x$ by (6.6). The element u^t has the immediate \ll -predecessor u_p^t in T , i.e., $u_p^t \ll u^t$. So $z \vee u_p^t \ll z \vee u^t = z \vee t$, as \ll is an **H**-relation.

As $u_p^t \notin S_t$, we have $z \vee u_p^t \neq z \vee t$. So $z \vee u_p^t$ is the immediate \ll -predecessor of $z \vee t = z \vee u^t$ in $\Upsilon_z(T)$. Thus $\Upsilon_z(T)$ is a lower \ll -gap chain from $z \vee y$ to $z \vee x$.

By Proposition 6.6, $\Upsilon_z(T)$ is a \vee -complete. Hence, by Lemma 4.5(iii), it is complete. Thus $z \vee x \ll^{\triangleleft} z \vee y$, so that \ll^{\triangleleft} is an **H**-order. The proof of part (ii) is similar. ■

Theorem 6.10 *If a modular lattice Q has properties (JIDC) and (MIDC) then $<_c$ is an **RR**-order and Q is a union of disjoint sets each of which has no gaps.*

Proof. Let $x <_c y$ and C be a complete continuous chain from x to y . Let $z \in Q$. By Proposition 6.6, $\lambda_z(C)$ is a \wedge -complete, continuous chain from $z \wedge x$ to $z \wedge y$. Let $\Gamma = \lambda_z(G) \subseteq \lambda_z(C)$ for some $G \subseteq C$. Then $\vee G \in C$, as C is complete and, by (6.5),

$$\vee \Gamma = \vee \lambda_z(G) = \vee \{z \wedge g : g \in G\} = z \wedge (\vee G) \in \lambda_z(C).$$

So $\lambda_z(C)$ is \vee -complete. Thus $\lambda_z(C)$ is complete, so that $z \wedge x <_c z \wedge y$. Hence \leq_c is a dual **H**-order. Similarly, $<_c$ is an **H**-order. By Proposition 6.5, $<_c = <_c^{\triangleleft} = <_c^{\triangleright}$. Thus, by Lemma 6.9, \leq_c is an **RR**-order.

By Theorem 2.12, $Q = \cup_{\lambda \in Q_r} [\mathfrak{p}(\lambda), \lambda]$, all $[\mathfrak{p}(\lambda), \lambda]$, $\lambda \in Q_r$, are mutually disjoint and $<_c = \leq$ in each $[\mathfrak{p}(\lambda), \lambda]$. By Proposition 6.2, $[\mathfrak{p}(\lambda), \lambda]$ has no gaps. ■

As complete chains are modular lattices and have properties (JID) and (MID), we have

Corollary 6.11 *Let Q be a complete chain. Then $<_g$ is an **HH**-relation, $<_g^{\triangleleft}$ is an **H**-order and a dual **R**-order, $<_g^{\triangleright}$ is an **R**-order and a dual **H**-order. The relation $<_c$ is an **RR**-order.*

7 Maps and relations in lattices

In this section we study special subsets in complete lattices – enveloping and inscribing sets. In particular, we consider the lattices $\text{Ref}(Q)$ of relations and $\text{Map}(Q)$ of all maps on a complete lattice Q . We describe some of their enveloping and inscribing sets and construct the corresponding radical maps.

7.1 Enveloping and inscribing sets in lattices, radical maps and topology

Recall that a subset L is a *sublattice* of (Q, \leq) if $x \vee y, x \wedge y \in L$ for all $x, y \in L$. It is \wedge -complete, if $\wedge G \in L$; and \vee -complete, if $\vee G \in L$ for all $G \subseteq L$. It is *complete* if it is \wedge - and \vee -complete.

On the other hand, L can be a lattice in its own right with respect to the restriction of \leq to L , not being a sublattice of Q . For example, the smallest element in L majorizing $x, y \in L$ may differ from $x \vee y$. Denote it $x \vee^L y$. Similarly, we use the notation $x \wedge^L y$. A lattice L is \vee^L -complete if each $G \subseteq L$ has the smallest element $\vee^L G$ in L majorizing all $x \in G$. In general, $\vee G \leq \vee^L G$. Similarly, if L is \wedge^L -complete then $\wedge^L G \leq \wedge G$. If L is \vee^L - and \wedge^L -complete, it is a *complete lattice*.

If L is a *sublattice* of Q then $x \vee y = x \vee^L y$ and $x \wedge y = x \wedge^L y$ for $x, y \in L$. It is \wedge -complete (\vee -complete), if and only if $\wedge^L G = \wedge G$ ($\vee^L G = \vee G$) for all $G \subseteq L$.

For $a \in Q$ and $G \subseteq Q$, write $G \leq a$ ($a \leq G$), if a majorizes (minorizes) all $x \in G$. Set

$$L^G = \{a \in L : G \leq a\} \text{ and } L_G = \{a \in L : a \leq G\}. \quad (7.1)$$

Lemma 7.1 (i) Let L be \wedge^L -complete. If L has the largest element, it is also \vee^L -complete, i.e., L is a complete lattice; and $\vee^L G = \wedge^L L^G$ is the smallest in L^G for all $G \subseteq L$.

(ii) Let L be \wedge -complete and $\mathbf{1} \in L$. For $G \subseteq Q$, $\wedge^L L^G = \wedge L^G$ is the smallest element in L^G .

(iii) Let $L \subseteq Q$ be \vee^L -complete. If L has the smallest element, then L is also \wedge^L -complete, i.e., L is a complete lattice, and $\wedge^L G = \vee^L L_G$ is the largest in L_G for all $G \subseteq L$.

(iv) Let L be \vee -complete and $\mathbf{0} \in L$. For $G \subseteq Q$, $\vee^L L_G = \vee L_G$ is the largest element in L_G .

Proof. Part (i) is proved in Lemma 34 [G]. The proof of (iii) is similar.

(ii) The set L^G contains $\mathbf{1}$. As L is \wedge -complete, it follows that $h := \wedge^L L^G = \wedge L^G \in L$ is the largest element in Q minorizing L^G . As $g \leq x$ for all $g \in G$ and $x \in L^G$, we have $g \leq h$. Thus $h \in L^G$ and is the smallest element in L^G . The proof of part (iv) is similar. ■

Definition 7.2 (cf. Definition 27 [G]) A set L in Q is **enveloping** (Cld in [G]) if, for each $x \in Q$, the set $\{l \in L: x \leq l\}$ has the smallest element \bar{x} , so that $x \leq \bar{x}$.

It is **inscribing** if, for each $x \in Q$, the set $\{l \in L: l \leq x\}$ has the largest element \underline{x} .

The following theorem refines Corollary 29 [G] and characterizes enveloping and inscribing sets in terms of lattice operations.

Theorem 7.3 (i) For $L \subseteq Q$, the following conditions are equivalent.

- 1) L is enveloping;
- 2) L is \wedge -complete and $\mathbf{1} \in L$;
- 3) L is a complete lattice, \wedge -complete and $\mathbf{1} \in L$;
- 4) For each $G \subseteq Q$, the set L^G (see (7.1)) has the smallest element m_G and

$$m_G = \wedge^L L^G = \wedge L^G = \vee^L \{\bar{x}: x \in G\}; \quad \text{if } G \subseteq L \text{ then } m_G = \vee^L G. \quad (7.2)$$

(ii) For $N \subseteq Q$, the following conditions are equivalent.

- 1) N is inscribing;
- 2) N is \vee -complete and $\mathbf{0} \in N$;
- 3) N is a complete lattice, \vee -complete and $\mathbf{0} \in N$;
- 4) For each $G \subseteq Q$, the set N_G (see (7.1)) has the largest element n_G and

$$n_G = \vee^N N_G = \vee N_G = \wedge^N \{\underline{x}: x \in G\}; \quad \text{if } G \subseteq N \text{ then } n_G = \wedge^N G.$$

Proof. (i) 1) \Rightarrow 2) As L is enveloping, $\bar{\mathbf{1}} = \mathbf{1} \in L$. For $G \subseteq L$, let $t = \wedge G$. As $t \leq g$ for all $g \in G$, and \bar{t} is the smallest element in L majorizing t , $t \leq \bar{t} \leq g$. So $t \leq \bar{t} \leq \wedge G = t$. Thus $t = \bar{t} \in L$ and L is \wedge -complete.

2) \Rightarrow 3) If L is \wedge -complete and $\mathbf{1} \in L$, it is a complete lattice by Lemma 7.1(i).

3) \Rightarrow 4) Let $G \subseteq Q$. By Lemma 7.1(ii), $\wedge^L L^G$ exists, $\wedge^L L^G = \wedge L^G$ and it is the smallest element in L^G . So $m_G = \wedge^L L^G = \wedge L^G$. If $G \subseteq L$ then $m_G = \vee^L G$ by Lemma 7.1(i).

For each $x \in G$, the set $\{l \in L: x \leq l\}$ majorizing x has the smallest element \bar{x} . As L is a complete lattice, it has the smallest element $s = \vee^L \{\bar{x}: x \in G\}$ majorizing all \bar{x} . Thus $x \leq \bar{x} \leq s$ for all $x \in G$. So $s \in L^G$.

If $y \in L$ majorizes all $x \in G$, then $\bar{x} \leq y$ for all $x \in G$ (by definition of \bar{x}), i.e., y majorizes all \bar{x} . Therefore $s \leq y$. Hence s is the smallest element in L majorizing all $x \in G$, i.e., s is the smallest element in L^G . Thus $s = m_G = \wedge^L L^G = \wedge L^G = \vee^L \{\bar{x} : x \in G\}$.

4) \Rightarrow 1) For each $x \in Q$, take $G = \{x\}$.

The proof of (ii) is similar. ■

Let $\{L_\lambda\}_{\lambda \in \Lambda}$ be subsets of Q . Consider the set $L_\Lambda = \{x = \{x_\lambda\}_{\lambda \in \Lambda} : x_\lambda \in L_\lambda\}$ and let

$$\widehat{L}_\Lambda = \{\wedge x := \wedge_{\lambda \in \Lambda} x_\lambda \text{ for } x \in L_\Lambda\} \text{ and } \check{L}_\Lambda = \{\vee x := \vee_{\lambda \in \Lambda} x_\lambda \text{ for } x \in L_\Lambda\}. \quad (7.3)$$

Corollary 7.4 *Let all $\{L_\lambda\}_{\lambda \in \Lambda}$ be enveloping and $\{N_\lambda\}_{\lambda \in \Lambda}$ inscribing sets in Q . Then*

(i) *The sets $\cap_{\lambda \in \Lambda} L_\lambda$, \widehat{L}_Λ and $\cup_{\lambda \in \Lambda} L_\lambda$ are enveloping in Q .*

(ii) *The sets $\cup_{\lambda \in \Lambda} N_\lambda$, \check{N}_Λ and $\cap_{\lambda \in \Lambda} N_\lambda$ are inscribing in Q .*

Proof. (i) All L_λ are \wedge -complete by Theorem 7.3. So $\cap_{\lambda \in \Lambda} L_\lambda$ is \wedge -complete. As $\mathbf{1} \in L_\lambda$ for all $\lambda \in \Lambda$, $\mathbf{1} \in \cap_{\lambda \in \Lambda} L_\lambda$. By Theorem 7.3, $\cap_{\lambda \in \Lambda} L_\lambda$ is enveloping.

Let $g \in G \subseteq \widehat{L}_\Lambda$. By (7.3), $g = \wedge_{\lambda \in \Lambda} g_\lambda$ for some $\{g_\lambda\}_{\lambda \in \Lambda} \in L_\Lambda$, and

$$\wedge G = \wedge_{g \in G} g = \wedge_{g \in G} (\wedge_{\lambda \in \Lambda} g_\lambda).$$

For $\lambda \in \Lambda$, set $y_\lambda = \wedge_{g \in G} g_\lambda$. As $g \leq g_\lambda$, we have $\wedge_{g \in G} g \leq \wedge_{g \in G} g_\lambda = y_\lambda$. So $\wedge G \leq \wedge_{\lambda \in \Lambda} y_\lambda$. On the other hand, as $y_\lambda \leq g_\lambda$ for each $g \in G$, we have $\wedge_{\lambda \in \Lambda} y_\lambda \leq \wedge_{\lambda \in \Lambda} g_\lambda = g$. Thus

$$\wedge G \leq \wedge_{\lambda \in \Lambda} y_\lambda \leq \wedge_{g \in G} g = \wedge G, \text{ so that } \wedge G = \wedge_{\lambda \in \Lambda} y_\lambda.$$

As all L_λ are \wedge -complete by Theorem 7.3, all $y_\lambda \in L_\lambda$. So $\{y_\lambda\}_{\lambda \in \Lambda} \in L_\Lambda$ and $\wedge_{\lambda \in \Lambda} y_\lambda \in \widehat{L}_\Lambda$. Hence $\wedge G \in \widehat{L}_\Lambda$. Thus \widehat{L}_Λ is \wedge -complete and $\mathbf{1} \in \widehat{L}_\Lambda$ as $\mathbf{1} \in L_\lambda$ for all λ . By Theorem 7.3, \widehat{L}_Λ is enveloping.

Now let $G \subseteq \cup_{\lambda \in \Lambda} L_\lambda$. Set $G_\lambda = G \cap L_\lambda$ for $\lambda \in \Lambda$. Then $\wedge G_\lambda \in L_\lambda$ for all $\lambda \in \Lambda$. Hence

$$\{\wedge G_\lambda\}_{\lambda \in \Lambda} \in L_\Lambda \text{ and } \wedge G = \wedge (\cup_{\lambda \in \Lambda} G_\lambda) \stackrel{(5.4)}{=} \wedge_{\lambda \in \Lambda} (\wedge G_\lambda) \stackrel{(7.3)}{\in} \widehat{L}_\Lambda.$$

So $\cup_{\lambda \in \Lambda} L_\lambda$ is \wedge -complete. As it contains $\mathbf{1}$, it is enveloping by Theorem 7.3.

Part (ii) can be proved similarly. ■

A map $g: Q \rightarrow Q$ is called

$$\text{a } T\text{-radical map if } x \leq g(x) = g(g(x)) \text{ and } g(x \vee y) = g(x) \vee g(y), \quad (7.4)$$

$$\text{a dual } T\text{-radical map if } g(g(x)) = g(x) \leq x \text{ and } g(x \wedge y) = g(x) \wedge g(y) \text{ for } x, y \in Q.$$

T -radical maps are radical (see (2.13)). Indeed, if $x \leq y$ then $g(x) \leq g(x) \vee g(y) = g(x \vee y) = g(y)$. Similarly, dual T -radical maps are dual radical maps.

For a map $g: Q \rightarrow Q$, we denote by $\text{Fix}(g)$ the set of all g -fixed points:

$$\text{Fix}(g) = \{x \in Q : g(x) = x\}.$$

For an enveloping set L and an inscribing set N , define the maps f_L and g_N on Q by

$$f_L : x \in Q \mapsto \bar{x} \in L, \text{ and } g_N : x \in Q \mapsto \underline{x} \in N. \quad (7.5)$$

Proposition 7.5 (i) $\theta: L \mapsto f_L$ is a bijection from the class of all enveloping sets in Q onto the class of all radical maps on Q . Its inverse $\theta^{-1}: g \mapsto \text{Fix}(g)$. Moreover, $L = \text{Fix}(f_L)$ and $g = f_{\text{Fix}(g)}$.

(ii) θ is a bijection from the class of all enveloping sublattices of Q onto the class of all T -radical maps on Q .

(iii) $\phi: N \mapsto g_N$ is a bijection from the class of all inscribing sets in Q onto the class of all dual radical maps on Q . Its inverse $\phi^{-1}: f \mapsto \text{Fix}(f)$. Moreover, $N = \text{Fix}(g_N)$ and $f = g_{\text{Fix}(f)}$.

(iv) ϕ is a bijection from the class of all inscribing sublattices of Q onto the class of all dual T -radical maps on Q .

Proof. Part (i) was proved in [G, Lemma 28]. The proof of (iii) is similar.

(ii) Let L be an enveloping sublattice of Q . Set $f = f_L$. By (i), f is a radical map. Hence, for $x, y \in Q$, $x \leq f(x) \leq f(x \vee y)$ and $y \leq f(y) \leq f(x \vee y)$. So $x \vee y \leq f(x) \vee f(y) \leq f(x \vee y)$. Thus $f(x \vee y) \leq f(f(x) \vee f(y)) \leq f(f(x \vee y)) = f(x \vee y)$, so that $f(x \vee y) = f(f(x) \vee f(y))$. As L is a sublattice, $f(x) \vee f(y) \in L = \text{Fix}(f)$. Hence $f(x \vee y) = f(x) \vee f(y)$. Thus f is a T -radical map.

Conversely, let g be a T -radical map. By (i), $\text{Fix}(g)$ is an enveloping set. By Theorem 7.3, $\text{Fix}(g)$ is \wedge -complete. If $x, y \in \text{Fix}(g)$ then $g(x \vee y) = g(x) \vee g(y) = x \vee y$. So $\text{Fix}(g)$ is an enveloping sublattice of Q which completes the proof of (ii). Part (iv) can be proved similarly. ■

Consider the following link between T -radical maps and topology. For a set X , the set $P(X)$ of all subsets of X is a complete lattice with order $\leq = \subseteq$, $\mathbf{0} = \emptyset$, $\mathbf{1} = X$, $\wedge = \cap$ and $\vee = \cup$.

For a *topological space* (X, τ) , denote by τ^{op} the \vee -complete sublattice of all open subsets in $P(X)$ and by τ^{cl} the \wedge -complete sublattice of all closed subsets in $P(X)$.

Proposition 7.6 (i) Let (X, τ) be a topological space and $Q = P(X)$.

1) If $f: Q \rightarrow Q$ maps each $G \in Q$ into its τ -closure then f is a T -radical map.

2) If $g: Q \rightarrow Q$ maps each $G \in Q$ into its τ -interior then g is a dual T -radical map.

(ii) Let X be a set and $Q = P(X)$. Let $f: Q \rightarrow Q$ be a map.

1) If f is a T -radical map and $f(\emptyset) = \emptyset$, then (X, τ) with $\tau^{\text{cl}} = \{f(G) : G \in Q\}$ is a topological space.

2) If f is a dual T -radical map and $f(X) = X$, then (X, τ) with $\tau^{\text{op}} = \{f(G) : G \in Q\}$ is a topological space.

Proof. (i) 1) For $G \in Q$, its τ -closure $f(G) = \cap \{F \in \tau^{\text{cl}} : G \subseteq F\} \in \tau^{\text{cl}}$. Hence $G \subseteq f(G) = f(f(G))$. Let $K \in Q$. As τ^{cl} is a sublattice of Q , $G \cup K \subseteq f(G) \cup f(K) \in \tau^{\text{cl}}$. Hence $f(G \cup K) \subseteq f(G) \cup f(K) \subseteq f(G \cup K)$. Thus $f(G \cup K) = f(G) \cup f(K)$, so that f is a T -radical map. Part 2) follows from duality.

