APPLICATION OF GALERKIN FINITE ELEMENT METHOD IN THE SOLUTION OF 3D DIFFUSION IN SOLIDS

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ABSTRACT

This paper presents the numerical solution by the Galerkin Finite Element Method, on the three-dimensional Laplace and Helmholtz equations, which represent the heat diffusion in solids. For the two applications proposed, the analytical solutions found in the literature review were used in comparison with the numerical solution. The results analysis was made based on the the L_2 Norm (average error throughout the domain) and L_∞ Norm (maximum error in the entire domain). The two application results, one of the Laplace equation and the Helmholtz equation, are presented and discussed in order to to test the efficiency of the method.

Keywords: Finite Element Method, Galerkin Method, Diffusion, Solid, Poisson Equation, Helmholtz Equation.

NOMENCLATURE

f	source term
k	thermal conductivity
N_j	interpolation function
N_{nodes}	number of nodes in each finite element
T	temperature
$\hat{T}_j^e$	temperature approximation in the finite element
v	weight function
v_i^e	weight function in the element

Greek symbols

Ω	three-dimensional domain
Ω^e	three-dimensional domain in the element
Γ	contour of a domain
Γ^e	contour of an element

Subscripts

node	identification of a node

1. INTRODUCTION

The first publications in finite element method appeared in 1950´s with the works written by Turner et al. (1956), Clough (1960) e Argyris (1963). These were used to solve problems in structural analysis. Some decades later, Zienkiewicz and Cheung (1965), Oden and Wellford (1972), Chung (1978) e Baker (1983), among other publications, treated the heat transfer and fluid flow problems solutions. The classical finite element method is known as Bubnov-Galerkin Finite Element Method (GFEM). There are other variants of the finite element such as those by Petrov-Galerkin Finite Element Method and the Least-Squares Finite Element Method (LSFEM), both developed in order to compensate the limitations of the GFEM when applied for heat transfer and fluid flow problems. The GFEM applied for this kind of problems generally produces oscillating solutions for high Péclet and Reynolds numbers.

Recently, several authors have presented applications of the finite element method for two and tridimensional problems, among them Camprub et al. (2000), Romão et al. (2008a), Romão et al. (2008b) e Hannukainen et al. (2010).

In this work it is presented an application of Galerkin Finite Element Method to the numerical solution of Laplace tridimensional equations for diffusion in solids and the Helmholtz´s equation for diffusion with generation dependable of temperature in solids. In this study, analytical solutions validate the numerical results by the analysis of the L_2 Norm of the error that represents an average of the error in the solution and $L\infty$ Norm that represents the maximum error in the solution.

2. MODEL EQUATION

It´s presented the tridimensional diffusion equation with generation dependable of the temperature in closed limited solid domains designed as $\Omega \subset \Re^3$. The model equation has the following form:

$$k\frac{\partial^2 T}{\partial x^2}+k\frac{\partial^2 T}{\partial y^2}+k\frac{\partial^2 T}{\partial z^2}+B\cdot T+f=0 \qquad (1)$$

where k is a positive constant; $T=T(x,y,y)$, $B=B(x,y,z)$ and $f=f(x,y,z)$ are functions of the space $x,y,z\in\Re$. The boundary conditions are of the first e second kind.

2.1 Discretization – Galerkin Method

The GFEM is applied for discretization of the integral equations. In this method an approximation of unknown variable is a function $\hat{T}$ that when substituted in the Eq. (1) produces a null residual. So, the approximation form is:

$$T\approx\hat{T}^e=\sum_{j=1}^{N_{nodes}}N_j\hat{T}_j^e \qquad (2)$$

where N_{nodes} is number of nodes inside a finite element, N_j are the interpolation functions for the element and $\hat{T}_j^e$ are nodal values of *T* in the element. The residual is determined by substituting the approximation $\hat{T}$ in Eq. (1) and is defined as:

$$R=k\frac{\partial^2\hat{T}}{\partial x^2}+k\frac{\partial^2\hat{T}}{\partial y^2}+k\frac{\partial^2\hat{T}}{\partial z^2}+B\cdot\hat{T}+f \qquad (3)$$

The solution is found by forcing the pondered residual to be null. In other words, it must be found as a function of $\hat{T}^e\in V^e$, $V^e\in C^2(\Omega)$, such as:

$$\int_{\Omega^e}R\,v_i^e\,d\Omega=0\,,\ \forall v_i^e\in V^e\,,\ i=1,2,...,N_{nodes} \qquad (4)$$

where $\Omega\subset\Re^3$ is a limited and closed domain.

In the Galerkin Finite Element Method, the weight function is the same interpolation function, i.e., $v_i^e=N_i$, $i=1,2,...,N_{nodes}$. After integration of Eq. (4) the result is an algebraic system of equations written in matricial form as follow:

$$[K]\{\hat{T}\}=\{F\} \qquad (5)$$

in which the matrix coefficients are;

$$K_{ij}=-\int_{\Omega^e}k\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,d\Omega-\int_{\Omega^e}k\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\,d\Omega-\int_{\Omega^e}k\frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\,d\Omega+\int_{\Omega^e}BN_iN_j\,d\Omega \qquad (6a)$$

$$F_i=-\int_{\Omega^e}f\,N_i\,d\Omega \qquad (6b)$$

with $i,j=1,2,...,Nnodes$.

