Germs of diffeomorphisms and their Taylor expansions

David John Marsay

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List of Contents.

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Declaration:

None of the material contained in this thesis has previously been published and is, except where indicated, my own work.

Summary

The thesis examines the relationship between the germ of a C^∞ diffeomorphism $f: R^n, \underline{O} \to R^n, \underline{O}$ which is tangent to the identity at $\underline{O}$ and its Taylor expansion. The case in which n is one is already well understood. For n greater than one some normal forms for germs are already known. These are germs with the property that any other germ having the same Taylor expansion is conjugate to the normal form. Conjugation may be thought of as a change of variables. The idea is that the Taylor expansion determines what the germ 'looks like'. The above concept is extended in the thesis in a new way to deal with the common situation where the Taylor expansion only partially determines what the germ 'looks like', for example the Taylor expansion may determine what the germ looks like near one axis, but not away from that axis. Examples are given.

The importance of the extended concept is highlighted by a construction (using the new idea) of a large class of germs which do not have normal forms in the old, limited, sense.

The theory allows one to study the centralisers of such germs, and to describe what their invariant curves 'look like', for example: can the germs be embedded in one-parameter groups, and do they have invariant curves which may be thought of as graphs of C^∞ functions?

Key to Abbreviations and Symbols.

C^n	differentiable n times, with continuous n-th derivative
C^∞	differentiable infinitely many times
G	the group of germs of C^∞ diffeomorphisms $R^n, 0 \to R^n, 0$
G_n	those members of G which are n-tangent to the identity
G_∞	those members of G which are in G_n for all integers n
R	the real numbers
R_+	the non-negative real numbers including zero
R_+^*	the positive real numbers excluding zero
R^n	the Euclidean product of R with itself n times
$Z(f)$	the centraliser of the germ f in G
exp	the exponential map from R into R_+^*
log	the inverse of exp from R_+^* into R
(,)	the open interval
(,]	half open interval
[,]	closed interval
Σ	summation
Π	product
$\cup$	union of sets
$\cap$	intersection of sets
$\subseteq$	inclusion of one set in another
$\in$	membership of a set
[ ]	reference number
( )	reference to another part of the text
$\bar{X}$	closure of the set X
$\hat{X}$	union of all segments between members of X
$f^{(m)}$	the m-th derivative of the diffeomorphism f

1 Introduction.

Consider the group of C^∞ diffeomorphisms $R^n,\underline{0} \to R^n,\underline{0}$ under functional composition, for some fixed integer, n. How close is the relationship between a germ at $\underline{0}$ and its Taylor expansion? The situation is well understood when the number of variables, n, is one, but not in higher dimensions.

First I shall give the well known results for one dimension and then outline the generalisations and counter-examples for higher dimensions which are proven in the main body.

1.1 Any germ whose linear term is neither the identity nor minus the identity is conjugate to the unique linear germ corresponding to its linear term. Moreover there is only one conjugating germ whose linear term is the identity.

As is very common in this field, the proof divides into three complementary parts: Firstly there is the corresponding result for formal power series, which in this case is easily proven. Secondly there is an existence theorem for the case when the Taylor expansions are identical, in this case given by Sternberg [9], and thirdly there is a uniqueness result, in this case provided by Kopell [6].

As an example of an application of this result, consider the following: The group of linear germs is a one-parameter group. But by 1.1 every germ (except those given there) is a linear germ in the appropriate co-ordinate system, hence every germ embeds canonically in a one-parameter group.

The linear germs are called 'normal forms' because of the central role they play in the proof of results like the one above. It is the relative simplicity of these proofs which motivates the study for such forms in higher dimensions.

Takens [11] has studied germs whose linear term is the identity. His results may be summarised (and slightly expanded upon) as follows:

1.2 For each strictly positive integer m, any infinite power series in one variable whose m-jet is the identity but whose m+1-jet is <u>not</u> the identity is conjugate to exactly one power series of the form

$$x - x^{m+1} + t.x^{2m+1},$$

for some fixed real number t.

Takens showed existence. Uniqueness of t is a case of simple algebra. The complementary result for germs is as follows:

1.3 For each strictly positive integer m and each real number t, any two germs whose Taylor expansions are

$$x - x^{m+1} + t.x^{2m+1}$$

are conjugate. Moreover there exists exactly one conjugating germ whose Taylor expansion is the identity.

Takens proves existence. A careful reading of his proof shows that the conjugating germ is unique. An alternative proof is given in the text [5.2]. An application follows:

1.4 For each strictly positive integer m and each real number t, the centraliser of the germ given by

$$x - x^{m+1} + t.x^{2m+1}$$

is either isomorphic to R (if m is odd) or RxZ_2 (if m is even). Moreover the m+1-jet of every element of the centraliser is of the form

$$\pm x + s.x^{m+1} \text{ for some real number, s.}$$

The necessary algebra is done in A2.3 & A2.8.

Thus when the number of variables, n, is one, the Taylor expansion determines the germ up to changes of co-ordinates, and the necessary change is essentially unique.

The only exceptions are the obvious ones, when the Taylor expansions are either plus or minus the identity. Sergeraert [8] has recently studied such germs, with identity Taylor expansion, showing, for example, that many such germs do not have 'square roots' in the sense that they cannot be formed by composing another germ with itself.

Sternberg [9] and Kopell [6] have generalised 1.1 to higher dimensions, leaving germs whose linear term is the identity to be studied here. Accordingly some generalisations will be given, together with some novel counter-examples where appropriate.

The formal normal form of 1.2 is extremely useful, in that the results 1.3 & 1.4 need only be proven for these normal forms. However in higher dimensions, except for the linear case, there is no finite-parameter family of normal forms for power series. In section 2 though 'standard quasi-contractions' are introduced, and these play a similar role in the developement of the theory. That is a large number of germs can be conjugated into these forms, and these germs are 'stable' in the sense of 1.3. More formally, the following notation is required:

1.5 Given a positive integer m, let G_m be the group of germs of diffeomorphisms $R^n, \underline{0} \mapsto R^n, \underline{0}$ whose m-jet is the identity. Let $G = G_0$ and $G_\infty = \bigcap_m G_m$. Give each G_m the topology provided by the m+1-jets.

Thus a typical neighbourhood of the identity consists of all germs in G_m whose m+1-linear part has sufficiently small coefficients.

Thus by a 'large number' of germs is meant on open set in the above topology, and 'almost all' means on all except some meagre subset. Note that almost all germs can be represented by normal forms in the one dimensional case, an ideal which the standard quasi-contractions fall far short of. It will in fact be shown that for m even there are no normal forms.

Beličkii [3] has recently partially generalised 1.3 by finding open sets with the following property:

1.6 An element $g \in G$ is <u>flat stable</u> if and only if for every germ $f \in G$ which is infinitely tangent to g at 0 (i.e. has the same Taylor expansion) there exists a germ $h \in G$ (with identity Taylor expansion) such that

$$g_{\circ}h = h_{\circ}f \ .$$

Beličkii writes in Russian, and the term he uses for the above notion has variously been translated as 'plane stable' or 'horizontally stable', but the term 'flat stable' seems more appropriate in English, since the germs are essentially stable under flat perturbations. Beličkii's result may now be summarised as follows:

1.7 For each even integer m which is at least two there exists a non-empty open set, QC, of G_m (which he calls the quasi-contractions), every member of which is flat stable.

A slightly different concept will prove useful later. The formal definition is given in section 2, together with examples. The relationship between the two notions is explained in section 3. Beličkii's theory is outlined in Appendix 1. This is done because the formal proof he gives is not quite general enough to provide the result required, namely that the 'new' quasi-contractions also have a type of flat stabiltity. It is also shown that the element 'h' of 1.6 is unique when g is a quasi-contraction. Moreover it follows from some algebraic results given in Appendix 2 that:

1.8 For each even integer m, at least two, there exists an open dense subset, QC' of QC, such that for all $f \in QC'$ the centraliser of f is either isomorphic to R or $R \times Z_2$.

The proof of this is done in two parts. Firstly it is shown in 5.2 that there exist integers k, depending continuously on the m+1-terms, such that only the identity germ has identity k-jet and commutes with the given germ. So 1.8 is reduced to a problem of algebra, which is covered by A2.9.

So far Belickiǐ's result has simply been generalised and extended slightly. In section 6 these results are applied to show that for each even integer m greater than zero there exists a non-empty open subset of G_m (see 1.5) every member of which has the same Taylor series as some other member which is yet not conjugate to it, in contrast to 1.3. This shows that Belickiǐ's 1.7 is inevitably limited. In other words the Taylor expansions are often insufficient to determine the behaviour of a germ. To illustrate this an open set is constructed, every member of which has the same Taylor series as a pair of germs, one of which has a centraliser isomorphic to R and the other a centraliser isomorphic to Z.

For the sake of completeness section 7 describes open subsets of each G_m, members of which have 'large' centralisers in the sense that their centralisers have non-trivial intersections with G_∞. This generalises a result of Kopell [6] for the case when m is zero (i.e. for germs with non-identity linear term), and shows that the element h in 1.6 is not always unique. In particular Belickiǐ has shown that what he calls quasi-hyperbolic germs [2] are flat stable, and the result of section 7 applies to these. They do not actually form an open set in the appropriate space G_m, but it is easily shown that there exist open sets of each G_m, every member of which is conjugate to a quasi-hyperbolic germ, and consequently every member of which is flat stable but has a large centraliser. As usual this is proven in two parts: Firstly formal algebra and secondly an application of the quasi-hyperbolic germs' stability.

Finally, in section 8 some applications of these results are given. These are made more interesting by including some applications of sections 4 and 5 which give results similar to those of section 3, namely on stability and the smallness of centralisers, for germs which are not necessarily infinitely differentiable. These sections are entirely elementary, for example the conjugating germs are constructed explicitly as limits rather than being merely shown to exist as fixed points of an operator. Besides widening the scope of the applications this also gives an insight into the C^∞ case, particularly for the quasi-contractions of Belickii when the two maps are unique, and hence the same.

To summarise: the contractions and hyperbolic maps of the linear theory have their anologues, but these by no means exhaust the open types of germ, some of which exhibit a new phenomenum, namely that the Taylor series does <u>not</u> determine what the centraliser looks like.

