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GROUND STATES AND GIBBS MEASURES OF λ-MODEL ON CAYLEY TREE OF ORDER TWO

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This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Prof. Farrukh Mukhamedov

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Declaration of Original Work

I, Rauda Humaid Al Shamsi, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Ground States and Gibbs Measures of λ-Model on Cayley Tree of Order Two"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Farrukh Mukhamedove, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

Statistical mechanics deals with the average properties of a mechanical system. Some examples are; the water in a kettle, the atmosphere inside a room and the number of atoms in a magnet bar. These kinds of systems are made up of a large number of components, usually molecules. The observer has restricted power to consider all the components. All that can be done is to specify a few average quantities of the system such as its density, pressure or temperature. The main objective of statistical mechanics is to predict the relationship between the observable macroscopic properties of the system, given only a knowledge of the microscopic interactions between the components. The present thesis is devoted to a model whose interacting molecules are located on nearest neighbor vertices of a Cayley tree. In this thesis, ground states and Gibbs measures of λ-model on a Cayley tree of order two are investigated. This investigation is closely related to the phase transitions phenomenon for lattice models on trees, by considering the model where spin has only three values. This kind of model aims to describe all its ground states and study phase transition phenomena by using Gibbs measures.

Keywords: Cayley tree, Gibbs measures, statistical mechanics.

Title and Abstract (in Arabic)

الحالات الدنيا و مقاييس غيبس لنموذج لمدا على شجرة كايلي من الرتبة الثانية

الملخص

من المعروف أن الميكانيكا الإحصائية تهتم بمتوسط خصائص النظام الميكانيكي. بعض الأمثلة مثل الماء في غلاية، والغلاف الجوي داخل الغرفة وعدد الذرات في قضيب مغناطيسي. تتكون هذه الأنواع من الأنظمة من عدد كبير من المكونات، وعادةً ما تكون جزيئات. وقد قيد المراقب القدرة على النظر في جميع المكونات. كل ما يمكننا القيام به هو تحديد عدد قليل من كمية النظام مثل الكثافة أو الضغط أو درجة الحرارة. الهدف الرئيسي للميكانيكا الإحصائية هو التنبؤ بالعلاقة بين الخواص العيانية التي يمكن ملاحظتها في النظام، مع العلم فقط بالتفاعلات المجهرية بين المكونات. تُخصص هذه الأطروحة لنموذج توجد جزيئات التفاعل فيه على أقرب الجيران من شجرة كايلي. في هذه الأطروحة، نقوم بالتحقيق في الحالات الدنيا ومقاييس جيبس لنموذج λ على شجرة كايلي حسب الطلب. يرتبط هذا التحقيق ارتباطًا وثيقًا بظاهرة التحولات الطورية للنماذج الشبكية على الأشجار. نحن نعتبر النموذج الذي يحتوي على رقم مغزلي المكون من ثلاث قيم فقط. بالنسبة لهذا النوع من النماذج، سنقوم بوصف جميع حالاتها الدنيا ودروس المرحلة الانتقالية من خلال إجراءات جيبس.

مفاهيم البحث الرئيسية: شجرة كايلي، مقاييس غيبس، الميكانيكا الاحصائية.

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I would like to thank UAEU Mathematics department for accepting me in mathematics master program even though I have different specialization.

I must also acknowledge the acquisition of new knowledge that helped me to realize innovative solutions that serve many topics in mathematical physics.

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Dedication

To my mother

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Chapter 1: Preface

1.1 Introduction

In a microscopic system of matter that has infinitely many degrees of freedom, only a mathematical framework can explain the microscopic behavior of the system (matter). Statistical mechanics laws play an important role by describing the microscopic forces between particles and predicting the relations between observable macroscopic properties of the system [1]. For example, the microscopic system's behavior such as freezing and boiling of water can well-described by simplifying the structure model of a system and atoms interactions contained in the object itself. The main problem in equilibrium statistical mechanics is to describe all limiting Gibbs distributions corresponding to a given Hamiltonian. The existence of many Gibbs measures means that phase transition exists. Also the Gibbs measures of the phase diagram of a Hamiltonian is close to the phase diagram of isolated ground states of this Hamiltonian [2]. The phase transition is closely connected to the description of ground states of Hamiltonian; namely the fortress of the ground state is the first indicator of the phase transition. Many mathematical models can describe the phenomenon of phase transition. Most of them are represented in a square lattice. Point out that the investigations of models on square lattices are very complicated and tricky. Therefore, it is natural to study these models over lattices, which approximate the original ones. One of the simplest lattices is the Cayley tree. This tree is mainly not physical, but it predicts physical phenomena, which exhibit unlike in integer lattices. The main question if this theory is to establish a phase transition for given Hamiltonian on the tree.

Ising model is the simplest model in statistical mechanics which is the most theoretical interest and has many practical applications. There are many papers [3, 4] describing the Ising model on the Cayley tree, but the complete result about all Gibbs measures is lacking. In [5] Ising model considered with next-nearest neighbor interactions on the Cayley tree that describe the phase diagram for a system that contains only two values of spin.

Later on, the Potts model is considered a more complicated structure then Ising one. It is describing the phase transition for many particles system. The Potts model is a generalization of the Ising model, but it is not well studied on the Cayley tree. For example, for periodic Gibbs measures, there are no result about the full description of it in the Potts model on the Cayley tree [3].

On the other hand, it is natural to consider a model that is is more complicated than Potts's one. Therefore, in Mukhamedov[6] it was proposed to study the so-called λ-model on Cayley tree see [7, 6, 5, 8]. In that model, many possible interactions (nearest-neighbor) are taken into account. In the mentioned paper of especial kind of λ-interactions, its disordered phase has been studied [7] and some of its algebraic properties were investigated. Furthermore, Mukhamedov[6] has described ground states of the λ-model on the Cayley tree of order two. This model is much more general than the Potts model and exhibits an interesting structure of ground states. In the mentioned work Mukhamedov[6], the interaction was taken as

$$\lambda(i,j) = \begin{cases} a & : |i-j| = 2 \\ b & : |i-j| = 1 \\ c & : i = j \end{cases} \tag{1.1}$$

with a set of all possible values of spin $\Phi = \{1,2,3\}$. In this present thesis, the main objective is to consider a more general λ-interaction than (1.1). noticing that $\lambda - model$ is a generalization model of Ising and Potts models and the function of this model is more complex than others on the Cayley tree. The λ-interactions in this research is more general than (1.1) and has established the phase transitions. A detailed definition will be given in the next chapter, which will allow the establishment of the existence of phase transitions.

1.2 Objectives

There are two main objectives of this thesis.

- To describe ground states of λ-model on Cayley tree of order two.
- To establish rigorously phase transition for λ-model by using Gibbs measures.

1.3 Overview

This thesis consists of three chapters.

As a preliminary, Chapter 1 is about the basic notations and setting that are used in this research. It explains the main definitions of a phase transition, Cayley tree and Gibbs measure. Also its comperes between Ising and Potts model on the Cayley tree and the importance of λ-model.

Chapter 2 describes the ground states on λ-model to establish the phase transition. In Chapter 3, Gibbs states are constructed and associated with the model, also translation invariant Gibbs measure is described. By using Gibbs measures the phase transition problem studied in this chapter; Namely, the existence of translation invariant Gibbs measures found, which allows to establish the phase transitions.

Chapter 2: Cayley Tree and λ-Model

2.1 Phase Transition

Phase transition can only occur from one phase to another phase. It is the change of the thermodynamic system of matter from one phase to another for example the transition between solid and liquid states of matter. It can be known at what specific system properties the change of matter from its state to another state occurs. According to thermodynamics laws, it must occur only on a lower energy "maximum entropy" of the state of a system. In the study of the theory of phase transition, the description of Gibbs measure of a given Hamiltonian has brought us to a fundamental problem of equilibrium statistical physics, which is describing all limiting Gibbs measures of a given Hamiltonian on a lattice.

Phase transition will occur if there is the existence of multiple Gibbs measures by a given Hamiltonian from this equilibrium state. Mathematically, it can be said that when there exists non-uniqueness of Gibbs measure, the phase transition occurs. It is mostly occurs at low temperature. So if the exact value of temperature T^c was found then phase transition occurs for all $T < T^c$. Note that T^c is called a critical value of temperature.

In this research, it was proposed a model that can describe the phase transition of matter that consists of particles having three different possible values of spin distributed among Cayley tree.

2.2 Cayley Tree

In general, Cayley tree is a simple finite undirected connected graph with no cycles $G = (V, L)$, where V is a set of all vertices and L is a set of all edges [3, 6] (Figure 2.1).

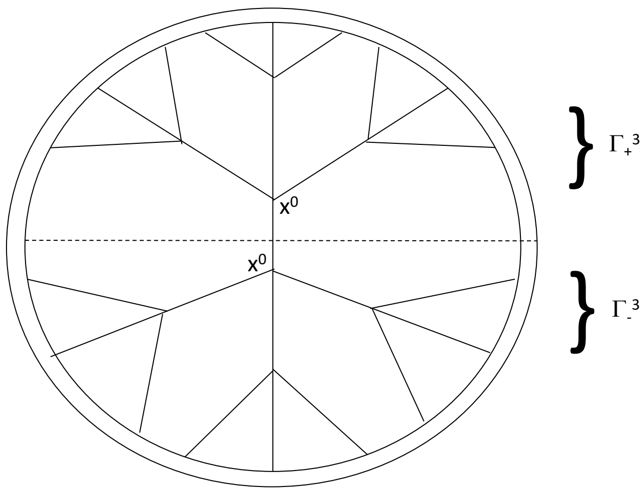


Figure 2.1: Cayley tree of order 3 for G^3

In present study, the assumption of the simi-infinite tree of order two is taken into account denoting by $\Gamma_+^k = (V, L)$ as shown in Figure 2.2 where:

k: is the order of tree which is equal to two.

$+$: simi-infinite tree.

V: the set of all vertices.

L: the set of all edges.

x^0: the root vertix of tree that have k=2 edges. Where all other vertices have k+1 edges.

x, y: is the nearest neighbors vertices. $\langle x, y \rangle = l$: the edge connecting x and y "if there exist". The path x to y is the collection of the pair $< x, x_1, < x_1, x_2 >, .., < x_{d-1}, y >$. $d(x, y)$: the length of the shortest path from x,y

$$W_n = \{x \in V \mid d(x, x^0) = n\}$$

$$V_n = \Sigma_{m=1}^n W_m$$

$$L_n = \{l =< x, y > \in L \mid x, y \in V_n\}$$

The definition of the set of direct successors is:

$$S(x) = \{y \in W_{n+1} : d(x,y) = 1\}, x \in W_n$$

Note that any vertex $x \neq x^0$ has k direct successors and x^0 has k+1.

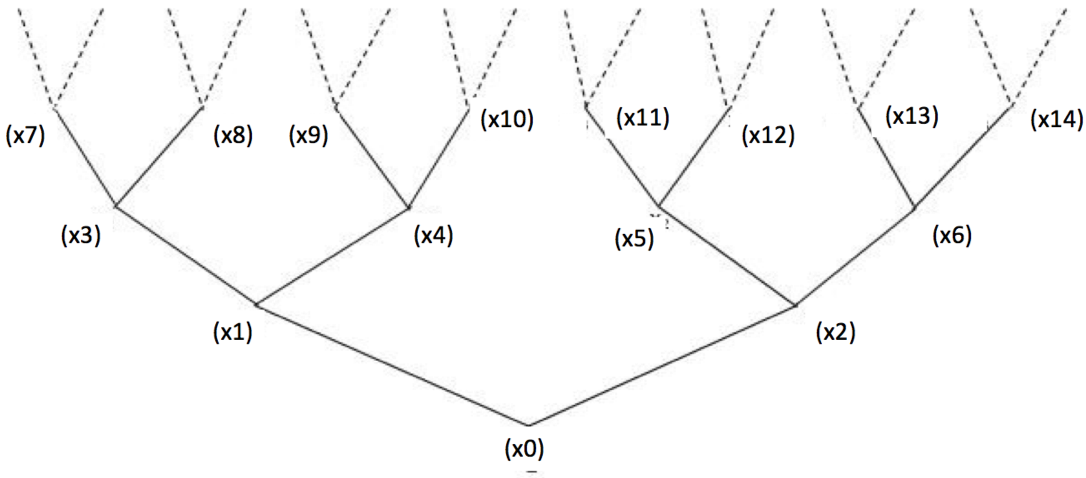


Figure 2.2: Γ^2_+

Now the coordinate structure in Γ^k_+ was introduced as follow. Every vertex x (except for x^0) of Γ^k_+ has coordinates $(i_1,\ldots,i_n)$, here $i_m \in \{1,\ldots,k\},\ 1 \le m \le n$ and for the vertex x^0 put (0) (Figure 2.3). Namely, the symbol (0) constitutes level 0 and the sites $i_1,\ldots,i_n$ form level n of the lattice. In this notation for $x \in \Gamma^k_+,\ x = \{i_1,\ldots,i_n\}$ having $S(x) = \{(x,i) : 1 \le i \le k\}$, here (x,i) means that $(i_1,\ldots,i_n,i)$.

