Some results for Volterra integro-differential equations depending on derivative in unbounded domains

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Abstract

In this paper we study the existence of continuous solutions of an integro-differential equation in unbounded interval depending on derivative. This paper extends some results obtained by the authors using the technique developed in their previous paper. This technique consists in introducing, in the given problems, a function q, belonging to a suitable space, instead of the state variable x. The fixed points of this function are the solutions of the original problem. In this investigation we use a fixed point theorem in Fréchet spaces.
Keywords: Fréchet spaces, semi-norms, acyclic sets, Ascoli-Arzelà theorem.
2010 AMS subject classification: 45G10, 47H09, 47H30*

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1 Introduction

In this paper we study, in abstract setting, the solvability of a nonlinear integro-differential equation of Volterra type with implicit derivative, defined in unbounded interval, like

$$(1) \quad x'(t) = \int_0^t k(t,s)f\big(s,x(s),x'(s)\big)ds \qquad x(0) = 0, \quad t \in J = [0,+\infty)$$

We will look for solutions of this equation in the Fréchet space of all real C^1 functions defined in the real unbounded interval $J = [0,+\infty)$.

Equation (1) is a special case of integro-differential equations. These equations have been seen as an important tool in the study of many boundary problems that we can encounter in various applications, like, for exemple, heat flow in material, kinetic theory, electrical ingeneering, vehicular traffic theory, biology, population dynamics, control theory, mechanics, mathematical economics.

The integro-differential equations have been studied in various papers with the help of several tools of functional analysis, topology and fixed point theory. For istance we can refer to [1], [2], [3], [4], [5], [11], [12] and the references therein. In [8] an Hammerstein equation, similar to (1), is consideren in the multivalued setting and bounded intervals.

Our paper extend some results obtained by the authors Anichini and Conti, using the techniques developed in previous paper (see to [1], [2], [3], [4], [5]).

The crucial key of our approach, in order to find solutions of equation (1), consists in the use of a very useful fixed point theorem for multivalued, compact, uppersemicontinuous maps with acyclic values in a Fréchet space.

2 Preliminaries and Notations

Let $C^1(J, \mathbb{R})$ be the Fréchet the of all real C^1 functions defined in the real unbounded interval $J = [0,+\infty) \subset \mathbb{R}$, equipped with the following family of semi-norms

$$\|x\|_{1,n} = \max\{\|x\|_n, \|x'\|_n\}$$

where $\|x\|_n = \sup\{|x(t)|, t \in [0,n]\}$ and $\|x'\|_n = \sup\{|x'(t)|, t \in [0,n]\}$.
We recall that the topology of $C^1(J, \mathbb{R})$ coincides with the topology of a complete metric space $\{F, d\}$ where

$$d(x,y) \ = \ \sum_{n=1}^{+\infty} \frac{2^{-n}\|x-y\|_{1,n}}{1+\|x-y\|_{1,n}}$$

A subset $A \subset C^1(J, \mathbb{R})$ is said to be bounded if, for every natural number n, there exists $M_n > 0$ such that $\|x\|_{1,n} \leq M_n \quad \forall \, x \in C^1(J, \mathbb{R})$.

A subset $A \subset C^1(J, \mathbb{R})$ is relatively compact set if and only if the functions of the set A are equicontinuous and uniformly bounded (with their derivatives) in any interval $[0, n]$.

We will denote by $C(F)$ the family of all nonempty and compact subset of a Fréchet space F.

Let M be a subset of a Fréchet space F; a multivalued map $S: M \to C(F)$ is said to be uppersemicontinuous (u.s.c.) if the graph is closed in $M \times F$, i.e. for any sequence $\{x_n\} \subset M$, $x_n \to x_0$ and $y_n \in S(x_n)$, $y_n \to y_0$, we have $y_0 \in S(x_0)$.

A multivalued map $S: M \to C(F)$ is said to be compact if it sends bounded sets into relatively compact sets. We apply the same definition for singlevalued maps.

A subset A of a metric space E is said to be an R_δ - set if A is the intersection of a countable decreasing sequence of absolute retracts contained in E (see [10]).

It is known that an R_δ - set is an acyclic set, i.e. it is acyclic with respect to any cohomology theory (see [7]).

Let M be a subset of the Fréchet space $C^1(J, \mathbb{R})$ and consider an operator $T: M \to C^1(J, \mathbb{R})$. Let $\{\epsilon_n\}$ be an infinitesimal sequence of real numbers.

A sequence $\{T_n\}$ of maps $T_n: M \to C^1(J, \mathbb{R})$ is said to be an ϵ_n-approximation of T on M if $\|T_n(x) - T(x)\|_{1,n} \leq \epsilon_n$ for every $x \in M$ and for any natural number n.

