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Laplacian Spectral Properties of Signed Circular Caterpillars

Maurizio Brunetti
University of Naples `Federico II' Italy, mbrunett@unina.it

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Abstract

A circular caterpillar of girth n is a graph such that the removal of all pendant vertices yields a cycle C_n of order n. A signed graph is a pair $\Gamma = (G, \sigma)$, where G is a simple graph and $\sigma : E(G) \to \{+1, -1\}$ is the sign function defined on the set $E(G)$ of edges of G. The signed graph Γ is said to be balanced if the number of negatively signed edges in each cycle is even, and it is said to be unbalanced otherwise. We determine some bounds for the first n Laplacian eigenvalues of any signed circular caterpillar. As an application, we prove that each signed spiked triangle $(G(3; p, q, r), \sigma)$, i. e. a signed circular caterpillar of girth 3 and degree sequence $\pi_{p,q,r} = (p+2, q+2, r+2, 1, \ldots, 1)$, is determined by its Laplacian spectrum up to switching isomorphism. Moreover, in the set of signed spiked triangles of order N, we identify the extremal graphs with respect to the Laplacian spectral radius and the first two Zagreb indices. It turns out that the unbalanced spiked triangle with degree sequence $\pi_{N-3,0,0}$ and the balanced spike triangle $(G(3; \hat{p}, \hat{q}, \hat{r}), +)$, where each pair in $\{\hat{p}, \hat{q}, \hat{r}\}$ differs at most by 1, respectively maximizes and minimizes the Laplacian spectral radius and both the Zagreb indices.

1 Introduction

A *signed graph* Γ is a pair (G, σ), where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is a sign function (or *signature*) on the edges of G. The (unsigned) graph G of $\Gamma = (G, \sigma)$ is called the *underlying graph*. Each cycle C in Γ has a *sign* given by $\sigma(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (resp. -1) is called *positive* (resp. *negative*). A signed graph (G, σ) (and its signature σ as well) is said to be *unbalanced* if it contains at least one negative cycle, and *balanced* otherwise. In particular, the null graph K_0 with one vertex and 0 edges is a balanced signed graph. The signed graph obtained from Γ by switching signs to all its edges is denoted by $-\Gamma$. If all edges in Γ are positive, we write $\Gamma = (G, +)$, and set $(G, -) = -(G, +)$.

The reader is referred to [14] for basic results on the graph spectra and to [21] for basic results on the spectra of signed graphs.

Many familiar notions related to unsigned graphs directly extend to signed graphs. For example, $\Gamma = (G, \sigma)$ is said to be k-*cyclic* if the underlying graph G is k-cyclic. This means that G is connected and its *cyclomatic number* $|E(G)| - |V(G)| + 1$ is equal to k. The words *unicyclic* and *bicyclic* stand as synonyms for 1-*cyclic* and 2-*cyclic* respectively. Moreover, if G is neither a tree or a forest, the *girth* (resp. *circumference*) of Γ is the length of the shortest (resp. longest) cycle contained in G.

The degree sequence $(d_1, \ldots, d_N)$ of Γ is simply the non-increasing sequence of vertex degrees in G. We recall that a vertex v is said to be *pendant* (resp. *internal*) if its vertex degree $d_G(v)$ equals 1 (resp. $d_G(v) > 1$). A *quasi-pendant vertex* is instead a vertex adjacent to a pendant vertex.

The signed adjacency matrix $A(\Gamma)$ is obtained from the usual adjacency matrix of the underlying graph G by replacing 1 with -1 whenever the corresponding edge is negative.

The Laplacian matrix of Γ is $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the diagonal matrix of vertex degrees. We denote the spectrum of $L(\Gamma)$ by $\mathrm{Spec}_L(\Gamma)$.

Given a signed graph $\Gamma = (G, \sigma)$, every subset $U \subset V(G)$ determines a *switching*, i.e. the operation that replaces Γ with the signed graph Γ^U obtained from Γ by changing the sign of all outgoing edges of U. Switchings give rise to an equivalence relation among all possible signatures on G. We write $\sigma \sim \sigma'$ to say that the two signatures σ and σ' on the same underlying graph are equivalent. Thus, $\sigma \sim +$ means that σ is balanced.

If a signed graph can be switched into an isomorphic copy of another signed graph, the two signed graphs are called switching isomorphic. It is worthy to observe that the signatures of two switching isomorphic graphs are not necessarily switching equivalent. In fact, for every $N \geqslant 5$, the graph $\Gamma_{13}(N)$ depicted in Fig. 3 and $-\Gamma_{13}(N)$ are switching isomorphic; nevertheless, for any subset of vertices U of their common underlying graph, $-\Gamma_{13}(N) \neq \Gamma_{13}(N)^U$.

When two signed graphs share the adjacency (resp. Laplacian) spectrum, we say that they are A- (resp. L-)cospectral. Since $(A(\Gamma), A(\Gamma^U))$ and $(L(\Gamma), L(\Gamma^U))$ are pairs of similar matrices, switching isomorphic graphs are both A- and L-cospectral.

Finally, we say that a signed graph Γ is DLS if it is determined by its Laplacian spectrum, i.e. if every signed graph L-cospectral to Γ is necessarily switching isomorphic to it.

In recent years, many scholars of spectral graph theory explored two interrelated topics. The first one consists in finding bounds for adjacency or Laplacian eigenvalues (see for instance [4, 8, 11, 19]) The second topic is known as the spectral determination problem: for any Γ belonging to a fixed class $\mathcal{G}$ of signed graphs, find all signed graphs which are A- or L-cospectral to Γ. Such problem has been recently investigated for $\mathcal{G}$ being the set of signed cycles [1], of signed paths [2], of signed lollipops [10], of signed ∞-graphs [7, 16], and of signed sun graphs [18].

Results in this paper concern both topics mentioned above. In fact, we consider the class $\mathcal{CC}_n$ of signed circular caterpillars of girth n. These are unicyclic signed graphs characterized by the following property: the removal of all pendant vertices yields a signed cycle with n vertices. Elements in $\mathcal{CC}_3$ are also called signed spiked triangles. We first determine in Section 3 the Laplacian spectrum of those objects in $\mathcal{CC}_n$ whose internal vertices all have the same degree. Such spectra give bounds for the first n Laplacian eigenvalues of the remaining graphs in $\mathcal{CC}_n$, and allow us to prove in Section 4 that each signed spiked triangle is DLS. Extremal graphs in $\mathcal{CC}_n$ with respect to the Laplacian spectral radius and the first two Zagreb indices are determined in Section 5.

Even for unsigned circular caterpillar, the spectral determination problem is still wide open, and only partial results have been obtained so far (see, for instance, [12, 15]). Since the Laplacian spectral properties of signed circular caterpillar detected by our Corollary 3.5 seem quite peculiar, and turned out to be decisive to solve the case $n = 3$, the author intends to study in a subsequent paper the structural conditions on a signed graph ensuring their occurrence, in order to further restrict the range of possible L-cospectral mates of items in $\mathcal{CC}_n$ $(n > 3)$.

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2 Preliminaries

We start by recalling a characterization for switching equivalent signatures.

Proposition 2.1. [20, Proposition 3.2] *Two signatures σ and σ' on G are switching equivalent if and only if (G,σ) and (G,σ') have the same list of balanced cycles.*

Let $\Gamma = (G,\sigma)$ be a signed graph of order N. The Laplacian eigenvalues, i.e. the roots of the *Laplacian polynomial* $\psi(\Gamma,x) = \det(xI - L(\Gamma))$, are all real since $L(\Gamma)$ is symmetric and are denoted by

$$\mu_1(\Gamma) \geqslant \mu_2(\Gamma) \geqslant \cdots \geqslant \mu_N(\Gamma) \geqslant 0.$$

The last inequality holds since the Laplacian matrix is positive semidefinite. The following two results are surely known to the experts. We provide a proof for both of them for sake of completeness.

Proposition 2.2. *Let $\Gamma = (G,\sigma)$ be any signed graph of order N, and let $\Gamma' = (H,\sigma|_H)$ be any subgraph of Γ. The following inequalities hold:*

$$\mu_i(\Gamma) \geqslant \mu_i(\Gamma') \qquad \forall\, i = 1,\dots,|V(H)|. \tag{1}$$

Proof. Let $\Gamma - E$ denote the signed graph obtained from Γ by deleting the edges in the set $E \subseteq E(G)$. From the ordinary vertex variant interlacing theorem for the adjacency matrix combined with [10, Theorem 2.3(ii)] we deduce that, for every $\{e\} \subseteq E(G)$, the L-eigenvalues of Γ and those of $\Gamma - \{e\}$ interlace as follows:

$$\mu_1(\Gamma) \geqslant \mu_1(\Gamma - \{e\}) \geqslant \mu_2(\Gamma) \geqslant \mu_2(\Gamma - \{e\}) \geqslant \cdots \geqslant \mu_N(\Gamma) \geqslant \mu_N(\Gamma - \{e\}). \tag{2}$$

Inequalities 2 are stated, for instance, in [10, Theorem 2.5]. Note now that Γ' shares the same non-zero eigenvalues of a suitable graph of type $\Gamma - E$. Hence, (1) comes from (2) used $|E(G)| - |E(H)|$ times. □