(ii) 1) Let $L = \text{Fix}(f) = \{G \in Q : f(G) = G\}$. Then $L = \{f(G) : G \in Q\}$. By Proposition 7.5, L is an enveloping sublattice. By Theorem 7.3, it is \cap -complete. By (7.4), $f(G) \cup f(K) = f(G \cup K) \in L$ for $G, K \in Q$. Hence L is a \cap -complete sublattice of Q . As $X \subseteq f(X) \subseteq X$ by (7.4), $X = f(X) \in L$. As $f(\emptyset) = \emptyset$, $\emptyset \in L$. Then $L = \tau^{\text{cl}}$ for some topology τ in X . Part 2) follows from duality. ■

7.2 Enveloping and inscribing sets in $\text{Ref}(Q)$

For a complete lattice (Q, \leq) , $\text{Ref}(Q)$ is also a complete lattice with the order \subseteq (see (1.1)). For a subset $F \subseteq \text{Ref}(Q)$, the relations $\wedge F = \ll_{\wedge F} := \bigcap_{\ll \in F} \ll$ and $\vee F = \ll_{\vee F} := \bigcup_{\ll \in F} \ll$ are defined in (5.6). Consider the following subsets of $\text{Ref}(Q)$:

$$\begin{aligned} L_{\mathbf{H}} &= \{\text{all } \mathbf{H}\text{-relations in } \text{Ref}(Q)\}, & L_{d\mathbf{H}} &= \{\text{all dual } \mathbf{H}\text{-relations in } \text{Ref}(Q)\}, \\ L_{\text{uc}} &= \{\text{all up-contiguous } \ll \text{ in } \text{Ref}(Q)\}, & L_{\text{dc}} &= \{\text{all down-contiguous } \ll \text{ in } \text{Ref}(Q)\}, \\ L_{\text{ue}} &= \{\text{all up-expanded } \ll \text{ in } \text{Ref}(Q)\}, & L_{\text{de}} &= \{\text{all down-expanded } \ll \text{ in } \text{Ref}(Q)\}, \\ L_{\triangleright} &= \{\ll \text{ in } \text{Ref}(Q) : \ll = \ll^{\triangleright}\}, & L_{\triangleleft} &= \{\ll \text{ in } \text{Ref}(Q) : \ll = \ll^{\triangleleft}\}. \end{aligned}$$

Theorem 7.7 (i) *The sets $L_{\mathbf{H}}, L_{d\mathbf{H}}, L_{\text{uc}}, L_{\text{dc}}$ are inscribing and enveloping sublattices of $\text{Ref}(Q)$.*
(ii) *The sets $L_{\triangleright}, L_{\triangleleft}, L_{\text{ue}}, L_{\text{de}}$ and the set $L_{\mathbf{o}}$ of all orders are enveloping in $\text{Ref}(Q)$.*

Proof. (i) Let $F \subseteq L_{\mathbf{H}}$. If $a \ll_{\wedge F} b$ (see (5.6)) then $a \ll b$ for all \ll in F . As all \ll are \mathbf{H} -relations, $a \vee x \ll b \vee x$ for all $x \in Q$. Hence $a \vee x \ll_{\wedge F} b \vee x$. So $\wedge F$ is an \mathbf{H} -relation. Let now $a \ll_{\vee F} b$ (see (5.6)). Then $a \ll b$ for some \ll in F , and $a \vee x \ll b \vee x$ for all $x \in Q$. Hence $a \vee x \ll_{\vee F} b \vee x$. So $\vee F$ is an \mathbf{H} -relation. Thus $L_{\mathbf{H}}$ is complete. As $\mathbf{0}, \mathbf{1} \in L_{\mathbf{H}}$, $L_{\mathbf{H}}$ is an inscribing and enveloping lattice by Theorem 7.3. The proof for $L_{d\mathbf{H}}$ is similar.

Let $F \subseteq L_{\text{uc}}$. If $a \ll_{\wedge F} b$ then $a \ll b$ for all \ll in F . So

$$[a, \ll_{\wedge F}] = \bigcap_{\ll \in F} [a, \ll] \text{ and } [\ll_{\wedge F}, b] = \bigcap_{\ll \in F} [\ll, b]. \quad (7.6)$$

As all \ll are up-contiguous, $[a, b] \subseteq [\ll, b]$ for all \ll in F . By (7.6), $[a, b] \subseteq [\ll_{\wedge F}, b]$. So $\wedge F$ is up-contiguous. Let now $a \ll_{\vee F} b$. Then $a \ll b$ for some \ll in F . Hence $[a, b] \subseteq [\ll, b]$. Thus $[a, b] \subseteq [\ll_{\vee F}, b]$. So $\vee F$ is up-contiguous. Thus L_{uc} is complete. As $\mathbf{1}$ and $\mathbf{0}$ are up-contiguous, L_{uc} is an inscribing and enveloping lattice by Theorem 7.3. The proof for L_{dc} is similar.

(ii) For $F \subseteq L_{\triangleright}$, let $a \ll_{\wedge F}^{\triangleright} b$. By (4.6), there is a complete, upper $\ll_{\wedge F}$ -gap chain C from a to b , i.e., each $x \in C \setminus \{b\}$ has an immediate $\ll_{\wedge F}$ -successor s_x : $x \ll_{\wedge F} s_x$ and $[x, s_x] \cap C = \{x, s_x\}$. By (5.6), $x \ll s_x$ for all \ll in F . So s_x is an immediate \ll -successor of x in C . Thus C is a complete, upper \ll -gap chain for all \ll in F , i.e., $a \ll^{\triangleright} b$. As $\ll = \ll^{\triangleright}$, $a \ll b$ for all \ll in F . Hence $a \ll_{\wedge F} b$. So $\ll_{\wedge F}^{\triangleright}$ is stronger than $\ll_{\wedge F}$. By (4.7), $\ll_{\wedge F}$ is stronger than $\ll_{\wedge F}^{\triangleright}$. Hence $\ll_{\wedge F}^{\triangleright} = \ll_{\wedge F}$. Thus L_{\triangleright} is \wedge -complete. As $\mathbf{1} \in L_{\triangleright}$, L_{\triangleright} is enveloping by Theorem 7.3. The proof for L_{\triangleleft} is similar.

For $F \subseteq L_{\text{ue}}$ and $a \in Q$, let $G \subseteq [a, \ll_{\wedge F}]$. By (7.6), $G \subseteq [a, \ll]$ for all \ll in F . As all \ll are up-expanded, $\vee G \in [a, \ll]$ for all \ll in F . Hence $\vee G \in [a, \ll_{\wedge F}]$. So $\ll_{\wedge F}$ is up-expanded. Thus L_{ue} is \wedge -complete. As $\mathbf{1} \in L_{\text{ue}}$, L_{ue} is enveloping by Theorem 7.3. The proof for L_{de} is similar.

For $F \subseteq L_{\mathbf{o}}$, let $a \ll_{\wedge F} b \ll_{\wedge F} c$. Then $a \ll b \ll c$ for all \ll in F , whence $a \ll c$. So $a \ll_{\wedge F} c$. Thus $\ll_{\wedge F}$ is an order. Thus $L_{\mathbf{o}}$ is \wedge -complete. As $\mathbf{1} \in L_{\mathbf{o}}$, $L_{\mathbf{o}}$ is enveloping by Theorem 7.3. ■

Corollary 7.8 *The sets of all \mathbf{T} -orders, of all dual \mathbf{T} -orders, of all \mathbf{R} -orders, of all dual \mathbf{R} -orders are enveloping in $\text{Ref}(Q)$.*

Proof. By Definition 2.3, the set of all \mathbf{T} -orders is the intersection of $L_{\mathbf{o}}, L_{\text{uc}}, L_{\text{ue}}$ which are enveloping. So it is enveloping by Corollary 7.4. The rest of the proof is similar. ■

By Proposition 7.5, there is a bijection between enveloping sets L and radical maps f_L . It is, however, often difficult to determine the action of f_L . Below we do it for some sets in $\text{Ref}(Q)$.

Lemma 7.9 *The radical maps $f_{L_{\triangleright}}, f_{L_{\triangleleft}}$ on $\text{Ref}(Q)$ act by $f_{L_{\triangleright}}: \ll \mapsto \ll^{\triangleright}$ and $f_{L_{\triangleleft}}: \ll \mapsto \ll^{\triangleleft}$.*

Proof. Set $g: \ll \mapsto \ll^{\triangleright}$. Then $g(g(\ll)) = (\ll^{\triangleright})^{\triangleright} \stackrel{(4.8)}{=} \ll^{\triangleright} = g(\ll) \supseteq \ll$. It is easy to check that $\ll \subseteq \prec$ implies $g(\ll) = \ll^{\triangleright} \subseteq \prec^{\triangleright} = g(\prec)$, as each complete upper \ll -gap chain is also a complete upper \prec -gap chain. Thus (2.13) holds. So g is a radical map from $\text{Ref}(Q)$ onto L_{\triangleright} . As $\text{Fix}(g) = L_{\triangleright}$, we have from Proposition 7.5 that $g = f_{\text{Fix}(g)} = f_{L_{\triangleright}}$. The proof for $f_{L_{\triangleleft}}$ is similar. ■

As $L_{\mathbf{H}}$ and $L_{d\mathbf{H}}$ are inscribing and enveloping sublattices of $\text{Ref}(Q)$, the maps $f_{L_{\mathbf{H}}}$ and $f_{L_{d\mathbf{H}}}$ are T -radical, and $g_{L_{\mathbf{H}}}$ and $g_{L_{d\mathbf{H}}}$ (see (7.5)) are dual T -radical. We describe their action below.

Proposition 7.10 (i) *Set $\prec_1 = g_{L_{\mathbf{H}}}(\ll)$. Then $a \prec_1 b$ if $a \vee x \ll b \vee x$ for all $x \in Q$.*

(ii) *Set $\prec_2 = g_{L_{d\mathbf{H}}}(\ll)$. Then $a \prec_2 b$ if $a \wedge x \ll b \wedge x$ for all $x \in Q$.*

(iii) *Set $\prec_3 = f_{L_{\mathbf{H}}}(\ll)$. Then $a \prec_3 b$ if there are $x \ll y$ such that $x \leq a$ and $b = a \vee y$.*

(iv) *Set $\prec_4 = f_{L_{d\mathbf{H}}}(\ll)$. Then $a \prec_4 b$ if there are $x \ll y$ such that $a = b \wedge x$ and $b \leq y$.*

Proof. (i) By Lemma 3.1, \prec_1 is an \mathbf{H} -relation. Set $x = \mathbf{0}$. We get that \prec_1 stronger than \ll : $\prec_1 \subseteq \ll$. If an \mathbf{H} -relation \prec is stronger than \ll , $a \prec b$ implies $a \vee x \prec b \vee x$ for all $x \in Q$, so that $a \vee x \ll b \vee x$. Thus $a \prec_1 b$. So \prec is stronger than \prec_1 : $\prec \subseteq \prec_1$. Hence \prec_1 is the largest \mathbf{H} -relation minorizing \ll . By Definition 7.2 and (7.5), $g_{L_{\mathbf{H}}}(\ll) = \prec_1$. The proof of (ii) is similar.

(iii) Let $a \prec_3 b$. Then there are

$$x, y \in Q \text{ such that } x \ll y, x \leq a \text{ and } b = a \vee y. \quad (7.7)$$

For $z \in Q$, $x \leq a \vee z$ and $b \vee z = (a \vee y) \vee z = (a \vee z) \vee y$. Hence $(a \vee z) \prec_3 (b \vee z)$. Thus \prec_3 is an \mathbf{H} -relation by Lemma 3.1. It majorizes \ll ($\ll \subseteq \prec_3$), as $a \ll b$ implies $a \prec_3 b$ (set $x = a, y = b$).

Let an \mathbf{H} -relation \prec majorize \ll ($\ll \subseteq \prec$). If $a \prec_3 b$ then (7.7) holds. As $x \ll y$ and $\ll \subseteq \prec$, we have $x \prec y$. As \prec is an \mathbf{H} -relation, it follows from (7.7) that $a = (a \vee x) \prec (a \vee y) = b$. Thus \prec_3 is stronger than \prec : $\prec_3 \subseteq \prec$. Hence \prec_3 is the smallest \mathbf{H} -relation majorizing \ll , i.e., $\prec_3 = f_{L_{\mathbf{H}}}(\ll)$. The proof of (iv) is similar. ■

For each \ll in $\text{Ref}(Q)$, define the relations \ll_{uc} and \ll_{dc} as follows:

$$a \ll_{\text{uc}} b \text{ if and only if there is } c \in Q \text{ satisfying } c \ll b \text{ and } a \in [c, b]; \quad (7.8)$$

$$a \ll_{\text{dc}} b \text{ if and only if there is } c \in Q \text{ such that } a \ll c \text{ and } b \in [a, c].$$

Proposition 7.11 *For each \ll in $\text{Ref}(Q)$, $\ll_{\text{uc}} = f_{L_{\text{uc}}}(\ll)$ and $\ll_{\text{dc}} = f_{L_{\text{dc}}}(\ll)$.*

Proof. If $a \ll_{\text{uc}} b$ then $x \in [c, b]$ for $x \in [a, b]$. Then, by (7.8), $x \ll_{\text{uc}} b$. Hence $[a, b] \subseteq [\ll_{\text{uc}}, b]$. So (see Definition 2.1) \ll_{uc} is up-contiguous.

If $a \ll b$ then $a \ll_{\text{uc}} b$ (set $c = a$ in (7.8)). So \ll_{uc} majorizes \ll ($\ll \subseteq \ll_{\text{uc}}$ see (1.1)).

Let an up-contiguous relation \prec majorize \ll ($\ll \subseteq \prec$). If $a \ll_{\text{uc}} b$ then $c \ll b$ and $a \in [c, b]$ for some $c \in Q$. Hence $c \prec b$. As \prec is up-contiguous, $[c, b] \subseteq [\prec, b]$, so that $a \prec b$. Thus \ll_{uc} is stronger than \prec . So \ll_{uc} is the smallest element of L_{uc} majorizing \ll , i.e., $\ll_{\text{uc}} = f_{L_{\text{uc}}}(\ll)$. The equality $f_{L_{\text{dc}}}(\ll) = \ll_{\text{dc}}$ is proved similarly. ■

7.3 Enveloping and inscribing sets of maps on lattices

For a complete lattice (Q, \leq) , denote by $\text{Map}(Q)$ the set of all maps $Q \rightarrow Q$. For $f, g \in \text{Map}(Q)$,

$$g \lesssim f \text{ if } g(x) \leq f(x) \text{ for all } x \in Q; \text{ and set } \mathbf{0}_{\text{Map}}: x \mapsto \mathbf{0} \text{ and } \mathbf{1}_{\text{Map}}: x \mapsto \mathbf{1}. \quad (7.9)$$

Then $(\text{Map}(Q), \lesssim)$ is a complete lattice and, for each $G \subseteq \text{Map}(Q)$,

$$(\vee G)(x) = \vee_{f \in G} f(x) \text{ and } (\wedge G)(x) = \wedge_{f \in G} f(x). \quad (7.10)$$

Let Rad_Q and dRad_Q be the sets of all radical and dual radical maps, respectively, on Q (we write Rad and dRad). They are partially ordered with respect to the order \lesssim ,

$$\begin{aligned} \mathbf{0}_{\text{Rad}}: x \mapsto x \text{ and } \mathbf{1}_{\text{Rad}} = \mathbf{1}_{\text{Map}}: x \mapsto \mathbf{1} \text{ for all } x \in Q; \\ \mathbf{0}_{\text{dRad}} = \mathbf{0}_{\text{Map}}: x \mapsto \mathbf{0} \text{ and } \mathbf{1}_{\text{dRad}} = \mathbf{0}_{\text{Rad}}: x \mapsto x \text{ for all } x \in Q. \end{aligned}$$

Lemma 7.12 (i) *Let f and g be radical maps in Q . The following conditions are equivalent:*

$$1) g \lesssim f; \quad 2) f(g(x)) = f(x) \text{ for all } x \in Q; \quad 3) \text{Fix}(f) \subseteq \text{Fix}(g).$$

(ii) *For $G \subseteq \text{Rad}$, the set $\bigcap_{g \in G} \text{Fix}(g)$ is enveloping.*

Proof. (i) 1) \Rightarrow 2) Let $x \in Q$. By (2.13), $x \leq g(x) \leq f(x)$, so that $f(x) \leq f(g(x)) \leq f(f(x)) = f(x)$. Thus $f(g(x)) = f(x)$.

2) \Rightarrow 3) If $x \in \text{Fix}(f)$ then, by (2.13), $x \leq g(x) \leq f(g(x)) = f(x) = x$. So $g(x) = x$ and $\text{Fix}(f) \subseteq \text{Fix}(g)$.

3) \Rightarrow 1) For all $x \in Q$, $f(x) \in \text{Fix}(f) \subseteq \text{Fix}(g)$. So, by (2.13), $g(x) \leq g(f(x)) = f(x)$.

(ii) By Proposition 7.5, $\text{Fix}(g)$ are enveloping sets for $g \in G$. So $\bigcap_{g \in G} \text{Fix}(g)$ is enveloping by Corollary 7.4. ■

Theorem 7.13 (i) *Rad is an enveloping set in $\text{Map}(Q)$.*

(ii) *For $G \subseteq \text{Rad}$, set $\mathfrak{h} = \vee^{\text{Rad}} G$ and $\mathfrak{g} = \wedge G$. Then $\mathfrak{h}, \mathfrak{g} \in \text{Rad}$, $\text{Fix}(\mathfrak{h}) = \bigcap_{g \in G} \text{Fix}(g)$,*

$$\text{Fix}(\mathfrak{g}) = (\bigcup_{g \in G} \text{Fix}(g)) \cup \widehat{K_G}, \text{ where } K_G = \{\{x_g\}_{g \in G}: x_g \in \text{Fix}(g)\} \text{ (see (7.3))}. \quad (7.11)$$

The set $\text{Fix}(\mathfrak{g})$ is the smallest enveloping set in Q containing $\bigcup_{g \in G} \text{Fix}(g)$.

Proof. (i) For $G \subseteq \text{Rad}$, set $\gamma = \wedge G \in \text{Map}(Q)$. As $x \leq f(x)$ for all $x \in Q$, $f \in G$ by (2.13), we have $x \leq \gamma(x) \leq f(x)$ by (7.10). From this and (2.13) it follows that for all $x \in Q$,

$$\gamma(x) \leq \gamma(\gamma(x)) \leq f(\gamma(x)) \leq f(f(x)) = f(x) \text{ for all } f \in G.$$

Hence $\gamma(x) \leq \gamma(\gamma(x)) \leq \wedge_{f \in G} f(x) = \gamma(x)$. Thus $x \leq \gamma(x) = \gamma(\gamma(x))$ for all $x \in Q$.

Let $x \leq y$. Then $f(x) \leq f(y)$ for all $f \in G$. Hence $\gamma(x) = \wedge_{f \in G} f(x) \leq \wedge_{f \in G} f(y) = \gamma(y)$. Thus $\gamma \in \text{Rad}$. So Rad is \wedge -complete. As $\mathbf{1}_{\text{Map}} \in \text{Rad}$, Rad is an enveloping set by Theorem 7.3.

(ii) As Rad is enveloping, it is complete and \wedge -complete by Theorem 7.3. Thus $\mathfrak{h}, \mathfrak{g} \in \text{Rad}$.

By Lemma 7.12, $L := \bigcap_{g \in G} \text{Fix}(g)$ is enveloping. By Proposition 7.5, $L = \text{Fix}(f_L)$ for some radical map f_L . As $\text{Fix}(f_L) \subseteq \text{Fix}(g)$ for all $g \in G$, $g \lesssim f_L$ by Lemma 7.12. As \mathfrak{h} is the smallest radical map majorizing all $g \in G$, $g \lesssim \mathfrak{h} \lesssim f_L$. So, by Lemma 7.12, $L = \text{Fix}(f_L) \subseteq \text{Fix}(\mathfrak{h}) \subseteq \bigcap_{g \in G} \text{Fix}(g) = L$. Thus $\text{Fix}(\mathfrak{h}) = L = \bigcap_{g \in G} \text{Fix}(g)$.

As $\mathfrak{g} \in \text{Rad}$, it is the largest radical map minorizing all $g \in G$. Hence, by Proposition 7.5 and Lemma 7.12, $\text{Fix}(\mathfrak{g})$ is an enveloping set in Q and $F := \cup_{g \in G} \text{Fix}(g) \subseteq \text{Fix}(\mathfrak{g})$. Let S be an enveloping set such that $F \subseteq S$. Then $S = \text{Fix}(f_S)$ for a radical map f_S , and $f_S \lesssim g$ for $g \in G$. Hence $f_S \lesssim \mathfrak{g}$, so that $\text{Fix}(\mathfrak{g}) \subseteq S$. Thus $\text{Fix}(\mathfrak{g})$ is the smallest enveloping set containing F .

