Existence and nonexistence of solutions to nonlinear parabolic problems

著者	Hisa Kotaro
学位授与機関	Tohoku University
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博　士　論　文

Existence and nonexistence of solutions to nonlinear parabolic problems

(非線形放物型問題の解の存在·非存在)

比佐　幸太郎

令和 元 年

Contents

Notation

- Let us denote by $\mathbf{N}$ and $\mathbf{R}$ the sets of natural numbers and real numbers, respectively.
- For $N \in \mathbf{N}$, we denote by $\mathbf{R}^N$ the N-dimensional Euclidean space.
- $\mathbf{R}^N_+ := \{x = (x', x_N) : x' \in \mathbf{R}^{N-1}, x_N > 0\}, \qquad D := \overline{\mathbf{R}^N_+}$.
- For any $x \in \mathbf{R}^N$ and $r > 0$, let

  $$B(x,r) := \{y \in \mathbf{R}^N \;:\; |x-y| < r\}, \quad B_+(x,r) := \{(y', y_N) \in B(x,r) \;:\; y_N \geq 0\}.$$

- Differential operators

  $$\partial_t := \frac{\partial}{\partial t}, \qquad \Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

- For $0 < \alpha < 1$, $(-\Delta)^\alpha$ denotes the fractional power of the Laplace operator $-\Delta$ in $\mathbf{R}^N$ and this is defined by

  $$(-\Delta)^\alpha \phi(x) := \mathcal{F}^{-1}[|\xi|^{2\alpha} \mathcal{F}[\phi](\xi)](x)$$

  for any $x \in \mathbf{R}^N$ and $\phi \in \mathcal{S}(\mathbf{R}^N)$, where $\mathcal{F}[v]$ is the Fourier transform of v.
- Let Ω be a open set in $\mathbf{R}^N$. For $1 \leq r \leq \infty$, $L^r(\Omega)$ denotes the usual Lebesgue space equipped with the norm

  $$\|u\|_{L^r(\Omega)} := \begin{cases} \left(\displaystyle\int_\Omega |u(x)|^r \, dx \right)^{\frac{1}{r}} & \text{if} \quad 1 \leq r < \infty, \\ \displaystyle\operatorname*{ess\,sup}_{x \in \Omega} |u(x)| & \text{if} \quad r = \infty. \end{cases}$$

- For $1 \leq r < \infty$, $L^r_{loc}(\mathbf{R}^N)$ denotes

  $$L^r_{loc}(\mathbf{R}^N) := \left\{ f : \mathbf{R}^N \to \mathbf{R} : \int_K |f(x)|^r \, dx < \infty \text{ for any compact set } K \subset \mathbf{R}^N \right\}$$

- We denote by $C_0(\mathbf{R}^N)$ the set of continuous function with compact support in $\mathbf{R}^N$.
- For $0 < \alpha < 1$, we denote by $C^\alpha(\mathbf{R}^N)$ the set of α-th Hölder continuous function in $\mathbf{R}^N$.

- For $k \in \mathbf{N}$, we denote by $C^k(\Omega)$ the set of functions whose derivatives up to k-th order are continuous on Ω, where $\Omega \subset \mathbf{R}^N$.

- Let $\Omega \subset \mathbf{R}^N$.
$$C^\infty(\Omega) := \bigcap_{k\in\mathbf{N}} C^k(\Omega).$$

- We denote by $C_0^\infty(\mathbf{R}^N)$ the set of $C^\infty(\mathbf{R}^N)$ functions with compact support in $\mathbf{R}^N$.

- For any $L^1_{loc}(\mathbf{R}^N)$ function f, we set
$$⨍_K f(y)\,dy := \frac{1}{|K|}\int_K f(y)\,dy,$$
where K is a measurable set in $\mathbf{R}^N$ and $|K|$ is volume of K.

- Let $G_\alpha = G_\alpha(x,t)$ be the fundamental solution to
$$\partial_t u + (-\Delta)^\alpha u = 0 \quad \text{in} \quad \mathbf{R}^N \times (0,\infty),$$
where $0 < \alpha \le 1$.

- For any Radon measure μ in $\mathbf{R}^N$, we define
$$[S_\alpha(t)\mu](x) := \int_{\mathbf{R}^N} G_\alpha(x-y,t)\,d\mu(y), \quad x \in \mathbf{R}^N,\ t>0.$$

- Let $\Gamma_N = \Gamma_N(x,t)$ be the Gauss kernel on $\mathbf{R}^N$, that is,
$$\Gamma_N(x,t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad (x,t) \in \mathbf{R}^N \times (0,\infty).$$
We notice that $G_1 = \Gamma_N$.

- Let $G = G(x,y,t)$ be the Green function for the heat equation on $\mathbf{R}^N_+$ with the homogeneous Neumann boundary condition, that is,
$$G(x,y,t) := \Gamma_N(x-y,t) + \Gamma_N(x-y_*,t), \quad x,y \in D,\ t>0,$$
where $y_* = (y', -y_N)$ for $y = (y', y_N) \in D$.

- For any Radon measure μ in $\mathbf{R}^N$ with $\operatorname{supp}\mu \subset D$, define
$$[S(t)\mu](x) := \int_D G(x,y,t)\,d\mu(y), \quad x \in D,\ t>0.$$

- For any set E, let χ_E be the characteristic function which has value 1 in E and value 0 outside E.
- For any set Λ, let f and g be maps from Λ to $(0, \infty)$. We say that

$$f(t) \asymp g(t) \quad \text{for all} \quad t \in \Lambda$$

if there exists a constant $C > 0$ such that $C^{-1}g(t) \leq f(t) \leq Cg(t)$ for all $t \in \Lambda$.

Summary

1.1 Introduction

One of main subjects of research on partial differential equation is the well-posedness of Cauchy problems, that is, existence of solutions, uniqueness of solutions and dependence of initial data. In particular, it is significant to ask whether Cauchy problems have solutions or not. Indeed, this question has attracted many interests in the mathematical literature. The purpose of this thesis is to investigate the threshold of the existence and the nonexistence of solutions to the Cauchy problems for several nonlinear parabolic equations.

It is well-known that nonlinear parabolic problems often appear in the various mathematical models such as heat transfer, chemical concentration, nonlinear radiation law and so on. The solvability of the nonlinear parabolic problems is taking on complicated aspects; the solvability depends on a lot of factors such as diffusion effect, nonlinearity of equations, boundary conditions and the shape of initial functions. This may be a reason why the issue has attracted much attention from many mathematicians with development of the nonlinear analysis. Here, we give an example:

Let us consider nonnegative solutions to semilinear parabolic heat equation

$$\partial_t u = \Delta u + u^p, \qquad x \in \mathbf{R}^N,\ t > 0, \tag{P}$$

where $N \geq 1$ and $p > 1$. This equation is one of the simplest nonlinear parabolic equation and has been studied extensively by many mathematicians since the pioneering work due to Fujita [29] (see, for example, [60], which is a book including a good list of references for problem (P)). It is known that the singularity of the initial function is one of the factors that determine the existence of solutions to problem (P). Precisely, if the singularity of the initial value is too strong, problem (P) has no local-in-time solutions. Now, what is the strongest singularity of the initial value for problem (P0) to possess a solution? There is a partial answer to this question:

In the case of $p > 1 + 2/N$ if the initial value $u(0)$ satisfies

$$u(x,0) \geq c_1 |x|^{-\frac{2}{p-1}}$$

in a neighborhood of the origin, where $c_1 > 0$ is a constant, then problem (P) has no local-in-time solutions for sufficiently large $c_1 > 0$ (See [10]). On the other hand, if the initial value $u(0)$ satisfies

$$0 \le u(x,0) \le c_2 |x|^{-\frac{2}{p-1}}, \qquad x \in \mathbf{R}^N,$$

where $c_2 > 0$ is a constant, then problem (P) has a local-in-time solution for sufficiently small $c_2 > 0$ (see [41]). See also [39, 50, 61].

By combining these facts, we see that this singularity is the strongest one for the solvability of problem (P) in the case of $p > 1 + 2/N$. However, in other cases the strongest singularity has not been obtained yet.