Define $U_n = \{x \in F : \|x\|_{1,n} < 1\}$.

Let T be a compact map $T: M \to C^1(J, \mathbb{R})$, where M is a closed set of the Fréchet space $C^1(J, \mathbb{R})$, and let $\{T_n\}$ be a ϵ_n-approximation of T on M, where $T_n: M \to C^1(J, \mathbb{R})$ are compact maps; then the set of fixed point of T is a compact R_δ - set if the equation $x - T_n(x) = y$ has at most a solution for every $y \in \varepsilon_n U_n$ for any natural number n (see [5]).

In the sequel we will use the following result (see [9]).

Proposition 1 (Kirszebraun's Theorem)
Let $F : M \to \mathbb{R}$ be a Lipschitz map defined on arbitrary subset M of $\mathbb{R}^n$. Then F admits a Lipschitz extension $\mathfrak{J}: \mathbb{R}^n \to \mathbb{R}$ with the same Lipschtiz constant.

The well known Gronwall's Lemma, from the standard theory of Ordinary Differential Equations, will be used.

Proposition 2 (Gronwall's Lemma)
Let $g, h : J \to J$ be continuous functions such that the following inequality:

$$g(t) \leq u(t) + \int_0^t h(s)g(s)ds \quad t \in J,$$

holds, where $u : J \to J$ is a continuous nondecreasing function.
Then we have:

$$g(t) \leq u(t)\exp\left(\int_0^t h(s)ds\right) \quad t \in J.$$

In the sequel we will use the following proposition that can be deduced from Theorem 1 of [6].

Proposition 3 (a fixed poin theorem)
Let F be a Fréchet space and $M \subset X$ be a bounded, closed and convex subset; let $S: F \to M$ be a multivalued, uppersemicontinuous map with acyclic values. If $S(F)$ is (relatively) compact, then S has a fixed point.

3 Main result

The following result holds.

Theorem
Consider integral equation (1). Assume that
i) $k : J \times J \to \mathbb{R}$ is a C^1 function; moreover we assume that there exists a continuous function $h : J \to J$ with

$$|k(t,s)| \leq h(s) \text{ and } \left|\frac{\partial k(t,s)}{\partial t}\right| \leq h(s).$$

ii) $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function; moreover we assume that there exist continuous functions $a, b : J \to J$, with $\int_0^{+\infty} a(s)ds = A < +\infty$ and $\int_0^{+\infty} b(s)ds = B < +\infty$, such that:

$$|f(s,x,y)| \leq a(s) + b(s)|y|.$$

iii)   Assume that $\int_0^{+\infty} h(s)b(s)ds = \beta < 1$.

Then equation (1) has at least one solution in the space $C^1(J, \mathbb{R})$.

Proof

Let q be a function belonging to $C^1(J, \mathbb{R})$ and consider the following integral equation:

$$(2) \quad y(t) = \int_0^t k(t,s) f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds \quad t \in J = [0, +\infty)$$

Let $S : C^1(J, \mathbb{R}) \to C^1(J, \mathbb{R})$ be the multivalued map which associates to every $q \in C^1(J, \mathbb{R})$ the set of solutions of equation (2).

Clearly, putting $x(t) = \int_0^t y(s)ds$ (hence $x'(t) = y(t)$ and $x(0) = 0$), we have that the fixed points of the map S are the solution of equation (1).

In order to find the fixed points of multivalued map S, the following steps in the proof have to be established (Proposition 3):

a) There exists a bounded, closed and convex set $M \subset C^1(J, \mathbb{R})$ such that $S(C^1(J, \mathbb{R})) \subset M$.

b) The set $S(C^1(J, \mathbb{R}))$ is relatively compact.

c) The map S is uppersemicontinuous.

d) The set $S(q)$ is an acyclic set for every $q \in C^1(J, \mathbb{R})$.

a) Let $q \in C^1(J, \mathbb{R})$ and consider equation (2); assume that $t \in [0, n]$, from hypotheses we have, :

$$|y(t)| = \left| \int_0^t k(t,s) f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds \right| \le$$

$$\le \left| \int_0^t h(s)(a(s) + b(s)|y(s)|)ds \right| \le$$

$$\le \left| \int_0^t h(s)a(s)ds \right| + \left| \int_0^t h(s)b(s)|y(s)|ds \right|$$

$$\le \|h\|_n A + \beta \|y\|_n.$$

So that, since $\beta < 1$, we have $\|y\|_n \le \frac{\|h\|_n A}{1-\beta}$.