Corollary 2.3. *Let Δ be the largest vertex degree of a signed graph $\Gamma = (G,\sigma)$. Then,*

$$\mu_1(\Gamma) \geqslant \Delta + 1.$$

Proof. The star $K_{1,\Delta}$ is a subgraph of G. Since $K_{1,\Delta}$ is a tree, Proposition 2.1 implies that all signatures defined on it are switching equivalent. Hence, we get $\mu_1(K_{1,\Delta}, \sigma|_{K_{1,\Delta}}) = \mu_1(K_{1,\Delta}, +) = \Delta + 1$. The statement now follows from Proposition 2.2. □

In [9] Belardo and Simić contrived a geometric-combinatorial way to compute the several coefficients of $\psi(\Gamma,x)$. In order to describe such achievement, we need to recall that a *TU-subgraph* of any fixed signed graph Γ is a subgraph whose components are trees or unbalanced unicyclic graphs. In other words, a TU-subgraph H admits a vertex disjoint decomposition $\left(\bigcup_{i=1}^{t} T_i\right) \cup \left(\bigcup_{j=1}^{c} U_j\right)$, where, if any, the T_i's are trees and the U_j's are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as

$$\gamma(\mathrm{H}) = \begin{cases} 4^c & \text{if } t = 0, \\ 4^c \prod_{i=1}^{t} |V(T_i)| & \text{if } t > 0. \end{cases} \tag{3}$$

Theorem 2.4. [9, Theorem 3.9] *Let* $\psi(\Gamma,x) = x^N + b_1x^{N-1} + \cdots + b_{N-1}x + b_N$ *be the* L*-polynomial of a signed graph* Γ*. Then, the equality*

$$b_i = (-1)^i \sum_{\mathrm{H}\in\mathcal{H}_i} \gamma(\mathrm{H}),$$

where $\mathcal{H}_i$ *denotes the set of signed TU-subgraphs of* Γ *containing* i *edges, holds for all* $i = 1,2,\dots,N$.

Let $\{v_1,\dots,v_N\}$ the vertex set of a signed graph $\Gamma = (G,\sigma)$, and let t^+_Γ and t^-_Γ respectively denote the number of balanced triangles and unbalanced triangles contained in Γ. We set

$$f_1(\Gamma) = \sum_{i=1}^N d_G(v_i)^2, \quad \text{and} \quad f_2(\Gamma) = 6(t^-_\Gamma - t^+_\Gamma) + \sum_{i=1}^N d_G(v_i)^3.$$

Theorem 2.5 reveals to be very helpful to detect possible L-cospectral mates of Γ.

Theorem 2.5. [10, Theorem 3.5] *Let* $\Gamma = (G,\sigma)$ *be a signed graph of order* N*, and let* $\Lambda = (H,\sigma')$ *be a signed graph* L*-cospectral to* Γ*. Then,*

(i) Γ *and* Λ *have the same number of vertices and edges;*

(ii) Γ *and* Λ *have the same number of balanced components;*

(iii) Γ *and* Λ *have the same Laplacian spectral moments* $T_k = \sum_{i=1}^N \mu_i^k$*, for all non-negative integers* k*;*

(iv) Γ *and* Λ *have the same sum of squares of degrees, i.e.* $f_1(\Gamma) = f_1(\Lambda)$*;*

(v) $f_2(\Gamma) = f_2(\Lambda)$.

Given any signed cycle (C,σ), we set

$$\omega(C) = (-1)^{|V(C)|}\sigma(C). \tag{4}$$

In the statement of Proposition 2.6 below, $u \sim v$ means that two vertices u and v in a signed graph Γ are adjacent; uv denotes the edge connecting them; $\mathcal{C}_v$ is the set of cycles in G passing through a fixed vertex v; and $\mathcal{C}_{uv}$ is the set of cycles in G containing uv among their edges.

Proposition 2.6. *Let* $\Gamma = (G,\sigma)$ *be any signed graph. The following equations hold:*

$$\psi(\Gamma,x) = (x - d_G(v))\,\psi(\Gamma\smallsetminus\{v\},x) - \sum_{u\sim v}\psi(\Gamma\smallsetminus\{u,v\},x) - 2\sum_{C\in\mathcal{C}_v}\omega(C)\,\psi(\Gamma\smallsetminus V(C),x); \tag{5}$$

$$\psi(\Gamma,x) = \psi(\Gamma - uv,x) - \psi(\Gamma - u - v,x) - 2\sum_{C\in\mathcal{C}_{uv}}\omega(C)\,\psi(\Gamma\smallsetminus V(C),x). \tag{6}$$

Proof. As explained at the end of Section 2 in [10], the Laplacian matrix of a signed graph $\Gamma = (G, \sigma)$ can be regarded as the adjacency matrix of a weighted multigraph Γ^* sharing with Γ the vertex set $V(G)$. The positive (resp. negative) edges of Γ correspond to the (-1)-weighted (resp. $(+1)$-weighted) edges of Γ^*. Moreover, if a vertex v of Γ has degree k, the graph Γ^* has a k-weighted loop at v. Formulæ 5 and 6 now come from [10, Theorem 2.9] applied to Γ^*. $\square$

We end this section of preliminaries by recalling a very well-known result on the determinant of a 2×2 block matrix.

Proposition 2.7. *Let us consider a block matrix M of size $(n+m) \times (n+m)$ of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D have size $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively. If A is invertible, then

$$\det(M) = \det(A) \det(D - CA^{-1}B).$$

3 Circular caterpillars

As recalled in Section 1, a graph G is said to be a circular caterpillar if its internal vertices induce a cycle. We say that a circular caterpillar is *complete* if each internal vertex is quasi-pendant. It is immediately seen that a circular caterpillar is complete if and only if no vertices have degree 2. Let $\bar{\sigma}$ be any fixed unbalanced signature on a circular caterpillar G. As a consequence of Proposition 2.1, every circular caterpillar, being unicyclic (and, hence, containing just one cycle), admits only two different non-equivalent signatures. In other words, all unbalanced signatures are equivalent to $\bar{\sigma}$. It not restrictive to assume that $(G, \bar{\sigma})$ has just one negative edge. Such edge necessarily connects two internal vertices.

Our first result concern the *bidegreed* circular caterpillar $U(\Delta, n)$. This is the only circular caterpillar of girth n whose internal vertices have all degree Δ. The graph $U(\Delta, n)$ has $n(\Delta - 1)$ vertices. We are borrowing the notation from [3, 5, 6], where extremality of the adjacency spectral radius of $U(\Delta, n)$ with respect to a suitable class of (unsigned) graphs is discussed. Graphs in the set $\{U(3, n) \mid \forall\, n \geqslant 3\}$ are also known as *sun graphs.*

To lighten notation we shall denote by C_n^σ the subgraph of $(U(\Delta, n), \sigma)$ induced by its internal vertices.

Theorem 3.1. *Let $\Delta \geqslant 3$. The Laplacian spectrum of $(U(\Delta, n), \sigma)$ contains 1 with multiplicity $n(\Delta - 3)$. The remaining $2n$ L-eigenvalues are given by*

$$\phi_\pm(\lambda, \Delta) = \frac{(\lambda + \Delta - 1) \pm \sqrt{(\lambda + \Delta - 1)^2 - 4\lambda}}{2}, \qquad \text{where } \lambda \in \mathrm{Spec}_L(C_n^\sigma). \tag{7}$$

Proof. Up to possibly replacing σ with a switching equivalent signature, we can assume that all edges not belonging to the cycle C_n are positive. Let $k = \Delta - 2$. We label the vertices of $(U(\Delta, n), \sigma)$ assigning the highest labels to the internal vertices. Thus, vertices $v_{nk+1}, \dots, v_{nk+n}$ belong to the subgraph C_n. Moreover, we suppose that, for each $h \in \{1, ..., n\}$, the k vertices $v_{(h-1)k+1}, \dots, v_{hk}$ of degree 1 are adjacent to v_{nk+h} (see Fig. 1).

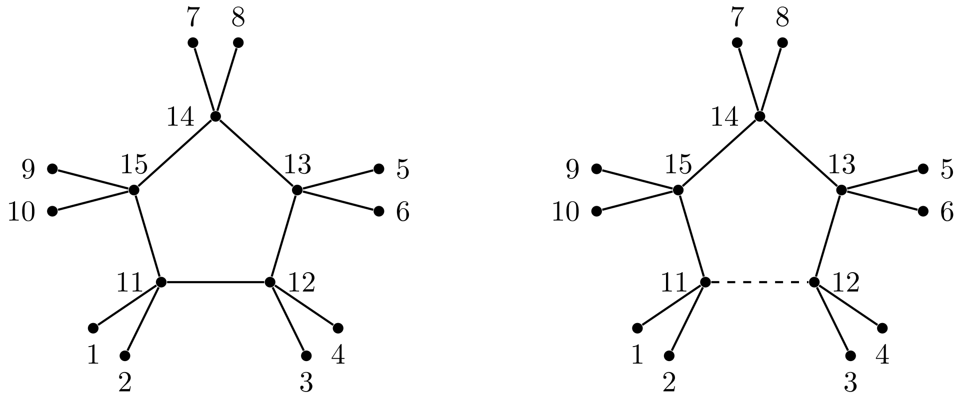


Fig. 1: Two non-equivalent signatures on $U(4,5)$. All edges, except the dashed one, are positively signed.

