GALERKIN FINITE ELEMENT METHOD AND FINITE DIFFERENCE METHOD FOR SOLVING CONVECTIVE NON-LINEAR EQUATION

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ABSTRACT

The fast progress has been observed in the development of numerical and analytical techniques for solving convection-diffusion and fluid mechanics problems. Here, a numerical approach, based in Galerkin Finite Element Method with Finite Difference Method is presented for the solution of a class of non-linear transient convection-diffusion problems. Using the analytical solutions and the L_2 and L_∞ error norms, some applications is carried and valuated with the literature.

Keywords: Numerical simulation, Burgers' equation, Galerkin Finite Element Method, Finite Difference Method, Cranck-Nicolson Method.

NOMENCLATURE

x e y	space coordinates
t	time coordinate
N_j	interpolation function
N_{nodes}	number of nodes in each finite element
u	velocity
$\hat{u}_j^e$	velocity approximation in the finite element

Greek symbols

Ω	one-dimensional domain
Ω^e	one-dimensional domain in the element
α	coefficient of Crank-Nicholson Method
ξ	local coordinate

Subscripts

node	identification of a node

1. INTRODUCTION

The majority of problems in fluid mechanics and heat transfer are represented by nonlinear partial differential equations. A classical nonlinear first-order hyperbolic equation inicially proposed by Bateman in 1915, later on was treated by Burgers in 1948, after whom this equation is widely referred to as *Burgers' equation*, a important role in the study of nonlinear waves since it can be used as mathematical model of free turbulence (Kevorkian, 1990). Besides, is also a model example for important applications in physical phenomena such as acoustics, continuous stochastic processes, dispersive water waves, gas dynamics, heat conduction, longitudinalelastic waves in an isotropic solid, number theory, shock waves, turbulence and so forth (Polyanin and Zaitsev, 2004). In recent years, many researchers have proposed various kinds of numerical methods for Burgers' equation (Dogan, 2004; Dag *et al.*, 2005; Zhang *et*

al., 2009; Xu *et al*., 2011), mainly concerned with the development of computational algorithms where, in a continuous drive to demonstrate the superior accuracy and stability properties of their latest numerical method, for numerical solution.

It is one of a few well-known nonlinear partial differential equations, which can be solved analytically on the restricted initial conditions (Guo-Zhong *et al.,* 2010).

The Burgers' equation is give by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

In 1950 and 1951, E. Hopf and J. D. Cole independently showed that Eq. (1) can be transformed in the linear diffusion equation (Kevorkian, 1990). Thus, we can rewrite the Eq.(1) as follows

$$\frac{\partial u}{\partial t}=-\frac{\partial E}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} \tag{2}$$

where $E=\frac{1}{2}u^2$ (Hoffmann and Chiang, 2000). This technique to linearize the equation has been widely used in finite difference method, here will be used with the finite element method.

2. DISCRETIZATION OF GOVERNING EQUATION

For the temporal discretization, considering the parabolic equation given by Eq. (2), we use the method called *α family of approximation* (Reddy, 1993), in which a weighted average of the time derivative of a dependent variable is approximated on two consecutive time steps by linear interpolation of the values of the variable at two steps:

$$\frac{\partial u}{\partial t}\approx\frac{\{\widetilde{u}\}^{n+1}-\{\widetilde{u}\}^{n}}{\Delta t^{n+1}}=(1-\alpha)\left\{\frac{\partial u}{\partial t}\right\}^{n}+\alpha\left\{\frac{\partial u}{\partial t}\right\}^{n+1} \tag{3}$$

where $0\le\alpha\le 1$, $t\in[t^n,t^{n+1}]$, $n=0,1,2,...,mt$, where *mt* is the number of steps in time, and $\{\ \}^n$ refers to the value of the enclosed quantity at time *n* and $\Delta t^{n+1}=t^{n+1}-t^n$ is the (n+1) i^{th} time step. For different values of α, we obtain well-known numerical integration schemes, described in (Reddy, 1993).