In this thesis we identify the strongest singularity of the initial function of the inhomogeneous term for the solvability of several nonlinear parabolic problems by studying the existence and nonexistence of solutions and as an application of the main results of this thesis, we obtain optimal estimates of the life span of solutions. In the following, we consider three nonlinear parabolic problems (which include problem (P)):

(1) Cauchy problem for the fractional semilinear heat equation,

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p, & x \in \mathbf{R}^N, \ t > 0, \\ u(0) = \mu \ge 0 & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{P1}$$

where $N \ge 1$, $0 < \alpha \le 1$, $p > 1$ and μ is a Radon measure or a measurable function in $\mathbf{R}^N$;

(2) Cauchy problem for the fractional semilinear heat equation with an inhomogeneous term,

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p + \mu, & x \in \mathbf{R}^N, \ t > 0, \\ u(0) = 0 & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{P2}$$

where $N \ge 1$, $0 < \alpha \le 1$, $p > 1$ and μ is a nonnegative Radon measure in $\mathbf{R}^N$ or a nonnegative measurable function in $\mathbf{R}^N$;

(3) Cauchy problem for the heat equation with a nonlinear boundary condition,

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+, \ t > 0, \\ \partial_\nu u = u^p & x \in \partial\mathbf{R}^N_+, \ t > 0, \\ u(x,0) = \mu(x) \ge 0 & x \in D, \end{cases} \tag{P3}$$

where $N \ge 1$, $p > 1$ and μ is a nonnegative measurable function in $\mathbf{R}^N_+$ or a Radon measure in $\mathbf{R}^N$ with $\operatorname{supp} \mu \subset D$.

The background of each problem is described in Sections 1.2, 1.3 and 1.4, respectively. For these problems, we wish to investigate

- Necessary conditions on the initial value or the inhomogeneous term for the solvability (which lead the nonexistence of solutions);
- Sufficient conditions on the initial value or the inhomogeneous term for the solvability (which lead the existence of solutions);
- Optimal estimates of the life span of solutions.

The rest of this thesis is organized as follows: In Chapter 1 we review some known results on the solvability of problems (P1), (P2) and (P3) and state the main results of this thesis. Sections 1.2, 1.3 and 1.4 in Chapter 1 deal with problems (P1), (P2) and (P3), respectively. In Chapter 2 we prove the main results on the solvability of problem (P1) and in Subsection 2.1.1 we collect properties of the fundamental solution G_α to the linear fractional heat equation. In Chapter 3 we prove the main results on the solvability of problem (P2). In Chapter 4 we prove the main results on the solvability of problem (P3). In Chapter 5, as an application of the results on the solvability, we give optimal estimates to the life span of solutions to (P1), (P3) and the Cauchy problem for the higher order semilinear parabolic equation.

1.2 Existence and nonexistence of solutions to (P1)

1.2.1 Motivation

In this section we consider the fractional semilinear parabolic equation

$$\partial_t u + (-\Delta)^\alpha u = u^p, \qquad x \in \mathbf{R}^N,\ t > 0, \tag{1.2.1}$$

where $\partial_t := \partial/\partial t$, $N \geq 1$, $0 < \alpha \leq 1$ and $p > 1$.

We show that every nonnegative solution to (1.2.1) has a unique Radon measure in $\mathbf{R}^N$ as the initial trace and study qualitative properties of the initial trace. Furthermore, we give sufficient conditions for the existence of the solution to Cauchy problem (1.2.1).

Let us consider the case of $\alpha = 1$, that is, the semilinear parabolic equation

$$\partial_t u - \Delta u = u^p, \quad x \in \mathbf{R}^N,\ t > 0, \qquad u(0) = \mu \geq 0 \quad \text{in} \quad \mathbf{R}^N, \tag{1.2.2}$$

where $N \geq 1$, $p > 1$ and μ is a Radon measure or a measurable function in $\mathbf{R}^N$. The solvability of Cauchy problem (1.2.2) has been studied extensively by many mathematicians since the pioneering work due to Fujita [29] (see, e.g., [60]). Among others, in 1985, Baras and Pierre [10] proved the following by the use of the capacity of potentials of Meyers [56]:

Theorem 1.2.1 *Let u be a nonnegative local-in-time solution to* (1.2.2)*, where μ is a Radon measure in $\mathbf{R}^N$. Then μ must satisfy the following:*

If $1 < p < p_F$, then $\sup_{x\in\mathbf{R}^N} \mu(B(x,1)) < \infty$;

If $p = p_F$, then $\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma\, |\log\sigma|^{-\frac{N}{2}}$ for sufficiently small $\sigma > 0$;

If $p > p_F$, then $\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma\sigma^{N-\frac{2}{p-1}}$ for sufficiently small $\sigma > 0$.

Here $p_F := 1 + 2/N$ and γ is a constant depending only on N and p.

Then we can find a positive constant c_1 with the following property:

Remark 1.2.1 *Problem* (1.2.2) *possesses no local-in-time solutions if μ is a nonnegative measurable function in $\mathbf{R}^N$ satisfying*

$$
\begin{aligned}
&\mu(x) \ge c_1|x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2}-1} && \textit{for} \quad p = p_F,\\
&\mu(x) \ge c_1|x|^{-\frac{2}{p-1}} && \textit{for} \quad p > p_F,
\end{aligned}
$$

in a neighborhood of the origin.

For related results, see e.g., [3, 9]. On the other hand, Takahashi [67] proved that, in the case of $p \ge p_F$, for any $\gamma > 0$, Cauchy problem (1.2.2) possesses no local-in-time nonnegative solutions with some Radon measure μ satisfying

$$
\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma\sigma^{N-\frac{2}{p-1}}\left[\log\left(e+\frac{1}{\sigma}\right)\right]^{-\frac{1}{p-1}}
$$

for all $\sigma > 0$. See [67, Theorem 1, Proposition 1].

The local solvability of Cauchy problem (1.2.2) has been studied in many papers (see e.g., [3, 9, 17, 28, 39, 41, 50, 61, 64, 65, 67, 68, 69] and references therein). It is known that there exists a constant $c_2 > 0$ such that Cauchy problem (1.2.2) possesses a solution in $\mathbf{R}^N \times [0,\rho^2]$, where $\rho > 0$, if $p > p_F$ and

$$
\sup_{x\in\mathbf{R}^N} \|\mu\|_{L^{r,\infty}(B(x,\rho))} \le c_2 \quad \text{with} \quad r = \frac{N(p-1)}{2} \tag{1.2.3}
$$

(see [41]). See also [39, 50, 61]. This implies that, if $p > p_F$ and

$$
0 \le \mu(x) \le c|x|^{-\frac{2}{p-1}} \quad \text{in} \quad \mathbf{R}^N \quad \text{with sufficiently small } c > 0,
$$

then (1.2.3) holds for any $\rho > 0$ and problem (1.2.2) possesses a global-in-time solution. On the other hand, in the case of $p = p_F$, as far as we know, there are no results on the local solvability of Cauchy problem (1.2.2) under such an assumption as

$$0 \le \mu(x) \le c_3|x|^{-N}\left[\log\left(e+\frac{1}{|x|}\right)\right]^{-\frac{N}{2}-1} \text{ in } \mathbf{R}^N \text{ with sufficiently small } c_3>0. \tag{1.2.4}$$

Some of the results on the solvability of Cauchy problem (1.2.2) are available to fractional semilinear parabolic equations, however there are no results on necessary conditions such as Theorem 1.2.1.

In Chapter 2 we show the existence and the uniqueness of the initial trace of the solution to (1.2.1) and obtain a refinement of Theorem 1.2.1. Furthermore, we give sufficient conditions on the existence of the solution to

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p, & x \in \mathbf{R}^N,\ t>0,\\ u(0)=\mu \ge 0 & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{1.2.5}$$

where $N \ge 1$, $0<\alpha\le 1$, $p>1$ and μ is a Radon measure or a measurable function in $\mathbf{R}^N$. Even in the case of $\alpha=1$, our sufficient conditions are new and they ensure that Cauchy problem (1.2.2) with (1.2.4) possesses a local-in-time solution.

1.2.2 Main results on (P1)

We formulate the definition of solutions to (1.2.1).