Moreover, we have for $t \in [0, n]$:

$$y'(t) = \int_0^t \frac{\partial k(t,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds + k(t,t)f\left(t, \int_0^t q(s)ds, y(t)\right)$$

and we obtain:

$$|y'(t)| \leq \left|\int_0^t \frac{\partial k(t,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds\right|$$

$$+ \left|k(t,t)f\left(t, \int_0^t q(s)ds, y(t)\right)\right| \leq$$

$$\leq \int_0^t h(s)a(s)ds + \int_0^t h(s)b(s)|y(s)|ds + h(t)(a(t) + b(t)|y(t)|) \leq$$

$$\leq \|h\|_n A + \beta\|y\|_n + \|ha\|_n + \|hb\|_n\|y\|_n \leq$$

$$\leq \|h\|_n A + \|ha\|_n + \|y\|_n(\beta + \|hb\|_n)$$

$$\leq \|h\|_n A + \|ha\|_n + \frac{\|h\|_n A}{1-\beta}(\beta + \|hb\|_n).$$

So that there exists $M_n > 0$ such that $\|y\|_{1,n} \leq M_n$.
Then we have $S(C^1(J, \mathbb{R})) \subset M$, where

$$M = \{y \in C^1(J, \Re), \|y\|_{1,n} \leq M_n\}.$$

b) Now, we want to prove that the set $S(C^1(J, \mathbb{R}))$ is relatively compact.
Let $y \in S(C^1(J, \mathbb{R}))$ and fix $\varepsilon > 0$. For any $u, w \in [0, n]$ we have:

$$y'(w) - y'(u) =$$

$$\int_0^w \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds + k(w,w)f\left(w, \int_0^w q(s)ds, y(w)\right) -$$

$$\int_0^u \frac{\partial k(u,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds - k(u,u)f\left(u, \int_0^u q(s)ds, y(u)\right)$$

$$= \int_0^u \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds + k(w,w)f\left(w, \int_0^w q(s)ds, y(w)\right)$$

$$- \int_0^u \frac{\partial k(u,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds$$

$$-k(u,u)f\left(u, \int_0^u q(s)ds, y(u)\right) + \int_u^w \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds$$

It follows that

$$|y'(w) - y'(u)| \leq$$

$$\leq \int_0^u \left| \frac{\partial k(w,s)}{\partial t} - \frac{\partial k(u,s)}{\partial t} \right| (a(s) + b(s)|y(s)|)ds$$

$$+ \|h\|_n \left| f\left(w, \int_0^w q(\tau)d\tau, y(w)\right) - f\left(u, \int_0^u q(\tau)d\tau, y(u)\right) \right|$$

$$+ \|h\|_n \left| \int_u^w \frac{\partial k(w,s)}{\partial t} f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) ds \right|$$

By continuity of the functions q, h, f and $\dfrac{\partial k}{\partial t}$ it follows that there exists $\delta > 0$ such that for $|w - u| < \delta, u, w \in [0, n],$ we have

$$|y'(w) - y'(u)| < \varepsilon$$

Since $|y(w) - y(u)| \leq M_n|w - u|$, we can conclude that the set $S(C^1(J, \mathbb{R}))$ is relatively compact.

c) Let us now show that the map S is uppersemicontinuous.
Let $\{q_m\}$ be a sequence, $q_m \in C^1(J, \mathbb{R})$, with $\|q_m - q_0\|_{1,n} \to 0$,
$y_m \in S(q_m)$, i.e.

$$y_m(t) = \int_0^t k(t,s)f\left(s, \int_0^s q_m(\tau)d\tau, y_m(s)\right) ds \quad t \in [0,n]$$

Assume that $\|y_m - y_0\|_{1,n} \to 0$. We need to show that $y_0 \in S(q_0)$.
From the Dominated Lebesgue Convergence Theorem it follows:

$$\lim_{m \to +\infty} f\left(s, \int_0^s q_m(\tau)d\tau, y_m(s)\right) = f\left(s, \int_0^s q_0(\tau)d\tau, y_0(s)\right)$$

and

$$\lim_{m \to +\infty} y_m(t) =$$

$$= \lim_{m \to +\infty} \int_0^t k(t,s)f\left(s, \int_0^s q_m(\tau)d\tau, y_m(s)\right)ds =$$

$$= \int_0^t \lim_{m \to +\infty} k(t,s)f\left(s, \int_0^s q_m(\tau)d\tau, y_m(s)\right)ds =$$

$$= \int_0^t k(t,s)f\left(s, \int_0^s q_0(\tau)d\tau, y_0(s)\right)ds .$$

Hence, we obtain

$$y_0(t) = \int_0^t k(t,s)f\left(s, \int_0^s q_0(\tau)d\tau, y_0(s)\right)ds$$

i. e. $y_0 \in S(q_0)$

d) Now we want to show that, for every fixed $q \in C^1(J, \mathbb{R})$, the set $S(q)$ is acyclic. Consider equation (2) (with q fixed).