2 Quasi-Contractions on Invariant Sets.

A contraction is a germ of a diffeomorphism $R^n,\underline{O} \to R^n,\underline{O}$ such that any diffeomorphism having the same first derivative at $\underline{O}$ is a topological contraction near $\underline{O}$. The definition of a quasi-contraction extends this concept in two ways. Firstly, one considers higher order terms in the Taylor Expansion. Secondly, one considers local phenomena. The following will help to illustrate this:

Examples 2.1

Let $f,g: R^2,\underline{O} \to R^2,\underline{O}$ be given by the formulae

$$f: (x,y) \mapsto (x - x^3, y - y^3) \quad \text{and}$$

$$g: (x,y) \mapsto (x/2 \; , y/2 \;) .$$

Then any diffeomorphism with the same Taylor Expansion as f up to terms of order 3 is a topological contraction near $\underline{O}$, just as any diffeomorphism with the same Taylor Expansion as g up to terms of order 1 is a topological contraction near $\underline{O}$.

Consider now $h: R,0 \to R,0$ given by

$$h: x \mapsto x - x^2 .$$

This is contracting on one side of 0 but expanding on the other. This is the sense in which the concept of a contraction has been generalised to 'local phenomena': in this case the diffeomorphism is only a contraction on half of the neighbourhood of 0. In higher dimensions the situation becomes more complex. Consider $k: R^2,\underline{O} \to R^2,\underline{O}$ given by

$$k: (x,y) \mapsto (x - x^2, y - 2xy) .$$

This is a topological contraction on a neighbourhood of the positive semi-axis. More precisely, if

$$X = \{(x,y) \in R^2 \mid 0 \leqslant x \leqslant 1, |y| \leqslant x\}$$

then for any neighbourhood, U, of $\underline{O}$ there exists an integer

N such that for all integers $n \geq N$, $k^n(X) \in U \cap X$.

This sort of behaviour does not occur with ordinary contractions.

The formal definition of a quasi-contraction is as follows:

Definition 2.2

Let m be a strictly positive integer. Let $X \subseteq R^n$. A C^1 diffeomorphism $f: R^n,\underline{0} \to R^n,\underline{0}$ is said to be a quasi-contraction of degree m on X if and only if there exist strictly positive numbers r, a, b, c such that for all $\underline{x} \in \mathbb{R}^n$ with $0 < \|\underline{x}\| \leq r$ one has:

i) $f(\underline{x}) \in X$, if $\underline{x} \in X$

ii) $\|\underline{x}\| - a\|\underline{x}\|^{m+1} \leq \|f(\underline{x})\| \leq \|\underline{x}\| - b\|\underline{x}\|^{m+1}$, if $\underline{x} \in X$

iii) $\|Df(\underline{x})^{-1}\| \leq 1 + c\|\underline{x}\|^m$.

With this notation the function f of 2.1 is a quasi-contraction of degree 2 on R^2, h is a quasi-contraction of degree 1 on R^+, and k is a quasi-contraction of degree 1 on the specified triangular region, X. Notice that any perturbations of the above diffeomorphisms having the same Taylor Expansions are also quasi-contractions of the same degree on the same sets. Often, but not always, the quasi-contractiveness will be a property of the Taylor Expansion and not just the germ.

Example 2.3

Let $l: R^2,\underline{0} \to R^2,\underline{0}$ be given by

$$l: (x,y) \mapsto (x - x^2,\ y - 2xy + x^2y).$$

Then $X = \{(x,y) \in R^2 \,/\, 0 \leq x < 1,\ |y| \leq x^2\}$ is invariant under l, and l is a quasi-contraction of degree 1 on X. However X is not invariant under

$$l': (x,y) \mapsto (x - x^2,\ y - 2xy + x^2y + \exp(-x^{-2})),$$

which is infinitely tangent to l at $\underline{0}$.

The following lemmas give some ways in which quasi-contractions can be built up. They are easily proven.

<u>Lemmas 2.4</u>

a) Given $Y \subseteq X \subseteq R^n$, every quasi-contraction, f, of finite degree on X, with $f(Y) \subseteq Y$, is a quasi-contraction of the same degree on Y.

b) Given X, $Y \subseteq R^n$, every diffeomorphism which is a quasi-contraction of the same degree on X and Y is also a quasi-contraction of that degree on both $X \cup Y$ and $X \cap Y$. (Thus in general one would aim to find the union of the sets on which a given diffeomorphism was a quasi-contraction of a given degree.)

c) The composition of two quasi-contractions of the same degree on the same set is a quasi-contraction of that same degree on that same set.

d) The composition (in either order) of a quasi-contraction of finite degree on R^n and a rotation is a quasi-contraction of the same degree.

e) Let $X \subseteq R^n$, let $f: R^n, O \rightarrow R^n, O$ be a quasi-contraction of degree m on X and let $g: R^n, O \rightarrow R^n, O$ be a C^1 diffeomorphism with the property that for some real $e > 0$ there exists a strictly positive real number r such that for all $\underline{x} \in R^n$ with $\|\underline{x}\| \leqslant r$ one has both

$$\|Dg(\underline{x})\| \leqslant 1 + e\|\underline{x}\|^m \quad \text{, and}$$

$$\|Dg^{-1}(\underline{x})\| \leqslant 1 + e\|\underline{x}\|^m .$$

Then the function

$$f^g = g^{-1} . f . g$$

is a quasi-contraction of degree m on $g^{-1}(X)$.

The simplest examples of quasi-contractions are C^∞ diffeomorphisms in one dimension. These are described in the following:

<u>Theorem 2.5</u>

Let $f: R , 0 \rightarrow R , 0$ be a C^∞ diffeomorphism with $Df(0) = \text{identity}$. Unless f is infinitely tangent to the identity at 0 there exists a least integer m such that $D^{m+1}f(0) \neq 0$.

Moreover, either f or f^{-1} is a quasi-contraction of degree m on R^+. If m is even then either f or f^{-1} is a quasi-contraction on R. If m is odd then neither f nor f^{-1} is a quasi-contraction on R.

The proof is a straightforward application of the Mean Value Theorem. More generally, it is possible to show that one has the following characteristic of quasi-contractions in one dimension.

<u>Theorem 2.6</u>

A C^1 diffeomorphism $f: R , 0 \rightarrow R , 0$ is a quasi-contraction of degree m on R^+ if and only if there exists a C^∞ diffeomorphism $g: R , 0 \rightarrow R , 0$ (called a 'gauge' for f) with the following properties:

i) $Dg(0) = \text{identity}$

ii) $D^2g(0) = \ldots = D^m g(0) = 0$

iii) $D^{m+1}g(0) < 0$

iv) for some integer n, for some real number r , for all $x \in R^+$ with $x \leqslant r$,

$$Dg^n(x) < Df(x) \leqslant Dg(x) .$$

Conditions (i) and (ii) are simply that g be m-tangent to the identity function at 0. The notion of m-tangent may be generalised as follows

<u>Definition 2.7</u>

Let m be a strictly positive integer and $X \subseteq R^n$. A function $f: R^n, 0 \rightarrow R^n, 0$ is <u>m-flat on X at 0</u> if and only if

$$\lim_{r \to 0} \sup_{\underline{x} \in X_r} \{ \|f(\underline{x})\| / \|\underline{x}\|^{m+1} \} < \infty \text{ , where}$$

$$X_r = \{ \underline{x} \in X \mid 0 < \|\underline{x}\| \leqslant r \}.$$

Two functions whose difference is m-flat on X at $\underline{0}$ are m-tangent on X at 0. Infinitely flat and infinitely tangent mean m-flat and m-tangent for all integers m.

The formal definition of a quasi-contraction is tied in with the intuitive concept as follows:

Theorem 2.8

Let f: $R^n, \underline{0} \to R^n, \underline{0}$ be a C^{m+1} diffeomorphism which is m-tangent to the identity function.

Then f is a quasi-contraction of degree m on R^n if and only if every C^1 diffeomorphism which is (m+1)-tangent to f is a topological contraction near $\underline{0}$.

Proof

The only if part is clear. Conversely, suppose that f is not a quasi-contraction of degree m on R^n. Then since f is C^{m+1} and m-tangent to the identity function it can only fail to be a quasi-contraction in one respect, namely that there exist points $\underline{x}_i \in R^n$ with $\underline{x}_i \to \underline{0}$ as $i \to \infty$ such that

$$\|f(\underline{x}_i)\| > \|\underline{x}_i\| - \|\underline{x}_i\|^{m+1}/i \ .$$

Let $\underline{y}_i = \underline{x}_i / \|\underline{x}_i\|$.

These points accumulate on the unit sphere, so in that direction $D^{m+1}f(\underline{0})$ is trivial, and so f is (m+1)-tangent to a function which has fixed points on a line through $\underline{0}$.

Theorem 2.8 may be generalised as follows: (proof not given)

Theorem 2.9

A C^1 diffeomorphism f: $R^n, \underline{0} \to R^n, \underline{0}$ is a quasi-contraction of degree m on X if and only if there exists a C^∞ diffeomorphism g: $R^n, \underline{0} \to R^n, \underline{0}$ (called a 'gauge' for f, such that

i) g is a quasi-contraction of degree m on X,

ii) for some strictly positive real number r and some integer n, one has for all $\underline{x} \in X_r$, $f(\underline{x}) \in X$,
$\|g^n(\underline{x})\| \leqslant \|f(\underline{x})\| \leqslant \|g(\underline{x})\|$, & for all $\underline{x} \in R^n$ with $\|\underline{x}\| \leqslant r$,
$\|Df(\underline{x})^{-1}\| \leqslant \|Dg^n(\underline{x})^{-1}\|$.

The above two results characterise the quasi-contractions of a given degree on R^n. For more complicated invariant sets the characterisation becomes more difficult. The following standard examples will prove useful:

Definition 2.10

Let p be an n-tuple of homogenous polynomials of degree m+1 in n variables.

Then p is in standard form if and only if for every pair of integers i,j = 1...n with $i \neq j$ the $x_1^m x_i$ coefficient of the j-th component of p is zero.

Given such a polynomial, for each integer i = 1...n, denote by r_i the $x_1^m x_i$ coefficient of the i-th component of p.

Let $f: R^n, \underline{0} \rightarrow R^n, \underline{0}$ be a C^{m+1} diffeomorphism with (m+1)-jet equal to the identity minus p.

Then f is a standard quasi-contraction of degree m about the x_1-axis if and only if $0 < r_1 < r_2 \leqslant r_3 \ldots \leqslant r_n$.

