# Rational torsion of generalized Jacobians of modular and Drinfeld modular curves


| | |
|---|---|
| 著者 | Fu-Tsun Wei, Takao Yamazaki |
| journal or publication title | Forum Mathematicum |
| volume | 31 |
| number | 3 |
| page range | 647-659 |
| year | 2019-01-20 |
| URL | http://hdl.handle.net/10097/00128039 |

doi: 10.1515/forum-2018-0141

DE GRUYTER Forum Math. 2019; 31(3): 647–659

Research Article

Fu-Tsun Wei and Takao Yamazaki*

# Rational torsion of generalized Jacobians of modular and Drinfeld modular curves

https://doi.org/10.1515/forum-2018-0141
Received June 13, 2018; revised November 28, 2018

**Abstract:** We consider the generalized Jacobian $\widetilde{J}$ of the modular curve $X_0(N)$ of level $N$ with respect to a reduced divisor consisting of all cusps. Supposing $N$ is square free, we explicitly determine the structure of the $\mathbb{Q}$-rational torsion points on $\widetilde{J}$ up to 6-primary torsion. The result depicts a fuller picture than [18] where the case of prime power level was studied. We also obtain an analogous result for Drinfeld modular curves. Our proof relies on similar results for classical Jacobians due to Ohta, Papikian and the first author. We also discuss the Hecke action on $\widetilde{J}$ and its Eisenstein property.

**Keywords:** Generalized Jacobian, Modular curves, Drinfeld modular curves, cuspidal divisor group, Eisenstein ideal

**MSC 2010:** 11G09, 11G18, 11F03, 14H40, 14G35

**Communicated by:** Freydoon Shahidi

## 1 Introduction

### 1.1 Background and overview

Let $J$ be the Jacobian variety of the modular curve $X := X_0(p)$ over $\mathbb{Q}$ of prime level $p$. In the celebrated paper [9], Mazur proved that the group of $\mathbb{Q}$-rational torsion points $J(\mathbb{Q})_{\mathrm{Tor}}$ on $J$ is a cyclic group of order $(p-1)/(p-1, 12)$. (Here and henceforth we denote by $(a, b)$ the greatest common divisor of $a$ and $b$.) This result has been generalized to prime power level by Lorenzini [7] and Ling [6], as well as to square free level by Ohta [11]. The latter result will be recalled in Theorem 1.2 below.

A research toward a different direction was started in [18]. Let $C$ be the set of all cusps on $X$, which we regard as a reduced effective divisor on $X$. We consider the generalized Jacobian $\widetilde{J}$ of $X$ with respect to modulus $C$, in the sense of Rosenlicht–Serre [15]. Since $C$ consists of two $\mathbb{Q}$-rational points in this case, $\widetilde{J}$ is an extension of $J$ by $\mathbb{G}_m$ so that we have an exact sequence (cf. (3.2))

$$1 \to \{\pm 1\} \xrightarrow{i} \widetilde{J}(\mathbb{Q})_{\mathrm{Tor}} \to J(\mathbb{Q})_{\mathrm{Tor}}.$$

In [18], it is shown that $i$ is an isomorphism. It is also proved in loc. cit. that a similar bijectivity result holds when the level is a power of a prime $\geq 5$, conditional to a folklore conjecture “the cupsidal divisor classes cover all torsion rational points”(cf. [10, Conjecture 2]). On the contrary, it is observed that $i$ is far from being an isomorphism when the level is a product of two different primes of certain type [18, Proposition 1.3.2].

***Corresponding author: Takao Yamazaki,** Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan, e-mail: ytakao@math.tohoku.ac.jp
**Fu-Tsun Wei,** Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan, e-mail: ftwei@math.nthu.edu.tw

The purpose of the present article is twofold. One is to clarify what happens in the case of square free level. The other is to develop a parallel story for the rank two Drinfeld modular curves, again for square free level. Our main result, explained in the next subsection, pinpoints where $i$ fails to be an isomorphism (see Remark 1.4). Our proof relies on the study of classical Jacobians due to Ohta [11] and to Papikian–Wei [14]. We also discuss the action of Hecke operators on $\widetilde{J}(F)_{\mathrm{Tor}}$ and study its Eisenstein property, see Section 1.3.

## 1.2 Main results

We work in either of the following setting:

(NF) Set $F = \mathbb{Q}$ and $A = \mathbb{Z}$. Let $p_1, \dots, p_s$ be distinct primes, and put $N = p_1 \cdots p_s \in A$. Let $X := X_0(N)$ be the modular curve with respect to

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A) \,\middle|\, c \equiv 0 \bmod N \right\}.$$

(FF) Let $\mathbb{F}_q$ be a finite field with $q$ elements. Set $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$. Let $p_1, \dots, p_s \in A$ be distinct irreducible monic polynomials, and put $N = p_1 \cdots p_s \in A$. Let $X := X_0(N)$ be the rank two Drinfeld modular curve with respect to

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \,\middle|\, c \equiv 0 \bmod N \right\}.$$

We regard $X$ as a smooth projective absolutely integral curve over $F$. Let $C \subset X$ be the reduced closed subscheme consisting of all cusps on $X$. All points of $C$ are $F$-rational and $|C| = 2^s$. As before, we denote by $J$ and $\widetilde{J}$ the Jacobian variety of $X$ and the generalized Jacobian of $X$ with modulus $C$, respectively. Then $\widetilde{J}$ is an extension of $J$ by the product of $2^s - 1$ copies of $\mathbb{G}_m$ so that we have an exact sequence (cf. (3.2))

$$1 \to \mu_F^{\oplus(2^s-1)} \xrightarrow{i} \widetilde{J}(F)_{\mathrm{Tor}} \to J(F)_{\mathrm{Tor}}, \tag{1.1}$$

where $\mu_F$ consists of the roots of unity in $F$. Our result can be stated easily when $s = 1$ (that is, $N = p_1 \in A$ is a prime element).

**Theorem 1.1.** If $s = 1$, then the map $i$ is an isomorphism.

As explained above, the case (NF) has been proved in [18]. Note also that in the case (FF), Pál showed that $J(F)_{\mathrm{Tor}}$ is a cyclic group of order $q^d/(q^2 - 1, q^d - 1)$ with $d = \deg(p)$ in [12].

For general $s$, we need to introduce notations. For $j \in \mathbb{Z}_{\geq 0}$, we define

$$\mathbb{E}_j := \{e\colon \{1, \dots, s\} \to \{\pm 1\} \mid |e^{-1}(-1)| \geq j\}, \tag{1.2}$$

$$\mathcal{M}_j := \bigoplus_{e \in \mathbb{E}_j} (\mathbb{Z}/d(e)\mathbb{Z}), \quad d(e) := \prod_{i=1}^{s} (|p_i| + e(i)). \tag{1.3}$$

Here $|e^{-1}(-1)|$ denotes the cardinality of $\{c \in \{1, \dots, s\} \mid e(c) = -1\}$, and in the case (FF), we write $|p_i| := q^{\deg(p_i)}$. We set

$$a = \begin{cases} 6 & \text{in the case (NF),} \\ q(q^2 - 1) & \text{in the case (FF).} \end{cases}$$

The group of $F$-rational torsion points on $J$ is studied by Ohta and Papikian–Wei (see Theorem 2.3 below for more detailed results).

**Theorem 1.2** ([11, Theorem (3.6.2)] and [14, Theorem 4.3]). There is an isomorphism

$$J(F)_{\mathrm{Tor}} \otimes \mathbb{Z}\left[\frac{1}{a}\right] \cong \mathcal{M}_1 \otimes \mathbb{Z}\left[\frac{1}{a}\right].$$

Our main result for general $s$ is the following.