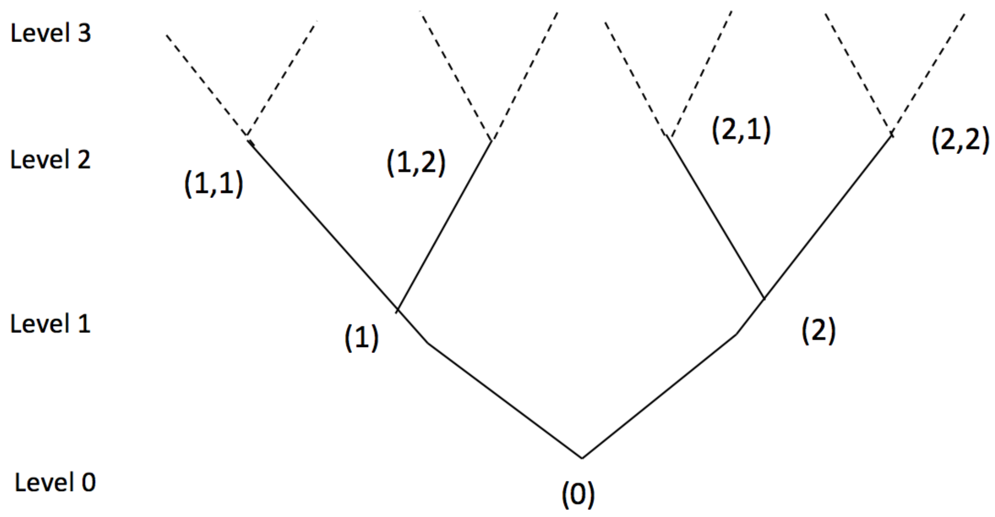


Figure 2.3: The first levels of Γ^2_+

Let define on Γ^k_+ a binary operation $\circ : \Gamma^k_+ \times \Gamma^k_+ \to \Gamma^k_+$ as follows: for any two elements $x = (i_1, \ldots, i_n)$ and $y = (j_1, \ldots, j_m)$ put

$$x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m)$$

and $$y \circ x = (j_1, \ldots, j_m) \circ (i_1, \ldots, i_n) = (j_1, \ldots, j_m, i_1, \ldots, i_n).$$

Through the defined operation, Γ^k_+ becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations $\tau_g : \Gamma^k_+ \to \Gamma^k_+,\ g \in \Gamma^k_+$ by

$$\tau_g(x) = g \circ x.$$

Let $G \subset \Gamma^k_+$ be a sub-semigroup of Γ^k_+ and $h : V \to \mathbb{R}$ be a function. Saying that h is a G-periodic if $h(\tau_g(x)) = h(x)$ for all $x \in V$, $g \in G$ and $l \in L$. Any Γ^k_+-periodic function is called translation-invariant.

Put

$$G_m = \left\{ x \in \Gamma^k_+ : d(x, x^0) \equiv 0 (\mathrm{mod}\ m) \right\}, \quad m \geq 2.$$

One can check that G_m is a sub-semigroup. Now, consider some examples. Let m=2, k=2, then G_2 can be written as follows:

$$G_2 = \{(0), (i_1, i_2, \dots, i_{2n}), n \in \mathbb{N}\}$$

In this case, G_2-periodic function h has the following form:

$$h(x) = \begin{cases} h_1, & x = (i_1, i_2, \dots, i_{2n}), \\ h_2, & x = (i_1, i_2, \dots, i_{2n+1}) \end{cases} \tag{2.1}$$

for $i_k \in \{1, 2\}$ and $k \in V$ see Figure 2.4 [6].

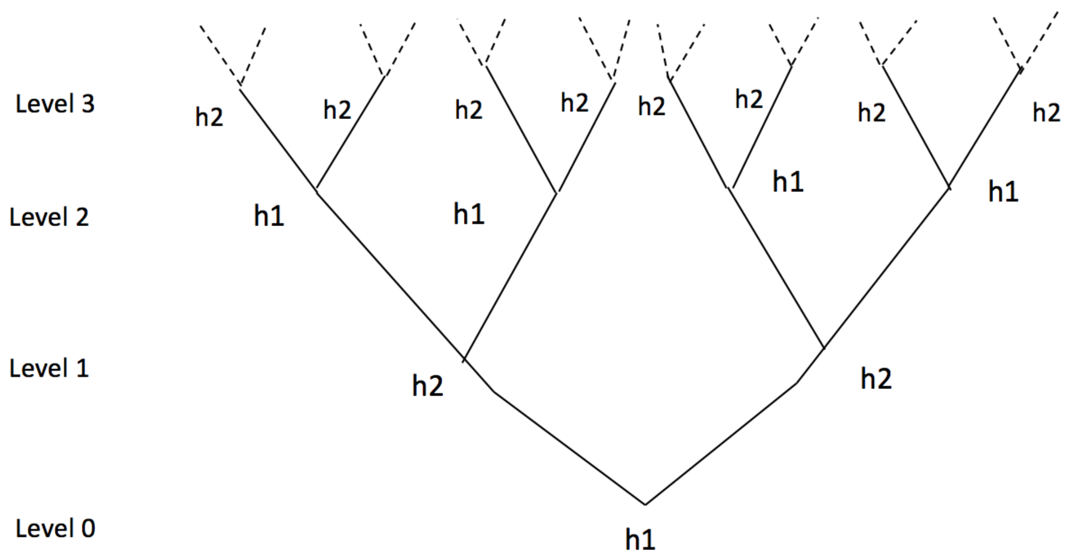


Figure 2.4: Cayley tree for G_2

2.2.1 Configuration space

For $A \subseteq V$ a spin configuration σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi = \{1,2,3,..\}$. The set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

Let G_k^* be a subgroup of the group G_k. A configuration $\sigma \in \Omega$ is called G_k^*-periodic if $\sigma(xy) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$. A configuration that is invariant with respect to all shifts is called translation-invariant [3].

2.2.2 Hamiltonian

The configuration energy $\sigma \in \Omega$ is given by

$$H(\sigma) = \sum_{A \subset V \ diam(A) \leq r} I(\sigma_A) \tag{2.2}$$

where $r \in N$, $diam(A) = max_{x,y \in A} d(x,y), I(\sigma_A) : \Omega_A \to R$ is a given potential.
For a finite domain $D \subset V$ with the boundary condition φ_{D^c} given on its complement $D^c = V \setminus D$, the conditional Hamiltonain is

$$H(\sigma_D \mid \varphi_{D^c}) = \sum_{A \subset V : A \cap D \neq \phi diam(A) \leq r} I(\sigma_A) \tag{2.3}$$

where

$$\sigma_A(x) = \begin{cases} \sigma(x) \ if & : x \in A \cap D \\ \varphi(x) \ if & : x \in A \cap D^c \end{cases}$$

A ground state of (3.2) is a configuration φ in Γ^k such that $H(\varphi) \leq H(\sigma)$ for all $\sigma \in \Omega$ [3, 9].

2.2.3 Gibbs Measure

Gibbs measure (with Hamiltonian H) is a probability measure μ on $(\Omega, \mathscr{F})$ (where $\mathscr{F}$ is σ-algebra generated by cylinder subset of (Ω)) if it satisfies the Dobrushine-Lanford-Ruelle (DLR) equation [10, 3] for all finite $D \subset V$ and $\sigma_D \in \Omega_D$:

$$\mu(\{w \in \Omega : w\,|_D = \sigma_D\}) = \int_{\Omega} \mu(d\varphi)\nu_{\varphi}^{D}(\sigma_D) \tag{2.4}$$

where ν_{φ}^{D} is the conditional probability:

$$\nu_{\varphi}^{D}(\sigma_D) = \frac{1}{Z_{D,\varphi}} e^{-\beta H(\sigma_D|\varphi_{D^c})} \tag{2.5}$$

Here $\beta = \frac{1}{T}, T > 0$: temperature and $Z_{D,\varphi}$ refers to the partition function in D, with the boundary condition φ:

$$Z_{D,\varphi} = \sum_{\widehat{\sigma_D} \in \Omega_D} e^{\beta H(\widehat{\sigma_D}|\varphi_{D^c})} \tag{2.6}$$

The studies focus on the following two basic problems:

- To investigate the case of existing at least one of Gibbs measure for a given Hamiltonian.
- To study the structure of the set G(H) of all Gibbs measures corresponding to a given Hamiltonian.

Note that if H is a continuous Hamiltonian then it is known that $G(H)$ is a non-empty, compact convex subset of all probability measures defined on (Ω, γ). A point $\mu \in G(H)$ is called extreme point of $G(H)$ if there does not exist $\nu_1, \nu_2 \in G(H)$ with $\nu_1 \neq \nu_2$ and $\mu = \frac{1}{2}(\nu_1 + \nu_2)$ [3].

2.3 Ising Model

Consider the Ising model where the spin takes values in the set $\Phi = \{-1, 1\}$, and is assigned to the vertices of the tree. A configuration σ on V is defined as a function $x \in V \to \sigma(x) \in \Phi$, the set of all configurations is Φ^V. The formal Hamiltonian is

$$H(\sigma) = -J \sum_{<x,y>\in L} \sigma(x)\sigma(y) \tag{2.7}$$

where $J > 0$ is coupling constant and <x,y> stands for nearest neighbor vertices. The finite-dimensional distribution of a probability measure μ in the volume V_n is

$$\mu_n(\sigma_n) = Z_n^{-1} e^{\{-\beta H_n(\sigma_n) + \sum_{x\in W_n} h_x \sigma(x)\}} \tag{2.8}$$

where $\beta = \dfrac{1}{T}$, $T > 0$-temperature, Z_n^{-1} is the normalizing factor,$\{h_x \in R, x \in V\}$ is a collection of real numbers [1] and

$$H_n(\sigma_n) = -J \sum_{<x,y>\in L_n} \sigma(x)\sigma(y)$$

For $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$ the probability distribution is called compatible if

$$\sum_{w_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee w_n) = \mu_{n-1}(\sigma_{n-1})$$

here $\sigma_{n-1} \vee \omega_n$ is the union of all configurations. In this case according to the Kolmogorov theorem [11], there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$

$$\mu(\{\sigma \mid v_n = \sigma_n\}) = \mu_n(\sigma_n)$$

such a measure is called a splitting Gibbs measure corresponding to the Hamiltonian (2.7) and function$\{h_x, x \in V\}$.

The following Theorem describes conditions on h_x guaranteeing compatibility of $\mu_n(\sigma_n)$.

Theorem 2.3.1. *Probability distributions $\mu_n(\sigma_n), n = 1,2,..,$ in (2.8) are compatible if and only if for any $x \in V$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} f(h_y, \theta) \tag{2.9}$$

where $\theta = tanh(J\beta), f(h,\theta) = arctanh(\theta tanhh)$ and $S(x)$ is the set of direct successors of x on Cayley tree of order k.

From Theorem 2.3.1, it is clear that for any $h = \{h_x, x \in V\}$ satisfying the functional equation (2.9) there exists a unique Gibbs measure μ and vice versa. However, the analysis of solutions to (2.9) is not easy [12, 3].

Theorem 2.3.2. *For the ferromagnetic Ising model on the Cayley tree of order $k \geq 2$ the following statements are true*

- *If $T \geq T_{c,k}$ then there is unique translation-invariant Gibbs measure μ_0.*
- *If $T < T_{c,k}$ then there are three translation-invariant Gibbs measures μ_-, μ_0, μ_+.*

2.4 Potts Model

Consider Potts model on Cayley tree where the spin takes values in the set $\Phi = \{1,2,3,..,q\}$, and is assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$, the set of all configurations is Φ^V. The formal Hamiltonian is

$$H(\sigma) = -J \sum_{<x,y> \in L} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V} \delta_{1\sigma(x)} \tag{2.10}$$

where $J \in R$ is coupling constant and $< x,y >$ stands for nearest neighbor vertices, $\alpha \in R$ is an external field and δ_{ij} is the Kroneker's symbol.

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{2.11}$$

The finite-dimensional distribution of a probability measure μ in the volume V_n is

$$\mu_n(\sigma_n) = Z_n^{-1} e^{\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\}} \tag{2.12}$$

where $\beta = \dfrac{1}{T}, T > 0$-temperature, Z_n^{-1} is the normalizing factor, $\{h_x = (h_{1,x,\ldots}, h_{q,x} \in R^q, x \in V\}$ is a collection of vectors [13, 14] and

$$H_n(\sigma_n) = -J \sum_{<x,y> \in L_n} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V_n} \delta_{1\sigma(x)}$$

.

Theorem 2.4.1. *Probability distributions $\mu_n(\sigma_n), n = 1, 2, ..,$ in (2.12) are compatible if and only if for any $x \in V$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta, \alpha) \tag{2.13}$$

where $F : h(h_1, .., h_{q-1}) \in R^{q-1} \to F(h, \theta, \alpha) = (F_1, ..., F_{q-1}) \in R^{q-1}$ be defined as

$$F_i = \alpha\beta\delta_{1i} + \ln \frac{(\theta - 1)e^{h_i} + \sum_{i=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{i=1}^{q-1} e^{h_j}}$$

and $\theta = tanh(JB)$ and $S(x)$ is the set of direct successores of x [3].

From Theorem 2.4.1, it follows that for any $h = \{h_x, x \in V\}$ satisfying the functional equation (2.13) there exists a unique Gibbs measure μ and vice versa. However, the analysis of solutions to (2.13) is not easy too [3].

Theorem 2.4.2. *For any $k>1, q>1, J<0, \alpha \in R$ the anti-ferromagnetic Potts model has unique translation-invariant Gibbs measure [15, 3].*

2.5 λ-Model

Consider a model where the spins take values in the set $\phi = \{1,2,3\}$ and is assigned to the vertices of the tree $\phi^{\Gamma^2_+}$. A configuration σ on V defined as a function $x \in V$ such as $\sigma(x) \in \phi$.

A set of all σ configurations are coincides with $\Omega = \phi^{\Gamma^2_+}$. In this case, the Hamiltonian of λ-model is

$$H(\sigma) = \sum_{<x,y>\in L} \lambda(\sigma(x), \sigma(y)) \tag{2.14}$$

where $<x,y>$ are pairs of nearest neighbor vertices $\sigma \in \Omega$.