Let $\mathbf{1}_k$ and I_c be the all 1's column vector of size k and the $(c \times c)$-identity matrix respectively. The Laplacian matrix for $U(\Delta, n)$ takes the following form:

$$L(U(\Delta, n), \sigma) = \begin{bmatrix} I_{nk} & P_{nk \times n} \\ (P_{nk \times n})^T & \Delta I_n - A(C_n^\sigma) \end{bmatrix},$$

where

$$P_{nk \times n} = \begin{bmatrix} \mathbf{1}_k & 0 & \dots & 0 \\ 0 & \mathbf{1}_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{1}_k \end{bmatrix}.$$

Since $(P_{nk \times n})^T P_{nk \times n} = kI_n$, Proposition 2.7 applied to

$$\det\left(xI_{(k+1)n} - L(U(\Delta, n), \sigma)\right) = \det \begin{bmatrix} (x-1)I_{nk} & -P_{nk \times n} \\ -(P_{nk \times n})^T & (x-\Delta)I_n + A(C_n^\sigma) \end{bmatrix}$$

gives

$$\begin{aligned} \det\left(xI_{(k+1)n} - L(U(\Delta, n), \sigma)\right) &= (x-1)^{kn} \det\left(\left(x - (k+2) - \frac{k}{x-1}\right) I_n + A(C_n^\sigma)\right). \\ &= (x-1)^{kn} \det\left(\left(x - k - \frac{k}{x-1}\right) I_n - L(C_n^\sigma)\right). \end{aligned}$$

This is the reason why every root of

$$x - k - \frac{k}{x-1} = \lambda \qquad \text{with} \qquad \lambda \in \mathrm{Spec}_L(C_n^\sigma)$$

is an L-eigenvalue of $(U(\Delta,n),\sigma)$. So far, we have proved that the L-spectrum of $(U(\Delta,n),\sigma)$ contains the roots of the polynomials

$$\Phi_\lambda(x) = x^2 - (\lambda+\Delta-1)x + \lambda, \qquad \text{with} \qquad \lambda \in \mathrm{Spec}_L(C_n^\sigma). \tag{8}$$

It is now elementary to check that such roots are given by (7). Since $\Phi_\lambda(1) = 2-\Delta \neq 0$, the multiplicity of 1 in $\mathrm{Spec}_L(U(\Delta,n),\sigma)$ is precisely $n(\Delta-1)-2n$ as claimed. □

As an immediate consequence of Theorem 3.1, the absence of 1 in the Laplacian spectrum characterizes the sun graphs among all bidegreed circular caterpillars. The following two results help to locate on the real line the several elements of $\mathrm{Spec}_L(U(\Delta,n),\sigma)$.

Proposition 3.2. *Let C_n^σ be a signed cycle.*

$$\mathrm{Spec}_L(C_n^\sigma) = \begin{cases} \left\{2 + 2\cos\dfrac{2k}{n}\pi,\ k=0,1,\dots,n-1\right\} & \text{if } \omega(C_n^\sigma)=1, \\ \left\{2+2\cos\dfrac{2k+1}{n}\pi,\ k=0,1,\dots,n-1\right\} & \text{if } \omega(C_n^\sigma)=-1. \end{cases} \tag{9}$$

Proof. Equation (9) summarizes some of the results stated in [10, Lemma 4.4]. □

Theorem 3.3. *Let $(U(\Delta,n),\sigma)$ be any signed bidegreed circular caterpillar with $\Delta \geqslant 3$, C_n^σ be the only signed cycle contained in it, and $\phi_+(\lambda,\Delta)$ be the polynomial defined in (7).*

i) The Laplacian spectral radius is given by:

$$\mu_1(U(\Delta,n),\sigma) = \begin{cases} \phi_+(4,\Delta) = \dfrac{\Delta+3+\sqrt{\Delta^2+6\Delta-7}}{2} & \text{if } \omega(C_n^\sigma)=1, \\ \phi_+(2+2\cos\dfrac{\pi}{n},\Delta) & \text{if } \omega(C_n^\sigma)=-1; \end{cases} \tag{10}$$

In the latter case, the Laplacian spectral radius has multiplicity 2.

ii) The $2n$ L-eigenvalues not equal to 1 are distributed as follows: half of them are in the interval $[0,1)$; the remaining ones are in the interval $[\Delta-1,\Delta+3)$. Moreover, 0 and $\Delta-1$ belong to $\mathrm{Spec}(U(\Delta,n),\sigma)$ if and only if $\sigma\sim+$.

Proof. For each fixed $\Delta\geqslant 3$, the number $\phi_+(\lambda,\Delta)$ defined in (7), thought as a function in λ, strictly increases in the interval $[0,+\infty)$. That is why

$$\mu_1(U(\Delta,n),\sigma) = \phi_+(\bar\lambda,\Delta), \qquad \text{where } \bar\lambda = \max \mathrm{Spec}_L(C_n^\sigma).$$

Part i) of the statement now comes from Proposition 3.2. In fact, in the case $\omega(C_n^\sigma)=-1$ the maximum in $\mathrm{Spec}_L(C_n^\sigma)$ is given by

$$2+2\cos\frac{2\cdot 0+1}{n}\pi = 2+2\cos\frac{2(n-1)+1}{n}\pi$$

which has multiplicity 2.

We now prove Part ii). The Descartes' rule of signs confirms that the two roots $\phi_-(\lambda,\Delta)$ and $\phi_+(\lambda,\Delta)$ of the polynomial $\Phi_\lambda(x)$ in (8) are non-negative for each $\lambda \in \mathrm{Spec}_L(C_n^\sigma)$. From

$$\Phi_\lambda(0) \geqslant 0, \qquad \Phi_\lambda(1) = 2-\Delta < 0, \qquad \text{and} \qquad \Phi_\lambda(\Delta-1) = -(\Delta-2)\lambda \leqslant 0$$

we deduce that $\phi_-(\lambda,\Delta) < 1$ and $\phi_+(\lambda,\Delta) \geqslant \Delta - 1$, where the equality holds if and only if $\lambda = 0$. Such value belongs to $\mathrm{Spec}_L(C_n^\sigma)$ if and only if the circular caterpillar is balanced. The proof ends once you note that, by Equation (10),

$$\mu_1(U(\Delta,n),\sigma) \leqslant \phi_+(4,\Delta) = \frac{\Delta+3+\sqrt{\Delta^2+6\Delta-7}}{2} < \Delta+3. \qquad \square$$

Let Γ be any signed circular caterpillar in $\mathcal{CC}_n$ not equal to a signed cycle. We denote by Δ_Γ its maximum vertex degree, and by δ_Γ its least vertex degree bigger than 2. Next theorem gives bounds for the first n Laplacian eigenvalues of Γ.

Theorem 3.4. *For each $\Gamma = (G,\sigma) \in \mathcal{CC}_n$ not equal to C_n^σ, we have*

$$\mu_i(\Gamma) \leqslant \begin{cases} \mu_i(U(\Delta_\Gamma,n),+) & \text{if } \Gamma \text{ is balanced,} \\ \mu_i(U(\Delta_\Gamma,n),\bar{\sigma}) & \text{otherwise,} \end{cases} \qquad \forall\, i = 1,\dots,|V(G)|.$$

In particular,

$$\mu_1(\Gamma) \leqslant \frac{\Delta_\Gamma+3+\sqrt{\Delta_\Gamma^2+6\Delta_\Gamma-7}}{2}.$$

Moreover, if Γ is complete, then

$$\mu_i(\Gamma) \geqslant \begin{cases} \mu_i(U(\delta_\Gamma,n),+) & \text{if } \Gamma \text{ is balanced,} \\ \mu_i(U(\delta_\Gamma,n),\bar{\sigma}) & \text{otherwise,} \end{cases} \qquad \forall\, i = 1,\dots,(n-1)\delta_\Gamma.$$

Proof. In our hypotheses G is a subgraph of $U(\Delta_\Gamma,n)$ and $3 \leqslant \delta_\Gamma \leqslant \Delta_\Gamma$. If Γ is additionally complete, $U(\delta_\Gamma,n)$ is a subgraph of G. Whichever signature we choose on $U(\Delta_\Gamma,n)$ to extend σ, balancedness (resp. unbalancedness) is preserved. The inequalities of the statement now come from Proposition 2.2, Theorem 3.3, and the fact that all unbalanced signatures on unicyclic graphs are equivalent. $\square$

Let (G,σ) be a signed graph, and let G be a subgraph of H. The more natural way to extend σ to H is to define

$$\sigma^\uparrow : e \in E(H) = \begin{cases} \sigma(e) & \text{if } e \in E(G); \\ 1 & \text{otherwise.} \end{cases}$$

For any signed graph $\Gamma = (G,\sigma)$, we denote by $\theta(\Gamma)$ the number of its Laplacian eigenvalues bigger that 1.

Corollary 3.5. *For each $\Gamma \in \mathcal{CC}_n$, we have $\theta(\Gamma) \leqslant n$. If, additionally, Γ is complete, then $\theta(\Gamma) = n$, and $\mu_n(\Gamma) \geqslant \delta_\Gamma - 1$.*

Proof. By Theorems 3.3 and 3.4 we get

$$\mu_{n+1}(\Gamma) \leqslant \mu_{n+1}(U(\Delta_\Gamma, n), \sigma^{\uparrow}) = 1.$$

If, additionally Γ is complete, then

$$1 = \mu_{n+1}(U(\delta_\Gamma, n), \sigma|_{U(\delta_\Gamma, n)}) \leqslant \mu_{n+1}(\Gamma)$$

and

$$1 < \delta_\Gamma - 1 \leqslant \mu_n(U(\delta_\Gamma, n), \sigma|_{U(\delta_\Gamma, n)}) \leqslant \mu_n(\Gamma).$$ □

For the rest of this section we collect some more results giving further constraints on a signed graph Λ in order to be L-cospectral to a circular caterpillar.