Note that the derivative at $\underline{0}$ of the m-th partial derivative with respect to the first variable is a diagonal linear map with entries along the diagonal of m! times $(m+1).r_1, r_2, r_3, \ldots r_n$. This fact will be used later. First, however, the aim is to show that these standard quasi-contractions are indeed quasi-contractions. Later it will be shown that they represent a large number of germs. The first step, then, is to find f-invariant sets. Such sets will be described as follows:

Definition 2.11

Let r be a strictly positive real number. Let $l_2,\dots l_n$ be strictly positive real numbers, and let $l_{n+2},\dots l_{2n}$ be strictly negative real numbers.

The pyramid associated with $r, l_2,\dots l_n, l_{n+2},\dots l_{2n}$ is

$$X = \left\{ (x_1,\dots,x_n) \in R^n \;\middle|\; 0 \leqslant x_1 \leqslant r \text{ and, for } i=2\dots n, \; x_1 l_{n+i} \leqslant x_i \leqslant x_1 l_i \right\} .$$

It has one vertex at $\underline{0}$ and the others at the points $(r, rl_2,\dots,rl_n)$, $(r, rl_2,\dots,rl_{n-1}, rl_{2n})$ etc.

One may now state the following result:

Theorem 2.12

Let $f: R^n,\underline{0} \to R^n,\underline{0}$ be a standard quasi-contraction of degree m about the x_1-axis. Let $g: R^n,\underline{0} \to R^n,\underline{0}$ be a C^∞ diffeomorphism, m-tangent to f at $\underline{0}$. If $g^{(m+1)}(\underline{0})$ is sufficiently close to $f^{(m+1)}(\underline{0})$ then for $r, l_2,\dots l_n$, $l_{n+2},\dots l_{2n}$ sufficiently small, the associated pyramid, X, is mapped onto itself by g, and g is a quasi-contraction of degree m on the pyramid X.

Proof

It will be shown that g maps X formally into a proper subset of itself, and this dominates the higher order terms.

Let X be as in 2.11, for some $r, l_2,\dots l_n, l_{n+2},\dots l_{2n}$. Consider $\underline{x} = (x_1, x_2,\dots x_n) \neq \underline{0}$ on the boundary of X. Then for each integer $i=1\dots n$ one has the decomposition

$$g(\underline{x})_i = x_i - r' x_1 x_i^m + p_i(\underline{x}) + q_i(\underline{x}) + r_i(\underline{x}) , \qquad (A)$$

where p_i, q_i are homogenous polynomials of degree m+1, p_i has no terms of the form $x_1^m x_j$, q_i has only $x_1^m x_j$ terms (with $j \neq i$), and the function r_i' is (m+1)-flat at $\underline{0}$.

Now, for $g^{(m+1)}(\underline{0})$ sufficiently close to $f^{(m+1)}(\underline{0})$,

$$0 < r_1' < r_j' \quad \text{for } j=2\dots n ,$$

and every term of each p_i has a factor $x_j x_k$ with $j,k \neq 1$.

Let $L = \max_{i=2\ldots n} \{ l_i, -l_{i+n} \}$. Then there exists a constant K such that, for r and L sufficiently small,

$$|p_i(\underline{x}) + r_i'(\underline{x})| \leqslant K.L^2|x_1|^{m+1} .$$

Moreover, for each real $e > 0$, for $g^{(m+1)}(\underline{O})$ sufficiently close to $f^{(m+1)}(\underline{O})$ and $\|x\|$ sufficiently small, one has

$$|q_i(\underline{x})| \leqslant eL|x_1|^{m+1} .$$

Thus, under the above conditions, (A) gives the inequalities

$$0 < g(\underline{x})_1 < x_1 \ \& \ g(\underline{x})_1 l_{n+i} < g(\underline{x})_i < g(\underline{x})_1 l_i \text{ for } i=2\ldots n.$$

In other words, the boundary of X minus $\underline{O}$ is mapped into the interior of X by g. But g is a homeomorphism, so the whole of X must be mapped into itself by g, as required.

By taking e, L sufficiently small it can be seen that the constants a,b of 2.2 exist and can be made to be arbitrarily close to r_1'. By proceeding as above, but using Dg^{-1}, it can be shown that c of 2.2 can be taken to be arbitrarily close to $\max\{(m+1)r_1', r_n'\}$.

Note: $a \geqslant r_1' \geqslant b$ and $c \geqslant \max\{(m+1)r_1', r_n'\}$.

Example 2.13

Let $f: R^2,\underline{O} \rightarrow R^2,\underline{O}$ be defined by

$$(x,y) \mapsto (x - x^2, y - 2xy).$$

This is a standard quasi-contraction of degree 1 about the x-axis. However f has fixed points along the entire y-axis, so cannot be a quasi-contraction on any open set containing $\underline{O}$. Also, Df is singular at the point $(\frac{1}{2},0)$ on the x-axis, so one has restrictions on the set on which f is a quasi-contraction.

For standard quasi-contractions it does not matter which norm is used, as the pyramids of 2.11 may be taken to be so narrow that the norm on the x_1-axis dominates the total norm, and there is essentially only one norm on R^1.

More generally, however, the following example shows that the choice of norm can be critical.

Example 2.14

Let $f: R^n,0 \to R^n,0$ be given by the formula

$$(x,y) \mapsto (x - 2.xy^2 - x^3 , y + 2.x^2y - y^3) .$$

Then using the usual Euclidean norm given by

$$\|(x,y)\| = (x^2+y^2)^{\frac{1}{2}} ,$$

one has the Taylor expansion

$$\|f(x,y)\|^2 = x^2+y^2 -2.x.(2.xy^2+x^3) +2.y.(2.x^2y-y^3) +h.o.ts.,$$
$$= \|(x,y)\|^2 - 2.(x^4+y^4) + \text{higher order terms}.$$

Thus f is a quasi-contraction of degree 2 on R^2.

However if one uses the supremum norm given by

$$\|(x,y)\|_s = \max\{|x|, |y|\} ,$$

then for $0 < x < y < 2^{\frac{1}{2}}.x < 0.1$ one has that

$$\|f(x,y)\|_s = \max\{x.(1-2.y^2-x^2, y.(1+2.x^2-y^2)\} ,$$
$$> y = \|(x,y)\|_s ,$$

and so f is not a quasi-contraction of any degree on R^2 with respect to the supremum norm.

From now on only the Euclidean norm will be used.

In the case studied by Sternberg et al., where the derivative alone forces the germs to be topological contractions one has

$\|Df\| < 1$ near 0. However for quasi-contractions one may have

$\|Df\| > 1$ arbitrarily near 0, as is shown below:

Example 2.15

Let $f: R^n,0 \to R^n,0$ be given by the formula

$$(x,y) \mapsto (x - x^3 , y - y^3 + 1\tfrac{1}{2}.x^2y) .$$

Then $\|f(x,y)\|^2 = x^2+y^2 - 2.(x^4-1\frac{1}{2}.x^2y^2+y^4) + h.o.ts.,$

but $x^4-1\frac{1}{2}.x^2y^2+y^4 = (x^2-y^2)^2 + \frac{1}{2}.x^2y^2 > 0$ unless $x=y=0$,

so f is a quasi-contraction of degree 2 on R^2.

However the following calculation shows that

$\|Df(x,y)\| > 1$ for x,y arbitrarily small:

Let $d_2f(\underline{x})_2$ represent the derivative of the second co-ordinate of f with respect to the second variable. Then

$$d_2f(x,y)_2 = 1 - 3.y^2 + 1\tfrac{1}{2}.x^2$$
$$= 1 + 3.y^2 \quad \text{for} \quad x = 2.y \ ,$$

so $\quad \|Df(x,y)\| \geqslant 1 + 3.y^2 > 1 \quad$ for $\quad x = 2.y \ , \ y \neq 0$.

As here, any C^∞ diffeomorphism which is m+1-tangent to the identity at $\underline{0}$ must have a derivative which is m-tangent to the identity at $\underline{0}$. However this is not the case for C^1 diffeomorphisms, as is shown below:

<u>Example 2.16</u>

Let $f: R,0 \to R,0$ be given by the formula

$$f: x \mapsto x + x^3.\cos(1/x) \ .$$

Then $\quad Df: x \mapsto 1 + 3.x^2.\cos(1/x) + x.\sin(1/x)$.

Hence f is a C^1 diffeomorphism near 0 and is 2-tangent to the identity at 0, but Df is <u>not</u> 1-tangent to the identity at 0, but only 0-tangent.

To unify the treatment from now on definition 2.7 is modified as follows:

<u>Definition 2.17</u>

Let k be a strictly positive integer and let $X \subseteq R^n$. Then C^1 functions $f,g: R^n,\underline{0} \to R^n,\underline{0}$ are <u>k-tangent at $\underline{0}$</u> if and only if

$$\lim_{r>0} \sup_{\underline{x} \in X_r} \{\|Df(\underline{x})-Dg(\underline{x})\| / \|\underline{x}\|^k\} < \infty \ .$$

By the M.V.T., if $\underline{0}$ is a star-centre of some X_r then any pair of functions satisfying this definition also satisfy 2.7.

In Theorem 2.12 it was shown that perturbations of standard quasi-contractions are quasi-contractions. Example 2.3 shows that perturbations of more general quasi-contractions need not be quasi-contractions. However, as the following result shows, if the invariant map is formally mapped properly into itself, then perturbations of the quasi-contraction are also quasi-contractions.

Theorem 2.18

Let $f,g: R^n,\underline{O} \longrightarrow R^n,\underline{O}$ be C^1 diffeomorphisms which preserve some subset $X \subsetneq R^n$. (In particular, if $f(X) \subsetneq X$ and f,g agree outside X then $g(X) \subsetneq X$ also.)

Suppose that f is a quasi-contraction of degree m on X, g is m-tangent to f at $\underline{O}$ and $g^{(m+1)}(\underline{O})$ is sufficiently close to $f^{(m+1)}(\underline{O})$. Then g is a quasi-contraction of degree m on X.

Moreover, by taking $g^{(m+1)}(\underline{O})$ sufficiently close to $f^{(m+1)}(\underline{O})$, the characteristic constants a, b, c of definition 2.2 may be chosen for g to be arbitrarily close to their values for f.

