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Functional relations for elliptic polylogarithms

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Abstract

Numerous examples of functional relations for multiple polylogarithms are
known. For elliptic polylogarithms, however, tools for the exploration of func-
tional relations are available, but only very few relations are identified. Starting
from an approach of Zagier and Gangl, which in turn is based on considerations
about an elliptic version of the Bloch group, we explore functional relations
between elliptic polylogarithms and link them to the relations which can be
derived using the elliptic symbol formalism. The elliptic symbol formalism in
turn allows for an alternative proof of the validity of the elliptic Bloch relation.
While the five-term identity is the prime example of a functional identity for
multiple polylogarithms and implies many dilogarithm identities, the situation
in the elliptic setup is more involved: there is no simple elliptic analogue, but
rather a whole class of elliptic identities.

Keywords: elliptic polylogarithms, iterated integrals, functional relations, Bloch
group

(Some figures may appear in colour only in the online journal)

1. Introduction

The majority of calculations in quantum field theory, in particular when considering quan-
tum chromodynamics, is based on the evaluation of integrals associated to Feynman graphs.
Using Feynman parameters one can rewrite integrations over loop variables into integrations

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over Feynman parameters in a formalised manner. Along with several further advantages, the reparametrisation allows to bring the integrals to an iterated form.

This is a rather general concept: Feynman integrals seem to be expressible in terms of iterated integrals over a suitably chosen set of differential forms on Riemann surfaces of various genera. The exploration of classes of these iterated integrals and the utilisation of their algebraic properties did not only change the way calculations are performed, but simultaneously leads to convenient representations: once a proper class of functions is identified, one can find functional relations and thus reduce to a basis of integrals.

It turns out that suitable differential forms defining classes of iterated integrals can be identified starting from geometrical considerations: taking the first abelian differential on the simplest genus-zero surface, the Riemann sphere, leads to the class of multiple polylogarithms [1–5] while abelian differentials on a genus-one Riemann surface are the starting point for the elliptic polylogarithms [6, 7] to be discussed in this article.

Genus zero: multiple polylogarithms have been a very active field of research in the last years: since their motivic version constitute a graded Hopf algebra [3, 8, 9], with the shuffle product as algebra multiplication and the deconcatenation coproduct, there are very strong tools available [10] allowing in particular to derive functional relations. While the Duval algorithm [11] delivers a basis with respect to the shuffle product, further relations between different arguments of polylogarithms can be explored using the coproduct, which is usually referred to as the symbol map. A non-exhaustive list of examples, where such relations are investigated, is references [1, 12–17]. Examples involving the evaluation of Feynman integrals include references [18–22]. We are mainly interested in functional relations of the dilogarithm. Of particular importance hereby is the so-called five-term identity

$$\mathrm{D}\left(t\right) + \mathrm{D}\left(s\right) + \mathrm{D}\left(\frac{1-t}{1-ts}\right) + \mathrm{D}\left(1-ts\right) + \mathrm{D}\left(\frac{1-s}{1-ts}\right) = 0, \tag{1.1}$$

where $\mathrm{D}(t) = \mathrm{Im}\left(Li_2(t) - \log\left(|t|\right) Li_1(t)\right)$ is the Bloch–Wigner function, the single-valued version of the dilogarithm. The five-term identity has a beautiful interpretation in terms of a volume decomposition in hyperbolic space into (hyperbolic) tetrahedra. In addition, it is known [14, 23] to create a large class of functional equations for the dilogarithm which are linear combinations of Bloch–Wigner functions where the arguments are rational functions of one variable and satisfy a particular condition, to be explained below. Similar statements are conjectured to hold in more general situations where the arguments are allowed to be algebraic functions or rational functions of more than one variable [16]. On the physics side, the idea of splitting a given volume into several polyhedra has been used to interpret and reformulate the calculation of various Feynman diagrams, see for example references [20, 21]. Linear combinations of values of the Bloch–Wigner function which satisfy the mentioned condition above and which are equal modulo finitely many applications of functional relations of the Bloch–Wigner function are identified in the Bloch group [24–26]. Similarly, higher Bloch groups have been investigated in the context of higher order polylogarithms.

Genus one: while elliptic polylogarithms have been explored for a long time [6, 27, 28], it is only recently that they have been facilitated in the calculation of scattering amplitudes in physics [29, 30]. However, as became apparent, many of the structures inherent in multiple polylogarithms can be taken to genus one easily: iterated integrals on genus one allow for a natural shuffle multiplication and an associated coaction or symbol map [31].

Given the existence of the symbol map for elliptic iterated integrals, it is a natural problem to investigate functional relations for elliptic polylogarithms. In particular, an elliptic analogue of the Bloch group has been considered in reference [32], which is based on a class of functional

relations for an elliptic generalisation of the Bloch–Wigner function, the elliptic Bloch–Wigner function D^E, and given by relations of the form

$$D^E(\eta_F) = 0 \tag{1.2}$$

where the object η_F is parametrised by (some of the zeros and singularities of) any non-constant elliptic function F [33]. However, a similar functional relation for the elliptic Bloch–Wigner function and the construction of an elliptic analogue of the Bloch group has already been discussed in reference [34]. In contrast to the genus-zero case, where the five-term identity suffices to represent a large class of functional identities of the dilogarithm, a whole class of functional identities given by equation (1.2) needs to be investigated in the genus-one case [32]. The considerations therein, however, remain on the level of a few particular examples, e.g. an implicitly defined elliptic analogue of the five-term identity. As will be described in detail below, the answer to the question of an explicit elliptic five-term identity and the explicit description of the other elliptic functional identities generated by equation (1.2) requires substantially more technical effort than for classical polylogarithms.

In this article, we are going to put Zagier's and Gangl's method to work in order to find several examples of functional identities between simple elliptic polylogarithms. The resulting relations are going to be contrasted with relations derived using the elliptic symbol map. In order to compare the two types of relations, one has to translate between different formulations of the elliptic curve, and thus different types of (iterated) integrals, which is a source of the complexity of the problem. Despite those difficulties we find several relations connecting elliptic polylogarithms of rather complicated arguments. In some cases, the relations found can be trivially accounted to known symmetry relations for the elliptic Bloch–Wigner function.

The translation of the elliptic Bloch–Wigner function to the torus, represented as the complex plane modulo a two-dimensional lattice $\mathbb{C}/\Lambda$, allows a new perspective on the elliptic Bloch relation: the condition encoded in equation (1.2) above translates into rather simple relations between iterated elliptic integrals on the torus, whose correctness is not difficult to show. Thus the translation combined with the elliptic symbol calculus provides an alternative proof of the elliptic Bloch relation.

As an aside, we are going to translate Ramakrishnan's generalisations of the elliptic Bloch–Wigner dilogarithm [25, 35] as well as Zagier's generalised single-valued elliptic polylogarithms [36] to the torus formulation of the elliptic curve. These representations will be serving as a starting point for the investigation of relations between higher elliptic functions in a forthcoming project.

Given the general structure of the elliptic curve, it was not to be expected that functional relations are at the same level of simplicity as their genus-zero cousins. On the one hand, the calculation of zeros and poles of elliptic functions is more complicated than in the case of rational functions on the Riemann sphere. On the other hand, the translation from the projective formulation of the elliptic curve, where the mentioned zeros and poles may be described in terms of rational functions, to the torus given by Abel's map is not algebraic and highly nontrivial.

This article is structured in the following way: in section 2 we present some of the well-known results for functional relations of the Bloch–Wigner function and in particular the construction of the Bloch group and the Bloch relation. In section 3 we review several known concepts: we set the notation for different formulations of elliptic curves as well as elliptic functions and review known results about the Bloch group in the genus-one situation, which are mostly formulated on the Tate curve describing the corresponding elliptic curve. Section 4 is devoted to the translation of the above and further concepts to the torus and the

projective elliptic curve. In particular, notions of (conjecturally) single-valued elliptic generalisations of polylogarithms defined on the Tate curve are related to the elliptic multiple polylogarithms as holomorphic iterated integrals on the torus, which further allows to formulate (and prove) the elliptic Bloch relation (1.2) on the torus and the projective elliptic curve, respectively.

2. Bloch groups for polylogarithms

The description of functional relations of polylogarithms and in particular of the single-valued dilogarithm—the Bloch–Wigner function—can be formalised using the concept of (higher) Bloch groups. These are certain (abelian) groups $\mathcal{B}_m$ which capture functional relations satisfied by single-valued polylogarithms of order m.

In subsection 2.1 we are going to review the geometric construction and interpretation of $\mathcal{B}_2$ in terms of hyperbolic three-manifolds. Afterwards, in subsection 2.2 we introduce the Bloch relation of the Bloch–Wigner function, which generates functional identities such as the five-term identity. In the subsequent section this Bloch relation will be generalised to the elliptic curve and will be used to define the elliptic analogue of $\mathcal{B}_2$, the elliptic Bloch group, which is discussed in subsection 3.4.

2.1. The Bloch group

The functional relations of the dilogarithm Li_2 often take a very simple form when expressed in terms of the Bloch–Wigner function

$$D(t) = \operatorname{Im}\left(Li_2(t) - \log\left(|t|\right) Li_1(t)\right), \qquad (2.1)$$

which is the single-valued version of the dilogarithm (see reference [16] for an extensive review of the Bloch–Wigner function). The Bloch–Wigner function is continuous on the Riemann sphere and real analytic except at the points 0, 1 and ∞, where it is defined to vanish.

The Bloch–Wigner function and its functional relations admit a broad variety of mathematical interpretations and applications, ranging from periodicities of a cluster algebra [37–39], volumes in hyperbolic space [40, 41] and the symbol calculus [42, 43] to functional identities generated by rational functions on the Riemann sphere [33], the latter is the main focus of our considerations.

The Bloch–Wigner function satisfies the symmetry relations

$$D(t) = D\left(1 - \frac{1}{t}\right) = D\left(\frac{1}{1-t}\right) = -D\left(\frac{1}{t}\right) = -D\left(1 - t\right) = -D\left(\frac{-t}{1-t}\right) \quad (2.2)$$

and the duplication relation

$$D(t^2) = 2D\left(t\right) + 2D\left(-t\right), \qquad (2.3)$$

which can be easily proven using the properties of the logarithm and Li_2. In addition, there is the famous five-term identity already mentioned in the introduction, which can be described as a consequence of the periodicity of the A_2 cluster algebra [39]. It reads

$$D\left(t\right) + D\left(s\right) + D\left(\frac{1-t}{1-ts}\right) + D\left(1-ts\right) + D\left(\frac{1-s}{1-ts}\right) = 0. \qquad (2.4)$$

In order for the above equation to yield a valid new relation, t and s are numbers chosen such that neither of the arguments yields 0, 1 or ∞, i.e. $s, t \neq 0, 1$ and $st \neq 1$. In those special cases, however, equation (2.4) degenerates to the symmetry relations in equation (2.2) above.

Alternatively, one can interpret the five-term identity as a relation between volumes of hyperbolic three-simplices in the so-called Poincaré half-space model [40, 41]. As it is this volume interpretation of the Bloch–Wigner function which leads to an illustrative geometric construction of the Bloch group[4], let us describe this construction in a little more detail following the lines of references [16, 32]. The volume of a complete, finite, hyperbolic three-manifold M can be triangulated and, thus, expressed as the sum over the volumes of a finite number of three-simplices

$$\text{Vol}(M) = \sum_i \text{D}(t_i), \tag{2.5}$$

each of which can be labelled by a cross ratio $t_i \in \mathbb{C}$ such that its volume is given by $\text{D}(t_i)$. Considering the geometric properties of such a triangulation, one can show that the associated coordinates t_i in equation (2.5) have to satisfy the following algebraic constraint [40]:

$$\sum_i t_i \wedge (1 - t_i) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*. \tag{2.6}$$

Correspondingly, one can in general express the volume of M as

$$\text{Vol}(M) = \sum_i \text{D}(t_i) = \text{D}(\xi), \tag{2.7}$$

for an element $\xi \in \mathcal{A}_2(\mathbb{C})$, where

$$\mathcal{A}_2(\mathbb{C}) = \left\{ \sum_{i=1}^n n_i(t_i) \,|\, t_i \in \mathbb{C}^* \setminus \{1\}, n \in \mathbb{N}, \, n_i \in \mathbb{Z}, \, \sum_{i=1}^n n_i \, (t_i \wedge (1 - t_i)) = 0 \right\} \subset \mathcal{F}_{\mathbb{C}}, \tag{2.8}$$

$\mathcal{F}_{\mathbb{C}}$ is the free abelian group[5] generated by $\mathbb{C}$ and the Bloch–Wigner function is extended by linearity to $\mathcal{F}_{\mathbb{C}}$, i.e.

$$\text{D}\left(\sum_i n_i(t_i) \right) = \sum_i n_i \text{D}(t_i). \tag{2.9}$$

Let us briefly discuss the definition (2.8) of $\mathcal{A}_2(\mathbb{C})$. The condition $t_i \notin \{0, 1\}$ corresponds to the definition of $\text{D}(0) = 0 = \text{D}(1)$. The fact that now, we allow in $\sum_{i=1}^n n_i \, (t_i \wedge (1 - t_i)) = 0$ the coefficients n_i to be any integer and not only to equal 1, as in equation (2.6), is required to turn $\mathcal{A}_2(\mathbb{C})$ into a subgroup of $\mathcal{F}_{\mathbb{C}}$ and to (uniquely) shorten the sum (2.7) in the case of $t_i = t_j$ for $i \neq j$.

The geometric interpretation of the five-term identity corresponds to a change of triangulation: it describes two distinct triangulations of a volume defined by five vertices and the edges

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[4] The Bloch group $\mathcal{B}_2$ has originally been introduced in reference [24] and has been extended in references [25, 26] to higher orders.

[5] The free abelian group generated by a set S is the group of formal finite sums $\sum_{s \in S} n_s(s)$ with $n_s \in \mathbb{Z}$, all but finitely many equal to zero. The group operation is defined by $\sum_{s \in S} n_s(s) + \sum_{s \in S} m_s(s) = \sum_{s \in S} (n_s + m_s)(s)$ and the identity element is the empty sum. Note that in contrast to the usual notation where square brackets are used to denote an element of the free abelian group, we use parentheses in agreement with the notation of divisors introduced in subsection 2.2.

being geodesics, which can either be described by a disjoint union of two hyperbolic three-simplices or of three such simplices. The five-term identity expresses the equality of the sum of the volumes of the former two and the latter three simplices. The change of triangulation and the associated applications of the five-term relation motivate the definition of the following subgroup of $\mathcal{A}_2(\mathbb{C})$, which can be thought of as constituting the group of relations of the Bloch–Wigner function with the generator[6] being the arguments occurring in the five-term identity,

$$\mathcal{C}_2(\mathbb{C}) = \left\langle (t) + (s) + \left(\frac{1-t}{1-ts} \right) + (1-ts) + \left(\frac{1-s}{1-ts} \right) \, | \, s,t \in \mathbb{C}^* \backslash \{1\}, st \neq 1 \right\rangle. \quad (2.10)$$

Thus, the volume of M can be expressed as the value

$$\mathrm{Vol}(M) = \mathrm{D}(\xi_M) \qquad (2.11)$$

for a canonical $\xi_M \in \mathcal{B}_2(\mathbb{C})$ associated to M with

$$\mathcal{B}_2(\mathbb{C}) = \frac{\mathcal{A}_2(\mathbb{C})}{\mathcal{C}_2(\mathbb{C})} \qquad (2.12)$$

being the Bloch group[7].

Besides this geometric construction, the Bloch group $\mathcal{B}_2(\mathbb{C})$ is an elementary algebraic structure for the description of dilogarithmic functional relations, i.e. identities of finite sums such as $\sum_i n_i \mathrm{D}(t_i(s_j)) = c$, for rational or algebraic functions t_i of one or more variables s_j and some constant $c \in \mathbb{C}$. In the case of only one variable s and rational functions $t_i(s) \in \mathbb{C}(s)$, the element $\xi = \sum_i n_i(t_i(s))$ evaluates under D to a constant if and only if $\sum_i n_i (t_i(s)) \wedge (1 - t_i(s))$ is independent of s [25]. For the particular condition $\sum_i n_i (t_i(s)) \wedge (1 - t_i(s)) = 0$, the element ξ belongs to the Bloch group $\mathcal{B}_2(\mathbb{C}(s))$ of the field of rational functions $\mathbb{C}(s)$. As proven in reference [14], all such elements are equal to zero in $\mathcal{B}_2(\mathbb{C}(s))$. Thus, in this case the functional equation $\sum_i n_i \mathrm{D}(t_i(s)) = 0$ is indeed obtained by a finite number of applications of the five-term identity. Similar statements are not known in the case of algebraic functions or rational functions in more than one variable, but they are expected to exist, see e.g. [16].

2.2. Bloch's dilogarithm relations

Bloch describes a concept to formalise the generation of functional identities for the Bloch–Wigner function and of its generalisation to elliptic curves [33]. In this subsection we state his results in the classical situation and generalise it to the elliptic case in section 3 below.

In the following, we are going to make use of the concept of a *divisor*: for any meromorphic function g defined on a compact Riemann surface X, the divisor of g is defined as

$$\mathrm{Div}(g) = \sum_{p \in X} \mathrm{ord}_p(g)\,(p), \qquad (2.13)$$

where $\mathrm{ord}_p(g)$ is the order of the pole (a negative integer) or the order of the zero (a positive integer), respectively, of g at p. If p is neither a pole nor a zero of g, $\mathrm{ord}_p(f) = 0$, which renders

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[6] For $T \subset S$, the subgroup $\langle t | t \in T \rangle$ of $\mathcal{F}_{\mathbb{C}}$ generated by T is the group of formal finite sums $\sum_{t \in T} n_t(t)$ with $n_t \in \mathbb{Z}$, all but finitely many equal to zero.

[7] Higher Bloch groups $\mathcal{B}_m(\mathbb{C})$ for $m > 2$ can be constructed recursively [32]. In analogy to the case $\mathcal{C}_2(\mathbb{C})$ considered above, the subgroup $\mathcal{C}_m(\mathbb{C})$ of the group of 'allowable' points $\mathcal{A}_m(\mathbb{C})$ (where allowable can be defined recursively and corresponds to the condition $\sum_{i=1}^n n_i (t_i \wedge (1 - t_i)) = 0$ in the case $m = 2$) is constructed to be the span of functional relations among polylogarithms of order m.

the number of terms in the above sum finite. In the definition above, divisors are elements of the free abelian group generated by the Riemann surface X.

Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a non-trivial rational function on the Riemann sphere satisfying

$$f(0) = f(\infty) = 1, \tag{2.14}$$

which can be realized by representing f as a finite product

$$f(t) = \prod_i (t - a_i)^{d_i}, \qquad \sum_i d_i = 0, \qquad \prod_i a_i^{d_i} = 1, \tag{2.15}$$

where $a_i \in \mathbb{C}$ and $d_i \in \mathbb{Z}$. Furthermore, let us write

$$1 - f(t) = b \prod_j (t - b_j)^{e_j}, \tag{2.16}$$

where $b, b_j \in \mathbb{C}$ and $e_j \in \mathbb{Z}$. The divisor of the function f defined in equation (2.15) reads

$$\mathrm{Div}(f) = \sum_i \mathrm{ord}_{a_i}(f)(a_i) = \sum_i d_i(a_i), \qquad \mathrm{Div}(1 - f) = \sum_j e_j(b_j). \tag{2.17}$$

In reference [33] Bloch proves that for any rational function f as defined above, the Bloch–Wigner function satisfies

$$\sum_{i,j} d_i e_j \mathrm{D}\left(\frac{a_i}{b_j}\right) = 0, \tag{2.18}$$

abbreviated in terms of the element

$$\eta_f = \sum_{i,j} d_i e_j \left(\frac{a_i}{b_j}\right) \tag{2.19}$$

of the free abelian group $\mathcal{F}_\mathbb{C}$ and the Bloch–Wigner function extended by linearity as in equation (2.9), this so-called *classical Bloch relation* reads

$$\mathrm{D}\left(\eta_f\right) = 0. \tag{2.20}$$

Letting the zeros a_i of f vary subject to the conditions in equation (2.15), this dilogarithm relation of Bloch becomes a functional relation. Choosing different rational functions satisfying equation (2.14) in the first place, equation (2.20) yields a whole class of functional relations for the Bloch–Wigner function parametrized by rational functions f on the Riemann sphere, which is however not independent. In fact, it is conjecturally generated by the single example of the five-term identity (see the discussion at the end of subsection 2.1), which is discussed in the following paragraph.

As the most fundamental example and an application of the Bloch relation, let us discuss how to recover the five-term identity equation (2.4) from equation (2.20) following the lines of reference [32]: let $a, b \in \mathbb{C}$, $a' = 1 - a$, $b' = 1 - b$ and consider the rational function

$$f(t) = \frac{(t - a)(t - a')(t - bb')}{(t - b)(t - b')(t - aa')}. \tag{2.21}$$

It satisfies $f(0) = f(\infty) = 1$ and

$$1 - f(t) = \frac{(bb' - aa')t^2}{(t - b)(t - b')(t - aa')}, \tag{2.22}$$

such that Bloch's relation can be applied, which yields the identity

$$
\mathrm{D}\left(\frac{a}{b}\right) + \mathrm{D}\left(\frac{a'}{b'}\right) + \mathrm{D}\left(\frac{a}{b'}\right) + \mathrm{D}\left(\frac{bb'}{aa'}\right) + \mathrm{D}\left(\frac{a'}{b}\right) = 0, \tag{2.23}
$$

where we have used that $\mathrm{D}(0) = \mathrm{D}(1) = \mathrm{D}(\infty) = 0$ and the symmetry relations (2.2). Changing variables to $t = \frac{a}{b}$, $s = \frac{a'}{b'}$ finally leads to the five-term identity in the usual form (2.4).

3. Elliptic curves, the divisor function and Bloch's relation

The aim of this section is twofold: after reviewing mathematical tools for the description of elliptic curves in various formulations and a particular type of elliptic iterated integrals in subsections 3.1 and 3.2 we are going to discuss and exemplify the generalisation of the concepts of the divisor function and the Bloch relation from the previous section to the genus-one Riemann surfaces/elliptic curves in subsections 3.3 and 3.4. In particular, subsection 3.4 contains three examples of functional relations on the elliptic curve parametrised by various rational functions.

