A note regarding extensions of fixed point theorems involving two metrics via an analysis of iterated functions

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Abstract

The purpose of this work is to advance the current state of mathematical knowledge regarding fixed point theorems of functions. Such ideas have historically enjoyed many applications, for example, to the qualitative and quantitative understanding of differential, difference and integral equations. Herein, we extend an established result due to Rus [Studia Univ. Babeş-Bolyai Math., 22, 1977, 40–42] that involves two metrics to ensure wider classes of functions admit a unique fixed point. In contrast to the literature, a key strategy herein involves placing assumptions on the iterations of the function under consideration, rather than on the function itself. In taking this approach

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we form new advances in fixed point theory under two metrics and establish interesting connections between previously distinct theorems, including those of Rus [Studia Univ. Babeş-Bolyai Math., 22, 1977, 40–42], Caccioppoli [Rend. Acad. Naz. Linzei. 11, 1930, 31–49] and Bryant [Am. Math. Month. 75, 1968, 399–400]. Our results make progress towards a fuller theory of fixed points of functions under two metrics. Our work lays the foundations for others to potentially explore applications of our new results to form existence and uniqueness of solutions to boundary value problems, integral equations and initial value problems.

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1 Introduction

In mathematics, a fixed point of a function is a point that remains unchanged when the function acts on it; that is, we call x a fixed point of a function f whenever $f(x) = x$. The field of fixed point theory aims to establish conditions under which certain classes of functions will admit one, or more, fixed points [18, 19].

Fixed point theory and its applications have a rich history that dates back at least 100 years. A full review is beyond the scope of the this article, but the work of Banach [3], Leray and Schauder [8], Schaefer [11] and Caccioppoli [6] have been influential. More details on fixed point theory may be found in the books of Goebel and Kirk [7] and Zeidler [20].

One of the motivations for developing fixed point theory lies in its applications to various branches of pure and applied mathematics. For example, an important challenge from applied analysis involves developing an understanding of the nature of solutions to certain classes of nonlinear boundary value problems (BVPs). These problems can be equivalently recast as an integral equation and, by defining an appropriate operator between spaces, the problems are reduced to the existence, uniqueness and approximation of fixed points of the operator. Any such knowledge regarding fixed points then delivers insight into the existence, uniqueness and approximation of solutions of the original boundary value problem.

In the 1970s, Rus [10] formulated and developed the following fixed point theorem involving two metrics that are of particular interest for the present article.

Theorem 1 (Rus [10]). *Let X be a nonempty set and let d and δ be two metrics on X such that (X, d) forms a complete metric space. If the mapping $f : X \to X$ is continuous with respect to d on X and:*

$$d(f(x), f(y)) \leqslant c\delta(x, y)\,, \quad \text{for some } c > 0 \text{ and all } x, y \in X; \tag{1.1}$$
$$\delta(f(x), f(y)) \leqslant \alpha\delta(x, y)\,, \quad \text{for some } 0 < \alpha < 1 \text{ and all } x, y \in X; \tag{1.2}$$

then there is a unique $z \in X$ such that $f(z) = z$. In addition, by recursively defining a sequence x_i via $x_0 \in X$ and $x_{i+1} := f(x_i) = f^i(x_0) = f(f^{i-1}(x_0))$, then we have

$$z = \lim_{i\to\infty} f^i(x_0)\,,$$

with respect to d for any $x_0 \in X$.

The two metrics d and δ in Theorem 1 may not necessarily be equivalent and the set X therein is assumed to be complete with respect to the first of these metrics d, but not necessarily complete with respect to the second metric δ. An important example in the study of differential equations involves the set of continuous, real-valued functions on $[0,1]$ denoted by $X = C([0,1])$ and the metrics $d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|$ and

$$\delta(x,y) = \left(\int_0^1 [x(s) - y(s)]^2 \, ds\right)^{1/2} .$$

It is well known [2] that in this case (X, d) is complete, whereas (X, δ) is not complete.

Inequality (1.2) is interpreted as the function f being contractive with respect to the second metric. It is the above properties that have proven to be fruitful in advancing recent results on existence and uniqueness of solutions to BVPs [2, e.g.,]. Recently, Almuthaybiri and Tisdell [1] have discussed applications of fixed point theory with two metrics.

The recursively defined sequence x_i in Theorem 1 provides a mechanism for approximating (or theoretically calculating) the fixed point of f. The sequence x_i is generated by starting with a point $x_0 \in X$ and is expressed in terms of iterations f^i, which denotes the ith iterate of f; that is: $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f \circ f$ and so on.