Proof

For each $e > 0$, for $\|\underline{x}\|$ sufficiently small, and $g^{(m+1)}(\underline{O})$ sufficiently close to $f^{(m+1)}(\underline{O})$, one has

$$\|g(\underline{x})-f(\underline{x})\| \leqslant e\|\underline{x}\|^{m+1} ,$$

so $$\|\underline{x}\| - (a+e)\|\underline{x}\|^{m+1} \leqslant \|g(\underline{x})\| \leqslant \|\underline{x}\| - (b-e)\|\underline{x}\|^{m+1} .$$

Let $h = g-f$ and $k = g^{-1}-f^{-1}$. Then one has the following:

$$id = Dg_{\circ}Dg^{-1} = (Df+Dh)_{\circ}(Df^{-1}+Dk)$$

$$= id + (Df+Dh)_{\circ}Dk + Dh_{\circ}Df^{-1} . \qquad (A)$$

Now, Df^{-1} is less than 2 near $\underline{O}$, so for $\|\underline{x}\|$ sufficiently small,

$$\inf_{\underline{y}\in R^n:\|\underline{y}\|=1} \{\|Df(\underline{x})(\underline{y})\|\} \geqslant \tfrac{1}{2} , \text{ since}$$
$$\|Df(\underline{x})(\underline{y})\|\cdot\|Df(\underline{x})^{-1}(\underline{y})\| \geqslant \|\underline{y}\|^2 = 1 . \qquad (B)$$

Hence $\|(Df+Dh)(\underline{x})(\underline{y})\| \geqslant \tfrac{1}{4}\|\underline{y}\|$ near $\underline{O}$, and so

$$\|Dk(\underline{x})\| \leqslant 8\|Dh(\underline{x})\| \text{ by (A).}$$

But for all real numbers $e, > 0$, for $g^{(m+1)}(\underline{0})$ sufficiently close to $f^{(m+1)}(\underline{0})$ and $\|\underline{x}\|$ sufficiently small,

$$\|Dh(\underline{x})\| \leq (e/8).\|\underline{x}\|^m ,$$

so

$$\|Dg^{-1}(\underline{x})\| \leq \|Df^{-1}(\underline{x})\| + \|Dh(\underline{x})\|$$
$$\leq 1 + (c+e)\|\underline{x}\|^m , \qquad \text{as required.}$$

Note that from (B) above,

$$\|Df(\underline{x})(\underline{y})\| \geq \|\underline{y}\|^2/\|Df(\underline{x})^{-1}(\underline{y})\| \qquad \text{for all } \underline{y} ,$$

but from 2.2

$$\|Df(\underline{x})^{-1}\| \leq 1 + c.\|\underline{x}\|^m ,$$

so

$$\|Df(\underline{x})(\underline{y})\| \geq \|\underline{y}\|/(1+c.\|\underline{x}\|^m) ,$$

whence for $\|x\|$ sufficiently small, $g^{(m+1)}(\underline{0})$ sufficiently close to $f^{(m+1)}(\underline{0})$, one has

$$\|Df(\underline{x})\| \geq 1 - (c+e).\|\underline{x}\|^m .$$

<u>Lemma 2.19</u>

Let $f: R^n,\underline{0} \rightarrow R^n,\underline{0}$ be a C^1 quasi-contraction of degree m on some subset $X \subseteq R^n$.

Given a positive integer $i > m$, a strictly positive real number s and a point $\underline{x} \in R^n$,

let $U(i,s,\underline{x}) = \{ \underline{y} \in R^n \mid \|\underline{x}-\underline{y}\| \leq s\|\underline{x}\|^{i-m} \}$.

Let $g: R^n,\underline{0} \rightarrow R^n,\underline{0}$ be a C^1 diffeomorphism which is k-tangent to f at $\underline{0}$ and equal to f outside X. Let a,b,c be characteristic constants for f, as in 2.2.

Then for both

i) $i = k > m+c/b$ & $\|\underline{x}\|$, $\|\underline{x}\|/s$ sufficiently small

and ii) $i = k+1 > m+c/b$ & $\|\underline{x}\|$, $1/s$ sufficiently small,

one has

$$U(i,s,f(\underline{x})) \subset g(U(i,s,\underline{x})) .$$

Proof

Let $z \in U(i,s,f(\underline{x}))$. Then, by the Mean Value Theorem, for some strictly positive real number e and some point $\underline{u}$ between $\underline{z}$ and $g(\underline{x})$,

$$\|g^{-1}(\underline{z})-\underline{x}\| \leqslant \|Dg^{-1}(\underline{u})\|.\{\|\underline{z}-f(\underline{x})\|+\|f(\underline{x})-g(\underline{x})\|\}$$
$$\leqslant \|Dg(g^{-1}(\underline{u}))^{-1}\|.\{s.\|f(\underline{x})\|^{i-m}+e.\|\underline{x}\|^{k+1}\}.$$

Thus for any real number $c' > c$, for $\|\underline{x}\|$ sufficiently small,

$$\|g^{-1}(\underline{z})-\underline{x}\| \leqslant s.(1+c'.\|\underline{x}\|^{m}).\{(\|\underline{x}\|-b.\|\underline{x}\|^{m+1})^{1-m}+e.\|\underline{x}\|^{k}(\|\underline{x}\|/s)\}$$

Hence for either of the two conditions (i),(ii),

$$\|g^{-1}(\underline{z})-\underline{x}\| < s.\|\underline{x}\|^{i-m},$$

as required.

N.B. In the above application of the Mean Value Theorem Dg is controlled on the whole of R^{n}, so $\underline{u}$ need not lie in X. Consequently it is not necessary to suppose, for example, that X is convex or even star-shaped.

3 Quasi-Contractions on Invariant Sets are Flat-Stable on those Sets.

The notion of flat-stability goes back to Sternberg [9]. The notation $\bar{g}$ is used for the germ at $\underline{0}$ of a function $g: R^n,\underline{0} \to R^n,\underline{0}$. The conventional idea of flat-stability is as follows:

Definition 3.1

Let $f: R^n,\underline{0} \to R^n,\underline{0}$ be a C^∞ diffeomorphism.

Then f is flat-stable if and only if for every C^∞ diffeomorphism $g: R^n,\underline{0} \to R^n,\underline{0}$ which is infinitely tangent to f at $\underline{0}$ there exists a C^∞ diffeomorphism $h: R^n,\underline{0} \to R^n,\underline{0}$, infinitely tangent to the identity at $\underline{0}$, such that

$$gh = hf .$$

Takens [11] proved that quasi-contractions are flat-stable for the special case when the number of variables, 'n', is one. Belickiǐ [2] has recently proved that quasi-contractions on R^n are flat-stable. The notion of a 'local' quasi-contraction is too general for the above definition to apply: an appropriately 'local' definition is required. This involves the following concept:

Definition 3.2

Given X, $Y \subseteq R^n$, Y is a core of X if and only if both

a) for each positive real number r, Y_r is non-empty (that is, Y has a non-trivial germ), and

b) there exists an integer s and strictly positive real numbers e and r such that for all $\underline{y} \in Y_r$
$$\{\underline{x} \in R^n \mid \|\underline{x}-\underline{y}\| \leqslant e\|\underline{y}\|^s\} \subseteq X ,$$

A set $X \subseteq R^n$ is significant if and only if it has a core.

Note that any set which contains a significant set is itself significant, and every significant set contains $\underline{0}$ in its closure. Conversely, in the case where n = 1 any interval which contains 0 in its closure is itself significant. Also the 'pyramids' of 2.11 are significant. The local definition of flat-stability can now be given.

<u>Definition 3.3</u>

Let $f: R^n, \underline{0} \to R^n, \underline{0}$ be a C^∞ diffeomorphism. Let $X \subseteq R^n$ be significant. Suppose that for each core Y_r of X with $Y_r \subset f(Y)$ for some $r > 0$* and each C^∞ diffeomorphism $g: R^n, \underline{0} \to R^n, \underline{0}$ which is infinitely tangent to f at $\underline{0}$ there exists a C^∞ diffeomorphism $h: R^n, \underline{0} \to R^n, \underline{0}$, infinitely tangent to the identity at $\underline{0}$, and a strictly positive real number r such that for all $y \in Y_r$,

$$gh(y) = hf(y) .$$

Then f is <u>flat-stable on X</u>.†

The following theorem may now be stated:

<u>Theorem 3.4</u>

Let $f: R^n, \underline{0} \to R^n, \underline{0}$ be a C^∞ quasi-contraction of finite degree on $X \subseteq R^n$, with Df (m-1)-tangent to the identity on X. If X is significant then f is flat-stable on X.

In particular this result applies to Takens' 1-dimensional maps, Belickiǐ's 'global' quasi-contractions and, a new result, the standard quasi-contractions of 2.10.

The following illustrates why only significant sets are considered:

<u>Example 3.5</u>

Let $f: R^2, \underline{0} \to R^2, \underline{0}$ be given by

$$(x,y) \mapsto (x - x^2, y) .$$

This is a quasi-contraction of degree 1 on the x-axis, but it is not flat-stable in any meaningful sense.

*These exist by 2.19.

†Said to be <u>uniquely</u> flat-stable when the diffeo. h is unique.

The proof of Theorem 3.4 follows Beličkii's lead. Another type of stability is used to link Beličkii's rigid notion with the more general notion introduced above.

Definition 3.6

Let J_0 be the space of germs of C^∞ maps $R^n,\underline{0} \to R^n,\underline{0}$ which are infinitely flat at $\underline{0}$. Let $J^1 \subseteq J_0$.

Then a map $f: R^n,\underline{0} \to R^n,\underline{0}$ is J^1-stable if and only if for all germs $h \in J^1$ there exists a germ $k \in J^1$ such that

$$(id+k)\circ f = (f+h)\circ(id+k) \ .$$

Beličkii considers only J_0-stability. The case of interest will be when for some subset $Z \subseteq R^n$

$$J^1 = \{ j \in J_0 \mid \text{for all } \underline{z} \in Z,\ j(\underline{z}) = 0 \} ,$$

in which case J^1 will be denoted by $J(Z)$.

For the usual topology (see Appendix 1), $J(Z)$ is complete. It is also closed under the following: subtraction; multiplication by C^∞ maps; and the maps induced by composition and the taking of inverses in $\{id\} + J^1$. These are vital to Beličkii's method. Any set $J^1 \subseteq J_0$ satisfying these conditions will be said to be 'suitable', and only such sets will be considered. Details are in Appendix 1.

Theorem 3.7

Let $f: R^n,\underline{0} \to R^n,\underline{0}$ be a quasi-contraction of degree m on a set $X \subseteq R^n$, with Df (m-1)-tangent to the identity on X. Suppose $f(Z) \subseteq Z$ and $Z \cup X = R^n$. Then f is a J(Z)-quasi-contraction of degree m (in the sense of A1.10), and hence uniquely J(Z)-stable (in the sense of 3.6).