3. NUMERICAL APPLICATIONS

The matrix coefficients are obtained by numerical integration using Gauss Method (Reddy, 1993) and mapping the real elements in the master element in the local coordinates ξ, η e ζ ($-1\le\xi,\eta,\zeta\le 1$). The interpolation functions and their derivatives for the hexahedral element can be found towards Dhatt et al. (1984).

The system of algebraic equations represent by Eq. (5) was solved by the Gauss-Seidel method and criteria of stop with maximum error $E_{max}\le 10^{-10}$. The computational code was developed in FORTRAN language. The meshes were refined until the limit of the computer´s memory capacity. Both linear (eight nodes) and quadratic hexahedra (twenty seven nodes) were used, with h representing the size of the element (cubic element).

The L_2 norm of the error was defined like in (Zlhmal, 1978): $\|e\|=\left[\left(\sum_{i=1}^{Nnost}e_i^2\right)/Nnost\right]^{1/2}$.

In this equation, *Nnost* is the total number of nodes in the mesh and $e_i=|T_{(num)_i}-T_{(an)_i}|$, where $T_{(num)}$ is the result from the numerical solution and $T_{(an)}$ is the result form the analytical solution respectively.

Application 1. Poisson Equation – Diffusion in Solids

In this application the coefficient B in Eq. (3) is null and the domain is an unitary cube $\Omega=[0,1]^3$. The governing equation is reduced to;

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \qquad (7)$$

where $T=T(x,y,z)$.

The analytical solution of Eq. (7) is of the form:

$$T(x,y,z)=\frac{\text{sen}(\pi.y)\,\text{sen}(\pi.z)}{\text{senh}(\pi\sqrt{2})}\times \\ \times\left[2\,\text{senh}(\pi\sqrt{2}x)+\text{senh}(\pi\sqrt{2}(1-x)\right] \qquad (8)$$

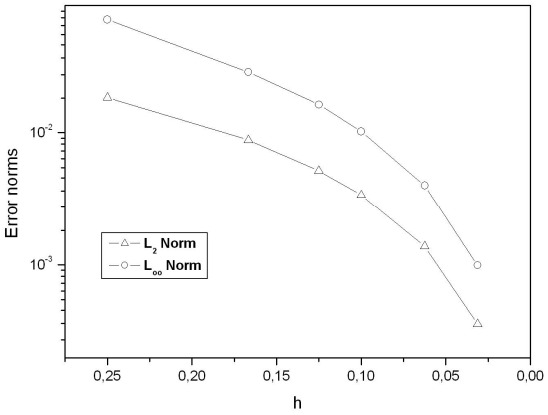

Figure 1. L_2 and L_∞ Norms for meshes with hexahedral of 8 nodes – application 1.

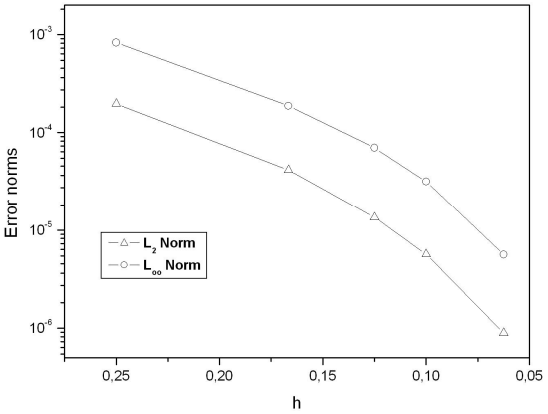

Figure 2. L_2 and L_∞ Norms for meshes with hexahedral of 27 nodes – application 1.

These boundary conditions were chosen to satisfy the analytical solution of the proposed problem. The results for the medium and maximum errors by using linear and quadratic elements are presented in Figures 1 and 2 respectively, where h represents the refinement of the mesh. It is observed that the errors are higher to gross meshes, as expected.

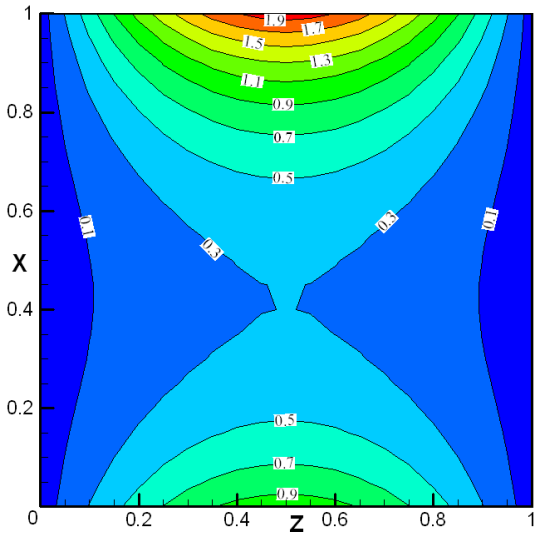

Figure 3. Temperature profile in plane xz with y = 0,5 by LSFEM from a mesh with 4913 nodes (h = 1/16) using hexaedrals with 8 nodes, Application 1.