**Theorem 1.3.** There is an isomorphism

$$\widetilde{J}(F)_{\mathrm{Tor}} \otimes \mathbb{Z}\left[\frac{1}{a}\right] \cong \mathcal{M}_2 \otimes \mathbb{Z}\left[\frac{1}{a}\right].$$

**Remark 1.4.** Both $J$ and $\widetilde{J}$ admit an action of $\mathbb{W} := (\mathbb{Z}/2\mathbb{Z})^s$ through the Atkin–Lehner involutions. We identify $\mathbb{E}_0$ with the character group of $\mathbb{W}$. Theorems 1.2 and 1.3 actually describe the decomposition of $M := J(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/a]$ and $\widetilde{M} := \widetilde{J}(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/a]$ according to characters of $\mathbb{W}$. If we write $\widetilde{M}^e$ and $M^e$ for the $e$-part of $\widetilde{M}$ and $M$ for $e \in \mathbb{E}_0$, then they say

- $\widetilde{M}^e = M^e = 0$ if $e = 1_{\mathbb{E}_0}$,
- $\widetilde{M}^e = 0$ and $M^e \cong (\mathbb{Z}/d(e)\mathbb{Z}) \otimes \mathbb{Z}[1/a]$ if $|e^{-1}(-1)| = 1$,
- $\widetilde{M}^e \cong M^e \cong (\mathbb{Z}/d(e)\mathbb{Z}) \otimes \mathbb{Z}[1/a]$ if $|e^{-1}(-1)| \geq 2$.

The referee pointed out that, in view of [14, Remark 4.4], $\widetilde{M}^e$ admits an interpretation as the kernel of the specialization map $M^e \to \prod_{i=1}^s \Phi_{p_i}$, where $\Phi_{p_i}$ is the group of components of the reduction of $J_0(N)$ at $p_i$. A direct proof of this statement would possibly lead to a more geometric proof of Theorem 1.3.

**Remark 1.5.** In Theorems 1.2 and 1.3, it is sometimes possible to describe the 3-part (resp. $(q+1)$-part) in the case (NF) (resp. (FF)), see Theorem 2.3 and Corollary 3.3.

## 1.3 Eisenstein property

Let $\mathbb{T}$ be the $\mathbb{Z}$-algebra generated by the Hecke correspondences $\tau_p$ on $X$ for all $p \nmid N$. Then $J$ and $\widetilde{J}$ admit $\mathbb{T}$-module structures, and the natural homomorphism $\widetilde{J} \to J$ is actually $\mathbb{T}$-equivariant. Let $\mathcal{E}$ be the ideal of $\mathbb{T}$ generated by $\tau_p - |p| - 1$ for all $p \nmid N$. In the case of (NF), we also use a little smaller ideal $\mathcal{E}'$ generated by $\tau_p - |p| - 1$ for all $p \nmid 2N$. We have the following results in the literature (cf. Lemma 5.1).

- If $s = 1$, then $J(F)_{\mathrm{Tor}}$ is annihilated by $\mathcal{E}$.
- In (NF), $J(F)_{\mathrm{Tor}}$ is annihilated by $\mathcal{E}'$.
- In (FF), $J(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/q]$ is annihilated by $\mathcal{E}$.

They played a fundamental role in the works of Mazur [9, Proposition 11.1], Pál [12, Lemma 7.16], Ohta [11, p. 316] and Papikian–Wei [13, Lemma 7.1]. We now ask an analogous question for $\widetilde{J}(F)_{\mathrm{Tor}}$. In Section 5, we will study this problem and prove the following results in Proposition 5.4 (see also Corollary 5.7).

**Proposition.** (1) If $s = 1$, then $\widetilde{J}(F)_{\mathrm{Tor}}$ is annihilated by $\mathcal{E}$.
(2) In (NF), $\widetilde{J}(F)_{\mathrm{Tor}}$ is annihilated by $\mathcal{E}'^2$.
(3) In (FF), $\widetilde{J}(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/q]$ is annihilated by $\mathcal{E}^2$.

## 1.4 Organization of the paper

We recall known facts and prove easy lemmas on Jacobian varieties (resp. generalized Jacobians) of (Drinfeld) modular curves in Section 2 (resp. Section 3). The proofs of Theorems 1.1 and 1.3 are reduced to a key technical result, Theorem 3.2, whose proof occupies Section 4. We discuss the Hecke action and the Eisenstein property of $\widetilde{J}(F)_{\mathrm{Tor}}$ in Section 5.

# 2 Jacobian varieties

## 2.1 Structure of cusps

We continue to use the notation introduced in Section 1.2. Put $A_+ = \mathbb{Z}_{>0}$ in the case (NF), and let $A_+ \subset A$ be the set of all monic polynomials in the case (FF). Recall that $C \subset X = X_0(N)$ denotes the set of all cusps, which admits a standard description

$$C \cong \Gamma_0(N) \setminus \mathbb{P}^1(F). \tag{2.1}$$

For $x \in \mathbb{P}^1(F)$, we denote by $[x] \in C$ the point corresponding to the $\Gamma_0(N)$-orbit of $x$. We shall constantly use two bijections

$$\mathbb{W} := (\mathbb{Z}/2\mathbb{Z})^s \xrightarrow{m} \{m \in A_+ : m|N\} \xrightarrow{[(-)^{-1}]} C. \tag{2.2}$$

Here the first map is given by $m(w) = \prod_{i=1}^s p_i^{\tilde{w}_i}$ for $w = (w_i)_{i=1}^s \in \mathbb{W}$, where for $x \in \mathbb{Z}/2\mathbb{Z}$, we write $\tilde{x} = 0 \in \mathbb{Z}$ if $x = 0$ and $\tilde{x} = 1 \in \mathbb{Z}$ if $x = 1$. Note that $m$ becomes an isomorphism of groups if we equip a group structure on $\{m \in A_+ : m|N\}$ by $m * m' = mm'/(m, m')^2$. The second map is given by sending $m$ to $[1/m]$. We shall abbreviate $[w] := [1/m(w)]$ for $w \in \mathbb{W}$.

We identify $\mathbb{E} := \{\pm 1\}^s$ with $\mathbb{E}_0$ from (1.2). Let

$$\langle\, ,\rangle : \mathbb{E} \times \mathbb{W} \to \{\pm 1\}$$

be the following canonical biadditive pairing:

$$\langle e, w\rangle = \prod_{i=1}^s e_i^{w_i} \in \{\pm 1\} \quad \text{for } e = (e_i)_{i=1}^s \in \mathbb{E},\ w = (w_i)_{i=1}^s \in \mathbb{W}.$$

Given $e = (e_i)_{i=1}^s \in \mathbb{E}$, we define

$$D^e := \sum_{w \in \mathbb{W}} \langle e, w\rangle [w] \in \operatorname{Div}(X). \tag{2.3}$$

The degree of $D^e$ is zero if $e \neq 1_{\mathbb{E}}$ and $2^s$ if $e = 1_{\mathbb{E}}$.

## 2.2 Atkin–Lehner involution

For each $m \in A_+$ with $m|N$, we let $W_m : X \to X$ be the Atkin–Lehner involution associated to $m$ (cf. [11, equation (1.1.5)] in the case (NF) and [14, Definition 2.11] in the case (FF)). To ease the notation, we write $W_w = W_{m(w)}$ for $w \in \mathbb{W}$. Recall that $W_m$ restricts to an $F$-automorphism of $C$.