3.1. Elliptic curves and functions

This subsection begins with the introduction of the torus description of elliptic curves: being a Riemann surface of genus one, the torus is the natural geometry underlying an elliptic curve due to its two periodicities. Along with the discussion of the torus formulation, several properties of elliptic functions are reviewed. Afterwards, two isomorphisms are discussed, where the first one relates the torus to the projective (elliptic) curve and the second one maps the torus to the so-called Tate curve given by the exponential map. These are well-known mathematical concepts, but in particular the map from the torus to the Tate curve is rarely mentioned in the physics literature. A thorough introduction which relates to the common physics language can e.g. be found in reference [44], which is the basis for the discussion in this subsection.

A torus can be described as the quotient $\mathbb{C}/\Lambda$ of the complex plane and a lattice

$$
\Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \tag{3.1}
$$

where the periods ω_1 and ω_2 are complex numbers and taken to be linearly independent over the real numbers. The domain $P_\Lambda = \{a\omega_1 + b\omega_2 | 0 \leqslant a, b < 1\}$ is called the *fundamental parallelogram* of $\mathbb{C}/\Lambda$ which defines the torus upon identifying the opposite sides of its closure. Due to this immediate relation, we will simply refer to $\mathbb{C}/\Lambda$ as the torus itself. The torus is often scaled such that $\tau = \omega_2/\omega_1$ and 1 are its periods and without loss of generality τ is assumed to be an element of the upper half plane, $\mathrm{Im}(\tau) > 0$, in this case the fundamental parallelogram can be depicted as in figure 1.

A function is called *elliptic* on $\mathbb{C}$ if it is Λ-periodic, i.e. a function defined on $\mathbb{C}/\Lambda$, and meromorphic. However, in the case of generalisations of multiple polylogarithms to the elliptic curve, we sometimes also refer to multi-valued functions on the torus $\mathbb{C}/\Lambda$ (i.e. not necessarily Λ-periodic functions), as elliptic functions if they are meromorphic. This is in particular the case for the elliptic multiple polylogarithms introduced in subsection 3.2.

Two explicit examples of elliptic functions are the even Weierstrass $\wp$ function

$$
\wp(z) = \wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right) \tag{3.2}
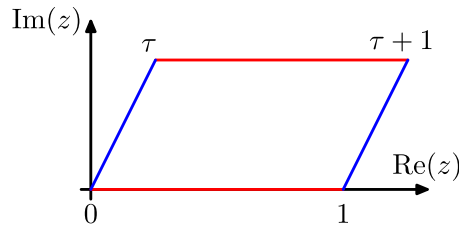$$

Figure 1. Fundamental domain of the torus $\mathbb{C}/\Lambda$.

and its odd derivative $\wp'(z)$. Note that $\wp$ has a double pole at any lattice point, whereas $\wp'$ has a triple pole at the lattice points. Closed expressions of zeros of $\wp$ are generally complicated, while the zeros of $\wp'$ are exactly the half periods $\omega_i/2$, for $i = 1, 2, 3$ and $\omega_3 = \omega_2 - \omega_1$. Moreover, these elliptic functions satisfy the differential equation

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{3.3}$$

where the three roots e_i are defined by

$$e_i = \wp(\omega_i/2) \tag{3.4}$$

and sum to zero. The Weierstrass invariants g_2 and g_3 in the above equation can be expressed in terms of Eisenstein series

$$g_2 = 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4}, \qquad g_3 = 140 \sum_{(m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6} \tag{3.5}$$

and are related to the roots by

$$e_1 + e_2 + e_3 = 0, \qquad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2, \qquad e_1e_2e_3 = \frac{1}{4}g_3. \tag{3.6}$$

It turns out that the notion of ellipticity is quite restrictive: for example, the zeros and poles of an elliptic function F are subject to the conditions[8]

$$\sum_{z \in P_\Lambda} \mathrm{ord}_z(F) = 0, \qquad \sum_{z \in P_\Lambda} \mathrm{ord}_z(F)\, z \in \Lambda, \tag{3.7}$$

where the order $\mathrm{ord}_z(F)$ of F at z is the usual order of zeros and poles of meromorphic functions, in analogy to the definition of the order in the context of rational functions on the Riemann sphere used in equation (2.13). In particular, points which are neither zeros nor poles are of order zero. Thus, the sums over the fundamental parallelogram in equation (3.7) are finite and include the non-vanishing terms at zeros and poles of F only.

Moreover, an elliptic function can not have a single simple pole: using Cauchy's residue theorem and integration along the fundamental parallelogram, where the (reversed) parallel paths cancel pairwise due to the Λ-periodicity, the sum of the residues has to vanish, which can not be satisfied by a single simple pole alone. The conditions in equation (3.7) follow from the

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[8] See e.g. the lecture notes [45] for the derivation of equation (3.7) and the following statements about Weierstrass functions.

same cancelation of the integration along the fundamental parallelogram in the (generalised) argument principle.

Furthermore, any elliptic function is determined up to scaling by its zeros and poles: the quotient of two elliptic functions with the same zeros and poles, counting multiplicities, is bounded on the fundamental parallelogram P_Λ and hence, by Λ-periodicity, is a bounded entire function, such that Liouville's theorem implies that these two elliptic functions are proportional to each other. This fact, in turn, implies that any elliptic function on $\mathbb{C}/\Lambda$ is a rational function in $\wp$ and $\wp'$: those rational functions are elliptic by construction and can be combined to have the same zeros and poles as any given elliptic function.

Alternatively, any elliptic function can be expressed in terms of the Weierstrass σ function

$$\sigma(z) = s_C \, \exp\left(\int_{z_0}^z \mathrm{d}z' \, \zeta(z')\right), \tag{3.8}$$

where the scaling factor s_C and the base point z_0 are chosen[9] such that $\sigma'(0) = 1$. The logarithmic derivative (and thus the integrand in equation (3.8)) of the Weierstrass σ function is the Weierstrass ζ function

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{z - m\omega_1 - n\omega_2} + \frac{1}{m\omega_1 + n\omega_2} + \frac{z}{(m\omega_1 + n\omega_2)^2}\right), \tag{3.9}$$

which itself is the negative odd primitive of $\wp$.

The Weierstrass σ function has no poles and one simple zero at the lattice points, hence, it can not be elliptic. In fact, neither ζ nor σ is Λ-periodic. For the Weierstrass ζ function and a lattice period ω_i, integrating the equation $\wp(z + \omega_i) = \wp(z)$ implies that ζ changes by a z-independent integration constant

$$\zeta(z + \omega_i) = \zeta(z) + 2\eta(\omega_i) \tag{3.10}$$

with the quasi-period $\eta(\omega_i) = \zeta(\omega_i/2)$, which follows from the evaluation of equation (3.10) at $z = -\omega_1/2$. In a similar manner one can determine the transformation behaviour of the Weierstrass σ function, which reads

$$\sigma(z + \omega_i) = \exp\left(2\eta(\omega_i)z + \xi(\omega_i)\right)\sigma(z), \tag{3.11}$$

where $\xi(\omega_i)$ is yet another integration constant (see e.g. reference [45]). This shows explicitly that σ is indeed not elliptic. The transformation (3.11) of σ and the fact that it has one simple zero at any lattice point and no poles at all leads to the alternative representation of a given elliptic function F mentioned above: one can always choose particular representatives A_i of the zeros and poles of F in $\mathbb{C}/\Lambda$ (not necessarily in the fundamental domain) such that

$$\sum_i d_i = 0, \qquad \sum_i d_i A_i = 0, \tag{3.12}$$

---

[9] Both, s_C and z_0, can be chosen canonically by adjusting the integration constant $\xi(\omega_i)$ in equation (3.11).

where $d_i = \mathrm{ord}_{A_i}(F)$. It is then the set of conditions (3.12), satisfying the natural constraints (3.7) for the zeros and poles of an elliptic function, which ensures that the combination[10]

$$\prod_i \sigma(z - A_i)^{d_i} = \exp\left(\sum_i d_i \int_0^{z - A_i} \mathrm{d}z' \zeta(z')\right) \tag{3.13}$$

is elliptic. Indeed, under a lattice displacement the exponential proportionality factor in equation (3.11) from the transformation of the individual factors $\sigma(z - A_i)^{d_i}$ in equation (3.13) form a product with an exponent which is a linear combination of the left-hand sides of the two conditions (3.12), such that the overall proportionality constant evaluates to one. Since σ has only one simple zero at the lattice points and no pole, the above product has exactly the same zeros and poles including multiplicities as the function F. Correspondingly, any elliptic function F can be written as

$$F(z) = s_A \prod_i \sigma(z - A_i)^{d_i} = s_A \exp\left(\sum_i d_i \int_0^{z - A_i} \mathrm{d}z' \zeta(z')\right) \tag{3.14}$$

for some scaling factor $s_A \in \mathbb{C}$. The behaviour of the zeros and poles of an elliptic function can be conveniently captured in terms of divisors, which are introduced in subsection 3.3.

The fact that all elliptic functions can be expressed as rational functions of $\wp$ and $\wp'$ facilitates their description in terms of rational functions on a complex projective algebraic curve. The Weierstrass $\wp$ function induces an isomorphism between $\mathbb{C}/\Lambda$ and the complex projective algebraic curve

$$E(\mathbb{C}) = \{[x : y : 1] \,|\, y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)\} \cup \{[0 : 1 : 0]\}, \tag{3.15}$$

where $[0 : 1 : 0]$ is denoted by infinity ∞. Note that the cubic equation in x and y of the curve in definition (3.15) is of the same form as the differential equation (3.3) for $\wp$: this representation of the constraint equation on the projective formulation of the elliptic curve is called the *Weierstrass form* or *Weierstrass equation*. Furthermore, the projective algebraic curve $E(\mathbb{C})$ is often called the projective formulation of the elliptic curve or the projective elliptic curve.

The isomorphism of Riemann surfaces is given by

$$\xi_{\Lambda,E} : \mathbb{C}/\Lambda \to E(\mathbb{C}), \qquad 0 \neq z \mapsto \left[\wp(z) : \wp(z)' : 1\right], \qquad 0 \mapsto [0 : 1 : 0] = \infty, \tag{3.16}$$

see e.g. reference [44] for more details. The addition on $E(\mathbb{C})$ is provided by the so-called chord-tangent construction with the additive unity being ∞. It has a nice geometric interpretation, which is described in appendix A.

The inverse of the isomorphism $\xi_{\Lambda,E}$ is called Abel's map and can be determined from the differential equation (3.3). Given a point $P = [x_P : y_P : 1]$ with $y_P \neq 0$, one finds

$$z = \pm \int_\infty^{x_P} \frac{\mathrm{d}x}{y} \mathrm{mod}\Lambda, \tag{3.17}$$

---

[10] Note that compared to the definition of the Weierstrass σ function (3.8), the factors of s_C from the product on the left-hand side of equation (3.13) multiply to one and the base point z_0 of the integrals in the exponential can be shifted to zero due to the condition $\sum_i d_i = 0$ in equation (3.12).

where the correct sign is determined by the requirement that $\wp'(z) = y_P$, and $\xi_{\Lambda,E}^{-1}(e_i) = \omega_i/2$ for $P = [e_i : 0 : 1]$. Upon identifying

$$x = \wp(z), \qquad y = \wp'(z) \tag{3.18}$$

as well as using the fact that these two functions generate any elliptic function on the torus in terms of rational functions, it follows that the elliptic functions can be described as the rational functions in x and y on the projective elliptic curve $E(\mathbb{C})$.

The above choice of signs in Abel's map (3.17) is not the only issue that needs some care if a translation from a given projective elliptic curve $E(\mathbb{C})$ with elliptic invariants g_2 and g_3 to the torus has to be implemented explicitly.

A first ambiguity has to be addressed by making a choice for the periods ω_1 and ω_2 associated to the elliptic curve with Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$. The roots e_i are defined by g_2 and g_3 up to relabelling according to equation (3.6). Simultaneously, Abel's map together with equation (3.4) implies

$$\frac{\omega_1}{2} = \frac{\omega_2}{2} - \frac{\omega_3}{2} = \left(\int_{e_3}^{e_2} \frac{dx}{y} \right) \mod\Lambda, \tag{3.19}$$

where the ω_i, or the fundamental parallelogram, respectively, are chosen such that the periods are given by the integrals [46]

$$\omega_1 = 2 \int_{e_3}^{e_2} \frac{dx}{y}, \qquad \omega_2 = 2 \int_{e_3}^{e_1} \frac{dx}{y}, \qquad \omega_3 = \omega_2 - \omega_1 = 2 \int_{e_2}^{e_1} \frac{dx}{y}. \tag{3.20}$$

Any other choice of labelling the roots will yield an integer linear combination of the periods defined in equation (3.20) above, i.e.

$$2 \int_{e_i}^{e_j} = m_{ij}\omega_1 + n_{ij}\omega_2 \tag{3.21}$$

with $m_{ij}, n_{ij} \in \mathbb{Z}$. Hence, the choice of periods corresponds to choosing different basis vectors for spanning the lattice Λ. Correspondingly, the six possible labellings of the roots define six pairs of periods (ω_1, ω_2), whereas the associated different tori are isomorphic to a particular elliptic curve.

The second, but related issue is that the complex plane may always be rescaled by ω_1. Hence, only the ratio $\tau = \omega_2/\omega_1$ matters when dealing with the Λ-periodicity, i.e. the geometry of the torus. Therefore, a torus is usually only defined by the modular parameter τ with positive imaginary part $\text{Im}(\tau) > 0$ while the second period is chosen to be one. Under scaling ω_1, the Weierstrass $\wp$ function rescales as

$$\wp(z; 1, \tau) = \omega_1^2 \, \wp(\omega_1 z; \omega_1, \omega_2). \tag{3.22}$$

Choosing τ in the upper half-plane means that three possible labellings of the roots e_i are disregarded. The remaining three period ratios obtained from the different labellings of the roots e_i are related to the τ in the upper half plane by modular transformations.

In summary, the Weierstrass invariants of an elliptic curve completely define the torus up to modular transformations. Conversely, given two tori with period ratios τ and τ' related by a modular transformation, the Weierstrass equations of the projective elliptic curves obtained by $\xi_{\tau\mathbb{Z}+\mathbb{Z},E}$ and $\xi_{\tau'\mathbb{Z}+\mathbb{Z},E'}$, respectively, are related by a coordinate transformation of the form

$$x \mapsto a^2 x + b, \quad y \mapsto a^3 y + c a^2 x + d, \quad \text{with } a, b, c, d \in \mathbb{C}, \, a \neq 0. \tag{3.23}$$

Two elliptic curves are called isomorphic if they are related in this way. For example, the transformation $x \mapsto a^2 x$, $y \mapsto a^3 y$ only changes the roots by the constant rescaling $e_i \mapsto a^{-2} e_i$, which is an isomorphism on the complex plane (with respect to addition). Thus, the period ratio τ modulo modular transformations uniquely defines the isomorphism class of elliptic curves defined by $\xi_{\tau \mathbb{Z} + \mathbb{Z}, E}$ and vice versa.

Note that the computer algebra system `Mathematica`[11] offers built-in functions for translating from the projective formulation of elliptic curves to the torus. In those functions, however, the ambiguities described above are chosen implicitly. For example, the determination of the half periods $\omega_1/2$, $\omega_2/2$ with the `Mathematica` function `WeierstrassHalfPeriods[{g2,g3}]` relies on a choice of the labellings of the roots which is selected depending on the signs of the Weierstrass invariants and the modular discriminant $\Delta = g_2^3 - 27 g_3^2$.

In order to define a third formulation of the elliptic curve, let us consider a torus defined by the modular parameter τ and define $q = e^{2\pi i \tau}$. The exponential map induces another isomorphism

$$\xi_{\tau, q} : \mathbb{C} / (\tau \mathbb{Z} + \mathbb{Z}) \to \mathbb{C}^* / q^{\mathbb{Z}}, \qquad z \mapsto e^{2\pi i z}. \tag{3.24}$$

where the codomain $\mathbb{C}^* / q^{\mathbb{Z}}$ is called Tate curve[12] and is endowed with the multiplicative group structure inherited by the exponential map from addition on the torus. For example, the representatives $z_1 + n_1 + m_1 \tau$ and $z_2 + n_2 + m_2 \tau$ of z_1 and z_2 modulo lattice displacements in $\mathbb{C}/\Lambda$, where $n_i, m_i \in \mathbb{Z}$, are mapped to the elements

$$\xi_{\tau, q}(z_i + n_i + m_i \tau) = e^{2\pi i (z_i + n_i + m_i \tau)} = e^{2\pi i z_i} q^{m_i}, \tag{3.25}$$

which are representatives of $t_1 = e^{2\pi i z_1}$ and $t_2 = e^{2\pi i z_2}$, respectively, modulo integer powers of q. Similarly, the sum $z_1 + z_2$ modulo lattice displacements is mapped to the product $t_1 t_2$ modulo integer powers of q on the Tate curve.

The description of elliptic functions on the Tate curve offers a connection to rational functions on the Riemann sphere $\mathbb{C}P^1$ and, in particular, admits a convenient tool to take the classical limit $q \to 0$. In order to reveal this connection to functions on the Riemann sphere, let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a non-trivial rational function on the Riemann sphere satisfying the condition

$$f(0) = f(\infty) = 1, \tag{3.26}$$

which will be justified in a moment. Note that this class of functions was already discussed in the context of the classical Bloch relation in subsection 2.2; the two approaches will be related below. For now, recall from the discussion of the classical Bloch relation that this ensures that f is of the form (2.15), i.e.

$$f(t) = \prod_i (t - a_i)^{d_i}, \tag{3.27}$$

with

$$\sum_i d_i = 0, \qquad \prod_i a_i^{d_i} = 1. \tag{3.28}$$

---

[11] See e.g. reference [47] for a guideline of the use and the choices of `Mathematica`'s built-in conversions from the projective elliptic curve to the torus, which is based on the conventions of [46].

[12] See reference [48], appendix A.1.2, or reference [49], section 4.3, for a more recent introduction to the Tate curve.

Averaging f multiplicatively as follows over the Tate curve yields a function

$$F(t) = \prod_{l \in \mathbb{Z}} f(tq^l), \tag{3.29}$$

which obeys the transformed Λ-periodicity condition $F(tq) = F(t)$, cf equation (3.25) for the transformation behaviour of lattice displacements under the isomorphism $\xi_{\tau,q}$, and, can therefore be called elliptic on the Tate curve. A discussion of the properties of such elliptic functions on the Tate curve can be found in reference [33].

The so far unexplained condition (3.26) can be justified as follows: on the one hand it ensures that in the limit $q \to 0$ we recover $f(t)$, on the other hand it implies the condition (3.12) on the zeros and poles a_i of the elliptic generalisation F of f from equation (3.29) after the application of the isomorphism $\xi_{\tau,q}^{-1}$ and the identification $a_i = e^{2\pi i A_i}$.[13] As we will see in subsection 3.3, these two conditions (modulo lattice displacements) are not only necessary, but also sufficient to be the zeros and poles of some elliptic function. Therefore, we can summarize that the function F is the elliptic generalisation of f on the Tate curve and all elliptic functions on the Tate curve can be obtained by this method up to scaling.

3.2. Elliptic multiple polylogarithms

There are several descriptions of elliptic generalisations of multiple polylogarithms, so-called elliptic multiple polylogarithms. Based on the fact that there is no elliptic function on the torus with just one simple pole, such generalisations are either not meromorphic or not Λ-periodic.

However, in Feynman integral calculations one usually chooses to work with meromorphic rather than single-valued functions. Motivated by this physical reason, we focus on the holomorphic iterated integrals $\tilde{\Gamma}$ on the torus described in reference [44] and relate some other notions of (single-valued but non-holomorphic) elliptic multiple polylogarithms to these iterated integrals in subsection 4.1. In analogy with the multi-valuedness of the logarithm function, we still refer to these holomorphic iterated integrals as elliptic multiple polylogarithms defined on the torus.

Consider a torus with periods 1 and τ, where $\text{Im}(\tau) > 0$ as described in subsection 3.1 above, and denote

$$t = e^{2\pi i z}, \qquad q = e^{2\pi i \tau} \quad \text{and} \quad w = e^{2\pi i \alpha}. \tag{3.30}$$

The holomorphic functions $g^{(n)}(z, \tau)$, which satisfy $g^{(n)}(z, \tau) = g^{(n)}(z + 1, \tau)$, constitute the integration kernels of the holomorphic iterated integrals $\tilde{\Gamma}$ described in reference [44]. They are generated by the Eisenstein–Kronecker series [50, 51]

$$F(z, \alpha, \tau) = \frac{\theta_1'(0, \tau)\theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\theta_1(\alpha, \tau)} = \frac{1}{\alpha}\sum_{n \geqslant 0} g^{(n)}(z, \tau)\alpha^n, \tag{3.31}$$

---

[13] We generally denote an elliptic function by a capital Latin letter F (while functions on the Riemann sphere are denoted by small Latin letters) and its zeros and poles on the Tate curve by small letters a_i, while their images on the torus are denoted by the corresponding capital letters A_i. However, for a point on the torus, which corresponds to a point t on the Tate curve, we usually write z_t. The same applies for a given point P on the elliptic curve and its image z_P on the torus.

where $\theta_1(z, \tau)$ is the odd Jacobi θ function. The corresponding iterated integrals are defined via

$$\tilde{\Gamma}\left(\begin{matrix} n_1 \ldots n_k \\ z_1 \ldots z_k \end{matrix}; z, \tau\right) = \int_0^z dz'\, g^{(n_1)}(z' - z_1, \tau)\tilde{\Gamma}\left(\begin{matrix} n_2 \ldots n_k \\ z_2 \ldots z_k \end{matrix}; z', \tau\right), \qquad \tilde{\Gamma}(; z, \tau) = 1. \quad (3.32)$$

The integration kernels $g^{(n)}(z, \tau)$ are not as abstract as they might seem at first glance: they are closely related to the doubly-periodic kernels introduced and used in reference [6]. Furthermore, the kernel $g^{(1)}(z, \tau)$ can be expressed as

$$g^{(1)}(z, \tau) = \zeta(z) - 2\eta_1 z, \quad (3.33)$$

where $\zeta(z)$ is the Weierstrass ζ function introduced in equation (3.9) and $\eta_1 = \zeta(1/2)$ is a quasi-period of the elliptic curve (cf equation (3.10)). For $n > 1$, the integration kernels can be expressed as polynomials of degree n in $g^{(1)}(z, \tau)$ and the coefficients depend polynomially on $\wp(z)$ and $\wp'(z)$, where the first two examples are

$$g^{(2)}(z) = \frac{1}{2}\left(g^{(1)}(z)\right)^2 - \frac{1}{2}\wp(z), \qquad g^{(3)}(z) = \frac{1}{6}\left(g^{(1)}(z)\right)^3 - \frac{1}{2}\wp(z)g^{(1)}(z) - \frac{1}{6}\wp'(z). \quad (3.34)$$

More suitable for numerical evaluation is the description of the integration kernels $g^n(z, \tau)$ by their q-expansions, which are stated in the appendix B. Furthermore, since the Eisenstein–Kronecker series satisfies the mixed heat equation $2\pi i \frac{\partial}{\partial \tau}F(z, \alpha, \tau) = \frac{\partial^2}{\partial z \partial \alpha}F(z, \alpha, \tau)$ [6], the integration kernels solve the partial differential equation

$$2\pi i \frac{\partial}{\partial \tau}g^{(n)}(z, \tau) = n\frac{\partial}{\partial z}g^{(n+1)}(z, \tau). \quad (3.35)$$

At this point, some facts about the regularisation of those iterated integrals need to be mentioned. Considering the q-expansions (B.1)–(B.4) it is obvious that only the kernel $g^{(1)}$ has a singularity at $z = 0$. This singularity is a simple pole, which renders the iterated integrals $\tilde{\Gamma}\left(\begin{matrix} n_1 \ldots n_k \\ z_1 \ldots z_k \end{matrix}; z, \tau\right)$ with $z_k = 0, n_k = 1$ singular.