On this manner, in the temporal discretization give by Eq. (1), we can approximate the function *u* by $\widetilde{u}$, and obtain

$$\frac{\partial u}{\partial t}\approx\frac{\{\widetilde{u}\}^{n+1}-\{\widetilde{u}\}^{n}}{\Delta t^{n+1}}=$$
$$(1-\alpha)\left\{\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}\right\}^{n}+\alpha\left\{\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}\right\}^{n+1}-\frac{\partial E}{\partial x}\Rightarrow$$
$$\left\{\frac{\widetilde{u}}{\Delta t}-\frac{\alpha}{\mathrm{Re}}\frac{\partial^2\widetilde{u}}{\partial x^2}\right\}^{n+1}+f=0 \tag{4}$$

where

$$f=\frac{\partial E}{\partial x}+\left\{-\frac{\widetilde{u}}{\Delta t}-\frac{(1-\alpha)}{\mathrm{Re}}\frac{\partial^2\widetilde{u}}{\partial x^2}\right\}^{n} \tag{5}$$

The first term on the right side of Eq. (5), was discretized using the *Forward Difference Formula with second order accuracy* for $i=1,2,...,Nnost-2$, $\left.\frac{\partial E}{\partial x}\right|_i\approx\frac{-3E_i+4E_{i+1}-E_{i+2}}{2\Delta x}$ (Chung, 2002) and the *Backward Difference Formula with second order accuracy* for $i=Nnost-1$ and $i=Nnost$, $\left.\frac{\partial E}{\partial x}\right|_i\approx\frac{3E_i-4E_{i-1}+E_{i-2}}{2\Delta x}$ (Chung, 2002), where *Nnost* is total number of nodes in the mesh and Δx the distance between the nodes, which, in this work, was adopted as the same for all neighboring mesh points.

Now, for the second term on the right side of Eq. (5), was used the Finite Element approximation, as follows

$$\left\{-\frac{\widetilde{u}}{\Delta t}-\frac{(1-\alpha)}{\mathrm{Re}}\frac{\partial^2\widetilde{u}}{\partial x^2}\right\}^{n}\Bigg|_i=$$
$$\left\{-\frac{\widetilde{u}}{\Delta t}-\frac{(1-\alpha)}{\mathrm{Re}}\sum_{j=1}^{Nnodes}\frac{\partial^2 N_j}{\partial x^2}\widetilde{u}_j\right\}^{n}\Bigg|_i$$

for $i=1,2,...,Nnost$ and *Nnodes* is the number of nodes in element and *N* are the Lagrange interpolation functions (Dhatt and Touzot, 1984).

For the spatial discretization, using the Galerkin Method, we have

$$\int_\Omega\left\{\frac{\widetilde{u}}{\Delta t}-\frac{\alpha}{\mathrm{Re}}\frac{\partial^2\widetilde{u}}{\partial x^2}\right\}^{n+1}N_i d\Omega=-\int_\Omega f.N_i d\Omega \tag{6}$$

Using integration by parts to lower the order of the highest derivatives contained in the integrand (Dhatt and Touzot, 1984), we obtain

$$\int_\Omega\left\{\frac{\widetilde{u}}{\Delta t}N_i+\frac{\alpha}{\mathrm{Re}}\frac{\partial N_i}{\partial x}\frac{\partial\widetilde{u}}{\partial x}\right\}^{n+1}d\Omega=$$
$$-\int_\Omega f.N_i d\Omega+\frac{1}{\mathrm{Re}}N_i\left.\frac{\partial\widetilde{u}}{\partial x}\right|_\Gamma \tag{7}$$

Approximating the function $\widetilde{u}$ by $\hat{u}^e$, in element, we have $\widetilde{u} \approx \hat{u}^e = \sum_{i=1}^{Nnodes} N_i \hat{u}_i^e$. Then, incorporating this in Eq. (7), we have

$$\left\{\int_{\Omega^e}\left\{\frac{N_i N_j}{\Delta t}+\frac{\alpha}{\mathrm{Re}}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right\}d\Omega^e\right\}\hat{u}_j^{n+1,e} = -\int_{\Omega^e} f.N_i d\Omega^e + \frac{1}{\mathrm{Re}} N_i \left.\frac{\partial \widetilde{u}}{\partial x}\right|_{\Gamma^e} \quad (8)$$

with i,j=1,2,…,*Nnodes*, Ω^e and Γ^e the domain and boundary element, respectively.