**Lemma 2.1.** (1) For $w, w' \in \mathbb{W}$ and $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$, we have

$$W_w([w']) = [w + w'], \quad W_w(D^e) = \langle e, w\rangle D^e.$$

(2) Let us define subgroups of $\operatorname{Div}(X)$ by

$$\mathcal{D}_1 := \bigoplus_{e \in \mathbb{E}\setminus\{1_{\mathbb{E}}\}} \mathbb{Z}D^e \subset \mathcal{D}_2 := \mathcal{D} \cap \operatorname{Div}^0(X) \subset \mathcal{D} := \bigoplus_{w \in \mathbb{W}} \mathbb{Z}[w],$$

which are all stable under the action of $\mathbb{W}$. Then the index $[\mathcal{D}_2 : \mathcal{D}_1]$ is given by $2^{(2^{s-1}-1)s}$.

*Proof.* The first statement of (1) follows from the definition (cf. loc. cit.), and the second follows from [14, Proposition 4.2]. To show (2), we consider a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}_1 & \longrightarrow & \bigoplus_{e \in \mathbb{E}} \mathbb{Z}D^e & \longrightarrow & \mathbb{Z}D^{1_{\mathbb{E}}} & \longrightarrow & 0 \\ & & \alpha \big\downarrow & & \beta \big\downarrow & & \gamma \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{D}_2 & \longrightarrow & \mathcal{D} & \xrightarrow[\text{aug}]{} & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

where $\alpha$ and $\beta$ are the inclusions, aug is the augmentation, and $\gamma$ is induced by $\operatorname{aug} \circ \beta$. Since $\gamma$ is injective and its cokernel is cyclic of order $2^s$, we are reduced to showing $|\operatorname{Coker}(\beta)| = 2^{2^{s-1}s}$. Define a matrix $A_s \in M_{2^s}(\mathbb{Z})$ by $A_s := (\langle e, w\rangle)_{e \in \mathbb{E}, w \in \mathbb{W}}$ so that we have $|\operatorname{Coker}(\beta)| = |\det A_s|$. Then we have $A_s = \left(\begin{smallmatrix} A_{s-1} & A_{s-1} \\ A_{s-1} & -A_{s-1} \end{smallmatrix}\right)$ for any $s \geq 1$. Now the claim follows by induction. □

**Remark 2.2.** The composition of canonical homomorphisms

$$\mathcal{D}_2 \hookrightarrow \mathbb{Z}[w] \twoheadrightarrow \mathcal{D}_3 := \mathcal{D}\Big/\Big\langle \sum_{w \in \mathbb{W}} [w] \Big\rangle_{\mathbb{Z}} \tag{2.4}$$

is injective and its cokernel is of order $2^s$. Hence, we have

$$\mathcal{D}_1 \otimes \mathbb{Z}\Big[\frac{1}{2}\Big] = \mathcal{D}_2 \otimes \mathbb{Z}\Big[\frac{1}{2}\Big] = \mathcal{D}_3 \otimes \mathbb{Z}\Big[\frac{1}{2}\Big].$$

For a $\mathbb{Z}[\mathbb{W}]$-module $M$ and $e \in \mathbb{E}$, we write

$$M^e = \{x \in M \mid wx = \langle e, w\rangle x \text{ for all } w \in \mathbb{W}\}. \tag{2.5}$$

Then, for any $\mathbb{Z}[1/2]$-module $B$ and $i = 1, 2, 3$, we have

$$\mathcal{D}_i \otimes B \cong \bigoplus_{e \in \mathbb{E}\setminus\{1\}} (\mathcal{D}_i \otimes B)^e$$

and, moreover,

$$(\mathcal{D}_1 \otimes B)^e = (\mathcal{D}_2 \otimes B)^e = (\mathcal{D}_3 \otimes B)^e = D^e \otimes B. \tag{2.6}$$

## 2.3 Jacobian variety

Let $J$ be the Jacobian variety of $X$. We define $\mathcal{C}$ to be the cuspidal divisor subgroup, i.e.,

$$\mathcal{C} := \mathrm{Im}(\mathcal{D}_2 \to J(F)) \subset J(F) \tag{2.7}$$

where $\mathcal{D}_2$ is from Lemma 2.1. It is known that $\mathcal{C}$ is contained in $J(F)_{\mathrm{Tor}}$ by [3, 8]. Since $J(F)_{\mathrm{Tor}}$ is finite, we can decompose

$$J(F)_{\mathrm{Tor}} = \bigoplus_{\ell} J(F)\{\ell\}, \quad \mathcal{C} = \bigoplus_{\ell} \mathcal{C}\{\ell\},$$

where $\ell$ runs through all primes and $\{\ell\}$ denotes the $\ell$-primary torsion part. If $\ell \neq 2$, we may further decompose (see (2.5))

$$J(F)\{\ell\} \cong \bigoplus_{e \in \mathbb{E}} J(F)\{\ell\}^e, \quad \mathcal{C}\{\ell\} \cong \bigoplus_{e \in \mathbb{E}} \mathcal{C}\{\ell\}^e.$$

Suppose either that we are in the case (NF) and $(3, N) = 1$, or that we are in the case (FF). We define $e_H \in \mathbb{E}$ by

$$e_H := \begin{cases} ((\frac{p_1}{3}), \dots, (\frac{p_s}{3})) & \text{in (NF), } (3, N) = 1, \\ ((-1)^{\deg p_1}, \dots, (-1)^{\deg p_s}) & \text{in (FF).} \end{cases} \tag{2.8}$$

Note that, for $e \in \mathbb{E}$, we have $(3, d(e)) = 3$ if and only if $e \neq e_H$ in the case (NF) (see (1.3) for the definition of $d(e)$). In the case (FF), we have $(q + 1, d(e)) = q + 1$ for any $e \neq e_H$, and $(q + 1, d(e_H))$ is a power of two [14, Remark 3.6]. We also put

$$k := \begin{cases} 12 & \text{in (NF),} \\ q^2 - 1 & \text{in (FF),} \end{cases} \qquad b := \begin{cases} 3 & \text{in (NF),} \\ q + 1 & \text{in (FF).} \end{cases} \tag{2.9}$$

In the case (FF), we write $|a| = q^{\deg a}$ for $a \in A \setminus \{0\}$.

**Theorem 2.3.** (1) ([9, Theorem 1] and [12, Theorems 1.2, 1.4]) Suppose $s = 1$. Then we have $\mathcal{C} = J(F)_{\mathrm{Tor}}$ and it is a cyclic group of order $(|N| - 1)/(k, |N| - 1)$.

(2) ([11, Theorem (3.6.2)] and [14, Theorem 4.3]) Let $\ell$ be an odd prime. In (NF) and $\ell = 3$, we further assume $(N, 3) = 1$. If we are in (FF), assume $(\ell, q(q - 1)) = 1$. Then, for any $e \in \mathbb{E}$, we have $\mathcal{C}\{\ell\}^e = J(F)\{\ell\}^e$ and it is a cyclic group whose order is the $\ell$-part of

$$\begin{cases} 1 & \text{if } e = 1_{\mathbb{E}}, \\ d(e) & \text{if } e = e_H \neq 1_{\mathbb{E}}, \\ d(e)/b & \text{if } e \neq 1_{\mathbb{E}}, e_H. \end{cases}$$

It is generated by the class of $D^e$ unless $e = 1_{\mathbb{E}}$.

Note that we do not need $e_H^{\pm}$ appearing in [11], because we assume $\ell \neq 2$ and $(3, N) = 1$ if $\ell = 3$.