Definition 1.2.1 *Let u be a nonnegative measurable function in $\mathbf{R}^N \times (0,T)$, where $0<T\le\infty$.*
(i) *We say that u is a solution to* (1.2.1) *in $\mathbf{R}^N\times(0,T)$ if u satisfies*

$$\infty > u(x,t) = \int_{\mathbf{R}^N} G_\alpha(x-y,t-\tau)u(y,\tau)\,dy + \int_\tau^t\int_{\mathbf{R}^N} G_\alpha(x-y,t-s)u(y,s)^p\,dy\,ds$$

for almost all $x\in\mathbf{R}^N$ and $0<\tau<t<T$.
(ii) *Let μ be a Radon measure in $\mathbf{R}^N$. We say that u is a solution to* (1.2.5) *in $\mathbf{R}^N\times[0,T)$ if u satisfies*

$$\infty > u(x,t) = \int_{\mathbf{R}^N} G_\alpha(x-y,t)\,d\mu(y) + \int_0^t\int_{\mathbf{R}^N} G_\alpha(x-y,t-s)u(y,s)^p\,dy\,ds \tag{1.2.6}$$

for almost all $x\in\mathbf{R}^N$ and $0<t<T$. If u satisfies (1.2.6) *with $=$ replaced by $\ge$, then u is said to be a supersolution to* (1.2.5) *in $\mathbf{R}^N\times[0,T)$.*
(iii) *Let u be a solution to* (1.2.5) *in $\mathbf{R}^N\times[0,T)$. We say that u is a minimal solution to* (1.2.5) *in $\mathbf{R}^N\times[0,T)$ if*

$$u(x,t) \le v(x,t) \quad \textit{for almost all } x\in\mathbf{R}^N \textit{ and } 0<t<T$$

for any solution v to (1.2.5) *in $\mathbf{R}^N\times[0,T)$.*

The definition of G_α is contained in Notation.

Now we are ready to state the main results on the solvability of problem (P1). In the first theorem we show the existence and the uniqueness of the initial trace of the solution to (1.2.1) and obtain a refinement of Theorem 1.2.1.

Theorem 1.2.2 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $p > 1$. Let u be a solution to* (1.2.1) *in $\mathbf{R}^N \times (0, T)$, where $0 < T < \infty$. Then there exists a unique Radon measure μ such that*

$$\operatorname*{ess\,lim}_{t \to +0} \int_{\mathbf{R}^N} u(y,t)\phi(y)\,dy = \int_{\mathbf{R}^N} \phi(y)\,d\mu(y) \tag{1.2.7}$$

for all $\phi \in C_0(\mathbf{R}^N)$. Furthermore, there exists $\gamma_1 > 0$ depending only on N, α and p such that

(1) $\sup_{x \in \mathbf{R}^N} \mu(B(x, T^{\frac{1}{2\alpha}})) \leq \gamma_1 T^{\frac{N}{2\alpha} - \frac{1}{p-1}}$ *if $1 < p < p_\alpha$;*

(2) $\sup_{x \in \mathbf{R}^N} \mu(B(x, \sigma)) \leq \gamma_1 \left[\log\left(e + \dfrac{T^{\frac{1}{2\alpha}}}{\sigma}\right)\right]^{-\frac{N}{2\alpha}}$ *for all $0 < \sigma \leq T^{\frac{1}{2\alpha}}$ if $p = p_\alpha$;*

(3) $\sup_{x \in \mathbf{R}^N} \mu(B(x, \sigma)) \leq \gamma_1 \sigma^{N - \frac{2\alpha}{p-1}}$ *for all $0 < \sigma \leq T^{\frac{1}{2\alpha}}$ if $p > p_\alpha$.*

Here $p_\alpha := 1 + 2\alpha/N$.

Remark 1.2.2 (i) *Sugitani* [66] *showed that, if $1 < p \leq p_\alpha$ and $\mu \not\equiv 0$ in $\mathbf{R}^N$, then problem* (1.2.5) *possesses no nonnegative global-in-time solutions.*
(ii) *Let u be a solution to* (1.2.1) *in $\mathbf{R}^N \times [0, \infty)$ and $1 < p \leq p_\alpha$. It follows from assertions* (1) *and* (2) *that the initial trace of u must be identically zero in $\mathbf{R}^N$. Then Theorem* 1.2.2 *leads the same conclusion as in Remark* 1.2.2 (i).

As a corollary of Theorem 1.2.2, we have

Corollary 1.2.1 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $p > 1$. Let u be a solution to* (1.2.1) *in $\mathbf{R}^N \times (0, T)$, where $0 < T < \infty$. Then there exists $\gamma > 0$ depending only on N, α and p such that*

$$\sup_{x \in \mathbf{R}^N} \fint_{B(x,(T-t)^{1/2\alpha})} u(y,t)\,dy \leq \gamma (T-t)^{-\frac{1}{p-1}}$$

for almost all $0 < t < T$.

Corollary 1.2.1 in the case of $\alpha = 1$ has been already obtained in [55, Lemma 4.4 (i)].

Our argument in the proof of Theorem 1.2.2 is completely different from those in [3, 9, 10]. Let u be a solution to (1.2.1) in $\mathbf{R}^N \times (0, T)$, where $0 < T < \infty$. We first prove the existence and the uniqueness of the initial trace of the solution u.

Next, in the case of $p \neq p_\alpha$ we apply the iteration argument in [68, Theorem 5] to obtain an $L^\infty(\mathbf{R}^N)$ estimate of the solution u. This yields a uniform estimate of $\|u(\tau)\|_{L^1(B(z,\rho))}$ with respect to $z \in \mathbf{R}^N$ and $\tau \in (0, T/2)$ for all small enough $\rho > 0$, and we complete the proof of Theorem 1.2.2. In the case of $p = p_\alpha$ we follow the argument in [29, 31, 66] and obtain an inequality related to

$$\int_{\mathbf{R}^N} u(x,t) G_\alpha(x,t)\, dx.$$

Then, applying the iteration argument in [52, Section 2], we prove Theorem 1.2.2. Furthermore, by Theorem 1.2.2 we obtain

Theorem 1.2.3 *Assume the same conditions as in Theorem* 1.2.2. *Let μ be a Radon measure satisfying* (1.2.7). *Then u is a solution to* (1.2.5) *in $\mathbf{R}^N \times [0,T)$.*

We give sufficient conditions for the solvability of problem (1.2.5). We modify the arguments in [41, 61] and prove the following two theorems.

Theorem 1.2.4 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $1 < p < p_\alpha$. Then there exists $\gamma_2 > 0$ such that, if μ is a Radon measure in $\mathbf{R}^N$ satisfying*

$$\sup_{x \in \mathbf{R}^N} \mu(B(x, T^{\frac{1}{2\alpha}})) \leq \gamma_2 T^{\frac{N}{2\alpha} - \frac{1}{p-1}} \quad \textit{for some } T > 0, \tag{1.2.8}$$

then problem (1.2.5) *possesses a solution in $\mathbf{R}^N \times [0,T)$.*

Theorem 1.2.5 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $1 < \theta < p$. Then there exists $\gamma_3 > 0$ such that, if μ is a nonnegative measurable function in $\mathbf{R}^N$ satisfying*

$$\sup_{x \in \mathbf{R}^N} \left[⨍_{B(x,\sigma)} \mu(y)^\theta \, dy \right]^{\frac{1}{\theta}} \leq \gamma_3 \sigma^{-\frac{2\alpha}{p-1}}, \qquad 0 < \sigma \leq T^{\frac{1}{2\alpha}}, \tag{1.2.9}$$

for some $T > 0$, then problem (1.2.5) *possesses a solution in $\mathbf{R}^N \times [0,T)$.*

Furthermore, we state the following theorem, which is a refinement of Theorem 1.2.5 in the case of $p = p_\alpha$ and enables us to prove the existence of the solution to (1.2.5) under assumption (1.2.4).

Theorem 1.2.6 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $p = p_\alpha$. For $s > 0$, set*

$$\Psi_\alpha(s) := s[\log(e+s)]^{\frac{N}{2\alpha}}, \qquad \rho_\alpha(s) := s^{-N} \left[\log\left(e + \frac{1}{s}\right)\right]^{-\frac{N}{2\alpha}}. \tag{1.2.10}$$

Then there exists $\gamma_4 > 0$ such that, if μ is a nonnegative measurable function in $\mathbf{R}^N$ satisfying

$$\sup_{x \in \mathbf{R}^N} \Psi_\alpha^{-1} \left[⨍_{B(x,\sigma)} \Psi_\alpha(T^{\frac{1}{p-1}} \mu(y))\, dy \right] \leq \gamma_4 \rho(\sigma T^{-\frac{1}{2\alpha}}), \qquad 0 < \sigma \leq T^{\frac{1}{2\alpha}}, \tag{1.2.11}$$

for some $T > 0$, then problem (1.2.5) *possesses a solution in $\mathbf{R}^N \times [0,T)$.*

As a corollary of Theorems 1.2.2, 1.2.5 and 1.2.6, we have

Corollary 1.2.2 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $p \geq p_\alpha$. Then there exists $\gamma_* > 0$ with the following properties:*

(i) *If $p = p_\alpha$ and*
$$\mu(x) = \gamma |x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2\alpha}-1} + C$$
for some $\gamma \geq 0$ and $C \geq 0$, then

- *problem* (1.2.5) *possesses a local-in-time solution if $0 \leq \gamma < \gamma_*$;*
- *problem* (1.2.5) *possesses no local-in-time solutions if $\gamma > \gamma_*$.*

(ii) *If $p > p_\alpha$ and*
$$\mu(x) = \gamma |x|^{-\frac{2\alpha}{p-1}} + C$$
for some $\gamma \geq 0$ and $C \geq 0$, then the same conclusion as in assertion (i) *holds. Furthermore, if $C = 0$ and γ is small enough, then problem* (1.2.5) *possesses a global-in-time solution.*

The proofs of the main results on problem (P1) are contained in Chapter 2.