While Theorem 1 has enjoyed some extensions and variations, for example, in what has been coined as a 'continuation method for contractive maps' by O'Regan and Precup [9], we are unaware of any significant body of fixed point theory that involves extensions of Theorem 1 via an exploration of the iterations of the function f under two metrics, rather than the function f itself.

As we show in Section 2, such a strategy of examining the iterations and introducing appropriate conditions will generate new theorems that advance the state of fixed point theory in novel directions. In addition, our approach forms interesting connections between some of the previous work of Rus [10], Bryant [5] and Caccioppoli [6] that has been seen as being separate. Furthermore, Section 3 shows that our results open up potential applications to differential, difference and integral equations regarding existence, uniqueness and approximations of solutions.

2 Main results

In this section we state, prove and discuss our main results.

The following result forms an extension of Theorem 1 by placing certain conditions on the iterations of f, as opposed to the function f itself. Our theorem was motivated through a reading and analysis of Bryant's paper [5] where iterations were utilized within the context of Banach's fixed point theorem with a single metric. Our result connects the previously separate work of Rus [10] and Bryant [5] in a new and interesting way, and for this reason we name the theorem a 'Rus–Bryant' fixed point theorem.

Theorem 2 (Rus–Bryant). *Let X be a non-empty set, let d and δ be two metrics on X with the pair (X, d) forming a complete metric space. In addition, let $f : X \to X$ be a mapping. If there exist positive numbers c and $\alpha < 1$ and some integer $n \geqslant 1$ such that f^n is continuous on X with respect to d and:*

$$d(f^n(x), f^n(y)) \leqslant c\delta(x, y)\,, \quad \textit{for all } x, y \in X; \tag{2.1}$$
$$\delta(f^n(x), f^n(y)) \leqslant \alpha\delta(x, y)\,, \quad \textit{for all } x, y \in X; \tag{2.2}$$

then f has a unique fixed point $z \in X$. In addition, by recursively defining a sequence x_k via $x_0 \in X$ and $x_{k+1} := f^n(x_k) = f^{kn}(x_0)$, then we have

$$z = \lim_{k\to\infty} f^{kn}(x_0)$$

with respect to d for any $x_0 \in X$.

Proof: We draw on Theorem 1 to prove Theorem 2.

We first show that f^n satisfies the conditions of Theorem 1. Let f and f^n satisfy the conditions of Theorem 2 for some $n \geqslant 1$. Our assumption $f : X \to X$ ensures that for this (and every) value of n we have $f^n : X \to X$. In addition, f^n is continuous with respect to d on X by assumption.

The inequalities (2.1) and (2.2) ensure that f^n satisfies (1.1) and (1.2), respectively.

Thus the function f^n satisfies all of the conditions of Theorem 1 and so f^n has a unique fixed point $z \in X$; that is

$$f^n(z) = z\,. \tag{2.3}$$

Secondly, we show that $f(z)$ is also a fixed point of the function f^n. By applying f to both sides of (2.3) we obtain

$$f(f^n(z)) = f(z)\,,$$

and so

$$f^n(f(z)) = f(z)\,.$$

Thus $f(z) \in X$ is also a fixed point of f^n.

Thirdly, we show our two fixed points of f^n are one and the same. Since Theorem 1 ensures the existence of a *unique* fixed point to our f^n, by uniqueness of solutions it follows that z and $f(z)$ must be identical; that is

$$z = f(z)\,.$$

Hence a fixed point $z \in X$ of f does exist.

Finally, we show that the fixed point $z \in X$ of f is unique. Suppose, on the contrary, that two fixed points to f exist in X and denote them by x and y with $x \neq y$. Thus

$$f(x) = x \quad \text{and} \quad f(y) = y\,. \tag{2.4}$$

On repeatedly applying f to both equations in (2.4) we obtain

$$f^n(x) = x \quad \text{and} \quad f^n(y) = y\,. \tag{2.5}$$

However, we know from the first three parts of the proof that each of the equations in (2.5) have a unique solution. Thus $x = y$, which contradicts our assumption that $x \neq y$. Hence there cannot be two fixed points of f, and thus there must be only one. ♠

Let us now provide some additional insight into Theorem 2.

Remark 3. On comparing the conditions of Theorem 2 with those of Theorem 1, we see that Theorem 1 forms a special case of Theorem 2 when $n = 1$. This provides one aspect of the generality of Theorem 2 and illustrates the advancement when $n \geqslant 2$.

The contraction condition (2.2) on f^n for the case when $n \geqslant 2$ in Theorem 2 is non-trivial as the following example illustrates.