λ is a symmetric function such that $\lambda(\upsilon,\nu) = \lambda(\nu,\upsilon)$ for every $\upsilon,\nu \in$ R, because the interaction between particles do not depend on their locations .

Now, for $\phi = 1,2,3$ in the case of Γ^2_+

$$\lambda(i,j) = \begin{cases} c_1 & : i=j=1 \\ c_2 & : i=j=2 \\ c_3 & : i=j=3 \\ a & : |i-j|=2 \\ b & : |i-j|=1 \end{cases} \tag{2.15}$$

Note that λ-model considered as a generalization of Potts-model corresponds to the choice $\lambda(x,y) = -J\delta_{(x,y)}$, where x,y and $J \in R$ [6].

Considering more explicit example of λ-interaction, which is related to Hardcore and softcore Widom-Roufinson model [16]. For this model, the interaction has the following form

$$\lambda(u,v) = J\delta_{(u-2)(v-2),-1} + \frac{\lambda}{2}((u-2)^2(v-2)^2)$$

$$= J\delta_{uv-2(v-10)+1,-1} + \frac{\lambda}{2}(u^2+v^2-4(u+8)+8)$$

This shows the λ-model (which is considered in this thesis) is more general than the mentioned ones. Therefore, the obtained result will still be valid for all those models.

Chapter 3: Ground States

In this chapter, the ground state of λ-Model on Cayley tree described by consider pair of configurations σ and ψ that are coinciding almost every where so the relative Hamiltonian of these two configurations will be the energy difference of σ and ψ:

$$H(\sigma,\psi) = \frac{1}{2}\sum_{<x,y>x,y\in V}[\lambda(\sigma(x),\sigma(y)) - \lambda(\psi(x),\psi(y))]$$

The set $\{x,S(x)\}$ such that all $x\in V$ represented a ball B where the set of all balls defined as M. The energy configuration σ_B on B is defined by:

$$U(\sigma_B) = \frac{1}{2}\sum_{x,y\in B}(\lambda(\sigma(x),\sigma(y))$$

Consider the following Formula from [6]:

$$H(\sigma,\psi) = \sum_{B\in M}[U(\sigma_B) - U(\psi_B)] \tag{3.1}$$

where the inclusion

$$U(\psi_B) = \{\frac{\alpha+\varepsilon}{2}, \forall\alpha,\varepsilon\in\{c_1,c_2,c_3,a,b\}\} \tag{3.2}$$

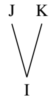
holds for any $B\in M$.

A configuration ψ is called a ground state if $H(\sigma)\geqslant H(\psi)$.

Now for any configuration σ_B on B

$$U(\sigma_B)\in\{U_1,U_2,U_3,U_4,U_5,U_6,U_7,U_8,U_9,U_{10},U_{11}\}$$

where $\{U_k\}$ is defined in Table 3.1.

Denote $\sigma_B = \sigma_{i,j,k} =$ 

and $U(\sigma_{i,j,k}) = U_{i,j,k}$.

The possible energies on σ_B configuration shown in the following table:

Table 3.1: Table of possible configurations

$U(\sigma_B)$	$U(\sigma_{(i,j,k)})$	$\frac{1}{2}[\lambda(i,j)+\lambda(i,k)]$
U_1	$U_{(1,1,1)}$	c_1
U_2	$U_{(2,2,2)}$	c_2
U_3	$U_{(3,3,3)}$	c_3
U_4	$U_{(1,3,3)}$ $U_{(3,1,1)}$	a
U_5	$U_{(2,1,1)}$ $U_{(2,3,1)}$ $U_{(2,3,3)}$ $U_{(1,2,2)}$ $U_{(2,1,3)}$ $U_{(3,2,2)}$	b
U_6	$U_{(1,3,1)}$ $U_{(1,1,3)}$ $U_{(1,3,3)}$ $U_{(3,1,1)}$ $U_{(1,1,1)}$	$\frac{1}{2}(c_1+a)$
U_7	$U_{(1,2,1)}$ $U_{(1,1,2)}$ $U_{(2,1,1)}$ $U_{(2,3,1)}$ $U_{(2,3,3)}$ $U_{(1,2,2)}$ $U_{(2,1,3)}$ $U_{(3,2,2)}$ $U_{(1,1,1)}$	$\frac{1}{2}(c_1+b)$
U_8	$U_{(2,2,1)}$ $U_{(2,3,3)}$ $U_{(2,1,2)}$ $U_{(2,3,2)}$ $U_{(2,1,1)}$ $U_{(2,3,1)}$ $U_{(2,3,3)}$ $U_{(1,2,2)}$ $U_{(2,1,3)}$ $U_{(3,2,2)}$ $U_{(2,2,2)}$	$\frac{1}{2}(c_2+b)$
U_9	$U_{(3,3,1)}$ $U_{(3,1,3)}$ $U_{(1,3,3)}$ $U_{(3,1,1)}$ $U_{(3,3,3)}$	$\frac{1}{2}(c_3+a)$
U_{10}	$U_{(3,3,2)}$ $U_{(3,2,3)}$ $U_{(2,1,1)}$ $U_{(2,3,1)}$ $U_{(2,3,3)}$ $U_{(1,2,2)}$ $U_{(2,1,3)}$ $U_{(3,2,2)}$ $U_{(3,3,3)}$	$\frac{1}{2}(c_3+b)$
U_{11}	$U_{(3,2,1)}$ $U_{(1,2,3)}$ $U_{(1,3,2)}$ $U_{(3,1,2)}$ $U_{(1,3,3)}$ $U_{(3,1,1)}$ $U_{(2,1,1)}$ $U_{(2,3,1)}$ $U_{(2,3,3)}$ $U_{(1,2,2)}$ $U_{(2,1,3)}$ $U_{(3,2,2)}$	$\frac{1}{2}(a+b)$

Denote that $A_k = \{(c_1,c_2,c_3,a,b) \mid U_k = \min_{1\le l\le 11} U_l \}$.

According to the table different cases was found as follows:

$A_1 = \{c_1,c_2,c_3,a,b \mid c_1 < c_2, c_1 < c_3, c_1 < a, c_1 < b\}$

$A_2 = \{c_1,c_2,c_3,a,b \mid c_2 < c_1, c_2 < c_3, c_2 < a, c_2 < b\}$

$A_3 = \{c_1,c_2,c_3,a,b \mid c_3 < c_2, c_3 < c_1, c_3 < a, c_3 < b\}$

$A_4 = \{c_1,c_2,c_3,a,b \mid a < c_1, a < c_2, a < c_3, a < b\}$

$A_5 = \{c_1,c_2,c_3,a,b \mid b < c_1, b < c_2, b < c_3, b < a\}$

$A_6 = \{c_1,c_2,c_3,a,b \mid a = c_1, a < c_2, a < c_3, a < b\}$

$A_7 = \{c_1, c_2, c_3, a, b \mid b = c_1, b < c_2, b < c_3, b < a\}$

$A_8 = \{c_1, c_2, c_3, a, b \mid b = c_2, b < c_1, b < c_3, b < a\}$

$A_9 = \{c_1, c_2, c_3, a, b \mid c_3 = a, a < c_1, a < c_2, a < b\}$

$A_{10} = \{c_1, c_2, c_3, a, b \mid c_3 = b, b < c_1, b < c_2, b < a\}$

$A_{11} = \{c_1, c_2, c_3, a, b \mid a = b, a < c_1, a < c_2, a < c_3\}$.

For every $i \in \{1, 2, 3\}$ by σ^i denote the following configuration on Ω defined by $\sigma^i(x) = i, \forall x \in \Gamma$ as shown in Figure 3.1.

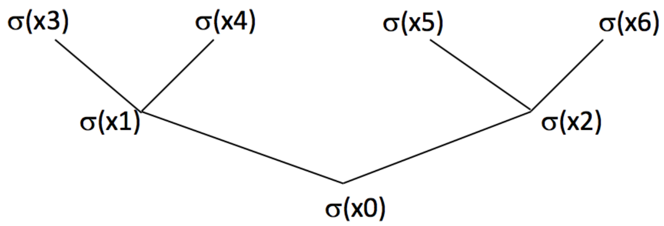


Figure 3.1: Ω configuration

The investigation of ground states for the above cases will present in the following subsections.

3.1 A$_1$-Case

Theorem 3.1.1. *Let* $(c_1, c_2, c_3, a, b) \in A_1$, *then there is only one translation invariant ground state* σ^1.

Proof. Let $(c_1, c_2, c_3, a, b) \in A_1$, then $\min_{1 \le l \le 11}(U_l) = U_1$, the minimum is achieved at configuration σ^1.

Now, construct a ground state configuration on Ω such that its restoration to any ball equal to σ_1, where $\sigma_1 =$  (see Figure 3.2)

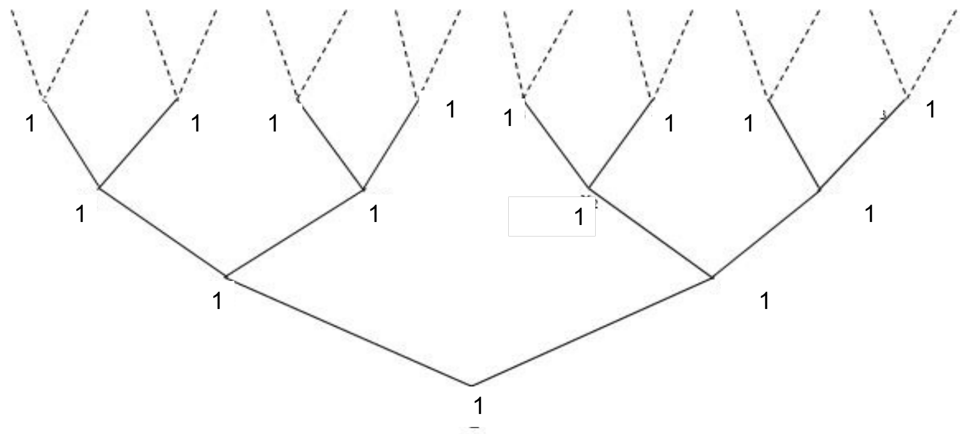


Figure 3.2: Translation invariant ground state σ^1

Clearly, such a configuration exists and it is unique, moreover for every $B \in M$, $\psi_B = \min U(\sigma)$

3.2 A$_2$-Case

Theorem 3.2.1. *Let $(c_1, c_2, c_3, a, b) \in A_2$, then there is only one translation invariant ground state σ^2.*

Proof. Let $(c_1, c_2, c_3, a, b) \in A_2$, then $\min\limits_{1 \leq l \leq 11} (U_l) = U_2$, the minimum is achieved at configuration σ^2.

Now, construct a ground state configuration on Ω such that its restoration to any ball equal to σ_2, where σ_2 = 

(see Figure 3.3)

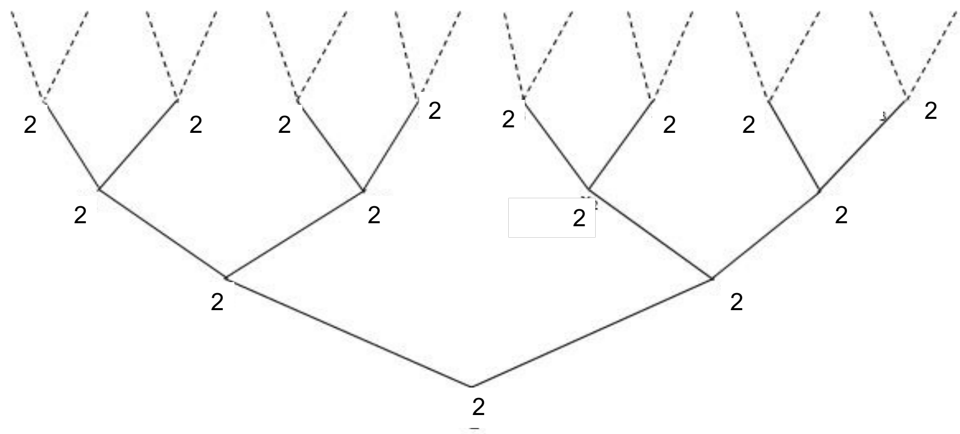


Figure 3.3: Translation invariant ground state σ^2

Clearly, such a configuration exists and it is unique, moreover for every $B \in M$, $\psi_B = \min U(\sigma)$.

3.3 A$_3$-Case

Theorem 3.3.1. *Let $(c_1, c_2, c_3, a, b) \in A_3$, then there is only one translation invariant ground state σ^3.*

Proof. Let $(c_1, c_2, c_3, a, b) \in A_3$, then $\min\limits_{1 \le l \le 11} (U_l) = U_3$, the minimum is achieved at configuration σ^3.

Now, construct a ground state configuration on Ω such that its restoration to any ball equal to σ_3, where $\sigma_3 =$ 

(see Figure 3.4).

Clearly, such a configuration exists and it is unique Moreover for every $B \in M$, $\psi_B = \min U(\sigma)$.