Proposition 3.6. *Let $\Gamma = (G, \sigma)$ be a circular caterpillar of order N and girth n. The last two coefficients of the Laplacian polynomial $\psi(\Gamma, x)$ can be read on Table 1.*

Table 1

	$(-1)^N b_N$	$(-1)^{N-1} b_{N-1}$
$\sigma \sim +$	0	nN
$\sigma \sim \bar{\sigma}$	4	$nN + 4(N-n)$

Proof. Note that $|E(G)| = N$, G being unicyclic. According to Theorem 2.4, we get

$$(-1)^N b_N = \sum_{\mathrm{H} \in \mathcal{H}_N} \gamma(\mathrm{H}), \qquad \text{and} \qquad (-1)^{N-1} b_{N-1} = \sum_{\mathrm{H} \in \mathcal{H}_{N-1}} \gamma(\mathrm{H}),$$

where $\mathcal{H}_k$ is the set of TU-subgraphs with k edges contained in Γ. Now,

$$\mathcal{H}_N = \begin{cases} \varnothing & \text{if } \sigma \sim +, \\ \{\Gamma\} & \text{if } \sigma \sim \bar{\sigma}, \end{cases}$$

and $\gamma(\Gamma) = 4$ by (3). This justifies the last column of Table 1. We are left to investigate the number and the geometric nature of elements in $\mathcal{H}_{N-1}$. When Γ is balanced, it does not contain any unbalanced unicyclic subgraph, Hence $\mathcal{H}_{N-1}$ just contains the pairwise distinct trees in $\{T_i \,|\, 1 \leqslant i \leqslant n\}$ of order N which are obtained by removing exactly one edge of the cycle C_n inside G. Thus, by (3),

$$(-1)^{N-1} b_{N-1} = \sum_{i=1}^{n} \gamma(T_i) = nN.$$

If instead Γ is unbalanced, in addition to the trees T_i's, $\mathcal{H}_{N-1}$ also contains the unbalanced unicyclic graphs U_j's obtained from Γ by removing an edge not belonging to C_n. therefore, we find $N-n$ pairwise different U_j's, and, by (3), $\gamma(U_j) = 4$ for $1 \leqslant j \leqslant N-n$. Hence,

$$(-1)^{N-1} b_{N-1} = \sum_{i=1}^{n} \gamma(T_i) + \sum_{j=1}^{N-n} \gamma(U_j) = nN + 4(N-n)$$

as claimed. □

Corollary 3.7. *A signed graph Λ which is L-cospectral to an unbalanced circular caterpillar is necessarily connected.*

Proof. Let c be the number of connected components of Λ. By Theorem 2.5 (ii) each component of Λ is unbalanced. It follows that Λ has no trees among its components. Together with Theorem 2.5 (i), this implies that each component of Λ is also unicyclic. We now use Theorem 2.4 and Proposition 3.6 to get

$$4^c = b_N(\Lambda) = b_N(\Gamma) = 4;$$

hence, $c = 1$ as claimed. □

Lemma 3.8. *Let P_s be the path with s vertices. Whatever signature σ we choose on P_s, we have $\theta(P_s, \sigma) = \lceil \frac{2s}{3} \rceil - 1$.*

Proof. Since P_s is a tree, $\sigma \sim +$. By [10, Lemma 4.4] we get

$$\operatorname{Spec}(P_s, \sigma) = \operatorname{Spec}(P_s, +) = \left\{ 2 + 2\cos\frac{k}{s}\pi,\ \ k = 1, \dots, s \right\};$$

hence, $\theta(P_s, \sigma)$ counts the number of positive integers less that $2s/3$. □

Proposition 3.9. *Let $\Lambda = (H, \tau)$ be a graph which is L-cospectral to a graph Γ in $\mathcal{CC}_n$. Then,*

$$\operatorname{diam}(\Lambda) \leqslant \frac{3}{2}(n+1).$$

Proof. Let $d = \operatorname{diam}(\Lambda)$. By definition, H contains the path P_d has a subgraph. Thus, by Lemma 3.8 and Corollary 3.5,

$$\left\lceil \frac{2d}{3} \right\rceil - 1 = \theta(P_d, +) = \theta(P_d, \tau|_{P_d}) \leqslant \theta(\Lambda) = \theta(\Gamma) \leqslant n.$$

By analyzing the several mod 3 cases for d, we get the statement. □

4 Spiked triangles

We now focus our attention on elements in $\mathcal{CC}_3$, i.e. on signed spiked triangles. For every $N \geqslant 3$, we denote by $\mathcal{CC}_3(N)$ the set of signed spiked triangles of order N, and by $G(3; p, q, r)$ the unique circular caterpillars of girth 3 and degree sequence $\pi_{p,q,r} = (p{+}2, q{+}2, r{+}2, 1, \dots, 1)$. Obviously, the graph $G((3; p, q, r), \sigma)$ belongs to $\mathcal{CC}_3(p{+}q{+}r{+}3)$. As noted at the beginning of Section 3, all unbalanced signatures on $G(3; p, q, r)$ give rise to the same Laplacian spectrum. In particular, the matrices $L(G(3; p, q, r), \bar{\sigma})$ and $L(G(3; p, q, r), -)$ are similar, and the latter is equal to $D(G(3; p, q, r)) + A(G(3; p, q, r))$. This is the reason why the Laplacian spectrum of $L(G(3; p, q, r), \bar{\sigma})$ is precisely the ordinary signless Laplacian spectrum of $G(3; p, q, r)$.

Theorem 4.1. *Let $\Gamma = (G(3; p, q, r), \sigma)$ be a signed circular caterpillar. The Laplacian polynomial $\psi(\Gamma, x)$ is given by the following formulæ:*

$$\psi(\Gamma, x) = \begin{cases} (x-1)^{p+q+r-3}\psi_+(p,q,r)(x) & \text{if } \sigma \sim +; \\ (x-1)^{p+q+r-3}\psi_-(p,q,r)(x) & \text{if } \sigma \sim -; \end{cases} \tag{11}$$

where $\psi_+(p, q, r)(x)$ is the polynomial

$$\begin{aligned} x^6 - (p+q+r+9)x^5 + (pq+pr+qr+6(p+q+r)+30)x^4 \\ - (pqr + 3(pq+pr+qr) + 12(p+q+r) + 46)x^3 + \\ (2(pq+pr+qr) + 10(p+q+r) + 33)x^2 - 3(p+q+r+3)x, \end{aligned} \tag{12}$$

whereas $\psi_-(p, q, r)(x)$ is instead equal to

$$\begin{aligned} x^6 - (p+q+r+9)x^5 + (pq+pr+qr+6(p+q+r)+30)x^4 \\ - (pqr + 3(pq+pr+qr) + 12(p+q+r) + 50)x^3 + \\ (2(pq+pr+qr) + 10(p+q+r) + 45)x^2 - 3(p+q+r+7)x + 4. \end{aligned}$$

Proof. Let $P(k) := x^2 - (k+3)x + 2$. The equality (11) can be reached using Proposition 2.6 through the following steps: employ first (5) for v being one of the three vertices of the cycle C_3 in $G(3; p, q, r)$. Afterwards, use either (5) or (6) on the resulting summands yet to expand. It turns out that $\psi(\Gamma, x)$ is equal to

$$(x-1)^{p+q+r-3}\left(P(p)P(q)P(r) - (P(p)+P(q)+P(r))(x-1)^2 - 2\omega(C_3^\sigma)(x-1)^3)\right).$$

The statement now comes by the definition of ω (see (4) above) and $P(k)$. □

Corollary 4.2. *Two signed circular caterpillar of girth* 3 *are L-cospectral if and only if they are switching isomorphic.*

Proof. In the light of the remarks made in Section 1, we just need to prove the 'only if' part. Let $\Gamma = (G(3; p, q, r), \sigma)$ and $\Gamma' = (G(3; p', q', r'), \sigma')$ be two L-cospectral circular caterpillars.

The presence or the absence of 0 in their common Laplacian spectrum allows to establish whether Γ and Γ' are both balanced or both unbalanced.