Since the boundary conditions proposed for this problem are invariants in time, then

$$-\frac{1}{\mathrm{Re}} N_i \left.\frac{\partial \widetilde{u}^n}{\partial x}\right|_{\Gamma} = -\frac{1}{\mathrm{Re}} N_i \left.\frac{\partial \widetilde{u}^{n+1}}{\partial x}\right|_{\Gamma} \quad (9)$$

Notice that the term $\frac{1}{\mathrm{Re}} N_i \left.\frac{\partial \widetilde{u}}{\partial x}\right|_{\Gamma}$ is related with the boundary conditions of this problem and has the following properties:

$$\frac{1}{\mathrm{Re}} N_i \left.\frac{\partial \widetilde{u}}{\partial x}\right|_{\Gamma} = \begin{cases} 0, \text{ out of } \Gamma \\ c, \text{ in } \Gamma, c \in \Re \end{cases} \quad (10)$$

where

$$c = \begin{cases} 0, \text{ if the boundary condition is prescribed function.} \\ q_x, \text{ for the prescribed boundary flow.} \end{cases}$$

From the Eq. (8), we obtain the following matricial system:

$$[K]\left\{\hat{u}_j^{n+1,e}\right\} = \{F\} \quad (11)$$

in which

$$K_{ij} = \int_{\Omega^e}\left\{\frac{N_i N_j}{\Delta t}-\frac{\alpha}{\mathrm{Re}}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right\}d\Omega^e \quad (12\text{-a})$$

$$F_i = -\int_{\Omega^e} f.N_i d\Omega^e \quad (12\text{-b})$$

To transform the global for local coordinates, we have that

$$d\Omega = \frac{d\Omega}{d\xi} d\xi = \det(J) d\xi, \quad (13)$$

being $d\Omega$ and $d\xi$ the differential elements in the global and local coordinates systems, respectively, and J is the Jacobian matrix of the transformation (Reddy, 1993). Hence,

$$\det(J) = \frac{d\Omega}{d\xi} = \frac{d}{d\xi}\left(\sum_{i=1}^{Nnodes} \Omega_i^e N_i^e\right) = \sum_{i=1}^{Nnodes} \Omega_i^e \frac{dN_i^e}{d\xi} \quad (14)$$

where Ω_i^e is the global coordinate of the i^{th} node of the element Ω^e and N_i^e are the Lagrange interpolation functions of degree N_{nodes}-1, being N_{nodes} is the nodes number in each element. Also, defining the differential transformation of interpolation function in relation to the global for the local coordinate, as follow,

$$\frac{dN_i^e}{d\Omega} = \frac{dN_i^e}{d\xi}\frac{d\xi}{d\Omega} = \frac{1}{\det(J)}\frac{dN_i^e}{d\xi} \quad (15)$$

Using the Eq.(13) and (15), the Eq. (12) can be rewritten in the form,

$$K_{ij} = \int_{\overline{\Omega}^e}\left\{\frac{\det(J)}{\Delta t} N_i N_j - \frac{\alpha}{\mathrm{Re}.\det(J)}\frac{dN_i}{d\xi}\frac{dN_j}{d\xi}\right\}d\overline{\Omega}^e \quad (16\text{-a})$$

$$F_i = -\int_{\overline{\Omega}^e} f.N_i \det(J) d\overline{\Omega}^e \quad (16\text{-b})$$

3. NUMERICAL EXAMPLES

In this section, are presented two numerical applications of Eq. (1). At first, from the analytical solution, is made a point-by-point analysis, i.e., a comparison, for some grid points, between the analytical with numerical solution. Also, an error analysis is made using two norms defined later. In the second application, which, *a priori*, has no analytical solution, is made an analysis through graphical results, showing the velocity profiles for a few moments of time.

3.1 Application 1

Considering the Burgers' equation defined by Eq. (1) with Re=100, the particular solution of Burgers' equation is given by (Ali *et al.*, 1992; Dogan, 2004; Dag *et al.*, 2005; Xu *et al.*, 2011):

$$u(x,t) = \frac{\gamma+\mu+(\mu-\gamma)e^{\eta}}{1+e^{\eta}}, \; 0 \le x \le 1, \; t \ge 0 \quad (17)$$

where $\eta=\gamma.R.(x-\mu t-\beta)$. It was chosen that the arbitrary constants γ, μ and β have 0.6, 0.4 and 0.125 as their respective values.

The Crank-Nicolson scheme was adopted with α=0.5.

The initial condition is found from Eq.(17) when t=0. The boundary conditions are $u(0,t)$=1, $u(1,t)$=0.2, $t \ge 0$ (Ali *et al.*, 1992).

To measure the accuracy of the methodology, we compute the error using the L_2 norm, which is the average error throughout the domain, defined by $\|e\|_2 = \left[\left(\sum_{i=1}^{Nnost} e_i^2\right)/Nnost\right]^{1/2}$ (Zlhmal, 1978), where $e_i = |T_{(num)_i} - T_{(an)_i}|$, in which the term $T_{(num)}$ and $T_{(an)}$ is the result from the numerical solution and the result from the analytical solution, respectively, and the L_∞ norm, defined by $\|e\|_\infty = |T_{(num)} - T_{(an)}|$, which is the maximum error in the entire domain.