Let $R = F \cup \widehat{K_G}$ (see (7.3) and (7.11)). By Corollary 7.4, R is enveloping. Hence, as above, $F \subseteq \text{Fix}(\mathfrak{g}) \subseteq R$. For $x = \{x_g\} \in K_G$, $\wedge x = \wedge_{g \in G} x_g \in \widehat{K_G}$ and $\wedge x \leq x_g$ for all $g \in G$. Hence $\mathfrak{g}(\wedge x) \leq \mathfrak{g}(x_g) = x_g$, as $\text{Fix}(g) \subseteq \text{Fix}(\mathfrak{g})$. So $\mathfrak{g}(\wedge x) \leq \wedge_{g \in G} x_g = \wedge x$. As \mathfrak{g} is radical $\mathfrak{g}(\wedge x) = \wedge x$. Thus $\wedge x \in \text{Fix}(\mathfrak{g})$, so that $\widehat{K_G} \subseteq \text{Fix}(\mathfrak{g})$. Hence $R = F \cup \widehat{K_G} \subseteq \text{Fix}(\mathfrak{g}) \subseteq R$. So $R = \text{Fix}(\mathfrak{g})$. ■

By duality, similarly to Theorem 7.13, we get

Theorem 7.14 (i) dRad is an inscribing set in $\text{Map}(Q)$.

(ii) For each $G \subseteq \text{dRad}$, we have $\text{Fix}(\wedge^{\text{Rad}} G) = \cup_{g \in G} \text{Fix}(g)$ and

$$\text{Fix}(\vee G) = (\cap_{g \in G} \text{Fix}(g)) \cap \widehat{K_G}, \text{ where } K_G = \{\{x_g\}_{g \in G} : x_g \in \text{Fix}(g)\}$$

(see (7.3)). The set $\text{Fix}(\vee G)$ is the largest inscribing set in Q contained in $\cap_{g \in G} \text{Fix}(g)$.

The subset \mathcal{T} of Rad of all T -radical maps on Q is partially ordered with respect to \lesssim . It has the smallest element $\mathbf{0}_{\mathcal{T}} = \mathbf{0}_{\text{Rad}}: x \mapsto x$, and the largest element $\mathbf{1}_{\mathcal{T}} = \mathbf{1}_{\text{Map}}: x \mapsto \mathbf{1}$.

Theorem 7.15 (i) \mathcal{T} is a complete lattice, i.e., $\vee^{\mathcal{T}} G$ and $\wedge^{\mathcal{T}} G$ exist for each $G \subseteq \mathcal{T}$. Moreover,

(a) $\vee^{\mathcal{T}} G = \vee^{\text{Rad}} G = \wedge^{\mathcal{T}} \mathcal{T}^G$ (see (7.1)) is the smallest element in \mathcal{T}^G ;

(b) $\wedge^{\mathcal{T}} G = \vee^{\mathcal{T}} \mathcal{T}_G$ (see (7.1)) is the largest element in \mathcal{T}_G ;

(c) $\text{Fix}(\wedge^{\mathcal{T}} G)$ is the smallest enveloping sublattice of Q containing $\cup_{g \in G} \text{Fix}(g)$.

(ii) \mathcal{T} is an inscribing sublattice of Rad_Q .

Proof. (i) Let $G \subseteq \mathcal{T}$. By Theorem 7.13, there is $\mathfrak{h} = \vee^{\text{Rad}} G$ in Rad_Q and $L := \text{Fix}(\mathfrak{h}) = \cap_{g \in G} \text{Fix}(g)$ is enveloping in Q . As each $g \in G$ is T -radical, $\text{Fix}(g)$ is a sublattice of Q . Hence L is an enveloping sublattice of Q . By Proposition 7.5, \mathfrak{h} a T -radical map. As \mathfrak{h} is the smallest radical map majorizing all $g \in G$, it is also the smallest T -radical map majorizing all $g \in G$, i.e., $\mathfrak{h} = \vee^{\mathcal{T}} G$. Thus \mathcal{T} is $\vee^{\mathcal{T}}$ -complete and $\vee^{\mathcal{T}} G = \vee^{\text{Rad}} G$.

As $\mathbf{0}_{\mathcal{T}}$ is the smallest element in \mathcal{T} , it follows from Lemma 7.1(iii) that \mathcal{T} is a complete lattice and $\mathfrak{n} := \wedge^{\mathcal{T}} G = \vee^{\mathcal{T}} \mathcal{T}_G$ is the largest element in \mathcal{T}_G (see (7.1)). As \mathcal{T} is a complete lattice, Lemma 7.1(i) implies that $\vee^{\mathcal{T}} G = \wedge^{\mathcal{T}} \mathcal{T}^G$ is the smallest in \mathcal{T}^G . This completes the proof of (a) and (b).

(c) As \mathfrak{n} is T -radical, $\text{Fix}(\mathfrak{n})$ is an enveloping sublattice of Q by Proposition 7.5. As $\mathfrak{n} \leq g$ for all $g \in G$, $\text{Fix}(\mathfrak{n})$ contains $\cup_{g \in G} \text{Fix}(g)$ by Lemma 7.12. If $\cup_{g \in G} \text{Fix}(g) \subseteq L$ for some enveloping sublattice L , then $L = \text{Fix}(f_L)$ for a T -radical map f_L . By Lemma 7.12, $f_L \leq g$ for $g \in G$. Hence $f_L \leq \mathfrak{n}$, as \mathfrak{n} is the largest T -radical map minorizing all $g \in G$. By Lemma 7.12, $\text{Fix}(\mathfrak{n}) \subseteq L$.

(ii) By Theorem 7.13, Rad_Q is a complete lattice. As $\vee^{\mathcal{T}} G = \vee^{\text{Rad}} G$ for all $G \subseteq \mathcal{T}$, \mathcal{T} is a \vee^{Rad} -complete sublattice of Rad . As $\mathbf{0}_{\mathcal{T}} \in \mathcal{T}$, \mathcal{T} is inscribing in Rad by Theorem 7.3. ■

7.4 Transfinite extensions of relations

For $f, g \in \text{Map}(Q)$, their superposition $g \circ f$ is defined by $(g \circ f)(x) = g(f(x))$. If g is a radical map then $f \lesssim g \circ f$. If f, g are radical then $g \circ f$ is radical. Consider the following process of transfinite superposition that has wide applications in the theory of radicals of rings and algebras.

Let $G \subseteq \text{Rad}$. For an interval $[1, \gamma]$ of ordinals, the set $(h_\alpha)_{1 \leq \alpha \leq \gamma}$ in Rad is an *ascending superposition G -series* if

$$\begin{aligned} h_1 &\in G, \quad h_{\alpha+1} = g^\alpha \circ h_\alpha \text{ for each } \alpha \in [1, \gamma) \text{ and some } g^\alpha \in G, \\ h_\beta &= \bigvee_{\alpha < \beta} h_\alpha \text{ for each limit ordinal } \beta \text{ (cf. (4.4)).} \end{aligned} \quad (7.12)$$

$$\text{If } g \circ h_\gamma = h_\gamma \text{ for all } g \in G, \text{ then the series is } \textit{maximal}. \quad (7.13)$$

Let $\mathbf{0}_{\text{Rad}} \notin G$. As all $g \in G$ are radical maps, $h_{\alpha+1}$ is larger than h_α for all α . So the cardinality of $(h_\alpha)_{1 \leq \alpha \leq \gamma}$ is not larger than the cardinality of Rad . Thus each G -series extends to a maximal one.

Proposition 7.16 *Let $G \subseteq \text{Rad}$ and $L = \bigcap_{g \in G} \text{Fix}(g)$. Then L is an enveloping set. Moreover,*

(i) *for each maximal ascending superposition G -series $(h_\alpha)_{\alpha \leq \gamma}$,*

$$f_L = h_\gamma = \bigvee^{\text{Rad}} G \in \text{Rad}, \quad h_\gamma = g \circ h_\gamma \text{ for all } g \in G \text{ and } \text{Fix}(h_\gamma) = L;$$

(ii) *if f is a radical map and $g \circ f = f$ for all $g \in G$, then $h_\gamma \lesssim f$;*

(iii) *if $x \leq y \in L$ then $h_\gamma(x) \leq y$.*

Proof. (i) Set $\mathfrak{h} = \bigvee^{\text{Rad}} G$. By Theorem 7.13, $\mathfrak{h} \in \text{Rad}$ and $\text{Fix}(\mathfrak{h}) = L$. As $(h_\alpha)_{1 \leq \alpha \leq \gamma}$ is maximal, $h_\gamma = g \circ h_\gamma$ for all $g \in G$. So $h_\gamma(x) = g(h_\gamma(x)) \in L$ for all $x \in Q$. Hence $h_\gamma(Q) \subseteq L$ and $\text{Fix}(h_\gamma) \subseteq L$.

If $z \in L$ then $g(z) = z$ for all $g \in G$. As $h_1 \in G$, $h_1(z) = z$ and, by induction, $h_2(z) = z, \dots, h_\alpha(z) = z$. Thus $L \subseteq \text{Fix}(h_\gamma)$. Hence $L = \text{Fix}(h_\gamma)$ and $h_\gamma(h_\gamma(x)) = h_\gamma(x)$ for all $x \in Q$.

As $x \leq g(x)$ for all $x \in Q$ and $g \in G$, we have $x \leq h_1(x) \leq h_2(x) \dots$. By induction, $x \leq h_\gamma(x) = h_\gamma(h_\gamma(x))$. If $x \leq y$ then, similarly by induction, $h_\gamma(x) \leq h_\gamma(y)$. So $h_\gamma \in \text{Rad}$. As $\text{Fix}(\mathfrak{h}) = L = \text{Fix}(h_\gamma)$, we have $\mathfrak{h} = h_\gamma = f_L$ by Lemma 7.12 and Proposition 7.5.

(ii) Let f be a radical map and $g \circ f = f$ for all $g \in G$. As $x \leq f(x)$ for all $x \in Q$, $g(x) \leq g(f(x)) = f(x)$ for all $g \in G$. Thus $h_1 \lesssim f$. Similarly, we prove by induction that $h_\gamma \lesssim f$.

(iii) As h_γ is a radical map, $h_\gamma(x) \leq h_\gamma(y)$ by (2.13), and $h_\gamma(y) = y$ by (i). ■

By Proposition 7.16, the radical map $f_L = \bigvee^{\text{Rad}} G = h_\gamma$ for $L = \bigcap_{g \in G} \text{Fix}(g)$, is obtained "step by step" via maximal ascending superposition G -series. Note that, if for some \lll in $\text{Ref}(Q)$,

$$g(\lll) = \lll \text{ for all } g \in G, \text{ then } \lll \text{ belongs to } L \text{ and } f_L(\lll) = \lll. \quad (7.14)$$

By Theorem 7.7, $L_{\text{uc}}, L_{\text{dc}}, L_{\text{ue}}, L_{\text{de}}, L_{\text{o}}$ are enveloping sets in $\text{Ref}(Q)$. So, by Proposition 7.5, $f_{L_{\text{uc}}}, f_{L_{\text{dc}}}, f_{L_{\text{ue}}}, f_{L_{\text{de}}}, f_{L_{\text{o}}}$ are radical maps. Set $G = \{f_{L_{\text{uc}}}, f_{L_{\text{dc}}}, f_{L_{\text{ue}}}, f_{L_{\text{de}}}, f_{L_{\text{o}}}\}$. By Corollary 7.4, $L = L_{\text{uc}} \cap L_{\text{dc}} \cap L_{\text{ue}} \cap L_{\text{de}} \cap L_{\text{o}} = \bigcap_{g \in G} \text{Fix}(g)$ is enveloping. As above, $f_L = \bigvee^{\text{Rad}} G = h_\gamma$ for any maximal ascending superposition G -series $(h_\alpha)_{1 \leq \alpha \leq \gamma}$ (see (7.12)). Set

$$\widetilde{\lll} = h_\gamma(\lll) = f_L(\lll) \text{ for } \lll \text{ from } \text{Ref}(Q). \quad (7.15)$$

Proposition 7.17 For each \ll from $\text{Ref}(Q)$, $\widetilde{\ll}$ is the smallest **TT**-order majorizing \ll .

Proof. By Proposition 7.16, $h_\gamma \in \text{Rad}_{\text{Ref}(Q)}$ and $h_\gamma = g \circ h_\gamma$ for all $g \in G$. Hence $g(\widetilde{\ll}) = g(h_\gamma(\ll)) = h_\gamma(\ll) = \widetilde{\ll}$ for all $g \in G$. So, by (7.14), $\widetilde{\ll}$ belongs to L , so that it is an up- and down-contiguous, and up- and down-expanded order, i.e., it is an **TT**-order.

If \prec is a **TT**-order then it is contiguous and expanded (see Definition 2.3). So \prec belongs to $L = \bigcap_{g \in G} \text{Fix}(g)$. Thus $g(\prec) = \prec$ for all $g \in G$. If $\ll \subseteq \prec$ (see (1.1)) then $\widetilde{\ll} = h_\gamma(\ll) \subseteq \prec$ by Proposition 7.16(iii). ■

By Theorem 7.7, $L_\triangleright, L_\triangleleft$ are enveloping sets in $\text{Ref}(Q)$. Hence $G_{\triangleright\triangleleft} = \{f_{L_\triangleright}, f_{L_\triangleleft}\}$ consists of radical maps, so that $L_{\triangleright\triangleleft} = L_\triangleright \cap L_\triangleleft$ is enveloping. The radical map $f_L = \sqrt{\text{Rad}} G_{\triangleright\triangleleft}$ is obtained by constructing a maximal ascending superposition $G_{\triangleright\triangleleft}$ -series $(h_\alpha)_{1 \leq \alpha \leq \gamma}$ (see (7.12)) and $f_{L_{\triangleright\triangleleft}} = h_\gamma$. For \ll from $\text{Ref}(Q)$, set

$$\overline{\ll} = h_\gamma(\ll) = f_{L_{\triangleright\triangleleft}}(\ll). \quad (7.16)$$

Example 7.18 The relation $\overline{\ll}$ in Q can be neither contiguous, nor expanded. Indeed,

- 1) Let $Q = \{\mathbf{0}, a, \mathbf{1}\}$, $\mathbf{0} < a < \mathbf{1}$ and $\mathbf{0} \ll \mathbf{1}$. Then $\overline{\ll} = \ll$ is not contiguous.
- 2) Let Q and \ll be as in Example 4.12. Then $\overline{\ll} = \ll$ is not expanded.

Proposition 7.19 (i) $\overline{\ll} = \overline{\ll}^\triangleright = \overline{\ll}^\triangleleft$ and $\overline{\ll}$ is the smallest of all relations \prec satisfying $\ll \subseteq \prec = \prec^\triangleright = \prec^\triangleleft$.

(ii) $\overline{\ll} \subseteq \widetilde{\ll}$; if $\overline{\ll}$ is a **TT**-order then $\overline{\ll} = \widetilde{\ll}$.

(iii) If \ll is an expanded order then $\overline{\ll} = \ll$.

(iv) Let Q be a chain. If \ll is up- or down-contiguous, $\overline{\ll}$ is up- or down-contiguous, respectively.

Proof. The proof of (i) is the same as in Proposition 7.17.

(ii) As $\widetilde{\ll}$ is a **TT**-order, $\ll \subseteq \widetilde{\ll} = \overline{\ll}^\triangleright = \overline{\ll}^\triangleleft$ by Theorem 4.9. Hence $\overline{\ll} \subseteq \widetilde{\ll}$ by (i).

By Proposition 7.17, $\widetilde{\ll}$ is the smallest **TT**-order majorizing \ll . So, if $\overline{\ll}$ is a **TT**-order then $\widetilde{\ll} \subseteq \overline{\ll}$. Thus $\overline{\ll} = \widetilde{\ll}$.

(iii) If \ll is an expanded order, it follows from Theorem 4.9 that $\ll = \ll^\triangleleft = \ll^\triangleright$. Thus $g(\ll) = \ll$ for all $g \in G_{\triangleright\triangleleft}$. By (7.14), $\overline{\ll} = h_\gamma(\ll) = \ll$.

(iv) If Q is a chain then, by Lemma 4.11, if \ll is up-contiguous, the relations \ll^\triangleleft and \ll^\triangleright are up-contiguous. By induction, $\overline{\ll}$ is up-contiguous. The down-contiguous case is similar. ■

By Proposition 7.19, $\overline{\ll} \subseteq \widetilde{\ll}$. Below we consider the case when they coincide.

Proposition 7.20 Let Q have properties (JIDC) and (MIDC) (see Definition 6.8). If \ll is an **HH**-relation then $\overline{\ll} = \widetilde{\ll}$ is an **RR**-order.

Proof. Let Q have properties (JIDC) and (MIDC) and let $G_{\triangleright\triangleleft} = \{f_{L_\triangleright}, f_{L_\triangleleft}\}$. By Lemma 7.9, the radical maps $f_{L_\triangleright}, f_{L_\triangleleft}$ on Q act by $f_{L_\triangleright}(\ll) = \ll^\triangleright$ and $f_{L_\triangleleft}(\ll) = \ll^\triangleleft$. Let \ll be an **HH**-relation. By Lemma 6.9, \ll^\triangleleft is an **H**-order and a dual **R**-order, and \ll^\triangleright is an **R**-order and a dual **H**-order. So $f_{L_\triangleleft}(\ll) = \ll^\triangleleft$ and $f_{L_\triangleright}(\ll) = \ll^\triangleright$ are **HH**-orders.

Let $(h_\alpha)_{1 \leq \alpha \leq \gamma}$ be a maximal ascending superposition $G_{\triangleright\triangleleft}$ -series Set $\ll_\alpha = h_\alpha(\ll)$. If \ll_α is an **HH**-order, then $\ll_{\alpha+1}$ is either $f_{L_\triangleright}(\ll_\alpha)$, or $f_{L_\triangleleft}(\ll_\alpha)$, so that it is an **HH**-order by above.

Let β be a limit ordinal. By (7.12), $\ll_{\beta} = h_{\beta}(\ll) \stackrel{(7.12)}{=} \bigvee_{\alpha < \beta} h_{\alpha}(\ll) = \bigvee_{\alpha < \beta} \ll_{\alpha}$. If $a \ll_{\beta} b$ in Q , it follows from (5.6) that $a \ll_{\alpha} b$ for some $\alpha < \beta$. As \ll_{α} is an **HH**-order, $a \wedge z \ll_{\alpha} b \wedge z$ and $a \vee z \ll_{\alpha} b \vee z$ for all $z \in Q$. Hence, by (5.6), $a \wedge z \ll_{\beta} b \wedge z$ and $a \vee z \ll_{\beta} b \vee z$. Thus \ll_{β} is an **HH**-order. So, by induction, $\overline{\ll} = h_{\gamma}(\ll)$ is an **HH**-order.

By (7.13), $\overline{\ll}^{\triangleleft} = f_{L^{\triangleleft}}(\overline{\ll}) = f_{L^{\triangleleft}}(h_{\gamma}(\ll)) = h_{\gamma}(\ll) = \overline{\ll}$ and $\overline{\ll}^{\triangleright} = f_{L^{\triangleright}}(\overline{\ll}) = f_{L^{\triangleright}}(h_{\gamma}(\ll)) = h_{\gamma}(\ll) = \overline{\ll}$. Then, by Corollary 5.6, $\overline{\ll}$ is an **RR**-order. So $\overline{\ll}$ is a **TT**-order. By Proposition 7.19, $\overline{\ll} = \widetilde{\ll}$. ■

Corollary 7.21 *Let a modular complete lattice Q have properties (JIDC) and (MIDC) (in particular, Q is a complete chain). Let $<_{\mathfrak{g}}$ and $<_{\mathfrak{c}}$ be the relations in Q defined in Definition 6.1. Then $\overline{<_{\mathfrak{g}}} = \widetilde{<_{\mathfrak{g}}}$ and $<_{\mathfrak{c}} = \overline{<_{\mathfrak{c}}} = \widetilde{<_{\mathfrak{c}}}$ are **RR**-orders.*

Proof. By Corollary 6.7, $<_{\mathfrak{g}}$ is an **HH**-relation, as Q is modular. Since Q has properties (JIDC) and (MIDC), Proposition 7.20 implies that $\overline{<_{\mathfrak{g}}} = \widetilde{<_{\mathfrak{g}}}$ is an **RR**-order.