Employing the shuffle product of iterated integrals, any singular integral can be rewritten such that the only singular terms are of the form $\tilde{\Gamma}\left(\begin{matrix} 1 \ldots 1 \\ \underbrace{0 \ldots 0}_{n} \end{matrix}; z, \tau\right)$. Those singular terms can then be regularised in a way that preserves the shuffle algebra. Following the prescription described in reference [31] the logarithmic singularity at the lower integration boundary of the integral $\tilde{\Gamma}\left(\begin{matrix} 1 \\ 0 \end{matrix}; z, \tau\right)$ for $z \neq 0$ can be subtracted by defining its regularised value as follows:

$$\begin{aligned} \tilde{\Gamma}_{\text{reg}}\left(\begin{matrix} 1 \\ 0 \end{matrix}; z, \tau\right) &= \lim_{\epsilon \to 0}\int_\epsilon^z dz'\, g^{(1)}(z', \tau) + \log(1 - e^{2\pi i \epsilon}) \\ &= \log(1 - e^{2\pi i z}) - \pi i z + 4\pi \sum_{k, l > 0} \frac{1}{2\pi k}\left(1 - \cos(2\pi k z)\right)q^{kl}. \end{aligned}$$

$$(3.36)$$

Note that while the original integral $\tilde{\Gamma}\left(\begin{matrix} 1 \\ 0 \end{matrix}; z, \tau\right)$ vanishes at $z = 0$ and is divergent at any other value of z, the regularised version $\tilde{\Gamma}_{\text{reg}}\left(\begin{matrix} 1 \\ 0 \end{matrix}; z, \tau\right)$ is finite at any $z \neq 0$, but has a logarith-

mic divergence at $z = 0$. The prescription can be easily generalized to iterated integrals with multiple successive divergent entries: with the generalisation

$$\tilde{\Gamma}_{\text{reg}} \left(\underbrace{\begin{matrix} 1 \cdots 1 \\ 0 \cdots 0 \end{matrix}}_{n} ; z, \tau \right) = \frac{1}{n!} \left(\tilde{\Gamma}_{\text{reg}} (\begin{matrix} 1 \\ 0 \end{matrix} ; z, \tau) \right)^{n}. \tag{3.37}$$

From here on—unless stated otherwise—we denote by $\tilde{\Gamma} \left(\underbrace{\begin{matrix} 1 \cdots 1 \\ 0 \cdots 0 \end{matrix}}_{n} ; z, \tau \right)$ its regularised value

and refer to the unregularised version as follows

$$\tilde{\Gamma}_{\text{unreg}} \left(\underbrace{\begin{matrix} 1 \cdots 1 \\ 0 \cdots 0 \end{matrix}}_{n} ; z, \tau \right) = \int_{0}^{z} dz' g^{(1)}(z', \tau) \tilde{\Gamma}_{\text{unreg}} \left(\underbrace{\begin{matrix} 1 \cdots 1 \\ 0 \cdots 0 \end{matrix}}_{n-1} ; z', \tau \right). \tag{3.38}$$

In subsection 4.1 below, a particular class of the iterated integrals $\tilde{\Gamma}$ is discussed in detail, which is the one given by the regularised elliptic polylogarithms of the form

$$\tilde{\Gamma} \left(\underbrace{\begin{matrix} 0 \cdots 0\, m \\ 0 \cdots 0\, 0 \end{matrix}}_{n} ; z, \tau \right), \tag{3.39}$$

where $n, m \geqslant 1$. The numerical evaluation of this class of functions is particularly simple, since their q-expansions can directly by given by n-fold integration of the q-expansions (B.3) and (B.4) of the integration kernels $g^{(m)}(z, \tau)$ for $m > 1$ and the q-expansion (3.36) of $\tilde{\Gamma}_{\text{reg}}(\begin{matrix} 1 \\ 0 \end{matrix} ; z, \tau)$. The results are given in equations (B.10)–(B.15).

The values of the (regularised) iterated integrals at $z = 1$ are particularly interesting since they can be used to define a class of elliptic multiple zeta values [52, 53]. Ordinary zeta values ζ_m, for $m > 1$, are defined as the values of the corresponding polylogarithms evaluated at one

$$\zeta_m = \text{Li}_m(1). \tag{3.40}$$

Analogously, we consider the elliptic zeta values defined by evaluation at one of the above class of elliptic polylogarithms

$$\omega_n(m; \tau) = \tilde{\Gamma} \left(\underbrace{\begin{matrix} 0 \cdots 0\, m \\ 0 \cdots 0\, 0 \end{matrix}}_{n} ; 1, \tau \right). \tag{3.41}$$

Note that this class of elliptic zeta values agrees with the definition of elliptic multiple zeta values in reference [52]. Furthermore, the even zeta values are related to the elliptic zeta values according to

$$\omega_1(2m; \tau) = -2\zeta_{2m}, \tag{3.42}$$

which can be seen from the q-expansion (B.13).

3.3. The divisor function

The last paragraph in subsection 3.1 was devoted to illuminating the relation between an elliptic function F on the Tate curve and the corresponding rational function f on the Riemann sphere. As mentioned at that point, this is closely connected to the formulation of the classical Bloch relation in terms of divisors of such rational functions f discussed in subsection 2.2. The combination of these two considerations leads to the formulation of the elliptic Bloch relation using the concept of divisors of elliptic functions.

The group of divisors $\mathrm{Div}(\mathbb{C}/\Lambda)$ of the torus $\mathbb{C}/\Lambda$ is the free abelian group $\mathcal{F}_{\mathbb{C}/\Lambda}$ generated by the points on the torus $\mathbb{C}/\Lambda$ and similarly for the projective elliptic curve as well as the Tate curve, which are related via the isomorphisms introduced in equations (3.16) and (3.24) above. Hence, a generic divisor is a finite sum of the form

$$\sum_i n_i(z_i) \in \mathrm{Div}(\mathbb{C}/\Lambda), \quad \sum_i n_i(P_i) \in \mathrm{Div}(E(\mathbb{C})) \quad \text{or} \quad \sum_i n_i(t_i) \in \mathrm{Div}(\mathbb{C}^*/q^{\mathbb{Z}}), \quad (3.43)$$

respectively, with $n_i \in \mathbb{Z}$, $z_i \in \mathbb{C}/\Lambda$, $\xi_{\Lambda,E}(z_i) = P_i \in E(\mathbb{C})$ and $\xi_{q,E}(z_i) = t_i \in \mathbb{C}^*/q^{\mathbb{Z}}$. Analogously to the case of rational functions on the Riemann sphere, cf equation (2.17), and according to the general definition (2.13), the divisor of an elliptic function F captures the structure of the zeros and poles of F and is defined by

$$\mathrm{Div}(F) = \sum_{z \in P_\Lambda} \mathrm{ord}_z(F)(z) \in \mathrm{Div}(\mathbb{C}/\Lambda) \tag{3.44}$$

where the sum runs over all points in the fundamental domain P_Λ of $\mathbb{C}/\Lambda$.

According to the identification of elliptic functions on the torus with rational functions on the projective elliptic curve and elliptic functions on the Tate curve alluded to above, the divisor (3.44) of an elliptic function F can be translated by the usual isomorphisms to the projective formulation and the Tate curve via

$$\mathrm{Div}(F) = \sum_{P \in E(\mathbb{C})} \mathrm{ord}_P(F)(P) \in \mathrm{Div}(E(\mathbb{C})) \tag{3.45}$$

and

$$\mathrm{Div}(F) = \sum_{t \in \mathbb{C}^*/q^{\mathbb{Z}}:|q|<|t|\leqslant 1} \mathrm{ord}_t(F)(t) \in \mathrm{Div}(\mathbb{C}^*/q^{\mathbb{Z}}), \tag{3.46}$$

where the orders of the rational function $F(x,y)$ and the elliptic function on the Tate curve $F(t)$ are defined by the order of the elliptic function $F(z)$ on the torus at the corresponding points. For two divisors $D = \sum_i d_i(P_i)$ and $E = \sum_j e_j(Q_j)$, a new divisor

$$D^- = \sum_i d_i(-P_i) \tag{3.47}$$

and the binary product

$$D * E = \sum_{i,j} d_i e_j(P_i + Q_j) \tag{3.48}$$

can be defined, such that a divisor of the form

$$\eta_F^\kappa = \mathrm{Div}(F) * \mathrm{Div}(\kappa - F)^-, \tag{3.49}$$

can be associated to any rational function F, and similarly on the torus and the Tate curve, respectively. In the above definition, $\kappa \in \mathbb{C}$ is a scaling parameter, which needs to equal one in the classical situation described in subsection 2.2. The divisor η_F^κ associated to an elliptic function F plays an important role in later sections of this article and in particular in the formulation of the elliptic Bloch relation in subsection 3.4.

The fact that two elliptic functions are equal up to scaling if they have the same zeros and poles (counted with multiplicities) translates to the condition of having the same divisors. On the other hand, a divisor D is said to be *principal*, if there exists an elliptic function F such that $D = \text{Div}(F)$. Now, we can properly rephrase the last two sentences in subsection 3.1: it turns out that a divisor D is principal if and only if it is of the form $D = \sum_i d_i(A_i)$ where

$$\sum_i d_i = 0, \qquad \sum_i d_i A_i \in \Lambda. \tag{3.50}$$

A proof of this equivalence can be outlined as follows (cf reference [45]): the necessary implication follows from the conditions (3.7) on the zeros and poles of an elliptic function. In order to prove sufficiency, first note that any divisor $D = \sum_i d_i(A_i)$ satisfying equation (3.50) can be written as a linear combination of divisors of the form $(A_1) + (A_2) - (0) - (A_1 + A_2)$. Now, consider elliptic functions of the form

$$F_\lambda(z) = (1 - \lambda)\frac{\wp'(z) - \wp'(-A_1 - A_2)}{\wp(z) - \wp(-A_1 - A_2)} + \lambda. \tag{3.51}$$

In reference [45] it is shown that one can always find a complex parameter λ such that the divisor associated to the above function reads:

$$\text{Div}(F_\lambda) = (A_1) + (A_2) - (0) - (A_1 + A_2) \tag{3.52}$$

and D can indeed be written as a divisor of an elliptic function $D = \sum_j e_j \text{Div}(F_{\lambda_j}) = \text{Div}\left(\prod_j F_{\lambda_j}^{e_j}\right)$, since the divisor function satisfies $\text{Div}(F_1 F_2) = \text{Div}(F_1) + \text{Div}(F_2)$ for two elliptic functions F_1 and F_2. Alternatively, the elliptic function F such that $\text{Div}(F) = D$ can be constructed by means of the Weierstrass σ function as in equation (3.14).

3.4. The elliptic Bloch relation

After having introduced the mathematical background for elliptic curves and elliptic functions in the previous subsections, the elliptic version of Bloch's dilogarithm identity (2.18) can be discussed: in order to do so, an elliptic generalisation of the Bloch–Wigner function D defined in equation (2.1) is required. Since the Bloch–Wigner function satisfies $D(0) = D(\infty) = 0$, the elliptic generalisation on the Tate curve in terms of an infinite product as in equation (3.29) is not applicable. However, an additive average over the Tate curve yields

$$D^E(t, q) = \sum_{l \in \mathbb{Z}} D(tq^l). \tag{3.53}$$

This function D^E is referred to as the *elliptic Bloch–Wigner function* [33]. It inherits some symmetry properties from the classical Bloch–Wigner function D, in particular the inversion relation

$$D^E(t^{-1}, q) = -D^E(t, q) \tag{3.54}$$

and the duplication relation

$$D^E(t^2, q) = 2\left(D^E(t, q) + D^E(t\sqrt{q}, q) + D^E(-t, q) + D^E(-t\sqrt{q}, q)\right) \qquad (3.55)$$

from equations (2.2) and (2.3), respectively. The elliptic version of Bloch's dilogarithm identity (2.18) is that for any elliptic function F, any $\kappa \in \mathbb{C}$ and the divisors $\mathrm{Div}\,(F) = \sum_i d_i(a_i)$ and $\mathrm{Div}\,(\kappa - F) = \sum_j e_j(b_j)$ expressed on the Tate curve, the following identity holds

$$\sum_{i,j} d_i e_j D^E\left(\frac{a_i}{b_j}, q\right) = 0, \qquad (3.56)$$

which takes the form

$$D^E\left(\eta_F^\kappa, q\right) = 0, \qquad (3.57)$$

when expressed in terms of η_F^κ, the divisor defined in equation (3.49). Here again D^E is extended by linearity to the group of divisors. The above identity is referred to as the *elliptic Bloch relation*.

Bloch proves this statement starting from the classical case with a rational function f satisfying $f(0) = f(\infty) = 1$, approximating its elliptic generalisation F on the Tate curve, constructed according to equation (3.29), by $F_N = \prod_{|l| \leqslant N} f(tq^l)$ and an error estimation as $N \to \infty$ [33]. Note that in contrast to the classical case, cf the second equation in (2.17), the constant κ defined in equation (3.49) does not need to equal one. But since any scaling of the elliptic function F is allowed, this condition is redundant and the elliptic Bloch relation can be stated without loss of generality with $\kappa = 1$. In the classical limit $q \to 0$ the Tate curve $\mathbb{C}^*/q^{\mathbb{Z}}$ degenerates to $\mathbb{C}^*$, and, simultaneously, the elliptic Bloch–Wigner function degenerates to its classical version, the Bloch–Wigner function. Finally, for an elliptic function on the Tate curve of the form

$$F = \lim_{N \to \infty} F_N \qquad (3.58)$$

with F_N and f (scaled) as before, the elliptic Bloch relation degenerates to the classical Bloch relation, cf (2.18),

$$D^E\left(\mathrm{Div}(F) * \mathrm{Div}(1 - F)^-, q\right) \to \sum_{i,j} \mathrm{ord}_{a_i}(f)\mathrm{ord}_{b_j}(1 - f)D\left(\frac{a_i}{b_j}\right). \qquad (3.59)$$

However, the above limit, i.e. the transition from elliptic to classical and vice-versa, is subtle, as it can be seen by Bloch's careful proof of the elliptic Bloch relation in terms of the classical Bloch relation. Another hint for this subtlety is the following: if on the left-hand side in the limit (3.59) instead of $1 - F$, the difference $\kappa - F$ for $\kappa \neq 1$ is chosen, the left-hand side still vanishes identically according to equation (3.57). But the right-hand side of (3.59) in general only vanishes for $1 - f$, but not for $\kappa - f$. Therefore, in such a case the elliptic Bloch relation does not degenerate to its classical analogue.

Analogously to the classical Bloch group $\mathcal{B}_2(\mathbb{C})$, Zagier and Gangl define the group of functional relations $\mathcal{C}_2(E)$ in the construction of the elliptic Bloch group $\mathcal{B}_2(E) = \mathcal{A}_2(E)/\mathcal{C}_2(E)$ as a subgroup of the group $\mathcal{A}_2(E)$ of 'allowable' elements in the free abelian group generated by points on the elliptic curve E, the precise meaning of allowable is reviewed in [32]. It is generated by the elliptic Bloch relation (3.57), the inversion relation (3.54) and the duplication relation (3.55), which are expected to form a full set of relations for the elliptic Bloch–Wigner function on the points in $\mathcal{A}_2(E)$ [32]. Thus, in contrast to the classical case

discussed in subsection 2.1, where the five-term identity is sufficient to generate the subgroup of functional relations for the dilogarithm on the points in $\mathcal{A}_2(\mathbb{C})$, the elliptic analogue may require a larger class of functional relations generated by the elliptic Bloch relation. Using the construction of elliptic functions on the Tate curve described in subsection 3.1, the class of functional relations of the elliptic Bloch–Wigner function is parametrised by rational functions on the Riemann sphere and the complex number κ. We refer to this procedure to generate functional identities for the elliptic Bloch–Wigner function as Zagier and Gangl's method. In the following three subsections we discuss some examples and show explicit calculations which use the above concepts and in particular the elliptic Bloch relation.

3.4.1. First example: a divisor on $y^2 = 4x^3 - 4x + 1$. Let us consider the following example[14] of reference [32] to approve the elliptic Bloch relation. Take the elliptic curve with Weierstrass equation $y^2 = 4x^3 - 4x + 1$, i.e. $g_2 = 4$ and $g_3 = -1$, and the rational function $F(x, y) = \frac{y+1}{2}$ on $E(\mathbb{C})$. The three zeros of F are $P = [0 : -1 : 1]$, $P_1 = [1 : -1 : 1]$ and $P_2 = [-1 : -1 : 1]$ and since $F(\wp(z), \wp'(z)) = \frac{\wp'(z)+1}{2}$, the (pull-back of the) rational function F as an elliptic function on the torus has a triple pole at the lattice points, such that on the elliptic curve $\mathrm{ord}_\infty(F) = -3$. Using the group addition on the elliptic curve described in appendix A, one obtains $P_1 = 2P$ and $P_2 = -3P$ and more generally

$$-3P = [-1 : -1 : 1], \quad -2P = [1 : 1 : 1], \quad -P = [0 : 1 : 1],$$
$$P = [0 : -1 : 1], \quad 2P = [1 : -1 : 1], \quad 3P = [-1 : 1 : 1], \tag{3.60}$$
$$4P = [2 : 5 : 1], \quad 5P = \left[\frac{1}{4} : \frac{1}{4} : 1\right], \quad 6P = [6 : -29 : 1].$$

Therefore, the divisor of F on the projective elliptic curve is

$$\mathrm{Div}(F) = (P) + (2P) + (-3P) - 3(\infty). \tag{3.61}$$

Similarly, the divisor of $1 - F$ is given by

$$\mathrm{Div}(1 - F) = (-P) + (-2P) + (3P) - 3(\infty), \tag{3.62}$$

such that the associated divisor η_F^1 of $F(x, y) = \frac{y+1}{2}$ defined in equation (3.49) is

$$\eta_F^1 = (-6P) - 6(-3P) + 2(-2P) + 2(-P) + 9(\infty) - 6(P) - 5(2P) + 2(3P) + (4P). \tag{3.63}$$

The roots of the elliptic curve $y^2 = 4x^3 - 4x + 1 = 4(x - e_1)(x - e_2)(x - e_3)$ are[15]

$$e_1 = 0.837\,565\,4352, \quad e_2 = 0.269\,594\,4364, \quad e_3 = -1.107\,159\,8716, \tag{3.64}$$

such that according to equation (3.20) the periods of the corresponding tori are given by

$$\omega_1 = 2.993\,458\,6462, \quad \omega_3 = 2.451\,389\,3819i, \quad \omega_2 = \omega_1 + \omega_3 \tag{3.65}$$

---

[14] This is the example $E_{37} : y^2 - y = x^3 - x$ in reference [32]. However, we directly work in the Weierstrass form, which can be obtained from the original example by the coordinate transformation $y \mapsto \frac{y+1}{2}$.

[15] We have chosen to display the first ten digits of numbers only. Of course, all calculations have been performed with much higher precision.

with the period ratio

$$\tau = \frac{\omega_2}{\omega_1} = 1 + 0.818\,915\,3991i. \tag{3.66}$$

The point z on the torus with lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ which corresponds to $P = [0 : -1 : 1]$ is determined by Abel's map (3.17)

$$\tilde{z}_P = \int_0^\infty \frac{\mathrm{d}x}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}$$
$$= 2.0638\,659\,408 + 1.225\,694\,7056i. \tag{3.67}$$

The rescaled point corresponding to P in the fundamental parallelogram of the torus defined by τ is

$$z_P = \frac{\tilde{z}_P}{\omega_1} \mod(\mathbb{Z} + \tau\mathbb{Z}) = 0.689\,458\,6481 + 0.409\,457\,7022i \tag{3.68}$$

which maps to

$$t_P = \mathrm{e}^{2\pi i z_P} = -0.0283\,399\,159 - 0.070\,873\,1874i \tag{3.69}$$

on the Tate curve, while the parameter q takes the value

$$q = \mathrm{e}^{2\pi i \tau} = 0.005\,826\,1597. \tag{3.70}$$

For practical purposes, let us define the following approximation of $\mathrm{D}^{\mathrm{E}}(t, q)$:

$$\mathrm{D}_k^{\mathrm{E}}(t, q) = \sum_{l=-k}^{k} \mathrm{D}(tq^l), \tag{3.71}$$

which allows to control the accuracy of convergence depending on the number of terms $2k + 1$. According to the elliptic Bloch relation, our example η_F^1 from equation (3.63) is expected to satisfy

$$-8\mathrm{D}^{\mathrm{E}}(t_P, q) - 7\mathrm{D}^{\mathrm{E}}\left(t_P^2, q\right) + 8\mathrm{D}^{\mathrm{E}}\left(t_P^3, q\right) + \mathrm{D}^{\mathrm{E}}\left(t_P^4, q\right) - \mathrm{D}^{\mathrm{E}}\left(t_P^6, q\right) = 0, \tag{3.72}$$

where we already used the inversion relation (3.54) to simplify the evaluation of the divisor η_F^1.

Using the approximation (3.71) for numerical evaluation of the above equation, we find agreement up to 10^{-7} already for $k = 10$. For other permutations of labelling the roots e_i the elliptic Bloch relation holds as well, as can be tested numerically.

