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An Iterative Method for Solving the Matrix Equation $X - A^*XA - B^*X^{-1}B = I$

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Abstract

In this paper we study iterative computing a positive definite solution of the matrix equation $X - A^*XA - B^*X^{-1}B = I$. We propose an iterative method for finding a positive definite solution of the considered equation. The theoretical results are illustrated by numerical examples.

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1 Introduction

We investigate the nonlinear matrix equation

$$X - A^*XA - B^*X^{-1}B = I\,, \tag{1}$$

where A, B are $n \times n$ complex matrices, I (or I_n) is the $n \times n$ identity matrix, and A^* denotes the conjugate transpose of A.

Eq. (1) has been introduced by Ali in [1], where an iterative method for computation a positive definite solution is proposed. A necessary and sufficient condition for the existence of a positive definite solution of Eq. (1) has been derived and a basic fixed point iteration has been proposed in [2]. In [3] by using the fixed point theorem for mixed monotone operator in a normal cone Gao has proved that the equation $X - A^*X^pA - B^*X^{-q}B = I$ with $0 < p, q < 1$ always has the unique positive definite solution. In [4] the similar equation

$$X - A^*XA + B^*X^{-1}B = I \tag{2}$$

has been investigated. We interpret (1) and (2) as linearly perturbed equations of the well-known and studied equations $X - B^*X^{-1}B = I$ [5, 6] and $X + B^*X^{-1}B = I$ [6, 7, 8, 9, 10], respectively.

In addition, there are some contributions in the literature to the solvability and numerical solutions of the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = I$ [11, 12, 13, 14].

Motivated by [1, 2, 3, 4], we study Eq. (1) for finding a positive definite solution as propose an iterative method.

Throughout this paper, we denote by $\mathcal{C}^{n\times n}$ the set of $n \times n$ complex matrices, by $\mathcal{H}^n$ the set of $n \times n$ Hermitian matrices, by $\rho(A)$ the spectral radius, by $\|A\|$ the spectral norm $\left(\|A\| = \sqrt{\rho(A^*A)}\right)$. $A > 0$ ($A \geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If $A - B > 0$ (or $A - B \geq 0$) we write $A > B$ (or $A \geq B$). For $N \geq M > 0$ we use $[M, N]$ to denote the set of matrices $\{X : M \leq X \leq N\}$.

2 Preliminaries

In this section we give some preliminary results.

In [1], the necessary conditions for existence of a positive definite solution and its lower bound have been obtained.

Theorem 1 *[1, Theorem 2.1.] Let X be a positive definite solution of Eq. (1). Then*

(a) $\rho(A) < 1$,

(b) $\rho(X^{-1}B) < 1$,

(c) $X \geq M$, *where M is the unique positive definite solution of the equation $X - A^*XA = I$.*

In [2], it has been proven that $\rho(A) < 1$ is a necessary and sufficient condition for the existence of a positive definite solution of Eq. (1). Moreover, it has been obtained an upper bound of all the solutions.

Theorem 2 *[2, Theorem 2.] Eq. (1) has a positive definite solution X, if and only if $\rho(A) < 1$. Moreover, the all positive definite solutions are in $[M, N]$, where M and N are the unique solutions of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively.*

Ali in [1] has investigated the iterative method

$$\begin{cases} X_0 = I, \quad Y_0 = \beta I, \quad \beta > 1 \\ X_{k+1} = I + A^*X_kA + B^*Y_k^{-1}B, \quad k = 0, 1, \ldots \\ Y_{k+1} = I + A^*Y_kA + B^*X_k^{-1}B \end{cases} \tag{3}$$

for computing a positive definite solution of Eq. (1) based on the mixed monotone operator $G(X, Y) = I + A^*XA + B^*Y^{-1}B$. The sequences $\{X_k\}$ and $\{Y_k\}$ defined by (3) have the following properties

$$X_0 \leq X_1 \leq \cdots \leq X_k \leq Y_k \leq \cdots \leq Y_1 \leq Y_0. \tag{4}$$

Moreover, it was proven that $\{X_k\}$ and $\{Y_k\}$ with $\beta \geq \frac{1+\|B\|^2}{1-\|A\|^2}$ converge to a unique positive definite solution of Eq. (1) under condition $\|A\|^2 + \|B\|^2 < 1$.