# 3 Generalized Jacobian

## 3.1 An exact sequence

Let $\widetilde{J}$ be the generalized Jacobian variety of $X$ with modulus $C$ (see (2.1) for $C$). Here we quickly recall some basic results from [15] and [18, Section 2], with which the readers may consult for details. We have (almost by definition)

$$\widetilde{J}(F) = \mathrm{Div}^0(X \setminus C)/\{\mathrm{div}(f) \mid f \in F(X)^\times,\ f \equiv 1 \bmod C\}. \tag{3.1}$$

Here by $f \equiv 1 \bmod C$, we mean $\mathrm{ord}_x(f-1) > 0$ for all $x \in C$. It follows that there is an exact sequence

$$0 \to \mathcal{D}_3 \otimes (F^\times)_{\mathrm{Tor}} \to \widetilde{J}(F)_{\mathrm{Tor}} \to J(F)_{\mathrm{Tor}} \xrightarrow{\delta} \mathcal{D}_3 \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}, \tag{3.2}$$

where $\mathcal{D}_3$ is from (2.4). All maps in this sequence are compatible with the action of Atkin–Lehner involutions. In particular, we can decompose the map $\delta$, up to 2-primary torsion, into the direct sum of

$$\delta_\ell^e : J(F)\{\ell\}^e \to D^e \otimes F^\times \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $\ell$ and $e$ ranges over all odd primes and elements of $\mathbb{E}$, respectively, see (2.6).

## 3.2 Connecting map

We recall a description of the map $\delta$. Let $\mathcal{D}_2$ and $\mathcal{D}_3$ be groups defined in Lemma 2.1 and Remark 2.2, respectively. We shall identify $\mathcal{D}_3 \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}$ with the quotient of

$$\bigoplus_{w\in\mathbb{W}} \mathbb{Z}[w] \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}$$

by $\langle \sum_{w\in\mathbb{W}}[w]\rangle_{\mathbb{Z}} \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}$.

**Lemma 3.1** ([18, Lemma 2.3.1]). Suppose that the class $[D] \in J(F)$ of $D = \sum_{w\in\mathbb{W}} a_w[w] \in \mathcal{D}_2$ is killed by $m \in \mathbb{Z}_{>0}$ so that there is an $f \in F(X)^\times$ such that $\mathrm{div}(f) = mD$. Then $\delta([D])$ is given by the image in $\mathcal{D}_3 \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}$ of

$$\sum_{w\in\mathbb{W}} [w] \otimes \frac{f}{t_{[w]}^{ma_w}}([w]) \otimes \frac{1}{m} \in \bigoplus_{w\in\mathbb{W}} \mathbb{Z}[w] \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}, \tag{3.3}$$

where $t_{[w]}$ is a uniformizer at $[w] \in C$. (Note that the image of (3.3) in $\mathcal{D}_3 \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}$ depends only on $[D] \in J(F)$ and is independent of the choices of $m$, $f$ and $t_{[w]}$.)

## 3.3 Main result

For $i = 1, \dots, s$, define $e^{(i)} = (e_j^{(i)})_j \in \mathbb{E}$ by

$$e_j^{(i)} = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

We now arrive at our main result (see (2.9) and (1.3) for the definitions of $k$ and $d(e)$).

**Theorem 3.2.** Let $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$. The order of $\delta([D^e])$ is given by

$$\begin{cases} d(e^{(i)})/(d(e^{(i)}), 2^{s-1}k) & \text{if } e = e^{(i)} \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

The proof of this theorem is given in the next section. Combined with Theorem 2.3, it implies Theorems 1.1 and 1.3. For the latter, we can also deduce the following supplementary results. We define

$$\mathcal{M}_2' := \bigoplus_e (\mathbb{Z}/(d(e)/b)\mathbb{Z}).$$

where $e$ ranges over $\mathbb{E} \setminus \{1_{\mathbb{E}}, e_H, e^{(1)}, \dots, e^{(s)}\}$ (see (2.9) and (2.8) for $b$ and $e_H$).

**Corollary 3.3.** If we are in the case (NF), we assume $(3, N) = 1$ and put $\ell = 3$. If we are in the case (FF), we assume $\ell$ is an odd prime divisor of $q + 1$. Then there is an isomorphism

$$\tilde{\mathcal{J}}(F)\{\ell\} \cong \mathcal{M}_2' \otimes \mathbb{Z}_{(\ell)}.$$

*Proof.* Using the remarks after (2.8) and $\mathrm{ord}_\ell(k) = \mathrm{ord}_\ell(b)$, this follows by comparing Theorem 2.3 with Theorem 3.2. □

# 4 Proof of Theorem 3.2

## 4.1 Discriminant functions

Let $F_\infty$ be the completion of $F$ with respect to the absolute value $|\cdot|$ on $F$ (which means $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{F}_q(t)_\infty = \mathbb{F}_q(\!(t^{-1})\!)$). Let $\mathbb{C}_\infty$ be the completion of a chosen algebraic closure of $F_\infty$. Let

$$\mathfrak{H} := \begin{cases} \{z \in \mathbb{C}_\infty \mid \mathrm{Im}(z) > 0\} & \text{in the case (NF)}, \\ \mathbb{C}_\infty - F_\infty & \text{in the case (FF)}. \end{cases}$$

Let $\Delta(z)$ be the *modular discriminant function* [17, Example 3.4.3] (resp. *Drinfeld discriminant function* [3, equation (1.2)]) on $\mathfrak{H}$, which satisfies

$$\Delta\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^k \cdot \Delta(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

where $k$ is from (2.9) and $G = \mathrm{SL}_2(A)$ (resp. $G = \mathrm{GL}_2(A)$) in the case (NF) (resp. (FF)). Given $e = (e_i)_{i=1}^s \in \mathbb{E}$, we define

$$\Delta^e(z) := \prod_{w \in \mathbb{W}} \Delta(m(w)z)^{\langle e,w \rangle}. \tag{4.1}$$

For $w \in \mathbb{W}$ and $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$, we define

$$c(e, w) = \begin{cases} p_i^{-2^{s-1}k} & \text{if } e = e^{(i)} \text{ and } w_i = 1, \\ 1 & \text{otherwise}, \end{cases} \tag{4.2}$$

where $k$ is from (2.9) and $e^{(i)}$ is from (3.4).

**Lemma 4.1.** Given $w \in \mathbb{W}$ and $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$, we have

$$\Delta^e(W_w z) = c(e, w)\Delta^e(z)^{\langle e,w \rangle}. \tag{4.3}$$

*Proof.* From the transformation law of $\Delta$, it is straightforward that

$$\Delta^e(W_w z) = \Delta^e(z)^{\langle e,w \rangle} \cdot \prod_{w' \in \mathbb{W}} (m(w'), m(w))^{-\langle e,w' \rangle \cdot k}.$$

For $i = 1, \dots, s$, one has (see the sentence after (2.2) for the notation˜)

$$\mathrm{ord}_{p_i}\Big(\prod_{w' \in \mathbb{W}} (m(w'), m(w))^{\langle e,w' \rangle}\Big) = \tilde{w}_i \cdot \sum_{\substack{w' \in \mathbb{W} \\ w_i' = 1}} \langle e, w' \rangle = \begin{cases} 2^{s-1} & \text{if } e = e^{(i)} \text{ and } w_i = 1, \\ 0 & \text{otherwise}, \end{cases}$$

proving the lemma. □

**Lemma 4.2.** Let $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$.
(1) The function $\Delta^e(z)$ from (4.1) is a rational function on $X$ defined over $F$.
(2) We have $\operatorname{div}(\Delta^e) = d(e)D^e$ (see (1.3) and (2.3)).

*Proof.* (1) The transformation law of $\Delta$ implies that $\Delta^e$ lies in $\mathbb{C}_\infty(X)$. Note that the "$q$-expansion of $\Delta$ at $\infty$" has $F$-rational coefficients (cf. [2] in the case (FF)). Regarding $\mathbb{C}_\infty(X)$ as a subfield of $\mathbb{C}_\infty(\!(q)\!)$, the result follows from the fact that $F(X) = \mathbb{C}_\infty(X) \cap F(\!(q)\!)$.