3.4.2. Second example: lines on the projective elliptic curve. As a second example, consider a line on the projective elliptic curve, i.e. a rational function of the form

$$L_{a,b,c}(x, y) = ax + by + c \tag{3.73}$$

with a or b not equal to zero, x and y satisfying $y^2 = 4x^3 - g_2 x - g_3$. The poles of $L_{a,b,c}$ are located at ∞ with multiplicities 2 if $b = 0$ and 3 otherwise (they correspond to the double and triple pole of $x = \wp(z)$ and $y = \wp'(z)$, respectively. See the discussion around equation (3.46)).

The zeros of $L_{a,b,c}$ can be determined explicitly as algebraic functions of the coefficients a, b and c, they satisfy the cubic equation

$$\left(\frac{a}{b}x + \frac{c}{b}\right)^2 = 4x^3 - g_2 x - g_3, \qquad y = -\frac{a}{b}x - \frac{c}{b}. \tag{3.74}$$

Similarly, the zeros of $1 - L_{a,b,c}$ satisfy

$$\left(\frac{a}{b}x + \frac{c-1}{b}\right)^2 = 4x^3 - g_2 x - g_3, \qquad y = -\frac{a}{b}x - \frac{c-1}{b}. \tag{3.75}$$

These cubic equations can be solved by radicals, such that $\mathrm{Div}\left(L_{a,b,c}\right)$ and $\mathrm{Div}\left(1 - L_{a,b,c}\right)$ depend algebraically on the coefficients a, b and c. Furthermore, since the group addition on the projective formulation $\mathrm{E}(\mathbb{C})$ of the elliptic curve also only involves algebraic operations, as can be seen from the explicit equations in the appendix A, the resulting divisor $\eta^1_{L_{a,b,c}}$ expressed on the projective elliptic curve is algebraic in a, b and c. However, applying Abel's map and translating the solutions to the Tate curve, where the elliptic Bloch–Wigner relation (3.57) is defined (so far), generally turns the zeros into integral expressions of the variables a, b and c. Thus, in order to obtain a functional relation with algebraic arguments, the elliptic Bloch–Wigner relation has to be expressed on the torus and ultimately on the projective elliptic curve, which is done in section 4.

Alternatively, instead of solving for the zeros starting from a particular choice of parameters a, b and c, one could as well choose three zeros directly and obtain another three from equations (3.74) and (3.75). However, since there are only three free parameters (the lines $L_{a,b,c}$ and $1 - L_{a,b,c}$ have the same slope), i.e. three roots which determine the remaining roots in terms of at least one non-linear equation, this still involves some non-trivial algebraic dependencies of the arguments in the final functional relation induced by the elliptic Bloch–Wigner relation on the Tate curve.

Let us illustrate the above argumentation by an example: if e.g. the zeros P_1, P_2 of $L_{a,b,c}$ and one zero Q_1 of $1 - L_{a,b,c}$ are called $[x_1 : y_1 : 1], [x_2 : y_2 : 1]$ and $[x_3 : y_3 : 1]$, respectively, the third zero of $L_{a,b,c}$ is $P_3 = -P_1 - P_2$ according to the definition of addition in appendix A and thus algebraic in $x_i, y_i, i = 1, 2$. This ensures that the x-coordinates of the divisor of $L_{a,b,c}$ are very simple. One of the two remaining zeros, Q_2 and Q_3, of $1 - L_{a,b,c}$ is defined by $Q_3 = -Q_1 - Q_2$. But the last zero, Q_2, is still determined by a quadratic equation in terms of Q_1, such that mapping the divisor to the Tate curve yields again a non-algebraic functional relation.

3.4.3. Third example: the five-term identity. As a last example, consider f given by equation (2.21) which is the rational function generating the classical five-term identity when inserted in the classical Bloch relation (2.18). It satisfies $f(0) = f(\infty) = 1$, such that its elliptic generalisation on the Tate curve $F(t) = \prod_{l \in \mathbb{Z}} f(tq^l)$ (cf equation (3.29)) with the following associated divisor can be formed

$$\mathrm{Div}(F) = (a) + (a') + (bb') - (b) - (b') - (aa'), \tag{3.76}$$

where all variables have been defined after equation (2.21). Since the elliptic Bloch relation $\mathrm{D}^{\mathrm{E}}(\eta^1_F, q) = 0$ degenerates to the classical one $\mathrm{D}(\eta_F) = 0$ for $q \to 0$, it can be expected that the elliptic Bloch relation evaluated for F generates an elliptic analogue of the five-term identity [32]: it is an elliptic dilogarithm identity generated by the same rational function f on the Riemann sphere, which implies the five-term identity. In order to write it down explicitly, the zeros and poles of $1 - F$ need to be known. While the poles are the same as the ones

of F, finding the zeros of $1 - F$ is a major obstacle, which was already encountered in the example with the lines in the previous paragraph. While the cubic equation for the zeros of $1 - F$ in the line example could be solved by radicals, the current situation involves a quintic equation in the x-coordinate of the projective elliptic curve, which cannot be solved in general. Correspondingly, an elliptic analogue of the five-term identity can in general not be written down explicitly in terms of algebraic arguments.

The quintic equation is obtained as follows: following the argumentation at the end of subsection 3.3, there exist $\lambda_a, \lambda_b \in \mathbb{C}$, such that

$$\mathrm{Div}\left(\frac{F_{\lambda_a}}{F_{\lambda_b}}\right) = \mathrm{Div}(F), \tag{3.77}$$

where F_λ is a rational function on the projective elliptic curve of the form (3.51) and the divisor of F is expressed on the projective elliptic curve via the usual isomorphisms

$$a \in \mathbb{C}^*/q^{\mathbb{Z}} \mapsto \xi_{\tau,q}^{-1}(a) = A \in \mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z}) \mapsto \xi_{\tau\mathbb{Z}+\mathbb{Z},E}(A) = P_a = [x_a : y_a : 1] \in E(\mathbb{C}). \tag{3.78}$$

Performing the translation, F can be expressed on the projective elliptic curve as the rational function

$$F(x,y) = \frac{1}{\kappa}\frac{F_{\lambda_a}(x,y)}{F_{\lambda_b}(x,y)} = \frac{(1-\lambda_a)\left(y - y_{\frac{1}{aa'}}\right)\left(x - x_{\frac{1}{bb'}}\right) + \lambda_a\left(x - x_{\frac{1}{bb'}}\right)\left(x - x_{\frac{1}{aa'}}\right)}{\kappa(1-\lambda_b)\left(y - y_{\frac{1}{bb'}}\right)\left(x - x_{\frac{1}{aa'}}\right) + \kappa\lambda_b\left(x - x_{\frac{1}{bb'}}\right)\left(x - x_{\frac{1}{aa'}}\right)} \tag{3.79}$$

for some scaling factor $\kappa \in \mathbb{C}$, $x_a = \wp(A)$ and $y_a = \wp'(A)$. The poles of $1 - F$ are the same as the ones of F, i.e. P_a, $P_{a'}$ and $P_{bb'}$. The zeros of $1 - F$ are determined by $\kappa F_{\lambda_b} - F_{\lambda_a} = 0$, which translates by the Weierstrass equation to the quintic equation mentioned above. Since a and b are variables as well as based on the fact that λ_a and λ_b depend non-trivially on a and b the resulting quintic equation is not solvable by radicals in general. Even though the elliptic analogue of the five-term identity generated by the elliptic Bloch relation can not be written down explicitly, it may however be described implicitly as above.

In summary, the elliptic Bloch relation (3.57) generates many (conjecturally all) functional relations of the elliptic Bloch–Wigner function. But for most of these relations, the relevant divisor η_F^1 can not be expressed as a linear combination of variables depending algebraically on each other. The most notable exceptions are the divisors $\eta_{L_{a,b,c}}^1$ generated by lines expressed on the projective elliptic curve. It is this situation, it would still be possible to explicitly write down functional relations. However, they are by no means nice and elucidating and we will thus refrain from doing so. Instead, all relations are going to be formulated on the torus in order to be contrasted with relations between elliptic polylogarithms on the torus introduced in subsection 3.2.

4. Elliptic multiple polylogarithms: connecting two languages

The aim of this section is to translate the elliptic Bloch relation (3.57) from the Tate curve to the torus and to the projective elliptic curve, respectively. From the previous section, it is known how elliptic functions and their divisors can be translated between the three descriptions of an

elliptic curve. Hence, we are left with the translation of the elliptic Bloch–Wigner function D^E, defined in equation (3.53), to the iterated integrals $\tilde{\Gamma}$ on the torus, which will be performed in subsection 4.1. Moreover, a further translation will allow to express the Bloch–Wigner function in the projective formulation of the elliptic curve.

In subsection 4.2 we show how these translations can be generalised to two families of elliptic polylogarithms of higher weight, both of which include the elliptic Bloch–Wigner function. Finally, in subsection 4.3 we combine our results and write down the elliptic Bloch relation on the torus and on the projective elliptic curve explicitly. Moreover, we discover some holomorphic functional relations on the torus which imply the elliptic Bloch relation and thereby give an interpretation of the elliptic Bloch relation in terms of the elliptic symbol calculus.

4.1. The elliptic dilogarithm: from the Tate curve to the torus

We begin with establishing a connection between the iterated integrals $\tilde{\Gamma}$ defined in equation (3.32) above and the sum

$$\mathrm{E}_{n,m}(t, s, q) = -\left(\mathrm{ELi}_{n,m}(t, s, q) - (-1)^{n+m}\mathrm{ELi}_{n,m}(t^{-1}, s^{-1}, q)\right), \tag{4.1}$$

where the objects

$$\mathrm{ELi}_{n,m}(t, s, q) = \sum_{k,l>0} \frac{t^k}{k^n} \frac{s^l}{l^m} q^{kl} \tag{4.2}$$

have been introduced and described in reference [54].

In the end, it will turn out that the value $E_{n,-m}(t, 1, q) = E_{n,-m}(e^{2\pi i z}, 1, q)$ of $E_{n,-m}$ defined on the Tate curve is, up to polynomials in z, equal to the n-fold iterated integral of the integration kernel $g^{(m+1)}(z, \tau)$, i.e. $\tilde{\Gamma}\left(\begin{matrix} \underbrace{0 \ldots m+1 \\ 0 \ldots 0}_{n} \end{matrix}; z, \tau\right)$, which is an iterated integral defined on the torus.

In order to show this, the case $m = 0$ is discussed first, for which the definition

$$\mathrm{E}_n(t, s, q) = -\left(\frac{1}{2}\mathrm{Li}_n(t) - (-1)^n \frac{1}{2}\mathrm{Li}_n(t^{-1})\right) + \mathrm{E}_{n,0}(t, s, q) \tag{4.3}$$

turns out to be useful. In terms of the variables t, q and w defined in (3.30), the Eisenstein–Kronecker series (3.31) can be rewritten as [55]

$$F(t, w, q) = -2\pi i \left(\frac{t}{1-t} + \frac{1}{1-w} + \sum_{k,l>0}(t^k w^l - t^{-k} w^{-l})q^{kl}\right), \tag{4.4}$$

such that from the limit $g^{(1)}(z, \tau) = \lim_{\alpha \to 0}\left(F(z, \alpha, \tau) - \frac{g^{(0)}(z,\tau)}{\alpha}\right)$ a straightforward calculation implies[16]

$$\mathrm{E}_0(t, 1, q) = \frac{1}{2\pi i} g^{(1)}(z, \tau). \tag{4.5}$$

---

[16] This calculation has been pointed out in reference [56] and motivated to consider the generalisations for $E_{n,-m}(t, 1, q)$ with $n, m > 0$ described in the following parts of this subsection. Similar considerations can be found in reference [57].

The iterated integrals $\tilde{\Gamma}$ on the torus may be recovered using the partial differential equation

$$\frac{\partial}{\partial z}E_n(z, 1, \tau) = 2\pi iE_{n-1}(z, 1, \tau), \tag{4.6}$$

where the function E_n is pulled-back to the torus by the exponential map. This leads to the following integral representation of $E_1(t, 1, q)$

$$E_1(t, 1, q) = \lim_{\epsilon \to 0} \int_\epsilon^z dz' \frac{\partial}{\partial z'}E_1(z', 1, \tau) + E_1(e^{2\pi i\epsilon}, 1, q)$$
$$= \tilde{\Gamma}\left(\begin{matrix}1\\0\end{matrix}; z, \tau\right) - 2ELi_{1,0}(1, 1, q) + \frac{\pi i}{2}, \tag{4.7}$$

where $\tilde{\Gamma}\left(\begin{matrix}1\\0\end{matrix}; z, \tau\right)$ is the regularised integral (see subsection 3.2). Note that the logarithmic singularity of $\tilde{\Gamma}_{unreg}\left(\begin{matrix}1\\0\end{matrix}; z, \tau\right) = \int_0^z dz' g^{(1)}(z', \tau)$ cancels the singular contribution $Li_1(1)$ of $E_1(1, 1, q)$, leaving only a phase shift $\frac{\pi i}{2}$ caused by the different directions of the paths approaching the singularity of $Li_1(1)$. For $n > 1$, there is no singularity at all if the regularised iterated integrals are used, since for $n > 1$

$$E_n(1, 1, q) = -\left(\frac{1}{2}(1 - (-1)^n)Li_n(1) + (1 - (-1)^n)ELi_{n,0}(1, 1, q)\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ -\zeta_n - 2ELi_{n,0}(1, 1, q) & n \text{ odd} \end{cases} \tag{4.8}$$

is finite as well. This can be seen by considering equation (4.1) for $s, t = 1$:

$$ELi_{n,0}(1, 1, q) = \sum_{k>0}\frac{q^k}{(1 - q^k)k^n} = -2i\sum_{k>0}\frac{e^{k\pi i\tau}}{\sin(k\pi\tau)k^n}. \tag{4.9}$$

Fortunately, the calculation of the above series can be circumvented by considering the integral representation of E_2 on the torus: taking into account that $E_2(1, 1, q) = 0$, a representation of $E_2(t, 1, q)$ can be obtained by the following calculation

$$E_2(t, 1, q) = 2\pi i\int_0^z dz'\, E_1(z', 1, \tau)$$
$$= 2\pi i\tilde{\Gamma}\left(\begin{matrix}0 & 1\\0 & 0\end{matrix}; z, \tau\right) + 2\pi i\left(\frac{\pi i}{2} - 2ELi_{1,0}(1, 1, q)\right)z. \tag{4.10}$$

Evaluation at $z = 1$ of equation (4.12) together with equation (4.8) yields the value of $ELi_{1,0}(1, 1, q)$ in terms of the regularised iterated integrals

$$2ELi_{1,0}(1, 1, q) = \omega_2(1; \tau) + \frac{\pi i}{2}, \tag{4.11}$$

such that

$$E_2(t, 1, q) = 2\pi i\int_0^z dz'\, E_1(z', 1, \tau) = 2\pi i\left(\tilde{\Gamma}\left(\begin{matrix}0 & 1\\0 & 0\end{matrix}; z, \tau\right) - \omega_2(1; \tau)z\right). \tag{4.12}$$

Turning back to the functions $E_n(t, 1, q)$, one finds recursively that for $n \geqslant 1$

$$E_n(t, 1, q) = (2\pi i)^{n-1} \tilde{\Gamma} \left(\underbrace{\begin{matrix} 0 \dots 0\, 1 \\ 0 \dots 0\, 0 \end{matrix}}_{n} ; z, \tau \right) + P_n(z, q), \tag{4.13}$$

where $P_n(z, q)$ is the polynomial of degree $n - 1$ in z

$$P_n(z, q) = -(2\pi i)^{n-1} \omega_2(1; \tau) \frac{z^{n-1}}{(n-1)!} + \sum_{j=2}^{n} (2\pi i)^{n-j} E_j(1, 1, q) \frac{z^{n-j}}{(n-j)!}. \tag{4.14}$$

In equation (4.19), the whole z dependence of $E_n(t, 1, q)$ is expressed solely in terms of (polynomials of) the regularised iterated integrals $\tilde{\Gamma}$ with at most weight one, since $z = \tilde{\Gamma} \left(\begin{matrix} 0 \\ 0 \end{matrix} ; z, \tau \right)$. The integration constants $E_j(1, 1, q)$, given in (4.8) and appearing in the polynomial $P_n(z, q)$, can be expressed as a linear combination of elliptic zeta values. The result can be obtained recursively by evaluation of equation (4.19) at one. The full calculation is shown in the appendix C and results in the explicit expression for $n > 1$

$$E_n(1, 1, q) = - \left(\frac{1}{2}(1 - (-1)^n) \text{Li}_n(1) + (1 - (-1)^n) \text{ELi}_{n,0}(1, 1, q) \right)$$

$$= \begin{cases} (2\pi i)^{n-1} \displaystyle\sum_{k=0}^{\frac{(n-1)}{2}} d_{2k+1} \omega_{n+1-2k}(1; \tau) & n \text{ odd} \\ 0 & n \text{ even,} \end{cases} \tag{4.15}$$

cf equation (C.14), where d_k is the sequence defined by

$$d_k = \begin{cases} -1 & k = 1 \\ 0 & k \text{ even} \\ -\dfrac{d_1}{k!} - \dfrac{d_3}{(k-2)!} - \dots - \dfrac{d_{k-2}}{3!} & k \text{ odd,} \end{cases} \tag{4.16}$$

such that e.g.

$$d_1 = -1, \qquad d_3 = \frac{1}{3!}, \qquad d_5 = \frac{1}{5!} - \frac{1}{3!3!}, \qquad d_7 = \frac{1}{7!} - \frac{1}{5!3!} - \frac{1}{3!5!} + \frac{1}{3!3!3!}. \tag{4.17}$$

Therefore, the polynomial $P_n(z, q)$ can be rewritten in terms of elliptic zeta values as

$$P_n(z, q) = (2\pi i)^{n-1} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{j} d_{2k+1} \omega_{2j+2-2k}(1; \tau) \frac{z^{n-1-2j}}{(n-1-2j)!} \tag{4.18}$$

and the sums $E_n(t, 1, q)$ for $n \geqslant 2$ can entirely be expressed by means of the elliptic polyloga-rithms on the torus

$$
E_n(t, 1, q) = (2\pi i)^{n-1} \left(\tilde{\Gamma} \begin{pmatrix} 0 \dots 0 \, 1 \\ \underbrace{0 \dots 0 \, 0}_{n} \end{pmatrix} ; z, \tau \end{pmatrix} + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{j} d_{2k+1} \omega_{2j+2-2k}(1; \tau) \frac{z^{n-1-2j}}{(n-1-2j)!} \right).
$$

(4.19)

Employing similar calculations, it is possible to relate iterated integrals $\tilde{\Gamma}$ of weight higher than one to the ELi-functions. The q-expansions (B.3) and (B.4) of $g^{(m+1)}$ for $m > 0$ lead to

$$
E_{0,-m}(t, 1, q) = \frac{m!}{(2\pi i)^{m+1}} \left(g^{(m+1)}(z, \tau) + (1 + (-1)^{m+1}) \zeta_{m+1} \right)
$$

(4.20)

and therefore, since $E_{n,m}$ satisfies the same partial differential equation as E_n,

$$
\frac{\partial}{\partial z} E_{n,m}(z, 1, \tau) = 2\pi i E_{n-1,m}(z, 1, \tau),
$$

(4.21)

the following relations can be identified: for $n = 1, m > 0$

$$
\begin{aligned}
E_{1,-m}(t, 1, q) &= \int_0^z dz' \frac{\partial}{\partial z'} E_{1,-m}(z', 1, \tau) + E_{1,-m}(1, 1, q) \\
&= \frac{m!}{(2\pi i)^m} \tilde{\Gamma} \begin{pmatrix} m+1 \\ 0 \end{pmatrix} ; z, \tau \end{pmatrix} + \frac{m!}{(2\pi i)^m} (1 + (-1)^{m+1}) \zeta_{m+1} z + E_{1,-m}(1, 1, q),
\end{aligned}
$$

(4.22)

for $n = 2, m > 0$

$$
\begin{aligned}
E_{2,-m}(t, 1, q) &= \int_0^z dz' \frac{\partial}{\partial z'} E_{2,-m}(z', 1, \tau) + E_{2,-m}(1, 1, q) \\
&= \frac{m!}{(2\pi i)^{m-1}} \tilde{\Gamma} \begin{pmatrix} 0 \, m+1 \\ 0 \quad 0 \end{pmatrix} ; z, \tau \end{pmatrix} + \frac{m!}{(2\pi i)^{m-1}} (1 + (-1)^{m+1}) \zeta_{m+1} \frac{z^2}{2} \\
&\quad + 2\pi i E_{1,-m}(1, 1, q) z + E_{2,-m}(1, 1, q).
\end{aligned}
$$

(4.23)

A recursion leads to the general formula for $n > 0, m > 0$

$$
E_{n,-m}(t, 1, q) = m! (2\pi i)^{n-m-1} \tilde{\Gamma} \begin{pmatrix} 0 \dots 0 \, m+1 \\ \underbrace{0 \dots 0 \quad 0}_{n} \end{pmatrix} ; z, \tau \end{pmatrix} + P_{n,m}(z, q),
$$

(4.24)

where

$$
P_{n,m}(z, q) = m! (2\pi i)^{n-m-1} (1 + (-1)^{m+1}) \zeta_{m+1} \frac{z^n}{n!} + \sum_{j=1}^{n} (2\pi i)^{n-j} E_{j,-m}(1, 1, q) \frac{z^{n-j}}{(n-j)!}.
$$

(4.25)

As in the case $m = 0$, evaluation of $E_{n,-m}(t, 1, q)$ given in equation (4.26) and the fact that $E_{n,-m}(1, 1, q)$ vanishes for $n + m$ even leads to an expression of the integration constants

$E_{n,-m}(1, 1, q)$ in terms of elliptic zeta values. The calculation is shown in appendix C and the result is given in equation (C.35), i.e.

$$
E_{n,-m}(1, 1, q) = \begin{cases} m!(2\pi i)^{n-1-m} \displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} d_{2k+1}\omega_{n+1-2k}(m+1; \tau) & n+m \text{ odd} \\[6mm] 0 & n+m \text{ even,} \end{cases}
\tag{4.26}
$$

such that

$$
E_{n,-m}(t, 1, q) = \begin{cases} m!(2\pi i)^{n-1-m}\left(\tilde{\Gamma}\begin{pmatrix} 0 \ldots 0 \; m+1 \\ \underbrace{0 \ldots 0 \quad 0}_{n} \end{pmatrix}; z, \tau) + \displaystyle\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{k=0}^{j} d_{2k+1}\omega_{2j+1-2k}(m+1; \tau)\frac{z^{n-2j}}{(n-2j)!} \right) & m \text{ odd} \\[10mm] m!(2\pi i)^{n-1-m}\left(\tilde{\Gamma}\begin{pmatrix} 0 \ldots 0 \; m+1 \\ \underbrace{0 \ldots 0 \quad 0}_{n} \end{pmatrix}; z, \tau) + \displaystyle\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor}\sum_{k=0}^{j} d_{2k+1}\omega_{2j+2-2k}(m+1; \tau)\frac{z^{n-1-2j}}{(n-1-2j)!} \right) & m \text{ even.} \end{cases}
\tag{4.27}
$$

For example and latter purposes, we find in particular the relations

$$
E_{1,0}(t, 1, q) = \tilde{\Gamma}\begin{pmatrix} 1 \\ 0 \end{pmatrix}; z, \tau) - \omega_2(1; \tau) + \frac{1}{2}\left(\text{Li}_1(t) + \text{Li}_1(t^{-1}) \right),
\tag{4.28}
$$

$$
E_{2,0}(t, 1, q) = 2\pi i \left(\tilde{\Gamma}\begin{pmatrix} 0 \; 1 \\ 0 \; 0 \end{pmatrix}; z, \tau) - \omega_2(1; \tau)z \right) + \frac{1}{2}\left(\text{Li}_2(t) - \text{Li}_2(t^{-1}) \right)
\tag{4.29}
$$

and

$$
E_{1,-1}(t, 1, q) = \tfrac{1}{2\pi i}\left(\tilde{\Gamma}\begin{pmatrix} 2 \\ 0 \end{pmatrix}; z, \tau) - \omega_1(2; \tau)z \right) = \tfrac{1}{2\pi i}\tilde{\Gamma}\begin{pmatrix} 2 \\ 0 \end{pmatrix}; z, \tau) + \tfrac{1}{\pi i}\zeta_2 z.
\tag{4.30}
$$

Thus, we have established a direct connection between the functions $E_{n,-m}$ on the Tate curve and the iterated integrals of the form $\tilde{\Gamma}\begin{pmatrix} 0 \ldots m \\ \underbrace{0 \ldots 0}_{n} \end{pmatrix}; z, \tau)$ for $n, m > 0$, which are defined on the torus.