Put $f\left(s, \int_0^s q(\tau)d\tau, y(s)\right) = l(s,y)$.

Then equation (2) can be written in the following way:

$$y(t) = \int_0^t k(t,s)l(s,y(s))ds \quad t \in [0, +\infty)$$

We have:

$$|y(t)| \leq \left|\int_0^t k(t,s)l(s,y(s))ds\right| \leq \int_0^t h(s)a(s)ds + \int_0^t h(s)b(s)|y(s)|ds.$$

From Gronwall's Lemma it follows that:

$$|y(t)| \leq \int_0^t h(s)a(s)ds \, \exp(\int_0^t h(s)b(s)ds) = m(s)$$

where *m* is a continuous function.

Let $U: \mathbb{R} \to [0,1]$ the Uryshon (continuous) function defined by

$$U(z) = 1 \text{ if } |z| \leq 1 \text{ and } U(z) = 0 \text{ if } |z| \geq 2.$$

Now we define the function

$$g(s,y) = U\left(\frac{y}{m(s)+1}\right)l(s,y).$$

Clearly $g(s,y) = l(s,y)$ when $|y| \leq m(s)$. Hence the set of solutions of the following equation

$$y(t) = \int_0^t k(t,s)g(s,y(s))ds \quad t \in [0,+\infty)$$

coincides with the set of solutions of equation (2) with q fixed.

Consider now the integral operator $H: C^1(J,\mathbb{R}) \to C^1(J,\mathbb{R})$:

$$\big(H(y)\big)(t) = \int_0^t k(t,s)g\big(s,y(s)\big)ds \quad t \in [0,+\infty)$$

If $z = H(y)$, we have

$$z(t) = \int_0^t k(t,s)U\left(\frac{y(s)}{m(s)+1}\right)l(s,y(s))ds \,.$$

Notice that

$$U\left(\tfrac{y(s)}{m(s)+1}\right)l(s,y(s)) = l(s,y(s)) \text{ if } y(s) \leq m(s)+1$$

and

$$U\left(\frac{y(s)}{m(s)+1}\right)l(s,y(s)) = 0 \text{ if } y(s) \geq 2m(s)+2.$$

So that:

$$\|z\|_n \leq \|h\|_n A + 2\beta(\|m\|_n + 1).$$

Moreover we obtain:

$$|z'(t)| \leq \int_0^t h(s)U\left(\frac{y(s)}{m(s)+1}\right)|l(s,y(s))|ds$$
$$+ h(t)U\left(\frac{y(t)}{m(t)+1}\right)|l(t,y(t))|$$

Hence

$$\|z'\|_n \leq \|h\|_n A + 2\beta(\|m\|_n + 1) + \|ha\|_n + 2\|hb\|_n(\|m\|_n + 1) = A_n$$

It follows that $\|z\|_{1,n} \leq A_n$, where $z = H(y)$.

So that the set of solutions of equation

$$y(t) = \int_0^t k(t,s)g(s,y(s))ds \quad t \in [0,+\infty)$$

coincides with the set of fixed points of operator H in the set

$$A = \left\{z \in C^1(J,\mathbb{R}), \|z\|_{1,n} \leq A_n\right\}.$$

It is easy to see (again as consequence of the Ascoli- Arzelà Theorem) that the set $H(A)$ is relatively compact set.

Moreover H is a continuous operator; to show the last assertion, let us take $y_0, y_m \in A, \|y_m - y_0\|_{1,n} \to 0, z_m \in H(y_m), \|z_m - z_0\|_{1,n} \to 0$; we are going to prove that $z_0 \in H(y_0)$.

For every $t \in [0,n]$ we have:

$$\lim_{m \to +\infty} \left| \int_0^t k(t,s)g(s,y_m(s))ds - \int_0^t k(t,s)g(s,y_0(s))ds \right| \le$$

(from the Dominated Lebesgue Convergence Theorem and the continuity of function g)

$$\le \int_0^t \lim_{m \to +\infty} h(s) \left| g(s,y_m(s)) - g(s,y_0(s)) \right| ds.$$