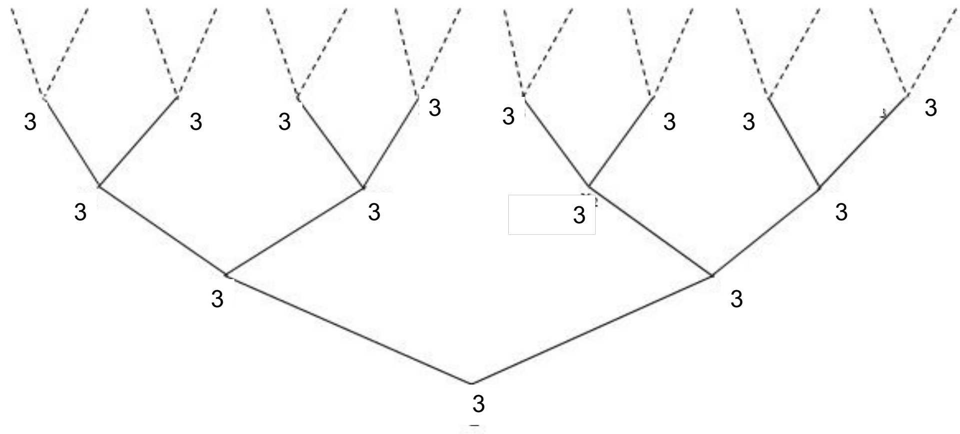


Figure 3.4: Translation invariant ground state σ^3

Now, for any $i, j \in \{1,2,3\}$, denote $\sigma^{(i,j)}$ as follows:

$$\sigma^{(i,j)}(x) = \begin{cases} i & x \in w_{2n}, \\ j & x \in w_{2n+1}, \quad n \geq 0 \end{cases}$$

3.4 A$_4$-Case

Theorem 3.4.1. *Let* $(c_1, c_2, c_3, a, b) \in A_4$*, then there are only two periodic ground states* $\sigma^{1,3}$ *and* $\sigma^{3,1}$.

Proof. For $(c_1, c_2, c_3, a, b) \in A_4$, infer that $\min\limits_{1 \leq l \leq 11}(U_l) = U_4$, where the minimum is achieve at:

$\sigma_{(1,3)} = \begin{smallmatrix} 3 \quad 3 \\ \diagdown\diagup \\ 1 \end{smallmatrix}$. Therefore, a configuration σ on Ω with $\sigma\,|_B = \sigma_{(1,3)}$is equal to $\sigma^{(1,3)}$ as shown in Figure 3.5.

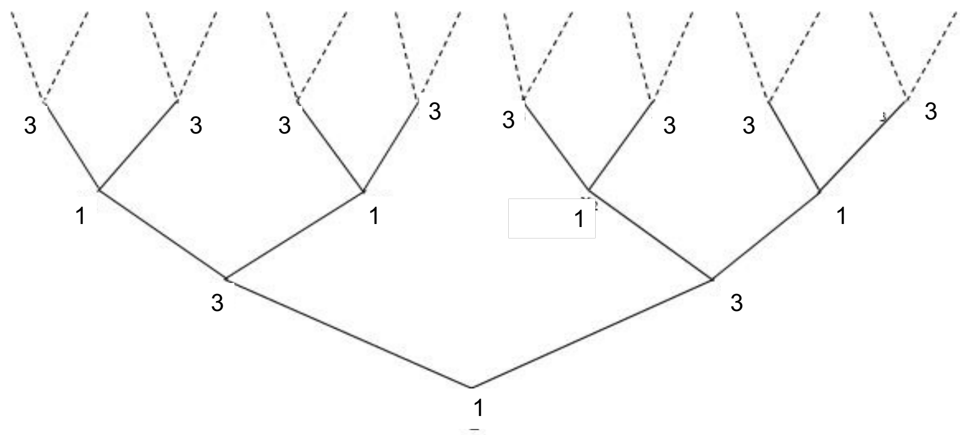


Figure 3.5: $\sigma^{(1,3)}$ configuration

Similarly, $\sigma^{(3,1)}$ is a ground state as well since for any $B \in M$, $\min_{1\leq l\leq 11} U_l = U(\sigma^{(1,3)})$,$U(\sigma^{(3,1)})$ hence there are two ground states which are 2-periodic by construction.

3.5 A$_5$-Case

Theorem 3.5.1. *Let* $(c_1, c_2, c_3, a, b) \in A_5$,*then:*

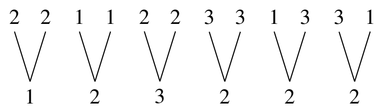
1. *For every* $n \in N$*, there is* $2n$*-periodic ground state*

2. *There is uncountable number of ground state.*

Proof. Fist let us prove part (1):

For $(c_1, c_2, c_3, a, b) \in A_5$ then,

$U(\sigma) = \min_{1\leq l\leq 11} U_l = U_5$ in this case, the minimum is achieve at the following configuration:



Now, construct the σ configuration on Ω, let $\{k_1,k_2,k_3,..,k_n\} \in \{1,2,3\}, n \in N$ define $\sigma(x) = k_l$ if $x \in w_l, l \geq 0$ as shown in Figure 3.6 the sequence $\{k_1,k_2,k_3,..,k_n\}$ is 2n periodic if $k_{l+2n} = k_l, \forall n \in N$ so, $\sigma_2 = [1,2,3,2]$ as shown in Figure 3.7.

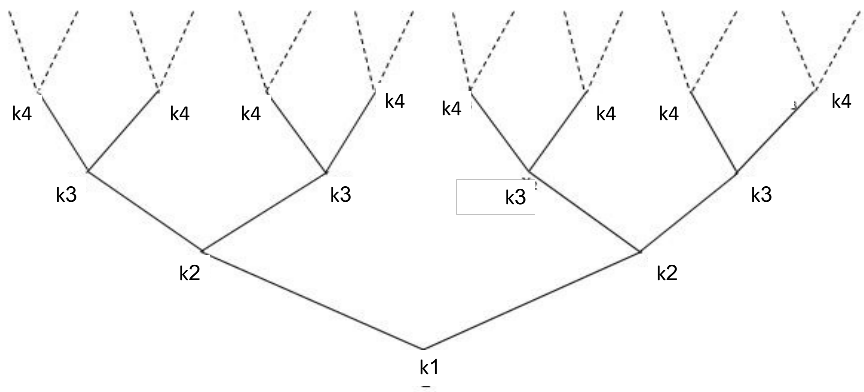


Figure 3.6: $\sigma(x) = k_n$ configuration

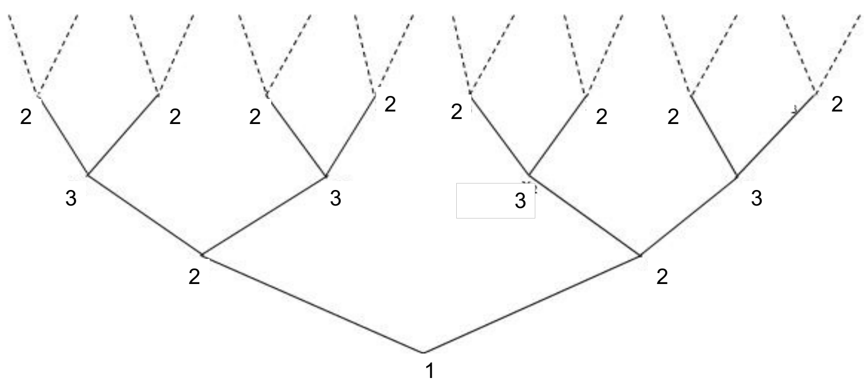


Figure 3.7: σ^{2n} configuration for A_5 case

For any $n \in N$, define $\sigma_{2n} = [1,2,1,2,...,2,3,2]$.

Hence, for any $B \in M$, $U(\sigma_{2n}) = \min_{1 \leq l \leq 11} U_l = b$ and $\sigma_{2n} = \psi_n$

therefore,this case have 2n-periodic ground states.

part(2):

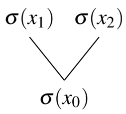
Now, several notations introduced by:

A ball b on Γ^2 identified as follows:

$B =$ 

Denote $B \downarrow = x_0$

A configuration σ on B is defined by:

$\sigma(B) =$ 

Take $B_1, B_2, B_3 \in M$ such that $B_2 \downarrow, B_3 \downarrow \in B_1$ as shown in Figures 3.8 and 3.9:

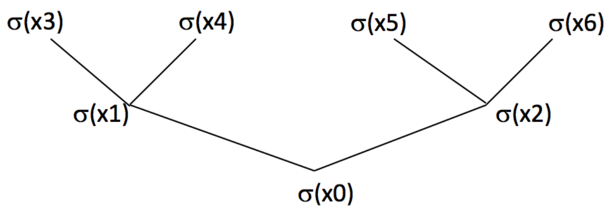


Figure 3.8: $\sigma(B)$ configuration

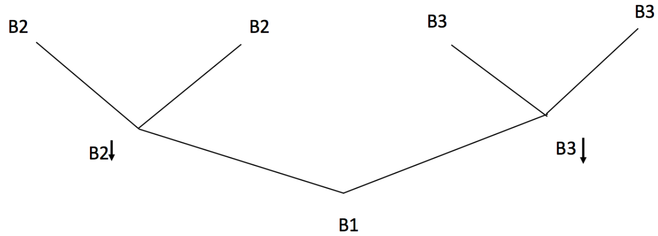


Figure 3.9: M configuration

Configurations $\sigma_1, \sigma_2, \sigma_3$ on B_1, B_2, B_3 respectively are called admissible if $\sigma(x_2) = \sigma(B_2 \downarrow), \sigma_1(x_1) = \sigma_3(B_3 \downarrow)$

Denote,

$$\sigma_1 = \underset{1}{\overset{2\ \ 2}{\bigvee}} \quad \sigma_2 = \underset{2}{\overset{1\ \ 1}{\bigvee}} \quad \sigma_3 = \underset{3}{\overset{2\ \ 2}{\bigvee}} \quad \sigma_4 = \underset{2}{\overset{3\ \ 3}{\bigvee}} \quad \sigma_5 = \underset{2}{\overset{1\ \ 3}{\bigvee}} \quad \sigma_6 = \underset{2}{\overset{3\ \ 1}{\bigvee}}$$

now construct ground states as follows:

Denote:

$\alpha = \{\sigma_1, \sigma_2, ..., \sigma_6\}$, one can see that for any ground state ψ (on Ω) its restriction to any ball B is admissible with α.

Now, denote

$\Sigma_{1,0} = \{(t_n) | t_n \in \{0,1\}, n \in N\}$.

It is well-known that $\Sigma_{1,0}$ is an uncountable set [17].

Now take any sequence $(w_n) \in \Sigma_{1,0}$ and associate configurations as follows:

if $w_n = 0$ take a configuration $\overline{\sigma_0}$ on B one of $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$

if $w_n = 1$ take a configuration $\overline{\sigma_1}$ on B one of $\{\sigma_5, \sigma_6\}$

Now take any admissible balls in $\{B_1, B_2, B_3\}$ and provide a construction of a configuration on these balls.

Case 1:

Let $\sigma(B \downarrow) = \sigma_1$

then, $\sigma(B_3 \downarrow) = 2, \sigma(B_2 \downarrow) = 2$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either one of $\{\sigma_2, \sigma_4, \sigma_5, \sigma_6\}$.

Case 2:

Let $\sigma(B \downarrow) = \sigma_2$

then, $\sigma(B_3 \downarrow) = 1, \sigma(B_2 \downarrow) = 1$

in this case, for $\sigma(B_3), \sigma(B_2)$ the choice will be σ_1.

Case 3:

Let $\sigma(B_1 \downarrow) = \sigma_3$

then, $\sigma(B_3 \downarrow) = 2, \sigma(B_2 \downarrow) = 2$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either one of $\{\sigma_5, \sigma_6\}$.

Case 4:

Let $\sigma(B_1 \downarrow) = \sigma_5$

then, $\sigma(B_3 \downarrow) = 1, \sigma(B_2 \downarrow) = 3$

so, for $\sigma(B_3)$ the choice will be σ_1

and for $\sigma(B_2)$ the choice will be σ_3.

Case 5:

Let $\sigma(B \downarrow) = \sigma_6$

then, $\sigma(B_3 \downarrow) = 3, \sigma(B_2 \downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be σ_3

and for $\sigma(B_2)$ the choice will be σ_1.

The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n is:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable. Therefore, the constructed ground states are uncountable this completes the proof.

3.6 A$_6$-Case

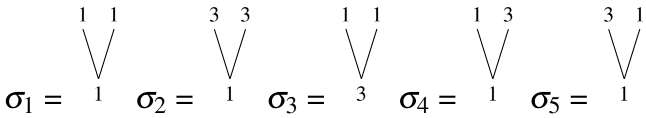
Theorem 3.6.1. *Let $(c_1, c_2, c_3, a, b) \in A_6$, then the following statements hold:*

1. *There is only one translation invariant ground state σ^1.*
2. *For any $n \in N$ there exist Gn periodic ground states.*
3. *There is uncountable number of ground states.*

Proof.

Part (1):

Let $(c_1, c_2, c_3, a, b) \in A_6$, then $\min\limits_{1 \leq l \leq 11} (U_l) = U_6$. In this case, $A_1 \subset A_6$ and $A_4 \subset A_6$. Where the minimum is achieved at the following configurations:



Due to Theorem 3.1.1 there is only one translation invariant ground state σ^1.

Part (2):

By following same steps of the proof of Theorem 3.5.1 part 1 then, for each $n \in N$ the sequence $\{k_1, k_2, k_3, ...\}$ is n-peiodic if $k_{n+l} = k_l$, one can construct a configuration (see Figure 3.10) on Ω defined by $\sigma^n = \sigma\underbrace{[3,1,1,1,...]}_{n}$, then conclude that for any $b \in M$ one has $\min_{1 \le l \le 11}(U_l) = U_{\sigma^n}$, which means σ^n is a Gn-periodic ground state.