The proof utilises A1.11, but mainly consists of showing that A1.3, a modification of a result of Beličkii's, may be applied.

Theorem A1.14 shows how a diffeomorphism may be proven to be stable in the sense of 3.7 by dividing the space of perturbations into subspaces, and dealing with them seperately. The same cannot be done for type of stability of 3.3, as will be shown in section 6.

Proof of Theorem 3.4

Let Y be a core of X, with e,s,r as in definition 3.2. Then for each point $\underline{y} \in Y_r$, let

$$U(\underline{y}) = \{ \underline{x} \in R^n \mid \| \underline{x}-\underline{y} \| \leq \tfrac{1}{2} e . \| \underline{y} \|^s \} \ .$$

Then for any sufficiently small r, each subset $U(\underline{y})$ is also a core of X (with $e' = \frac{1}{2}e$, say.)

Now , for r' sufficiently small,

$$Y_{r'} \subset f(Y) \ ,$$

and for all $\underline{y} \in Y_{r'}$, one has

$$f^{-1}(\underline{y}) \in Y_r \ ,$$

where r is as above.

Hence by Lemma 2.19, for s sufficiently large and r sufficiently small,

$$U(\underline{y}) \subset f(U(f^{-1}(\underline{y}))) \ ,$$

Let $Y^* = \bigcup_{\underline{y} \in Y_r} U(\underline{y})$.

Then for some real number $r^* > 0$,

$$Y^*_{r^*} \subset f(Y^*) \ ,$$

and Y* is a core of X, since the choice of e' above is independent of $\underline{y}$.

Now, take

$$Z = R^n \setminus \bigcup_{n \in N, n \geq 0} f^{-n}(Y^*) \quad ,$$

$$X^* = X \cup \bigcup_{n \in N} f^n(Y^*) \quad .$$

Then X* and X agree near $\underline{0}$, so f is a quasi-contraction of X*. Moreover $Z \cup X^* = R^n$ and $f(Z) \subset Z$, so Theorem 3.7 applies.

Let $B: R^n, \underline{O} \longrightarrow R^1, O$ be C^∞ on $R^n - \{\underline{O}\}$ with

$$B = \begin{cases} 0 & \text{on } Y^*_{\frac{1}{2}r} \\ 1 & \text{on } Z , \end{cases}$$

and with bounded derivatives. This is possible since Y^* is a core of $R^n \setminus Z$.

Given a C^∞ diffeomorphism $g: R^n, \underline{O} \rightarrow R^n, \underline{O}$ which is infinitely tangent to f on X, let

$$g' = B.f + (1-B).g .$$

Then g' is C^∞, equals g on $Y^*_{\frac{1}{2}r}$, and equals f on Z_r . Hence by Theorem 3.7 there exists a C^∞ function $h: R^n, \underline{O} \rightarrow R^n, \underline{O}$, equal to the identity outside Z_r, such that

$$h_\circ g' = f_\circ h \qquad \text{near } \underline{O} \text{ , and so}$$

$$h_\circ g = f_\circ h \quad \text{on } Y \text{ near } \underline{O} \text{ , as required.}$$

To complete the proof of Theorem 3.7 it suffices to prove:

Lemma 3.8

Let $f: R^n, \underline{O} \mapsto R^n, \underline{O}$ be a C^∞ quasi-contraction of degree m on X, with Df (m-1)-tangent to the identity on X. Let $Z \subseteq R^n$ with $f(Z) \subseteq Z$ and $Z \cup X = R^n$. Let the operator $T: J(Z) \rightarrow J(Z)$ be given by the following formula:

$$h \mapsto -(Df)^{-1}.h_\circ f .$$

Then there exists a real number c such that

for all non-negative integers i and all sufficiently large integers k there exists an integer k' such that

for all sufficiently small strictly positive real numbers e there exists a real number d such that

for all integers M and all $h \in J_e(Z)^*$,

$$\sum_{j=0}^{M} \|T^j h\|^i_k \leqslant c.\|h\|^i_{k+m} + i.d.\|h\|^{i-1}_{k'} .$$

This is sufficient because by A1.3 the map ID+T has an inverse, L, and so the map of A1.10 given by

*See Appendix 1 for the definition.

$$Df.(id+T): h \mapsto Df.h - h_\circ f$$

has an inverse, $(Df)^{-1}.L$. Thus f is a J(Z)-quasi-contraction.

Proof of Lemma 3.8 (after A.Masson's thesis [7].)

The multi-index notation of Theorem A3.2 is used. Also, given $\underline{x}$, $A_j = \prod_{q=0}^{j-1} \|Df(f^q(\underline{x}))^{-1}\|$, $D_j^i = \prod_{q=0}^{j-1} \|Df(f^q(\underline{x}))\|^i$, $H_j^i = \max_{0 \leq q \leq i} \{\|h^{(q)}(f^j(\underline{x}))\|\}$, and for increasing sequences of strictly positive reals, $a_0, a_1, \ldots$, $P_j^i = \prod_{q=0}^{j-1} a_{i-q}$, $X_{j,1}^i = H_1^i + \sum_{p=1}^{j} C_p^j . P_p^i . H_1^{i-p}$. Note that these increase with i.

Also, $A_1 . A_{j\circ} f = A_{j+1}$, $D_1^i . D_{j\circ}^i f = D_{j+1}^i$, $H_{j\circ}^i f = H_{j+1}^i$, and $a_i P_j^{i-1} = P_{j+1}^i$.

By Leibniz's Formula (A3.1) and the Chain Rule (A3.2),

$$(Th)^{(i)} = -A_1^{(i)} . h_\circ f - \sum_{p=1}^{i} C_p^i A_1^{(i-p)} . \sum_{q=1}^{p} h^{(q)} f . \sum_{\underline{u} \in I_q^p} c_{\underline{u}} f^{\underline{u}}, \text{ for all } i > 0.$$

Df and A_1 are bounded away from $\underline{0}$, and all derivatives of f are bounded, so for some increasing sequence $a_0, a_1, \ldots$, for each i,

$$\|(Th)^{(i)}(\underline{x})\| \leq A_1 . D_1^i (H_1^i + a_i H_1^{i-1}) = A_1 . D_1^i . X_{1,1}^i .$$

This is the case j=1 of the more general inequality below:

$$\|(T^j h)^{(i)}(\underline{x})\| \leq A_j . D_j^i . X_{j,j}^i . \qquad (A)$$

For larger values of j the induction step is as follows:

$$\begin{aligned} \|(T^{j+1}h)^{(i)}(\underline{x})\| &= \|(T(T^j h))^{(i)}(\underline{x})\| \\ &\leq A_1 . D_1^i . A_{j\circ} f . D_{j\circ}^i f . (X_{j,j+1}^i + a_i . \max_{0 \leq q < i} \{X_{j,j+1}^q\}) , \\ &= A_{j+1} . D_{j+1}^i . Y , \text{ where} \end{aligned}$$

$$Y = H_{j+1}^i + a_i H_{j+1}^{i-1} + \sum_{p=2}^{j} (C_p^j + C_{p-1}^j) . P_p^i . H_{j+1}^{i-p} + C_j^j . a_i P_j^{i-1} . H_{j+1}^{i-1-j} + a_i C_1^j . H_{j+1}^{i-1},$$

$$= H_{j+1}^i + a_i C_1^{j+1} . H_{j+1}^{i-1} + \sum_{p=2}^{j} C_p^{j+1} . P_p^i . H_{j+1}^{i-p} + C_{j+1}^{j+1} . P_{j+1}^i . H_{j+1}^{i-(j+1)} ,$$

$$= X_{j+1,j+1}^i ,$$ as required to prove the inequality (A).

Now, let $l(\underline{x}) = A_1 . (\|f(\underline{x})\| / \|\underline{x}\|)^{k+m} . D_1^i$.

Since each $C_p^j \leq j^i$, there exist constants c_i such that

$$\frac{\|(T^j h)^{(i)}(\underline{x})\|}{\|\underline{x}\|^{k+m}} \leq l(\underline{x}) \ldots l(f^{j-1}(\underline{x})) . (H_j^i + c_i j^i H_j^{i-1}) / (\|f^j(\underline{x})\|^{k+m}) . \qquad (B)$$

For $k'=k+(i+1).m$, the right hand side of (B) is bounded by

$$l(\underline{x})\ldots l(f^{j-1}(\underline{x}))\left\{\|h\|^{i}_{k+m}.\|\underline{x}\|^{m}+c_i j^{i}\|h\|^{i-1}_{k'}.\|\underline{x}\|^{k'-m}\right\}.$$

Thus it suffices to prove the following result:

Lemma 3.9

Let i be a positive integer. Let $f: R^n,\underline{0} \longrightarrow R^n,\underline{0}$ be a C^i quasi-contraction of degree m on X. For $i\neq 0$ suppose that Df is (m-1)-tangent to the identity at $\underline{0}$.

For each integer k, let

$$l(\underline{x}) = \|Df(\underline{x})^{-1}\|.(\|f(\underline{x})\|/\|\underline{x}\|)^{k}.\|Df(\underline{x})\|^{i} .$$

Then for k sufficiently large, for all $\underline{x}\in X$, one has

$$\sum_{j=0}^{\infty} l(\underline{x})\ldots l(f^{j-1}(\underline{x})).j^{i} \leq D/\|\underline{x}\|^{(i+1).m}, \text{ for some D.}$$

Moreover, when $i=0$, any $k > c^*+a^*.m$ will do, where c^*, a^* are the greatest lower bounds of c/b, a/b respectively, where a, b, and c are as in Definition 2.2.

Proof

Let a, b, c be as in Definition 2.2. Let d be such that

$\|Df(\underline{x})\| \leq 1+d.\|\underline{x}\|^{m}$, and choose k so that

$$k.b > c+a.m(i+1)+i.d .$$

Then for all $\underline{x}\in X_r$ (with r sufficiently small) one has

$$l(\underline{x}) \leq (1-b\|\underline{x}\|^{m})^{k}.(1+c\|\underline{x}\|^{m}).(1+d\|\underline{x}\|^{m})^{i} .$$

A real number u may now be chosen such that

$$a.m.(i+1) < u < k.b-c-i.d .$$

Let $p(\underline{x}) = 1-u.\|\underline{x}\|^{m}$. Then for r sufficiently small,

$$l(\underline{x}) \leq p(\underline{x}) \quad \text{for all } \underline{x}\in X_r .$$

Let $H_j(\underline{x}) = p(\underline{x})\ldots p(f^{j}(\underline{x}))$ for each positive integer j.