By Theorem 6.10, $\leq_{\mathfrak{c}}$ is an **RR**-order, as Q is modular and has properties (JIDC) and (MIDC). Hence $<_{\mathfrak{c}} = <_{\mathfrak{c}}^{\triangleleft} = <_{\mathfrak{c}}^{\triangleright}$ by Corollary 5.6. So $<_{\mathfrak{c}} = \overline{<_{\mathfrak{c}}}$ by (7.14). It also follows from Proposition 7.20 that $\overline{\ll_{\mathfrak{c}}} = \widetilde{\ll_{\mathfrak{c}}}$. ■

Problem 7.22 *Let a modular lattice do not have properties (JIDC) and (MIDC). Will $\overline{<_{\mathfrak{g}}} = \widetilde{<_{\mathfrak{g}}}$ and $<_{\mathfrak{c}} = \overline{<_{\mathfrak{c}}} = \widetilde{<_{\mathfrak{c}}}$? Will they still be **RR**-orders?*

Let \ll belong to $\text{Ref}(Q)$. We say that a chain C in Q is *complete \ll -gap dense* if,

$$\text{for all } z < w \text{ in } C \text{ there are } u \ll v \text{ in } [z, w]_C \text{ such that } [u, v]_C \text{ is a gap in } C. \quad (7.17)$$

Consider the following reflexive relation $\ll_{\mathfrak{g-d}}$ on Q : for $a < b$, we write

$$a \ll_{\mathfrak{g-d}} b \text{ if there is a complete } \ll \text{-gap dense chain from } a \text{ to } b. \quad (7.18)$$

Example 7.23 *The relation $\ll_{\mathfrak{g-d}}$ in Q can be neither contiguous, nor expanded. Indeed,*

1) Let $Q = \{\mathbf{0}, a, b, \mathbf{1}\}$, $\mathbf{0} < a < \mathbf{1}$, $\mathbf{0} < b < \mathbf{1}$. Let \ll be a reflexive order in Q such that $\mathbf{0} \ll a \ll \mathbf{1}$. Then $\ll_{\mathfrak{g-d}} = \ll$ and $\mathbf{0} \ll_{\mathfrak{g-d}} \mathbf{1}$. However $[\mathbf{0}, \mathbf{1}] \not\subseteq [\ll_{\mathfrak{g-d}}, \mathbf{1}]$. So $\ll_{\mathfrak{g-d}}$ is not contiguous.

2) Let \ll be the relation considered in Example 4.12. Then $\ll_{\mathfrak{g-d}} = \ll$ is not expanded. ■

Note that $\ll_{\mathfrak{g-d}}$ is an order. Each complete upper (lower) \ll -gap chain is \ll -gap dense.

The proof of the following lemma is evident.

Lemma 7.24 *Let T be a chain from a to b such that each $t \in T \setminus \{b\}$ has an immediate successor t_s in T , i.e., $[t, t_s]_T = \{t, t_s\}$ is a gap in T . Let \ll belong to $\text{Ref}(Q)$ and suppose that, for each $t \in T \setminus \{b\}$, there is a \ll -gap dense chain C_t in Q from t to t_s . Set $C_b = \{b\}$. Then the chain $C = \cup_{t \in T} C_t$ is \ll -gap dense.*

Corollary 7.25 (i) *For each \ll from $\text{Ref}(Q)$, $\ll \subseteq \ll_{\mathfrak{g-d}} = (\ll_{\mathfrak{g-d}})^{\triangleright} = (\ll_{\mathfrak{g-d}})^{\triangleleft}$. So $\overline{\ll_{\mathfrak{g-d}}} = \ll_{\mathfrak{g-d}}$.*

(ii) *For each \ll from $\text{Ref}(Q)$, $\overline{\ll} \subseteq \ll_{\mathfrak{g-d}}$.*

Proof. (i) Clearly, $\ll \subseteq \ll_{g-d}$ and \ll_{g-d} is an order. Set $\prec = \ll_{g-d}$. If $a \prec^{\triangleright} b$ then there is a complete upper \prec -gap chain T from a to b , i.e., each $t \in T \setminus \{b\}$ has the immediate \prec -successor t_s in T . Then there is a complete \ll -gap dense chain C^t in Q from t to t_s . Set $C^b = \{b\}$. By Lemma 7.24, $C = \cup_{t \in T} C^t$ is a complete \ll -gap dense chain from a to b . Thus $a \prec b$, so that $\prec^{\triangleright} \subseteq \prec$. As $\prec \subseteq \prec^{\triangleright}$ by (4.7), we have $\prec^{\triangleright} = \prec$. Similarly, $\prec^{\triangleleft} = \prec$. By (7.14), $\overline{\prec} = \prec$.

(ii) By Proposition 7.19(i), $\overline{\ll}$ is the smallest of all relations \prec satisfying the condition $\ll \subseteq \prec = \prec^{\triangleright} = \prec^{\triangleleft}$. As the relation \ll_{g-d} satisfies this condition by (i), $\overline{\ll} \subseteq \ll_{g-d}$. ■

In general, $\overline{\ll} \neq \ll_{g-d}$ in Q , even if Q is a chain. To show this, consider the gap relation $<_g$ (see Definition 6.1). First, it should be noted that, although the relations $<_g$ and $<_c$ are not compatible (if $a <_c b$ then $a \not<_g b$ and vice versa), the relations $\overline{<_g}$ and $<_c$ can be compatible, as the lattice Q in Example 6.3 shows: $\mathbf{0} <_c \mathbf{1}$ and $\mathbf{0} \overline{<_g} \mathbf{1}$, as $\mathbf{0} <_g a$ and $a <_g \mathbf{1}$.

Proposition 7.26 *If Q has no continuous chains (see Definition 6.1) then $(<_g)_{g-d} = \leq$.*

Proof. For $a < b$ in Q , let C be a maximal chain from a to b . By Lemma 4.5, C is complete. By (7.18), C is $<_g$ -gap dense if

$$\text{each interval } [x, y]_C \subseteq C \text{ contains a gap } [u, v]_Q. \quad (7.19)$$

Hence, if C is not \ll_{g-d} -gap dense, then there is an interval $[x, y]_C$ in C without gaps. So, by Definition 6.1, $[x, y]_C$ is a continuous chain, a contradiction. Thus C is $<_g$ -gap dense, so that $a (<_g)_{g-d} b$. Hence $\leq \subseteq (<_g)_{g-d}$. As we always have $(<_g)_{g-d} \subseteq \leq$, so $(<_g)_{g-d} = \leq$. ■

We shall now consider an example of a chain, for which $\overline{<_g} \neq (<_g)_{g-d}$.

Example 7.27 Let $Q = \{(t; n) : t \in [0, 1] \subset \mathbb{R}, n = 0, 1\}$. Let $(t; n) < (s; m)$, if either $t < s$, or $t = s$, $n = 0$ and $m = 1$. Then Q is a complete chain from $\mathbf{0} = (0; 0)$ to $\mathbf{1} = (1; 1)$.

As each interval $[(t, 0), (t, 1)]$ in Q is a gap, all intervals in Q have gaps. So, by (7.19), Q is $<_g$ -gap dense. By Proposition 7.26, $(<_g)_{g-d} = \leq$. Hence $\mathbf{0} (<_g)_{g-d} \mathbf{1}$.

On the hand, it is easy to see that $<_g = \overline{<_g}$. So $\mathbf{0} \not<_g \mathbf{1}$. Thus $\overline{<_g} \neq (<_g)_{g-d}$. ■

Although the relations $\overline{\ll}$ and \ll_{g-d} do not coincide, they are closely linked.

Theorem 7.28 *If $a \overline{\ll} b$ then there is a complete \ll -gap dense chain C from a to b such that $x \overline{\ll} y$ for all $x < y$ in C .*

Proof. Let $G_{\triangleright\triangleleft} = \{f_{L_{\triangleright}}, f_{L_{\triangleleft}}\}$ and $(h_{\alpha})_{1 \leq \alpha \leq \gamma}$ be a maximal ascending superposition $G_{\triangleright\triangleleft}$ -series, $h_1 = f_{L_{\triangleleft}}$ and $h_{\gamma} = f_{L_{\triangleright}}$. Set $\ll_{\alpha} = h_{\alpha}(\ll)$ and $\overline{\ll} = \ll_{\gamma}$. Consider the following induction.

Let $a \ll_1 b$, i.e., $a \ll^{\triangleleft} b$ for some $a, b \in Q$. Then there is a complete lower \ll -gap chain C^1 from a to b . As each $x \in C^1 \setminus \{a\}$ has an immediate \ll -predecessor x_p : $[x_p, x]_C$ is a gap in C^1 and $x_p \ll x$. So C^1 is \ll -gap dense. For all $x < y$ in C^1 , $[x, y]_{C^1}$ is a complete lower \ll -gap chain from x to y . Thus $x \ll^{\triangleleft} y$.

Assume that if $a \ll_{\alpha} b$, for some $a, b \in Q$ and some $\alpha < \gamma$, then

$$1) \text{ there is a complete } \ll\text{-gap dense chain } C^{\alpha} \text{ from } a \text{ to } b \text{ and} \quad (7.20)$$

$$2) x \ll_{\alpha} y \text{ for all } x < y \text{ in } C^{\alpha}. \quad (7.21)$$

Let now $a \ll_{\alpha+1} b$, say, $a (\ll_{\alpha})^{\triangleright} b$ for some $a, b \in Q$. Then there is a complete upper gap \ll_{α} -chain T from a to b , i.e., each $t \in T \setminus \{b\}$ has an immediate \ll_{α} -successor $t_s \in T - [t, t_s]_T = \{t, t_s\}$ is a gap in T and $t \ll_{\alpha} t_s$. By (7.20), for each $t \in T \setminus \{b\}$, there is a complete \ll -gap dense chain C_t from t to t_s satisfying (7.21). Set $C_b = \{b\}$. By Lemma 7.24, $C^{\alpha+1} = \cup_{t \in T} C_t$ is a complete \ll -gap dense chain from a to b .

Let $x < y$ in $C^{\alpha+1}$. Then $x \in C_t, y \in C_{t'}$ for some $t \leq t'$ in T . If $t = t'$ then $x, y \in C_t$ and $x \ll_{\alpha} y$, by (7.21). Thus $x \ll_{\alpha+1} y$ by (4.7). If $t \neq t'$ then either $x < t_s < y$, or $x = t_s < y$, or $x < t_s = y$. Let $x < t_s < y$. As $C_t, C_{t'}$ satisfy (7.21), $x \ll_{\alpha} t_s$ and $t' \ll_{\alpha} y$. Hence $x \ll_{\alpha+1} t_s$ and $t' \ll_{\alpha+1} y$ by (4.7). As T is a complete upper gap \ll_{α} -chain, $t_s \ll_{\alpha+1} t'$. As $\ll_{\alpha+1}$ is an order, $x \ll_{\alpha+1} y$. Similarly, $x \ll_{\alpha+1} y$ when $x = t_s < y$, or $x < t_s = y$. Thus (7.21) holds for all $x < y$ in $C^{\alpha+1}$ and $\ll_{\alpha+1}$.

Let β be a limit ordinal and let, for each $\alpha < \beta$, $a \ll_{\alpha} b$ imply (7.20) and (7.21). Let now $a \ll_{\beta} b$. As $\ll_{\beta} = h_{\beta}(\ll) \stackrel{(7.12)}{=} \vee_{\alpha < \beta} h_{\alpha}(\ll) = \vee_{\alpha < \beta} \ll_{\alpha}$, it follows from (5.6) that $a \ll_{\alpha} b$ for some $\alpha < \beta$. Hence there is a complete \ll -gap dense chain C^{α} from a to b satisfying (7.21). Set $C^{\beta} = C^{\alpha}$. Then C^{β} satisfies (7.20). Let $x < y$ in C^{β} . By (7.21), $x \ll_{\alpha} y$. Hence, by (5.6), $x \ll_{\beta} y$. Thus (7.20) and (7.21) hold for β . By induction, if $a \ll_{\gamma} b$ then there is a complete \ll -gap dense chain C^{γ} from a to b satisfying (7.21) for $\alpha = \gamma$. As $\ll_{\gamma} = \ll$, the proof is complete. ■

8 Relations in lattices of subspaces of Banach spaces

Let X be a Banach space. The set $\text{Ln}(X)$ of all linear subspaces of X and the set $\text{Cl}(X)$ of all closed subspaces of X are complete lattices with order \subseteq ,

$$\begin{aligned} \mathbf{0} &= \{0\}, \quad \mathbf{1} = X, \quad \wedge G = \bigcap_{Y \in G} Y \text{ for a subset } G \neq \emptyset \text{ in } \text{Ln}(X) \text{ and } \text{Cl}(X), \\ \vee G &= \sum_{Y \in G} Y \text{ for } G \subseteq \text{Ln}(X) \text{ and } \vee G = \overline{\sum_{Y \in G} Y} \text{ for } G \subseteq \text{Cl}(X). \end{aligned} \quad (8.1)$$

As before, for $L \subseteq M$ in Q , we write

$$L <_{\mathfrak{g}} M \text{ if either } L = M, \text{ or } [L, M]_Q \text{ is a gap in } Q. \quad (8.2)$$

In this section we concentrate on the relation $<_{\mathfrak{g}}$ in the sublattices of $\text{Ln}(X)$ and $\text{Cl}(X)$. We show that all sublattices of $\text{Ln}(X)$ are modular, so that $<_{\mathfrak{g}}$ is an **HH**-relation in all of them. The lattice $\text{Cl}(X)$ is not modular. Although $<_{\mathfrak{g}}$ is an **HH**-relation in many of its sublattices (Corollary 8.24), there are sublattices of $\text{Cl}(X)$, where it is neither an **H**-, nor a dual **H**-relation (Corollary 8.21).

Let Q be a complete sublattice of $\text{Ln}(X)$, or $\text{Cl}(X)$. As in (2.2) and (5.7), for \ll from $\text{Ref}(Q)$,

$$\begin{aligned} [\ll, L]_Q &= \{K \in Q: K \ll L\}, \quad [L, \ll]_Q = \{K \in Q: L \ll K\} \text{ for } L \in Q; \\ \sigma_{\ll}(L) &= \wedge[\ll, L]_Q = \bigcap \{K: K \in [\ll, L]_Q\} \text{ and } s^{\ll}(L) = \vee[L, \ll]_Q, \text{ so that} \end{aligned} \quad (8.3)$$

$$s^{\ll}(L) = \sum_{K \in [L, \ll]_Q} K \text{ in } \text{Ln}(X), \text{ and } s^{\ll}(L) = \overline{\sum_{K \in [L, \ll]_Q} K} \text{ in } \text{Cl}(X). \quad (8.4)$$

It follows from Theorem 5.9 that if \ll is an **H**-relation then \ll^{\triangleright} is an **R**-order in Q and $\mathfrak{r}_L = s^{\ll^{\triangleright}}(L)$ is the unique \ll^{\triangleright} -radical in $[L, X]_Q$, i.e., $L \ll^{\triangleright} \mathfrak{r}_L \ll^{\triangleright} X$.

If \ll is a dual \mathbf{H} -relation then \ll^\triangleleft is a dual \mathbf{R} -order in Q and $\mathfrak{p}_L = \sigma_{\ll^\triangleleft}(L)$ is the unique dual \ll^\triangleleft -radical in $[\{0\}, L]_Q$, i.e., $\{0\} \xrightarrow{\ll^\triangleleft} \mathfrak{p}_L \ll^\triangleleft L$. By (5.8) and (5.9),

$$s^{\ll}(L) \subseteq \mathfrak{r}_L \text{ for an } \mathbf{H}\text{-relation } \ll, \text{ and } \mathfrak{p}_L \subseteq \sigma_{\ll}(L) \text{ for a dual } \mathbf{H}\text{-relation } \ll. \quad (8.5)$$

Thus, for a complete sublattice Q of $\text{Ln}(X)$, or of $\text{Cl}(X)$, Theorem 5.9 yields

Theorem 8.1 (i) *Let \ll be an \mathbf{H} -relation and $L \subseteq K$ in Q .*

- 1) *If $K \subseteq \mathfrak{r}_L$ then there is an ascending \ll -series of spaces in Q from K to \mathfrak{r}_L .*
- 2) *If $\mathfrak{r}_L \not\subseteq K \neq X$ then K has a \ll -successor $S \in Q$: $K \ll S$; \mathfrak{r}_L has no \ll -successor.*
- 3) *If there is an ascending \ll -series of spaces in Q from L to K then $K \subseteq \mathfrak{r}_L$.*

(ii) *Let \ll be a dual \mathbf{H} -relation and $K \subseteq L$ in Q .*

- 1) *If $\mathfrak{p}_L \subseteq K$ then there is a descending \ll -series of spaces in Q from K to \mathfrak{p}_L .*
- 2) *If $\{0\} \neq K \not\subseteq \mathfrak{p}_L$ then K has a \ll -predecessor $P \in Q$: $P \ll K$; \mathfrak{p}_L has no \ll -predecessor.*
- 3) *If there is a descending \ll -series of spaces in Q from L to K then $\mathfrak{p}_L \subseteq K$.*

If X is separable then the \ll -series in Theorem 8.1 are isomorphic to \mathbb{N} .

Proposition 8.2 *Let X be a separable Banach space and Q be a complete sublattice of $\text{Cl}(X)$.*

(i) *If \ll is a dual \mathbf{H} -relation in Q then there are spaces*

$$\dots \ll Y_n \ll \dots \ll Y_1 \ll X \text{ in } Q \text{ such that } \bigcap_{n=1}^{\infty} Y_n = \sigma_{\ll}(X). \quad (8.6)$$

There are also spaces $\dots \ll^\triangleleft Z_n \ll^\triangleleft \dots \ll^\triangleleft Z_1 \ll^\triangleleft X$ in Q such that

$$\bigcap_{n=1}^{\infty} Z_n = \mathfrak{p}_X \text{ is the dual } \ll^\triangleleft\text{-radical in } X. \quad (8.7)$$

(ii) *If \ll is an \mathbf{H} -relation then there are spaces $\{0\} \ll Y_1 \ll \dots \ll Y_n \ll \dots$ in Q and spaces $\{0\} \ll^\triangleright Z_1 \ll^\triangleright \dots \ll^\triangleright Z_n \ll^\triangleright \dots$ in Q such that*

$$\overline{\sum_{n=1}^{\infty} Y_n} = s^{\ll}(\{0\}) \text{ and } \overline{\sum_{n=1}^{\infty} Z_n} = \mathfrak{r}_{\{0\}} \text{ is the } \ll^\triangleright\text{-radical in } X.$$

Proof. (i) Set $\omega = \sigma_{\ll}(X)$. Let \mathbf{B} be the unit ball of X^* . For each $L \in [\ll, X]_Q$, set $W_L = \{f \in \mathbf{B} : L \subseteq \ker(f)\}$. Set $W = \cup_{L \ll X} W_L$. Then $W \subseteq \mathbf{B}$. As X is separable, \mathbf{B} is a separable metric space in the $\sigma(X^*, X)$ -topology (see [Sch, Section 4.1.7]). Hence W has a $\sigma(X^*, X)$ dense sequence $\{f_k : k \in \mathbb{N}\}$. As each $L = \cap_{f \in W_L} \ker f$, we have $\cap_k \ker(f_k) = \cap_{f \in W} \ker(f) = \cap_{L \ll X} L = \omega$.

