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On commutativity of prime and semiprime rings with generalized derivations

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Abstract

Let R be a prime ring, extended centroid C and $m, n, k \geq 1$ are fixed integers. If R admits a generalized derivation F associated with a derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ or $(F(x) \circ_m y)^k + x \circ_n d(y)$=0 for all $x, y \in I$, where I is a nonzero ideal of R, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$. Moreover, we also examine the case R is a semiprime ring.

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1 Introduction

R is always an associative ring with centre $Z(R)$, extended centroid C and Utumi quotient ring U. For further information, definitions and properties on these concepts refer to Beidar and Martindale [1996]. For any $x, y \in R$, the symbols $[x, y]$ and $x \circ y$ stand for the Lie commutator $xy - yx$ and Jordan commutator $xy + yx$ respectively. Given $x, y \in R$, we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$ and inductively $x \circ_m y = (x \circ_{m-1} y) \circ y$ for $m \geqslant 2$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies that $a = 0$. Every prime ring is a semiprime ring but semiprime ring need not be prime ring. The socle of a ring R denoted by $Soc(R)$ is the sum of the minimal left (right) ideals of R, if R has minimal left

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(right) ideals, otherwise $Soc(R) = (0)$. The goal of this paper is to establish that there is a relationship between the structure of the ring R and the behaviour of suitable additive mappings defined on R that satisfy certain special identities. In particular we study the case when the map is a generalized derivation of R. We recall that an additive map $d : R \to R$ is a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $I_a(x) = [a, x]$ for all $x \in R$ and d is outer derivation if it is not inner derivation. An additive map $F : R \to R$ is said to be a generalized derivation if there is a derivation d of R such that for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. All derivations are generalized derivations but generalized derivation need to be derivation. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see Ashraf and Rehman [2002] , where further references can be found). In Argaç and Inceboz [2009], Argaç and Inceboz proved that: If R is a prime ring, I a nonzero ideal of R, k a fixed positive integer and R admits a nonzero derivation d with the property $(d(x \circ y))^k = x \circ y$ for all $x, y \in I$, then R is commutative. In [Quadri et al., 2003, Theorem 2.3], Quadri et al. discussed the commutativity of prime rings with generalized derivations. More precisely, Quadri et al. proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In Ashraf and Rehman [2002], Ashraf and Rehman proved that if R is a 2-torsion free prime ring, I a nonzero ideal of R and d a nonzero derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative.

The present paper is motivated by the previous results and we here generalized the result obtained in Ashraf and Rehman [2002] . Moreover, we continue this line of investigation by examining what happens if a ring R satisfies the following identities:

$$(i)(F(x) \circ y)^m + (x \circ d(y))^n = 0, (ii)(F(x) \circ_m y)^k + x \circ_n d(y) = 0$$

for all $x, y \in I$, a nonzero ideal of R. We obtain some analogous results for semiprime ring in the case $I = R$. More precisely, we shall prove the following theorems:

Theorem 1.1. Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in I$, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let R be a prime ring, I a nonzero ideal of R and m, n, k are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in I$, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Theorem 1.3. Let R be a semiprime ring, U the left Utumi quotient ring of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in R$, then R is commutative.

Theorem 1.4. Let R be a semiprime ring, U the left Utumi quotient ring of R and m, n, k are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in R$, then R is commutative.

2 Preliminary results

Before starting our results, we state some well known facts which are very crucial for developing the proof of our main result. In particular, we will make frequent use of the following facts.

Fact 2.1. *[Lee, 1992, Theorem 2] If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities.*

Fact 2.2. *[Lee, 1999, Theorem 4] Let R be a semiprime ring. Then every generalized derivation F on a dense right ideal of R is uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Moreover, a and d are uniquely determined by the generalized derivation F.*

Fact 2.3. *[Chuang, 1988, Theorem 2] If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities with coefficients in U.*

Fact 2.4. *Let R be a prime ring and d a nonzero derivation on R and I be a nonzero ideal of R. By Kharchenko's Theorem [Kharchenko, 1978, Theorem 2], if I satisfies the differential polynomial identity $P(x_1, x_2, ..., x_n, d(x_1), d(x_2), ..., d(x_n)) = 0$, then either d is an inner derivation or d is outer derivation and I satisfies the generalized polynomial identity $P(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) = 0$.*

Fact 2.5. *[Chuang, 1994, Page no. 38] If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 2.6. *[Lee, 1992, Lemma 2] Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.*

Fact 2.7. *[Chuang, 1994, Page no. 42] Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a $B - algebra$ orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any nonzero derivation of R.*