(2) Let $w_\infty := (-1, \dots, -1) \in \mathbb{W}$. Then $[w_\infty] = [1/N] \in C$, corresponding to the "cusp at $\infty$". Considering the $q$-expansion of $\Delta$, it is observed that

$$\operatorname{ord}_{[w_\infty]}(\Delta^e) = d(e) \cdot \langle e, w_\infty \rangle = d(e) \cdot \operatorname{ord}_{[w_\infty]} D^e.$$

Thus, the result follows from Lemma 2.1 (1) and Lemma 4.1. □

## 4.2 Evaluation of the connecting map

Theorem 3.2 immediately follows from the following result.

**Proposition 4.3.** Let $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$. If $e = e^{(i)}$ for some $i = 1, \dots, s$ (see (3.4)), then we have

$$\delta(D^{e^{(i)}}) = \sum_{\substack{w \in \mathbb{W} \\ w_i = 0}} [w] \otimes p_i \otimes \frac{-2^{s-1}k}{d(e^{(i)})} = D^{e^{(i)}} \otimes p_i \otimes \frac{-2^{s-2}k}{d(e^{(i)})} \quad \text{in } \mathcal{D}_3 \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}.$$

Otherwise, we have $\delta(D^e) = 0$.

*Proof.* The second equality follows from

$$D^{e^{(i)}} = \sum_{\substack{w \in \mathbb{W} \\ w_i = 0}} [w] - \sum_{\substack{w \in \mathbb{W} \\ w_i = 1}} [w] = 2 \sum_{\substack{w \in \mathbb{W} \\ w_i = 0}} [w] - \sum_{w \in \mathbb{W}} [w] \quad \text{in } \bigoplus_{w \in \mathbb{W}} \mathbb{Z}[w].$$

To show the first equality, by Lemma 3.1, it suffices to take suitable uniformizers $t_{[w]} \in F(X)$ for $w \in \mathbb{W}$ so that

$$\frac{\Delta^e}{t_{[w]}^{\langle e, w \rangle \cdot d(e)}}([w]) = c(e, \bar{w}) \quad \text{for all } e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}, w \in \mathbb{W}, \tag{4.4}$$

where $c(e, \bar{w})$ is from (4.2) and we put $\bar{w} := w_\infty - w (= w_\infty + w) \in \mathbb{W}$, with $w_\infty := (1, \dots, 1) \in \mathbb{W}$. We first take $t_{[w_\infty]} \in F(X)^\times$ to be any uniformizer at $w_\infty$ which has 1 as the leading term of its $q$-expansion. (For instance, we may take the pull-back of the reciprocal of the $j$-function $X_0(1) \xrightarrow{\cong} \mathbb{P}^1$ along the morphism $\pi\colon X = X_0(N) \to X_0(1)$ given by forgetting the level structure.) Then we have

$$\left(\frac{\Delta^e}{t_{[w_\infty]}^{\langle e, w_\infty \rangle \cdot d(e)}}\right)([w_\infty]) = 1$$

for any $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$. For general $w \in \mathbb{W}$, let

$$t_{[w]} := t_{[w_\infty]} \circ W_{\bar{w}} \in F(X).$$

Then we have

$$\left(\frac{(\Delta^e)^{\langle e, \bar{w} \rangle}}{t_{[w_\infty]}^{\langle e, w \rangle \cdot d(e)}}\right)([w_\infty]) = \left(\frac{\Delta^e}{t_{[w_\infty]}^{\langle e, w_\infty \rangle \cdot d(e)}}\right)^{\langle e, \bar{w} \rangle}([w_\infty]) = 1.$$

Therefore, for any $e \in \mathbb{E} \setminus \{1_{\mathbb{E}}\}$ and $w \in \mathbb{W}$, applying (4.3), we obtain

$$\begin{aligned}
\left(\frac{\Delta^e}{t_{[w]}^{\langle e, w \rangle \cdot d(e)}}\right)([w]) &= \left(\frac{\Delta^e \circ W_{\bar{w}}}{t_{[w_\infty]}^{\langle e, w \rangle \cdot d(e)}}\right)([w_\infty]) \\
&= c(e, \bar{w}) \cdot \left(\frac{(\Delta^e)^{\langle e, \bar{w} \rangle}}{t_{[w_\infty]}^{\langle e, w \rangle \cdot d(e)}}\right)([w_\infty]) \\
&= c(e, \bar{w}),
\end{aligned}$$

and (4.4) holds. □

### 4.3 Generators of $\mathcal{C} \cap \ker \delta$

For $i = 1, \ldots, s$, we have

$$2^{s-1} \cdot ([1] - [1/p_i]) = \sum_{\substack{e \in \mathbb{E} \\ e_i = -1}} D^e.$$

Let $d(\mathbb{E}, i) := \prod_{e \in \mathbb{E}, e_i = -1} d(e)$. Then

$$d(\mathbb{E}, i) 2^{s-1} \cdot ([1] - [1/p_i]) = \operatorname{div}\Bigg( \prod_{\substack{e \in \mathbb{E} \\ e_i = -1}} (\Delta^e)^{\frac{d(\mathbb{E}, i)}{d(e)}} \Bigg) \quad \text{in } \operatorname{Div}^0(X).$$

Thus, (3.3) and (4.4) imply

$$\begin{aligned} \delta([1] - [1/p_i]) &= \sum_{\substack{w \in \mathbb{W} \\ w_i = 0}} [w] \otimes p_i^{-2^{s-1} k \cdot \frac{d(\mathbb{E}, i)}{d(e^{(i)})}} \otimes \frac{1}{2^{s-1} d(\mathbb{E}, i)} \\ &= \sum_{\substack{w \in \mathbb{W} \\ w_i = 0}} [w] \otimes p_i \otimes \frac{-k}{d(e^{(i)})} \\ &= D^{e^{(i)}} \otimes p_i \otimes \frac{-k}{2d(e^{(i)})} \quad \text{in } \mathcal{D}_3 \otimes F^\times \otimes \mathbb{Q}/\mathbb{Z}, \end{aligned} \tag{4.5}$$

which is of order $d(e^{(i)})/(d(e^{(i)}), k)$. In particular, for $m \mid (N/p_i)$, we get

$$\begin{aligned} \delta([1/m] - [1/(mp_i)]) &= \delta(W_m([1] - [1/p_i])) \\ &= W_m(D^{e^{(i)}}) \otimes p_i \otimes \frac{-k}{2d(e^{(i)})} \\ &= D^{e^{(i)}} \otimes p_i \otimes \frac{-k}{2d(e^{(i)})} = \delta([1] - [1/p_i]). \end{aligned} \tag{4.6}$$

Observe that

$$\{[1/m] - [1/(mp_i)] : i = 1, \ldots, s \text{ and } m \mid (N/p_1 \cdots p_i)\}$$

is a $\mathbb{Z}$-basis of $\mathcal{D}_2$. Therefore, equalities (4.5) and (4.6) lead us to following result.