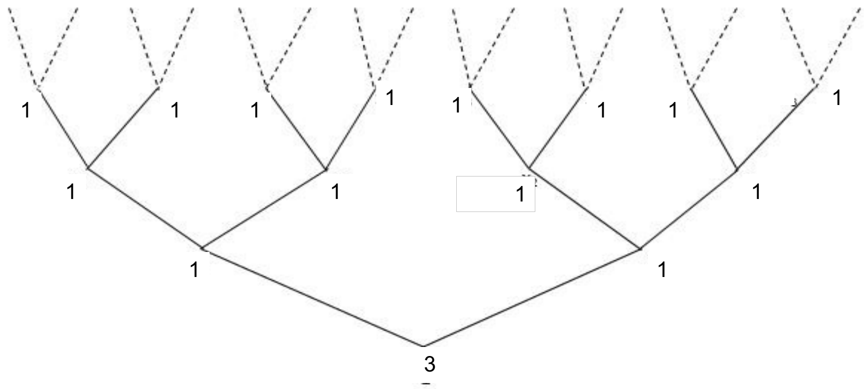


Figure 3.10: σ^n configuration for A_6 case

Part (3):

By following the argument of the proof of Theorem 3.5.1 part 2.

Denote:

$\alpha = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ one can see for any ground state ψ on (Ω) its restriction to any ball B is admissible with α.

Now, let $\Sigma_{1,0} = \{t_n | t_n \in 0, 1, n \in N\}$ take any sequence $(w_n) \in \Sigma_{1,0}$ and construct ground state as follows:

If $w_n = 0$ take one of $\{\sigma_1, \sigma_2\} = \overline{\sigma_0}$

If $w_n = 1$ take one of $\{\sigma_3, \sigma_4, \sigma_5\} = \overline{\sigma_1}$

Case 1:

let $\sigma(B\downarrow) = \sigma_1$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_4, \sigma_5\}$.

Case 2:

let $\sigma(B\downarrow) = \sigma_2$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

in this case, for $\sigma(B_3), \sigma(B_2)$ the choice is $\{\sigma_3\}$.

Case 3:

let $\sigma(B_1\downarrow) = \sigma_3$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5\}$.

Case 4:

let $\sigma(B_1\downarrow) = \sigma_4$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice is $\{\sigma_3\}$ and for $\sigma(B_2)$ the choice will be one of $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5\}$

Case 5:

let $\sigma(B\downarrow) = \sigma_5$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ the choice will be either one of $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5\}$

and for $\sigma(B_2)$ the choice is $\{\sigma_3\}$.

The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n is:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable. Therefore, the constructed ground states are uncountable this completes the proof.

3.7 A_7-Case

Theorem 3.7.1. *Let $(c_1, c_2, c_3, a, b) \in A_7$, then the following statements hold:*

1. *There is only one translation invariant ground state σ^1.*
2. *For any $n \in N$ there exist Gn periodic ground states.*
3. *There is uncountable number of ground states.*

Proof. Part (1): Let $(c_1, c_2, c_3, a, b) \in A_7$, then $\min\limits_{1 \le l \le 11} (U_l) = U_7$. In this case, $A_1 \subset A_7$ and $A_5 \subset A_7$, where the minimum is achieved at the following configurations:

$$\sigma_1 = \begin{smallmatrix}1 & & 1\\ & \bigvee & \\ & 1 & \end{smallmatrix} \quad \sigma_2 = \begin{smallmatrix}1 & & 1\\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_3 = \begin{smallmatrix}3 & & 1\\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_4 = \begin{smallmatrix}3 & & 3\\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_5 = \begin{smallmatrix}2 & & 2\\ & \bigvee & \\ & 1 & \end{smallmatrix} \quad \sigma_6 = \begin{smallmatrix}1 & & 3\\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_7 = \begin{smallmatrix}1 & & 2\\ & \bigvee & \\ & 1 & \end{smallmatrix}$$

$$\sigma_8 = \begin{smallmatrix}2 & & 2\\ & \bigvee & \\ & 3 & \end{smallmatrix} \quad \sigma_9 = \begin{smallmatrix}2 & & 1\\ & \bigvee & \\ & 1 & \end{smallmatrix}$$

Due to Theorem 3.1.1 there is only one translation invariant ground state σ^1.

Part (2):

For each $n \in N$, one can construct a configuration (see Figure 3.11) on Ω defined by $\sigma^n = \sigma\underbrace{[2, 1, 1, 1, ...]}_{n}$, then it is clear that for any $B \in M$ one has $\min\limits_{1 \le l \le 11} (U_l) = U_{\sigma^n}$, which means σ^n is a Gn-periodic ground state.

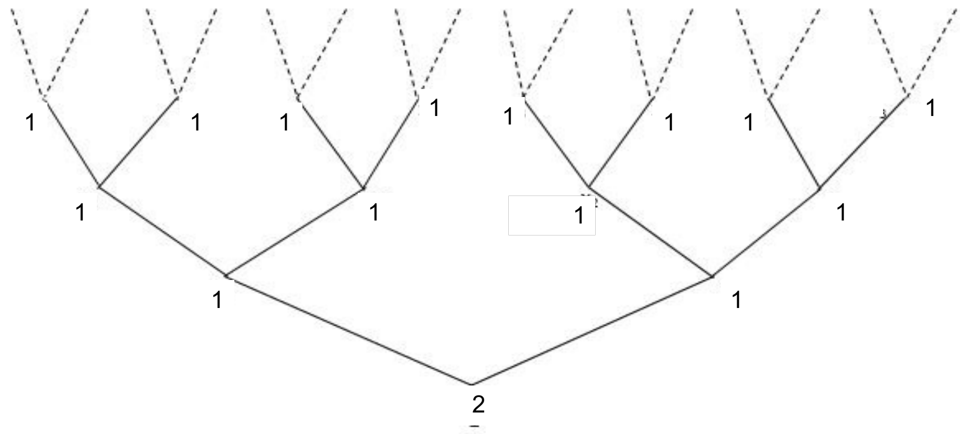


Figure 3.11: σ^n configuration for A_7 case

Part (3):

By following the argument of the proof of Theorem 3.5.1.

Denote:

$\alpha = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9\}$ one can see for any ground state ψ on (Ω) its restriction to any ball B is admissible with α.

Now, let $\Sigma_{1,0} = \{t_n | t_n \in 0, 1, n \in N\}$ take any sequence $(w_n) \in \Sigma_{1,0}$ and construct ground state as follows:

if $w_n = 0$ take one of $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_7, \sigma_8, \sigma_9\} = \overline{\sigma_0}$

if $w_n = 1$ take one of $\{\sigma_3, \sigma_6\} = \overline{\sigma_1}$

Case 1:

let $\sigma(B\downarrow) = \sigma_1$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_5, \sigma_7, \sigma_9\}$.

Case 2:

let $\sigma(B\downarrow) = \sigma_2$

Then, $\sigma(B_3\downarrow)=1, \sigma(B_2\downarrow)=1$

In this case, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_1, \sigma_5\sigma_7, \sigma_9\}$.

Case 3:

let $\sigma(B_1\downarrow)=\sigma_3$

Then, $\sigma(B_3\downarrow)=3, \sigma(B_2\downarrow)=1$

So, for $\sigma(B_3)$ the choice is $\{\sigma_8\}$ and for $\sigma(B_2)$ the choice is one of $\{\sigma_2, \sigma_4\}$

Case 4:

let $\sigma(B_1\downarrow)=\sigma_4$

then, $\sigma(B_3\downarrow)=3, \sigma(B_2\downarrow)=3$

so, for $\sigma(B_3)$ and $\sigma(B_2$ the choice will be $\{\sigma_8\}$

Case 5:

let $\sigma(B\downarrow)=\sigma_5$

then, $\sigma(B_3\downarrow)=2, \sigma(B_2\downarrow)=2$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice will be either one of $\{\sigma_2, \sigma_4\}$ or $\{\sigma_3, \sigma_6\}$

Case 6:

let $\sigma(B_1\downarrow)=\sigma_6$

Then, $\sigma(B_3\downarrow)=1, \sigma(B_2\downarrow)=3$

so, for $\sigma(B_3)$ the choice will be one of $\{\sigma_2, \sigma_4\}$ and for$\sigma(B_2$ the choice is $\{\sigma_8\}$

Case 7:

let $\sigma(B_1\downarrow)=\sigma_7$

then, $\sigma(B_3\downarrow)=1, \sigma(B_2\downarrow)=2$

so, for $\sigma(B_3)$ the choice is one of $\{\sigma_1, \sigma_5, \sigma_9\}$ and for $\sigma(B_2)$ the choice will be either one of $\{\sigma_2, \sigma_4\}$ or $\{\sigma_3, \sigma_6\}$

Case 8:

let $\sigma(B\downarrow)=\sigma_8$

then, $\sigma(B_3\downarrow)=2, \sigma(B_2\downarrow)=2$

In this case, for $\sigma(B_3), \sigma(B_2)$ the choice will be either one of $\{\sigma_2, \sigma_4\}$ or $\{\sigma_3, \sigma_6\}$

Case 9:

let $\sigma(B_1\downarrow)=\sigma_9$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be either one of $\{\sigma_2, \sigma_4\}$ or $\{\sigma_3, \sigma_6\}$ and for $\sigma(B_2$ the choice will be one of $\{\sigma_1, \sigma_5, \sigma_9\}$.

The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n follows these conditions:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable. Therefore, the constructed ground states are uncountable this completes the proof.

3.8 A$_8$-Case

Theorem 3.8.1. *Let $(c_1, c_2, c_3, a, b) \in A_8$, then the following statements hold:*

1. *There is only one translation invariant ground state σ^2.*

2. *For any $n \in N$ there exist Gn periodic ground states.*

3. *There is uncountable number of ground states.*

Proof. Part (1): Let $(c_1, c_2, c_3, a, b) \in A_8$, then $\min\limits_{1 \le l \le 11}(U_l) = U_8$. In this case, $A_2 \subset A_8$ and $A_5 \subset A_8$, where the minimum is achieved at the following configurations:

$$\sigma_1 = \begin{array}{c}2\ \ 2\\ \bigvee\\ 2\end{array} \quad \sigma_2 = \begin{array}{c}2\ \ 1\\ \bigvee\\ 2\end{array} \quad \sigma_3 = \begin{array}{c}3\ \ 3\\ \bigvee\\ 2\end{array} \quad \sigma_4 = \begin{array}{c}1\ \ 2\\ \bigvee\\ 2\end{array} \quad \sigma_5 = \begin{array}{c}3\ \ 2\\ \bigvee\\ 2\end{array} \quad \sigma_6 = \begin{array}{c}1\ \ 1\\ \bigvee\\ 2\end{array}$$

$$\sigma_7 = \begin{array}{c}2\ \ 1\\ \bigvee\\ 2\end{array} \quad \sigma_8 = \begin{array}{c}2\ \ 2\\ \bigvee\\ 1\end{array} \quad \sigma_9 = \begin{array}{c}1\ \ 3\\ \bigvee\\ 2\end{array} \quad \sigma_{10} = \begin{array}{c}2\ \ 2\\ \bigvee\\ 3\end{array}$$

due to Theorem 3.2.1 there is only one translation invariant ground state σ^2.

Part (2):

For each $n \in N$, one can construct a configuration on Ω defined by $\sigma^n = \sigma\underbrace{[1, 2, 2, 2, \ldots]}_{n}$, then it is clear that for any $B \in M$ one has $\min\limits_{1 \le l \le 11}(U_l) = U_{\sigma^n}$, which means σ^n is a Gn-periodic ground state.

Part (3):

By following the argument of the proof of Theorem 3.5.1.

Denote:

$\alpha = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$ one can see for any ground state ψ on (Ω) its restriction to any ball B is admissible with α.

Now, let $\Sigma_{1,0} = \{t_n | t_n \in 0, 1, n \in N\}$ take any sequence $(w_n) \in \Sigma_{1,0}$ then construct ground state as follows:

if $w_n = 0$ take one of $\{\sigma_1, \sigma_8, \sigma_{10}\} = \overline{\sigma_0}$

if $w_n = 1$ take one of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\} = \overline{\sigma_1}$

Case 1:

let $\sigma(B\downarrow) = \sigma_1$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\}$ or $\{\sigma_1\}$.

Case 2:

let $\sigma(B\downarrow) = \sigma_2$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\}$ or $\{\sigma_1\}$ and for $\sigma(B_2)$ the choice is $\{\sigma_8\}$

Case 3:

let $\sigma(B_1\downarrow) = \sigma_3$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice $\{\sigma_{10}\}$

Case 4:

let $\sigma(B_1\downarrow) = \sigma_4$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3)$ the choice is $\{\sigma_8\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\}$ or $\{\sigma_1\}$

Case 5:

let $\sigma(B\downarrow) = \sigma_5$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2)\downarrow) = 2$

so, for $\sigma(B_3)$ the choice is $\{\sigma_{10}\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\}$ or $\{\sigma_1\}$

Case 6:

let $\sigma(B_1\downarrow) = \sigma_6$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2)\downarrow) = 1$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice is $\{\sigma_8\}$

Case 7:

let $\sigma(B_1\downarrow) = \sigma_7$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2)\downarrow) = 1$

so, for $\sigma(B_3)$ the choice is $\{\sigma_{10}\}$ and for $\sigma(B_2)$ the choice is $\{\sigma_8\}$.

Case 8:

let $\sigma(B\downarrow) = \sigma_8$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\}$ or $\{\sigma_1\}$.

Case 9:

let $\sigma(B_1\downarrow) = \sigma_9$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ the choice is $\{\sigma_8\}$ and for $\sigma(B_2)$ the choice is $\{\sigma_{10}\}$

Case 10:

let $\sigma(B\downarrow) = \sigma_{10}$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_9\}$ or $\{\sigma_1\}$. The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n follows these conditions:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable.

Therefore, the constructed ground states are uncountable this completes the proof.