1.3 Existence and nonexistence of solutions to (P2)

1.3.1 Motivation

In this section we consider the Cauchy problem for a fractional semilinear heat equation with an inhomogeneous term

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p + \mu, & x \in \mathbf{R}^N, \ t > 0, \\ u(0) = 0 & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{1.3.1}$$

where $\partial_t := \partial/\partial t$, $N \geq 1$, $0 < \alpha \leq 1$, $p > 1$ and μ is a nonnegative Radon measure in $\mathbf{R}^N$ or a nonnegative measurable function in $\mathbf{R}^N$.

We study necessary conditions and sufficient conditions on the inhomogeneous term μ for the existence of nonnegative solutions to problem (1.3.1) and identify the strongest singularity of μ for the solvability of problem (1.3.1). Our identification is new even for $\alpha = 1$.

Before considering problem (1.3.1), we recall some results on Cauchy problem (1.2.5). In [34] the author of this thesis and Ishige studied necessary conditions and sufficient conditions on the initial data for the solvability of problem (1.2.5) and identified the singularity of the initial data

$$\mu(x) = \begin{cases} c|x|^{-\frac{2\alpha}{p-1}} & \text{if} \quad p > 1 + 2\alpha/N, \\ c|x|^{-N}\left[\log\left(e + \dfrac{1}{|x|}\right)\right]^{-\frac{N}{2\alpha}-1} & \text{if} \quad p = 1 + 2\alpha/N, \end{cases} \tag{1.3.2}$$

as the strongest one for the solvability of problem (1.2.5), where $c > 0$ is a constant. Precisely, if μ satisfies (1.3.2) in a neighborhood of the origin, problem (1.2.5) has no local-in-time solutions for sufficiently large $c > 0$. On the other hand, if μ satisfies (1.3.2) in $\mathbf{R}^N$, problem (1.2.5) has a local-in-time solution for sufficiently small $c > 0$. See Theorem 1.2.2, Theorem 1.2.4 and Corollary 1.2.2.

The existence of solutions to nonlinear parabolic equations with inhomogeneous terms has been studied in many papers, see e.g. [8, 10, 12, 46, 47, 48, 49, 51, 74, 75, 76, 77] and references therein. However, there are no results concerning the identification of the strongest spatial singularity of the inhomogeneous term for the existence of solutions.

In Chapter 3, motivated by [34], we study necessary conditions and sufficient conditions on the inhomogeneous term μ for the existence of solutions to problem (1.3.1) and identify the strongest singularity of the inhomogeneous term μ for the solvability of (1.3.1).

1.3.2 Main results on (P2)

We formulate the definition of solutions to problem (1.3.1) and state our main results.

Definition 1.3.1 *Let u be a nonnegative measurable function in $\mathbf{R}^N \times (0, T)$, where $0 < T \le \infty$. We say that u is a solution to problem* (1.3.1) *in $\mathbf{R}^N \times [0, T)$ if u satisfies*

$$\int_0^T \int_{\mathbf{R}^N} u(-\partial_t \varphi + (-\Delta)^\alpha \varphi)\, dx\, dt = \int_0^T \int_{\mathbf{R}^N} u^p \varphi\, dx\, dt + \int_0^T \int_{\mathbf{R}^N} \varphi\, d\mu(x)\, dt$$

for $\varphi \in C_0^\infty(\mathbf{R}^N \times [0, T))$.

The first theorem is concerned with necessary conditions on the inhomogeneous term μ for the solvability of problem (1.3.1). Set

$$p_* := \frac{N}{N - 2\alpha} \quad \text{if} \quad 0 < 2\alpha < N \quad \text{and} \quad p_* := \infty \quad \text{if} \quad 2\alpha \ge N.$$

Theorem 1.3.1 *Let $N \ge 1$, $0 < \alpha \le 1$ and $p > 1$. Let u be a solution to problem* (1.3.1) *in $\mathbf{R}^N \times [0, T)$, where $0 < T < \infty$. Then there exists $\gamma = \gamma(N, \alpha, p) > 0$ such that*

$$\sup_{x \in \mathbf{R}^N} \mu(B(x, \sigma)) \le \gamma \sigma^{N - \frac{2\alpha p}{p-1}} \tag{1.3.3}$$

for $0 < \sigma \le T^{1/2\alpha}$. *Furthermore, if* $p = p_*$, *then there exists* $\gamma' = \gamma'(N,\alpha) > 0$ *such that*

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma' \left[\log\left(e + \frac{T^{\frac{1}{2\alpha}}}{\sigma}\right)\right]^{-\frac{N}{2\alpha}+1} \tag{1.3.4}$$

for $0 < \sigma \le T^{1/2\alpha}$.

If $1 < p < p_*$, then the function $\sigma \mapsto \sigma^{N-2\alpha p/(p-1)}$ is decreasing for $\sigma > 0$. This means that (1.3.3) is equivalent to

$$\sup_{x \in \mathbf{R}^N} \mu(B(x, T^{\frac{1}{2\alpha}})) \le \gamma T^{\frac{N}{2\alpha} - \frac{p}{p-1}}$$

in the case of $1 < p < p_*$. As corollaries of Theorem 1.3.1, we have

Corollary 1.3.1 *Let* $N \ge 1$, $0 < \alpha \le 1$ *and* $p \ge p_*$. *Then there exists* $\gamma = \gamma(N,\alpha,p) > 0$ *such that, if a nonnegative measurable function* μ *in* $\mathbf{R}^N$ *satisfies*

$$\mu(x) \ge \begin{cases} \gamma |x|^{-\frac{2\alpha p}{p-1}} & \text{if} \quad p > p_*, \\ \gamma |x|^{-N} \left[\log\left(e + \dfrac{1}{|x|}\right)\right]^{-\frac{N}{2\alpha}} & \text{if} \quad p = p_*, \end{cases}$$

in a neighborhood of the origin, then problem (1.3.1) *possesses no local-in-time solutions.*

Corollary 1.3.2 *Let* $N \ge 1$ *and* $0 < \alpha \le 1$.

(1) *Let* $1 < p \le p_*$ *and* $\mu \not\equiv 0$ *in* $\mathbf{R}^N$. *Then problem* (1.3.1) *possesses no global-in-time solutions.*

(2) *Let* $p > p_*$ *and* μ *be a nonnegative measurable function in* $\mathbf{R}^N$. *Then there exists* $\gamma = \gamma(N,\alpha,p) > 0$ *with the following property: If there exists* $R > 0$ *such that*

$$\mu(x) \ge \gamma |x|^{-\frac{2\alpha p}{p-1}}$$

for almost all $x \in \mathbf{R}^N \setminus B(0,R)$, *then problem* (1.3.1) *possesses no global-in-time solutions.*

Next we state our results on sufficient conditions for the solvability.

Theorem 1.3.2 *Let* $N \ge 1$, $0 < \alpha \le 1$ *and* $1 < p < p_*$. *Then there exists* $\gamma = \gamma(N,\alpha,p) > 0$ *such that, if a nonnegative Radon measure* μ *in* $\mathbf{R}^N$ *satisfies*

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma \sigma^{N - \frac{2\alpha p}{p-1}} \quad \text{for some } \sigma > 0,$$

then problem (1.3.1) *possesses a solution in* $\mathbf{R}^N \times [0,T)$ *with* $T = \sigma^{2\alpha}$.