In Figure 3 is presented the temperature profile in a transversal section of the domain.

Application 2. Helmholtz Equation – Diffusion with Generation in Solids

In this application the coefficient B in Eq. (3) is non null and the domain is an unitary cube $\Omega=[0,1]^3$. So the governing equation is of the form:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+T=0\,. \qquad (9)$$

where $T=T(x,y,z)$.

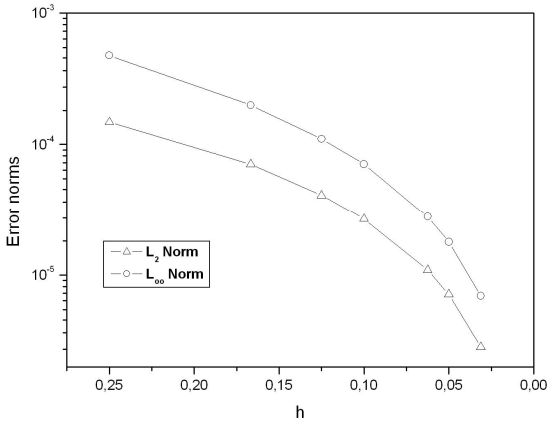

Figure 4. L_2 and L_∞ Norms for meshes with hexahedral of 8 nodes – application 2.

The Eq. (9) has an analytical solution, this solution is:

$$T(x,y,z)=\text{sen}\,x+\text{sen}\,y+\text{sen}\,z\,. \qquad (10)$$

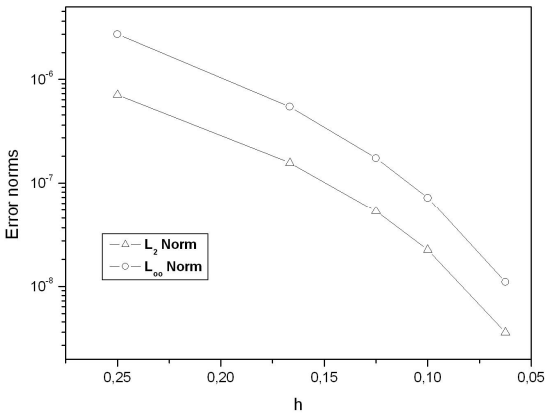

Figure 5. L_2 and L_∞ Norms for meshes with hexahedral of 27 nodes – application 2.

As seem in the Application 1, the boundary conditions were chosen to satisfy the analytical solution in the proposed problem. The results for the medium and maximum errors by using linear and quadratic elements are presented in Figures 4 and 5 respectively, where h represents the refinement of the mesh. Similar to those results of Application 1, it is observed that the errors are higher to gross meshes, as expected.

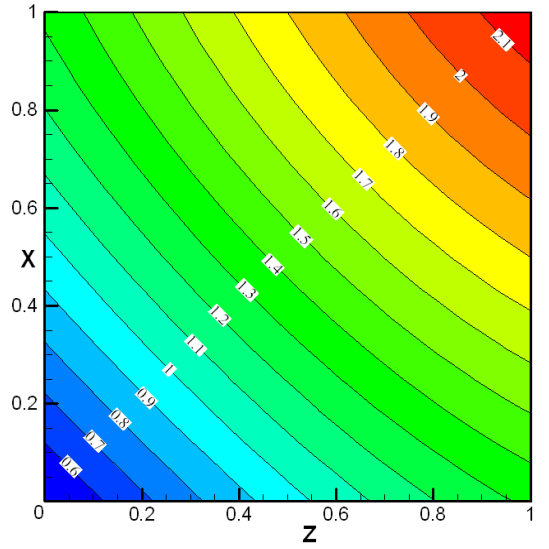

Figure 6. Temperature profile in plane xz with y = 0,5 by LSFEM from a mesh with 4913 nodes (h = 1/16) utilizing hexaedrals with 8 nodes, Application 2.

In Figure 3 is presented the temperature profile in a transversal section of the domain.

4. CONCLUSION

In the proposed applications, the Galerkin Finite Element Method shown good results, mainly when quadratic elements were used, even for the mesh with h = 1/4 the quadratic element better results were reached than the linear element. For pure diffusion the quadratic mesh with h = 1/4 presents better results than the more refined mesh of linear elements with h = 1/32. The same behavior was obtained in the Application 2, where neither the more refined mesh of linear elements presents better results than the h = 1/4 mesh of quadratic elements.

5. ACKNOWLEDGMENTS

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6. REFERENCES

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