It now suffices to show that for each point $\underline{x}\in X_r$ one has

$$1+\sum_{j=0}^{\infty} H_j(\underline{x}).j^{i} \leq D/\|\underline{x}\|^{(i+1).m} \text{ for some constant D.}$$

Let $t\in(0,1)$. Let $\underline{x}\in X$ with $\|\underline{x}\|=r$. Then for each positive integer p, define the set of integers E_p by the formula

$$E_p = \left\{ q\in N \mid f^{q}(\underline{x}) \in (t^{p}r, t^{p-1}r] \right\},$$

and let C_p be the number of elements in E_p. This division will be used to calculate the sum above.

For all $\underline{y} \in X$ with $\|\underline{y}\| \in (t^p r,\ t^{p-1}]$ one has

$$\|\underline{y} - f(\underline{y})\| > b(t^p r)^{m+1},$$

hence one derives the following upper bounds for C_p:

$$C_p < r(t^{p-1} - tp)/(b(t^p r)^{(m+1)}) + 1$$
$$< (1-t)(t^{-pm-1} r^{-m} b^{-1}) + 1$$
$$\leqslant K/(b t^{pm+1} r^m), \qquad (1)$$

where $K = 1 + br^m$.

Similarly one has the following lower bound for C_p:

$$C_p > (t^{-(p-1)m}(1-t)a^{-1} r^{-m}) - 1 \qquad (2).$$

Hence with $t \in (0,1)$ and r sufficiently small compared with $t(1-t)$, C_p is non-zero. For each positive integer p, let b_p be the largest integer in E_p, let $P_p(\underline{x}) = H_{b_p}(\underline{x})$, and let $P_0(\underline{x}) = 1$. Then

$$P_p(\underline{x}) = P_{p-1}(\underline{x}) . \prod_{i \in E_p} pf^i(\underline{x})$$

$$< P_{p-1}(\underline{x}) . p(t^p \underline{x})^{C_p} \qquad (3).$$

Now, $p(t^p \underline{x}) = (1 - u t^{pm} r^m)$, so by (2)

$$p(t^p \underline{x})^{C_p} \leqslant (\exp(-1))^{((1-t)t^m - ar^m t^{mp})a^{-1}u}, \qquad (4),$$

provided that r is sufficiently small compared with t.

The last inequality uses the fact that

$$(1-z)^{1/z} \rightarrow \exp(-1) \text{ from below as } z \rightarrow 0 \text{ from above.}$$

Let $v = (t^m(t-1) + ar^m)u/(a.\log(t))$. Then for r sufficiently small compared with $t(1-t)$, v is strictly positive and (4) gives

$$p(t^p \underline{x})^{C_p} < t^v.$$

Note that v is independent of p.

By induction on p, the above inequality and (3) lead to

$$P_p(\underline{x}) < t^{pv} . \qquad (5).$$

Note that

$$(t-1)/\log(t) \rightarrow 1 \text{ from above as}$$

$$t \rightarrow 1 \text{ from below.}$$

By taking t sufficiently close to 1 and taking r sufficiently small compared with (1-t) so that $r^m/\log(t)$ is sufficiently small, v may be made arbitrarily close to u/a. Let $E = v - m(i+1)$. Then t and r may be chosen so as to make E strictly positive.

Now $H_p(\underline{x})$ decreases as p increases, so

$$\sum_{j=0}^{\infty} H_j(\underline{x}).j^i \leqslant \sum_{p=1}^{\infty} c_p P_{p-1}(\underline{x}).b_p{}^i \quad .$$

Now from (1)

$$b_p = \sum_{j=1}^{p} c_j \leq \sum_{j=1}^{p} \left(\frac{K}{bt^{pm+1} r^m} \right) \leqslant \frac{C}{r^m} \cdot \frac{1}{t^{mp}} \quad ,$$

for some constant C.

Using (1) and (5) one gets

$$\sum_{j=0}^{\infty} H_j(\underline{x}).j^i \leqslant \left(\frac{K}{br^m t} \cdot \frac{C^i}{r^{im}} \right) \sum_{p=1}^{\infty} t^{-mp} t^{(p-1)v} . t^{-mpi}$$

$$= \left(\frac{KC^i}{br^{m(i+1)} t^{m(i+1)+1}} \right) \sum_{p=1}^{\infty} (t^E)^{(p-1)}$$

$$= \left(\frac{KC^i}{bt^{m(i+1)+1}} \right) \cdot \frac{1}{\|\underline{x}\|^{m(i+1)}} \cdot \frac{1}{1-t^E} \quad ,$$

since $E > 0$ and $0 < t < 1$.

This is the inequality required, and completes the proof of theorem 3.7.

4 Finite Stability.

In the previous sections the main motivation was to study the structure of G, the group of germs of C^{∞} diffeomorphisms $R^n, \underline{0} \to R^n, \underline{0}$. Later some applications of this theory will be made, and it will be clear that the notion of flat stability is at times too restrictive. Accordingly the next concept is introduced:

Definition 4.1

Given positive integers k, m and a set $X \subseteq R^n$, a C^1 diffeomorphism $f: R^n, \underline{0} \to R^n, \underline{0}$ is said to be (k,m)-stable on X if and only if for all C^1 diffeomorphisms $g: R^n, \underline{0} \to R^n, \underline{0}$ which are k-tangent to f at $\underline{0}$ and equal to f outside X there exists a C^0 function $h: R^n, \underline{0} \to R^n, \underline{0}$ (k-m)-tangent to the identity at $\underline{0}$ and equal to the identity outside X such that

$$gh = hf \quad \text{and}$$

$$fh^{-1} = h^{-1}g \quad \text{near } \underline{0}.$$

As in the previous section the quasi-contractions in particular display this type of stability, but this time the proof is elementary.

In order to make precise the dependence of the permissible k values of the parameters of the quasi-contraction the following notation is introduced:

Definition 4.2

Given a quasi-contraction $f: R^n, \underline{0} \to R^n, \underline{0}$ of degree m on $X \subseteq R^n$, let a*, c* be the greatest lower bounds for the numbers a/b, c/b of 2.2.

Then (a*,c*) is called the characteristic of f.

Note that a* is at least 1.

The main result may now be stated:

Theorem 4.3

Given a strictly positive integer, m, and an open set $X \subseteq R^n$, let $f: R^n, \underline{0} \to R^n, \underline{0}$ be a quasi-contraction of degree m and characteristic (a^*, c^*) on X, with Df m-tangent to the identity at $\underline{0}$.

Then for each point $\underline{x} \in X$ there exists an open subset $X' \subsetneq X$ with $\underline{x} \in X'$ such that $f(X') = X'$ near $\underline{0}$.

Moreover, for all integers $k > m.a^* + c^*$, the diffeomorphism f is (k,m)-stable on all such open sets.

Proof

Firstly, given $\underline{x} \in X$, let $U \subsetneq X$ be a neighbourhood of $\underline{x}$. Then the set $X' = \bigcup_{i=0}^{\infty} f^i(U) \subseteq X$ is open, invariant under f and contains $\underline{x}$, as required.

Secondly, let $g: R^n, \underline{0} \to R^n, \underline{0}$ be a C^1 diffeomorphism which is k-tangent to f at $\underline{0}$ and equal to f outside X'. Then $X' = g(X')$ and q is a quasi-contraction of degree m and characteristic (a^*, c^*) on both X and X'.

For each positive integer i and point $\underline{x} \in R^n$, let

$$\underline{x}_i = f^i(\underline{x}) .$$

Notice that if $\underline{x}_{i+1} \in X'$ then $\underline{x}_i \in X'$ and $\|\underline{x}_{i+1}\| \leq \|\underline{x}_i\|$.

Now, g is k-tangent to f, so for any strictly positive real number e, any $\underline{x} \in X$, and for any sufficiently large i,

$$\|f(\underline{x}_i) - g(\underline{x}_i)\| \leq e.\|\underline{x}_i\|^k .$$

By the Mean Value Theorem, for some point $\underline{u}$ between $f(\underline{x}_i)$ and $g(\underline{x}_i)$, not necessarily in X,

$$\|g^{-i-1}f^{i+1}(\underline{x}) - g^{-i}f^i(\underline{x})\| \leq \|Dg^{-i-1}(\underline{u})\|.\|f(\underline{x}_i) - g(\underline{x}_i)\|$$
$$\leq \|Dg^{i+1}(\underline{y})^{-1}\|.e.\|\underline{x}_i\|^k ,$$

where $\underline{y} = g^{-i-1}(\underline{u})$.

By Lemma 2.19, for s sufficiently large and $\|\underline{x}\|$ sufficiently small,

$$\|\underline{y} - \underline{x}\| \leq s.\|\underline{x}\|^{k-m} .$$

On the other hand, if $\underline{x} \notin X$ then g and f agree at $\underline{x}_i$, so

$$\|g^{-i-1}f^{i+1}(\underline{x})-g^{-i}f^{i}(\underline{x})\| = 0 .$$

Hence, in either case, one has by Theorem 2.18 and Lemma 3.9 that for some constant D, independent of $\underline{x}$, for all positive integers i, j, with i sufficiently large,

$$\|g^{-i-j}f^{i+j}(\underline{x})-g^{-i}f^{i}(\underline{x})\|$$

$$\leqslant (D/(\|\underline{x}\|-s.\|\underline{x}\|^{k-m})^{m}).e.(\|\underline{x}\|+s.\|\underline{x}\|^{k-m})^{k}.M^{k} \quad , \quad \text{(A)}$$

where $M = \sup_{i:\underline{x}_i \in X'} \{(\|\underline{x}_i\|/(\|\underline{x}_i\|-s.\|\underline{x}_i\|^{k-m}))\}$ is finite for

$\|\underline{x}\|$ sufficiently small, depending on s.

Hence the sequence $\{g^{-i}f^{i}(\underline{x})\}_{i \in N}$ is Cauchy for all $\underline{x} \in R^n$ sufficiently close to $\underline{0}$. Let $h(\underline{x})$ denote the limit. Then h is the identity outside X, since f and g agree there.

From (A) it is clear that the function h is (k-m)-tangent to the identity at $\underline{0}$ (in the sense for C^0 functions), and in particular is continuous at $\underline{0}$.