Theorem 1.3.3 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $p > p_*$. Let $1 < r < \infty$ be such that*

$$r > r_* := \frac{N(p-1)}{2\alpha p}.$$

Then there exists $\gamma = \gamma(N, \alpha, p, r) > 0$ such that, if a nonnegative measurable function μ in $\mathbf{R}^N$ satisfies

$$\sup_{x \in \mathbf{R}^N} \|\mu\|_{L^r(B(x,\sigma))} \leq \gamma \sigma^{\frac{N}{r} - \frac{2\alpha p}{p-1}} \quad \text{for some } \sigma > 0, \tag{1.3.5}$$

then problem (1.3.1) possesses a solution in $\mathbf{R}^N \times [0, T)$ with $T = \sigma^{2\alpha}$.

Theorem 1.3.4 *Let $N \geq 1$, $0 < \alpha \leq 1$ and $p \geq p_*$. Let μ be a nonnegative measurable function in $\mathbf{R}^N$ such that*

$$0 \leq \mu(x) \leq \begin{cases} \gamma |x|^{-\frac{2\alpha p}{p-1}} + C_0 & \text{if} \quad p > p_*, \\ \gamma |x|^{-N} \left[\log\left(e + \dfrac{1}{|x|} \right) \right]^{-\frac{N}{2\alpha}} + C_0 & \text{if} \quad p = p_*, \end{cases} \tag{1.3.6}$$

for almost all $x \in \mathbf{R}^N$, where $\gamma > 0$ and $C_0 \geq 0$. Then there exists $\gamma_* = \gamma_*(N, \alpha, p) > 0$ such that problem (1.3.1) possesses a local-in-time solution if $\gamma \leq \gamma_*$ and a global-in-time solution if $\gamma \leq \gamma_*$, $C_0 = 0$ and $p > p_*$.

By Theorems 1.3.1, 1.3.2 and 1.3.4 we can identify the strongest spatial singularity of μ for the solvability of problem (1.3.1). Furthermore, by Theorems 1.3.1 and 1.3.2 we easily obtain

Corollary 1.3.3 *Let δ be the Dirac delta function in $\mathbf{R}^N$. Then problem (1.3.1) possesses a local-in-time solution with $\mu = D\delta$ for some $D > 0$ if and only if $1 < p < p_*$.*

Remark 1.3.1 (i) *Corollary 1.3.2 (1) and Theorem 1.3.4 imply the following properties.*

- (a) *If $1 < p \leq p_*$ and $\mu \not\equiv 0$, then problem (1.3.1) possesses no global-in-time solutions;*
- (b) *If $p > p_*$, then problem (1.3.1) possesses a global-in-time solution for some $\mu\,(\not\equiv 0)$.*

(ii) *In the case of $\alpha = 1$, assertions (a) and (b) were first obtained by [75] and they have been extended to various nonlinear parabolic equations with inhomogeneous terms. See e.g. [8, 48, 74, 75, 76, 77] and references therein. In the case of $0 < \alpha < 1$, see [49].*

(iii) *Necessary conditions and sufficient conditions for the existence of solutions to the problem*

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u + \delta \otimes \nu, & x \in \mathbf{R}^N, \ t > 0, \\ u(0) = 0 & in \quad \mathbf{R}^N, \end{cases}$$

were discussed in [46, 47], *where* $p > 1$ *and* ν *is a Radon measure in* $[0, \infty)$. *Corollary* 1.3.3 *with* $\alpha = 1$ *follows from* [46, *Theorem* 2.2] *and* [47, *Theorem* 2.1].

We explain the idea of proving our theorems. Kartsatos and Kurta [48] obtained necessary conditions on the existence of global-in-time solutions to problem (1.3.1) with $\alpha = 1$. Except for the case of $0 < \alpha < 1$ and $p = p_*$, their arguments are available for the proof of Theorem 1.3.1. Indeed, the proof of Theorem 1.3.1 except for such a case is given as a modification of the arguments in [48]. In the case of $0 < \alpha < 1$ and $p = p_*$, using a fractional Poisson equation, we modify arguments in [48] to prove Theorem 1.3.1. The regularity of solutions to the fractional Poisson equation plays an important role in the proof. On the other hand, the proofs of Theorems 1.3.2 and 1.3.3 are based on the contraction mapping theorem in uniformly local Lebesgue spaces. Theorem 1.3.4 is proved by the construction of supersolutions to problem (1.3.1). This requires delicate estimates of volume potentials associated with the fundamental solution to the fractional heat equation. The proofs of these results are contained in Chapter 3.

1.4 Existence and nonexistence of solutions to (P3)

1.4.1 Motivation

We are interested in finding necessary conditions and sufficient conditions on the initial data for the solvability of problem

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+, \ t > 0, \\ \partial_\nu u = u^p & x \in \partial\mathbf{R}^N_+, \ t > 0, \end{cases} \tag{1.4.1}$$

with the initial condition

$$u(x, 0) = \mu(x) \geq 0, \qquad x \in D, \tag{1.4.2}$$

where $N \geq 1$, $p > 1$ and μ is a nonnegative measurable function in $\mathbf{R}^N_+$ or a Radon measure in $\mathbf{R}^N$ with $\operatorname{supp}\mu \subset \mathrm{D}$.

For the solvability of problem (1.4.1) with (1.4.2), sufficient conditions have been studied in many papers (see e.g., [6], [7], [19], [26], [30], [38], [43] and [44]). However little is known concerning necessary conditions and the strongest singularity of initial data for which problem (1.4.1) possesses a local-in-time nonnegative solution is still open as far as we know.

This section is motivated by the results on the solvability of the Cauchy problem for semilinear parabolic equations. In 1985, Baras and Pierre [10] studied necessary conditions on the initial data for the existence of nonnegative solutions to

$$\partial_t u = \Delta u + u^q, \quad x \in \mathbf{R}^N,\ t > 0,$$

where $N \geq 1$ and $q > 1$. (See Theorem 1.2.1). Subsequently, the author of this thesis and Ishige [34] proved the existence and the uniqueness of the initial trace of a nonnegative solution to a fractional semilinear heat equation (1.2.5). Furthermore, they obtained necessary conditions on the initial data for the existence of the solution to (1.2.5) (See Theorem 1.2.2). In [34], developing the arguments in [41] and [61], they also obtained sufficient conditions on the initial data for the existence of the solution to (1.2.5) and identified the strongest singularity of the initial data for the solvability of Cauchy problem (1.2.5) (See Corollary 1.2.2). We are interested in finding similar necessary conditions for the existence of solutions to (1.4.1) and identifying the strongest singularity of the initial data for the solvability of problem (1.4.1).

The study of the initial traces of solutions is a classical subject and it has been investigated for various parabolic equations, for example, the heat equation (see [4, 70]), the porous medium equation (see [5, 11, 33]), the evolution of p-Laplacian (see [21, 22]), the doubly nonlinear parabolic equation (see [37, 38, 79]), the fractional diffusion equation (see [16]), the Finsler heat equation (see [2]), and parabolic equations with nonlinear terms (see e.g., [3, 10, 13, 34, 42, 54, 78]).

In Chapter 4 we show the existence and the uniqueness of the initial trace of a nonnegative solution to (1.4.1) and obtain necessary conditions on the existence of nonnegative solutions to (1.4.1) and (1.4.2). We also obtain new sufficient conditions on the existence of nonnegative solutions to (1.4.1) and (1.4.2). Our necessary conditions and sufficient conditions enable us to identify the strongest singularity of initial data for which problem (1.4.1) possesses a local-in-time nonnegative solution. Surprisingly, the strongest singularity depends on whether it exists on $\partial\mathbf{R}^N_+$ or not (see Corollary 1.4.1).