**Corollary 4.4.** For $i = 1, \ldots, s$ and $m \mid (N/p_i)$, we have

$$\delta([1] - [1/p_i] - [1/m] + [1/(mp_i)]) = 0.$$

Moreover, the intersection of $\mathcal{C} \cap \ker \delta$ is generated by

$$\{[1] - [1/p_i] - [1/m] + [1/(mp_i)] : i = 1, \ldots, s \text{ and } m \mid (N/p_1 \cdots p_i)\}$$

and

$$\left\{ \frac{d(e^{(i)})}{(d(e^{(i)}), k)} \cdot ([1] - [1/p_i]) : i = 1, \ldots, s \right\}.$$

## 5 Eisenstein property

### 5.1 Jacobian variety

Let $\mathbb{T}$ be the polynomial ring $\mathbb{Z}[\tau_p : p \nmid N]$ generated by variables $\tau_p$ for all prime elements $p \in A_+$ such that $p \nmid N$. Let $\mathcal{E}$ be the ideal generated by $\tau_p - |p| - 1$ for all $p \nmid N$, which is called the *Eisenstein ideal*. We say that a $\mathbb{T}$-module is *Eisenstein* if it is annihilated by $\mathcal{E}$. In the case (NF), we define $\mathcal{E}'$ to be the ideal generated by $\tau_p - p - 1$ for all $p \nmid 2N$, and we say a $\mathbb{T}$-module is *Eisenstein away from* 2 if it is annihilated by $\mathcal{E}'$. When $N$ is even, this is the same as saying Eisenstein.

Given $p \nmid N$, the Hecke correspondence on $X$ associated to $p$ induces an endomorphism $T_p$ of $J$. We obtain a $\mathbb{Z}$-algebra homomorphism $\mathbb{T} \to \mathrm{End}(J)$ sending $\tau_p$ to $T_p$. Then we may ask if $J(F)_{\mathrm{Tor}}$ is Eisenstein. We collect known results in the literature.

**Lemma 5.1.** (1) In (NF), $J(F)_{\mathrm{Tor}}$ is Eisenstein away from 2, and $J(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/2]$ is Eisenstein.
(2) In the case (FF), $J(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/q]$ is Eisenstein.
(3) In general, $\mathcal{C}$ is Eisenstein (see (2.7)). Consequently, $J(F)_{\mathrm{Tor}}$ is Eisenstein if $s = 1$, by Theorem 2.3 (1).

*Proof.* (1) is shown in the proof of [11, Theorem 3.6.2, p. 316]. For the completeness sake, we sketch its proof. If $\ell$ is an odd prime (resp. $\ell = 2$), then the reduction map $J(F) \to J_{/\mathbb{F}_\ell}(\mathbb{F}_\ell)$ restricted to $J(F)_{\mathrm{Tor}}$ (resp. $J(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/2]$) is injective by [4, Appendix] (resp. [16, Chapter IV, Proposition 3.1 (b)]), where $J_{/\mathbb{F}_\ell}$ is the reduction of $J$ over $\mathbb{F}_\ell$. On the other hand, the Eichler–Shimura congruence relation shows that the Hecke correspondence $T_\ell$ acts on $J_{/\mathbb{F}_\ell}(\mathbb{F}_\ell)$ by the multiplication by $\ell + 1$, whence the statement.

(2) is shown in [13, Lemma 7.1].

(3) is well known, and it also follows from Proposition 5.6 (3) below. □

## 5.2 Generalized Jacobian

We can play a similar game for $\widetilde{J}$. Given $p \nmid N$, the Hecke correspondence on $X$ associated to $p$ again induces an endomorphism $\widetilde{T}_p$ of $\widetilde{J}$ as follows. Let $f, g\colon X_0(pN) \to X_0(N) = X$ be the maps that send a pair $(E, Q)$ of an elliptic curve (or a Drinfeld module) $E$ and its $pN$-torsion subgroup (submodule) $Q$ to the pair $(E, Q[N])$ and $(E/Q[p], Q/Q[p])$, respectively. (We write $Q[N]$ for the $N$-torsion subgroup (submodule) of $Q$, and similarly for $Q[p]$.) Over $\mathbb{C}_\infty$, they are induced by the identity map and the multiplication by $p$ on $\mathbb{P}^1(\mathbb{C}_\infty) \sqcup \mathbb{P}^1(F)$, passing through the quotients by $\Gamma_0(N)$ and $\Gamma_0(pN)$. Recall that $C$ is the set of all cusps on $X = X_0(N)$. Let $C'$ be the set of all cusps on $X_0(pN)$ and $\widetilde{J}'$ the generalized Jacobian of $X_0(pN)$ with modulus $C'$. Since we have $f^{-1}(C) = g^{-1}(C) = C'$, there are the pull-back map $f^* : \widetilde{J} \to \widetilde{J}'$ and the push-forward map $g_* : \widetilde{J}' \to \widetilde{J}$. The action of the Hecke correspondence is given by

$$\widetilde{T}_p := g_* \circ f^* : \widetilde{J} \to \widetilde{J}.$$

We omit the proof of the following lemma, which is well known (and it is seen in the same way as $J$).

**Lemma 5.2.** We have $\widetilde{T}_{p_1}\widetilde{T}_{p_2} = \widetilde{T}_{p_2}\widetilde{T}_{p_1}$ for any $p_1, p_2 \nmid N$.

Hence, we obtain a $\mathbb{Z}$-algebra homomorphism $\mathbb{T} \to \mathrm{End}(\widetilde{J})$ sending $\tau_p$ to $\widetilde{T}_p$, and the canonical map $\widetilde{J} \to J$ is $\mathbb{T}$-equivariant. Then we may ask if $\widetilde{J}(F)_{\mathrm{Tor}}$ is Eisenstein. We provide a partial answer to this problem. The first result is the following.

**Lemma 5.3.** (1) In (NF), $\widetilde{J}(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/2]$ is Eisenstein.
(2) In (FF), $\widetilde{J}(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/q(q-1)]$ is Eisenstein.

*Proof.* The Hecke algebra $\mathbb{T}$ acts on $\mathrm{Div}(X)$ as algebraic correspondences, which induces $\mathbb{T}$-module structures on the subgroup $\mathcal{D}_2 \subset \mathrm{Div}(X)$ from Lemma 2.1 (2), and, in turn, on the quotient $\mathcal{D}_3$ of $\mathcal{D}_2$ from (2.4). Under the identification $\mu_F^{\oplus(2^s-1)} = \mathcal{D}_3 \otimes \mu_F$, the maps in the exact sequence (1.1) are $\mathbb{T}$-equivariant. Now the lemma immediately follows from Lemma 5.1. □

We say a $\mathbb{T}$-module has *Eisenstein exponent two* (resp. *Eisenstein exponent two away from* 2 in the case (NF)) if it is annihilated by $\mathcal{E}^2$ (resp. $\mathcal{E}'^2$). As before, the two notions are identical when $N$ is even.

**Proposition 5.4.** (1) If $s = 1$, then $\widetilde{J}(F)_{\mathrm{Tor}}$ is Eisenstein.
(2) In (NF), $\widetilde{J}(F)_{\mathrm{Tor}}$ has Eisenstein exponent two away from 2.
(3) In (FF), $\widetilde{J}(F)_{\mathrm{Tor}} \otimes \mathbb{Z}[1/q]$ has Eisenstein exponent two.