On the other hand, the elliptic Bloch–Wigner function D^E can be rewritten in terms of the above examples $E_{1,0}$, $E_{2,0}$ and $E_{1,-1}$. This involves the identities

$$
\sum_{l>0}\left(\text{Li}_2(tq^l) - \text{Li}_2(t^{-1}q^l) \right) = -E_{2,0}(t, 1, q)
\tag{4.31}
$$

and

$$
\sum_{l>0}\log(|tq^l|)\text{Li}_1(tq^l) - \sum_{l>0}\log(|t^{-1}q^l|)\text{Li}_1(t^{-1}q^l)
$$

$$= -\log(|t|)\mathrm{E}_{1,0}(t,1,q) - \log(|q|)\mathrm{E}_{1,1}(t,1,q), \tag{4.32}$$

which follow straightforwardly from the definition (4.1) of $E_{n,m}(t,s,q)$. Therefore, the value of D^{E} can be expressed in terms of the iterated integrals $\tilde{\Gamma}$ on the torus as follows

$$
\begin{aligned}
\mathrm{D}^{\mathrm{E}}(t,q) &= \sum_{l>0} \mathrm{Im}\left(\mathrm{Li}_2(tq^l) - \mathrm{Li}_2(t^{-1}q^l)\right) \\
&\quad - \sum_{l>0} \mathrm{Im}\left(\log(|tq^l|)\mathrm{Li}_1(tq^l) - \log(|t^{-1}q^l|)\mathrm{Li}_1(t^{-1}q^l)\right) + \mathrm{D}(t) \\
&= -\mathrm{Im}\left(\mathrm{E}_2(t,1,q)\right) + \log(|t|)\,\mathrm{Im}\left(\mathrm{E}_1(t,1,q)\right) + \log(|q|)\,\mathrm{Im}\left(\mathrm{E}_{1,-1}(t,1,q)\right) \\
&= \mathrm{Im}(\tau)\,\mathrm{Re}\left(\tilde{\Gamma}\begin{pmatrix}2\\0\end{pmatrix};z,\tau\right)\right) + 2\pi\,\mathrm{Re}\left(\tilde{\Gamma}\begin{pmatrix}1\,0\\0\,0\end{pmatrix};z,\tau\right)\right) - 2\pi\,\mathrm{Re}(z)\,\mathrm{Re}\left(\tilde{\Gamma}\begin{pmatrix}1\\0\end{pmatrix};z,\tau\right)\right) \\
&\quad + 2\,\mathrm{Re}(z)\left(\pi\,\mathrm{Re}\left(\omega_2(1;\tau)\right) + \zeta_2\,\mathrm{Im}(\tau)\right),
\end{aligned}
\tag{4.33}
$$

where the q-independent term $\mathrm{D}(t)$ is absorbed in the second equality by going from $E_{n,m}(t,1,q)$ to $E_n(t,1,q)$ according to equation (4.3). The logarithmic factors with the absolute values of t and q, respectively, yield contributions of the imaginary parts of z and τ, respectively. The final expression explicitly involving the real part of z is obtained by using equations (4.28)–(4.30) and the identity $\mathrm{Re}(z_1 z_2) + \mathrm{Im}(z_1)\mathrm{Im}(z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2)$, where $z_1, z_2 \in \mathbb{C}$, for the last equality above. The translation of the elliptic Bloch–Wigner function D^{E} from the torus, as given by equation (4.33), to the projective elliptic curve is based on the results in reference [44]. The iterated integrals $\tilde{\Gamma}$ on the torus can be expressed via the isomorphism $\xi_{\tau\mathbb{Z}+\mathbb{Z},E}$ in terms of some iterated integrals on the projective elliptic curve, which are defined as follows

$$
\mathrm{E}_3\begin{pmatrix}n_1 \dots n_k\\c_1 \dots c_k\end{pmatrix};x,\vec{e}\right) = \int_0^x \mathrm{d}x'\,\varphi_{n_1}(c_1;x',\vec{a})\mathrm{E}_3\begin{pmatrix}n_2 \dots n_k\\c_2 \dots c_k\end{pmatrix};x',\vec{e}\right), \qquad \mathrm{E}_3(;x,\vec{e}) = 1,
\tag{4.34}
$$

with $c_i \in \mathbb{C} \cup \{\infty\}$, $\vec{e} = (e_1, e_2, e_3)$ is the vector of the roots[17] of the Weierstrass equation and the integration kernels $\varphi_n(c;x,\vec{e})$ are defined according to the construction of reference [44]. For example, the differential $\varphi_0(0,x,\vec{e})\,\mathrm{d}x$ is simply the holomorphic differential $\mathrm{d}x/y$ which itself is the differential $\mathrm{d}z$ on the torus

$$\varphi_0(0,x,\vec{e})\,\mathrm{d}x = \frac{\mathrm{d}x}{y} = \frac{\mathrm{d}\wp(z)}{\wp'(z)} = \mathrm{d}z. \tag{4.35}$$

The integration kernels $\varphi_n(\infty;x,\vec{e})$ for $n \geqslant 1$ are defined as follows: first, define the integral of x/y with an additional term as follows

$$Z_3(x_P,\vec{e}) = -\int_{e_1}^{x_P} \mathrm{d}x\left(\frac{x}{y} + 2\frac{\eta_1}{y}\right). \tag{4.36}$$

This defines the kernel for $n = 1$

$$\varphi_1(\infty;x,\vec{e}) = \frac{1}{y}Z_3(x,\vec{e}). \tag{4.37}$$

---

[17] Note that we use slightly different conventions than in reference [44], where the defining cubic equation of the projective curve is written in standard form $y^2 = (x - a_1)(x - a_2)(x - a_3)$ in contrast to our notation which only involves the Weierstrass form.

The kernels for higher n are defined by some polynomials $Z_3^{(n)}$, which are of degree n in $Z_3(x)$ with the coefficients being polynomials in x and y and that do not have any poles in x. For example in the case of $n = 2$, the integration kernel is defined as

$$\varphi_2(\infty; x, \vec{e}) = \frac{1}{y} Z_3^{(2)}(x, \vec{e}) = \frac{1}{y} \left(\frac{1}{8} Z_3(x, \vec{e})^2 - \frac{x}{2} \right). \tag{4.38}$$

The (explicit) construction of $Z_3^{(n)}$ is exactly the same as the construction of $g^{(n)}(z, \tau)$ as a polynomial in $g^{(1)}(z, \tau)$ with polynomial coefficients in $\wp(z)$ and $\wp'(z)$, see reference [44]. This leads to a very close relation between the kernels $\varphi_n(\infty; x, \vec{e})$ and $g^{(n)}(z, \tau)$. For $n = 0$, we first rewrite[18]

$$Z_3(x, \vec{e}) = \zeta(z) - 2\eta_1 z = g^{(1)}(z, \tau) \tag{4.39}$$

using equation (3.33), such that

$$\varphi_1(\infty; x, \vec{e}) \, \mathrm{d}x = g^{(1)}(z, \tau) \, \mathrm{d}z. \tag{4.40}$$

Thus, the construction of $Z_3^{(n)}$ ensures that the same result holds for $n \geqslant 1$

$$\varphi_n(\infty; x, \vec{e}) \, \mathrm{d}x = g^{(n)}(z, \tau) \, \mathrm{d}z, \tag{4.41}$$

which is all that is needed to rewrite $\mathrm{D^E}$. With z_0 being a zero of $\wp$ such that $\wp'(z_0) > 0$, the identification $x = \wp(z)$ and from the equations (4.35), (4.40) and (4.41) for the differentials, the iterated integrals in equation (4.33) can be expressed as follows on the projective elliptic curve

$$\tilde{\Gamma} \left(\begin{matrix} 1 \, 0 \\ 0 \, 0 \end{matrix} ; z, \tau \right) = \mathrm{E}_3 \left(\begin{matrix} 1 & 0 \\ \infty & 0 \end{matrix} ; x, \vec{e} \right) + \tilde{\Gamma} \left(\begin{matrix} 1 \, 0 \\ 0 \, 0 \end{matrix} ; z_0, \tau \right), \tag{4.42}$$

$$\tilde{\Gamma} \left(\begin{matrix} 2 \\ 0 \end{matrix} ; z, \tau \right) = \mathrm{E}_3 \left(\begin{matrix} 2 \\ \infty \end{matrix} ; x, \vec{e} \right) + \tilde{\Gamma} \left(\begin{matrix} 2 \\ 0 \end{matrix} ; z_0, \tau \right), \tag{4.43}$$

$$\tilde{\Gamma} \left(\begin{matrix} 1 \\ 0 \end{matrix} ; z, \tau \right) = \mathrm{E}_3 \left(\begin{matrix} 1 \\ \infty \end{matrix} ; x, \vec{e} \right) + \tilde{\Gamma} \left(\begin{matrix} 1 \\ 0 \end{matrix} ; z_0, \tau \right), \tag{4.44}$$

as well as

$$z = \mathrm{E}_3 \left(\begin{matrix} 0 \\ 0 \end{matrix} ; x, \vec{e} \right) + z_0. \tag{4.45}$$

---

[18] Note that for this calculation, we choose the sign of $y = \pm\sqrt{4x^3 - g_2 x - g_3}$ in Abel's map (3.17) such that we indeed obtain equation (4.39) and not the negative of the right-hand side, i.e. $Z_3(x, \vec{e}) = -g^{(1)}(z, \tau)$.

Therefore, the elliptic Bloch–Wigner function takes the following form on the projective elliptic curve

$$
\begin{aligned}
D^{E}(t,q) &= \operatorname{Im}(\tau)\operatorname{Re}\left(\tilde{\Gamma}\begin{pmatrix} 2 \\ 0 \end{pmatrix};z,\tau\right) + 2\pi\operatorname{Re}\left(\tilde{\Gamma}\begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix};z,\tau\right) - 2\pi\operatorname{Re}(z)\operatorname{Re}\left(\tilde{\Gamma}\begin{pmatrix} 1 \\ 0 \end{pmatrix};z,\tau\right) \\
&\quad + 2\operatorname{Re}(z)\left(\pi\operatorname{Re}(\omega_2(1;\tau)) + \zeta_2\operatorname{Im}(\tau)\right) \\
&= \operatorname{Im}(\tau)\operatorname{Re}\left(E_3\begin{pmatrix} 2 \\ \infty \end{pmatrix};x,\vec{e}\right) + \tilde{\Gamma}\begin{pmatrix} 2 \\ 0 \end{pmatrix};z_0,\tau\right) + 2\pi\operatorname{Re}\left(E_3\begin{pmatrix} 1\;\,0 \\ \infty\;0 \end{pmatrix};x,\vec{e}\right) + \tilde{\Gamma}\begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix};z_0,\tau\right) \\
&\quad - 2\pi\operatorname{Re}\left(E_3\begin{pmatrix} 0 \\ 0 \end{pmatrix};x,\vec{e}\right) + z_0\right)\operatorname{Re}\left(E_3\begin{pmatrix} 1 \\ \infty \end{pmatrix};x,\vec{e}\right) + \tilde{\Gamma}\begin{pmatrix} 1 \\ 0 \end{pmatrix};z_0,\tau\right) \\
&\quad + 2\operatorname{Re}\left(E_3\begin{pmatrix} 0 \\ 0 \end{pmatrix};x,\vec{e}\right) + z_0\right)\left(\pi\operatorname{Re}(\omega_2(1;\tau)) + \zeta_2\operatorname{Im}(\tau)\right).
\end{aligned}
$$

$$(4.46)$$

The constant terms involving the iterated integrals on the torus evaluated at z_0 and 1, respectively, drop out once the elliptic Bloch relation (3.57) is formed.

4.2. Higher elliptic polylogarithms

The translation procedure from the Tate curve to the torus described in the previous section is applicable to elliptic generalisations of higher polylogarithms. In this subsection we present two such families, both of which include the elliptic Bloch–Wigner function, and show how they can be expressed in terms of the elliptic integrals $\tilde{\Gamma}$ on the torus. These families of functions are not independent, the first one is actually a subclass of the second.

The first construction of higher elliptic polylogarithms is based on the averaging process over the Tate curve which was used to define the elliptic Bloch–Wigner function in equation (3.53). The single-valued polylogarithms that are to be averaged were first described by Ramakrishnan [35] and generalise the Bloch–Wigner function to higher orders. They are defined by

$$
\mathcal{L}_n(t) = \mathcal{R}_n\left(\sum_{k=0}^{n-1}\frac{2^k B_k}{k!}\log^k(|t|)\operatorname{Li}_{n-k}(t)\right),
$$

$$(4.47)$$

where $\mathcal{R}_n$ denotes the imaginary or real part if n is even or odd, respectively, and B_k the kth Bernoulli number. The Bloch–Wigner function D is obtained for $n=2$, and these functions also satisfy a similar inversion relation as D, namely

$$
\mathcal{L}_n(t^{-1}) = (-1)^{n-1}\mathcal{L}_n(t).
$$

$$(4.48)$$

The elliptic generalisation used in reference [32] and proposed in reference [36], as linear combinations of the more general class described below, is

$$
\begin{aligned}
\mathcal{L}_n^E(t,q) = \sum_{l\in\mathbb{Z}}\mathcal{L}_n(tq^l) = \sum_{k=0}^{n-1}\frac{2^k B_k}{k!}\mathcal{R}_n\Bigg(&\sum_{l>0}\log^k(|tq^l|)\operatorname{Li}_{n-k}(tq^l) \\
&+ (-1)^{n-1}\sum_{l>0}\log^k(|t^{-1}q^l|)\operatorname{Li}_{n-k}(t^{-1}q^l)\Bigg) + \mathcal{L}_n(t),
\end{aligned}
$$

$$(4.49)$$

such that in particular $\mathcal{L}_2^{\mathrm{E}} = \mathrm{D}^{\mathrm{E}}$. By a similar calculation as for D^{E}, which e.g. involves the identity

$$\sum_{l>0} \log^k(|tq^l|)\mathrm{Li}_{n-k}(tq^l) + (-1)^{n-1}\sum_{l>0} \log^k(|t^{-1}q^l|)\mathrm{Li}_{n-k}(t^{-1}q^l)$$

$$= -\sum_{m=0}^{k} \binom{k}{m} \log^{k-m}(|t|)\log^m(|q|)\mathrm{E}_{n-k,-m} \tag{4.50}$$

generalising equations (4.31) and (4.32), the elliptic polylogarithms $\mathcal{L}_n^{\mathrm{E}}$ turn out to be related to the functions $E_{n,-m}$ according to

$$\mathcal{L}_n^{\mathrm{E}}(t,q) = -\sum_{m=0}^{k} \binom{k}{m} \frac{2^k B_k}{k!} \log^{k-m}(|t|)\log^m(|q|)\mathcal{R}_n\left(\mathrm{E}_{n-k,-m}\right) + \mathcal{L}_n(t). \tag{4.51}$$

Just like in the dilogarithmic case of $n = 2$, this result can immediately be expressed in terms of the iterated integrals on the torus and the projective curve using the results of the previous section.

The more general class of single-valued elliptic polylogarithms, introduced in reference [36] and used in reference [58] in the context of modular graph functions for one-loop closed string amplitudes, can be constructed from the single-valued sum

$$\mathrm{D}_{a,b}(t) = (-1)^{a-1}\sum_{n=a}^{a+b-1} \binom{n-1}{a-1} \frac{(-2\,\log(|t|))^{a+b-1-n}}{(a+b-1-n)!}\mathrm{Li}_n(t)$$

$$+ (-1)^{b-1}\sum_{n=b}^{a+b-1} \binom{n-1}{b-1} \frac{(-2\,\log(|t|))^{a+b-1-n}}{(a+b-1-n)!}\overline{\mathrm{Li}_n(t)}, \tag{4.52}$$

which satisfies $\overline{\mathrm{D}_{a,b}(t)} = \mathrm{D}_{b,a}(t)$, where the overline denotes complex conjugation. The functions $\mathcal{L}_n$ above are linear combinations of $\mathrm{D}_{a,b}$ and hence, a subclass of the latter [36]. For example,

$$\mathrm{D}_{1,2}(t) = 2i\mathrm{D}(t) + 2\,\log(|t|)\log(|1-t|), \tag{4.53}$$

such that the Bloch–Wigner function can be written as $\mathrm{D}(t) = \frac{1}{4i}\left(\mathrm{D}_{1,2}(t) - \mathrm{D}_{2,1}(t)\right)$. The elliptic generalisation is similar to the previous average over the Tate curve and given by [36]

$$\mathrm{D}_{a,b}^{\mathrm{E}}(t,q) = \sum_{l\geqslant 0}\mathrm{D}_{a,b}(tq^l) + (-1)^{a+b}\sum_{l>0}\mathrm{D}_{a,b}(t^{-1}q^l) + \frac{(4\pi\,\mathrm{Im}(\tau))^{a+b-1}}{(a+b)!}B_{a+b}(u), \tag{4.54}$$

where B_n is the nth Bernoulli polynomial and $z = u\tau + v$ with $u, v \in [0, 1]$. For example, the elliptic Bloch–Wigner function can be expressed as

$$\mathrm{D}^{\mathrm{E}}(t,q) = -\frac{1}{2}\,\mathrm{Im}(\mathrm{D}_{2,1}^{\mathrm{E}}(t,q)). \tag{4.55}$$

In order to express the functions $\mathrm{D}_{a,b}^{\mathrm{E}}$ in terms of $E_{n,m}$, the relevant prefactor in $\mathrm{D}_{a,b}^{\mathrm{E}}(t,q)$ for the translation has to be determined. This is the factor obtained by plugging the right-hand side of

the definition (4.52) of $D_{a,b}$ into equation (4.54) and pushing the sum over l to the logarithmic functions depending on this summation index, i.e.

$$\sum_{l>0} \left(\log(|tq^l|)^{a+b-1-n} \mathrm{Li}_n(tq^l) + (-1)^{a+b} \log(|t^{-1}q^l|)^{a+b-1-n} \mathrm{Li}_n(t^{-1}q^l) \right)$$

$$= -\sum_{m=0}^{a+b-1-n} \binom{a+b-1-n}{m} \log(|t|)^{a+b-1-n-m} \log(|q|)^m \mathrm{E}_{n,-m}(t,1,q) \quad (4.56)$$

where we used equation (4.50). This leads to an expression of $D_{a,b}^{\mathrm{E}}(t,q)$ as a linear combination of terms of the form $E_{n,-m}$ and complex conjugates thereof, such that, according to the previous section, it is indeed a linear combination of (powers of) the iterated integrals $\tilde{\Gamma}$ and their complex conjugates. The explicit result is rather lengthy and can be found in appendix D. In particular, it matches the result for D^{E} given in equation (4.33).

Let us make a comment about the K-theoretic use of the elliptic Bloch–Wigner function D^{E} in the construction of a regulator map $R : K_2(E) \to \mathbb{C}$ in equation (8.1.1) of reference [33], where $K_2(E)$ is the second K-group associated to an elliptic curve E over $\mathbb{C}$. The non-elliptic version of the map R generalised to higher K-groups is of particular interest in the formulation of the conjectures of reference [32], which relate the Dedekind zeta function $\zeta_F(m)$ of a number field F to special values of the single-valued polylogarithms $\mathcal{L}_m$, and which are also used in the description of the mth Bloch group. The elliptic version R can be used in the construction of the second elliptic Bloch group, see e.g. reference [32], and its imaginary part is the elliptic Bloch–Wigner function D^{E}. In order to describe its real part, let

$$\mathrm{J}(t) = \log(|t|)\log(|1-t|), \quad (4.57)$$

such that the real part of the regulator map R is given by

$$\mathrm{J}^{\mathrm{E}}(t,q) = \sum_{l \geqslant 0} \mathrm{J}(tq^l) - \sum_{l>0} \mathrm{J}(t^{-1}q^l). \quad (4.58)$$

Comparing equations (4.53) and (4.57) as well as the definitions of their elliptic generalisations (4.54) and (4.58), leads to the conclusion that

$$\mathrm{J}^{\mathrm{E}}(t,q) = \frac{1}{2}\,\mathrm{Re}(D_{1,2}^{\mathrm{E}}(t,q)) + \frac{(4\pi\,\mathrm{Im}(\tau))^2}{6} B_3(u). \quad (4.59)$$

Therefore, according to equation (4.55), the regulator map R equals one half of $D_{1,2}^{\mathrm{E}}$ up to the last term in equation (4.59), such that, as for its imaginary part, i.e. the elliptic Bloch–Wigner function, the whole regulator map R can immediately be translated to the iterated integrals on the torus and the projective elliptic curve, as described above.