1.4.2 Main results on (P3)

We introduce some notation and define solutions to (1.4.1). Throughout this thesis we often identify $\mathbf{R}^{N-1}$ with $\partial\mathbf{R}^N_+$. For any $L \geq 0$, we set

$$\begin{aligned} D_L &:= \{(x', x_N) \,:\, x' \in \mathbf{R}^{N-1},\ x_N \geq L^{1/2}\}, \\ D'_L &:= \{(x', x_N) \,:\, x' \in \mathbf{R}^{N-1},\ 0 \leq x_N < L^{1/2}\}. \end{aligned}$$

We remark that $D = D_0 = \overline{\mathbf{R}^N_+}$. For any locally integrable nonnegative function ϕ on D, we often identify ϕ with the Radon measure $\phi\, dx$. It follows that

$$\lim_{t\to+0} \|S(t)\eta - \eta\|_{L^\infty(D)} = 0, \qquad \eta \in C_0(D : [0,\infty)). \tag{1.4.3}$$

Definition 1.4.1 *Let u be a nonnegative and continuous function in $D \times (0,T)$, where $0 < T < \infty$.*
(i) *We say that u is a solution to* (1.4.1) *in* $(0,T)$ *if u satisfies*

$$u(x,t) = \int_D G(x,y,t-\tau)u(y,\tau)\,dy + \int_\tau^t \int_{\mathbf{R}^{N-1}} G(x,y',0,t-s)u(y',0,s)^p\,dy'\,ds \tag{1.4.4}$$

for $(x,t) \in D \times (\tau,T)$ and $0 < \tau < T$.
(ii) *Let μ be a nonnegative measurable function in $\mathbf{R}^N_+$ or a Radon measure in $\mathbf{R}^N$ with $\operatorname{supp}\mu \subset D$. We say that u is a solution to* (1.4.1) *and* (1.4.2) *in* $[0,T)$ *if u satisfies*

$$u(x,t) = \int_D G(x,y,t)\,d\mu + \int_0^t \int_{\mathbf{R}^{N-1}} G(x,y',0,t-s)u(y',0,s)^p\,dy'\,ds \tag{1.4.5}$$

for $(x,t) \in D \times (0,T)$. If u satisfies (1.4.5) *with " $=$ " replaced by " $\geq$ ", then u is said to be a supersolution to* (1.4.1) *and* (1.4.2) *in* $[0,T)$.
(iii) *Let u be a solution to* (1.4.1) *and* (1.4.2) *in* $[0,T)$. *We say that u is a minimal solution to* (1.4.1) *and* (1.4.2) *in* $[0,T)$ *if $u(x,t) \leq v(x,t)$ in $D \times (0,T)$ for any solution v of* (1.4.1) *and* (1.4.2) *in* $[0,T)$.

The definition of G is contained in Notation.

Now we are ready to state our main results. In Theorem 1.4.1 we show the existence and the uniqueness of the initial trace of the solution to (1.4.1) and give necessary conditions on the initial trace. In what follows, we set $\overline{p} := 1 + 1/N$.

Theorem 1.4.1 *Let $p > 1$ and u be a solution to* (1.4.1) *in* $(0,T)$, *where $0 < T < \infty$. Then there exists a unique Radon measure μ in $\mathbf{R}^N$ with $\operatorname{supp}\mu \subset D$ such that*

$$\lim_{t \to +0} \int_D u(y,t)\phi(y)\,dy = \int_D \phi(y)\,d\mu(y), \qquad \phi \in C_0(\mathbf{R}^N). \tag{1.4.6}$$

Furthermore, for any $\delta > 0$, there exists $\gamma_1 = \gamma_1(N,p,\delta) > 0$ such that

$$\sup_{x \in D} \exp\left(-(1+\delta)\frac{x_N^2}{4\sigma^2}\right)\mu(B(x,\sigma)) \leq \gamma_1\,\sigma^{N-\frac{1}{p-1}} \tag{1.4.7}$$

for $0 < \sigma \leq T^{\frac{1}{2}}$. In particular, in the case of $p = \overline{p}$, there exists $\gamma_1' = \gamma_1'(N,\delta) > 0$ such that

$$\sup_{x \in D} \exp\left(-(1+\delta)\frac{x_N^2}{4\sigma^2}\right)\mu(B(x,\sigma)) \leq \gamma_1' \left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-N} \tag{1.4.8}$$

for $0 < \sigma \leq T^{\frac{1}{2}}$.

In Theorem 1.4.2 we show that the initial trace of the solution to (1.4.1) and (1.4.2) coincides with its initial data.

Theorem 1.4.2 *Let $p > 1$ and μ be a Radon measure in $\mathbf{R}^N$ with $\operatorname{supp}\mu \subset D$.*

(a) *Let u be a solution to* (1.4.1) *and* (1.4.2) *in $[0,T)$ for some $T > 0$. Then* (1.4.6) *holds.*

(b) *Let u be a solution to* (1.4.1) *in $(0,T)$ for some $T > 0$. Assume* (1.4.6). *Then u is a solution to* (1.4.1) *and* (1.4.2) *in $[0,T)$.*

Combining Theorem 1.4.1 with Theorem 1.4.2, we obtain necessary conditions on the initial data for the solvability of problem (1.4.1) with (1.4.2).

Remark 1.4.1 (i) *If $1 < p \le p_*$ and $\mu \not\equiv 0$ on D, then problem* (1.4.1) *possesses no nonnegative global-in-time solutions. See* [19] *and* [30].
(ii) *Let u be a solution to* (1.4.1) *in $[0,\infty)$ and $1 < p \le \overline{p}$. It follows from* (1.4.7) *and* (1.4.8) *that the initial trace of u must be identically zero in D. Then Theorem* 1.4.2 *leads the same conclusion as in Remark* 1.4.1 (i) *by taking $T \to \infty$.*

Next we state our main results on sufficient conditions for the solvability of problem (1.4.1) with (1.4.2).

Theorem 1.4.3 *Let $1 < p < \overline{p}$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1-\delta)/4T$. Then there exists $\gamma_2 = \gamma_2(N,p,\delta) > 0$ with the following property:*

- *If μ is a Radon measure in $\mathbf{R}^N$ with $\operatorname{supp}\mu \subset D$ satisfying*

$$\sup_{x\in D} \fint_{B(x,T^{1/2})} e^{-\lambda y_N^2}\, d\mu(y) \le \gamma_2 T^{-\frac{1}{2(p-1)}}, \tag{1.4.9}$$

then there exists a solution u of (1.4.1) *and* (1.4.2) *in $[0,T)$ such that*

$$0 \le u(x,t) \le 2[S(t)\mu](x), \quad (x,t) \in D \times (0,T).$$

Theorem 1.4.4 *Let $p > 1$, $\alpha \in (1,p)$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1-\delta)/4T$. Then there exists $\gamma_3 = \gamma_3(N,p,\alpha,\delta) > 0$ with the following property:*

- *Let μ_1 be a Radon measure in $\mathbf{R}^N$ such that $\operatorname{supp}\mu_1 \subset D_T$ and*

$$\sup_{x\in D_T} \fint_{B(x,T^{1/2})} e^{-\lambda y_N^2}\, d\mu_1(y) \le \gamma_3 T^{-\frac{1}{2(p-1)}}. \tag{1.4.10}$$

Let μ_2 be a nonnegative measurable function in $\mathbf{R}^N_+$ such that $\operatorname{supp}\mu_2 \subset D'_T$ and

$$\sup_{x\in D'_T} \left[\fint_{B(x,\sigma)} \mu_2(y)^\alpha\, dy \right]^{\frac{1}{\alpha}} \le \gamma_3 \sigma^{-\frac{1}{p-1}} \quad \textit{for } 0 < \sigma < T^{\frac{1}{2}}. \tag{1.4.11}$$

Then there exists a solution u of (1.4.1) *and* (1.4.2) *in* $[0,T)$ *with* $\mu = \mu_1 + \mu_2$ *such that*

$$0 \le u(x,t) \le 2[S(t)\mu_1](x) + 2\left([S(t)\mu_2^\alpha](x)\right)^{\frac{1}{\alpha}}, \quad (x,t) \in D \times (0,T).$$

Remark 1.4.2 *Let $p > \overline{p}$. Let $\gamma > 0$ be sufficiently small and*

$$\mu(x) = \gamma |x|^{-\frac{1}{p-1}} \quad in \quad D.$$

Then, for any $T > 0$, μ satisfies the assumptions in Theorem 1.4.4 *with some $\alpha > 1$. This implies that problem* (1.4.1) *with* (1.4.2) *possesses a global-in-time solution. On the other hand, if γ is sufficiently large, then Theorem* 1.5.1 *implies that problem* (1.4.1) *with* (1.4.2) *possesses no local-in-time solutions.*

Theorem 1.4.5 *Let $p = \overline{p}$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1-\delta)/4T$ and*

$$\Phi(s) := s[\log(e+s)]^N, \quad \rho(s) := s^{-N}\left[\log\left(e + \frac{1}{s}\right)\right]^{-N} \quad for \quad s > 0. \tag{1.4.12}$$

Then there exists $\gamma_4 = \gamma_4(N,\delta) > 0$ with the following property:

- *Let μ_1 be a Radon measure in $\mathbf{R}^N$ such that $\operatorname{supp}\mu_1 \subset D_T$ and*

$$\sup_{x \in D_T} \fint_{B(x,T^{1/2})} e^{-\lambda y_N^2}\, d\mu_1(y) \le \gamma_4 T^{-\frac{1}{2(p-1)}}. \tag{1.4.13}$$

  Let μ_2 be a nonnegative measurable function in $\mathbf{R}^N_+$ such that $\operatorname{supp}\mu_2 \subset D'_T$ and

$$\sup_{x \in D'_T} \Phi^{-1}\left[\fint_{B(x,\sigma)} \Phi(T^{\frac{1}{2(p-1)}}\mu_2(y))\, dy\right] \le \gamma_4 \rho(\sigma T^{-\frac{1}{2}}) \quad for\ 0 < \sigma < T^{\frac{1}{2}}. \tag{1.4.14}$$

  Then there exists a solution to (1.4.1) *and* (1.4.2) *in* $[0,T)$ *with* $\mu = \mu_1 + \mu_2$ *such that*

$$0 \le u(x,t) \le 2[S(t)\mu_1](x) + d\Phi^{-1}\left([S(t)\Phi(\mu_2)](x)\right), \quad (x,t) \in D \times (0,T),$$

  where d is a positive constant depending only on p and β.

Remark 1.4.3 *Let $p = \overline{p}$. Let $\gamma > 0$ be sufficiently small and*

$$\mu(x) = \gamma |x|^{-N} |\log|x||^{-N-1} \chi_{B(0,1/2)} \quad in \quad D.$$

Then, for any sufficiently small $\beta > 0$, μ satisfies the assumptions in Theorem 1.4.5 *with some $T > 0$. This implies that problem* (1.4.1) *with* (1.4.2) *possesses a local-in-time solution.* (*See also Remark* 1.4.1 (i).) *On the other hand, if γ is sufficiently large, then Theorem* 1.4.1 *implies that problem* (1.4.1) *with* (1.4.2) *possesses no local-in-time solutions.*

As a corollary of our theorems, we have:

Corollary 1.4.1 *Let δ be the Delta function in $\mathbf{R}^N$ and $x_0 \in D$. Let $\mu(y) = \delta(y-x_0)$ in $\mathbf{R}^N$. Then there exists a solution to* (1.4.1) *and* (1.4.2) *in* $[0,T)$ *for some* $T > 0$ *if and only if, either*

$$\text{(i)} \quad x_0 \in \partial\mathbf{R}^N_+ \quad \textit{and} \quad 1 < p < p_* \qquad \textit{or} \qquad \text{(ii)} \quad x_0 \in \mathbf{R}^N_+ \quad \textit{and} \quad p > 1.$$

We develop the arguments in [34] and prove our theorems. Let u be a solution to (1.4.1) in $(0,T)$ for some $T > 0$. By the same argument as in [34] we can prove the existence and the uniqueness of the initial trace of the solution u. Furthermore, we study a lower estimate of the solution u near the boundary ∂D by the use of $\|u(\tau)\|_{L^1(B_+(z,\rho))}$, where $z \in D$, $\rho \in (0,T^{1/2})$ and $\tau \in (0,T)$. Combining this lower estimate with [19, Lemma 2.1.2], we complete the proof of Theorem 1.5.1 in the case of $p \neq \overline{p}$. For the case of $p = \overline{p}$, we obtain an integral inequality with respect to the quantity

$$\int_{\partial D} \Gamma_{N-1}(y',t)u(y',0,t)\,dy'.$$

Then we apply a similar iteration argument as in [52, Section 2] to obtain $\|u(\tau)\|_{L^1(B_+(z,\rho))}$, where $z \in D$, $\rho \in (0,T^{1/2})$ and $\tau \in (0,T)$. This completes the proof of Theorem 1.4.1 in the case of $p = \overline{p}$. Theorem 1.4.2 is proved by a similar argument as in the proof of [34, Theorem 1.2] with the aid of Theorem 1.4.1. Furthermore, we prove a lemma on an estimate of an integral related to the nonlinear boundary condition and apply the arguments in [34, 41, 61] to prove Theorems 1.4.3–1.4.5. Chapter 4 contains the proofs of these results.

1.5 Application: Optimal estimates of the life span of solutions to nonlinear parabolic problems

1.5.1 Introduction

Consider the nonnegative solution to the heat equation with a nonlinear boundary condition

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+, \ \ t > 0, \\ \partial_\nu u = u^p, & x \in \partial\mathbf{R}^N_+, \ \ t > 0, \end{cases} \tag{1.5.1}$$

with the initial condition

$$u(x,0) = \kappa\psi(x), \qquad x \in D := \overline{\mathbf{R}^N_+}, \tag{1.5.2}$$

where $N \geq 1$, $p > 1$, $\kappa > 0$ and ψ is a nonnegative measurable function in $\mathbf{R}^N_+ := \{y \in \mathbf{R}^N : y_N > 0\}$. The aim of this section is to obtain an optimal estimate of the life span $T(\kappa\psi)$ of solutions to problem (1.5.1) with (1.5.2), as $\kappa \to \infty$ or $\kappa \to +0$. In general, the life span $T(\kappa\psi)$ is complicated and the research on the life span $T(\kappa\psi)$ has fascinated many mathematicians.

Problem (1.5.1) can be physically interpreted as a nonlinear radiation law and it has been studied in many papers (see e.g., [6, 7, 19, 25, 26, 30, 35, 38, 43, 44] and references therein). Among others, the author of this thesis and Ishige [35] obtained the necessary conditions and the sufficient conditions for the solvability of problem (1.5.1) and identified the strongest singularity. See Section 1.4. It follows from these conditions that the behavior of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ depends on the singularity of ψ and that of the life span $T(\kappa\psi)$ as $\kappa \to +0$ depends on that of ψ at the space infinity. In this section, we investigate these relationships and give an estimate to the life span $T(\kappa\psi)$ as $\kappa \to \infty$ and $\kappa \to +0$. Our results are optimal and give complete classifications of the behavior of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ and $\kappa \to +0$ (See Subsection 1.4).

The main idea is to apply the necessary conditions and the sufficient conditions for the solvability, which have been proved in [35] (see Section 1.4). Unfortunately, since these conditions have many parameters and are complicated, careful calculation is required to apply them.

1.5.2 Preliminaries

Before stating the main results of this section, we have to define the life span $T(\kappa\psi)$ of solutions to (1.5.1) with (1.5.2) strictly. To do that, we formulate the definition of minimal solutions to (1.5.1).

Definition 1.5.1 *Let u be a nonnegative and continuous function in $D \times (0,T)$, where $0 < T < \infty$.*

- *We say that u is a minimal solution to* (1.5.1) *in $[0,T)$ with $u(0) = \varphi$ (in the sense of Definition 1.4.1) if u is a solution to* (1.5.1) *in $[0,T)$ with $u(0) = \varphi$ and satisfies*
$$u(x,t) \le w(x,t) \quad \text{in} \quad D \times (0,T)$$
for any solution w to (1.5.1) *in $[0,T)$ with $w(0) = \varphi$.*

Since the minimal solution is unique, we can define the life span $T(\kappa\psi)$ as following:

Definition 1.5.2 *The life span $T(\kappa\psi)$ of solutions to* (1.5.1) *with* (1.5.2) *is defined by the maximal existence time of the minimal solution to* (1.5.1) *with* (1.5.2).

1.5.3 Main results

Now we are ready to state the main results of this section. In Theorem 1.5.1 we obtain the relationship between the singularity of ψ and the life span $T(\kappa\psi)$ as

$\kappa \to \infty$ and give an optimal estimate to the life span as $\kappa \to \infty$. Subsection 1.4 contains a brief summary of Theorem 1.5.1 (See Tables 1, 2 and 3).