*Proof.* In general, if $0 \to M' \to M \to M''$ is an exact sequence of $\mathbb{T}$-modules and if $M', M''$ are Eisenstein (away from 2), then $M$ has Eisenstein exponent two (away from 2). Hence, by Lemma 5.1, the proposition is reduced to the following lemma that shows $\mu_F^{\oplus(2^s-1)} = \mathcal{D}_3 \otimes \mu_F$ is Eisenstein in (1.1), where $\mathcal{D}_3$ is from (2.4). □

**Lemma 5.5.** The group $\mathcal{D}_3$ is Eisenstein.

We will prove this lemma as a part of Proposition 5.6 (3) below (although it can also be shown directly). To state the result, we need some preparation. We define

$$L := \bigoplus_{w \in \mathbb{W}} F(X)^\times_{[w]}/U^{(1)}_{[w]}, \quad U^{(1)}_{[w]} := \{f \in F(X)^\times_{[w]} \mid \operatorname{ord}_{[w]}(f-1) > 0\},$$
$$L^0 := \ker(d\colon L \to \mathbb{Z}), \qquad d((f_w)_w) = \sum_w \operatorname{ord}_{[w]}(f_w),$$

where $F(X)_{[w]}$ is the completion of $F(X)$ at $[w] \in X$ and $\operatorname{ord}_{[w]}\colon F(X)^\times_{[w]} \to \mathbb{Z}$ is the normalized discrete valuation. By definition, we have an exact sequence (see Lemma 2.1 (2) for $\mathcal{D}$ and $\mathcal{D}_2$)

$$0 \to \mathcal{D} \otimes F^\times \to L^0 \to \mathcal{D}_2 \to 0.$$

Using the approximation lemma, (3.1) can be rewritten as

$$\widetilde{J}(F) = \frac{\ker[(\deg, d) : \operatorname{Div}(X \setminus C) \oplus L \to \mathbb{Z}]}{\{(\operatorname{div}_{X\setminus C}(f), \Delta(f)) \mid f \in F(X)^\times\}},$$

where $\Delta\colon F(X)^\times \to L$ is induced by the diagonal embedding (see [18, Section 2.2] for more details). We define $\widetilde{\mathcal{C}} \subset \widetilde{J}(F)$ to be the image of the composition of the canonical maps $L^0 \hookrightarrow L \to \widetilde{J}(F)$. This is related to the cuspidal divisor class $\mathcal{C}$ from (2.7) by an exact sequence

$$0 \to \mathcal{D}_3 \otimes F^\times \to \widetilde{\mathcal{C}} \to \mathcal{C} \to 0.$$

**Proposition 5.6.** (1) In (NF), $\widetilde{\mathcal{C}}$ is Eisensetein away from 2 and has Eisenstein exponent two. It is Eisenstein if $s = 1$ (or if $N$ is even).
(2) In (FF), $\widetilde{\mathcal{C}}$ is Eisensetein.
(3) In any case, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{C}$ are Eisentein.

We obtain the following because $\mathcal{C} = J(F)_{\mathrm{Tor}}$ implies $\widetilde{J}(F)_{\mathrm{Tor}} \subset \widetilde{\mathcal{C}}$.

**Corollary 5.7.** Suppose $\mathcal{C} = J(F)_{\mathrm{Tor}}$.
(1) In (NF), $\widetilde{J}(F)_{\mathrm{Tor}}$ is Eisensetein away from 2 and has Eisenstein exponent two.
(2) In (FF), $\widetilde{J}(F)_{\mathrm{Tor}}$ is Eisensetein.

It remains to prove Proposition 5.6. Take $p \nmid N$. We use the notations introduced in the beginning of Section 5.2. For any $w \in \mathbb{W}$, we define two cusps $[w]', [w^*]' \in C'$ on $X' = X_0(pN)$ to be the $\Gamma_0(pN)$-orbits of $1/m(w)$ and $1/(pm(w)) \in \mathbb{P}^1(F)$, respectively (see (2.2) for $m(w)$). We have $f^{-1}([w]) = g^{-1}([w]) = \{[w]', [w^*]'\}$. The action of $\tau_p$ on $L$ is induced by the direct sum of

$$\phi_w\colon \frac{F(X)^\times_{[w]}}{U^{(1)}_{[w]}} \xrightarrow{f^*} \frac{F(X')^\times_{[w]'}}{U^{(1)}_{[w]'}} \oplus \frac{F(X')^\times_{[w^*]'}}{U^{(1)}_{[w^*]'}} \xrightarrow{g_*} \frac{F(X)^\times_{[w]}}{U^{(1)}_{[w]}}$$

over all $w \in \mathbb{W}$. We remark that both of $f$ and $g$ are of degree $|p| + 1$ and their ramification indexes are given by $e_f([w]') = |p|$, $e_f([w^*]') = 1$ and $e_g([w]') = 1$, $e_g([w^*]') = |p|$. The following is the key step in the proof of Proposition 5.6.

**Lemma 5.8.** For any $\alpha \in F(X)^\times_{[w]}$, we have

$$\phi_w(\alpha \bmod U^{(1)}_{[w]}) = (-1)^{(|p|+1)\operatorname{ord}_{[w]}(\alpha)}\alpha^{(|p|+1)} \bmod U^{(1)}_{[w]}. \tag{5.1}$$

In particular, we have

$$\operatorname{ord}_{[w]}(\phi_w(\alpha \bmod U^{(1)}_{[w]})) = (|p| + 1)\operatorname{ord}_{[w]}(\alpha \bmod U^{(1)}_{[w]}).$$

*Proof.* Note first that $F(X)^{\times}_{[w]}/U^{(1)}_{[w]}$ is generated by the classes of non-zero constants $F^{\times}$ and a single element $\alpha \in F(X)^{\times}$ such that $\mathrm{ord}_{[w]}(\alpha) = \pm 1$, and hence it suffices to verify (5.1) for such elements. We consider the map

$$\Phi_w \colon F(X)^{\times} \xrightarrow{f^*} F(X')^{\times} \xrightarrow{g_*} F(X)^{\times},$$

which induces $\phi_w$ in view of

$$F(X')_{[w]'} \times F(X')_{[w^*]'} = F(X') \otimes_{F(X)} F(X)_{[w]}.$$

Since $f$ and $g$ are of degree $|p| + 1$, one has $\Phi_w(\alpha) = \alpha^{|p|+1}$ for a constant $\alpha \in F^{\times}$, showing (5.1) in this case.

As for the other type of generators, we first consider the case $w = w_\infty = (-1, \dots, -1)$ (i.e., $[w] = [\infty]$). Let us consider the diagram

$$\begin{array}{ccccc} X = X(N) & \xleftarrow{\ g\ } & X' = X_0(pN) & \xrightarrow{\ f\ } & X = X_0(N) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X_0(1) & \xleftarrow[\ g'\ ]{} & X_0(p) & \xrightarrow[\ f'\ ]{} & X_0(1), \end{array}$$

where $f'$ and $g'$ are defined similarly as above, and $\pi$ and $\pi'$ are given by forgetting level structures (like $f$). We may take $\alpha := \pi^*(j)$ as the pull-back of the $j$-function $X_0(1) \xrightarrow{\cong} \mathbb{P}^1$ so that $\mathrm{ord}_{[w_\infty]}(\alpha) = -1$. Since the squares in the above diagram are Cartesian, we have

$$\Phi_{w_\infty}(\alpha) = g_* f^* \pi^*(j) = g_* \pi'^* f'^*(j) = \pi^* g'_* f'^*(j).$$

Therefore, (5.1) is reduced to the case $N = 1$, which we now assume. Let $W'_p : X' = X_0(p) \to X'$ be the Atkin–Lehner involution with respect to $p$ so that we have $g = f \circ W'_p$, and therefore $\Phi_{w_\infty}(j) = g_* f^*(j)$ is given by the norm of $j' := W'_p(f^*(j)) \in F(X')^{\times}$ with respect to the field extension $F(X')/F(X) = F(j)$ along $f$. Let $F_p(X, Y) \in A[X, Y]$ be the modular polynomial attached to $p$ so that we have $F_p(f^*(j), j') = 0$ and $F_p(f^*(j), Y)$ is an irreducible, monic and of degree $|p| + 1$ in $\mathbb{C}_\infty(f^*(j))[Y]$. (cf. [5, Chapter 5, Section 2, Theorem 3] and [1, Theorem 2.4]). Therefore, we have

$$\Phi_{w_\infty}(j) = (-1)^{|p|+1} F_p(j, 0) \equiv (-1)^{|p|+1} j^{|p|+1} \bmod U^{(1)}_{[w_\infty]}.$$

This proves (5.1) in this case.