3.9 A₉-Case

Theorem 3.9.1. *Let $(c_1, c_2, c_3, a, b) \in A_9$, then the following statements hold:*

1. *There is only one translation invariant ground state σ^3.*

2. *For any $n \in N$ there exist Gn periodic ground states.*

3. *There is uncountable number of ground states.*

Proof. Part (1): let $(c_1, c_2, c_3, a, b) \in A_9$, then $\min_{1 \leq l \leq 11} (U_l) = U_9$. In this case, $A_3 \subset A_9$ and $A_4 \subset A_9$, where the minimum is achieved at the following configurations:

$$\sigma_1 = \begin{matrix} 3 \quad 3 \\ \bigvee \\ 3 \end{matrix} \quad \sigma_2 = \begin{matrix} 3 \quad 3 \\ \bigvee \\ 1 \end{matrix} \quad \sigma_3 = \begin{matrix} 1 \quad 1 \\ \bigvee \\ 3 \end{matrix} \quad \sigma_4 = \begin{matrix} 3 \quad 1 \\ \bigvee \\ 3 \end{matrix} \quad \sigma_5 = \begin{matrix} 1 \quad 3 \\ \bigvee \\ 3 \end{matrix}$$

due to Theorem 3.3.1 there is only one translation invariant ground state σ^3.

Part (2):

For each $n \in N$, one can construct a configuration on Ω defined by $\sigma^n = \sigma \underbrace{[1, 3, 3, 3, ...]}_{n}$, then it is clear that for any $B \in M$ one has $\min_{1 \leq l \leq 11} (U_l) = U_{\sigma^n}$, which means σ^n is a Gn-periodic ground state.

Part (3):

By following the argument of the proof of Theorem 3.5.1 part (2).

Denote:

$\alpha = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ one can see for any ground state ψ on (Ω) its restriction to any ball B is admissible with α.

Now, let $\Sigma_{1,0} = t_n | t_n \in 0, 1, n \in N$ take any sequence $(w_n) \in \Sigma_{1,0}$ the construction of ground state will be as the following:

if $w_n = 0$ take one of $\{\sigma_1, \sigma_2\} = \overline{\sigma_0}$

if $w_n = 1$ take one of $\{\sigma_3, \sigma_4, \sigma_5\} = \overline{\sigma_1}$

Case 1:

let $\sigma(B\downarrow) = \sigma_1$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_3, \sigma_4, \sigma_5\}$ or $\{\sigma_1\}$.

Case 2:

let $\sigma(B\downarrow) = \sigma_2$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_3, \sigma_4, \sigma_5\}$ or $\{\sigma_1\}$.

Case 3:

let $\sigma(B_1\downarrow) = \sigma_3$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3), \sigma(B_2)$ the choice is $\{\sigma_2\}$.

Case 4:

let $\sigma(B_1\downarrow) = \sigma_4$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be one of $\{\sigma_3, \sigma_4, \sigma_5\}$ or $\{\sigma_1\}$ and for $\sigma(B_2$ the choice is σ_2

Case 5:

let $\sigma(B\downarrow) = \sigma_5$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ the choice is σ_2.

and for $\sigma(B_2)$ the choice will be either one of $\{\sigma_3, \sigma_4, \sigma_5\}$ or $\{\sigma_1\}$

The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n follows these conditions:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable. Therefore, the constructed ground states are uncountable this completes the proof.

3.10 A_{10}-Case

Theorem 3.10.1. *Let $(c_1, c_2, c_3, a, b) \in A_{10}$, then the following statements hold:*

1. *There is only one translation invariant ground state σ^3.*

2. *For any $n \in N$ there exist Gn periodic ground states.*

3. *There is uncountable number of ground states.*

Proof.

Part (1):

Let $(c_1, c_2, c_3, a, b) \in A_{10}$, then $\min\limits_{1 \le l \le 11} (U_l) = U_{10}$. In this case, $A_3 \subset A_{10}$ and $A_5 \subset A_{10}$. Where the minimum is achieved at the following configurations:

$$\sigma_1 = \begin{smallmatrix} 3 & & 3 \\ & \bigvee & \\ & 3 & \end{smallmatrix} \quad \sigma_2 = \begin{smallmatrix} 3 & & 2 \\ & \bigvee & \\ & 3 & \end{smallmatrix} \quad \sigma_3 = \begin{smallmatrix} 2 & & 3 \\ & \bigvee & \\ & 3 & \end{smallmatrix} \quad \sigma_4 = \begin{smallmatrix} 1 & & 1 \\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_5 = \begin{smallmatrix} 3 & & 1 \\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_6 = \begin{smallmatrix} 3 & & 3 \\ & \bigvee & \\ & 2 & \end{smallmatrix}$$

$$\sigma_7 = \begin{smallmatrix} 2 & & 2 \\ & \bigvee & \\ & 1 & \end{smallmatrix} \quad \sigma_8 = \begin{smallmatrix} 1 & & 3 \\ & \bigvee & \\ & 2 & \end{smallmatrix} \quad \sigma_9 = \begin{smallmatrix} 2 & & 2 \\ & \bigvee & \\ & 3 & \end{smallmatrix}$$

due to Theorem 3.3.1 there is only one translation invariant ground state $\sigma_{[3]}$.

Part (2):

For each $n \in N$, one can construct a configuration on Ω defined by $\sigma^n = \sigma \underbrace{[2, 3, 3, 3, ...]}_{n}$, then it is clear that for any $B \in M$ one has $\min\limits_{1 \le l \le 11} (U_l) = U_{\sigma^n}$. Which means σ^n is a Gn-periodic ground state.

Part (3):

By following the argument of the proof of Theorem 3.5.1.

Denote:

$\alpha = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9\}$ one can see for any ground state ψ on (Ω) it is restriction to any ball B is admissible with α.

Now, let $\Sigma_{1,0} = \{t_n | t_n \in 0, 1, n \in N\}$ take any sequence $(w_n) \in \Sigma_{1,0}$ the construction of the ground state will be as follow:

if $w_n = 0$ take one of $\{\sigma_2, \sigma_3, \sigma_9\} = \overline{\sigma_0}$

if $w_n = 1$ take one of $\{\sigma_1, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\} = \overline{\sigma_1}$

Case 1:

let $\sigma(B\downarrow) = \sigma_1$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3), \sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_9\}$ or $\{\sigma_1\}$.

Case 2:

let $\sigma(B\downarrow) = \sigma_2$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_9\}$ or $\{\sigma_1\}$ and for $\sigma(B_2)$ the choice is $\{\sigma_4, \sigma_5, \sigma_6\sigma_8\}$

Case 3:

let $\sigma(B_1\downarrow) = \sigma_3$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ the choice is $\{\sigma_4, \sigma_5, \sigma_6\sigma_8\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_9\}$ or$\{\sigma_1\}$

Case 4:

let $\sigma(B_1\downarrow) = \sigma_4$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice is $\{\sigma_7\}$

Case 5:

let $\sigma(B\downarrow) = \sigma_5$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_9\}$ or $\{\sigma_1\}$ and for $\sigma(B_2)$ the choice is $\{\sigma_7\}$

Case 6:

let $\sigma(B_1\downarrow) = \sigma_6$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ and $\sigma(B_2$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_9\}$ or $\{\sigma_1\}$

Case 7:

let $\sigma(B_1\downarrow)=\sigma_7$

then, $\sigma(B_3\downarrow)=2, \sigma(B_2\downarrow)=2$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice is $\{\sigma_4,\sigma_5,\sigma_6\sigma_8\}$

Case 8:

let $\sigma(B\downarrow)=\sigma_8$

then, $\sigma(B_3\downarrow)=1, \sigma(B_2\downarrow)=3$

so, for $\sigma(B_3)$ the choice is $\{\sigma_7\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_2,\sigma_3,\sigma_9\}$ or $\{\sigma_1\}$.

Case 9:

let $\sigma(B_1\downarrow)=\sigma_9$

then, $\sigma(B_3\downarrow)=2, \sigma(B_2\downarrow)=2$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice is $\{\sigma_4,\sigma_5,\sigma_6\sigma_8\}$.

The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n as follows:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable. Therefore, the constructed ground states are uncountable this completes the proof.

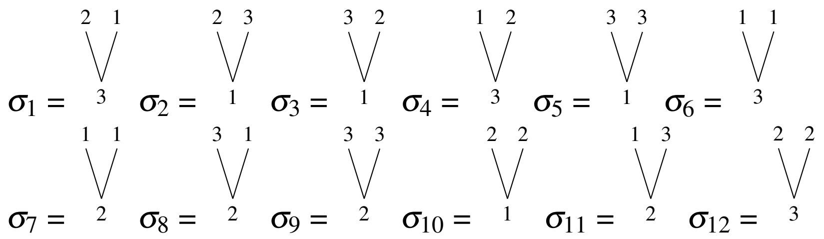
3.11 A$_{11}$-Case

Theorem 3.11.1. *Let $(c_1,c_2,c_3,a,b)\in A_{11}$, then the following statements hold:*

1. *There exist (Gn, 2n, n+2) periodic ground states.*

2. *There is uncountable number of ground states.*

Proof. Let $(c_1,c_2,c_3,a,b)\in A_{11}$, then $\min\limits_{1\leq l\leq 11}(U_l)=U_{11}$. In this case, $A_4\subset A_{11}$ and $A_5\subset A_{11}$, where the minimum is achieved at the following configurations:



Part (1):

Now construct the σ configuration on Ω, so let $\{k_1, k_2, k_3, .., k_n\} \in \{1,2,3\}, n \in N$, define $\sigma(x) = k_l$ if $x \in w_l, l \geq 0$ as shown in Figure 3.6 the sequence $\{k_1, k_2, k_3, .., k_n\}$ is n periodic if $k_{l+n} = k_l, \forall n \in N$ so, $\sigma_n = [,3,3,3,..]$ as shown in Figure 3.12
and the sequence $\{k_1, k_2, k_3, .., k_n\}$ is 2n periodic if $k_{l+2n} = k_l, \forall n \in N$ so, $\sigma_{2n} = [1,2,1,2,1,2,..]$ as shown in Figure 3.13.
Also the sequence $\{k_1, k_2, k_3, .., k_n\}$ is (n+2) periodic if $k_{l+(n+2)} = k_l, \forall n \in N$ so, $\sigma_{(n+2)} = [1,2,3,2,1,2,..]$ as shown in Figure 3.14.

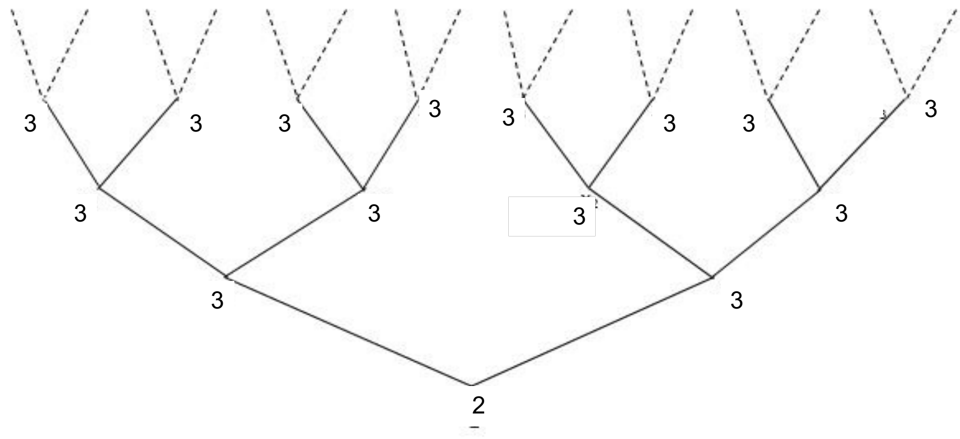


Figure 3.12: σ^n configuration for A_{11} case

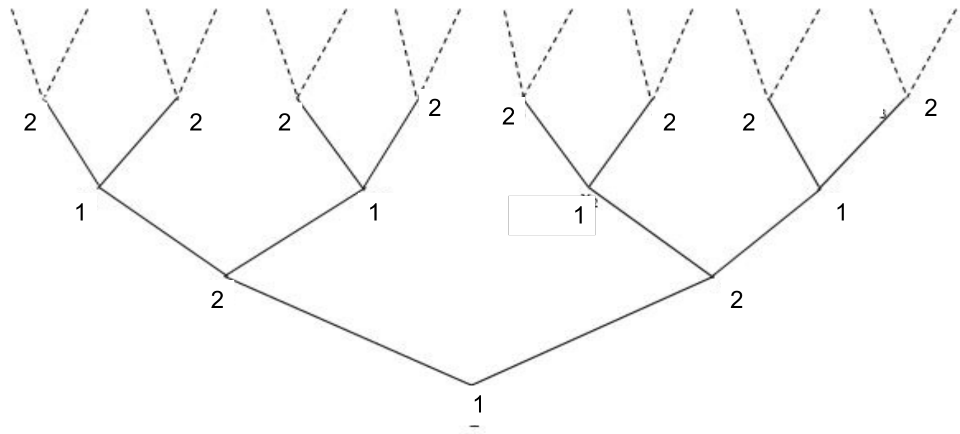


Figure 3.13: σ^{2n} configuration for A_{11} case

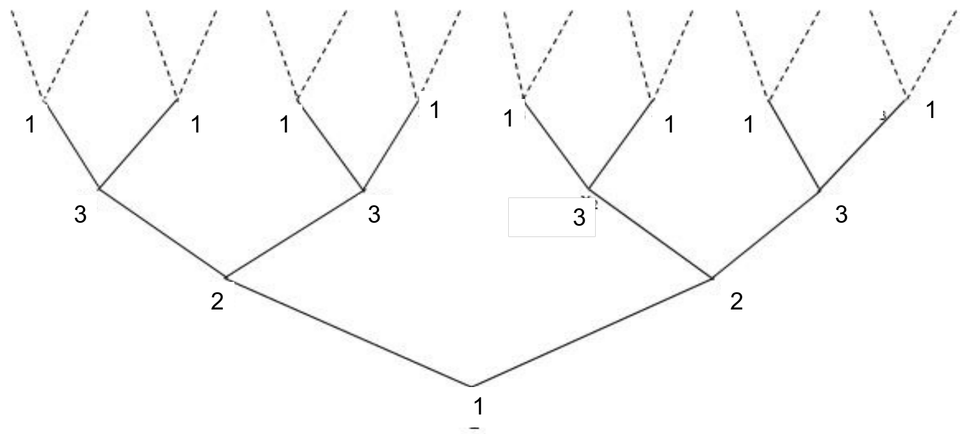


Figure 3.14: $\sigma^{(n+2)}$ configuration for A_{11} case

Part (2):

By following the argument of the proof of Theorem 3.5.1.