Let $\widetilde{\mathbb{N}}_0^3$ be the set of non-increasing triples of non-negative integers. The coefficients of $\psi_+(p,q,r)(x)$ and $\psi_-(p,q,r)(x)$ are peculiar linear combinations of the four elementary symmetric polynomials

$$s_1(X_1,X_2,X_3) = 1 \qquad s_2(X_1,X_2,X_3) = X_1 + X_2 + X_3$$
$$s_3(X_1,X_2,X_3) = X_1X_2 + X_1X_3 + X_2X_3 \qquad s_4(X_1,X_2,X_3) = X_1X_2X_3.$$

evaluated at $(p,q,r) \in \widetilde{\mathbb{N}}_0^3$. The two polynomial $\psi(\Gamma,x)$ and $\psi(\Gamma',x)$ are equal only if

$$s_i(p,q,r) = s_i(p',q',r') \qquad \forall\, i = 1,...,4.$$

Such equalities actually occur only if the non-increasing sequences (p,q,r) and (p',q',r') are equal. In order to see this, observe that the map

$$\Theta : (a,b,c) \in \tilde{\mathbb{N}}_0^3 \to (x-a)(x-b)(x-c) \in \mathbb{Z}[x]$$

is injective. In fact, Θ maps different non-increasing triples to cubic polynomials with different zero sets. The non-negative numbers $s_i(a,b,c)$ for $i = 1,\dots,4$ are precisely the moduli of the coefficients in $\Theta(a,b,c)$. □

Proposition 4.3. *Every signed spiked triangle of order $N \leqslant 6$ is DLS.*

Proof. The statement follows from a direct comparison between the Laplacian spectra of the relatively small list of signed graphs of order $N \leqslant 6$. □

Proposition 4.4. *Let $N \geqslant 7$. For every $\Gamma = (G(3;p,q,r),\sigma) \in \mathcal{CC}_3(N)$ we have $\mu_4(\Gamma) = 1$.*

Proof. Since $N \geqslant 7$, the number p is at least 2. It follows that $G(3;p,q,r)$ contains $K = G(3;2,0,0)$ as subgraph. Recalling Proposition 2.2,

$$1 = \mu_4(K,\sigma|_K) \leqslant \mu_4(\Gamma) \leqslant \mu_4(U(p+2,3),\sigma^{\uparrow}) = 1,$$

where the last equality is due to Theorem 3.3 ii). □

Recall that, given any signed graph Γ, the number $\theta(\Gamma)$ counts the number of eigenvalues bigger than 1 in $\mathrm{Spec}_L(\Gamma)$. Throughout the rest of the paper, Ω_k will denote the set of connected signed graphs Γ such that $\theta(\Gamma) = k$, and $\mathcal{U}$ is understood to be the set of all signed graphs which are L-cospectral to some signed spiked triangle.

Lemma 4.5. *Let $\Lambda = (H,\tau)$ be a graph in $\mathcal{U}$. Then,* $\mathrm{diam}(\Lambda) \leqslant 6$ *and* $\theta(\Lambda) \leqslant 3$.

Proof. Let $\Gamma \in \mathcal{CC}_3$ be a spiked triangle which is L-cospectral to Λ. By Proposition 3.9, it follows that $\mathrm{diam}(\Lambda) \leqslant \frac{3}{2}\cdot 4 = 6$, whereas Corollary 3.5 yields $\theta(\Lambda) \leqslant 3$. □

The following lemma, though known to the experts, has been inserted with a proof for sake of clarity.

Lemma 4.6. *The set Ω_0 just contains the null graph K_0. Up to switching equivalence, the only elements in Ω_1 are $(C_3,-)$ and the stars $(K_{1,\Delta},+)$ for all $\Delta \geqslant 1$.*

Proof. The graphs $(C_3,-)$ and $(K_{1,\Delta},+)$ for $\Delta \geqslant 1$ are in Ω_1, In fact,

$$\mathrm{Spec}_L(C_3,-) = \{1^{(2)},4\} \qquad \text{and} \qquad \mathrm{Spec}_L(K_{1,\Delta},+) = \{0,1^{(\Delta-1)},\Delta+1\}.$$

If Γ has at least one edge, surely $\Gamma \notin \Omega_0$. In fact, its underlying graph G contains the path P_2 as a subgraph. Therefore, $\mu_1(\Gamma) \geqslant \mu_1(P_2,+) = 2$ by Proposition 2.2 and Corollary 2.3. If G contains two independent edges, i.e. it has $2P_2$ among its subgraphs, surely $\Gamma \notin \Omega_1$. In fact, again by Proposition 2.2 and Corollary 2.3, we have $\mu_1(\Gamma) \geqslant \mu_2(\Gamma) \geqslant \mu_2(2P_2) = 2$. The proof is over since $\theta(C_3,+) = 2$. □

In order to achieve Proposition 4.9 ensuring the connectivity of all signed graphs in $\mathcal{U}$, the following intermediate lemma will be helpful.

Lemma 4.7. *Let $\Lambda_1,\ldots,\Lambda_h$ be the connected components of $\Lambda = (H,\tau) \in \mathcal{U}$ ordered in a non-decreasing fashion with respect to the cyclomatic number. If the Λ_i's are not all unicyclic, then $h > 1$; Λ_1 is a tree; Λ_h is bicyclic and unbalanced; and, when $h > 2$, Λ_i is unicyclic and unbalanced for $2 \leqslant i \leqslant h-1$.*

Proof. From Theorem 2.5 (ii), we know that $|V(H)| = |E(H)|$. Because of such equality, for each k-cyclic signed graph with $k > 1$ in the set $\Upsilon = \{\Lambda_i \,|\, 1 \leqslant i \leqslant h\}$, we also find in it $k-1$ trees. By Theorem 2.5 (ii), the set Υ contains at most one balanced component, and *a fortiori* at most one tree. Hence, $k-1 \leqslant 1$, and Λ has at most one bicyclic component. □

Proposition 4.8. *The circumference of every $\Lambda = (H,\tau) \in \mathcal{U}$ is 3.*

Proof. Lemma 4.7 guarantees that no signed graphs in $\mathcal{U}$ have k-cyclic components with $k \geqslant 3$. Moreover, by Theorem 2.5 (i), we know that $|V(H)| = |E(H)|$ for all $\Lambda = (H,\tau) \in \mathcal{U}$. This implies in particular that $\mathcal{U}$ does not contains trees or forests.

The rest of the proof is devoted to show that no signed graphs with circumference $c > 3$ lie in $\mathcal{U}$.

Consider the set $\mathcal{S}$ of all signed graphs switching isomorphic to either $(C_6,+)$; $(C_5,-)$; $(C_4,+)$; $(C_4,-)$ or one of the graphs in Fig. 2. If Γ is a k-cyclic signed graph with $k \in \{1,2\}$, circumference $c \geqslant 4$ and $\theta(\Gamma) \leqslant 3$, then Γ belongs to $\mathcal{S}$. This fact relies upon Proposition 2.2, once you check that:

- $\theta(C_5,+) = \theta(C_6,-) = 4$, $\theta(C_n,\sigma) \geqslant 4$ for all $n \geqslant 7$ (see Proposition 3.2);
- for every 1-cyclic or 2-cyclic signed graph Γ' obtained by adding an additional edge (and possibly an additional vertex) to a graph Γ in $\mathcal{S}$, we have $\theta(\Gamma') = 4$ unless $\Gamma' \in \mathcal{S}$.

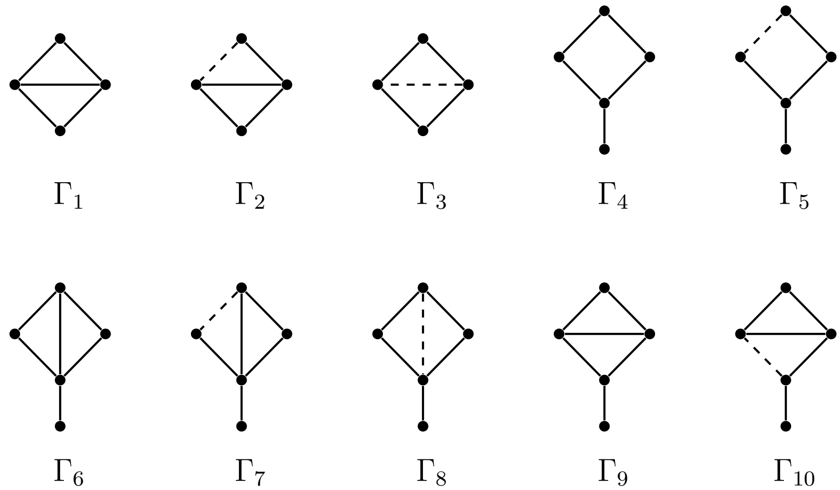


Fig. 2: Signed graphs Γ with $\theta(\Gamma) \leqslant 3$ and circumference $c \geqslant 4$. Dashed lines represent negative edges.

By Theorem 2.5 (i) and Proposition 4.3, the intersection $\mathcal{U} \cap \mathcal{S}$ is empty. We now claim that there are no non-connected graphs in $\mathcal{U}$ having elements in $\mathcal{S}$ among their components.

Assume by contradiction that there exists a non-connected $\Lambda \in \mathcal{U}$ such that one of its components, say Λ', is in $\mathcal{S}$, and consider the set of signed graphs $\mathcal{T}$ obtained from $\mathcal{S}$ by subtracting the switching isomorphic copies of $(C_4, -)$. Note that the function θ is additive with respect to the union of disjoint graphs, and $\theta(\Gamma) = 3$ for all graphs in $\mathcal{T}$. By Theorem 2.5 (ii), Lemma 4.5 and Lemma 4.6, we infer that if $\Lambda' \in \mathcal{T}$, then $\Lambda = \Lambda' \cup K_0$ and Λ' is unbalanced. It follows that Λ would have at most 6 vertices, but signed spiked triangles with at most 6 vertices are DLS; hence, up to switching equivalence, $\Lambda' = (C_4, -)$; thus, $\theta(\Lambda') = 2$.