For each $k \in \mathbb{N}$, choose $L_k \in [\ll, X]_Q$ such that $f_k \in W_{L_k}$. Then, by (8.4), $\omega \subseteq \cap_{k=1}^{\infty} L_k \subseteq \cap_k \ker(f_k) = \omega$. Hence $\cap_{k=1}^{\infty} L_k = \omega$.

Set $Y_n = L_1 \cap \dots \cap L_n \in Q$. Then $Y_{k+1} \subseteq Y_k$ for all k , and $\cap_{n=1}^{\infty} Y_n = \cap_{k=1}^{\infty} L_k = \omega$. Suppose, by induction, that $Y_k \ll Y_{k-1} \ll \dots \ll Y_1 \ll X$ for some k . As all $L_n \in [\ll, X]_Q$, we have $L_{k+1} \ll X$. Since \ll is a dual \mathbf{H} -order, it follows from Lemma 3.1(ii) that

$$Y_{k+1} = Y_k \cap L_{k+1} = Y_k \wedge L_{k+1} \ll Y_k \wedge X = Y_k \cap X = Y_k.$$

Hence (8.6) holds. As $\mathfrak{p}_X = \sigma_{\ll^\triangleleft}(X)$ and \ll^\triangleleft is a dual \mathbf{R} -order in Q , (8.7) follows from (8.6).

Part (ii) can be proved similarly. ■

8.1 Relations $<_{\mathfrak{g}}$ and \ll_n in sublattices of $\text{Ln}(X)$.

Proposition 8.3 $\text{Ln}(X)$ is modular and $<_{\mathfrak{g}}$ is an **HH**-relation in $\text{Ln}(X)$.

Proof. Let $I, J, K \in \text{Ln}(X)$ and $I \subseteq J$. Then $\gamma_I \lambda_J (K) = I + (J \cap K)$ and $\lambda_J \gamma_I (K) = J \cap (I + K)$. Clearly, $I + (J \cap K) \subseteq J \cap (I + K)$. Conversely, if $x \in J \cap (I + K)$ then $x = i + k \in J$, where $i \in I, k \in K$. Then $i \in J$, as $I \subseteq J$. Hence $k \in J \cap K$. So $x \in I + (J \cap K)$. Thus $J \cap (I + K) \subseteq I + (J \cap K)$. Hence $I + (J \cap K) = J \cap (I + K)$. So $\gamma_I \lambda_J (K) = \lambda_J \gamma_I (K)$. Thus $\text{Ln}(X)$ is modular by (6.2).

It follows from Corollary 6.7 that $<_{\mathfrak{g}}$ is an **HH**-relation in $\text{Ln}(X)$. ■

To introduce more **HH**-relations, we will use the following result proved in Theorem 2.2 [Di].

Lemma 8.4 Let $L \subset M$ in $\text{Ln}(X)$ and $n := \dim(M/L) < \infty$. Let $K \in \text{Ln}(X)$.

- (i) If $K \subseteq M$ then $\dim(K/(L \cap K)) = \dim((L + K)/L) \leq n$.
- (ii) If $L \subseteq K$ then $\dim(M/(M \cap K)) = \dim((M + K)/K) \leq n$.
- (iii) If K, L, M are closed subspaces, then $L + K$ in (i) and $M + K$ in (ii) are closed.

Consider the following relation in $\text{Ln}(X)$. For $n \in \mathbb{N} \cup \infty$ and $L \subseteq M$ in $\text{Ln}(X)$, we write

$$L \ll_n M \text{ if } \dim(M/L) < n. \quad (8.8)$$

Corollary 8.5 All \ll_n are **HH**-relations in each sublattice of $\text{Ln}(X)$ and \ll_{∞} is an **HH**-order.

Proof. Let $L \ll_n M$, i.e., $\dim(M/L) < n$. Let $L \subseteq K$. By (8.1) and Lemma 8.4(ii), $M \vee K = M + K$ and $\dim((M + K)/K) \leq n$. Hence $K \ll_n M \vee K$. So, by Lemma 3.1(i), \ll_n is an **H**-relation.

Let $K \subseteq M$. By (8.1) and Lemma 8.4(i), $L \wedge K = L \cap K$ and $\dim(K/(L \cap K)) \leq n$. Hence $L \wedge K \ll_n K$. So, by Lemma 3.1(ii), \ll_n is a dual **H**-relation. Clearly, \ll_{∞} is an **HH**-order. ■

8.2 Relations $\ll_n, <_{\mathfrak{g}}, \sqsubset_{\mathfrak{g}}$ and $\prec_{\mathfrak{g}}$ in sublattices of $\text{Cl}(X)$.

Note that the restrictions of **H**- and dual **H**-relations to sublattices Q of $\text{Cl}(X)$ also have the same properties. However, an **H**-, or a dual **H**-relation in Q is not necessarily a restriction of a relation in $\text{Cl}(X)$ with the same property.

Making use of Lemma 8.4(iii), and repeating the proof of Corollary 8.5, we get

Corollary 8.6 $\{\ll_n\}_{n=1}^{\infty}$ are **HH**-relations in all sublattices of $\text{Cl}(X)$ and \ll_{∞} is an **HH**-order.

The relation $<_{\mathfrak{g}}$ is an **HH**-relation in many sublattices of $\text{Cl}(X)$. Corollary 6.11 yields

Lemma 8.7 If Q is a nest (a complete linearly ordered set) in $\text{Cl}(X)$ then $<_{\mathfrak{g}}$ is an **HH**-relation.

However, there are sublattices of $\text{Cl}(X)$, where $<_{\mathfrak{g}}$ is neither an **H**-, nor a dual **H**-relation (see Corollary 8.21). The main obstacle is the fact that the sum of subspaces is not necessarily closed.

To avoid this, introduce relations \sqsubset and \prec in sublattices Q of $\text{Cl}(X)$. For $L \subseteq M$ in Q , we write

$$\begin{aligned} L \sqsubset M & \text{ if } M + K \text{ is closed for each } K \in Q \text{ such that } L \subseteq K, \\ L \prec M & \text{ if } L + K \text{ is closed for each } K \in Q \text{ such that } K \subseteq M. \end{aligned} \quad (8.9)$$

We will show that the intersection of $<_{\mathfrak{g}}$ with \sqsubset is an **H**-relation and with \prec is a dual **H**-relation.

Proposition 8.8 \sqsubset is an \mathbf{H} -relation and \prec is a dual \mathbf{H} -relation in any sublattice Q of $\text{Cl}(X)$.

Proof. Let $L \sqsubset M$ in Q . For $L \subseteq K \in Q$, let us show that $K \sqsubset M \vee K$. Indeed, for each $R \in Q$, $K \subseteq R$, we have $L \subseteq R$, so that $(M \vee K) + R = M + K + R = M + R$ is closed. Thus $K \sqsubset M \vee K$. Hence \sqsubset is an \mathbf{H} -relation by Lemma 3.1(i).

Let $L \prec M$ in Q . For $K \subseteq M$ in Q , let us show that $L \cap K \prec K$. Indeed, for each $R \in Q$, $R \subseteq K$, we have $R \subseteq M$, so that $L \cap K + R = (L + R) \cap K$ is closed, as $L + R$ is closed. Thus $L \cap K \prec K$. Hence \prec is a dual \mathbf{H} -relation by Lemma 3.1(ii). ■

In many important sublattices Q of $\text{Cl}(X)$ the relations \sqsubset , \prec , \subseteq coincide. For example, they coincide if Q is a nest (a linearly ordered set of subspaces), or a commutative subspace lattice, if X is a Hilbert space (Theorem 8.23).

Consider another example. Let $B(X)$ the algebra of all bounded operators on X . A projection p in $B(X)$ is an L -projection if $\|x\| = \|px\| + \|x - px\|$ for $x \in X$. The subspace pX of X is called an L -summand. A subspace J of X is an M -ideal if

$$J^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in J\} \text{ is an } L\text{-summand in } X^*. \quad (8.10)$$

Some spaces (for example, L^q -spaces, $1 < q < \infty$) have no non-trivial M -ideals. In Banach algebras M -ideals are closed subalgebras but not necessarily ideals [Har, Theorem V.2.3]. In C^* -algebras the sets of M -ideals and all closed two-sided ideals coincide.

If I, J are M -ideals of X then $I + J$ is an M -ideal [Har, Proposition 1.11]. This yields

Corollary 8.9 If a sublattice Q of $\text{Cl}(X)$ consists of M -ideals then $\sqsubset = \prec = \subseteq$ in Q .

For a subset $S \subseteq \text{Cl}(X)$, $\text{Alg } S$ is the algebra of all operators in $B(X)$ that leave all $L \in S$ invariant. For a subalgebra \mathcal{A} of $B(X)$, $\text{Lat } \mathcal{A}$ is the lattice of all \mathcal{A} -invariant subspaces of X . A sublattice Q of $\text{Cl}(X)$ is *reflexive* if $Q = \text{Lat}(\text{Alg } Q)$. Similarly, a subalgebra \mathcal{A} of $B(X)$ is *reflexive* if $\mathcal{A} = \text{Alg}(\text{Lat } \mathcal{A})$. $\text{Alg } Q$ is a reflexive subalgebra of $B(X)$ and $\text{Lat } \mathcal{A}$ is a strongly closed, complete reflexive sublattice of $\text{Cl}(X)$.

For $L \subseteq M$ in $\text{Lat } \mathcal{A}$, let $p: M \rightarrow M/L$ be the quotient map. Set $A_p p(x) = p(Ax)$ for $A \in \mathcal{A}$ and $x \in M$. Then $\mathcal{A}_p = \{A_p: A \in \mathcal{A}\}$ is a subalgebra of the algebra $B(M/L)$ of all operators acting on M/L . To proceed further we need the following lemma.

Lemma 8.10 Let $L \subset M$ in $\text{Lat } \mathcal{A}$, let $p: M \rightarrow M/L$ and $K \in \text{Lat } \mathcal{A}$.

(i) Suppose that $K \subseteq M$ and $L + K$ is closed. Set $q: K \rightarrow K/(L \cap K)$. For $x \in K$, let $Sq(x) = p(x)$. Then $p(K)$ is closed in $p(M)$, S is an invertible contraction from $q(K)$ onto $p(K)$ and

$$SA_q q(x) = A_p Sq(x) \text{ for all } A \in \mathcal{A} \text{ and } x \in K.$$

(ii) Suppose that $L \subseteq K$ and $N := M + K$ be closed. Set $q: N \rightarrow N/K$. For $x \in M$, let $Tp(x) = q(x)$. Then T is a contraction from $p(M)$ onto $q(N)$, $\ker T = p(M \cap K)$ and

$$TA_p p(x) = A_q Tp(x) \text{ for all } A \in \mathcal{A} \text{ and } x \in M.$$

Proof. (i) As $L + K$ is closed, $p(L + K)$ is closed. So $p(K) = p(L + K)$ is closed in $p(M)$.

If $q(x) = q(y)$ then $x - y \in L \cap K$. So $p(x) = p(y)$. Thus S is a well defined linear operator from $q(K)$ onto $p(K)$. If $Sq(x) = 0$ for $x \in K$, then $p(x) = 0$. So $x \in L$. Thus $x \in L \cap K$ and $q(x) = 0$. Hence $\ker S = \{0\}$. Moreover, $\|S\| \leq 1$, as

$$\|p(x)\|_{M/L} = \inf_{t \in L} \|x + t\| \leq \inf_{t \in L \cap K} \|x + t\| = \|q(x)\|_{K/(L \cap K)} \text{ for all } x \in K.$$

Hence S is an invertible contraction. For $A \in \mathcal{A}$ and $x \in K$, $A_q q(x) = q(Ax)$, so that

$$SA_q q(x) = Sq(Ax) = p(Ax) = A_p p(x) = A_p S q(x).$$

(ii) As N is closed, $q(M) = q(N)$ is a Banach space. If $p(x) = p(y)$ then $x - y \in L$. So $q(x) = q(y)$, as $L \subseteq K$. Thus T is a well defined linear operator from $p(M)$ onto $q(N)$. If $Tp(x) = q(x) = 0$ then $x \in M \cap K$. So $\ker T = p(M \cap K)$. Moreover, $\|T\| \leq 1$, since

$$\|q(x)\|_{N/K} = \inf_{t \in K} \|x + t\| \leq \inf_{t \in L} \|x + t\| = \|p(x)\|_{M/L} \text{ for all } x \in M.$$

We also have $TA_p p(x) = Tp(Ax) = q(Ax) = A_q q(x) = A_q Tp(x)$ for $A \in \mathcal{A}$ and $x \in M$. ■

Recall that a subalgebra \mathcal{A} of $B(X)$ is *irreducible* on $L \in \text{Lat } \mathcal{A}$, if L has no non-trivial invariant subspaces. It is *algebraically irreducible*, if L has no non-trivial invariant linear manifolds.

For a sublattice Q of $\text{Cl}(X)$, consider the following relations from $\text{Ref}(Q)$ stronger than $<_{\mathfrak{g}}$:

$$\sqsubset_{\mathfrak{g}} = <_{\mathfrak{g}} \cap \sqsubset \text{ and } \prec_{\mathfrak{g}} = <_{\mathfrak{g}} \cap \prec, \quad (8.11)$$

that is, $L \sqsubset_{\mathfrak{g}} M$ if $L <_{\mathfrak{g}} M$ and $L \sqsubset M$; and $L \prec_{\mathfrak{g}} M$ if $L <_{\mathfrak{g}} M$ and $L \prec M$ for $L, M \in Q$.

However, for some pairs (L, M) , $L <_{\mathfrak{g}} M$ implies $L \sqsubset_{\mathfrak{g}} M$, for some it implies $L \prec_{\mathfrak{g}} M$. For example, if $L <_{\mathfrak{g}} X$ then $L \sqsubset_{\mathfrak{g}} X$; if $\{0\} <_{\mathfrak{g}} M$ then $\{0\} \prec_{\mathfrak{g}} M$.

Theorem 8.11 $\sqsubset_{\mathfrak{g}}$ is an **H**-relation and $\prec_{\mathfrak{g}}$ is a dual **H**-relation in each reflexive sublattice.

Proof. If a sublattice Q of $\text{Cl}(X)$ is reflexive then $Q = \text{Lat } \mathcal{A}$, where $\mathcal{A} = \text{Alg } Q$.

Let $L \sqsubset_{\mathfrak{g}} M$ in Q and $L \subseteq K \in Q$. As $L \sqsubset M$, we have that $N := K + M = K \vee M$ is invariant and closed, i.e., $N \in Q$. If we show that $K \sqsubset_{\mathfrak{g}} N$ then, by Lemma 3.1(i), $\sqsubset_{\mathfrak{g}}$ is an **H**-relation.

By Proposition 8.8, \sqsubset is an **H**-relation. Hence it follows from Lemma 3.1(i) that $K \sqsubset N$. As $\sqsubset_{\mathfrak{g}} = <_{\mathfrak{g}} \cap \sqsubset$, we only need to prove $K <_{\mathfrak{g}} N$. As $L \subseteq M \cap K \subseteq M$ and $L <_{\mathfrak{g}} M$, either $L = M \cap K$ or $M \cap K = M$. In the second case $M \subseteq K$, so that $K = N$. Thus $K <_{\mathfrak{g}} N$ by (8.2).

Let now $L = M \cap K$ and $p: M \rightarrow M/L$, $q: N \rightarrow N/K$ be the quotient maps. The operator T in Lemma 8.10(ii) is a contraction from $p(M)$ onto $q(N)$ and $\ker T = p(M \cap K) = p(L) = \{0\}$. Thus T is invertible. Moreover, $TA_p p(x) = A_q T p(x)$ for all $A \in \mathcal{A}$ and $x \in M$.

As $L <_{\mathfrak{g}} M$, the algebra \mathcal{A}_p on M/L is irreducible. Hence, as T is invertible, the algebra \mathcal{A}_q on N/K is irreducible, i.e., $K <_{\mathfrak{g}} N$. Thus $K \sqsubset_{\mathfrak{g}} N$, so that $\sqsubset_{\mathfrak{g}}$ is an **H**-relation.

Similarly, one can prove that $\prec_{\mathfrak{g}}$ is a dual **H**-relation in Q . ■

Corollary 8.12 Let $Q = \text{Lat } \mathcal{A}$ be a reflexive sublattice of $\text{Cl}(X)$.

(i) $<_{\mathfrak{g}} \cap \ll_n$ is an **HH**-relation in Q for each $1 \leq n \leq \infty$.

(ii) Suppose that whenever $L <_{\mathfrak{g}} M$ in Q then \mathcal{A}_p is algebraically irreducible in M/L , where $p: M \rightarrow M/L$. Then $<_{\mathfrak{g}} = \prec_{\mathfrak{g}}$ is a dual **H**-relation.

Proof. (i) By Lemma 8.4, the **HH**-relation \ll_n is stronger than \sqsubset and than \prec . By Theorem 8.11, $\prec_g \cap \sqsubset = \sqsubset_g$ is an **H**-relation and $\prec_g \cap \prec = \prec_g$ is a dual **H**-relation. Hence, by Lemma 5.8, $\prec_g \cap \ll_n$ is an **HH**-relation.

(ii) If $K \subset M$ in Q then $L + K$ is an \mathcal{A} -invariant manifold and $L \subseteq L + K \subseteq M$. As \mathcal{A}_p is algebraically irreducible in M/L , either $L + K = L$, or $L + K = M$. Hence $L + K$ is closed. Thus $L \prec M$. So $L \prec_g M$. ■

Algebraically irreducible representations of Banach algebras were studied in the papers of Poguntke [P], Jeu and Tomiyama [JT], Radjavi [R], Barnes [Ba] and of many others.

8.3 Superinvariant subspaces in Lat \mathcal{A} .

A subspace $W \subseteq B(X)$ is a *Lie subalgebra* of $B(X)$ if the *commutator* $AB - BA \in W$ for all $A, B \in W$. Let \mathcal{A} be a closed subalgebra of $B(X)$ and $Q = \text{Lat } \mathcal{A}$. Then

$$\text{Nor } \mathcal{A} = \{S \in B(X) : SA - AS \in \mathcal{A} \text{ for all } A \in \mathcal{A}\} \quad (8.12)$$

is a closed Lie subalgebra of $B(X)$ and $\mathcal{A}' + \mathcal{A} \subseteq \text{Nor } \mathcal{A}$, where \mathcal{A}' is the commutant of \mathcal{A} :

$$\mathcal{A}' = \{B \in B(X) : AB - BA = 0 \text{ for all } A \in \mathcal{A}\}.$$

A space in Q is *superinvariant* ([K1]) if it is invariant for all operators in $\text{Nor } \mathcal{A}$.

For $S \in B(X)$, we define the map θ_S on $\text{Cl}(X)$ by the formula $\theta_S(L) = e^S L$. If $S \in \text{Nor } \mathcal{A}$ then (see [K2, Lemmas 2.1 and 2.3]) θ_S is an isomorphism of Q and

$$L \in Q \text{ is superinvariant if and only if } \theta_S(L) = L \text{ for all } S \in \text{Nor } \mathcal{A}. \quad (8.13)$$

Recall (see (5.11)) that θ_S preserves a relation \ll in Q if $\theta_S(L) \ll \theta_S(K) \Leftrightarrow L \ll K$ for $L, K \in Q$.