3 Main Results

Theorem 1.1 Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in I$, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Proof. By hypothesis

$$(F(x) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in I. \tag{1}$$

By the Fact 2.1 I, R and U satisfy the same generalized polynomial identity (GPI), we have

$$(F(x) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{2}$$

Since R is a prime ring and F a generalized derivation of R, by Fact 2.2, $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Then U satisfies

$$((ax + d(x)) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{3}$$

That is

$$(ax \circ y + d(x) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{4}$$

In the light of Kharchenko's theorem [Kharchenko, 1978, Theorem 2], we divide the proof into two cases:-
Case I Let d be an inner derivation of U, that is, $d(x) = [q, x]$ for all $x \in U$ and for some $q \in U$. Then U satisfies

$$F(x, y) = (ax \circ y + [q, x] \circ y)^m + (x \circ [q, y])^n = 0 \text{ for all } x, y \in U. \tag{5}$$

In case C is infinite, we have $F(x, y) = 0$ for all $x, y \in U \bigotimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \bar{C}$ are prime and centrally closed Erickson et al. [1975], we may replace R by U or $U \bigotimes_C \bar{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $F(x, y) = 0$ for all $x, y \in R$. By Martindale's theorem [Martindale, 1969, Theorem 3], R is then a primitive ring having nonzero socle, $soc(R)$ with D as associative division ring. Hence by Jacobson's theorem [Jacobson, 1956, p.75], R is isomorphic to dense ring of linear transformations of vector space V over C. Then the density of R on V implies that $R \cong M_k(D)$, where $k = dim_D V$. Assume that $dim_D V \geqslant 2$, otherwise we are done. Suppose that there exists $v \in V$ such that v and qv are linearly D-independent. Since $dim_D V \geqslant 2$, then there exists $w \in V$ such that $\{v, qv, w\}$ is linearly independent over D. By density of R there exist $x, y \in R$ such that

$$xv = 0, xqv = w, yw = v, xw = 0, yv = 0, yqv = v. \tag{6}$$

Multiplying equation (5) by v from right and using conditions in equation (6), we get $(-1)^m v = 0$, a contradiction. Now we want to show that $qv = v\beta$ for some $\beta \in D$. Let v, w be linearly independent. Then by the precedant argument, there exist $\beta_v, \beta_w, \beta_{v+w} \in D$, such that $qv = v\beta_v$, $qw = w\beta_w$, $q(v+w) = (v+w)\beta_{v+w}$. Moreover, $v\beta_v + w\beta_w = (v+w)\beta_{v+w}$ and hence $v(\beta_v - \beta_{v+w}) + w(\beta_w - \beta_{v+w}) = 0$. Since v, w are linearly independent, we have $\beta_v = \beta_w = \beta_{v+w}$ that is β does not depend on the choice of v.

Let now for $r \in R$, $v \in V$, by precedant calculation, $qv = v\alpha$, $r(qv) = r(v\alpha)$ and also $q(rv) = (rv)\alpha$. Thus $0 = [q, r]v$ for any $v \in V$, that is, $[q, r]V = 0$. Since V is a left faithful irreducible R -module, $[q, r] = 0$ for all $r \in R$ i.e., $q \in Z(R)$ and $d = 0$.

Case 2 Let d be an outer derivation of R. Then by Fact 2.4, I satisfies the generalized polynomial identity

$$(ax \circ y + t \circ y)^m + (x \circ t)^n = 0 \text{ for all } x, y, t \in I. \tag{7}$$

In particular, for $y = 0$, I satisfies $(xt + tx)^n = 0$. By Chuang [Chuang, 1988, Theorem 2], this polynomial identity is also satisfied by Q and hence R as well. By Lemma 1 Lanski [1993], there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over a field F, where $k \geq 1$. Moreover, R and $M_k(F)$ satisfy the same polynomial identity, that is $(xt + tx)^n = 0$ for all $t, x \in M_k(F)$. Let e_{ij} be the usual matrix unit with 1 in the (i, j) entry and zero elsewhere. By choosing $t = e_{12}, x = e_{21}$, we see that $(xt + tx)^n = (e_{11} + e_{22})^n \neq 0$, a contradiction.

Theorem 1.2 Let R be a prime ring, I a nonzero ideal of R and m, n, k are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in I$, then either

R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.
Proof. By hypothesis

$$(F(x) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in I. \tag{8}$$

By the Fact 2.1 I, R and U satisfy the same generalized polynomial identity (GPI), we have

$$(F(x) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in U. \tag{9}$$

By Fact 2.2, $F(x) = ax + d(x)$ for some $a \in U$ and derivation d on U. Then U satisfies

$$((ax + d(x)) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in U. \tag{10}$$

That is

$$(ax \circ_m y + d(x) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in U. \tag{11}$$

In the light of Kharchenko's theorem [Kharchenko, 1978, Theorem 2], we divide the proof into two cases:-
Case I Let d be an inner derivation of U that is $d(x) = [q, x]$ for all $x \in U$ and for some $q \in U$. Then U satisfies

$$(ax \circ_m y + [q, x] \circ_m y)^k + x \circ_n [q, y] = 0 \text{ for all } x, y \in U. \tag{12}$$