It also follows from (A) that for $\|\underline{x}\|$ sufficiently small, for each $e>0$, for all sufficiently large integers i,

$$\|g^{-i}f^{i}(\underline{x})-h(\underline{x})\| \leqslant \tfrac{1}{3}e .$$

Each function $g^{-i}f^{i}$ is uniformly continuous, so for $\underline{y}$ sufficiently close to $\underline{x}$,

$$\|g^{-i}f^{i}(\underline{y})-g^{-i}f^{i}(\underline{x})\| \leqslant \tfrac{1}{3}e .$$

Now by the triangle inequality,

$$\|h(\underline{y})-h(\underline{x})\| \leqslant e .$$

Thus h is continuous away from $\underline{0}$ also.

By swopping the functions f and g in the above calculations one obtains a continuous function h' which is easily seen to be the inverse of h. These two functions satisfy the equalities of Definition 4.1.

This completes the proof of Theorem 4.3.

In the next section it will be shown that the conjugating germ, h, is unique. Consequently whenever the two quasi-contractions, f and g, are C^{∞} and infinitely tangent to each other, the germ of the diffeomorphism h, constructed in this section as a limit, is the same as was shown in 3.4 to exist as a fixed point of a map.

The methods used in the proof of theorem 3.4 can be used to show that quasi-contractions also have a type of stability analogous to that of 3.4, namely that for perturbations which are only given to be 'small' on some pyramid-like invariant set, there exist conjugating germs, h, defined on 'cores' of the invariant set.

5 Quasi-Contractions have Small Centralisers.

In section 3 it was shown that C^{∞} quasi-contractions are not only flat stable but uniquely flat stable: their centralisers are small in the sense that they only contain one germ (the identity) whose Taylor series is the identity. Here a similar result appropriate to the finitely differentiable diffeomorphisms of the previous section will be proven which will also give a stronger result for the C^{∞} case. First the notion of a 'small centraliser' is made explicit:

Definition 5.1

The centraliser of a function $f: R^n,0 \rightarrow R^n,0$ is said to be small of order l on X if and only if all functions $g: R^n,0 \rightarrow R^n,0$ satisfying

i) g is the identity outside X,

ii) g is l-tangent to the identity, and

iii) g commutes with f near 0,

are the identity near 0.

The main result may now be stated.

Theorem 5.2

Let $f: R^n,0 \rightarrow R^n,0$ be a quasi-contraction of degree m and characteristic (a^*,c^*) on $X \subseteq R^n$.

Then for any integer $k > ma^* + c^*$, the centraliser of f is small of order $k-(m+1)$ on X.

Since f is m-tangent to the identity and commutes with f, the centraliser of f is not small of order m. In one dimension the above theorem asserts that the centraliser of a C^{m+1} function f is small of order m+1 so in this case the result is the best possible. Before proving it consider the following:

Example 5.3.

Let $f: R^n, \underline{O} \to R^n, \underline{O}$ be the standard quasi-contraction about the x_1-axis (see 2.10) given by the following formula:

$$(x_1, \ldots x_n) \mapsto (x_1(1-x_1^m)^{r_1}, \ldots x_n(1-x_1^m)^{r_n}),$$

where $0 < r_1 \leq r_2 \leq \ldots \leq r_n$.

Suppose that the quotient r_n/r_1 is an integer. Then the given function f commutes with the function given by:

$$(x_1, \ldots x_n) \mapsto (x_1, \ldots x_{n-1}, x_n + x_1^{r_n/r_1}) .$$

This latter function is a C^∞ diffeomorphism near $\underline{O}$, and is (r_n/r_1-1)-tangent to the identity at $\underline{O}$, and so the centraliser of the given function, f, is not small of order $\max\{m+1, r_n/r_1\}-1$. Theorem 5.2 asserts* that the centraliser is small of order $\max\{m+1, r_n/r_1\}$, and so in this case cannot be bettered.

More generally however the lower bound for k given in 5.2 is not the greatest lower bound.

Proof of 5.2.

Let the diffeomorphism f and the integer k be as in the statement of the theorem. Let $l=k-m$. By A3.5, $l > m+1$. Let $g: R^n, \underline{O} \to R^n, \underline{O}$ be a function which is the identity outside X, l-tangent to the identity, and commutes with f near $\underline{O}$, as in 5.1.

Let $Hg = Df.g - Df.id + f - f_\circ g$.

Then by the Mean Value Theorem, for all $\underline{x} \in R^n$ near $\underline{O}$ there exist a point $\underline{y}$ between $\underline{x}$ and $g(\underline{x})$ such that one has the following:

$$\begin{aligned} \|Hg(\underline{x})\| &\leq \|Df(\underline{x})-Df(\underline{y})\| . \|g(\underline{x})-\underline{x}\|, \\ &\leq L.\|\underline{x}-\underline{y}\|. \|g(\underline{x})-\underline{x}\|, \\ &\leq L.\|g(\underline{x})-\underline{x}\|^2, \end{aligned}$$

where L is a Lipschitz constant for the map Df near $\underline{O}$.

Let $h = g - id$, and

define $Th = Df.h - h_\circ f$, as in A1.10.

Then $Th = Hg$ near $\underline{O}$.

*N.B. $c^* \geq \max(m+1, r_n/r_1)$: See 2.12.

Now, Theorem 3.7 asserts that, under the additional hypothesis that Df is (m-1)-tangent to the identity, f is a Beličkii type quasi-contraction of degree m, and hence the map T restricted to the infinitely flat C^{∞} maps has an inverse set, L, of order m.

Thus: $\|h\|_1^0 = \|LTh\|_1^0 \leqslant c.\|Th\|_k^0$ for some constant c,

$$= c.\|Hg\|_k^0$$

$$\leqslant c.L.\|h\|_1^0.\|h\|_m^0 .$$

In fact, for this limited application of Lemma 3.9 (which is where the extra hypothesis in Theorem 3.7 is used), the perturbation function h need only be C^1 and 1-flat at $\underline{0}$, and Df need not be (m-1)-tangent to the identity at $\underline{0}$.

Now, h is 1-flat, and in particular is (m+1)-flat, so for any strictly positive real number e, one has, close to $\underline{0}$, that

$$\|h\|_m^0 \leqslant e .$$

Hence $\|h\|_1^0 \leqslant \frac{1}{2}.\|h\|_1^0$, (by taking $e = 1/(2cL)$),

and so $\|h\|_1^0 = 0$,

and the commuting function, $g = h+id$, is the identity near $\underline{0}$, as required.

Before stating a few easy corollaries, we need the following:

Notation.

Let G be the group of germs at $\underline{0}$ of C^{∞} diffeomorphisms $R^n,\underline{0} \to R^n,\underline{0}$. For each $f \in G$ let Z(f) be the centraliser of f,

$$Z(f) = \{g \in G \mid g_\circ f = f_\circ g\} .$$

For each integer l, let $G_l \subseteq G$ be the subgroup of G consisting of all those germs whose l-jet is the identity.

One can now state:

Corollary 5.4.

For any $f \in G$ which is the germ of a quasi-contraction of some finite degree on R^n there exists an integer l (as in 5.2) such that $G_l \cap Z(f) = \{id\}$.

Corollary 5.5.

For any diffeomorphism $f \in G$ which is the germ of a standard quasi-contraction of finite degree there exists an integer l (as in Theorem 5.2) such that every l-flat member of the centraliser of f is infinitely flat, i.e.:

$$G_l \cap Z(f) = G_\infty \cap Z(f) \ .$$

This second result follows from the same line of reasoning as Theorem 3.7, showing that each member of the left-hand group is equal to the identity on a 'pyramid' associated with f (see 2.11), and hence infinitely flat on R^n at $\underline{0}$.

In the next section a standard quasi-contraction, f, is constructed for which $G \cap Z(f') = \text{id}$ for any diffeomorphism f' which is infinitely tangent to f. This contrasts with a result of section 7, where $G \cap Z(f')$ is shown to be 'large' for any diffeomorphism f' which is in a specified open set.

6 Diffeomorphism Germs with Discrete Centralisers.

In the previous section it was shown that C^∞ quasi-contractions on R^n have small centralisers, in the sense that each Taylor series which formally commutes with the Taylor series of the given germ can be represented by exactly one germ which commutes with the given germ. (The 'at most one' part follows from 5.2 and the 'at least one' part from 5.7.) In this section a type of germ is constructed which has an even smaller centraliser, consisting only of integer powers of itself. These germs are common in the same sense that quasi-contractions are common: compare 2.14. They provide the first common examples of germs which are <u>not</u> flat stable. That they are not flat stable is a consequence of their having discrete centralisers and the following well known result:

<u>Theorem 6.1</u>.

Given any C^∞ diffeomorphism $f: R^n,\underline{0} \to R^n,\underline{0}$ with $Df(\underline{0})$ equal to the identity, there exists a one parameter group $\{f^t\}_{t\in R}$ of diffeomorphisms $R^n,\underline{0} \to R^n,\underline{0}$ such that f^1 is infinitely tangent to f at $\underline{0}$.

<u>Proof</u>.

By theorem A2.2 the Taylor expansion of f is embedded in a canonical one-parameter group of power series, associated to which is a unique formal vector field on R^n. By a well known theorem of E. Borel* there exists a representative vector field, V, say, for this formal vector field which has time t integrals for all times t. Call these integrals f^t. Then the Taylor expansion of f^1 is the formal time 1 integral of the formal part of V, and hence equal to the Taylor expansion of f, as required.

Before constructing the germs with discrete centralisers the following notation is introduced:

*In his Thesis.

Notation.

Let $f: R^n, \underline{0} \rightarrow R^n, \underline{0}$ be a C^∞ diffeomorphism.

The roots of f are those members of the centraliser of f whose Taylor series lie in the canonical one parameter group of A2.1.

The integral roots of f are those members of the centraliser of f whose Taylor series are integral powers of the Taylor series of f.

In section 7 some 'partly hyperbolic' germs will be given for which the integral roots are not all integral powers. In this section though the aim is to constuct a type of diffeomorphism whose only roots are integral powers. One of the key properties of such germs is as follows:

Definition 6.2.

Given a C^1 diffeomorphism $f: R^n, \underline{0} \rightarrow R^n, \underline{0}$ and a set $U \subseteq R^n$,

let $\quad orb(f,U) = \bigcup_{m\in Z} f^m(U)$ and

$\quad orb^+(f,U) = \bigcup_{m\in N} f^{m+1}(U)$.