In [2], it has been noted that the iterative method (3) can be used with $X_0 = M$ and $Y_0 = N$, where the matrices M and N are from Theorem 2. Moreover, it has been concluded that, if $\lim_{k\to\infty} \|Y_k - X_k\| = 0$, then Eq. (1) has a unique positive definite solution.

Hasanov in [2] has considered the basic fixed point iteration (BFPI):

$$Z_{k+1} = I + A^*Z_kA + B^*Z_k^{-1}B, \quad k = 0, 1, \ldots, \quad Z_0 \in [X_0, Y_0], \tag{5}$$

where X_0 and Y_0 are initial value in method (3). The sequences $\{Z_k\}$, $\{X_k\}$ and $\{Y_k\}$ defined by (5) and (3) have the following properties $X_k \leq Z_k \leq Y_k$, $k = 0, 1, \ldots$.

3 An iterative method

Here, we consider a modification of the iterative method (3), which is a partially inverse free variant.

Let M and N be the unique solutions of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively. We consider

$$\begin{cases} X_0 = M, \quad Y_0 = N, \text{ (or } X_0 = I, Y_0 = \beta I), \quad V_0 = Y_0^{-1}, \\ V_{k+1} = V_k(2I - Y_kV_k), \\ X_{k+1} = I + A^*X_kA + B^*V_{k+1}B, \quad k = 0, 1, \ldots \\ Y_{k+1} = I + A^*Y_kA + B^*X_k^{-1}B. \end{cases} \tag{6}$$

Lemma 1 *[9, Lemma 3.2] Let C and P be Hermitian matrices of the same order and let $P > 0$. Then $CPC + P^{-1} \geq 2C$.*

Theorem 3 *The sequences V_k, X_k and Y_k generated by iterative method (6) have the following properties*

(i) $X_0 \leq X_1 \leq \ldots \leq X_k \leq Y_k \leq \ldots \leq Y_1 = Y_0, \quad k = 0, 1, \ldots,$

(ii) $V_0 \leq V_1 \leq \ldots \leq V_{k+1} \leq Y_k^{-1}, \quad k = 0, 1, \ldots,$

(iii) $\lim_{k\to\infty} X_k = \bar{X} \leq \bar{Y} = \lim_{k\to\infty} Y_k, \quad \lim_{k\to\infty} V_k = \bar{Y}^{-1}.$

Proof. We prove the theorem by induction.

We have $X_0 = M \leq N = Y_0$ by Theorem 2. We compute

$$\begin{aligned} V_1 &= N^{-1}(2I - NN^{-1}) = N^{-1} = V_0, \\ X_1 &= I + A^*MA + B^*V_1B \geq I + A^*MA = M = X_0, \\ Y_1 &= I + A^*NA + B^*M^{-1}B = N = Y_0. \end{aligned}$$

We have by Lemma 1 that

$$V_1 = 2V_0 - V_0Y_0V_0 \leq Y_0^{-1} = N^{-1}$$

and

$$Y_1 - X_1 = A^*(N - M)A + B^*(M^{-1} - V_1)B \geq B^*(M^{-1} - N^{-1})B \geq 0.$$

Therefore, $V_0 \leq V_1 \leq Y_0^{-1}$, $X_0 \leq X_1 \leq Y_1 \leq Y_0$.

Assume that $V_{k-1} \leq V_k \leq Y_{k-1}^{-1}$ and $X_{k-1} \leq X_k \leq Y_k \leq Y_{k-1}$. Thus, we have

$$\begin{aligned} Y_{k+1} - Y_k &= A^*(Y_k - Y_{k-1})A + B^*(X_k^{-1} - X_{k-1}^{-1})B \leq 0, \\ V_{k+1} - V_k &= V_k(V_k^{-1} - Y_k)V_k \geq V_k(V_k^{-1} - Y_{k-1})V_k \geq 0, \end{aligned}$$

and

$$X_{k+1} - X_k = A^*(X_k - X_{k-1})A + B^*(V_{k+1} - V_k)B \geq 0.$$

By Lemma 1, we have

$$V_{k+1} = 2V_k - V_kY_kV_k \leq Y_k^{-1}.$$

Thus,

$$\begin{aligned} Y_{k+1} - X_{k+1} &= A^*(Y_k - X_k)A + B^*(X_k^{-1} - V_{k+1})B \\ &\geq B^*(Y_k^{-1} - V_{k+1})B \geq 0. \end{aligned}$$

Hence, $X_k \leq X_{k+1} \leq Y_{k+1} \leq Y_k$ and $V_k \leq V_{k+1} \leq Y_k^{-1}$ for $k = 1, 2, \ldots$. Thus, the limits $\lim_{k\to\infty} X_k$, $\lim_{k\to\infty} Y_k$, and $\lim_{k\to\infty} V_k$ exist, and $\lim_{k\to\infty} X_k \leq \lim_{k\to\infty} Y_k$, $\lim_{k\to\infty} V_k = (\lim_{k\to\infty} Y_k)^{-1}$.