Example 4. Consider the function $f(x) := \cos x$ with $X = (-\infty, \infty)$ and $d(x,y) := |x-y| = \delta(x,y)$. It is well-known that there is no $\alpha < 1$ such that the contraction condition (1.2) will hold for our f on X. However, if we consider the iteration $f^2(x) = \cos(\cos x)$ then (2.2) will hold on X for the choice $\alpha = 1/2$. Thus, the conditions of Theorem 2 hold in this case, but Theorem 1 does not hold.

Example 5. If $n \geqslant 2$, then the contractivity assumption (2.2) of f^n on X with respect to δ does not imply that f is continuous with respect to d or δ on X. To illustrate this, we construct the following example based on Bryant's discussion [5] with $X = (-\infty, \infty)$ and $d(x,y) := |x-y| = \delta(x,y)$. Let $a > 0$ be a constant and consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$$f(x) := \begin{cases} 0, & \text{if} \quad x \leqslant a\,, \\ a, & \text{if} \quad x > a\,. \end{cases}$$

Note that f is not continuous on $(-\infty, \infty)$. However, we calculate $f^2(x) \equiv 0$ on $(-\infty, \infty)$ and thus we see that for $n = 2$ our f^n is continuous and (trivially) contractive on the whole of $(-\infty, \infty)$.

Remark 6. If $d = \delta$ in Theorem 2, then we obtain the result of Bryant [5] as a special case.

Remark 7. In Theorem 2 our approximative sequence x_k is defined via

$$x_{k+1} = f^n(x_k) = f^{kn}(x_0)\,, \quad k = 0, 1, 2, \ldots\,, \quad x_0 \in X\,.$$

Let us briefly mention some aspects of convergence through the following *a priori* estimate on the error between z, the fixed point of f (and also a fixed point of f^n), and our x_k. We have

$$d(x_{k+1}, z) \leqslant c\frac{\alpha^k}{1-\alpha}\delta(x_1, x_0)\,, \quad \text{for } k = 0, 1, 2, \dots\,. \tag{2.6}$$

In addition, we have the *a posteriori* estimate

$$d(x_{k+1}, z) \leqslant \frac{c}{1-\alpha}\delta(x_k, x_{k-1})\,, \quad \text{for } k = 1, 2, \dots\,. \tag{2.7}$$

Both the inequalities (2.6) and (2.7) follow from the results of Rus [10] and so their proofs are omitted for brevity.

As we have seen in the above discussions, the condition (2.2) in Theorem 2 is not always satisfied for iterations f^n with $0 < \alpha < 1$. However, the following result forms one way of potentially navigating this restriction by replacing the constraint $0 < \alpha < 1$ with conditions on the infinite sum involving a sequence of constants. Our following theorem was motivated through a reading and analysis of Caccioppoli's result [6] on fixed points of functions which was developed separately to Banach's fixed point result [3]. Our result brings together the previously disconnected work of Rus [10] and Caccioppoli [6] in a new and interesting way, and for this reason we name the theorem as a 'Rus–Caccioppoli' fixed point theorem.

Theorem 8 (Rus–Caccioppoli). *Let X be a non-empty set, let d and δ be two metrics on X with the pair (X, d) forming a complete metric space. In addition, let $f : X \to X$ be a continuous mapping on X with respect to d. If there is a constant $c > 0$ and a sequence of positive terms α_m such that*

$$d(f^m(x), f^m(y)) \leqslant c\delta(x, y)\,, \quad \textit{for all } x, y \in X \textit{ and } m = 1, 2, \dots\,; \tag{2.8}$$

$$\delta(f^m(x), f^m(y)) \leqslant \alpha_m\delta(x, y)\,, \quad \textit{for all } x, y \in X \textit{ and } m = 1, 2, \dots\,; \tag{2.9}$$

$$\sum_{m=1}^{\infty} \alpha_m < \infty\,; \tag{2.10}$$

then f *has a unique fixed point* $z \in X$ *that satisfies*

$$z = \lim_{k\to\infty} f^{kn}(x_0))\,,$$

for any $x_0 \in X$ *where the limit is taken with respect to the* d *metric and* n *is sufficiently large.*

Proof: Perhaps the most straightforward approach to the proof is to illustrate that the conditions of Theorem 2 are satisfied for some integer $n \geqslant 1$.

Since our series of positive terms in (2.10) converges, we know by the mth term test that

$$\lim_{m\to\infty} \alpha_m = 0\,.$$

Thus, there must exist some positive integer n such that $\alpha_n < 1$.