As in the proof of Theorem 1.1, we have

$$(ax \circ_m y + [q, x] \circ_m y)^k + x \circ_n [q, y] = 0 \text{ for all } x, y \in R. \tag{13}$$

where R is a primitive ring with D as the associated division ring. If V is finite dimensional over D, then the density of R implies that $R \cong M_k(D)$, where $k = dim_D V$. Assume that $dim_D V \geqslant 2$, otherwise we are done. Suppose that there exists $v \in V$ such that v and qv are linearly D-independent. Since $dim_D V \geqslant 2$, then there exists $w \in V$ such that $\{v, qv, w\}$ is linearly independent over D. By density of R there exist $x, y \in R$ such that

$$xv = 0, xqv = w, yw = v, xw = 0, yv = 0, yqv = v.$$

Multiplying equation (13) by v from right, we get $(-1)^{mk}v = 0$ which is a contradiction to the linearly independent of the set $\{v, qv\}$. Therefore, $\{v, qv\}$ is linearly

dependent and so $q \in Z(R)$, i.e, $d = 0$.
Case 2 Let d be an outer derivation. Then

$$(ax \circ_m y + t \circ_m y)^k + x \circ_n s = 0 \text{ for all } x, y, s, t \in I. \tag{14}$$

In particular, choosing $y = 0$, we have $s \circ_n x = 0$. By Chuang [Chuang, 1988, Theorem 2], this polynomial identity is also satisfied by Q and hence R as well. By Lemma 1 Lanski [1993], there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over a field F, where $k \geq 1$. Moreover, R and $M_k(F)$ satisfy the same polynomial identity, that is, $s \circ_n x = 0$ for all $s, x \in M_k(F)$. Denote e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. By choosing $s = e_{12}, x = e_{11}$, we see that $s \circ_n x = e_{12} \neq 0$, a contradiction.

The following examples demonstrate that R to be $prime$ can not be omitted in the hypothesis of Theorem 1.1 and Theorem 1.2.

Example 3.1. *For any ring S, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in S \right\}$. Then R is a ring under usual addition and multiplication of matrices and I is a nonzero ideal of R. Define maps $F, d : R \to R$ by $F(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation on R associated with the nonzero derivation d satisfying $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in I$. However R is not commutative as well as F can not be written as $F(x) = bx$ for all $x \in R$ as d is nonzero. Hence Theorem 1.1 is not true for arbitrary rings.*

Example 3.2. *Let $R = \left\{ \begin{pmatrix} x & y \\ o & z \end{pmatrix} \mid x, y, z \in Z_2 \right\}$ and $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in Z_2 \right\}$. Then R is a ring under usual addition and multiplication of matrices and I is a nonzero ideal of R. Define maps $F, d : R \to R$ by $F(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation on R associated with the nonzero derivation d satisfying $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in I$. However R is not commutative as well as F can not be written as $F(x) = bx$ for all $x \in R$ as d is nonzero . Hence Theorem 1.2 is not true for arbitrary rings.*

Theorem 1.3 Let R be a semiprime ring, U the left Utumi quotient ring of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x)\circ y)^m+(x\circ d(y))^n=0$ for all $x,y\in R$, then R is commutative.

Proof. By Fact 2.6, any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U. Moreover, by Fact 2.3 I, R and U satisfy the same GPIs and by Fact 2.1 I, R, U satisfy same differential identities. Also by Fact 2.2, we have $F(x)=ax+d(x)$ for some $a\in U$ and derivation d of U. Then

$$((ax+d(x))\circ y)^m+(x\circ d(y))^n=0 \text{ for all } x,y\in U. \tag{15}$$

By Fact 2.5, we have $Z(U)=C$. Let $M(C)$ be the set of all maximal ideals of C and $P\in M(C)$. By Fact 2.7, we have PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P\in M(C)\ \}=0$. Set $\overline{U}=U/PU.$ Then derivation d canonically induce derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x})=\overline{d(x)}$ for all $x\in\bar{U}$. Therefore,

$$((\bar{a}\overline{x}+\overline{d(x)})\circ\bar{y})^m+(\overline{x}\circ\overline{d(y)})^n=0$$

for all $\overline{x},\overline{y}\in\overline{U}$. It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 1.1, we have for each $P\in M(C)$ either $[U,U]\subseteq PU$ or $d(U)\subseteq PU$. In any case $d(U)[U,U]\subseteq PU$ for all $P\in M(C)$. Thus $d(U)[U,U]\subseteq\bigcap\{PU\mid P\in M(C)\ \}=0$, we obtain $d(U)[U,U]=0$. Therefore, $[U,U]=0$ since $\bigcap\{PU\mid P\in M(C)\ \}=0$ and $d\neq 0$. Since R is subring of U, so in particular $[R,R]=0$. Hence R is commutative. This completes the proof of the theorem.

Using the similar arguments as used in the proof of the above theorem, we can prove the following theorem.

Theorem 1.4 Let R be a semiprime ring, U the left Utumi quotient ring of R and k,m,n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x)\circ_m y)^k+x\circ_n d(y)=0$ for all $x,y\in R$, then R is commutative.

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