4 Numerical experiments

In this section we carry out numerical experiments for computing the positive definite solution of Eq. (1) by iterative methods (3), (5), and (6) with $X_0 = I$, $Y_0 = \beta I$, and $Z_0 = \frac{X_0+Y_0}{2}$, where $\beta = \frac{1+\|B\|^2}{1-\|A\|^2}$ (or $X_0 = M$, $Y_0 = N$, where M and N are the unique solutions of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively).

For the stopping criterion we take $\|Y_k - X_k\| \leq 10^{-10}$ for methods (3) and (6), and $\|Z_k - Z_{k-1}\| \leq 10^{-10}$ for method (5), where k is the number of iterations. We use the notation $res(X) = \|X - A^*XA - B^*X^{-1}B - I\|$ and compute

- $res(\widetilde{X}_k)$ for methods (3) and (6), where $\widetilde{X}_k = \frac{Y_k+X_k}{2}$,
- $res(Z_k)$ for method (5).

Example 1 *We consider Eq. (1) with*

$$A = \frac{1}{56}\begin{pmatrix} 1 & 5 & 3 & 2 \\ -1 & -6 & 3 & 4 \\ -4 & 3 & 7 & 5 \\ 1 & 8 & 2 & 1 \end{pmatrix}, \quad B = \frac{1}{70}\begin{pmatrix} 7 & 9 & 6 & 8 \\ 7 & 5 & 8 & 3 \\ 9 & 8 & 6 & 7 \\ 11 & 5 & 9 & 3 \end{pmatrix}.$$

In Table 1 we report the results of experiment for Example 1 by using iterative methods (3), (5) and (6).

Table 1: Numerical results for Example 1.

Method	k	$\|Y_k - X_k\|$ or $\|Z_k - Z_{k-1}\|$	$res(\widetilde{X}_k)$ or $res(Z_k)$
by $X_0 = I$ and $Y_0 = \beta I$			
(3)	11	$8.8594e-11$	$4.5543e-13$
(5)	10	$4.0638e-12$	$4.3280e-11$
(6)	12	$1.4069e-11$	$3.0624e-14$
by $X_0 = M$ and $Y_0 = N$			
(3)	11	$5.8609e-11$	$3.6315e-13$
(5)	10	$3.7103e-11$	$3.4837e-12$
(6)	11	$6.6076e-11$	$3.0991e-13$

Example 2 *We consider Eq. (1) with*

$$A = \frac{1}{200}\begin{pmatrix} 41 & 15 & 23 & 35 & 66 \\ 25 & 12 & 27 & 45 & 21 \\ 23 & 27 & 28 & 16 & 24 \\ 15 & 45 & 16 & 52 & 65 \\ 66 & 21 & 24 & 65 & 35 \end{pmatrix}, B = \frac{1}{30}\begin{pmatrix} 23 & 21 & 23 & 25 & 32 \\ 21 & 45 & 60 & 42 & 33 \\ 23 & 24 & 34 & 18 & 17 \\ 13 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 26 \end{pmatrix}.$$

In Table 2 we report the results of experiment for Example 2 by using iterative methods (3), (5) and (6).

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Table 2: Numerical results for Example 2.

Method	k	$\|Y_k - X_k\|$ or $\|Z_k - Z_{k-1}\|$	$res(\widetilde{X}_k)$ or $res(Z_k)$
by $X_0 = I$ and $Y_0 = \beta I$			
(3)	148	$9.4222e-11$	$3.7667e-15$
(5)	90	$7.8252e-11$	$5.7602e-11$
(6)	151	$8.3939e-11$	$7.7527e-15$
by $X_0 = M$ and $Y_0 = N$			
(3)	141	$5.6876e-11$	$8.7894e-15$
(5)	73	$8.1558e-11$	$6.0036e-11$
(6)	141	$8.9929e-11$	$1.2240e-14$

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