Proposition 8.13 *Let all isomorphisms θ_S , $S \in \text{Nor } \mathcal{A}$, preserve a relation \ll in Q . If $L \in Q$ is superinvariant then*

- (i) *The subspaces $s^{\ll}(L)$ and $\sigma_{\ll}(L)$ in (8.3) and (8.4) are superinvariant. In particular, $s^{\ll}(\{0\})$ and $\sigma_{\ll}(X)$ are superinvariant.*
- (ii) *If \ll is an **H**-relation then the \ll^{\triangleright} -radical in $[L, X]_Q$ is superinvariant. In particular, the \ll^{\triangleright} -radical in Q is superinvariant.*
- (iii) *If \ll is a dual **H**-relation then the dual \ll^{\triangleleft} -radical in $[\{0\}, L]_Q$ is superinvariant. In particular, the dual \ll^{\triangleleft} -radical in Q is superinvariant.*

Proof. (i) As θ_S preserve \ll and as $\theta_S(L) = L$ for all $S \in \text{Nor } \mathcal{A}$ by (8.13), Proposition 5.10 gives $\theta_S(s^{\ll}(L)) = s^{\ll}(L)$ and $\theta_S(\sigma_{\ll}(L)) = \sigma_{\ll}(L)$. Then, by (8.13), $s^{\ll}(L)$ and $\sigma_{\ll}(L)$ are superinvariant. Parts (ii) and (iii) have similar proofs and follow from Proposition 5.10. ■

Combining Theorem 8.1, Propositions 8.2 and 8.13 and (8.5), we obtain

Corollary 8.14 Let \ll be a relation in $Q = \text{Lat } \mathcal{A}$ and let all $\theta_s, S \in \text{Nor } \mathcal{A}$, preserve \ll .

(i) Suppose that \mathcal{A} has no non-trivial superinvariant subspaces.

If $[\ll, X]_Q \neq \{X\}$ then $\sigma_{\ll}(X) = \{0\}$. If $[\{0\}, \ll]_Q \neq \{0\}$ then $s^{\ll}(\{0\}) = X$.

(ii) Let \ll be a dual **H**-relation. If $\sigma_{\ll}(X) = \{0\}$ then the dual \ll^{\triangleleft} -radical $\mathfrak{p} = \{0\}$ and

1) for each $L \neq \{0\}$ in Q , there is $K \subset L$ in Q such that $K \ll L$.

2) If X is separable, there are $\dots \ll Y_n \ll \dots \ll Y_1 \ll X$ in Q such that $\bigcap_{n=1}^{\infty} Y_n = \{0\}$.

(iii) Let \ll be an **H**-relation. If $s^{\ll}(\{0\}) = X$ then the \ll^{\triangleright} -radical $\mathfrak{r} = X$ and

1) for each $L \neq X$ in Q , there is $M \supset L$ in Q such that $L \ll M$.

2) If X is separable, there are $\{0\} \ll Y_1 \ll \dots \ll Y_n \ll \dots$ in Q such that $\overline{\sum_{n=1}^{\infty} Y_n} = X$.

Corollary 8.15 Isomorphisms $\theta_s, S \in \text{Nor } \mathcal{A}$, preserve the relations $<_{\mathfrak{g}}, \sqsubset, \prec, \ll_n$ for $n \in \mathbb{N} \cup \infty$. Thus all results of Propositions 8.2 and 8.13 and Corollary 8.14 hold for these relations.

Proof. For $S \in \text{Nor } \mathcal{A}$ and each $R \in Q$, set $R_s = e^S R$. Let $L <_{\mathfrak{g}} M$ in Q . If $L_s \subsetneq K \subsetneq M_s$ for some $K \in Q$, then $L \subsetneq K_{-s} \subsetneq M$ and $K_{-s} \in Q$, a contradiction. Thus $L_s <_{\mathfrak{g}} M_s$.

Let $L \sqsubset M$. Let $L_s \subset K \in Q$. Then $L \subset K_{-s}$, so that $M + K_{-s}$ is closed. Hence $(M + K_{-s})_s = M_s + K$ is closed. So $L_s \sqsubset M_s$. Conversely, if $L_s \sqsubset M_s$ then, by above, $L = (L_s)_{-s} \sqsubset (M_s)_{-s} = M$. So θ_s preserves the relation \sqsubset . Similarly, θ_s preserves the relation \prec .

If $L \subset M$ then $\dim(M/L) = \dim(M_s/L_s)$. So θ_s preserve all relations $\ll_n, n \in \mathbb{N} \cup \infty$. ■

Let us consider some cases where Corollary 8.14 can be applied.

Proposition 8.16 Let an operator $T \in B(X)$ have eigenvectors $\{e_{\lambda}\}_{\lambda \in \Lambda}$.

(i) If $X = \overline{\text{span}(e_{\lambda})_{\lambda \in \Lambda}}$ then the \ll_1^{\triangleright} -radical $\mathfrak{r} = s^{\ll_1}(\{0\}) = X$ in $\text{Lat } T$ and all results of Corollary 8.14(iii) hold for the **HH**-relation \ll_1 (see (8.8)).

(ii) Set $X_{\mu} = \overline{\text{span}(e_{\lambda})_{\mu \neq \lambda \in \Lambda}}$ for $\mu \in \Lambda$. If $\bigcap_{\mu \in \Lambda} X_{\mu} = \{0\}$ then the dual \ll_1^{\triangleleft} -radical $\mathfrak{p} = \sigma_{\ll_1}(X) = \{0\}$ in $\text{Lat } T$ and all results of Corollary 8.14(ii) hold for the **HH**-relation \ll_1 .

Proof. (i) As $\{0\} \ll_1 \mathbb{C}e_{\lambda}$, we have $\mathbb{C}e_{\lambda} \in [\{0\}, \ll_1]$ (see (8.3)) for all $\lambda \in \Lambda$. By (8.4), $X = \vee \mathbb{C}e_{\lambda} = s^{\ll_1}(\{0\})$. By Corollary 8.15, $\mathfrak{r} = X$ and Corollary 8.14(iii) holds for \ll_1 .

Part (ii) has a similar proof. ■

To illustrate Proposition 8.16, consider the following example.

Example 8.17 Let X be a Hilbert space with basis $\{e_n\}_{n=1}^{\infty}$ and T be the adjoint to the unilateral shift: $Te_n = e_{n-1}$. Let $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. For each $\lambda \in \Lambda$, $e_{\lambda} = (1, \lambda, \lambda^2, \dots) \in X$ is an eigenvector of T : $Te_{\lambda} = \lambda e_{\lambda}$, and $X = \overline{\text{span}(e_{\lambda})_{\lambda \in \Lambda}}$. Thus all results of Corollary 8.14(iii) hold. ■

Let the commutant S' of a set $S \subset B(X)$ contain projections $\{P_{\lambda}\}_{\lambda \in \Lambda}$, $\dim P_{\lambda} < \infty$. Set $Q = \text{Lat } S$. All subspaces $X_{\lambda} = P_{\lambda}X$ belong to $[\{0\}, \ll_{\infty}]_Q$ and all $(\mathbf{1} - P_{\lambda})X$ to $[\ll_{\infty}, X]_Q$. As in Proposition 8.16, we obtain

Proposition 8.18 (i) If $\overline{\sum_{\lambda \in \Lambda} X_{\lambda}} = X$ then the $\ll_{\infty}^{\triangleright}$ -radical $\mathfrak{r} = s^{\ll_{\infty}}(\{0\}) = X$ in Q and all results of Corollary 8.14(iii) hold for \ll_{∞} .

(ii) If $\bigcap_{\lambda \in \Lambda} (\mathbf{1} - P_{\lambda})X = \{0\}$ then the dual $\ll_{\infty}^{\triangleleft}$ -radical $\mathfrak{p} = \sigma_{\ll_{\infty}}(X) = \{0\}$ in Q and all results of Corollary 8.14(ii) hold for \ll_{∞} .

For example, suppose that the commutant S' contains a compact operator T . Then $\text{Sp}(T) = \{\lambda_n\}_{n=0}^m$ for $m \leq \infty$, where $\lambda_0 = 0$. The algebra S' contains projections P_n , $1 \leq n \leq m$, such that $\dim P_n < \infty$. The subspaces $X_n = P_n X$ and $V_n = (1 - P_n)X$ are T -invariant,

$$X = X_n \dot{+} V_n, \quad \text{Sp}(T|_{X_n}) = \{\lambda_n\} \text{ and } \text{Sp}(T|_{V_n}) = \text{Sp}(T) \setminus \{\lambda_n\} \text{ for } 1 \leq n \leq m. \quad (8.14)$$

Moreover, $X_n \subseteq V_k$ for $n \neq k$. So, for each $1 \leq k \leq m$, $X = Y_k \dot{+} \sum_{n=1}^k X_n$, where $Y_k = \bigcap_{n=1}^k V_k$, and all subspaces in the decompositions are S -invariant. Thus

- 1) if $\overline{\sum_{n \geq 1} X_n} = X$ then all results of Corollary 8.14(iii) hold for the relation \ll_∞ ;
- 2) if $\bigcap_{n \geq 1} V_n = \{0\}$ then all results of Corollary 8.14(ii) hold for the relation \ll_∞ .

Proposition 8.19 *The conditions $\overline{\sum_{n \geq 1} X_n} = X$ and $\bigcap_{n \geq 1} V_n = \{0\}$ above are not equivalent.*

Proof. To show that $\overline{\sum_{n \geq 1} X_n} = X \not\Rightarrow \bigcap_{n \geq 1} V_n = \{0\}$, consider the compact operator T constructed in [Ha] (see also [N, p. 262]) on a Hilbert space X . It has the following properties:

$$\text{Sp}(T) = \{\lambda_n\}_{n=0}^\infty, \quad \overline{\sum_{n \geq 1} X_n} = X, \quad \text{where } X_n = \text{Ker}(T - \lambda_n \mathbf{1}),$$

and there exists a T -invariant subspace $E \neq \{0\}$ satisfying $T|_E \neq 0$ and $\text{Sp}(T|_E) = 0$. Let P_n be the projections on X_n . Then (8.14) hold.

As the spectral radius $r(T|_E) = 0$, we have, for each $e \in E$,

$$\|T^k e\|^{1/k} \leq \|(T|_E)^k\|^{1/k} \longrightarrow r(T|_E) = 0, \quad \text{as } k \longrightarrow \infty. \quad (8.15)$$

Let $e \in E$ and $n \geq 1$. By (8.14), $e = x + y$, where $x \in X_n$ and $y \in V_n$. As T and P_n commute,

$$\|T^k x\|^{1/k} = \|T^k P_n e\|^{1/k} = \|P_n T^k e\|^{1/k} \leq \|P_n\|^{1/k} \|T^k e\|^{1/k} \xrightarrow{(8.15)} 0,$$

as $k \rightarrow \infty$. As $\dim X_n < \infty$ and $\text{Sp}(T|_{X_n}) = \lambda_n \neq 0$, it is easy to prove that $x = 0$. Thus $e \in V_n$. Hence $\{0\} \neq E \subseteq \bigcap_{n \geq 1} V_n$. Thus $\overline{\sum_{n \geq 1} X_n} = X$ does not imply $\bigcap_{n \geq 1} V_n = \{0\}$.

To prove conversely that $\bigcap_{n \geq 1} V_n = \{0\} \not\Rightarrow \overline{\sum_{n \geq 1} X_n} = X$, consider $A = T^*$. Then $\text{Sp}(A) = \{\lambda_n^*\}_{n=0}^\infty$. As in (8.14), let Y_n and U_n be A -invariant subspaces satisfying

$$X = Y_n \dot{+} U_n, \quad \text{Sp}(A|_{Y_n}) = \{\lambda_n^*\}, \quad \dim Y_n < \infty \text{ and } \text{Sp}(A|_{U_n}) = \text{Sp}(A) \setminus \{\lambda_n^*\},$$

for each n . Set $U = \bigcap_{n \geq 1} U_n$. Since $\text{Sp}(A|_U) = \{0\}$, we have as in (8.15) that $\|A^k u\|^{1/k} \longrightarrow 0$, as $k \rightarrow \infty$, for each $u \in U$. Let $x \in X_n$ and $Tx = \lambda_n x$ for $n \geq 1$. Then for all $u \in U$,

$$|\lambda_n| |(x, u)|^{1/k} = |(T^k x, u)|^{1/k} = |(x, A^k u)|^{1/k} \leq \|x\|^{1/k} \|A^k u\|^{1/k} \longrightarrow 0, \quad (8.16)$$

as $k \rightarrow \infty$. Hence $x \perp U$. If $Tx = \lambda_n x + z$ and $Tz = \lambda_n z$, then $z \perp U$ and, by (8.16), $x \perp U$. As $\dim X_n < \infty$, continuing this, we get that all $X_n \perp U$. As $\overline{\sum_{n \geq 1} X_n} = X$, we have $U = \{0\}$.

On the other hand, let $y \in Y_n$ be such that $Ay = \lambda_n^* y$. Then, for each $e \in E$,

$$|\lambda_n^* |(y, e)|^{1/k} = |(A^k y, e)|^{1/k} = |(y, T^k e)|^{1/k} \leq \|y\|^{1/k} \|T^k e\|^{1/k} \xrightarrow{(8.15)} 0, \quad (8.17)$$

as $k \rightarrow \infty$. Hence $y \perp E$. If $Ay = \lambda_n^* y + z$ and $Az = \lambda_n^* z$, then $z \perp U$ and, by (8.17), $y \perp U$. As $\dim Y_n < \infty$, continuing this, we get that all $Y_n \perp E$, so that $\overline{\sum_{n \geq 1} Y_n} \neq X$. Thus $\bigcap_{n \geq 1} U_n = \{0\}$ does not imply $\overline{\sum_{n \geq 1} X_n} = X$. ■

8.4 Relation $<_{\mathfrak{g}}$ in sublattices of $\text{Cl}(H)$, where H is a Hilbert space

In this section $X = H$ is a Hilbert space. First we consider reflexive sublattices of $\text{Cl}(H)$, $\dim H = \infty$, where $<_{\mathfrak{g}}$ is neither \mathbf{H} -, nor a dual \mathbf{H} -relation.

For $x, y \in H$, define the rank one operator $x \otimes y$ on H by

$$(x \otimes y)\xi = (\xi, x)y \text{ for } \xi \in H. \quad (8.18)$$

Let F be a closed symmetric operator on H with domain $D(F)$ and F^* be its adjoint. Let $F^* \neq F$. Then $D(F) \not\subseteq D(F^*)$. For $u, v \in H$, $y \in D(F)$ and $x \in D(F^*)$,

$$(x \otimes y)(u \otimes v) = (v, x)(u \otimes y) \text{ and } F(x \otimes y) = x \otimes Fy, (x \otimes y)F = F^*x \otimes y. \quad (8.19)$$

Set $X = H \oplus H$ and let $\Omega = \{\{0\}, H \oplus \{0\}, \{0\} \oplus H, X\}$. Consider

$$\mathcal{A} = \text{span} \left\{ A(x, y) = \begin{pmatrix} x \otimes Fy & 0 \\ 0 & Fx \otimes y \end{pmatrix} \text{ for } x, y \in D(F) \right\}.$$

By (8.19), \mathcal{A} is a subalgebra of $B(X)$. For each closed operator S on H with domain $D(S)$,

$$M_S = \left\{ \widehat{\xi} = S\xi \oplus \xi : \xi \in D(S) \right\} \text{ is a closed subspace of } X.$$

Lemma 8.20 *Let $\overline{FD(F)} = H$. Then $\text{Lat } \mathcal{A} = \Omega \cup (\cup_{t \in \mathbb{C} \setminus \{0\}} \mathcal{K}_t)$ and, for each t ,*

$$\mathcal{K}_t = \{M_{tT} : F \subseteq T \subseteq F^*, T \text{ is closed}\} = \{L : L \text{ is a space, } M_{tF} \subseteq L \subseteq M_{tF^*}\}.$$

Proof. Let $L \in \text{Lat } \mathcal{A}$ and $L \notin \Omega$. As $FD(F)$ is dense in H , the subspaces $H \oplus \{0\}$ and $\{0\} \oplus H$ have no non-trivial \mathcal{A} -invariant subspaces. Hence $L = M_S$ for some closed operator $S \neq 0$ on H . As $A(x, y)\widehat{\xi} \subseteq M_S$ for all $x, y \in D(F)$ and $\xi \in D(S)$,

$$(Fx \otimes y)\xi \stackrel{(8.18)}{=} \sum_{i=1}^n (\xi, Fx)y \in D(S) \text{ and } (x \otimes Fy)S = S(Fx \otimes y) \stackrel{(8.19)}{=} Fx \otimes Sy.$$

As $FD(F)$ is dense in H , we have $D(F) \subseteq D(S)$. As $(x \otimes Fy)S = Fx \otimes Sy$, we have

$$(x \otimes Fy)S\eta \stackrel{(8.18)}{=} (S\eta, x)Fy = (Fx \otimes Sy)\eta \stackrel{(8.18)}{=} (\eta, Fx)Sy \text{ for } \eta \in D(S). \quad (8.20)$$

Fix η and choose x such that $(\eta, Fx) \neq 0$. Then $Sy = tFy$ for $y \in D(F)$ with $t = (S\eta, x)/(\eta, Fx) \in \mathbb{C}$. As $0 \neq S$ is closed and $D(F) \subseteq D(S)$, we have $t \neq 0$. Thus $tF \subseteq S$.

Choose now $y \in D(F)$ in (8.20) such that $Fy \neq 0$. Then $(S\eta, x) = t(\eta, Fx)$ for all $x \in D(F)$ and $\eta \in D(S)$. Hence $\eta \in D(F^*)$ and $S\eta = tF^*\eta$. Thus $S \subseteq tF^*$ and $tF \subseteq S \subseteq tF^*$. Set $T = S/t$. Then $M_S = M_{tT}$ and $F \subseteq T \subseteq F^*$.

Conversely, let $F \subseteq T \subseteq F^*$. Then, for $A(x, y) \in \mathcal{A}$, $\xi \in D(T)$ and $t \in \mathbb{C} \setminus \{0\}$, we have

$$A(x, y)(tT\xi \oplus \xi) \stackrel{(8.18)}{=} t(T\xi, x)Fy \oplus (\xi, Fx)y = (T\xi, x)(tTy \oplus y) \in M_{tT}.$$

Thus $M_{tT} \in \text{Lat } \mathcal{A}$ for all $t \in \mathbb{C} \setminus \{0\}$.

If L is a subspace of X and $M_{tF} \subseteq L \subseteq M_{tF^*}$, then there is a linear operator T such that $D(F) \subseteq D(T) \subseteq D(F^*)$ and $L = M_{tT}$. As L is closed, T is closed. Thus $L \in \mathcal{K}_t$. ■

Corollary 8.21 *The sublattice $\text{Lat } \mathcal{A}$ of $\text{Cl}(H)$ in Lemma 8.20 is not modular and the relation $<_{\mathfrak{g}}$ in $\text{Lat } \mathcal{A}$ is neither an \mathbf{H} -, nor a dual \mathbf{H} -relation.*

Proof. As $\{0\}, H \oplus \{0\}, X, M_{F^*}, M_F$ form a pentagon in $\text{Lat } \mathcal{A}$ (see (6.3)), $\text{Lat } \mathcal{A}$ is not modular.

By Lemma 8.20, $\{0\} <_{\mathfrak{g}} H \oplus \{0\} <_{\mathfrak{g}} X$. If $<_{\mathfrak{g}}$ is an \mathbf{H} -relation then, by Lemma 3.1(iii), $M_F = \{0\} \vee M_F <_{\mathfrak{g}} (H \oplus \{0\}) \vee M_F = \overline{(H \oplus \{0\})} + M_F = X$. However, $M_F \not<_{\mathfrak{g}} X$, since $M_F \subsetneq M_{F^*} \subsetneq X$. Thus $<_{\mathfrak{g}}$ is not an \mathbf{H} -relation.

If $<_{\mathfrak{g}}$ is a dual \mathbf{H} -relation then $\{0\} = (H \oplus \{0\}) \cap M_{F^*} <_{\mathfrak{g}} X \cap M_{F^*} = M_{F^*}$ by Lemma 3.1(ii). However, $\{0\} \not<_{\mathfrak{g}} M_{F^*}$, since $\{0\} \subsetneq M_F \subsetneq M_{F^*}$. Thus $<_{\mathfrak{g}}$ is not a dual \mathbf{H} -relation. ■

As sublattices of a modular lattice are modular, it follows from Corollary 8.21 that $\text{Cl}(H)$ is not modular if $\dim H = \infty$. On the other hand, $\text{Cl}(H) = \text{Ln}(H)$ if $\dim H < \infty$, so that $\text{Cl}(H)$ is modular by Proposition 8.3. This yields (cf. Proposition 1.5.5 [K]).