4.3. The elliptic Bloch relation on the torus

The connections between the different notions of elliptic (multiple) polylogarithms found in the previous subsections 4.1 and 4.2 can be exploited to translate and to compare various concepts and structures among them. In this section we show how the elliptic Bloch relation (3.57) translates to the torus, discover more general relations thereon and hence, provide an alternative proof of the elliptic Bloch relation. In doing so, we will show, how the Bloch relation can be interpreted in terms of differentials of iterated integrals or, more generally, in terms of the elliptic symbol calculus introduced in reference [31].

Let F be an elliptic function on the Tate curve with the following divisor

$$\mathrm{Div}(F) = \sum_i d_i(a_i), \qquad \sum_i d_i = 0, \qquad \prod_i a_i^{d_i} = 1. \tag{4.60}$$

Formulated on the torus the above equation translates into,

$$\mathrm{Div}(F) = \sum_i d_i(A_i), \qquad \sum_i d_i = 0, \qquad \sum_i d_i A_i = 0. \tag{4.61}$$

where $a_i = \mathrm{e}^{2\pi i A_i}$. Using equation (3.14), one can express F in terms of a product of Weierstrass σ functions

$$F(z) = s_A \prod_i \sigma(z - A_i)^{d_i} = s_A \, \exp\left(\sum_i d_i \int_0^{z - A_i} \mathrm{d}z' \, \zeta(z') \right) \tag{4.62}$$

for some scaling $s_A \in \mathbb{C}^*$ of F. Similarly, for a given $\kappa \in \mathbb{C}^*$, $\kappa - F$ can be represented by[19]

$$\kappa - F(z) = s_B \prod_j \sigma(z - B_j)^{e_j} = s_B \, \exp\left(\sum_j e_j \int_0^{z - B_j} \mathrm{d}z' \zeta(z') \right), \tag{4.63}$$

where $s_B \in \mathbb{C}^*$. For notational convenience, let us split the set of zeros and poles of F and $\kappa - F$, denoted by I and J, respectively, into the zeros of F, $I' = \{A_i | d_i > 0\}$, the zeros of $\kappa - F$, $J' = \{B_j | e_j > 0\}$, and the common set of poles $K = \{A_i | d_i < 0\} = \{B_j | e_j < 0\}$. Using these conventions, the elliptic Bloch relation (3.57) can be rewritten by means of equation (4.33) as

$$0 = \sum_{i,j} d_i e_j \mathrm{D}^{\mathrm{E}} \left(\tfrac{a_i}{b_j}, q \right) = -2\pi \sum_{i,j} d_i e_j \left(\mathrm{Re} \left(\tilde{\Gamma} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; A_i - B_j, \tau \right) \right) \right. \tag{4.64a}$$

$$+ \, \mathrm{Re} \left(\frac{\tau}{2\pi i} \right) \mathrm{Re} \left(\tilde{\Gamma}(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}; A_i - B_j, \tau) \right) \tag{4.64b}$$

$$\left. -\mathrm{Re}(A_i - B_j) \, \mathrm{Re} \left(\tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau) \right) \right), \tag{4.64c}$$

where B_j is given by $b_j = \mathrm{e}^{2\pi i B_j}$ and the summation indices (i, j) run over $I \times J$, unless mentioned otherwise.

We give an alternative proof of equation (4.90), which we refer to as the elliptic Bloch relation on the torus, in the following paragraphs by showing that the sums over the single iterated integrals $\tilde{\Gamma}$ occurring in the above formula vanish separately (and for the first two also their imaginary parts, yielding two holomorphic analogues of the elliptic Bloch relation). Note that since we are interested in generating functional equations we consider the zeros and poles A_i and B_j as well as the scaling factors s_A and s_B to be (not independent) variables, e.g. depending on variable coefficients of the rational function on the elliptic curve that determine F, cf the examples in subsection 3.4.

Let us start with the first term of the elliptic Bloch relation on the torus, equation (4.64a): naturally, the zeros and poles satisfy the constraints $\sum_i d_i A_i = 0$ and $\sum_j e_j B_j = 0$ as functional

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[19] Note that here, the e_j do not denote the roots of a Weierstrass equation, but the orders of the zeros and poles of the elliptic function $\kappa - F$.

identities. Hence, the functional identity

$$\kappa = \kappa - F(A_i) = s_B \prod_j \sigma(A_i - B_j)^{e_j} \tag{4.65}$$

holds for $i \in I'$, such that taking the total differential of both sides and using equation (3.33), i.e. $\zeta(z) = g^{(1)}(z, \tau) + 2\eta_1 z$, as well as the representations (4.62) and (4.63) the differential equation

$$\sum_j e_j g^{(1)}(A_i - B_j) d(A_i - B_j) = -d \log(s_B) - c_1 \sum_j e_j B_j dB_j \tag{4.66}$$

can be obtained. For $k \in K$, a functional identity involving the residue instead of the infinite value $\kappa - F(A_k)$ can be used for a similar calculation: since by convention $\sigma'(0) = 1$, the residue of $\kappa - F$ at A_k is

$$\text{Res}_{A_k}(\kappa - F) = s_B \prod_{j \neq k} \sigma(A_k - B_j)^{e_j}, \tag{4.67}$$

which implies that

$$\sum_{j \neq k} e_j g^{(1)}(A_k - B_j) d(A_k - B_j) = d \log \left(\text{Res}_{A_k}(\kappa - F) \right) - d \log(s_B) - c_1 \sum_j e_j B_j dB_j. \tag{4.68}$$

Two similar differential equations for sums over I can be found, the first one starting from $\kappa = F(B_j)$, where $j \in J'$,

$$\sum_i d_i g^{(1)}(A_i - B_j) d(A_i - B_j) = -d \log(s_A) - c_1 \sum_i d_i A_i dA_i. \tag{4.69}$$

With $k \in K$ and using that $\text{Res}_{A_k}(F) = -\text{Res}_{A_k}(\kappa - F)$, the last such differential equation turns out to be

$$\sum_{i \neq k} d_i g^{(1)}(A_k - A_i) d(A_k - A_i) = d \log \left(\text{Res}_{A_k}(\kappa - F) \right) - d \log(s_A) - c_1 \sum_i d_i A_i dA_i. \tag{4.70}$$

Going through an elaborate calculation, whose details we have outsourced to appendix E, the four differential equations (4.66) and (4.68)–(4.70) can be combined into the differential equation

$$\sum_{i,j} d_i e_j (A_i - B_j) g^{(1)}(A_i - B_j, \tau) d(A_i - B_j) = 0. \tag{4.71}$$

For integration paths with $d\tau = 0$, the differential of the iterated integral $\tilde{\Gamma} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; z, \tau \right)$ is given by

$$d\tilde{\Gamma} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; z, \tau \right) = z g^1(z, \tau) dz. \tag{4.72}$$

Accordingly, equation (4.71) implies that

$$\sum_{i,j} d_i e_j \tilde{\Gamma} \left(\begin{matrix} 1\,0 \\ 0\,0 \end{matrix}; A_i - B_j, \tau \right) = c_2 \tag{4.73}$$

for some constant $c_2 \in \mathbb{C}$. In general, the zeros and poles of F are only constrained by $\sum_i d_i A_i = 0 = \sum_i d_i$, thus, it may be assumed that they can be split in a way such that the divisor of F consists of triplets with two of them being unconstrained and the third one being given by $A_3 = -A_1 - A_2$. An alternative way of saying this is that divisors of the form $(A_1) + (A_2) - (0) - (A_1 + A_2)$ span the set of principal divisors, which was encountered in subsection 3.3, cf equation (3.52). Thus, by continuity, the above equation can be evaluated at the point where all $A_i = 0$ to determine

$$c_2 = \sum_{j} e_j \tilde{\Gamma} \left(\begin{matrix} 1\,0 \\ 0\,0 \end{matrix}; -B_j, \tau \right) \sum_{i} d_i = 0. \tag{4.74}$$

Therefore, we find a holomorphic analogue of the elliptic Bloch relation

$$\sum_{i,j} d_i e_j \tilde{\Gamma} \left(\begin{matrix} 1\,0 \\ 0\,0 \end{matrix}; A_i - B_j, \tau \right) = 0. \tag{4.75}$$

Similar arguments apply for the term (4.64c) involving the iterated integral $z\,\tilde{\Gamma} \left(\begin{matrix} 1 \\ 0 \end{matrix}; z, \tau \right)$ in the elliptic Bloch relation on the torus (4.90). Let $i \in I'$ and write

$$\kappa = \kappa - F(A_i) = s_B \, \exp \left(\sum_{j} e_j \int_0^{A_i - B_j} dz \, g^{(1)}(z, \tau) + \frac{c_1}{2} \sum_{j} e_j B_j^2 \right), \tag{4.76}$$

such that

$$\sum_{j} e_j \tilde{\Gamma} \left(\begin{matrix} 1 \\ 0 \end{matrix}; A_i - B_j, \tau \right) = \log(\kappa) - \log(s_B) - \frac{c_1}{2} \sum_{j} e_j B_j^2 - 2\pi i m_1, \tag{4.77}$$

for some $m_1 \in \mathbb{Z}$, which holds for $\tilde{\Gamma}(\begin{matrix} 1 \\ 0 \end{matrix}; A_i - B_j, \tau)$ being the regularised or unregularised iterated integral, because the factor $\sum_j e_j = 0$ cancels the logarithmic singularity. For $k \in K$ and with $\sigma(z) = s_C \, \exp \left(\int_{z_0}^{z} dz' \zeta(z') \right)$ such that $\sigma'(0) = 1$, the same calculation as before leads to

$$\text{Res}_{A_k}(\kappa - F) = s_B s_C \, \exp \left(\sum_{j \neq k} e_j \int_{z_0}^{A_k - B_j} dz \, \zeta(z) \right)$$

$$= s_B s_C \, \exp \left(\sum_{j \neq k} e_j \int_0^{A_k - B_j} dz \, g^{(1)}(z, \tau) + \frac{c_1}{2} \sum_{j} e_j B_j^2 + \int_0^{z_0} dz \, \zeta(z) \right) \tag{4.78}$$

which implies that

$$\sum_{j\neq k} e_j \int_0^{A_k-B_j} \mathrm{d}z g^{(1)}(z,\tau) = \log(\mathrm{Res}_{A_k}(\kappa-F)) - \log(s_C) - \log(s_B) - \frac{c_1}{2}\sum_j e_j B_j^2$$

$$- \int_0^{z_0} \mathrm{d}z \zeta(z) - 2\pi i m_2, \tag{4.79}$$

where $m_2 \in \mathbb{Z}$, and analogously for the sum over $I\backslash\{k\}$

$$\sum_{i\neq k} d_i \int_0^{A_k-A_i} \mathrm{d}z g^{(1)}(z,\tau) = \log(\mathrm{Res}_{A_k}(F)) - \log(s_C) - \log(s_A) - \frac{c_1}{2}\sum_i d_i A_i^2$$

$$- \int_0^{z_0} \mathrm{d}z \zeta(z) - 2\pi i m_3, \tag{4.80}$$

for $m_3 \in \mathbb{Z}$. A similar result holds for $j \in J'$,

$$\sum_i d_i \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_i - B_j, \tau) = \log(\kappa) - \log(s_A) - \frac{c_1}{2}\sum_i d_i A_i^2 - 2\pi i m_4, \tag{4.81}$$

where $m_4 \in \mathbb{Z}$. Since $\log(\mathrm{Res}_{A_k}(F)) = \log(\mathrm{Res}_{A_k}(1-F)) + i\pi$, equations (4.79) and (4.80) lead to

$$\sum_j e_j \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_k - B_j, \tau) - \sum_i d_i \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_k - A_i, \tau)$$
$$= -i\pi(1 + 2m_2 - 2m_3) + \log(s_A) + \frac{c_1}{2}\sum_i d_i A_i^2 - \log(s_B) - \frac{c_1}{2}\sum_j e_j B_j^2. \tag{4.82}$$

Finally, using the equations (4.77) and (4.79)–(4.81) all together, the identities

$$\sum_{i,j} d_i e_j \mathrm{Re}\,(A_i - B_j)\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; A_i - B_j, \tau\right)\right) = 0 \tag{4.83}$$

and

$$\sum_{i,j} d_i e_j\,\mathrm{Im}\,(A_i - B_j)\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; A_i - B_j, \tau\right)\right) = 0. \tag{4.84}$$

can be obtained, see appendix E for the calculation.

Now, we are left with the term (4.64b) involving $\tilde{\Gamma}\left(\begin{smallmatrix}2\\0\end{smallmatrix}; z, \tau\right)$. Let us take the partial derivative of equation (4.77) with respect to τ and use the partial differential equation (3.35) of the integration kernel, i.e. $2\pi i \frac{\partial}{\partial \tau} g^{(1)}(z,\tau) = \frac{\partial}{\partial z} g^{(2)}(z,\tau)$, to find

$$\sum_j e_j g^{(2)}(A_i - B_j, \tau) = -2\pi i \frac{\partial}{\partial \tau} \frac{c_1}{2}\sum_j e_j B_j^2 - 2\pi i \sum_j e_j g^{(1)}(A_i - B_j, \tau)\frac{\partial}{\partial \tau}(A_i - B_j), \tag{4.85}$$

valid for $i \in I'$. A similar result holds for $j \in J'$

$$\sum_i d_i g^{(2)}(A_i - B_j, \tau) = -2\pi i \frac{\partial}{\partial \tau}\frac{c_1}{2}\sum_i d_i A_i^2 - 2\pi i \sum_i d_i g^{(1)}(A_i - B_j, \tau)\frac{\partial}{\partial \tau}(A_i - B_j) \tag{4.86}$$

and for $k \in K$

$$\sum_j e_j g^{(2)}(A_k - B_j) - \sum_i d_i g^{(2)}(A_k - A_i) = -2\pi i \frac{\partial}{\partial \tau} \frac{c_1}{2} \sum_i d_i A_i^2 - 2\pi i \frac{\partial}{\partial \tau} \frac{c_1}{2} \sum_j e_j B_j^2$$

$$- 2\pi i \sum_j e_j g^{(1)}(A_k - B_j) \frac{\partial}{\partial \tau}(A_k - B_j)$$

$$+ 2\pi i \sum_i d_i g^{(1)}(A_k - A_i) \frac{\partial}{\partial \tau}(A_k - B_j). \quad (4.87)$$

The equations (4.85)–(4.87) imply that for paths with $d\tau = 0$ the differential equation

$$d\sum_{ij} d_i e_j \tilde{\Gamma}\left(\begin{matrix} 2 \\ 0 \end{matrix}; A_i - B_j, \tau\right) = 0 \quad (4.88)$$

holds, the explicit calculation is shown in the appendix E. By the same argument as for equation (4.75), we therefore find another functional identity which can be interpreted as a holomorphic analogue of the elliptic Bloch relation on the torus

$$\sum_{i,j} d_i e_j \tilde{\Gamma}\left(\begin{matrix} 2 \\ 0 \end{matrix}; A_i - B_j, \tau\right) = 0. \quad (4.89)$$

To summarise, we managed to express the elliptic Bloch relation (4.90) in terms of iterated integrals on the torus.

Let us comment on the two holomorphic functional equations (4.75) and (4.89) respectively, in terms of the iterated integrals $\tilde{\Gamma}$ on the torus which have the same structure as the original elliptic Bloch relation: in the language of reference [33], it turns out that the iterated integrals $\tilde{\Gamma}(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; z, \tau)$ and $\tilde{\Gamma}(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}; z, \tau)$ are Steinberg functions. However, we have to be careful when using these iterated integrals are multi-valued and in order to reproduce equations (4.75) and (4.89) they have to be evaluated on the representatives of the zeros and poles of F and $\kappa - F$ which satisfy $\sum_i d_i A_i = 0 = \sum_j e_j B_j$, and not only such that these sums lie in the lattice Λ. These equations have been obtained by differential calculus of iterated integrals, which is simply the symbol calculus of an iterated integral with depth 1. Thus, together with equation (4.83) we provide an interpretation of the elliptic Bloch relation using the elliptic symbol calculus of the iterated integrals $\tilde{\Gamma}$ on the torus.

4.4. The elliptic Bloch relation in the projective formulation

By means of equations (4.42)–(4.45), the elliptic Bloch relation (4.90) can also be expressed on the projective elliptic curve

$$\begin{aligned}
0 &= \sum_{i,j} d_i e_j \mathrm{D}^{\mathrm{E}}\left(\frac{a_i}{b_j}, q\right) \\
&= -2\pi \sum_{i,j} d_i e_j \left(\mathrm{Re}\left(\tilde{\Gamma}(\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}; A_i - B_j, \tau)\right) + \mathrm{Re}\left(\frac{\tau}{2\pi i}\right) \mathrm{Re}\left(\tilde{\Gamma}(\begin{matrix} 2 \\ 0 \end{matrix}; A_i - B_j, \tau)\right) \right. \\
&\qquad \left. - \mathrm{Re}(A_i - B_j) \mathrm{Re}\left(\tilde{\Gamma}(\begin{matrix} 1 \\ 0 \end{matrix}; A_i - B_j, \tau)\right) \right) \\
&= -2\pi \sum_{i,j} d_i e_j \left(\mathrm{Re}\left(\mathrm{E}_3(\begin{matrix} 1 & 0 \\ \infty & 0 \end{matrix}; x_{ij}, \vec{e})\right) + \mathrm{Re}\left(\frac{\tau}{2\pi i}\right) \mathrm{Re}\left(\mathrm{E}_3(\begin{matrix} 2 \\ \infty \end{matrix}; x_{ij}, \vec{e})\right) \right. \\
&\qquad \left. - \mathrm{Re}(\mathrm{E}_3(\begin{matrix} 0 \\ 0 \end{matrix}; x_i, \vec{e}) - \mathrm{E}_3(\begin{matrix} 0 \\ 0 \end{matrix}; x_j, \vec{e})) \mathrm{Re}\left(\mathrm{E}_3(\begin{matrix} 1 \\ \infty \end{matrix}; x_{ij}, \vec{e})\right) \right),
\end{aligned} \quad (4.90)$$

where $x_i = \wp(A_i)$, $x_j = \wp(B_j)$ and $x_{ij} = \wp\left(A_i - B_j\right)$. Similarly, the holomorphic functional relations (4.75) and (4.89) translate to

$$\sum_{i,j} d_i e_j \mathrm{E}_3 \begin{pmatrix} 1 & 0 \\ \infty & 0 \end{pmatrix}; x_{ij}, \vec{e} \end{pmatrix} = 0 \tag{4.91}$$

and

$$\sum_{i,j} d_i e_j \mathrm{E}_3 \begin{pmatrix} 2 \\ \infty \end{pmatrix}; x_{ij}, \vec{e} \end{pmatrix} = 0. \tag{4.92}$$

5. Conclusions

In this article, we have investigated the elliptic Bloch–Wigner function D^{E} in order to obtain functional relations of the iterated integrals $\tilde{\Gamma}$ on the torus and especially to formulate an elliptic analogue of the five-term identity on the torus. This analysis led to several results:

- The elliptic Bloch–Wigner function D^{E}, which is usually defined on the Tate curve, has been translated into the language of iterated elliptic integrals $\tilde{\Gamma}$ on the torus. This was the precondition for the application of the elliptic symbol calculus.

- We have been extending the translation to the torus for two additional classes of functions: the first class are the sums $\mathrm{D}^{\mathrm{E}}_{a,b}$ [36] on the Tate curve defined in equation (4.54). These functions play a crucial role in the calculation of modular graph functions [58]. The final formulæ can be obtained by combining equation (D.1) with equations (4.5), (4.19), (4.20) and (4.26). The representation of functions $\mathrm{D}^{\mathrm{E}}_{a,b}$ in terms of $\tilde{\Gamma}$'s on the torus allows for series expansions and therefore the investigation of relations between different modular graph functions. In particular, those representations might shed some light on the explicit construction of a representation of a single-valued map for genus-one string amplitudes as suggested in reference [59].

  The second class is a particular subclass of the former and is given by the functions $\mathcal{L}^{\mathrm{E}}_n$, defined in equation (4.51), which are based on Ramakrishnan's single-valued polylogarithms: their representation on the torus follows from a combination of equation (4.51) with equations (4.5), (4.19), (4.20) and (4.26).

- Once representations on the torus do exist, it is just one further step to translate [44] those into representations in the projective formulation of the elliptic curve. In particular, we have chosen to express the elliptic Bloch–Wigner function in terms of iterated integrals E_3 on the projective elliptic curve. For the two general classes of functions mentioned above this can be done in a straightforward manner as well.

  The above expressions on the Tate curve and on the torus can be extended to the projective elliptic curve as well. In the case of the elliptic Bloch–Wigner function D^{E}, we have found equation (4.46), which expresses the value of D^{E} on the Tate curve, the torus and the projective curve.

- Employing the above translations, we have taken the elliptic Bloch relation (3.57), which is defined in terms of the elliptic Bloch–Wigner function D^{E}, from the Tate curve to the torus and the projective elliptic curve: the result is noted in equation (4.90).

- The investigation of the elliptic Bloch relation on the torus led to the holomorphic analogues of the elliptic Bloch relation given by equations (4.75) and (4.89) as well as the non-holomorphic equations (4.83) and (4.84). Since validity of those equations can be proved using the elliptic symbol calculus on the torus, we hereby found an alternative proof and interpretation of the elliptic Bloch relation.

- Translating the elliptic Bloch relation even one step further to the projective elliptic curve yields a possibility to write down functional equations in terms of algebraic arguments. However, this is possible only when restricting the parametrising rational function to lines. Beyond lines, the complexity of the calculation of zeros of the parametrising rational function prevents the corresponding functional relation from being purely algebraic in the arguments.

  In particular, this applies to the elliptic analogue of the five-term identity, i.e. the functional relation induced via the elliptic Bloch relation by the elliptic generalisation (3.79) of the rational function (2.21) parametrising the classical five-term identity.

  In general, due to the complexity of Abel's map, it can not be expected, that generic functional relations generated by the Bloch relation may be formulated explicitly in terms of algebraic arguments on the torus or the Tate curve, respectively.

In the classical case, the five-term identity is conjectured to generate all other functional identities between the dilogarithm. It would be interesting to investigate, whether a similar conjecture can be formulated for the elliptic case. As described at the end of subsection 3.4, Gangl and Zagier state [32] that the elliptic Bloch relation, the symmetry and the duplication relation are expected to generate all the functional relations of the elliptic Bloch group associated to the elliptic Bloch–Wigner function. However, the construction of higher elliptic Bloch groups and in particular the corresponding group of functional relations awaits further investigation.

In particular it would be nice, if this conjecture came along with a geometric interpretation of the elliptic analogue of the five-term identity, similar, but for sure more complicated, to the classical case.