Denote:

$\alpha = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}\}$ one can see for any ground state ψ on (Ω) its restriction to any ball B is admissible with α.

Now, let $\Sigma_{1,0} = \{t_n | t_n \in 0, 1, n \in N\}$ take any sequence $(w_n) \in \Sigma_{1,0}$ then construct ground state as follows:

if $w_n = 0$ take one of $\{\sigma_5, \sigma_6, \sigma_7, \sigma_9, \sigma_{10}, \sigma_{12}\} = \overline{\sigma_0}$

if $w_n = 1$ take one of $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_8, \sigma_{11}\} = \overline{\sigma_1}$

Case 1:

let $\sigma(B\downarrow) = \sigma_1$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_2, \sigma_3, \sigma_9\}$ or $\{\sigma_1\}$, and for $\sigma(B_2)$ the choice is $\{\sigma_7\}$.

Case 2:

let $\sigma(B\downarrow) = \sigma_2$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_7, \sigma_6\}$ or $\{\sigma_8, \sigma_{11}\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_5, \sigma_{10}\}$ or $\{\sigma_2, \sigma_3\}$

Case 3:

let $\sigma(B\downarrow) = \sigma_3$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_7, \sigma_6\}$ or $\{\sigma_8, \sigma_{11}\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_6, \sigma_{12}\}$ or$\{\sigma_1, \sigma_4\}$

Case 4:

let $\sigma(B\downarrow) = \sigma_4$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_5, \sigma_{10}\}$ or $\{\sigma_2, \sigma_3\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_7, \sigma_9\}$ or $\{\sigma_8, \sigma_{11}\}$

Case 5:

let $\sigma(B\downarrow) = \sigma_5$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(b_3)$ and $\sigma(B_2)$ the choice will be either on of $\{\sigma_6, \sigma_{12}\}$ or $\{\sigma_1, \sigma_4\}$

Case 6:

let $\sigma(B\downarrow) = \sigma_6$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ and $\sigma(B_2$ the choice will be either on of $\{\sigma_5, \sigma_{10}\}$ or $\{\sigma_2, \sigma_3\}$

Case 7:

let $\sigma(B\downarrow) = \sigma_7$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice will be either on of $\{\sigma_5, \sigma_{10}\}$ or $\{\sigma_2, \sigma_3\}$

Case 8:

let $\sigma(B\downarrow) = \sigma_8$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 1$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_6, \sigma_{11}\}$ or $\{\sigma_1, \sigma_4\}$ and for $\sigma(B_2)$ the choice will be either on of $\{\sigma_6, \sigma_{12}\}$ or $\{\sigma_1, \sigma_4\}$

Case 9:

let $\sigma(B_1\downarrow) = \sigma_9$

then, $\sigma(B_3\downarrow) = 3, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice will be either on of $\{\sigma_6, \sigma_{12}\}$ or $\{\sigma_1, \sigma_4\}$

Case 10:

let $\sigma(B_1\downarrow) = \sigma_{10}$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice will be either on of $\{\sigma_7, \sigma_9\}$ or $\{\sigma_8, \sigma_{11}\}$

Case 11:

let $\sigma(B\downarrow) = \sigma_{11}$

then, $\sigma(B_3\downarrow) = 1, \sigma(B_2\downarrow) = 3$

so, for $\sigma(B_3)$ the choice will be either on of $\{\sigma_5, \sigma_{10}\}$ or $\{\sigma_2, \sigma_3\}$ and for$\sigma(B_2)$ the choice will be either on of $\{\sigma_6, \sigma_{12}\}$ or $\{\sigma_1, \sigma_4\}$

Case 12:

let $\sigma(B_1\downarrow) = \sigma_{12}$

then, $\sigma(B_3\downarrow) = 2, \sigma(B_2\downarrow) = 2$

so, for $\sigma(B_3)$ and $\sigma(B_2)$ the choice will be either on of $\{\sigma_7, \sigma_9\}$ or $\{\sigma_8, \sigma_{11}\}$

The set of all balls M is numbered as shown in Figure 3.9.

Now by using this construction, the construction of a ground state ψ on Ω corresponding to the sequence w_n as follows:

If w_1 is 0 or 1 then ψ_{B_1} is taken from the set $\overline{\sigma_0}$ or $\overline{\sigma_1}$ then using above given construction and look to w and then look for w_2 and continue this procedure and soon correspondingly the needed configuration was obtained, knowing that the set $\Sigma_{0,1}$ is uncountable. Therefore, the constructed ground states are uncountable this completes the proof.

Chapter 4: Gibbs Measures

4.1 Contraction of Gibbs States for λ-Model

In this section, Gibbs measures associated to λ-model will investigated as before with the set $\Phi = \{1,2,3\}$. Let define a finite-dimensional distribution of probability measure μ^n in a volume V_n as:

$$\mu^n(\sigma_n) = \frac{1}{Z_n} e^{\beta H_n(\sigma_n) + \sum\limits_{x \in V} h(\sigma(x),x)} \tag{4.1}$$

where $\sigma_n \in \Phi^{V_n}$, and the partition function given by

$$Z_n = \sum_{\sigma(n) \in \Phi^{V_n}} e^{-[\sum h(\sigma(x),x) + H_n(\sigma_n)\beta]} \tag{4.2}$$

here $h_x = (h_{1,x}, h_{2,x}, h_{3,x}, ..., h_{q,x}) \in R, x \in V$ is the vector-valued function. In this case,

$$H_n(\sigma_n) = \sum_{<x,y> \in V_n} \lambda(\sigma(x), \sigma(y)) \tag{4.3}$$

Take a sequence of probability distribution $\{\mu^{(n)}\}$ if

$$\sum_{\omega_n \in \Phi} \mu^n(\sigma_{n-1} \vee \omega_n) = \mu^{n-1}(\sigma_{n-1}) \tag{4.4}$$

$\forall n \geqslant 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$. In this case, there is a unique measure μ on Φ^V such that $\forall n, \sigma_n \in \Phi^{V_n}$ having

$$\mu^{n+1}(\sigma_{n+1}) = \mu^n(\sigma_n) \tag{4.5}$$

this measure is called a splitting Gibbs measure corresponding to Hamiltonian (4.3) and to the vector valued function $h_x, x \in V$.

Now the next step is formulation of consistency condition for the measures. For the sake of completeness the proof is given [6].

Theorem 4.1.1. *The measures $\mu^n, n = 1,2,3,..$ satisfy the consistency condition if and only if for any $x \in V$ the following equation holds:*

$$u_{k,x} = \prod_{y \in s(x)} \frac{\sum_{j=1}^{q-1} e^{\beta\lambda(k,j)} u_{j,y} + e^{\beta\lambda(k,q)}}{\sum_{j=1}^{q-1} e^{\beta\lambda(q,j)} u_{j,y} + e^{\beta\lambda(q,q)}} \tag{4.6}$$

where $u_{k,x} = e^{h_{k,x} - h_{q,x}}, k = \{1,2,..,q-1\}$.

Proof. According to the consistency condition (4.5):

$$\sum_{\omega_n \in \Phi} \mu^n(\sigma_{n-1} \vee \omega_n) = \sum_{\omega \in \Phi} \frac{1}{Z_n} e^{\beta H_n(\sigma_{n-1} \vee \omega) + \sum_{x \in W_n} h(\omega(x),x)} = \frac{1}{Z_{n-1}} e^{\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h(\sigma(x),x)}$$

where

$$H_n(\sigma_{n-1} \vee \omega_n) = \sum_{\sigma \in \Phi^{V_{n-1}}} \lambda(\sigma(x), \sigma(y)) + \sum_{x \in W_{n-1} y \in S(x)} \lambda(\sigma(x), w(y))$$

$$= H_{n-1}(\sigma) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \lambda(\sigma(x), \sigma(y))$$

now let

$$\frac{Z_{n-1}}{Z_n} e^{\beta \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \lambda(\sigma(x),\omega(y)) + \beta \sum_{x \in W_{n-1}} \sum_{y \in S(x)} h_{\omega(x),(x)}} = e^{\sum_{x \in W_{n-1}} h_{\sigma(x),(x)}}$$

$$\frac{Z_{n-1}}{Z_n} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum_{\omega \in \Phi^{V_n}} e^{\beta\lambda(\sigma(x),\omega(y)) + h_{\omega(y),(y)}} = \prod_{x \in W_{n-1}} e^{h_{\sigma(x),(x)}} \tag{4.7}$$

considering configuration $\sigma^{(k)} \in \Phi^{V_{n-1}}, \Phi = \{1,2,3,..q\}$ such that $\sigma(x) = k$ for fixed $x \in V$

and $k = 1, q$, and dividing (4.7) at $\sigma^{(k)}$ by (4.7) at $\sigma^{(q)}$ one gets

$$\prod_{y \in S(x)} \frac{\sum_{\omega \in \Phi} e^{\beta\lambda(k,\omega(y))+h_{\omega(y),y}}}{\sum_{\omega \in \Phi} e^{\beta\lambda(q,\omega(y))+h_{\omega(y),y}}} = \frac{e^{h_{k,x}}}{e^{h_{q,x}}} \tag{4.8}$$

so,

$$\prod_{y \in S(x)} \frac{\sum_{\omega \in \Phi}^{q} e^{\beta\lambda(k,j)+h_{j,y}}}{\sum_{\omega \in \Phi}^{q} e^{\beta\lambda(q,j)+h_{j,y}}} = e^{h_{k,x}-h_{q,x}} \tag{4.9}$$

hence, by denoting $u_{k,x} = e^{h_{k,x}-h_{q,x}}$ from (2.9) one finds

$$\prod_{y \in S(x)} \frac{\sum_{j=1}^{q-1} e^{\beta\lambda(k,j)+u_{j,y}+e^{b\lambda(k,q)}}}{\sum_{j=1}^{q-1} e^{\beta\lambda(q,j)+u_{j,y}+e^{\beta\lambda(q,q)}}} = u_{k,x} \tag{4.10}$$

suppose that (4.5) holds, then will get (4.9), which yields that

$$\prod_{y \in S(x)} \sum_{j=1}^{q} e^{\beta\lambda(\sigma(x),j)+h_{j,y}} = a(x)e^{h_{k,x}}, k = 1, q x \in W_{n-1} \tag{4.11}$$

for some function $a(x) > 0, x \in V$.

Let us multiply (4.11) by x, $x \in W_{n-1}$, then by obtaining

$$\prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum_{j=1}^{q} e^{\beta\lambda(\sigma(x),j)+h_{j,y}} = \prod_{x \in w_{n-1}} (a(x)e^{h_{k,x}}) \tag{4.12}$$

for any configuration $\sigma \in \Phi_{n-1}^{V}$. Denoting $A_{n-1} = \prod_{x \in W_{n-1}} a(x)$, from (4.12), one finds

$$\prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum_{\omega \in \Phi} e^{\beta\lambda(\sigma(x),\omega(y))+h_{\omega(y),y}} = A_n \prod_{x \in W_{n-1}} e^{h_{\sigma(x),x}} \tag{4.13}$$

by multiplying both sides of (4.13) by $e^{\beta H_{n-1}(\sigma)}$,

$$e^{\beta H_{n-1}(\sigma)} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum_{\omega \in \Phi} e^{\beta \lambda(\sigma(x),\omega(y)) + h_{\omega(y),y}} = A_n e^{\beta H_{n-1}(\sigma)} \prod_{x \in W_{n-1}} e^{h_{\sigma(x),x}}$$

which yields

$$Z_n \sum_{\omega_n \in \Phi^{V_n}} \mu^n(\sigma \vee \omega_n) = A_{n-1} Z_{n-1} \mu^{(n-1)}(\sigma) \tag{4.14}$$

since $\mu^n(\sigma \vee \omega_n), n \geqslant 1$ is a probabilistic measure,

$$\sum_{\omega_n \in \Phi^{V_n}} \mu^n(\sigma \vee \omega_n) = \mu^{n-1}(\sigma) = 1$$

which form (4.14) yields

$$Z_n = A_{n-1} Z_{n-1}$$

□

The proved theorem allows us to find Gibbs measures by solving equation (4.6). In next section translation invariant solutions will described.