Using once again Theorem 2.5 (ii), Lemma 4.5 and Lemma 4.6, we see that $\Lambda = \Lambda' \cup \Lambda''$, where $\Lambda'' = (H'', \tau'')$ contains just one balanced component by Corollary 3.7, $\theta(\Lambda'') = 1$ and $|V(H'')| = |E(H'')|$. By Lemma 4.6 such graph does not exist, and the proof is complete. $\square$

Along the proofs of Proposition 4.9 and Theorem 4.10 we denote by $L_{3,N}$ the lollipop graph with girth g and order N, i.e. the unsigned graph obtained by attaching a pendant path P_{N-2} to a vertex of the triangle C_3.

Proposition 4.9. *Every signed graph in $\mathcal{U}$ is unicyclic.*

Proof. By Theorem 2.5 (i), it suffices to show that all signed graphs in $\mathcal{U}$ are connected. Connectedness is already ensured for items in $\mathcal{U}$ which are either L-cospectral to an unbalanced spiked triangle (see Corollary 3.7) or have at most 6 vertices (see Proposition 4.3).

Assume now that $\Lambda = (H, \tau) \in \mathcal{U}$ has at least 7 vertices and is switching isomorphic to a balanced spiked triangle. Thank to Theorem 2.5 (ii) and Proposition 4.8 respectively, we know that Λ has just one balanced component and circumference $c = 3$. Moreover, $\theta(\Lambda) \leqslant 3$ by Lemma 4.5.

We first prove that Λ cannot have a bicyclic component. Suppose the contrary. Fig. 3 describes all bicyclic graphs Γ of circumference 3 such that $\theta(\Lambda) \leqslant 3$, where it is understood that $N \geqslant 5$ is the order of the graphs $\Gamma_{12}(N)$, $\Gamma_{13}(N)$ and $\Gamma_{14}(N)$, all having $N-5$ pendant vertices.

From a direct computation, we get $\mathrm{Spec}_L(\Gamma_{11}) = \{1^{(3)}, 2, 4, 5\}$. The Schwenk-like formula (5) can be used to compute the Laplacian polynomials of $\Gamma_{12}(N)$, $\Gamma_{13}(N)$ and $\Gamma_{14}(N)$. In fact, we obtain

$$\psi(\Gamma_{12}(N), x) = x(x-1)^{N-4}(x-3)^2(x-N), \tag{13}$$

$$\psi(\Gamma_{13}(N), x) = (x-3)(x-1)^{N-4}\left(x^3 - (N+3)x^2 + 3Nx - 4\right), \tag{14}$$

and

$$\psi(\Gamma_{14}(N), x) = (x-3)(x-1)^{N-4}\left(x^3 - (N+3)x^2 + 3Nx - 8\right). \tag{15}$$

For all graphs Γ in Fig. 3, we have $\theta(\Gamma) = 3$. This is immediately seen for $\Gamma = \Gamma_{11}(N)$ since $\mathrm{Spec}_L(\Gamma_{11}(N)) = \{1^{(3)}, 2, 4, 5\}$; the equality $\theta(\Gamma_{12}(N)) = 3$ comes from (13). In order to see that $\theta(\Gamma_i(N)) = 3$ for $i \in \{13, 14\}$, we first show that this is true for $N = 5$ by a direct computation; then, fixed $N \geqslant 6$, we observe that the polynomials

$$\frac{1}{(x-3)(x-1)^{N-4}}\psi(\Gamma_i(N), x) \qquad \text{for } i \in \{13, 14\}$$

have a root in the interval $(0,1)$, (since, when evaluated in 0 and 1, they have opposite signs), and finally note that

$$3 = \theta(\Gamma_i(5)) \leqslant \theta(\Gamma_i(N)) \leqslant 3$$

by Proposition 2.2, (14) and (15).

It follows that Λ has precisely two components, and one of them is K_0. Now, $\mu_1(\Gamma_{11} \cup K_0) = 5$, while the spectral radius of all balanced spiked triangles in $\mathcal{CC}_3(7)$ is bigger. Graphs of type $\Gamma_{12}(N)$ are excluded since Λ cannot have two balanced components.

When evaluated in 0 and in 3, the polynomials

$$(x-3)^{-1}\psi(\Gamma_{13}(N), x) \qquad \text{and} \qquad (x-3)^{-1}\psi(\Gamma_{14}(N), x)$$

both give negative values. A straightforward calculus argument shows that

$$\mu_2(\Gamma_{13}(N)) = \mu_2(\Gamma_{14}(N)) = 3.$$

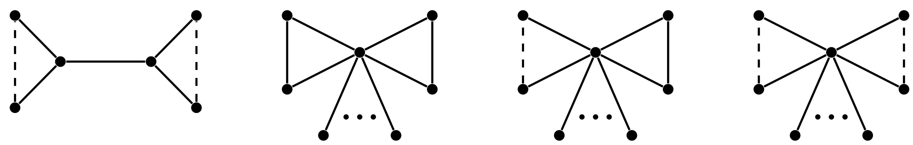
Since $\mu_2(G(3;1,1,0),+) > 3$, by Proposition 2.2 we deduce that only the spiked triangle $(G(3, N-3, 0, 0), +)$ can possibly be L-cospectral to Λ. In any case,

$$\mathrm{Spec}_L(G(3; N-3, 0, 0), +) = \{0, 1^{(N-3)}, 3, N\}. \tag{16}$$

We check directly that no pairs in the set of signed graphs

$$\{\Gamma_{13}(5), \Gamma_{13}(5), (G(3; 2, 0, 0), +)\}$$

are L-cospectral, and, for $N \geqslant 6$, we already observed that $\Gamma_{13}(N)$ and $\Gamma_{14}(N)$ have at least one eigenvalue belonging to the interval $(0,1)$. We conclude that there is not a $\Lambda \in \mathcal{U}$ with a bicyclic connected component.



Γ_{11} $\Gamma_{12}(N)$ $\Gamma_{13}(N)$ $\Gamma_{14}(N)$

Fig. 3: Bicyclic graphs Γ with $\theta(\Gamma) \leqslant 3$ and circumference $c = 3$. Dashed lines represent negative edges.

So far, we have proved that if $\Lambda \in \mathcal{U}$ is not connected, just one among its connected components $\Lambda_1, \dots, \Lambda_h$ is balanced, and none of them is bicyclic. By Lemmas 4.7 and 4.6 it follows that each Λ_i is unicyclic and has circumference $c = 3$.

With this information at hand, Lemma 4.6 implies that a non-connected $\Lambda \in \mathcal{U}$ has two components: up to switching equivalence, the unbalanced one Λ_1 is $(C_3, -)$, and the balanced Λ_2 belongs to Ω_2. The balanced unicyclic graphs in Ω_2 are switching isomorphic to $(G(3; p, 0, 0), +)$ for some $p \geqslant 0$; this is a consequence of Proposition 2.2, once you note $\theta(C_n, +) \geqslant 3$ for $n \geqslant 4$, and $\theta(G(3; 1, 1, 0), +) = \theta(L_{3,5}, +) = 3$. Since Λ has N vertices, Λ_2 is switching isomorphic to $((G(3; N-6, 0, 0), +)$. It follows that Λ is L-cospectral to $(C_3, -) \cup ((G(3; N-6, 0, 0), +)$ and $\mathrm{Spec}_L(\Lambda) = \{0, 1^{(N-4)}, 3, 4, N-6\}$.

Let Γ be any balanced spiked triangle. By (11), the eigenvalue 1 has multiplicity $N-4$ in $\mathrm{Spec}_L(\Gamma)$ only if $(x-1)^2$ divides $\psi_+(p,q,r)(x)$. But this happens only if Γ is switching isomorphic to $(G(3; N-3, 0, 0), +)$. In fact, from (12) we deduce that $\psi_+(p,q,r)(1) = -pqr$, and the polynomial $\chi(p,q)(x) \coloneqq (x-1)^{-1}\psi_+(p,q,0)(x)$,which is equal to

$$x(x^4 - (p+q+8)x^3 + (pq+5(p+q+22)x^2 - (2pq+7(p+q)+24)x + 3(p+q)+9),$$

gives $-pq$ when evaluated at $x = 1$. Looking at (16), we see that

$$\theta(G(3; N-3, 0, 0), +) = 2 \neq 3 = \theta(\Lambda).$$

Hence, there are no spiked triangles L-cospectral to a non-connected Λ, and the proof is over. $\square$

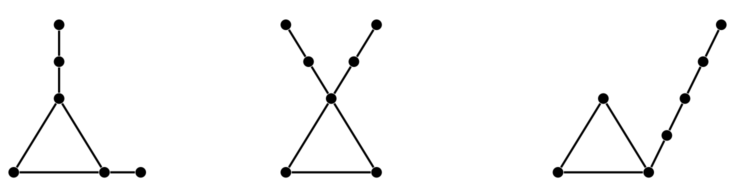


Fig. 4: Three underlying graphs of signed graphs Γ with $\theta(\Gamma) = 4$.

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Theorem 4.10. *Every signed spiked triangle is DLS.*

Proof. Let H be one of the unsigned graph depicted in Fig. 4. No matter which signature τ we choose on $E(G)$, a direct calculation shows that $\theta(H, \tau) = 4$ (since H is unicyclic of odd girth we have just to check the equality for $(H, +)$ and $(H, -)$, the former being balanced and the latter unbalanced). By Proposition 2.2 and Lemma 4.5, if a signed graph $\Gamma = (G, \sigma)$ contains (H, τ) as signed subgraph, then it cannot be in $\mathcal{U}$. This fact, together with Proposition 4.3, Proposition 4.8 and Proposition 4.9, implies that the only signed graphs which are not spiked triangles and possibly belong to $\mathcal{U}$ are of type $\Gamma(s,t) = (G(s,t), \sigma)$ (see Fig. 5) for some $s \geqslant 3$ and $t \geqslant 2$.