While the elliptic Bloch relation is capable of providing a large class of functional relations, it is not clear, whether there might exist other structures or similar mechanisms generating further relations, for example based on functions beyond the elliptic Bloch–Wigner function. Based on the experience gained from the current implementation and the investigations in this article, a straightforward generalisation seems unlikely.

In the context of Feynman diagrams leading to elliptic polylogarithms, the main recent focus was usually on evaluating the integrals. Despite the lack of a dedicated application of the functional relations investigated in this article, we hope that our results will facilitate simplifications and unravel new structures in further elliptic Feynman calculations to come.

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Appendix A. Group addition on $E(\mathbb{C})$

The geometric picture of the addition on the elliptic curve is that two distinct points $P_1 = [x_1 : y_1 : 1]$ and $P_2 = [x_2 : y_2 : 1]$ with $y_1 \neq \pm y_2$ form a line which intersects the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$ at a third point $-P_3 = [x_3 : -y_3 : 1]$. The sum $P_3 = P_1 + P_2$ is defined as being the projection of $-P_3 = [x_3 : -y_3 : 1]$ to the negative y-coordinate $P_3 = [x_3 : y_3 : 1]$. Thus, two points with their y-coordinate being of the opposite sign are indeed the inverse of each other with $\infty = [0 : 1 : 0]$ being the unit element since the line defined by P_3 and $-P_3$ intersects the elliptic curve only at infinity. The algebraic description is the following. For P_1 and P_2 as above, the line intersecting them is given by $y = \lambda x + \mu$, where

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}, \qquad \mu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}. \tag{A.1}$$

The x-coordinate of the third point $-P_3 = [x_3 : -y_3 : 1]$ intersecting the line and the elliptic curve is the third solution (besides x_1 and x_2) of the cubic equation $(\lambda x + \mu)^2 = 4x^3 - g_2 x - g_3$, which is in terms of x_1 and x_2

$$x_3 = -x_1 - x_2 + \frac{\lambda^2}{4}. \tag{A.2}$$

The y coordinate of P_3 is then simply the negative of the y coordinate determined by the line and x_3,

$$y_3 = -\lambda x_3 - \mu. \tag{A.3}$$

The last case we need to consider is if the points P_1 and P_2 are identical and not the unit element, i.e. $P_1 = P_2 = P = [x_P : y_P : 1]$. For $y_P \neq 0$, the above description of taking the line intersecting P_1 and P_2 degenerates to taking the tangent on the elliptic curve at P. The sum $2P = P + P = [x_{2P} : y_{2P} : 1]$ is then again the projection of the second point lying on this tangent and the elliptic curve with respect to the x-coordinate. Algebraically, this corresponds to

$$\lambda = \frac{12 x_P^2 - g_2}{2 y_P}, \qquad \mu = y_P - \lambda x_P \tag{A.4}$$

and

$$x_{2P} = -2 x_P + \frac{\lambda^2}{4}, \qquad y_{2P} = -\lambda x_{2P} - \mu \tag{A.5}$$

as before. In the case of $y_P = 0$, the point P is inverse to itself, such that $P + P = P - P = \infty$. These addition rules exactly agree with the well-known addition formula of the Weierstrass $\wp$ function

$$\wp(x_1 + x_2) = -\wp(x_1) - \wp(x_2) + \frac{1}{4}\left(\frac{\wp'(x_2) - \wp'(x_1)}{\wp(x_2) - \wp(x_1)}\right)^2 \tag{A.6}$$

for $x_1 \neq x_2$ and similar for its derivative. This ensures that $\xi_{\Lambda,E}$ defined in equation (3.16) is indeed a homomorphism.

Appendix B. *q*-expansion of integration kernels and elliptic polylogarithms

Starting from the q-expansion of the Jacobi θ function, the q-expansion of the integration kernels are obtained via the generating Eisenstein–Kronecker series [52] and are given by

$$g^{(0)}(z, \tau) = 1, \tag{B.1}$$

$$g^{(1)}(z, \tau) = \pi \cot(\pi z) + 4\pi \sum_{k,l>0} \sin(2\pi kz)q^{kl} \tag{B.2}$$

and for $m > 0$ by

$$g^{(2m)}(z, \tau) = -2\zeta_{2m} - 2\frac{(2\pi i)^{2m}}{(2m-1)!} \sum_{k,l>0} \cos(2\pi kz)l^{2m-1}q^{kl}, \tag{B.3}$$

as well as by

$$g^{(2m+1)}(z, \tau) = -2i\frac{(2\pi i)^{2m+1}}{(2m)!} \sum_{k,l>0} \sin(2\pi kz)l^{2m}q^{kl}. \tag{B.4}$$

The $(n-1)$-fold integration of the regularised integral (3.36), i.e.

$$\tilde{\Gamma}_{\text{reg}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}; z, \tau \end{pmatrix} = \log(1 - e^{2\pi iz}) - \pi iz + 4\pi \sum_{k,l>0} \frac{1}{2\pi k} \left(1 - \cos(2\pi kz)\right) q^{kl}, \tag{B.5}$$

and the n-fold integration of the above integration kernels $g^{(m)}(z, \tau)$ for $m > 1$ can be determined analytically. This yields the following efficient method to write down their q-expansion and, hence, for their numerical evaluation. The central observation is that for $n \geqslant 0$ the $2n$-fold integration of $\sin(2\pi kz)$ with $k \in \mathbb{Z}$ is given by

$$\int_0^z dz_1 \int_0^{z_1} dz_2 \dots \int_0^{z_{2n-1}} dz_{2n} \, \sin(2\pi kz_{2n}) = \frac{(-1)^n}{(2\pi k)^{2n}} \, \sin(2\pi kz)$$
$$+ \sum_{j=1}^n \frac{(-1)^{n-j}}{(2\pi k)^{2n+1-2j}} \frac{z^{2j-1}}{(2j-1)!} \tag{B.6}$$

and the $(2n + 1)$-fold integration by

$$\int_0^z dz_1 \int_0^{z_1} dz_2 \dots \int_0^{z_{2n}} dz_{2n+1} \, \sin(2\pi kz_{2n+1}) = \frac{(-1)^{n+1}}{(2\pi k)^{2n+1}} \, \cos(2\pi kz)$$
$$+ \sum_{j=0}^n \frac{(-1)^{n-j}}{(2\pi k)^{2n+1-2j}} \frac{z^{2j}}{(2j)!}. \tag{B.7}$$

A similar result holds for the iterative integration of $\cos(2\pi kz)$,

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$$\int_0^z dz_1 \int_0^{z_1} dz_2 \ldots \int_0^{z_{2n-1}} dz_{2n} \cos(2\pi k z_{2n}) = \frac{(-1)^n}{(2\pi k)^{2n}} \cos(2\pi k z)$$

$$+ \sum_{j=0}^{n-1} \frac{(-1)^{n+1-j}}{(2\pi k)^{2n-2j}} \frac{z^{2j}}{(2j)!} \qquad \text{(B.8)}$$

and

$$\int_0^z dz_1 \int_0^{z_1} dz_2 \ldots \int_0^{z_{2n}} dz_{2n+1} \cos(2\pi k z_{2n+1}) = \frac{(-1)^n}{(2\pi k)^{2n+1}} \sin(2\pi k z)$$

$$+ \sum_{j=1}^{n} \frac{(-1)^{n-j}}{(2\pi k)^{2n+2-2j}} \frac{z^{2j-1}}{(2j-1)!}. \quad \text{(B.9)}$$

Combining the above results yields the following q-expansions of the elliptic polylogarithms of the form $\tilde{\Gamma}\left(\begin{smallmatrix} 0 \ldots 0\, m \\ 0 \ldots 0\, 0 \end{smallmatrix}; z, \tau\right)$ for $n \geqslant 1$:

$$\tilde{\Gamma}\left(\underbrace{\begin{smallmatrix} 0 \ldots 0\, 1 \\ 0 \ldots 0\, 0 \end{smallmatrix}}_{2n}; z, \tau\right) = -\frac{1}{(2\pi i)^{2n-1}} \mathrm{Li}_{2n}(e^{2\pi i z}) + \sum_{j=1}^{2n-1} \frac{\zeta_{j+1}}{(2\pi i)^j} \frac{z^{2n-1-j}}{(2n-1-j)!} - \pi i \frac{z^{2n}}{(2n)!}$$

$$+ (-1)^n 4\pi \sum_{k,l>0} \frac{1}{(2\pi k)^{2n}} \left(\sin(2\pi k z) + \sum_{j=1}^{n} \frac{(-1)^j}{(2\pi k)^{1-2j}} \frac{z^{2j-1}}{(2j-1)!} \right) q^{kl},$$

$$\text{(B.10)}$$

$$\tilde{\Gamma}\left(\underbrace{\begin{smallmatrix} 0 \ldots 0\, 1 \\ 0 \ldots 0\, 0 \end{smallmatrix}}_{2n+1}; z, \tau\right) = -\frac{1}{(2\pi i)^{2n}} \mathrm{Li}_{2n+1}(e^{2\pi i z}) + \sum_{j=1}^{2n} \frac{\zeta_{j+1}}{(2\pi i)^j} \frac{z^{2n-j}}{(2n-j)!} - \pi i \frac{z^{2n+1}}{(2n+1)!}$$

$$+ (-1)^{n+1} 4\pi \sum_{k,l>0} \frac{1}{(2\pi k)^{2n+1}} \left(\cos(2\pi k z) + \sum_{j=0}^{n} \frac{(-1)^{j+1}}{(2\pi k)^{-2j}} \frac{z^{2j}}{(2j)!} \right) q^{kl}$$

$$\text{(B.11)}$$

and for $m > 1$ and $n \geqslant 0$

$$\tilde{\Gamma}\left(\underbrace{\begin{smallmatrix} 0 \ldots 0\, 2m \\ 0 \ldots 0\, 0 \end{smallmatrix}}_{2n}; z, \tau\right) = -2\zeta_{2m} \frac{z^{2n}}{(2n)!} + (-1)^{n+1} 2 \frac{(2\pi i)^{2m}}{(2m-1)!} \sum_{k,l>0} \frac{1}{(2\pi k)^{2n}}$$

$$\times \left(\cos(2\pi k z) + \sum_{j=0}^{n-1} \frac{(-1)^{1+j}}{(2\pi k)^{-2j}} \frac{z^{2j}}{(2j)!} \right) l^{2m-1} q^{kl}, \qquad \text{(B.12)}$$

$$\tilde{\Gamma}\left(\underbrace{\begin{smallmatrix} 0 \ldots 0\, 2m \\ 0 \ldots 0\, 0 \end{smallmatrix}}_{2n+1}; z, \tau\right) = -2\zeta_{2m} \frac{z^{2n+1}}{(2n+1)!} + (-1)^{n+1} 2 \frac{(2\pi i)^{2m}}{(2m-1)!} \sum_{k,l>0} \frac{1}{(2\pi k)^{2n+1}}$$

$$\times \left(\sin(2\pi k z) + \sum_{j=1}^{n} \frac{(-1)^j}{(2\pi k)^{1-2j}} \frac{z^{2j-1}}{(2j-1)!} \right) l^{2m-1} q^{kl}, \qquad \text{(B.13)}$$

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as well as

$$\tilde{\Gamma}\left(\begin{array}{c} 0\ldots 0\, 2m+1 \\ \underbrace{0\ldots 0\, 0}_{2n} \end{array}; z, \tau\right) = (-1)^{n+1}2i\frac{(2\pi i)^{2m+1}}{(2m)!}\sum_{k,l>0}\frac{1}{(2\pi k)^{2n}}$$
$$\times\left(\sin(2\pi kz) + \sum_{j=1}^{n}\frac{(-1)^j}{(2\pi k)^{1-2j}}\frac{z^{2j-1}}{(2j-1)!}\right)l^{2m}q^{kl}, \quad\text{(B.14)}$$

$$\tilde{\Gamma}\left(\begin{array}{c} 0\ldots 0\, 2m+1 \\ \underbrace{0\ldots 0\, 0}_{2n+1} \end{array}; z, \tau\right) = (-1)^{n}2i\frac{(2\pi i)^{2m+1}}{(2m)!}\sum_{k,l>0}\frac{1}{(2\pi k)^{2n+1}}$$
$$\times\left(\cos(2\pi kz) + \sum_{j=0}^{n}\frac{(-1)^{1+j}}{(2\pi k)^{-2j}}\frac{z^{2j}}{(2j)!}\right)l^{2m}q^{kl}$$
$$\text{(B.15)}$$

where, in the above formula, we denote the integration kernels by $\tilde{\Gamma}\left(\begin{array}{c} 0\ldots 0\, m \\ \underbrace{0\ldots 0\, 0}_{0} \end{array}; z, \tau\right) = g^{(m)}(z, \tau)$.

Appendix C. Integration constants as elliptic zeta values

This section is dedicated to the calculation of the integration constants from subsection 4.1, i.e.

$$E_{n,-m}(1, 1, q) = -(1 - (-1)^{n+m})\text{ELi}_{n,-m}(1, 1, q) \quad\text{(C.1)}$$

for $n \geqslant 1$ and $m \geqslant 0$, where

$$\text{ELi}_{n,m}(1, 1, q) = \sum_{k,l>0}\frac{1}{k^n}\frac{1}{l^m}q^{kl}, \quad\text{(C.2)}$$

in terms of the elliptic zeta values

$$\omega_n(m; \tau) = \tilde{\Gamma}\left(\begin{array}{c} 0\ldots 0\, m \\ \underbrace{0\ldots 0\, 0}_{n} \end{array}; 1, \tau\right) \quad\text{(C.3)}$$

defined in equation (3.41).

Let us begin with the case $m = 0$, where we consider for $n \geqslant 2$

$$E_n(1, 1, q) = -\frac{1}{2}(1 - (-1)^n)\zeta_n + E_{n,0}(1, 1, q) = \begin{cases} 0 & n\text{ even} \\ -\zeta_n - 2\text{ELi}_{n,0}(1, 1, q) & n\text{ odd} \end{cases} \quad\text{(C.4)}$$

as defined in equation (4.3). While $E_n(1, 1, q)$ vanishes for even n, for odd $n \geqslant 3$ the integration constants $E_n(1, 1, q)$ will turn out to be linear combinations of $\omega_1(1; \tau), \omega_3(1; \tau), \ldots, \omega_n(1; \tau)$, which we derive similarly as the result for $n = 1$ given by equation (4.11)

$$2\text{ELi}_{1,0}(1, 1, q) = \omega_2(1; \tau) + \frac{\pi i}{2}. \quad\text{(C.5)}$$

In order to do so, let $n \geqslant 4$ be even. In this case, the recursion given by equation (4.19) evaluated at one, which is based on the partial differential equation (4.6),

$$\frac{\partial}{\partial z} E_n(z, 1, \tau) = 2\pi i E_{n-1}(z, 1, \tau),$$
(C.6)

takes the explicit form

$$0 = E_n(1, 1, q) = 2\pi i \int_0^1 dz_0 E_{n-1}(z_0, 1, \tau)$$

$$= (2\pi i)^2 \int_0^1 dz_0 \int_0^{z_0} dz_1 E_{n-2}(z_1, 1, \tau) + 2\pi i E_{n-1}(1, 1, q)$$

$$= (2\pi i)^4 \int_0^1 dz_0 \int_0^{z_0} dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 E_{n-4}(z_3, 1, \tau)$$

$$+ \frac{(2\pi i)^3}{3!} E_{n-3}(1, 1, q) + 2\pi i E_{n-1}(1, 1, q)$$

$$= (2\pi i)^{n-2} \int_0^1 dz_0 \int_0^{z_0} dz_1 \ldots \int_0^{z_{n-4}} dz_{n-3} E_2(z_{n-3}, 1, \tau)$$

$$+ \frac{(2\pi i)^{n-3}}{(n-3)!} E_3(1, 1, q) + \cdots + 2\pi i E_{n-1}(1, 1, q).$$
(C.7)

Plugging in

$$E_2(t, 1, q) = 2\pi i \left(\tilde{\Gamma}(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}; z, \tau) - \omega_2(1; \tau) z \right)$$
(C.8)

from equation (4.12), and solving for $E_{n-1}(1, 1, q)$ leads to the recursive formula

$$E_{n-1}(1, 1, q) = (2\pi i)^{n-2} \left(\frac{\omega_2(1; \tau)}{(n-1)!} - \omega_n(1; \tau) \right)$$

$$- \frac{(2\pi i)^{n-4}}{(n-3)!} E_3(1, 1, q) - \cdots - \frac{(2\pi i)^2}{3!} E_{n-3}(1, 1, q).$$
(C.9)

The first examples are $n = 4$

$$E_3(1, 1, q) = (2\pi i)^2 \left(\frac{\omega_2(1; \tau)}{3!} - \omega_4(1; \tau) \right),$$
(C.10)

$n = 6$

$$E_5(1, 1, q) = (2\pi i)^4 \left(\frac{\omega_2(1; \tau)}{5!} - \omega_6(1; \tau) \right) - \frac{(2\pi i)^2}{3!} E_3(1, 1, q)$$

$$= (2\pi i)^4 \left(\frac{\omega_2(1; \tau)}{5!} - \omega_6(1; \tau) \right) - \frac{(2\pi i)^2}{3!} (2\pi i)^2 \left(\frac{\omega_2(1; \tau)}{3!} - \omega_4(1; \tau) \right)$$

$$= (2\pi i)^4 \left(\left(\frac{1}{5!} - \frac{1}{3!3!} \right) \omega_2(1; \tau) + \frac{1}{3!} \omega_4(1; \tau) - \omega_6(1; \tau) \right)$$
(C.11)

and $n = 8$

$$E_7(1,1,q) = (2\pi i)^6 \left(\frac{\omega_2(1;\tau)}{7!} - \omega_8(1;\tau) \right) - \frac{(2\pi i)^4}{5!} E_3(1,1,q) - \frac{(2\pi i)^2}{3!} E_5(1,1,q)$$

$$= (2\pi i)^6 \left(\frac{\omega_2(1;\tau)}{7!} - \omega_8(1;\tau) \right) - \frac{(2\pi i)^4}{5!} (2\pi i)^2 \left(\frac{\omega_2(1;\tau)}{3!} - \omega_4(1;\tau) \right)$$

$$- \frac{(2\pi i)^2}{3!} (2\pi i)^4 \left(\left(\frac{1}{5!} - \frac{1}{3!3!} \right) \omega_2(1;\tau) + \frac{1}{3!} \omega_4(1;\tau) - \omega_6(1;\tau) \right)$$

$$= (2\pi i)^6 \left(\left(\frac{1}{7!} - \frac{1}{5!3!} - \frac{1}{3!5!} + \frac{1}{3!3!3!} \right) \omega_2(1;\tau) \right.$$

$$\left. + \left(\frac{1}{5!} - \frac{1}{3!3!} \right) \omega_4(1;\tau) + \frac{1}{3!} \omega_6(1;\tau) - \omega_8(1;\tau) \right). \tag{C.12}$$

This recursive structure can be expressed explicitly in terms of the series

$$d_k = \begin{cases} -1 & k = 1 \\ 0 & k \text{ even} \\ -\dfrac{d_1}{k!} - \dfrac{d_3}{(k-2)!} - \cdots - \dfrac{d_{k-2}}{3!} & k \text{ odd} \end{cases} \tag{C.13}$$

with the final result being for any natural number $n \geqslant 1$

$$E_{2n+1}(1,1,q) = (2\pi i)^{2n} \sum_{k=0}^{n} d_{2k+1} \omega_{2n+2-2k}(1;\tau), \tag{C.14}$$

which can be checked inductively as follows: first, note that the series d_k begins with

$$d_1 = -1, \qquad d_3 = -\frac{1}{3!}(-1) = \frac{1}{3!}, \qquad d_5 = -\frac{1}{5!}(-1) - \frac{1}{3!}\frac{1}{3!} = \frac{1}{5!} - \frac{1}{3!3!} \tag{C.15}$$

and

$$d_7 = \frac{1}{7!} - \frac{1}{5!}\frac{1}{3!} - \frac{1}{3!} \left(\frac{1}{5!} - \frac{1}{3!3!} \right) = \frac{1}{7!} - \frac{1}{5!3!} - \frac{1}{3!5!} + \frac{1}{3!3!3!}, \tag{C.16}$$

such that for $n = 1, 2, 3$ the explicit formula (C.14) is indeed in agreement with the first three examples (C.10)–(C.12). For the general case, let $n > 1$ and assume that the explicit formula (C.14) holds for $n - 1$, such that the recursive formula (C.9) implies

$$E_{2n+1}(1,1,q) = (2\pi i)^{2n} \left(\frac{\omega_2(1;\tau)}{(2n+1)!} - \omega_{2n+2}(1;\tau) \right)$$

$$- \frac{(2\pi i)^{2n-3}}{(2n-1)!} E_3(1,1,q) - \cdots - \frac{(2\pi i)^2}{3!} E_{2n-1}(1,1,q)$$

$$= (2\pi i)^{2n} \left(\frac{\omega_2(1;\tau)}{(2n+1)!} - \omega_{2n+2}(1;\tau) \right)$$

$$- \frac{(2\pi i)^{2n-2}}{(2n-1)!} \left((2\pi i)^2 \sum_{k=0}^{1} d_{2k+1} \omega_{4-2k}(1;\tau) \right) - \ldots$$

$$- \frac{(2\pi i)^2}{3!} \left((2\pi i)^{2n-2} \sum_{k=0}^{n-1} d_{2k+1} \omega_{2n-2k}(1;\tau) \right)$$

$$= (2\pi i)^{2n} \left(\sum_{l=0}^{n-1} \sum_{k=0}^{l} \frac{(-d_{2k+1})}{(2n+1-2l)!} \omega_{2l+2-2k}(1;\tau) - \omega_{2n+2}(1;\tau) \right)$$

$$= (2\pi i)^{2n} \left(\sum_{k=0}^{n-1} \sum_{l=k}^{n-1} \frac{(-d_{2k+1})}{(2n+1-2l)!} \omega_{2(l-k)+2}(1;\tau) - \omega_{2n+2}(1;\tau) \right)$$

$$= (2\pi i)^{2n} \left(\sum_{k=0}^{n-1} \sum_{m=0}^{n-k-1} \frac{-d_{2k+1}}{(2(n-m)+1-2k)!} \omega_{2m+2}(1;\tau) - \omega_{2n+2}(1;\tau) \right)$$

$$= (2\pi i)^{2n} \left(\sum_{m=0}^{n-1} \left(\sum_{k=0}^{n-m-1} \frac{-d_{2k+1}}{(2(n-m)+1-2k)!} \right) \omega_{2m+2}(1;\tau) - \omega_{2n+2}(1;\tau) \right)$$

$$= (2\pi i)^{2n} \left(\sum_{m=0}^{n-1} d_{2(n-m)+1} \omega_{2m+2}(1;\tau) - \omega_{2n+2}(1;\tau) \right)$$

$$= (2\pi i)^{2n} \sum_{m=0}^{n} d_{2(n-m)+1} \omega_{2m+2}(1;\tau) = (2\pi i)^{2n} \sum_{k=0}^{n} d_{2k+1} \omega_{2n+2-2k}(1;\tau),$$

$$\tag{C.17}$$

where we used the definition (C.13) of d_{2n+1} for $n > 1$, i.e.

$$d_{2n+1} = \sum_{k=0}^{n-1} \frac{-d_{2k+1}}{(2n+1-2k)!}. \tag{C.18}$$

This calculation proves the explicit formula (C.14).