For the value of n for which $\alpha_n < 1$, our assumptions ensure that the mapping f^n satisfies the conditions of Theorem 2. The continuity of f^n on X is ensured by the assumption that f is continuous on X. Thus, by Theorem 2, f has a unique fixed point $z \in X$ such that $d(x_k, z) \to 0$ as $k \to \infty$, where the sequence $\{x_k\}_{k=0}^{\infty}$ is given by $x_k = f^n(x_{k-1})$ for $k = 1, 2, \ldots$ and x_0 is any arbitrary point in X. ♠

Remark 9. In Theorem 8 it is not necessary for inequalities (2.8) and (2.9) to hold for each $m = 1, 2, \ldots$ for the proof to work. For example, if there is some $i > 1$ such that inequalities (2.8) and (2.9) hold for all $m \geqslant i$, then we can always choose an $n \geqslant i$ sufficiently large such that $\alpha_n < 1$ with the proof running along the same lines as above.

Remark 10. If $d = \delta$ in Theorem 8, then the theorem reduces to the classical result of Caccioppoli [20, Ch. 1].

For a given $f : Y \to Y$ it is sometimes the case that f is not contractive on the whole of the set Y. Instead it may be possible that f is contractive only on a subset of Y. With this in mind, the following theorem forms a version of Theorem 2 for balls.

In what follows, we use the notation for a closed ball in X with radius $R > 0$ and centre $x_0 \in X$:

$$\overline{B_R(x_0)} := \{x \in X : d(x, x_0) \leqslant R\}.$$

Theorem 11 (Rus–Bryant Ball Theorem)**.** *Let* Y *be a non-empty set, let* d *and* δ *be two metrics on* Y *with the pair* (Y, d) *forming a complete metric space. In addition, let* $f : \overline{B_R(x_0)} \to Y$ *be a mapping. If there exists a positive constant* $\alpha < 1$ *and a positive integer* n *such that* f^n *is continuous on* $\overline{B_R(x_0)}$ *with respect to* d *and:*

$$\begin{aligned} \delta(x, x_0) &\leqslant R, \quad \textit{for all } x \in \overline{B_R(x_0)}; && (2.11)\\ \delta(f^n(x), f^n(y)) &\leqslant \alpha\delta(x, y), \quad \textit{for all } x, y \in \overline{B_R(x_0)}; && (2.12)\\ d(f^n(x_0), x_0) &\leqslant (1 - \alpha)R; && (2.13)\\ d(f^n(x), f^n(y)) &\leqslant \alpha\delta(x, y), \quad \textit{for all } x, y \in \overline{B_R(x_0)}; && (2.14) \end{aligned}$$

then f *has a unique fixed point in* $\overline{B_R(x_0)}$.

Proof: We show that the conditions of our Theorem 2 hold for $X = \overline{B_R(x_0)}$ and $c = \alpha$.

We first prove that f^n maps $\overline{B_R(x_0)}$ to itself. For any $x \in \overline{B_R(x_0)}$ consider

$$\begin{aligned} d(f^n(x), x_0) &\leqslant d(f^n(x), f^n(x_0)) + d(f^n(x_0), x_0)\\ &\leqslant \alpha\delta(x, x_0) + (1 - \alpha)R\\ &\leqslant \alpha R + (1 - \alpha)R\\ &= R. \end{aligned}$$

Since $(\overline{B_R(x_0)}, d)$ is a closed subset of a complete metric space (Y, d), the pair $(\overline{B_R(x_0)}, d)$ is also complete and thus all of the conditions of Theorem 2 hold with $X = \overline{B_R(x_0)}$ and f^n. Our result now follows from Theorem 2. ♠

Remark 12. The conclusion of Theorem 11 remains true if the conditions (2.11), (2.13), and (2.14) are respectively replaced by:

$$\begin{aligned}
\delta(x, x_0) &\leqslant c_1 d(x, x_0)\,, \quad \text{for all } x \in \overline{B_R(x_0)}\,; &(2.15)\\
d(f^n(x_0), x_0) &\leqslant (1 - cc_1)R\,; &(2.16)\\
d(f^n(x), f^n(y)) &\leqslant c\delta(x, y)\,, \quad \text{for all } x, y \in \overline{B_R(x_0)}\,; &(2.17)
\end{aligned}$$

where the positive constants c_1 and c satisfy $c_1 c < 1$. To see this, for any $x \in \overline{B_R(x_0)}$ consider

$$\begin{aligned}
d(f^n(x), x_0) &\leqslant d(f^n(x), f^n(x_0)) + d(f^n(x_0), x_0)\\
&\leqslant c\delta(x, x_0) + (1 - cc_1)R\\
&\leqslant cc_1 d(x, x_0) + (1 - cc_1)R\\
&\leqslant cc_1 R + (1 - cc_1)R\\
&= R\,.
\end{aligned}$$

Remark 13. If $n = 1$ in Theorem 11, then we obtain the standard version of Theorem 1 in balls.