We evoke once again Proposition 2.2 to affirm that $\theta(\Gamma(s,t)) \geqslant \theta(\Gamma(3,2))$, and the latter number is 3 by a direct computation. The order of $\Gamma(s,t)$ is $N = s + t$.

Suppose that $\Gamma(s,t) = (G(s,t), \sigma)$ is unbalanced, and let Γ be any unbalanced spiked triangle of order N. Theorem 2.4 and Proposition 3.6 give

$$|b_{N-1}(\Gamma(s,t))| = 7N - 12 + 4(t-1) > 7N - 12 = |b_{N-1}(\Gamma)|.$$

This proves that all unbalanced spiked triangles are DLS.

Consider now the balanced graph $\Gamma(s,t) = (G(s,t), +)$. The multiset $\mathrm{Spec}_L(\Gamma(s,t))$ contains 3 for all $s \geqslant 3$ and $t \geqslant 2$. This fact is due to the presence in $\Gamma(s,t)$ of a 'pendant triangle'. Through MATLAB or a manual polynomial long division, we discover that if 3 belongs to $\mathrm{Spec}_L(G(3;p,q,r), +)$, then

$$2(pq + pr + qr) = 3pqr. \tag{17}$$

Since $\mu_3(G(3;3,2,2), +) > 3$, we have $\mu_3(G(3;p,q,r), +) > 3$ for all non-increasing triples (p,q,r) such that $p \geqslant 3$, $q \geqslant 2$ and $r \geqslant 2$. Such information, together with Proposition 4.3, Equation (10) and Equation (17), helps to identify the set of spiked triangles possibly L-cospectral to a suitable balanced $\Gamma(s,t)$. Up to switching equivalence, such set just contains

$$(G(3;2,2,2), +); \qquad (G(3;4,4,1), +); \qquad (G(3;6,3,1), +); \tag{18}$$

and $(G(3;p,0,0), +)$ for all $p \geqslant 3$.

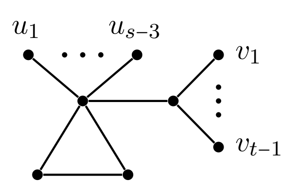

Fig. 5: The unsigned graph $G(s,t)$ with $s \geqslant 3$ and $t \geqslant 2$.

We discard the several $(G(3;p,0,0), +)$'s, since, by(16),

$$\theta(\Gamma(s,t)) = 3 > \theta(G(3;p,0,0), +) = 2.$$

The proof will be over, once we show that the signed graphs in (18) are not L-cospectral to a graph of type $\Gamma(s,t)$. To this aim we consider, for every signed graph $\Gamma = (G,\sigma)$, the triple of non negative integers

$$\Psi(\Gamma) = (|V(G)|, f_1(\Gamma), f_2(\Gamma)),$$

where the functions f_1 and f_2 have been defined just before the statement of Theorem 2.5. By Parts (i), (iv) and (v) of Theorem 2.5, the function Ψ should return the same triples when evaluated on pairs of L-cospectral graphs, and elementary algebraic manipulations show that no integers s and t exist such that

$$\Psi(\Gamma(s,t)) = (s+t,\ s^2+t^2+s+t+4,\ s^3+t^3+s+t+6)$$

is equal to one of the triples

$$\Psi(G(3;2,2,2),+) = (9,54,192), \qquad \Psi(G(3;4,4,1),+) = (12,90,462),$$

$$\Psi(G(3;6,3,1),+) = (13,108,668). \qquad \square$$

5 Extremal spiked triangles

Now that the Laplacian spectral characterization of signed spiked triangles is over, we solve the problem of finding extremal elements in $\mathcal{CC}_3(N)$ with respect to certain topological indices.

Lemma 5.1. *Let $p \geqslant q \geqslant r \geqslant 0$. The graph $G(3;p,q,r)$ is the only connected graph of order $N = p+q+r+3$ and degree sequence $\pi_{p,q,r} = (p+2, q+2, r+2, 1, \ldots, 1)$.*

Proof. Let G be a connected graph of order $N = p+q+r+3$ and degree sequence $\pi_{p,q,r} = (p+2, q+2, r+2, 1, \ldots, 1)$. The graph G is necessarily unicyclic; in fact the sum of vertex degrees gives $2|E(G)|$. Therefore,

$$|E(G)| = \frac{1}{2}\left(p+2+q+2+r+2+\underbrace{1+\cdots+1}_{p+q+r\text{ times}}\right) = N.$$

Since there are only 3 vertices whose degree is bigger than 1, the girth of G is necessarily 3, and $G = G(3;p,q,r)$. $\square$

Let $\tilde{\mathbb{N}}_0^n(k)$ be the set of non-increasing non-negative n-tuples $(p_1,\ldots,p_n)$ such that $\sum_{i=1}^n p_i = k$. Given $\pi = (p_1, p_2, \ldots, p_n)$ and $\pi' = (p'_1, p'_2, \ldots, p'_n)$ two n-tuples in $\tilde{\mathbb{N}}_0^n(k)$, we write $\pi \triangleleft \pi'$ if and only if $\pi \neq \pi'$, $\sum_i^n p_i = \sum_i^n p'_i$ and $\sum_i^h p_i \leqslant \sum_i^h p'_i$ whenever $1 \leqslant h \leqslant n$. Such an ordering is sometimes called *majorization* (see, for instance [17]). As an ordering, the just defined majorization is not total for $n > 2$. For instance the triples $(6,4,1)$ and $(7,2,2)$ are not comparable. We recall that a non-increasing sequence of *almost-equal values* $(p_1,\ldots,p_n)$ satisfies, by definition, the inequality $p_1 - p_n \leqslant 1$.

Lemma 5.2. *Let n and k be two positive integers. In the set $\tilde{\mathbb{N}}_0^n(k)$ of non-increasing non-negative n-tuples $(p_1, \dots, p_n)$ such that $\sum_{i=1}^n p_i = k$, there exists just one n-tuple $(\hat{p}_1, \dots, \hat{p}_n)$ of almost-equal integers. Moreover, the majorizations $(\hat{p}_1, \dots, \hat{p}_n) \triangleleft (p_1, \dots, p_n) \triangleleft (k, 0, \dots, 0)$ hold for every*

$$(p_1, \dots, p_n) \in \tilde{\mathbb{S}}^n(k) := \tilde{\mathbb{N}}_0^n(k) \smallsetminus \{(\hat{p}_1, \dots, \hat{p}_n), (k, 0, \dots, 0)\}.$$

Proof. Unicity of $(\hat{p}_1, \dots, \hat{p}_n)$ comes from the Division algorithm: since there exists a unique pair $(s, r) \in \mathbb{N}_0^2$ such that $k = ns + r$ and $0 \leqslant r < n$, we necessarily have

$$\hat{p}_1 = \dots = \hat{p}_r = \left\lceil \frac{k}{n} \right\rceil = s + 1, \qquad \text{and} \qquad \hat{p}_{r+1} = \dots = \hat{p}_n = \left\lfloor \frac{k}{n} \right\rfloor = s.$$

When either $n \leqslant 2$ or $k \leqslant 2$, there is nothing else to prove, since $\tilde{\mathbb{N}}_0^1(k)$ is a singleton, $\triangleleft$ is a total ordering on $\tilde{\mathbb{N}}_0^2(k)$ for all $k > 0$, and $\tilde{\mathbb{S}}^n(1)$ and $\tilde{\mathbb{S}}^n(2)$ are empty for all $n > 0$.

Let now $n > 2$ and $k \geqslant 3$. The second majorization is immediate. In order to prove the first one, we assume by contradiction that, for some fixed k, the set

$$\{(p_1, \dots, p_n) \in \tilde{\mathbb{S}}^n(k) \,|\, (\hat{p}_1, \dots, \hat{p}_n) \ntriangleleft (p_1, \dots, p_n)\}$$

is not empty and denote by $(q_1, \dots, q_n)$ its minimum with respect to the lexicographic order. Let us distinguish two cases.