Corollary 8.22 *$\text{Cl}(H)$ is a modular lattice if and only if $\dim(H) < \infty$.*

We will now consider some sublattices of $\text{Cl}(H)$, where $<_{\mathfrak{g}}$ is an \mathbf{HH} -relation. Denote by $P(H)$ the set of all orthogonal projections in $B(H)$. For $L \in \text{Cl}(H)$, p_L denotes the projection on L . So $\text{Cl}(H)$ can be identified with $P(H)$ and each sublattice of $\text{Cl}(H)$ with a sublattice of $P(H)$.

A complete sublattice Q of $\text{Cl}(H)$ is a commutative subspace lattice (CSL) if all projections p_L , $L \in Q$, commute. For $p \in P(H)$, let $p^\perp = \mathbf{1} - p$. If Q is CSL and $p, q \in Q$ then

$$p \wedge q = pq, \quad p \vee q = p + q - pq = p + p^\perp q. \quad (8.21)$$

Theorem 8.23 *Let Q be a CSL in $\text{Cl}(H)$. Then*

- (i) Q has properties (JID) and (MID) (see (6.5)), so that it is modular.
- (ii) $L + K \in Q$ for all $L, K \in Q$ and $<_{\mathfrak{g}}$ is an \mathbf{HH} -relation in Q .
- (iii) $\sqsubset = \prec = \subseteq$ and $\sqsubset_{\mathfrak{g}} = \prec_{\mathfrak{g}} = <_{\mathfrak{g}}$ in Q .
- (iv) $<_{\mathfrak{g}}^{\triangleleft}$ is an \mathbf{H} -order and a dual \mathbf{R} -order; $<_{\mathfrak{g}}^{\triangleright}$ is an \mathbf{R} -order and a dual \mathbf{H} -order.
- (v) $\widetilde{\prec}_{\mathfrak{g}} = \widetilde{\prec}_{\mathfrak{g}}$ (see (7.15) and (7.16)) is an \mathbf{RR} -order.
- (vi) Q is a union of disjoint intervals that have no gaps.

Proof. (i) It was noted in [Da, p. 357] that Q has properties (JID) and (MID). For completeness, we prove this here directly. Let $G \subseteq Q$ and $z \in Q$. Set $q = \vee G$ and $p = \vee \{zx: x \in G\}$.

As Q is a complete sublattice of $\text{Cl}(H)$, $q \in Q$. So

$$p = \vee \{zx: x \in G\} \stackrel{(8.21)}{=} \vee \{z \wedge x: x \in G\} \stackrel{(6.4)}{\leq} z \wedge (\vee G) = z \wedge q \stackrel{(8.21)}{=} zq. \quad (8.22)$$

We have $x = zx \oplus z^\perp x \leq p \oplus z^\perp q$ for all $x \in G$. Hence $q = \vee \{x: x \in G\} \leq p \oplus z^\perp q \stackrel{(8.22)}{\leq} zq \oplus z^\perp q = q$. So $zq = p$, i.e., $z \wedge (\vee G) = \vee \{z \wedge x: x \in G\}$. Thus Q has property (MID) (see (6.5)).

If $G = \{x, y\}$ then $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$. Thus Q is distributive. So it is modular.

Set now $r = \wedge G$ and $s = \wedge \{z^\perp x: x \in G\}$. As Q is complete, $r \in Q$. So

$$\begin{aligned} z \oplus z^\perp r &\stackrel{(8.21)}{=} z \vee r = z \vee (\wedge G) \stackrel{(6.4)}{\leq} \wedge \{z \vee x: x \in G\} \stackrel{(8.21)}{=} \wedge \{z \oplus z^\perp x: x \in G\} \\ &= z \oplus \wedge \{z^\perp x: x \in G\} = z \oplus s. \end{aligned} \quad (8.23)$$

So $z^\perp r \leq s$. For $x \in G$, we have $x = zx \oplus z^\perp x \geq zr \oplus s$. Therefore $r = \wedge\{x: x \in G\} \geq zr \oplus s$. So $z^\perp r \geq s$. Hence $z^\perp r = s$, so that $z \oplus z^\perp r = z \oplus s$. Thus

$$z \vee (\wedge G) \stackrel{(8.21)}{=} z \oplus z^\perp r = z \oplus s \stackrel{(8.23)}{=} \wedge\{z \vee x : x \in G\}.$$

Therefore (see (6.5)) Q has properties (JID).

(ii) Let $a = p_L, b = p_K$ for $L, K \in Q$. Then $L + K = L \oplus a^\perp K = L \oplus a^\perp bH$. As Q is CSL, a and b commute. So $a^\perp b$ is a projection. Thus $a^\perp bH$ is a closed subspace of K^\perp . So $L + K$ is closed.

As Q is modular, it follows from Corollary 6.7 that $<_{\mathfrak{g}}$ is an **HH**-relation in Q .

(iii) As $L + K \in Q$ for all $L, K \in Q$, it follows from (8.9) that $\sqsubset = \prec = \subseteq$. Hence, since $<_{\mathfrak{g}} = <_{\mathfrak{g}} \cap \subseteq$, $\sqsubset_{\mathfrak{g}} = <_{\mathfrak{g}} \cap \sqsubset$ and $\prec_{\mathfrak{g}} = <_{\mathfrak{g}} \cap \prec$, we have $<_{\mathfrak{g}} = \sqsubset_{\mathfrak{g}} = \prec_{\mathfrak{g}}$.

As Q is modular, has properties (JID) and (MID) (see (6.5)) and $<_{\mathfrak{g}}$ is an **HH**-relation in Q , (iv) and (v) follow from Lemma 6.9 and Corollary 7.21. Part (vi) follows from Theorem 6.10. ■

The lattice of all finite projections in a W^* -algebra is modular (Theorem V.1.37 [T]). So Corollary 6.7 yields

Corollary 8.24 $<_{\mathfrak{g}}$ is an **HH**-relation in the lattice of all finite projections in any W^* -algebra.

The lattice $\text{Cl}(H) \approx P(H)$ has a large variety of **H**- and dual **H**-relations which is difficult to describe. However, it is possible to describe all **HH**-relations in $P(H)$. To do this, recall that projections $p, q \in P(H)$ are equivalent ($p \sim q$) if

$$p = vv^* \text{ and } q = v^*v \text{ for some partial isometry } v. \quad (8.24)$$

To introduce a new class of **HH**-relations in $P(H)$, we need the following result.

Proposition 8.25 [K3] *Let $p \leq q$ in $P(H)$. Then, for each projection $r \in P(H)$,*

$$(q - p)^\perp \leq (q \vee r - p \vee r)^\perp; \quad (8.25)$$

$$(q - p)^\perp \sim \mathfrak{z} \text{ for some } \mathfrak{z} \in P(H), \text{ and } \mathfrak{z} \leq (q \wedge r - p \wedge r)^\perp. \quad (8.26)$$

Consider now the reflexive relations \ll_n^\perp in $P(H)$, for $0 \leq n \leq \infty$, defined by the condition

$$p \ll_n^\perp q \text{ if } p \leq q \text{ in } P(H) \text{ and } \dim(q - p)^\perp \geq n, \text{ so that } \ll_0^\perp = \leq. \quad (8.27)$$

Theorem 8.26 *The relations \ll_n^\perp , $0 \leq n \leq \infty$, are **HH**-relations in $P(H)$.*

Proof. Let $p, q, r \in Q$ and $p \ll_n^\perp q$ for some n . Then $p \leq q$ and $\dim(q - p)^\perp \geq n$.

We have that $p \vee r, q \vee r$ belong to Q and $p \vee r \leq q \vee r$. By (8.25), $(q - p)^\perp \leq (q \vee r - p \vee r)^\perp$. So $\dim(q \vee r - p \vee r)^\perp \geq n$. Hence $p \vee r \ll_n^\perp q \vee r$ by (8.27).

We have $p \wedge r, q \wedge r \in Q$ and $p \wedge r \leq q \wedge r$. By (8.26), $(q - p)^\perp \sim \mathfrak{z}$ and $\mathfrak{z} \leq (q \wedge r - p \wedge r)^\perp$ for some projection \mathfrak{z} in $P(H)$. It follows from (8.24) that $\dim(q - p)^\perp = \dim \mathfrak{z}$. Hence $\dim(q \wedge r - p \wedge r)^\perp \geq n$. So $p \wedge r \ll_n^\perp q \wedge r$ by (8.27). Thus, by Definition 3.2 \ll_n^\perp is an **HH**-relation. ■

By Corollary 8.6, all \ll_n , $1 \leq n \leq \infty$, are **HH**-relations in $\text{Cl}(H) \approx P(H)$. The following result obtained [K3] in describes all **HH**-relation in $P(H)$.

Theorem 8.27 *For a separable H , each **HH**-relation in $P(H)$ is either \ll_n , or \ll_n^\perp for $n \in \mathbb{N} \cup \infty$.*

9 H-relations in the lattices of ideals of Banach algebras

9.1 H-relations in the lattices of ideals of Banach algebras

Let A be a Banach algebra. Denote by Id_A the set of all closed two-sided ideals in A (we call them *ideals*). Let $LR(A)$ be the Banach algebra of operators on A generated by all operators of left and right multiplication by elements from A . Then $\text{Id}_A = \text{Lat } LR(A)$ is a sublattice of $\text{Cl}(A)$.

The following Proposition was obtained in [Ru1, Example 4.6] and rediscovered in [Ki, Lemma 4.9(iv)] (see also [W, Lemma 8.1] and [Di, Proposition 2.4]).

Proposition 9.1 *Let L and R be closed left and right ideals of A . If L has a bounded right approximate identity (a. i.), or R has a bounded left a. i., then $L + R$ is closed.*

Corollary 9.2 *Let A be a Banach algebra. If each $I \in \text{Id}_A$ has a bounded left or right a. i., then $\prec = \sqsubset = \subseteq$ and $\prec_{\mathfrak{g}} = \prec_{\mathfrak{g}} = \sqsubset_{\mathfrak{g}}$ are **HH**-relations in Id_A .*

Proof. It follows from (8.9) and Proposition 9.1 that $\prec = \sqsubset = \subseteq$. From this and from (8.2) and (8.11) we have $\prec_{\mathfrak{g}} = \prec_{\mathfrak{g}} = \sqsubset_{\mathfrak{g}}$ in Id_A . By Theorem 8.11, it is an **HH**-relation. ■

Let \ll be a relation in Id_A . As in (8.3), for $L \in \text{Id}_A$, let

$$\sigma_{\ll}(L) = \cap\{J \in \text{Id}_A: J \ll L\} \text{ and } s^{\ll}(L) = \overline{\sum\{J \in \text{Id}_A: L \ll J\}}. \quad (9.1)$$

For an **H**-relations \ll , \ll^{\triangleright} is an **R**-order in Id_A and the ideal

$$\mathfrak{r}_L \stackrel{(2.7)}{=} \overline{\text{span}\{J \in \text{Id}_A: L \ll^{\triangleright} J\}} \text{ is the unique } \ll^{\triangleright}\text{-radical in } [L, A] \subseteq \text{Id}_A.$$

If \ll is a dual **H**-relations then \ll^{\triangleleft} is a dual **R**-order in Id_A and the ideal

$$\mathfrak{p}_L \stackrel{(2.6)}{=} \cap\{J \in \text{Id}_A: J \ll^{\triangleleft} L\} \text{ is the unique dual } \ll^{\triangleleft}\text{-radical in } [\{0\}, L] \subseteq \text{Id}_A.$$

Corollary 9.3 *The results of Theorem 8.1 and Proposition 8.2 hold for $X = A$, where $Q = \text{Id}_A$ is a sublattice of $\text{Cl}(A)$ and the word "spaces" replaced by "ideals" of A .*

By Corollary 8.6, all \ll_n are **HH**-relations in Id_A and \ll_{∞} is an **HH**-order. Proposition 8.2 yields

Corollary 9.4 *Let A be a separable Banach algebra.*

(i) *Let $\sigma_{\ll_{\infty}}(A) = \{0\}$. Then there is a chain $\dots I_n \subset \dots \subset I_0 = A$ of ideals of finite codimension in A such that $\cap_n I_n = \{0\}$. Each $\{0\} \neq J \in \text{Id}_A$ contains $I \in \text{Id}_A$ such that $0 < \dim(J/I) < \infty$.*

(ii) *Let $s^{\ll_{\infty}}(\{0\}) = A$. Then there is a chain $\{0\} = I_0 \subset \dots \subset I_n \dots$ of ideals such that $\dim I_n < \infty$ and $\overline{\sum_{n=1}^{\infty} I_n} = A$. Each $A \neq J \in \text{Id}_A$ is contained in $I \in \text{Id}_A$ such that $0 < \dim(I/J) < \infty$.*

Since ideals of a Banach algebra A are precisely invariant subspaces of the algebra $LR(A)$, it is natural to call superinvariant subspaces of A by *superideals*. Thus $I \in \text{Id}_A$ is a superideal of A , if (see (8.12)) it is invariant for all operators S from the Lie subalgebra $\text{Nor } LR(A)$ of the algebra $B(A)$:

$$\text{Nor } LR(A) = \{S \in B(A): ST - TS \in LR(A) \text{ for all } T \in LR(A)\}. \quad (9.2)$$

A bounded operator $\delta \in B(A)$ is a derivation on A if $\delta(ab) = \delta(a)b + a\delta(b)$ for $a, b \in A$.

Theorem 9.5 (i) *Nor* $LR(A)$ contains all derivations of A .

(ii) If $I \in \text{Id}_A$ then each operator $S \in \text{Nor } LR(A)$ maps $\overline{I^2}$ in I : $Sx \in I$ for $x \in \overline{I^2}$.

(iii) If $\overline{I^2} = I$ (in particular, if I has a left (right) a. i.) then I is a superideal.

Proof. (i) Let L_a, R_a be the operators of left and right multiplication by $a \in A$ on A . Then

$$[\delta, L_a]x = \delta L_a x - L_a \delta(x) = \delta(ax) - a\delta(x) = \delta(a)x = L_{\delta(a)}x \text{ for all } x \in A.$$

Hence $[\delta, L_a] = L_{\delta(a)} \in LR(A)$. Similarly, $[\delta, R_a] = \delta R_a - R_a \delta = R_{\delta(a)} \in LR(A)$. Since the algebra $LR(A)$ is generated by the sums of products of the operators L_a and R_b and since $[\delta, TS] = [\delta, T]S + T[\delta, S]$ for all $T, S \in B(A)$, we have $\delta \in \text{Nor } LR(A)$.

(ii) Let $S \in \text{Nor } LR(A)$ and $a, b \in I$. Then

$$S(ab) = SR_b a = R_b S a + [S, R_b]a = (S a)b + [S, R_b]a.$$

As $S a \in A$ and $b \in I$, we have $(S a)b \in I$. As $[S, R_b] \in LR(A)$ by (9.2), we have $[S, R_b]a \in I$. Hence $S(ab) \in I$. Since S is linear and bounded, it maps $\overline{I^2}$ into I . Part (iii) follows from (ii). ■

To see an example of a Banach algebra that has no non-trivial superideal, consider a Banach space A with trivial multiplication: $ab = 0$ for $a, b \in A$. Then A is a Banach algebra, $LR(A) = \{0\}$ and $\text{Nor } LR(A) = B(A)$. As $B(A)$ has no non-trivial invariant subspaces, A has no superideals. On the other hand, all closed subspaces of A are ideals of A .

Denote by Σ_A the set of all closed subalgebras of A of finite codimension.

Proposition 9.6 For each Banach algebra A , $\sigma_{\ll\infty}(A) = \bigcap_{S \in \Sigma_A} S$.

Proof. As $\{J \in \text{Id}_A: J \ll\infty A\} \subseteq \Sigma_A$, we have $\bigcap_{S \in \Sigma_A} S \subseteq \sigma_{\ll\infty}(A)$ by (9.1). In [L] it was proved that each $S \in \Sigma_A$ contains an ideal of finite codimension. Thus $\sigma_{\ll\infty}(A) \subseteq \bigcap_{S \in \Sigma_A} S$. So $\sigma_{\ll\infty}(A) = \bigcap_{S \in \Sigma_A} S$. ■

9.2 H-relations in the lattices of all ideals of C*-algebras.

Let A be a C*-algebra. Since ideals of A have bounded a.i., $I + J \in \text{Id}_A$ for all $I, J \in \text{Id}_A$, by Proposition 9.1 (see [D]). Thus Id_A is a sublattice (not a complete sublattice) of $\text{Ln}(A)$. So Id_A is a modular lattice and $<_{\mathfrak{g}}$ is an **HH**-relation in Id_A by Proposition 8.3. It also follows from Corollary 9.2 that the relations \prec, \sqsubset and \subseteq coincide in Id_A and $<_{\mathfrak{g}} = \prec_{\mathfrak{g}} = \sqsubset_{\mathfrak{g}}$.

The complete lattice Id_A has many **H**-relations (some of them stronger than $<_{\mathfrak{g}}$) that depend on the nature of the quotient ideals J/I . A large variety of such relations was investigated in [KST4]. In this section we briefly consider some of them.

Let \mathfrak{A} be the set of all C*-algebras. We say that a subclass P of \mathfrak{A} is a *property*, if

$$\{0\} \in P \text{ and } A \in P \text{ implies } B \in P \text{ for all } B \approx A. \quad (9.3)$$

If $A \in P$, we say that A has property P , or A is a *P-algebra*. For example, the classes *CCR* of all CCR-algebras and *GCR* of all GCR-algebras are properties.

A standard way to define a relation in Id_A is to consider a property P and to write

$$I \ll_P J \text{ if } I \subseteq J \text{ in } \text{Id}_A \text{ and } J/I \text{ is a } P\text{-algebra, i.e., } J/I \in P. \quad (9.4)$$

A property P on \mathfrak{A} is **lower stable** if $A \in P$ implies that all $I \in \text{Id}_A$ belong to P ; P is **upper stable** if $A \in P$ implies that the quotients $A/I \in P$ for all $I \in \text{Id}_A$.

Theorem 9.7 (i) A property P is upper stable if and only if \ll_P is an **H**-relation.

Then \ll_P^\triangleright is an **R**-order and Id_A has the \ll_P^\triangleright -radical $\mathfrak{r}_P: \{0\} \ll_P^\triangleright \mathfrak{r}_P \xleftarrow{\ll_P^\triangleright} A$.

(ii) A property P is lower stable if and only if \ll_P is a dual **H**-relation.

Then \ll_P^\triangleleft is a dual **R**-order and Id_A has the dual \ll_P^\triangleleft -radical $\mathfrak{p}_P: \{0\} \xrightarrow{\ll_P^\triangleleft} \mathfrak{p}_P \ll_P^\triangleleft A$.

Proof. (i) Let $I \ll_P J$ in Id_A , $I \neq J$, and $I \subseteq K \in \text{Id}_A$. Then $(K \cap J)/I$ is an ideal of J/I . As $J/I \in P$ and P is upper stable, the quotient $(J/I)/((K \cap J)/I) \in P$. As $(J/I)/((K \cap J)/I) \approx J/(K \cap J)$, we have $J/(K \cap J) \in P$. By Corollary 1.8.4 [D], $J/(K \cap J) \approx (J + K)/K$, so that $(J + K)/K \in P$. Thus $K \ll_P (J + K) = J \vee K$. So, by Lemma 3.1, \ll_P is an **H**-relation.

Conversely, if \ll_P is an **H**-relation and $A \in P$ then $\{0\} \ll_P A$ by (9.4). Hence, for $I \in \text{Id}_A$, $I \ll_P (A + I) = A$ by Lemma 3.1. Thus $A/I \in P$ by (9.4). So P is upper stable.