4.2 Translation-Invariant Gibbs Measure

It is known now that $\{\mu^n\}$ consisted if $\mu^n(\sigma_{(n)}) = \mu^{n-1}(\sigma_{(n-1)})$ therefore, $h_{\sigma(x),x} = h_{\sigma(x)}$ in this case, $q = 3, k = 1, 2, .., q-1$ let $u_{k,x} = e^{h_{k,x} - h_{q-1,x}}$

$u_{(1,x)}, u_{(2,x)} = u_1, u_2$ so, $u_1 = e^{h_1 - h_2}$ and $u_2 = e^{h_2 - h_3}$

by Theorem 4.1.1:

$$u_{k,x} = \prod_{y \in S(x)} \frac{\sum_{j=1}^{q-1} e^{\beta \lambda(k,j)} u_{j,y} + e^{\beta \lambda(k,q)}}{\sum_{j=1}^{q-1} e^{\beta \lambda(q,j)} u_{j,y} + e^{\beta \lambda(q,q)}}$$

only for splitting Gibbs measure because its satisfy consistency condition. Now, by finding translation invariant Gibbs measure, the existence of phase transition will found, so by assume that in splitting Gibbs measure $\overline{u_x} = \overline{u_y}\ \forall\ x,y \in V$ such that $\overline{u_x} = \overline{u} = (u_1,u_2), u_1,u_2 > 0$

by using Theorem 4.1.1 for k=2

$$u_1 = \frac{e^{\beta\lambda(1,1)}u_1 + e^{\beta\lambda(1,2)}u_2 + e^{\beta\lambda(1,3)}}{e^{\beta\lambda(3,1)}u_1 + e^{\beta\lambda(3,2)}u_2 + e^{\beta\lambda(3,3)}} . \frac{e^{\beta\lambda(1,1)}u_1 + e^{\beta\lambda(1,2)}u_2 + e^{\beta\lambda(1,3)}}{e^{\beta\lambda(3,1)}u_1 + e^{\beta\lambda(3,2)}u_2 + e^{\beta\lambda(3,3)}}$$

$$u_1 = \left[\frac{e^{\beta\lambda(1,1)}u_1 + e^{\beta\lambda(1,2)}u_2 + e^{\beta\lambda(1,3)}}{e^{\beta\lambda(3,1)}u_1 + e^{\beta\lambda(3,2)}u_2 + e^{\beta\lambda(3,3)}}\right]^2$$

and

$$u_2 = \frac{e^{\beta\lambda(2,1)}u_1 + e^{\beta\lambda(2,2)}u_2 + e^{\beta\lambda(2,3)}}{e^{\beta\lambda(3,1)}u_1 + e^{\beta\lambda(3,2)}u_2 + e^{\beta\lambda(3,3)}} . \frac{e^{\beta\lambda(2,1)}u_1 + e^{\beta\lambda(2,2)}u_2 + e^{\beta\lambda(2,3)}}{e^{\beta\lambda(3,1)}u_1 + e^{\beta\lambda(3,2)}u_2 + e^{\beta\lambda(3,3)}}$$

$$u_2 = \left[\frac{e^{\beta\lambda(2,1)}u_1 + e^{\beta\lambda(2,2)}u_2 + e^{\beta\lambda(2,3)}}{e^{\beta\lambda(3,1)}u_1 + e^{\beta\lambda(3,2)}u_2 + e^{\beta\lambda(3,3)}}\right]^2$$

recall that:

$\lambda(2,1) = \lambda(1,2) = \lambda(3,2) = \lambda(2,3) = a$

$\lambda(3,1) = \lambda(1,3) = b$

$\lambda(1,1) = c_1$

$\lambda(2,2) = c_2$

$\lambda(3,3) = c_3$

denote: $\theta_1 = e^{\beta a}$, $\theta_2 = e^{\beta b}$, $\theta_3 = e^{\beta c_1}$, $\theta_4 = e^{\beta c_2}$, $\theta_5 = e^{\beta c_3}$.

It is obtained that

$$u_1 = \left[\frac{\theta_3 u_1 + \theta_1 u_2 + \theta_2}{\theta_2 u_1 + \theta_1 u_2 + \theta_5}\right]^2 \tag{4.15}$$

$$u_2 = \left[\frac{\theta_1 u_1 + \theta_4 u_2 + \theta_1}{\theta_2 u_1 + \theta_1 u_2 + \theta_5}\right]^2 \tag{4.16}$$

For now assume $\theta_5 = \theta_3$ and $u_1 = 1$ is the invariant line for

$$u_2 = \left[\frac{2\theta_1 + \theta_4 u_2}{\theta_2 + \theta_1 u_2 + \theta_3}\right]^2$$

after some calculation, the last one is reduced to

$$\frac{2\theta_1^3}{\theta_4^3}\frac{\theta_4}{2\theta_1}u_2 = \left[\frac{1 + \frac{\theta_4}{2\theta 1}u_2}{\theta_4\frac{(\theta_2+\theta_3)}{2\theta_1^2} + \frac{\theta_4}{2\theta_1}u_2}\right]^2$$

Let, $x = \frac{\theta_4}{2\theta_2}u_2,\ a = 2\left[\frac{\theta_1}{\theta_4}\right]^3,\ b = \frac{\theta_4(\theta_2+\theta_3)}{2\theta_1^2}$
such that

$$ax = \left[\frac{1+x}{b+x}\right]^2 \tag{4.17}$$

since x>0, k>0, a>0 and b>0 and according to proposition 10.7 [18] the next lemma is:

Lemma 4.2.1. *1. If $b \leq 9$ then a solution of (4.17) is unique.*

2. *If b>9 then there are $\eta_1(b)$ and $\eta_2(b)$ such that $0 < \eta_1 < \eta_2$*

   and if $\eta_1 < a < \eta_2$ then (4.17) has three solutions

3. *If $a = \eta_1$ or $a = \eta_2$ then (4.17) has two solutions.*

   The quantity η_1 and η_2 are determined from the formula

$$\eta_i(b) = \frac{1}{x_i}\left(\frac{1+x_i}{b+x_i}\right)^2, i = 1,2 \tag{4.18}$$

where x_1 and x_2 are solutions to the equation $x^2 + [2-(b-1)]x + b = 0$ where

$$xf'(x) = f(x) \text{ if } f(x) = \left[\frac{1+x}{b+x}\right]^2.$$

Now, the solutions of the equation $x^2 + [2-(b-1)]x + b = 0$ are:

$$x_1 = \frac{1}{2} + (-3 + b - \sqrt{9 - 10b + b^2})$$

$$x_2 = \frac{1}{2} + (-3 + b + \sqrt{9 - 10b + b^2})$$

Substituting x_1, x_2 into η_i yields:

$$\eta_1(b) = \frac{-2(5 - b + \sqrt{9 - 10b + b^2})^2}{(3 - b + \sqrt{9 - 10b + b^2})(3 - 3b + \sqrt{9 - 10b + b^2})^2} \tag{4.19}$$

$$\eta_2(b) = \frac{4(-5 - b + \sqrt{9 - 10b + b^2})^2}{(-3 + b + \sqrt{9 - 10b + b^2}^2)(-3 + 3b + \sqrt{9 - 10b + b^2})^2} \tag{4.20}$$

Since $b = \dfrac{\theta_4(\theta_2 + \theta_3)}{2\theta_1^2}$, conclude that $\eta_1(\theta_i), \eta_2(\theta_i)$ depend on θ_i where $i = 1, 2, 3, 4$.

Theorem 4.2.2. *Assume for the λ-model $\theta_3 = \theta_5$ and $b > 9$ and $\eta_1 < 2\left[\dfrac{\theta_1}{\theta_4}\right]^3 < \eta_2$ where $b = \dfrac{\theta_4(\theta_2 + \theta_3)}{2\theta_1^2}$ and η_1 and η_2 are given by (4.19) (4.20), then the phase transitions occur.*

Example 4.2.1. Let b=12. then $x_1 = 3.75$ and $x_2 = 15.24$. Then $\eta_1 = 0.0242$ and $\eta_2 = 0.0233$ so from theorem conclude that if $0.0242 < \dfrac{2\theta_2^3}{\theta_1^3} < 0.0233$, then the phase transition will occur.

Now, in study of the relation between different θ's under condition (2) in Lemma 4.2.1, suppose $\theta_1 = \theta_2 = \theta_3$, which implies $b = \dfrac{\theta_4}{\theta_1}, a = \dfrac{2}{b^3}$ since b>9 then $\theta_4 > 9\theta_1$ and according to the previous lemma $\eta_1(b) < a(b)$ and $\eta_2(b) > a(b)$. which is clear in Figure 4.1.

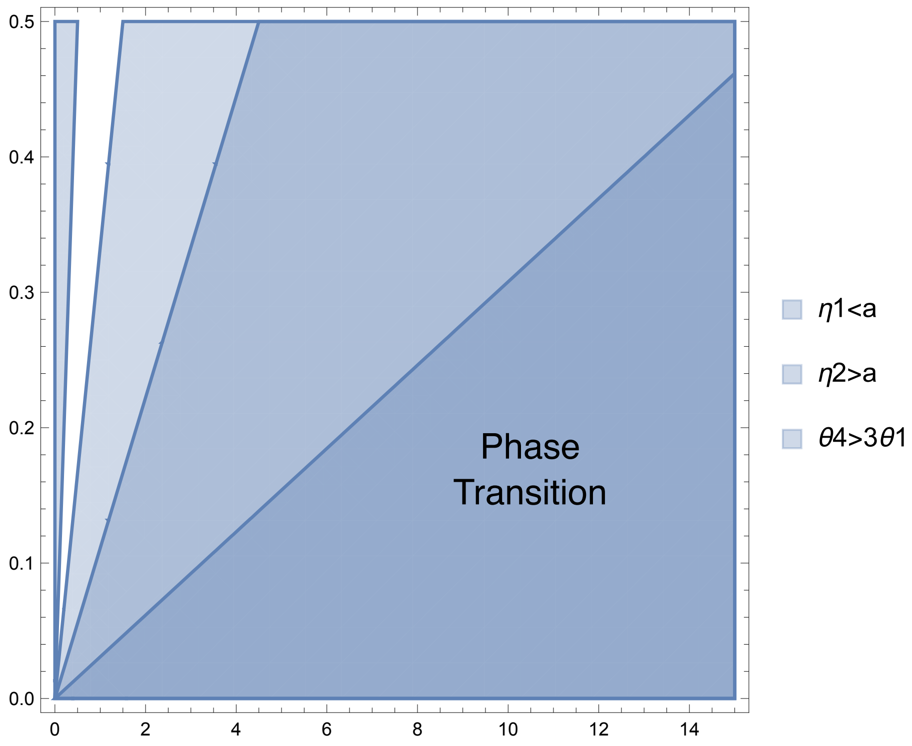


Figure 4.1: In the case of $\theta_2 = \theta_3 = \theta_1$

The graph below (Figure 4.2) shows the case of $\theta_4 = \theta_2 = \theta_3$, which implies $b = [\dfrac{\theta_4}{\theta_1}]^2$, $a = \dfrac{2}{b^{3/2}}$ finding that $\theta_4 > 3\theta_1$ and according to the previous lemma $\eta_1(b) < a(b)$ and $\eta_2(b) > a(b)$.

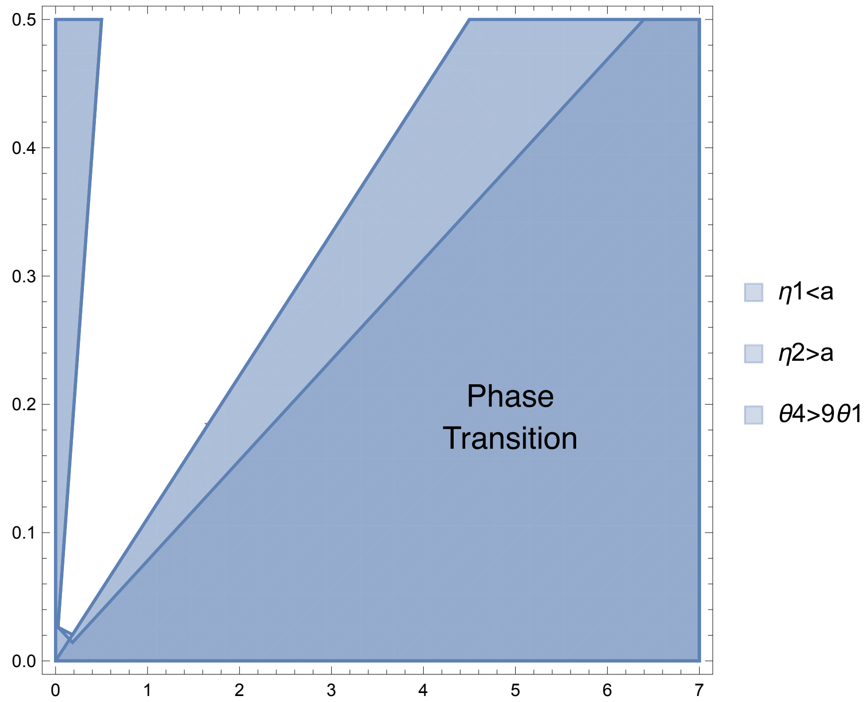


Figure 4.2: In the case of $\theta_4 = \theta_2 = \theta_3$

Chapter 5: Conclusion

This thesis, has two main objectives. The first objective is to study ground states of λ-model on the Cayley tree of order two. After applying Hamiltonian on λ function, 11 different possible cases of ground states were found. The Cases 1, 2 and 3 have one translation invariant ground state each case is different from others. Case 4 has 2-periodic ground states. In Case 5, there are 2n-periodic ground states and uncountable number of ground states. For Cases 7, 8, 9 and 10, there is only one translation invariant, Gn-periodic ground states and the uncountable number of ground states. In the last case, there are (Gn, 2n, n+2) periodic ground states and uncountable number of ground state.

From given Hamiltonian and vector-valued function, the Gibbs measures were constructed, also by using splitting Gibbs measure the translation invariant Gibbs measure of λ-model was described and showed that the existence of the phase transition.

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