Case 1: $q_1 > \hat{p}_1$. Since $(q_1, \dots, q_n) \neq (\hat{p}_1, \dots, \hat{p}_n)$, then $q_1 - q_n \geqslant 2$. Hence, the two integers

$$k_1 = \max\{i \mid q_1 = q_i\} \qquad \text{and} \qquad k_2 = \min\{i \mid q_1 - q_i = 2\}$$

are well-defined. The minimum integer h such that $\sum_{i=1}^h q_i < \sum_{i=1}^h \hat{p}_i$ is not less than k_2. In fact, since we are assuming $q_1 - 1 \geqslant \hat{p}_1$,

$$\sum_{i=1}^{\ell} q_i = \ell q_1 > \ell \hat{p}_1 \geqslant \sum_{i=1}^{\ell} \hat{p}_i \qquad \text{for } \ell \leqslant k_1,$$

$$\sum_{i=1}^{\ell} q_i = k_1 q_1 + (\ell - k_1)(q_1 - 1) > \ell \hat{p}_1 \geqslant \sum_{i=1}^{\ell} \hat{p}_i \qquad \text{for } k_1 < \ell < k_2,$$

and

$$\sum_{i=1}^{k_2} q_i = k_1 q_1 + (k_2 - k_1 - 1)(q_1 - 1) + (q_1 - 2) \geqslant k_2 \hat{p}_1 \geqslant \sum_{i=1}^{k_2} \hat{p}_i.$$

Consider now the n-tuple $(q_1', \dots, q_n')$ defined as follows

$$q_i' = \begin{cases} q_i - 1 & \text{if } i = k_1, \\ q_i + 1 & \text{if } i = k_2, \\ q_i & \text{otherwise.} \end{cases}$$

By definition, $(q_1, \dots, q_n)$ is bigger than $(q_1', \dots, q_n')$ with respect to the lexicographic order. Yet, $\sum_{i=1}^h q_i' = \sum_{i=1}^h q_i < \sum_{i=1}^h \hat{p}_i$; hence, $(\hat{p}_1, \dots, \hat{p}_n) \ntriangleleft (q_1', \dots, q_n')$, against the minimality of $(q_1, \dots, q_n)$.

Case 2: $q_1 = \hat{p}_1$. If this case occurs, then $k = ns + r$ with $r > 0$. Thus,

$$\hat{p}_1 = \cdots = \hat{p}_r = s + 1 \quad \text{and} \quad \hat{p}_{r+1} = \cdots = \hat{p}_n = s.$$

Now, $q_1 = \cdots = q_{r+1} = s + 1$, otherwise either $\sum_{i=1}^n q_i < n$ or $(q_1, \ldots, q_n) = (\hat{p}_1, \ldots, \hat{p}_n)$, against the definition of $(q_1, \ldots, q_n)$. It turns out that $(q_{r+1}, \ldots, q_n)$ is the minimum of the set

$$\{(p_{r+1}, \ldots, p_n) \in \tilde{\mathbb{S}}^{n-r}(s(n-r)) \,|\, (\hat{p}_{r+1}, \ldots, \hat{p}_n) = (s, \ldots, s) \ntriangleleft (p_{r+1}, \ldots, p_n)\}.$$

This is absurd, since such set is empty. Its emptiness follows by Case 1 if $n - r > 2$ (since $q_{r+1} = s + 1 > s = \hat{p}_{r+1}$), and by the fact that $\triangleleft$ is a total ordering if $n - r = 2$. □

By Lemma 5.2, it makes sense to denote by $\hat{\Gamma}^+(N)$ the all-positive signed spiked triangle in $\mathcal{CC}_3(N)$ whose triple of largest vertex degrees has almost-equal values. Our last theorem involves, among other things, the first two Zagreb indices. Recall that, given a signed graph $\Gamma = (G, \sigma)$,

- the first Zagreb index $M_1(\Gamma)$ of Γ is defined as $\sum_i^N d_G^2(v_i)$;
- the second Zagreb index $M_2(\Gamma)$ of Γ is given by $\sum_{uv \in E(G)} d_G(u) d_G(v)$.

.

Theorem 5.3. *Let $N \geqslant 3$. The graph $\hat{\Gamma}^+(N)$ (resp. $(G(3; N-3, 0, 0), -)$) minimizes (resp. maximizes) the Laplacian spectral radius and both the first two Zagreb indices in the set $\mathcal{CC}_3(N)$.*

Proof. Let G be any spiked triangle. We start by recalling that $\mu_1(G, -)$ also gives the signless spectral radius of G. Since G is not bipartite, Proposition 3.9.1 in [13] yields

$$\mu_1(G, +) < \mu_1(G, -). \tag{19}$$

Let $S(\pi)$ be the class of connected signed graphs with degree sequence π. and consider two different non-increasing degree sequences π and π' of signed unicyclic graphs such that $\pi \triangleleft \pi'$.

Al least three results in [17] concerning unsigned unicyclic graphs can be translated in the context of signed graphs. They are:

$$\max\{\, \mu_1(G, -) \,|\, (G, -) \in S(\pi)\,\} < \max\{\, \mu_1(G, -) \,|\, (G, -) \in S(\pi')\,\}; \tag{20}$$

$$\max\{\, M_1(\Gamma) \,|\, \Gamma \in S(\pi)\,\} < \min\{\, M_1(\Gamma) \,|\, \Gamma \in S(\pi')\,\}; \tag{21}$$

$$\max\{\, M_2(\Gamma) \,|\, \Gamma \in S(\pi)\,\} < \min\{\, M_2(\Gamma) \,|\, \Gamma \in S(\pi')\,\}. \tag{22}$$

Equations (20), (21) and (22) are respectively equivalent to [17, Theorem 2.2, i)], [17, Theorem 3.1] and [17, Theorem 3.5].

Let $\hat{\pi}$ denote the N-tuple $(\hat{p}_1, \hat{p}_2, \hat{p}_3, 1 \ldots, 1)$, where $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ is the unique non-increasing triple with almost-equal values in $\tilde{\mathbb{N}}_0^3(N+3)$. From Lemma 5.1, we deduce that $\hat{\pi}$ and

$(N-1,2,2,1,\dots,1)$ are the extremal degree sequences associated to spiked triangles in $\mathcal{CC}_3(N)$ with respect to majorization. Thus, by (21) and (22), $\hat{\Gamma}^+(N)$ (resp. $(G(3;N-3,0,0),-)$) minimizes (resp. maximizes) the first two Zagreb indices. Since, by (19) and (20), $(G(3;N-3,0,0),-)$ maximizes the Laplacian spectral radius as well, the proof will be over once we show that $\hat{\Gamma}^+(N)$ also minimizes the Laplacian-spectral radius in $\mathcal{CC}_3(N)$.

Inequality (19) says that minimizers with respect to μ_1 have to be searched among balanced spiked triangles. Our claim is surely true for $N \leqslant 5$. Infact, $\hat{\Gamma}^+(3)$ and $\hat{\Gamma}^+(4)$ are the only balanced graphs in $\mathcal{CC}_3(3)$ and in $\mathcal{CC}_3(4)$ respectively. In $\mathcal{CC}_3(5)$ there are only two balanced spiked triangle, and

$$\mu_1(\hat{\Gamma}^+(5)) < 4,31 < \mu_1((G(3;2,0,0),+) = 5.$$

Let now $N \geqslant 6$. We distinguish three cases.

Case 1: $N = 3s+3$ for some integer $s \geqslant 1$. In this case,

$$\hat{\Gamma}^+(N) = (G(3;s,s,s),+) = (U(s+2,3),+)$$

and any other balanced spiked triangle in $\mathcal{CC}_3(N)$ has largest vertex bigger than $s+2$.

From (7), (10) and Corollary 2.3 we get

$$\mu_1(\hat{\Gamma}^+(N)) = \phi_+(3,s+2) = \frac{s+4+\sqrt{(s+4)^2-12}}{2} < s+4 \leqslant \mu_1(\Gamma),$$

for every balanced $\Gamma \in \mathcal{CC}_3(N) \smallsetminus \{\hat{\Gamma}^+(N)\}$.

Case 2: $N = 3s+4$. We have $\hat{\Gamma}^+(N) = (G(3;s+1,s,s),+)$. Using Theorem 3.4 and Corollary 2.3, we see that

$$\mu_1(\hat{\Gamma}^+(N)) \leqslant \phi_+(3,s+3) < s+5 \leqslant \mu_1(\Gamma),$$

for every balanced $\Gamma \in \mathcal{CC}_3(N)$ such that $\Delta_\Gamma > s+3$. The only balanced graph in $\mathcal{CC}_3(N) \smallsetminus \{\hat{\Gamma}^+(N)\}$ whose largest Laplacian eigenvalue is not bigger than $s+3$ is $\breve{\Gamma} = (G(3;s+1,s+1,s-1),+)$. Now, from (11) and (12) we compute

$$\psi(\hat{\Gamma}^+(N), s+4) = -6(s+3)^{3s-2}s(s+4), \tag{23}$$

and

$$\psi(\breve{\Gamma}, s+4) = -8(s+3)^{3s-2}(s+1)(s+4). \tag{24}$$

Since the numbers (23) and (24) are both negative, we deduce that $\mu_1(\hat{\Gamma}^+(N))$ and $\mu_1(\breve{\Gamma})\}$ are both larger than $s+4$. We can also compute

$$\psi(\hat{\Gamma}^+(N),x) - \psi(\breve{\Gamma},x) = x^2(x-1)^{3s-2}(x^2-(s+4)x+2),$$

which is a positive number when evaluated at $x \geqslant s+4$. It follows that $\mu_1(\hat{\Gamma}^+(N)) < \mu_1(\breve{\Gamma})$, as wanted.

Case 3: $N = 3s+5$. We have $\hat{\Gamma}^+(N) = G(3;s+1,s+1,s)$. In this case, there is no other balanced spiked triangle in $\mathcal{CC}_3(N)$ with largest vertex degree $s+3$. Hence, by Corollary 2.3,

$$\mu_1(\hat{\Gamma}^+(N)) \leqslant \phi_+(3,s+3) < s+5 \leqslant \mu_1(\Gamma)$$

for all balanced $\Gamma \in \mathcal{CC}_3(N) \smallsetminus \{\hat{\Gamma}^+(N)\}$. □

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