The rest of (i) follows from Theorem 5.5. Part (ii) can be proved similarly. ■

Now we can treat Id_A as a lattice with two relations: \subseteq and \ll_P . With this approach we can construct a large variety of **H**- and dual **H**-relations and, using methods of the lattice theory, obtain many results in the theory of C^* -algebras. Thus Theorems 8.1 and 9.7 yield

Theorem 9.8 I. Let P be an upper stable property, so that \ll_P is an **H**-relation. Let $A \in \mathfrak{A}$.

(i) The radical \mathfrak{r}_P is the **largest** ideal in A such that there is an ascending \ll_P -series of ideals from $\{0\}$ to \mathfrak{r}_P .

(ii) If $\mathfrak{r}_P \not\subseteq I \neq A$, then there is $J \in \text{Id}_A$ such that J/I is a P -algebra.

(iii) \mathfrak{r}_P is the **smallest** out of all ideals J such that A/J has no P -ideals.

II. Let P be a lower stable property, so that \ll_P is a dual **H**-relation. Let $A \in \mathfrak{A}$.

(i) The dual radical \mathfrak{p}_P is the **smallest** ideal in A such that there is a descending \ll_P -series of ideals from A to \mathfrak{p}_P .

(ii) If $\{0\} \neq I \not\subseteq \mathfrak{p}_P$ then there is $J \in \text{Id}_A$ such that I/J is a P -algebra.

(iii) \mathfrak{p}_P is the **largest** ideal such that all its quotients are not P -algebras.

Proof. I. Parts (i) and (ii) follow from Theorem 8.1 (ii).

(iii). By Theorem 8.1 (ii), \mathfrak{r}_P has no \ll_P -successor. So A/\mathfrak{r}_P has no P -ideals.

Let A/J have no P -ideals for $J \in \text{Id}_A$. If $\mathfrak{r}_P \not\subseteq J$ then, by Theorem 8.1 (ii) 2), A/J has P -ideals. So $\mathfrak{r}_P \subseteq J$ which proves (iii). The proof of part II is similar. ■

Each automorphism ϕ of a C^* -algebra A generates a lattice automorphism $\tilde{\phi}$ of Id_A . For a property P , let $I \ll_P J$ in Id_A , i.e., $J/I \in P$. Then $\tilde{\phi}(I) \subseteq \tilde{\phi}(J)$ and the map $\hat{\phi}: J/I \rightarrow \tilde{\phi}(J)/\tilde{\phi}(I)$ defined by $\hat{\phi}(x + I) = \phi(x) + \tilde{\phi}(I)$ for $x \in J$, is an isomorphism, i.e., $J/I \approx \tilde{\phi}(J)/\tilde{\phi}(I)$, so that $\tilde{\phi}(J)/\tilde{\phi}(I) \in P$, i.e., $\tilde{\phi}(I) \ll_P \tilde{\phi}(J)$. As ϕ^{-1} is also an automorphism, $\tilde{\phi}(I) \ll_P \tilde{\phi}(J)$ implies $I \ll_P J$. Thus $\tilde{\phi}$ preserves \ll_P . So Proposition 5.10 yields

Corollary 9.9 Let \ll_P be an **H**-relation with \ll_P^\triangleright -radical \mathfrak{r}_P and \ll_S be a dual **H**-relation with dual \ll_S^\triangleleft -radical \mathfrak{p}_S . Then $\tilde{\phi}(\mathfrak{r}_P) = \mathfrak{r}_P$ and $\tilde{\phi}(\mathfrak{p}_S) = \mathfrak{p}_S$ for all automorphisms ϕ of A .

By Propositions 4.2.4 and 4.3.5 [D], the properties CCR and GCR of all CCR - and GCR -algebras are lower and upper stable. For the property CCR this gives a well-known result that

the radical \mathfrak{r}_{CCR} is the largest GCR-ideal in A and A/\mathfrak{r}_{CCR} has no CCR-ideals. Moreover, if $\mathfrak{r}_{CCR} \not\subseteq I \neq A$ then there is $J \in \text{Id}_A$ such that J/I is a CCR-algebra. However, the property of all NGCR-algebras is lower, but not upper stable (see 4.7.4 b) and c) [D]).

Consider now some more lower and upper stable properties. A unital $A \in \mathfrak{A}$ has real rank zero [BP] if its invertible selfadjoint elements are dense in the set of all selfadjoint elements of A . A non-unital algebra is real rank zero if its unitization is real rank zero. Denote by RZ the class of all real rank zero algebras. Then RZ is a property.

Let $A \in RZ$. By Corollary 2.8 [BP], each hereditary C^* -subalgebra of A also belongs to RZ . As each ideal I of A is hereditary (Theorem I.5.3 [Da1]), $I \in RZ$. Let $p: A \rightarrow A/I$. As $\|p(x)\|_{A/I} \leq \|x\|_A$, it follows that A/I also belongs to RZ . This yields

Corollary 9.10 *The property RZ is lower and upper stable.*

Recall that $A \in \mathfrak{A}$ is *approximately finite-dimensional* (AF algebra) if it is the closure of an increasing union of finite-dimensional $*$ -subalgebras. It is *nuclear* if, for each C^* -algebra B , $A \odot B$ has only one C^* -norm. Finite-dimensional and all commutative C^* -algebras, $C(\mathcal{H})$, all AF and all C^* -algebras of type I are nuclear. Then (see Section III.4 [Da1] and Corollary XV.3.4 [T])

Theorem 9.11 (i) *A C^* -algebra A is nuclear if and only if each ideal I and A/I are nuclear.*
(ii) *A C^* -algebra A is an AF algebra if and only if each ideal I and A/I are AF algebras.*

Denote by AF and NU the classes of all AF and of all nuclear C^* -algebras. By Theorem 9.11, they are lower and upper stable properties. Hence Theorem 9.7 yields

Corollary 9.12 *For each $A \in \mathfrak{A}$, $\ll_{CCR}, \ll_{GCR}, \ll_{RZ}, \ll_{AF}, \ll_{NU}$ are **HH**-relations in Id_A .*

We will now discuss transitivity of the relation \ll_P for a property P .

Lemma 9.13 *The relation \ll_P is transitive in all lattices Id_A , $A \in \mathfrak{A}$, if and only if, for all $B \in \mathfrak{A}$ and all $S \in \text{Id}_B$, the condition $S, B/S \in P$ implies $B \in P$.*

Proof. Let $B \in \mathfrak{A}$ and $S \in \text{Id}_B$. Let $S, B/S \in P$. Then $\{0\} \ll_P S \ll_P B$. If \ll_P is transitive in all lattices Id_A , $A \in \mathfrak{A}$, it is transitive in Id_B . So $\{0\} \ll_P B$. Thus $B \in P$.

Conversely, let $A \in \mathfrak{A}$ and $K \ll_P I \ll_P J$ in Id_A . Then $I/K, J/I \in P$ and $J/I \approx (J/K)/(I/K)$. Let, for all $B \in \mathfrak{A}$ and $S \in \text{Id}_B$, $S, B/S \in P$ implies $B \in P$. Set $B = J/K$ and $S = I/K$. As $S, B/S \approx J/I \in P$, we have $J/K = B \in P$. So $K \ll_P J$. Thus \ll_P is transitive in all Id_A , $A \in \mathfrak{A}$. ■

The relation \ll_{CCR} is not transitive. Indeed, let $B = C(H) + \mathbb{C}\mathbf{1}_H$. Then $C(H)$ and $B/C(H) \approx \mathbb{C}\mathbf{1}$ are CCR-algebras, but B is not a CCR-algebra. So, by Lemma 9.13, \ll_{CCR} is not transitive.

Similarly, \ll_{RZ} is not transitive, since $A \in RZ$ if and only if all $I \in \text{Id}_A$ and A/I are RZ -algebras and *all projections in A/I lift to projections in A* (Theorem 3.14 [BP]). On the other hand, \ll_{AF} and \ll_{NU} are transitive by Theorem 9.11. It is also easy to show that \ll_{GCR} is transitive.

If an **HH**-relation \ll_P is not transitive then, generally, the $\ll_P^>$ -radical \mathfrak{r}_P is not a P -algebra. For example, if $B = C(H) + \mathbb{C}\mathbf{1}_H$, we have $\{0\} \ll_{CCR} C(H) \ll_{CCR} B$ and $\mathfrak{r}_{CCR} = B$ is a GCR- and not a CCR-algebra. Even if \ll_P is transitive, \mathfrak{r}_P may still be not a P -algebra. It is a P -algebra if and only if the inductive limits of ascending series of P -ideals of A are P -algebras ([KST4]). For example, AF and nuclear algebras have this property (see [W, p. 17]).

Corollary 9.14 For $A \in \mathfrak{A}$, let \mathfrak{r}_{AF} be the $\ll_{AF}^{\triangleright}$ -radical and \mathfrak{r}_{NU} be the $\ll_{NU}^{\triangleright}$ -radical.

- (i) ([KST4]) The relations \ll_{AF} , \ll_{NU} are **R**-orders in Id_A : $\ll_{AF} = \ll_{AF}^{\triangleright}$ and $\ll_{NU} = \ll_{NU}^{\triangleright}$.
- (ii) ([ST], [KST4]) \mathfrak{r}_{AF} is the largest AF-algebra and \mathfrak{r}_{NU} is the largest nuclear algebra in A .

9.3 Relation \ll_{∞} in the lattices of Lie ideals of Banach Lie algebras.

A complex Lie algebra A with Lie multiplication $[\cdot, \cdot]$ is a Banach Lie algebra, if it is a Banach space in some norm $\|\cdot\|$ and $\|[a, b]\| \leq C \|a\| \|b\|$ for some $C > 0$ and for all $a, b \in A$.

A subspace I of A is a *Lie subalgebra* if $[a, b] \in I$ for all $a, b \in I$; it is a *Lie ideal* if $[a, b] \in I$ for all $a \in I, b \in A$. Banach algebras are Banach Lie algebras with Lie multiplication $[a, b] = ab - ba$.

The set Li_A of all closed Lie ideals of A is a complete lattice of invariant subspaces of the subalgebra of $B(A)$ generated by all operators $\text{ad}(a): x \rightarrow [a, x]$ on A . As in (8.8), the **HH**-relation \ll_{∞} in Li_A is defined by: $I \ll_{\infty} J$ if $\dim(J/I) < \infty$. Then Corollaries 9.3 and 9.4 hold in this setting with two-sided ideals replaced by Lie ideals.

A linear bounded operator δ on A is a *Lie derivation* if

$$\delta([a, b]) = [\delta(a), b] + [a, \delta(b)] \text{ for } a, b \in A. \quad (9.5)$$

The set $\mathfrak{D}(A)$ of all Lie derivations on A is a closed Lie subalgebra of the algebra $B(A)$ of all bounded operators on A . Each $a \in A$ defines a Lie derivation $\text{ad}(a)$ on A .

An ideal $I \in \text{Li}_A$ is called *characteristic* if $\delta(I) \subseteq I$ for all $\delta \in \mathfrak{D}(A)$. By (9.5), the centre of A is a characteristic Lie ideal. If A is commutative then $\{0\}$ and A are the only characteristic Lie ideals of A , as $\mathfrak{D}(A) = B(A)$ and only $\{0\}$ and A are invariant for $B(A)$.

If $\delta \in \mathfrak{D}(A)$ then $e^{t\delta} = \sum_{n=0}^{\infty} t^n \delta^n / n!$ for $t \in \mathbb{R}$, is a one-parameter group of bounded Lie automorphism of A : $e^{t\delta}([a, b]) = [e^{t\delta}(a), e^{t\delta}(b)]$ for $a, b \in A$. Hence

$$I \in \text{Li}_A \Rightarrow e^{t\delta}(I) \in \text{Li}_A \text{ for all } t \in \mathbb{R} \text{ and } \delta \in \mathfrak{D}(A). \quad (9.6)$$

Moreover, since $\delta(a) = \lim_{t \rightarrow 0} (e^{t\delta}(a) - a)/t \in I$ for all $a \in I$, it follows that

$$I \text{ is a characteristic Lie ideal if and only if } e^{t\delta}(I) = I \text{ for all } t \in \mathbb{R} \text{ and } \delta \in \mathfrak{D}(A). \quad (9.7)$$

Denote by Li_A^{ch} the subset of Li_A of all closed characteristic Lie ideals of A . We have that

$$\text{if } J \in \text{Li}_A^{\text{ch}} \text{ and } I \text{ is a characteristic Lie ideal of } J, \text{ then } I \in \text{Li}_A^{\text{ch}}. \quad (9.8)$$

Indeed, $\delta|_J \in \mathfrak{D}(J)$ for all $\delta \in \mathfrak{D}(A)$, as $J \in \text{Li}_A^{\text{ch}}$. Then $\delta(I) = \delta|_J(I) \subseteq I$, as $I \in \text{Li}_J^{\text{ch}}$. Thus, as $\text{ad}(a) \in \mathfrak{D}(A)$, for each $a \in A$, $[a, I] = \text{ad}(a)(I) \subseteq I$. Hence $I \in \text{Li}_A^{\text{ch}}$.

The existence of Lie and characteristic Lie ideals of finite codimension in Banach Lie algebras was studied in [KST2]. We will use below the following result obtained there.

Theorem 9.15 *If a Banach non-commutative Lie algebra has a closed proper Lie subalgebra of finite codimension, then it has a proper closed characteristic Lie ideal of finite codimension.*

As the intersection and the closed sum of any family of characteristic Lie ideals is a characteristic Lie ideal, Li_A^{ch} is a complete sublattice of Li_A . Denote by $\mathfrak{p}_{\infty}^{\text{ch}}(A)$ the dual $\ll_{\infty}^{\triangleleft}$ -radical in Li_A^{ch} and by Σ_A the set of all closed Lie subalgebras of finite codimension in A .

We now prove the following version of Proposition 9.6 for Banach Lie algebras.

Theorem 9.16 *Let Li_A^{ch} have no commutative infinite-dimensional Lie ideals and $\bigcap_{L \in \Sigma_A} L = \{0\}$. Then, for each $\{0\} \neq K \in \text{Li}_A^{\text{ch}}$, there is $I \in \text{Li}_A^{\text{ch}}$ such that $K \neq I \ll_{\infty} I$, and $\mathfrak{p}_{\infty}^{\text{ch}}(A) = \{0\}$.*

Proof. Set $\mathfrak{p} = \mathfrak{p}_{\infty}^{\text{ch}}(A)$. Let $\{0\} \neq K \in \text{Li}_A^{\text{ch}}$. If $\dim K < \infty$ then $\{0\} \ll_{\infty} K$ and $\{0\} \in \text{Li}_A^{\text{ch}}$.

Let $\dim K = \infty$. As $\bigcap_{L \in \Sigma_A} L = \{0\}$, there is $L \in \Sigma_A$ that does not contain K . Then $K \cap L$ is a proper closed Lie subalgebra of A . Replacing M by A in Lemma 8.4(i), we have that $K \cap L$ has finite codimension in K . By our assumption, K is non-commutative. So, by Theorem 9.15, there is $I \in \text{Li}_K^{\text{ch}}$ of finite codimension, i.e., $\dim(K/I) < \infty$. By (9.8), $I \in \text{Li}_A^{\text{ch}}$, so that $I \ll_{\infty} K$ in Li_A^{ch} .

If $\mathfrak{p} \neq \{0\}$ then, by above, $I \ll_{\infty} \mathfrak{p}$ for some $\mathfrak{p} \neq I \in \text{Li}_A^{\text{ch}}$ which contradicts Theorem 8.1(i). ■

The condition that Li_A^{ch} has no commutative infinite-dimensional Lie ideals in Theorem 9.16 is essential. To show this, consider the following construction of Banach Lie algebras [Bo, Sec 1.8].

Let X be a Banach space and \mathfrak{L} a closed Lie subalgebra of $B(X)$. The direct sum $A = \mathfrak{L} \oplus^{\text{id}} X$ (semidirect product of \mathfrak{L} and X) endowed with Lie multiplication and norm

$$[(a; x), (b; y)] = ([a, b]; ay - bx) \quad \text{and} \quad \|(a; x)\| = \max\{\|a\|, \|x\|\} \quad \text{for } a, b \in \mathfrak{L}, x, y \in X,$$

is a Banach Lie algebra. Clearly, $J = \{0\} \oplus^{\text{id}} X$ is a closed commutative Lie ideal of A .

Example 9.17 *Let $\dim X = \infty$, $\dim \mathfrak{L} < \infty$ and let \mathfrak{L} have no invariant subspaces in X . Then $\bigcap_{L \in \Sigma_A} L = \{0\}$. However, $\mathfrak{p}_{\infty}^{\text{ch}}(A) = J \neq \{0\}$, as Li_A^{ch} has a commutative Lie ideal J .*

Proof. If Y is a closed subspace of codimension 1 in X then $\{0\} \oplus^{\text{id}} Y \in \Sigma_A$, as $\dim \mathfrak{L} < \infty$. Since the intersection of all such Lie subalgebras is $\{0\}$, we have $\bigcap_{L \in \Sigma_A} L = \{0\}$.

As \mathfrak{L} is irreducible in X , each $K \in \text{Li}_A$ has form $K = I \oplus^{\text{id}} X$, where $I \in \text{Li}_{\mathfrak{L}}$; and J is the unique minimal non-zero closed Lie ideal of A . By (9.6), $e^{t\delta}(J)$ is a minimal Lie ideal for each $t \in \mathbb{R}$ and $\delta \in \mathfrak{D}(A)$, so that $e^{t\delta}(J) = J$. By (9.7), $J \in \text{Li}_A^{\text{ch}}$. Thus the condition in Theorem 9.16 fails and the theorem does not hold since $\{0\} \neq \mathfrak{p}_{\infty}^{\text{ch}}(A) = J \ll_{\infty} A$. ■

Let $\mathfrak{p}_{\infty}(\mathfrak{L})$ be the dual $\ll_{\infty}^{\triangleleft}$ -radicals in $\text{Li}_{\mathfrak{L}}$ and $\mathfrak{p}_{\infty}(X)$ be the dual $\ll_{\infty}^{\triangleleft}$ -radicals in $\text{Lat } \mathfrak{L}$ – the lattice of all \mathfrak{L} -invariant subspaces in X . Let $\mathfrak{p}_{\infty}(A)$ be the dual $\ll_{\infty}^{\triangleleft}$ -radical in Li_A .

Proposition 9.18 *Let $A = \mathfrak{L} \oplus^{\text{id}} X$. If $\mathfrak{p}_{\infty}(\mathfrak{L}) = \{0\}$ then $\mathfrak{p}_{\infty}(A) = \{0\} \oplus^{\text{id}} \mathfrak{p}_{\infty}(X)$.*

Proof. As $\mathfrak{p}_{\infty}(\mathfrak{L}) = \{0\}$, there is a descending \ll_{∞} -series of Lie ideals $(I_{\lambda})_{1 \leq \lambda \leq \gamma}$ of \mathfrak{L} from \mathfrak{L} to $\{0\}$. Then $C_{\mathfrak{L}} = \{I_{\lambda} \oplus^{\text{id}} X\}_{1 \leq \lambda \leq \gamma}$ is a descending \ll_{∞} -series of Lie ideals of A from A to $\{0\} \oplus^{\text{id}} X$.

Let $C_X = \{Y_{\omega}\}_{1 \leq \omega \leq \alpha}$ be a descending \ll_{∞} -series of subspaces in $\text{Lat } \mathfrak{L}$ from X to $\mathfrak{p}_{\infty}(X)$. As $\mathfrak{p}_{\infty}(X)$ has no invariant subspaces of finite codimension, the Lie ideal $J = \{0\} \oplus^{\text{id}} \mathfrak{p}_{\infty}(X)$ contains no Lie ideals of A of finite codimension and $C = C_{\mathfrak{L}} \cup C_X$ is a descending \ll_{∞} -series of Lie ideals from A to J . So, by Corollary 9.3, $\mathfrak{p}_{\infty}(A) = J$. ■

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