The numerical results for this example are presented in Table 1 and are compared with the analytical solution.

Table 1. Comparison of Application 1 at different time and mesh with 500 elements.

	***Nnos* (Δ*t* = 0,01)**		***Nnos* (Δ*t* = 0,005)**		***Nnos* (Δ*t* = 0,001)**		***Nnos* (Δ*t* = 0,0005)**		***Analytical solution***
x	***2***	***3***	***2***	***3***	***2***	***3***	***2***	***3***	
0.02	0.999923	0.999934	0.999999	0.999843	0.999992	0.999758	0.999995	0.999738	0.998913
0.05	0.100066	0.999136	0.999968	0.998472	0.999746	0.997910	0.999810	0.997787	0.996403
0.10	0.996726	0.984126	0.993343	0.979667	0.991163	0.976042	0.991945	0.975262	0.974164
0.15	0.929348	0.857383	0.911468	0.849411	0.904881	0.842986	0.912147	0.841394	0.841747
0.20	0.409301	0.465109	0.423914	0.475435	0.431388	0.483200	0.423133	0.484627	0.483475
0.25	0.227536	0.249012	0.227737	0.251834	0.225856	0.254814	0.223171	0.255741	0.255311
0.30	0.203609	0.207140	0.203570	0.207465	0.203277	0.207857	0.202942	0.208019	0.207961
0.35	0.200485	0.200983	0.200478	0.201022	0.200439	0.201071	0.200394	0.201094	0.201087
0.40	0.200065	0.200133	0.200064	0.200138	0.200059	0.200145	0.200053	0.200148	0.200147
0.50	0.200001	0.200002	0.200001	0.200002	0.200001	0.200002	0.200000	0.200002	0.200003
$\|e\|_2$	8.87E-02	1.86E-02	6.99E-02	8.53E-03	6.51E-02	1.94E-03	7.42E-02	1.55E-03	
$\|e\|_\infty$	2.30E-02	5.53E-03	1.93E-02	2.72E-03	1.85E-02	6.27E-04	2.09E-02	4.79E-04	

It is noted from Table 1, that the Galerkin method, when used with two nodes by element, i.e., using linear interpolation functions, does not suffer significant improvement in results, even with the refinement of Δt. The same is not true we used them to three nodes by element, when the refinement in the spacing of time results in improved outcomes, especially regarding the L_2 norm of the error, which represents the average error committed in all parts of the mesh.

3.2 Application 2

The Eq.(1) is now solved considering Re=1, i.e.,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

and the following initial condition $u(x,0) = 1$ and the boundary conditions $u(0,t) = 1$ and $u(1,t) = 0{,}1$ for t > 0. As before, was adopted α=0.5 in the Crank-Nicolson scheme. Here, 500 nodes are distributed uniformly in the domain, Δt = 0.01 and was used 3 nodes for each element.

The Fig. 1 illustrates the numerical results at different time. In order to demonstrate the stability of the method, the profiles of *u* at times 0.01, 0.05, 0.1, 0.2, 0.4 and 0.6 are shown, which exhibit the correct physical behavior of the problem.

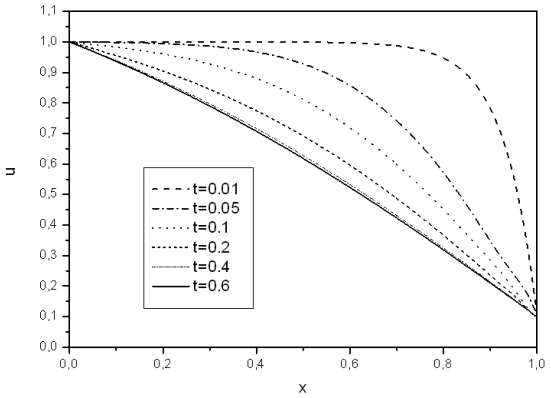

Figure 1. The profiles of *u* at different times for Re=1.

In Fig. (1), the curves showed no numerical oscillations and, for t = 0.6, the velocity profile is almost linear.

4. CONCLUSIONS

In this paper, we develop a numerical algorithm for solving Burgers' equation based on the Crank-Nicolson scheme for the temporal discretization and the Finite Element Method via Standard Galerkin's for the spatial discretization and the Difference Formula with second order accuracy for discretization of the nonlinear term of the Burgers' equation. The numerical examples show that our scheme could produce high accurate solution, demonstrating that this method can be extended to 2D and 3D domains.

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