Finally, let us go back to a general $N$ and consider other $w \in \mathbb{W}$. Then we may take $\alpha := W_w(j)$. Using the Atkin–Lehner involution $W'_w \colon X' \to X'$ acting on $X'$, we have the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\ g\ } & X' & \xrightarrow{\ f\ } & X \\ \downarrow{\scriptstyle W_w} & & \downarrow{\scriptstyle W'_w} & & \downarrow{\scriptstyle W_w} \\ X & \xleftarrow{\ g\ } & X' & \xrightarrow{\ f\ } & X. \end{array}$$

Hence, (5.1) follows from the case $w = w_\infty$ above. □

*Proof of Proposition 5.6.* The above lemma shows everything except the second statement of (1), which we now prove. Hence, we suppose that we are in the case (NF) and $s = 1$, $p = 2$. Then any element of $L^0$ is represented by $(\alpha, \beta) \in F(X)^{\times}_{[0]} \oplus F(X)^{\times}_{[\infty]}$ such that $\mathrm{ord}_{[0]}(\alpha) = -\mathrm{ord}_{[\infty]}(\beta) =: a$. By the lemma, we see that $\widetilde{T}_2$ sends it to the class of $((-1)^a \alpha^3, (-1)^a \beta^3)$, but the image of $(-1, -1)$ in $\widetilde{\mathcal{C}}$ is trivial because it is in the image of the diagonal map. This completes the proof. □

**Remark 5.9.** In the case (NF), we actually obtain that $\widetilde{J}(F)_{\mathrm{Tor}}$ is never Eisenstein when $N$ is odd and $s > 2$. To prove this, let $c \in \mathcal{C}$ be the divisor class represented by

$$[1] - [1/p_1] - [p_1/N] + [1/N] \in \mathcal{D}_2.$$

By Corollary 4.4, we have $\delta(c) = 0$. From the exact sequence (3.2), there exists $\widetilde{c} \in \widetilde{J}(F)_{\mathrm{Tor}}$ whose image in $\widetilde{J}(F)_{\mathrm{Tor}}$ is $c$. We shall show that $\widetilde{c}$ is not annihilated by $\widetilde{T}_2 - 3$. Take $\xi \in L^0$ so that the surjective map $L^0 \twoheadrightarrow \mathcal{D}_2$ sends $\xi$ to $[1] - [1/p_1] - [p_1/N] + [1/N]$. Here we identify $\mathbb{W}$ with $\{m \in A_+ : m|N\}$ as in (2.2). Let $\widetilde{c}' \in \widetilde{C} \subset \widetilde{J}(F)$ be the image of $\xi$ under the map $L^0 \twoheadrightarrow \widetilde{C}$. Then the canonical map from $\widetilde{J}(F)$ to $J(F)$ also sends $\widetilde{c}'$ to $c$. From the exact sequence

$$1 \to \mathcal{D}_3 \otimes F^\times \to \widetilde{J}(F) \to J(F) \to 0,$$

we may identify $\widetilde{c} - \widetilde{c}'$ with an element in $\mathcal{D}_3 \otimes F^\times$, which is Eisenstein. However, Lemma 5.8 indicates that $\widetilde{c}'$ cannot be killed by $\widetilde{T}_2 - 3$ when $s > 2$. (The action of $\widetilde{T}_2$ on the components $[1]$, $[1/p_1]$, $[p_1/N]$, $[1/N]$ is given by $x \mapsto -x^3$, while on other components like $[1/p_2]$ it is given by $x \mapsto x^3$.) Therefore, neither is $\widetilde{c}$.

**Remark 5.10.** Let $\mathbb{T}_{\widetilde{J}}$ (resp. $\mathcal{E}_{\widetilde{J}}$) be the image of $\mathbb{T}$ (resp. $\mathcal{E}$) in $\mathrm{End}(\widetilde{J})$. Then the Eisenstein property of $\mathcal{D}_3 \otimes \mathbb{G}_m \subset \widetilde{J}$ implies that

$$\mathbb{T}/\mathcal{E} \cong \mathbb{T}_{\widetilde{J}}/\mathcal{E}_{\widetilde{J}} \cong \mathbb{Z},$$

unless $N = 1$ (in which case we have $\widetilde{J} = J = 0$). Let $\mathbb{T}_J$ (resp. $\mathcal{E}_J$) be the image of $\mathbb{T}$ (resp. $\mathcal{E}$) in $\mathrm{End}(J)$. It is known that $\mathbb{T}_J/\mathcal{E}_J$ is always finite. This is a remarkable difference between $\mathbb{T}_{\widetilde{J}}$ and $\mathbb{T}_J$.

**Acknowledgment:** We thank the referee for careful reading which helped improving our manuscript.

**Funding:** The first author is supported by Ministry of Science and Technology, Taiwan (grant number 105-2115-M-007-018-MY2 and 107-2628-M-007-004-MY4). The second author is supported by Japan Society for the Promotion of Science KAKENHI Grant (grant number 15K04773).

## References

[1] S. Bae, On the modular equation for Drinfel'd modules of rank 2, *J. Number Theory* **42** (1992), no. 2, 123–133.
[2] E.-U. Gekeler, A product expansion for the discriminant function of Drinfel'd modules of rank two, *J. Number Theory* **21** (1985), no. 2, 135–140.
[3] E.-U. Gekeler, On the Drinfeld discriminant function, *Compos. Math.* **106** (1997), no. 2, 181–202.
[4] N. M. Katz, Galois properties of torsion points on abelian varieties, *Invent. Math.* **62** (1981), no. 3, 481–502.
[5] S. Lang, *Elliptic Functions*, 2nd ed., Grad. Texts in Math. 112, Springer, New York, 1987.
[6] S. Ling, On the **Q**-rational cuspidal subgroup and the component group of $J_0(p^r)$, *Israel J. Math.* **99** (1997), 29–54.
[7] D. J. Lorenzini, Torsion points on the modular Jacobian $J_0(N)$, *Compos. Math.* **96** (1995), no. 2, 149–172.
[8] J. I. Manin, Parabolic points and zeta functions of modular curves, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 19–66.
[9] B. Mazur, Modular curves and the Eisenstein ideal, *Inst. Hautes Études Sci. Publ. Math.* (1977), no. 47, 33–186.
[10] A. P. Ogg, Diophantine equations and modular forms, *Bull. Amer. Math. Soc.* **81** (1975), 14–27.
[11] M. Ohta, Eisenstein ideals and the rational torsion subgroups of modular Jacobian varieties II, *Tokyo J. Math.* **37** (2014), no. 2, 273–318.
[12] A. Pál, On the torsion of the Mordell-Weil group of the Jacobian of Drinfeld modular curves, *Doc. Math.* **10** (2005), 131–198.
[13] M. Papikian and F.-T. Wei, The Eisenstein ideal and Jacquet–Langlands isogeny over function fields, *Doc. Math.* **20** (2015), 551–629.
[14] M. Papikian and F.-T. Wei, The rational torsion subgroups of Drinfeld modular Jacobians and Eisenstein pseudo-harmonic cochains, *Math. Z.* **287** (2017), no. 1–2, 521–546.
[15] J.-P. Serre, *Algebraic Groups and Class Fields*, Grad. Texts in Math. 117, Springer, New York, 1988.
[16] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Grad. Texts in Math. 106, Springer, New York, 1986.
[17] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. 151, Springer, New York, 1994.
[18] T. Yamazaki and Y. Yang, Rational torsion on the generalized Jacobian of a modular curve with cuspidal modulus, *Doc. Math.* **21** (2016), 1669–1690.