Hence

$$z_0(t) = \int_0^t k(t,s)g(s,y_0(s))ds = (H(y_0))(t).$$

Fix now a natural number *n*. We know (Proposition 1) that there exists a Lipschitz function

$$g_n: [0,n] \times [-A_n, A_n] \to \mathbb{R}$$

such that, for every $(s,y) \in [0,n] \times [-A_n, A_n]$, we have:

$$|g_n(s,y) - g(s,y)| \le \frac{1}{(n+1)^2 \|h\|_n}$$

and

$$|g_n(s,y) - g_n(s,y_1)| \le L_n |y - y_1|$$

for every $(s,y), (s,y_1) \in [0,n] \times [-A_n, A_n]$,

Let $G_n: J \times \mathbb{R} \to \mathbb{R}$ be the Lipschitz extension of the function g_n; hence

$G_n(s,y) = g_n(s,y)$ for every $(s,y) \in [0,n] \times [-A_n, A_n]$

and $|G_n(s,y) - G_n(s,y_1)| \le L_n |y - y_1|$ for every $(s,y), (s,y_1) \in J \times \mathbb{R}$. Let $H_n : A \to C^1(J, \mathbb{R}))$ be the operator defined as follows:

$$(H_n(y))(t) = \int_0^t k(t,s)G_n(s,y(s))ds \quad t \in [0,+\infty)$$

Clearly this operator is compact for every natural number n.
Moreover, for every $t \in [0, n]$ and $y \in A$, we have:

$$\left| (H_n(y))(t) - (H(y))(t) \right| \le \int_0^t k(t,s) \left| G_n(s, y(s)) - g(s, y(s)) \right| ds \le$$

$$\le n\|h\|_n \, \frac{1}{(n+1)^2 \|h\|_n} < \frac{1}{n}.$$

So that $\|H_n(y) - H(y)\|_n < \frac{1}{n}$.
Moreover we have for every $y \in A$:

$$\left| (H'_n(y))(t) - (H'(y))(t) \right| \le$$

$$\le \left| \int_0^t \frac{\partial k(t,s)}{\partial t} G_n(s, y(s)) ds \, - k(t,t) G_n(t, y(t)) \right.$$

$$\left. + \int_0^t \frac{\partial k(t,s)}{\partial t} g(s, y(s)) ds \, - k(t,t) g(t, y(t)) \right| \le$$

$$\int_0^t h(s) \left| G_n(s, y(s)) - g(s, y(s)) \right| ds \, + h(t) \left| G_n(t, y(t)) - g(t, y(t)) \right| \le$$

$$\le n\|h\|_n \, \frac{1}{(n+1)^2 \|h\|_n} + \|h\|_n \, \frac{1}{(n+1)^2 \|h\|_n} = \frac{n+1}{(n+1)^2} < \frac{1}{n}.$$

Hence $\|H'_n(y) - H'(y)\|_n < \frac{1}{n}$.

Let now $b \in A$. We consider the equation $y - H_n(y) = b$. We want to prove that it has at most one solution. Consider the equation $z - H_n(z) = b$; then, for every $t \in J$ and by Gronwall's Lemma we have:

$$|y(t) - z(t)| \le \int_0^t h(s) \left| G_n(s, y(s)) - G_n(s, z(s)) \right| ds \le$$

$$\le \int_0^t h(s) L_n |y(s) - z(s)| ds \le 0.$$

So that we can say that $y(t) = z(t)$ for every $t \in J$.

Finally, we are able to conclude that, for every $q \in C^1(J, \mathbb{R})$, the set $S(q)$ is acyclic and the theorem is proved.

4 An example

Consider the following integro-differential equation:

$$(3) \qquad x'(t) = \int_0^t \frac{3t \, e^{-s+2}}{1+t^3} \left(\frac{3s^2 \, e^{-2s}}{1 + \left(\sin(x(s))\right)^2} + s \, e^{-s-2} x'(s) \right) ds$$

$$x(0) = 0, \quad t \in J = [0, +\infty).$$

We have

$$k(t,s) = \frac{3t \, e^{-s+2}}{1+t^3},$$

$$f\big(s, x(s), x'(s)\big) = \frac{3s^2 \, e^{-2s}}{1 + \left(\sin(x(s))\right)^2} + s \, e^{-s-2} x'(s)$$

$h(s) = 3 \, e^{-s+2}$, $a(s) = 3s^2 \, e^{-2s}$, $b(s) = s \, e^{-s-2}$.

Hence, we obtain:

$$\int_0^{+\infty} a(s)ds = \frac{3}{4}$$

$$\int_0^{+\infty} b(s)ds = e^{-2}$$

$$\int_0^{+\infty} h(s)b(s)ds = \int_0^{+\infty} 3se^{-2s}ds = \frac{3}{4} < 1$$

So that the assumptions of our theorem are satisfied and integro-differential equation (3) has solutions.

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