Remark 14. When $\delta \leqslant d$ on $\overline{B_R(x_0)} \times \overline{B_R(x_0)}$, then the conditions of Theorem 11 imply that f^n is also a contraction on $\overline{B_R(x_0)}$ with respect to d. To see this, consider (2.12) and (2.14) which lead to

$$\begin{aligned}
\delta(f^n(x), f^n(y)) &\leqslant d(f^n(x), f^n(y))\\
&\leqslant \alpha\delta(x, y)\\
&\leqslant \alpha d(x, y)\,,
\end{aligned}$$

and thus f^n is a contraction on $\overline{B_R(x_0)}$ with respect to d.

Remark 15. If we compare the conditions of Theorem 11 with those of Theorem 2, then we see an interesting trade-off occurring. The contraction condition is assumed to only hold on a ball (rather than, say, on an unbounded set), but this is counterbalanced with the introduction of the additional conditions (2.11), (2.13) and (2.14). In particular, (2.13) is interpreted as a condition on the iteration f^n not to move the centre x_0 of the ball too much. Essentially, this is to ensure the invariance of f^n; that is, $f^n : \overline{B_R(x_0)} \to \overline{B_R(x_0)}$.

Sometimes fixed point problems of interest involve a parameter p that comes from a 'parametric space' P. In what follows, the function f is replaced by a 'family' of mappings that depends on both P and X.

Theorem 16 (Rus–Bryant with parameter)**.** *Let X and P be non-empty sets, let d be a metric on X with the pair (X, d) forming a complete metric space, and let δ be a metric on X. In addition, let $f_p : P \times X \to X$ be a family of mappings. If, for each fixed $p \in P$, there exists some positive constants c and $\alpha < 1$ (α is assumed to be independent of p) and a positive integer n such that f_p^n is continuous on X with respect to d and:*

$$\begin{aligned} d(f_p^n(x), f_p^n(y)) &\leqslant c\delta(x,y)\,, \quad \textit{for all } x,y \in X; \\ \delta(f_p^n(x), f_p^n(y)) &\leqslant \alpha\delta(x,y)\,, \quad \textit{for all } \ x,y \in X; \end{aligned}$$

then, for each $p \in P$, our f_p has a unique fixed point x_p. In addition, if for the above value of n and some fixed $q \in P$ we have

$$\lim_{p\to q} f_p^n(x) = f_q^n(x)\,, \quad \textit{with respect to } \delta \textit{ for all } x \in X \tag{2.18}$$

then,

$$\lim_{p\to q} x_p = x_q$$

with respect to δ.

Proof: For each fixed $p \in P$, our assumptions ensure that the conditions of Theorem 2 are satisfied. Thus, for every $p \in P$ there exists a unique fixed point of x_p of f_p (and f_p^n).

Now, if x_p and x_q represent fixed points of our f_p^n for two values p and q from our parameter space P, then

$$\delta(x_p, x_q) \;=\; \delta(f_p^n(x_p), f_q^n(x_q))$$

$$\begin{aligned} &\leqslant \delta(f_p^n(x_p), f_p^n(x_q)) + \delta(f_p^n(x_q), f_q^n(x_q)) \\ &\leqslant \alpha\delta(x_p, x_q) + \delta(f_p^n(x_q), f_q^n(x_q)) \,. \end{aligned}$$

We therefore have

$$\delta(x_p, x_q) \leqslant \frac{\delta(f_p^n(x_q), f_q^n(x_q))}{1-\alpha} \,. \tag{2.19}$$

From (2.18), the right side of (2.19) tends to zero as $p \to q$, thus completing our proof. ♠

3 Concluding remarks

In this work we have made a contribution to fixed point theory through the use of iterations and two metrics. In doing so we have formed new connections between the classical results of Rus [10], Bryant [5] and Caccioppoli [6]. There are a number of variations of our fixed point approaches that currently remain open, including: a Rus–Caccioppoli result in balls; a Rus–Caccioppoli result with a parameter.

At the time of writing, the question of applications of our new fixed point theorems with iterations and two metrics remains open. There are opportunities for interested readers to explore how these results might be applied to differential, difference and integral equations, including boundary value problems and initial value problems. For related problems, we refer the reader to the works of Tisdell et al. [12, 13, 14, 15, 16, 17].

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