Theorem 1.5.1 *Assume that*

$$\psi(x) := |x|^A \left[\log\left(e + \frac{1}{|x|}\right)\right]^{-B} \chi_{B_+(0,1)}(x) \in L^1(\mathbf{R}^N_+) \setminus L^\infty(\mathbf{R}^N_+),$$

where $-N \le A \le 0$ *and*

$$B > 0 \quad \text{if} \quad A = 0, \qquad B \in \mathbf{R} \quad \text{if} \quad -N < A < 0, \qquad B > 1 \quad \text{if} \quad A = -N. \tag{1.5.3}$$

Then $T(\kappa\psi) \to 0$ *as* $\kappa \to \infty$ *and following holds:*

(i) $T(\kappa\psi)$ *behaves*

$$T(\kappa\psi) \sim \begin{cases} \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}} & \text{if} \quad A > -\min\left\{N, \dfrac{1}{p-1}\right\}, \\ \left[\kappa(\log\kappa)^{-B+1}\right]^{-\frac{2(p-1)}{A(p-1)+1}} & \text{if} \quad 1 < p < \overline{p},\ A = -N,\ B > 1, \end{cases}$$

and

$$|\log T(\kappa\psi)| \sim \begin{cases} \kappa^{\frac{1}{B}} & \text{if} \quad p > p_*,\ A = -\dfrac{1}{p-1},\ B > 0, \\ \kappa^{\frac{1}{B-N-1}} & \text{if} \quad p = p_*,\ A = -N,\ B > N+1, \end{cases}$$

as $\kappa \to \infty$;

(ii) *Let* $p > \overline{p}$. *If, either*

$$A < -1/(p-1) \quad \text{and} \quad B \in \mathbf{R} \qquad \text{or} \qquad A = -1/(p-1) \quad \text{and} \quad B < 0,$$

then problem (1.5.1) *with* (1.5.2) *possesses no local-in-time solutions for all* $\kappa > 0$. *If*

$$A = -1/(p-1) \quad \text{and} \quad B = 0,$$

then problem (1.5.1) *with* (1.5.2) *possesses no local-in-time solutions for sufficiently large* $\kappa > 0$;

(iii) *Let* $p = \overline{p}$. *If*

$$A = -N \quad \text{and} \quad B < N+1,$$

then problem (1.5.1) *with* (1.5.2) *possesses no local-in-time solutions for all* $\kappa > 0$. *If*

$$A = -N \quad \text{and} \quad B = N+1,$$

then problem (1.5.1) *with* (1.5.2) *possesses no local-in-time solutions for sufficiently large* $\kappa > 0$.

We remark that when ψ is as in Theorem 1.5.1, ψ satisfies (1.5.3) if and only if $\psi \in L^1_{loc}(\mathbf{R}^N_+)$. It is obvious that $T(\kappa\psi) = 0$ for all $\kappa > 0$ if (1.5.3) does not hold.

Remark 1.5.1 *Ishige and Sato* [43] *obtained following: if ψ satisfies*

$$\psi(x) = |x|^A$$

in a neighborhood of the origin, where

$$-N < A \le 0 \quad \text{if} \quad 1 < p < \overline{p} \qquad \text{and} \qquad -\frac{1}{p-1} < A \le 0 \quad \text{if} \quad p \ge \overline{p},$$

then

$$T(\kappa\psi) \sim \kappa^{-\frac{2(p-1)}{A(p-1)+1}}$$

for sufficiently large $\kappa > 0$. Compare with Theorem 1.5.1.

Theorem 1.5.2 gives an optimal estimate to the life span $T(\kappa\psi)$ as $\kappa \to +0$ with ψ behaving like $|x|^{-A}(A > 0)$ at the space infinity. Subsection 1.4 contains a brief summary of Theorem 1.5.2 (See Tables 4 and 5).

Theorem 1.5.2 *Let $A > 0$ and $\psi(x) = (1 + |x|)^{-A}$. Then $T(\kappa\psi) \to \infty$ as $\kappa \to 0$ and following holds:*

(1) *Let $1 < p < \overline{p}$ or $0 < A < 1/(p-1)$. Then*

$$T(\kappa\psi) \sim \begin{cases} \kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A,N\}\right)^{-1}} & \text{if} \quad A \ne N, \\ \left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}} & \text{if} \quad A = N, \end{cases}$$

as $\kappa \to +0$;

(2) *Let $p = \overline{p}$ and $A \ge 1/(p-1)$. Then*

$$\log T(\kappa\psi) \sim \begin{cases} \kappa^{-(p-1)} & \text{if} \quad A > N, \\ \kappa^{-\frac{p-1}{p}} & \text{if} \quad A = N, \end{cases}$$

as $\kappa \to +0$;

(3) *Let $p > \overline{p}$ and $A \ge 1/(p-1)$. Then problem* (1.5.1) *with* (1.5.2) *possesses a global-in-time solution if $\kappa > 0$ is sufficiently small.*

Remark 1.5.2 *An optimal estimate of the life span $T(\kappa\psi)$ as $\kappa \to +0$ have been already obtained in some cases. Specifically, if ψ satisfies*

$$\psi(x) = (1 + |x|)^{-A} \quad (A > 0)$$

for all $x \in D$, then the following holds:

$$T(\kappa\psi) \sim \begin{cases} \kappa^{-\left(\frac{1}{2(p-1)} - \frac{A}{2}\right)^{-1}} & \text{if} \quad p \geq \overline{p}, \quad 0 \leq A < 1/(p-1), \\ \kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A, N\}\right)^{-1}} & \text{if} \quad p < \overline{p}, \quad A \neq N, \\ \left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}} & \text{if} \quad p < \overline{p}, \quad A = N, \end{cases}$$

for sufficiently small $\kappa > 0$ (See also [43])

Finally, we show that $\lim_{\kappa \to 0} T_\kappa = \infty$ does not necessarily hold for problem (1.5.1) if ψ has an exponential growth as $x_N \to \infty$.

Theorem 1.5.3 *Let $p > 1$, $\lambda > 0$ and $\psi(x) := \exp(\lambda x_N^2)$. Then*

$$\lim_{\kappa \to +0} T(\kappa\psi) = (4\lambda)^{-1}. \tag{1.5.4}$$

1.5.4 Summary of Theorems 1.5.1 and 1.5.2

By Theorem 1.5.1, we obtain following tables. These tables show the behavior of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ when ψ is as in Theorem 1.5.1. For simplicity of notation, we write T_κ instead of $T(\kappa\psi)$.

Table 1.1: In the case of $1 < p < \overline{p}$ (as $\kappa \to \infty$)

B \ A	$A > -N$	$A = -N$
$B > 1$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B+1}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$
$B \leq 1$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$T_\kappa = 0$

Table 1.2: In the case of $p > \overline{p}$ (as $\kappa \to \infty$)

$B \backslash A$	$A > -\frac{1}{p-1}$	$A = -\frac{1}{p-1}$	$-N \le A < -\frac{1}{p-1}$
$B > 0$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$\lvert\log T_\kappa\rvert \sim \kappa^{\frac{1}{B}}$	$T_\kappa = 0$
$B = 0$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$T_\kappa = 0$	$T_\kappa = 0$
$B < 0$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$T_\kappa = 0$	$T_\kappa = 0$

Table 1.3: In the case of $p = \overline{p}$ (as $\kappa \to \infty$)

$B \backslash A$	$A > -N$	$A = -N$
$B > N+1$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$\lvert\log T_\kappa\rvert \sim \kappa^{\frac{1}{B-N-1}}$
$B = N+1$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{A(p-1)+1}}$	$T_\kappa = 0$
$B < N+1$	$T_\kappa = 0$	$T_\kappa = 0$

By Theorem 1.5.2, we obtain following tables. These tables show the behavior of the life span $T(\kappa\psi)$ as $\kappa \to +0$ when ψ is as in Theorem 1.5.2.

Table 1.4: In the case of $A \neq N$ (as $\kappa \to +0$)

$p \backslash A$	$A < \frac{1}{p-1}$	$A = \frac{1}{p-1}$	$A > \frac{1}{p-1}$
$p < \overline{p}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A,N\}\right)^{-1}}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}}$
$p = \overline{p}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{A}{2}\right)^{-1}}$	none	$\log T_\kappa \sim \kappa^{-(p-1)}$
$p > \overline{p}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{A}{2}\right)^{-1}}$	$T_\kappa = \infty$	$T_\kappa = \infty$

Table 1.5: In the case of $A = N$ (as $\kappa \to +0$)

p \ A	$A = N$
$p < \overline{p}$	$T_\kappa \sim \left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}}$
$p = \overline{p}$	$\log T_\kappa \sim \kappa^{-\frac{p-1}{p}}$
$p > \overline{p}$	$T_\kappa = \infty$

The proofs of Theorems 1.5.1, 1.5.2 and 1.5.3 are contained in Chapter 5. Furthermore, Chapter 5 deals with the optimal estimate of solutions to the Cauchy problem for the higher order semilinear parabolic equation (see Section 5.3).

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