For $m \neq 0$, the two trivial cases, where the $E_{n,-m}(1,1,q)$ vanish by definition, are n and m both being either even or both being odd and have to be distinguished. Starting with the former and using the partial differential equation (4.21)

$$\frac{\partial}{\partial z} E_{n,m}(z,1,\tau) = 2\pi i\, E_{n-1,m}(z,1,\tau), \tag{C.19}$$

a similar recursion formula as above, which corresponds to the evaluation of equation (4.26) at one, can be obtained for even $m \geqslant 1, n \geqslant 4$

$$0 = E_{n,-m}(1,1,q)$$

$$= 2\pi i \int_0^1 dz_0 E_{n-1,-m}(z_0,1,\tau)$$

$$= (2\pi i)^2 \int_0^1 dz_0 \int_0^{z_0} dz_1 E_{n-2,-m}(z_1,1,\tau) + 2\pi i E_{n-1,-m}(1,1,q)$$

$$= (2\pi i)^4 \int_0^1 dz_0 \int_0^{z_0} dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 E_{n-4,-m}(z_3, 1, \tau)$$

$$+ \frac{(2\pi i)^3}{3!} E_{n-3,-m}(1, 1, q) + 2\pi i E_{n-1,-m}(1, 1, q)$$

$$= (2\pi i)^{n-2} \int_0^1 dz_0 \int_0^{z_0} dz_1 \dots \int_0^{z_{n-4}} dz_{n-3} E_{2,-m}(z_{n-3}, 1, \tau)$$

$$+ \frac{(2\pi i)^{n-3}}{(n-3)!} E_{3,-m}(1, 1, q) + \cdots + 2\pi i E_{n-1,-m}(1, 1, q). \tag{C.20}$$

This can be solved for $E_{n-1,-m}(1, 1, q)$ using the result from equation (4.23) for m even, i.e.

$$E_{2,-m}(t, 1, q) = \frac{m!}{(2\pi i)^{m-1}} \tilde{\Gamma}\begin{pmatrix} 0 \, m+1 \\ 0 \, 0 \end{pmatrix}; z, \tau) + 2\pi i E_{1,-m}(1, 1, q)z, \tag{C.21}$$

which leads to

$$E_{1,-m}(1, 1, q) = -\frac{m!}{(2\pi i)^m} \omega_2(m+1; \tau) \tag{C.22}$$

upon evaluation at one, such that

$$E_{n-1,-m}(1, 1, q) = m!(2\pi i)^{n-m-2} \left(\frac{\omega_2(m+1; \tau)}{(n-1)!} - \omega_n(m+1; \tau) \right)$$

$$- \frac{(2\pi i)^{n-4}}{(n-3)!} E_{3,-m}(1, 1, q) - \cdots - \frac{(2\pi i)^2}{3!} E_{n-3,-m}(1, 1, q). \tag{C.23}$$

This evaluates e.g. for $n = 4$ to

$$E_{3,-m}(1, 1, q) = m!(2\pi i)^{2-m} \left(\frac{\omega_2(m+1; \tau)}{3!} - \omega_4(m+1; \tau) \right) \tag{C.24}$$

and for $n = 6$ to

$$E_{5,-m}(1, 1, q) = m!(2\pi i)^{4-m} \left(\frac{\omega_2(m+1; \tau)}{5!} - \omega_6(m+1; \tau) \right) - \frac{(2\pi i)^2}{3!} E_{3,-m}(1, 1, q)$$

$$= m!(2\pi i)^{4-m} \left(\frac{\omega_2(m+1; \tau)}{5!} - \omega_6(m+1; \tau) \right)$$

$$- \frac{(2\pi i)^2}{3!} m!(2\pi i)^{2-m} \left(\frac{\omega_2(m+1; \tau)}{3!} - \omega_4(m+1; \tau) \right)$$

$$= m!(2\pi i)^{4-m} \left(\left(\frac{1}{5!} - \frac{1}{3!3!} \right) \omega_2(m+1; \tau) + \frac{1}{3!} \omega_4(m+1; \tau) - \omega_6(m+1; \tau) \right). \tag{C.25}$$

Since this recursion is the same as the one for $m = 0$ up to the factor $m!(2\pi i)^{-m}$ and the higher elliptic zeta values, the explicit formula solving this recursion corresponds to the previous formula given in equation (C.14) and can immediately be written down and proven as

before. The result is that for any natural number $n \geqslant 0$ and even $m \geqslant 1$

$$E_{2n+1,-m}(1,1,q) = m!(2\pi i)^{2n-m} \sum_{k=0}^{n} a_{2k+1}\omega_{2n+2-2k}(m+1;\tau). \tag{C.26}$$

The remaining case is $m \geqslant 1$, $n \geqslant 3$ both odd. The recursive formula can be obtained as before

$$0 = E_{n,-m}(1,1,q) = 2\pi i \int_0^1 dz_0 E_{n-1,-m}(z_0,1,\tau)$$

$$= (2\pi i)^2 \int_0^1 dz_0 \int_0^{z_0} dz_1 E_{n-2,-m}(z_1,1,\tau) + 2\pi i E_{n-1,-m}(1,1,q)$$

$$= (2\pi i)^{n-1} \int_0^1 dz_0 \int_0^{z_0} dz_1 \dots \int_0^{z_{n-3}} dz_{n-2} E_{1,-m}(z_{n-2},1,\tau)$$

$$+ \frac{(2\pi i)^{n-2}}{(n-2)!} E_{2,-m}(1,1,q) + \dots + 2\pi i E_{n-1,-m}(1,1,q). \tag{C.27}$$

As above, we can plug in $E_{1,-m}(t,1,q)$ given by equation (4.22) for m odd, i.e.

$$E_{1,-m}(t,1,q) = \frac{m!}{(2\pi i)^m} \left(\tilde{\Gamma}\!\left(\begin{matrix} m+1 \\ 0 \end{matrix}; z, \tau\right) + 2\zeta_{m+1} z \right), \tag{C.28}$$

and solve for $E_{n-1,-m}(1,1,q)$, which yields the recursive formula

$$E_{n-1,-m}(1,1,q) = m!(2\pi i)^{n-m-2} \left(-2\frac{\zeta_{m+1}}{n!} - \omega_n(m+1;\tau) \right)$$

$$- \frac{(2\pi i)^{n-3}}{(n-2)!} E_{2,-m}(1,1,q) - \dots - \frac{(2\pi i)^2}{3!} E_{n-3,-m}(1,1,q). \tag{C.29}$$

Evaluation of equation (C.28) at one, or considering the q-expansion (B.13), leads to the following connection between the even zeta values and the elliptic zeta values

$$\omega_1(m+1;\tau) = -2\zeta_{m+1}, \tag{C.30}$$

such that the above recursion can be expressed in the more familiar form

$$E_{n-1,-m}(1,1,q) = m!(2\pi i)^{n-m-2} \left(\frac{\omega_1(m+1;\tau)}{n!} - \omega_n(m+1;\tau) \right)$$

$$- \frac{(2\pi i)^{n-3}}{(n-2)!} E_{2,-m}(1,1,q) - \dots - \frac{(2\pi i)^2}{3!} E_{n-3,-m}(1,1,q). \tag{C.31}$$

This yields for $n = 3$

$$E_{2,-m}(1,1,q) = m!(2\pi i)^{1-m} \left(\frac{\omega_1(m+1;\tau)}{3!} - \omega_3(m+1;\tau) \right) \tag{C.32}$$

and for $n = 5$

$$
\begin{aligned}
E_4(1, 1, q) &= m!(2\pi i)^{3-m}\left(\frac{\omega_1(m+1; \tau)}{5!} - \omega_5(m+1; \tau)\right) - \frac{(2\pi i)^2}{3!}E_{2,-m}(1, 1, q) \\
&= m!(2\pi i)^{3-m}\left(\frac{\omega_1(m+1; \tau)}{5!} - \omega_5(m+1; \tau)\right) \\
&\quad - \frac{(2\pi i)^2}{3!}m!(2\pi i)^{1-m}\left(\frac{\omega_1(m+1; \tau)}{3!} - \omega_3(m+1; \tau)\right) \\
&= m!(2\pi i)^{3-m}\left(\left(\frac{1}{5!} - \frac{1}{3!3!}\right)\omega_1(m+1; \tau) + \frac{1}{3!}\omega_3(m+1; \tau) - \omega_5(m+1; \tau)\right).
\end{aligned}
$$

$$(C.33)$$

Thus, the explicit solution can combinatorially be deduced as the ones above, which leads for $n \geqslant 1$ a natural number and $m \geqslant 1$ odd to

$$
E_{2n,-m}(1, 1, q) = m!(2\pi i)^{2n-1-m}\sum_{k=0}^{n}d_{2k+1}\omega_{2n+1-2k}(m+1; \tau). \tag{C.34}
$$

The above results (C.26) and (C.34) for $m \neq 0$ can conveniently be summarised in one single formula: the values $E_{n,-m}(1, 1, q)$ for $n, m \geqslant 1$ can be expressed as the following linear combinations of elliptic zeta values

$$
E_{n,-m}(1, 1, q) = \begin{cases} m!(2\pi i)^{n-1-m}\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}d_{2k+1}\omega_{n+1-2k}(m+1; \tau) & n+m \text{ odd} \\ 0 & n+m \text{ even.} \end{cases} \tag{C.35}
$$

Appendix D. $D_{a,b}^{E}$ on the torus

In the following equation the functions $D_{a,b}^{E}$, defined in equation (4.54), are related to the sums $E_{n,-m}$ as introduced in equation (4.1):

$$
\begin{aligned}
D_{a,b}^{E}(t, q) &= \sum_{l\geqslant 0}D_{a,b}(tq^l) + (-1)^{a+b}\sum_{l>0}D_{a,b}(t^{-1}q^l) + \frac{(4\pi\,\text{Im}(\tau))^{a+b-1}}{(a+b)!}B_{a+b}(u) \\
&= (-1)^{a-1}\sum_{n=a}^{a+b-1}\binom{n-1}{a-1}\frac{(-2)^{a+b-1-n}}{(a+b-1-n)!}\sum_{l>0}\left(\log(|tq^l|)^{a+b-1-n}\text{Li}_n(tq^l)\right. \\
&\quad + (-1)^{a+b}\log(|t^{-1}q^l|)^{a+b-1-n}\text{Li}_n(t^{-1}q^l)\bigg) \\
&\quad + (-1)^{b-1}\sum_{n=b}^{a+b-1}\binom{n-1}{b-1}\frac{(-2)^{a+b-1-n}}{(a+b-1-n)!}\sum_{l>0}\bigg(\log(|tq^l|)^{a+b-1-n} \\
&\quad \times \overline{\text{Li}_n(tq^l)} + (-1)^{a+b}\log(|t^{-1}q^l|)^{a+b-1-n}\overline{\text{Li}_n(t^{-1}q^l)}\bigg) \\
&\quad + D_{a,b}(t) + \frac{(4\pi\,\text{Im}(\tau))^{a+b-1}}{(a+b)!}B_{a+b}(u)
\end{aligned}
$$

$$
= (-1)^a \sum_{n=a}^{a+b-1} \binom{n-1}{a-1} \frac{(-2)^{a+b-1-n}}{(a+b-1-n)!}
$$

$$
\times \sum_{m=0}^{a+b-1-n} \binom{a+b-1-n}{m} \log(|t|)^{a+b-1-n-m} \log(|q|)^m E_{n,-m}(t,1,q)
$$

$$
+ (-1)^b \sum_{n=b}^{a+b-1} \binom{n-1}{b-1} \frac{(-2)^{a+b-1-n}}{(a+b-1-n)!}
$$

$$
\times \sum_{m=0}^{a+b-1-n} \binom{a+b-1-n}{m} \log(|t|)^{a+b-1-n-m} \log(|q|)^m \overline{E_{n,-m}}(t,1,q)
$$

$$
+ D_{a,b}(t) + \frac{(4\pi \ \mathrm{Im}(\tau))^{a+b-1}}{(a+b)!} B_{a+b}(u). \tag{D.1}
$$

The sums $E_{n,-m}$ can be written in terms of the iterated integrals on the torus as shown in subsection 4.1. This provides an explicit translation of $D_{a,b}^E$ on the Tate curve to the elliptic integrals $\tilde{\Gamma}$ on the torus.

Appendix E. Vanishing sums over integration kernels

In this section we show some explicit calculations used in the main part of this article. First, let us show how we can get from equations (4.66), (4.68)–(4.70) to equation (4.71), i.e.

$$
\sum_{i,j} d_i e_j (A_i - B_j) g^{(1)}(A_i - B_j, \tau) d(A_i - B_j) = 0. \tag{E.1}
$$

In order to apply the initial equations, the sum has to be split correctly, the equations have to be plugged in, and the sum pulled together again, such that $\sum_i d_i A_i = 0 = \sum_j e_j B_j$ can be used. Explicitly, this is the following calculation

$$
\sum_{i,j} d_i e_j (A_i - B_j) g^{(1)}(A_i - B_j, \tau) d(A_i - B_j)
$$

$$
= \sum_{i \in I'} d_i A_i \sum_{j \in J} e_j g^{(1)}(A_i - B_j, \tau) d(A_i - B_j) + \sum_{k \in K} d_k A_k
$$

$$
\times \sum_{j \in J \backslash \{k\}} e_j g^{(1)}(A_k - B_j, \tau) d(A_k - B_j)
$$

$$
- \sum_{j \in J'} e_j B_j \sum_{i \in I} d_i g^{(1)}(A_i - B_j, \tau) d(A_i - B_j) - \sum_{k \in K} d_k A_k
$$

$$
\times \sum_{i \in I \backslash \{k\}} d_i g^{(1)}(A_k - A_i, \tau) d(A_k - A_i)
$$

$$
= \sum_{i \in I'} d_i A_i \sum_{j \in J} e_j g^{(1)}(A_i - B_j, \tau) d(A_i - B_j) - \sum_{j \in J'} e_j B_j \sum_{i \in I} d_i g^{(1)}(A_i - B_j, \tau) d(A_i - B_j)
$$

$$+ \sum_{k\in K} d_k A_k \left(\sum_{j\in J\setminus\{k\}} e_j g^{(1)}(A_k - B_j, \tau) d(A_k - B_j) - \sum_{i\in I\setminus\{k\}} d_i g^{(1)}(A_k - A_i, \tau) d(A_k - A_i) \right)$$

$$= \sum_{i\in I'} d_i A_i \left(-d \log(s_B) - c_1 \sum_{j\in J} e_j B_j dB_j \right) - \sum_{j\in J'} e_j B_j \left(\sum_{i\in I} - d \log(s_A) - c_1 \sum_{i\in I} d_i A_i dA_i \right)$$

$$+ \sum_{k\in K} d_k A_k \left(d \log(s_A) - d \log(s_B) + c_1 \sum_{i\in I} d_i A_i dA_i - c_1 \sum_{j\in J} e_j B_j dB_j \right)$$

$$= - \sum_{i\in I} d_i A_i \left(d \log(s_B) + c_1 \sum_{j\in J} e_j B_j dB_j \right) + \sum_{j\in J} e_j B_j \left(\sum_{i\in I} d \log(s_A) + c_1 \sum_{i\in I} d_i A_i dA_i \right)$$

$$= 0. \tag{E.2}$$

A similar calculation leads from equations (4.77), (4.79)–(4.81) to the equation

$$\sum_{i,j} d_i e_j \mathrm{Re}\,(A_i - B_j)\, \tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_k - B_j, \tau)$$

$$= -i\pi\, \mathrm{Re} \left(2m_1 \sum_{i\in I'} d_i A_i - 2m_4 \sum_{j\in J'} e_j B_j + (1 + 2m_2 - 2m_3) \sum_{k\in K} d_k A_k \right), \tag{E.3}$$

which implies equation (4.83) upon taking the real part. Equation (4.84) can be obtained by the same calculation with $\mathrm{Re}(A_i - B_j)$ being replaced by $\mathrm{Im}(A_i - B_j)$. Note that the following sum is valid for the regularised as well as for the unregularised version of $\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau\right)$. With this in mind, let us calculate (E.3) and split the sum as before to findz

$$\sum_{i,j} d_i e_j \mathrm{Re}\,(A_i - B_j)\, \tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_k - B_j, \tau)$$

$$= \sum_{i\in I'} d_i \mathrm{Re}\,(A_i) \sum_{j\in J} e_j \tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau) + \sum_{k\in K} d_k \mathrm{Re}\,(A_k)$$

$$\times \sum_{j\in J\setminus\{k\}} e_j \tilde{\Gamma}_{\mathrm{unreg}}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_k - B_j, \tau) - \sum_{j\in J'} e_j \mathrm{Re}\,(B_j) \sum_{i\in I} d_i \tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau)$$

$$- \sum_{k\in K} d_k \mathrm{Re}\,(A_k) \sum_{i\in I\setminus\{k\}} d_i \tilde{\Gamma}_{\mathrm{unreg}}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_k - A_i, \tau)$$

$$= \sum_{i\in I'} d_i \mathrm{Re}\,(A_i) \left(-2\pi i m_1 + \log(\kappa) - \log(s_B) - \frac{c_1}{2} \sum_j e_j B_j^2 \right)$$

$$- \sum_{j\in J'} e_j \mathrm{Re}\,(B_j) \left(-2\pi i m_4 + \log(\kappa) - \log(s_A) - \frac{c_1}{2} \sum_i d_i A_i^2 \right)$$

$$+ \sum_{k\in K} d_k \mathrm{Re}\,(A_k) \left(-i\pi(1 + 2m_2 - 2m_3) - \log(\kappa) + \log(s_A)\right.$$

$$+ \frac{c_1}{2}\sum_i d_i A_i^2 + \log(\kappa) - \log(s_B) - \frac{c_1}{2}\sum_j e_j B_j^2 \Bigg)$$

$$= \text{Re}\left(\sum_{i\in I} d_i A_i\right)\left(\log(\kappa) - \log(s_B) - \frac{c_1}{2}\sum_{j\in J} e_j B_j^2\right)$$

$$-e_j \text{Re}\left(\sum_{j\in J} B_j\right)\left(\log(\kappa) - \log(s_A) - \frac{c_1}{2}\sum_{i\in I} d_i A_i^2\right)$$

$$-i\pi\,\text{Re}\left(2m_1 \sum_{i\in I'} d_i A_i - 2m_4 \sum_{j\in J'} e_j B_j + (1 + 2m_2 - 2m_3)\sum_{k\in K} d_k A_k\right)$$

$$= -i\pi\,\text{Re}\left(2m_1 \sum_{i\in I'} d_i A_i - 2m_4 \sum_{j\in J'} e_j B_j + (1 + 2m_2 - 2m_3)\sum_{k\in K} d_k A_k\right).$$

$$(\text{E.4})$$

The last calculation of this kind is the step getting from equations (4.85)–(4.87) to equation (4.88), i.e.

$$d\sum_{i,j} d_i e_j \tilde{\Gamma}\!\left(\genfrac{}{}{0pt}{}{2}{0}; A_i - B_j, \tau\right) = 0. \qquad (\text{E.5})$$

Here, we have to apply the above splitting of the sum twice to obtain for $d\tau = 0$

$$d\sum_{i,j} d_i e_j \tilde{\Gamma}\!\left(\genfrac{}{}{0pt}{}{2}{0}; A_i - B_j, \tau\right)$$

$$= \sum_{i,j} d_i e_j g^{(2)}(A_i - B_j, \tau) d(A_i - B_j)$$

$$= \sum_{i\in I'} d_i dA_i \sum_{j\in J} e_j g^{(2)}(A_i - B_j, \tau) - \sum_{j\in J'} e_j dB_j \sum_{i\in I} d_i g^{(2)}(A_i - B_j, \tau)$$

$$+ \sum_{k\in K} d_k dA_k \left(\sum_{j\in J} e_j g^{(2)}(A_k - B_j, \tau) - \sum_{i\in I} d_i g^{(2)}(A_k - A_i, \tau)\right)$$

$$= \sum_{i\in I'} d_i dA_i \left(-2\pi i \frac{\partial}{\partial\tau}\frac{c_1}{2}\sum_{j\in J} e_j B_j^2 - 2\pi i\right.$$

$$\times \sum_{j\in J} e_j g^{(1)}(A_i - B_j, \tau)\frac{\partial}{\partial\tau}(A_i - B_j)\Bigg) - \sum_{j\in J'} e_j dB_j$$

$$\times \left(-2\pi i \frac{\partial}{\partial\tau}\frac{c_1}{2}\sum_{i\in I} d_i A_i^2 - 2\pi i \sum_{i\in I} d_i g^{(1)}(A_i - B_j, \tau)\frac{\partial}{\partial\tau}(A_i - B_j)\right)$$

$$+ \sum_{k\in K} d_k dA_k \left(-2\pi i \frac{\partial}{\partial\tau}\frac{c_1}{2}\sum_{i\in I} d_i A_i^2 - 2\pi i \frac{\partial}{\partial\tau}\frac{c_1}{2}\sum_{j\in J} e_j B_j^2 \right.$$

$$\left. -2\pi i \sum_{j\in J} e_j g^{(1)}(A_k - B_j)\frac{\partial}{\partial\tau}(A_k - B_j) + 2\pi i \sum_{i\in I} d_i g^{(1)}(A_k - A_i)\frac{\partial}{\partial\tau}(A_k - B_j)\right)$$

$$
\begin{aligned}
&= \sum_{ij} d_i e_j \frac{\partial}{\partial \tau} (A_i - B_j) g^{(1)}(A_i - B_j, \tau) d(A_i - B_j) \\
&= 0,
\end{aligned}
\tag{E.6}
$$

where for the last equality, we split the sum once again and proceed as in the calculation of equation (E.2).

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