A point $\underline{x} \in R^n$ is said to be a forcing point for f if and only if there exists a neighbourhood, U, of $\underline{x}$ such that

a) $orb(f,U)$ is a subset of the ball of radius $2.\|\underline{x}\|$, centre $\underline{0}$, and

b) for some integer m, the diffeomorphism f is a quasi-contraction of finite degree on $orb^+(f,f^m(U))$.

If there exist points arbitrarily close to $\underline{0}$ which are simultaneously forcing points for f and f^{-1} then f is forced.

Note that the property of being forced depends only on the germ of of the diffeomorphism. Also the iterates $f^m(U)$ can be made to lie in any given neighbourhood of $\underline{0}$ simply by taking m to be sufficiently large. The following may now be stated:

Theorem 6.3.

Any forced diffeomorphism which lies in a one-parameter group of C^∞ diffeomorphisms is infinitely tangent to a diffeomorphism whose only roots are integral.

Proof.

Let $\{f^t\}_{t\in R}$ be a group of diffeomorphisms with f^1 forced. Then there exists a sequence $\{\underline{x}_i\}_{i\in N}$ of forcing points for f^1 and open sets $\{U_i\}_{i\in N}$ as in 6.2 such that one has the following: for all positive integers i,

a) $\underline{x}_i \mapsto \underline{0}$ as $i \mapsto \infty$,

b) $\underline{x}_i \in U_i$,

c) for all $\underline{y} \in U_i$, $\|\underline{y}\| > \|\underline{x}_i\|/2$, and

d) $f^m(U_i) \cap f^n(U_j)$ is empty unless m=n and i=j.

These are derived in two stages. Firstly by taking subsets if necessary, one has (c). Now, for m sufficiently large (or large and negative), $f^m(U_i)$ is contained in the ball of radius $\|\underline{x}_i\|/2$ centre $\underline{0}$, hence for only finitely many integers m does $f^m(U_i)$ intersect U_i non-trivially. By making U_i smaller, if necessary, one has (d) for i=j and n=0 . But f is a diffeomorphism, so (d) follows for any value of n, provided i=j. For the rest of (d), simply discard $\underline{x}_i$'s and U_i's until one has

$$\|\underline{x}_{i+1}\| < \|\underline{x}_i\|/6.$$

One then has the required property by (a) of 6.2.

Now for each integer m define the open set V_m by

$$V_m = \bigcup_{t\in(0,1)} f^t(U_m) .$$

Then for no integer m is $\underline{0}$ in the closure of V_m , and for each neighbourhood of $\underline{0}$ there exists an integer M such that for $|n| > M$ the set $f^n(V_m)$ is contained in the given neighbourhood. Hence for n sufficiently large

$$f^n(V_m) \cap V_m \text{ is empty .}$$

Now replace V_m by a subset $V'_m \subseteq V_m$, also open, such that

a) $f^n(V'_m) \cap V'_m$ is empty unless n=0,

b) $\{f^t(\underline{x}_m) \mid t \in (0,1)\} \subset V'_m$, and

c) $f^n(\underline{x}_m) \notin V'_m$ for all n.

Suppose that $V'_1, \ldots V'_m$ have been constructed as above. Then the set of integers i such that

$$\left(\bigcup_{t \in R} f^t(V'_i)\right) \cap \left(\bigcup_{j=1}^{m} V'_j\right) \text{ is non-empty}$$

is finite. Hence without loss of generality one may suppose that the sets V'_m have disjoint iterates. Let V be the union of these sets.

Let $h: R^n, \underline{0} \longrightarrow R^n, \underline{0}$ be a C^∞ diffeomorphism which is infinitely tangent to the identity, equal to the identity outside V, but which moves each point of the set given by:

$$\{f^t(\underline{x}_m) \mid t \in (0,1) , m \in N\} .$$

Such a diffeomorphism may be constructed inductively.

Let $f^* = f^1 \circ h$. Suppose that g is a non-integral root of f^*. Then $g \circ f^1 = f^1 \circ g$ on the set $R^n - (V \cup g^{-1}(V))$. (1)

Now, g is a root of f^*, and f^* is infinitely tangent to f^1, so the Taylor expansion of g lies in the canonical one-parameter group containing the Taylor expansion of f. So by Theorem 5.2 there exists a real number t, not an integer, such that

$$g = f^t \quad \text{on} \quad orb^+(f, f^{i'}(\underline{x}_m)), \tag{2}$$

for some integer i'. Since t is not an integer there exists an integer j such that $f^t(f^j(\underline{x}_m)) \in V$: i.e. such that $t+j \in (0,1)$. Without loss of generality suppose that j=0 . Then for $|n|$ large,

$$g = f^t \quad \text{on} \quad f^n(\underline{x}_m) .$$

Thus

$$\begin{aligned} g \circ f^{n+1}(\underline{x}_m) &= g \circ f \circ f^n(\underline{x}_m) = g \circ f^* \circ f^n(\underline{x}_m) \\ &= f^* \circ g \circ f^n(\underline{x}_m) = f \circ f^t \circ f^n(\underline{x}_m) \qquad (\text{for } n \neq 0) \\ &= f^t \circ f^{n+1}(\underline{x}_m) . \end{aligned}$$

Similarly for $n \neq 1$,

$$g \circ f^{n-1}(\underline{x}_m) = f^t(f^{n-1}(\underline{x}_m)) .$$

Thus, by a process of induction from the extreme values,

$$g = f^t \text{ on } f^n(\underline{x}_m) \text{ for } n \neq 0 .$$

But $f.g.f^{-1}(\underline{x}_m) = f.f^t.f^{-1}(\underline{x}_m) = f^t(\underline{x}_m)$,

and $f.g.f^{-1}(\underline{x}_m) = f^*.g.f^{*-1}(\underline{x}_m) = g(\underline{x}_m)$,

so $g(\underline{x}_m) = f^t(\underline{x}_m)$.

Hence
$$\begin{aligned} f.f^t(\underline{x}_m) &= f^t.f(\underline{x}_m) = g.f(\underline{x}_m) , \\ &= g.f^*(\underline{x}_m) = f^*.g(\underline{x}_m) , \\ &= f.h.f^t(\underline{x}_m) , \end{aligned}$$

so $h.f^t(\underline{x}_m) = f^t(\underline{x}_m)$,

contrary to hypothesis.

This completes the proof of Theorem 6.3.

<u>Theorem 6.4</u>

For each pair of integers n, m with m even and n, m ≥ 2 there exists a non-empty open subset of G_m, the vector space of germs at $\underline{0}$ of C^∞ diffeomorphisms which are m-tangent to the identity, every member of which is forced.

Recall that the sets G_m are topologised by the coefficients of order m+1. A germ is forced if and only if it has a forced representative, in which case every representative is forced.

For n=1 it is easy to show directly that no member of G_m is forced. For m=0 members of G_m are generically either contractions, the inverse of contractions, or hyperbolic, and none of these can be forced.

<u>Proof of Theorem 6.4</u>

Let $d_i = \begin{cases} 0 & \text{for each integer } i=1\ldots m-1 , \\ 1 & \text{for } i \geq m . \end{cases}$

Let $g:R^n,\underline{0} \to R^n,\underline{0}$ be the C^∞ diffeomorphism given by

$$g(x_1,\ldots x_n) = (x_1 - x_1L_1,\ldots,x_n - x_nL_n) ,$$

where for each integer $i=1\ldots n$ one has

$$L_i = ((1+d_i).x_1^2 - (n+1+d_n-i-d_i).x_n^2).\|(x_1,..x_n)\|^{m-2} \quad .$$

Note that the L_is increase with i unless $x_1=x_n=0$.

In standard form about the x_1-axis (see 2.10) one has

$$r_i = i+d_i \text{ for } i=1...n.$$

If one employs the linear change of co-ordinates

$$(x_1,x_2,...x_n) \mapsto (x_n,...x_2,x_1)$$

(i.e. reverse the order of the axes) then one gets the same standard form, so by 2.12 any diffeomorphism whose m+1-jet is sufficiently close to that of g is a quasi-contraction of degree m on some 'pyramid', X, about the x_1-axis and is the inverse of a quasi-contraction of degree m on some 'pyramid', Y, about the x_n-axis.

The following calculations hold for each function g in some fixed neighbourhood of the function specified above.

Let $\underline{x} = (x_1, l_2x_1,... l_nx_1)$ for some non-zero real number x_1 and some non-zero real numbers $l_2...l_n$. Then for i=2...n

$$|g(\underline{x})_i| < |g(\underline{x})_1|.|l_i| \quad , \quad \text{since} \quad L_i > L_1,$$

provided only that $\underline{x}$ is sufficiently close to the origin.

For each integer j, let x_j and $l_2^j...l_n^j$ be given by

$$g^j(\underline{x}) = (x_j, l_2^jx_j,...l_n^jx_j) \ .$$

Then for each integer i, the sequence $\{l_i^j\}_{j\in N}$ is monotone towards 0 and bounded by 0. Let the limit be l_i^*. Let k be a positive integer, and let e be a strictly positive real number. Then for

$$|l_n^k| < n^{-\frac{1}{2}} \text{ and } x_k \text{ sufficiently small,}$$

the sequence $\{x_j\}_{j\in N}$ is monotone after k, and so has a limit, x*.

Clearly $(x^*, l_2^*x^*,..l_n^*x^*)$ is a fixed point of g, and so

$$x^* = 0 \ .$$

Now, for i=2...n, for each integer j, for all e > 0 and for all x_j sufficiently small one has the following inequalities:

$$|l_i^{j+1} - l_i^j| \geqslant (1-e).|l_i^j|.|x_j|^m \quad \text{and}$$
$$|x_{j+1} - x_j| \leqslant (1-n.(l_n^*)^2+e).|x_j|^{m+1}.$$

Hence $|l_i^{j+1} - l_i^j|/|x_{j+1} - x_j| \geqslant (1-e).|l_i^j|/(|x_j|.(1+e-n.(l_n^*)^2))$.

Thus $l_i^* = 0$ for $i=2...n$.

Now let $\underline{x}' = (x,0,...0,n^{-\frac{1}{2}}.x)$ for some sufficiently small strictly positive real number, x. Then for each integer $i \geqslant 2$,

$$\|g_i(\underline{x}')\| \leqslant \|\underline{x}'\| \quad \text{and}$$
$$\|g(\underline{x}')\| \leqslant 2.\|\underline{x}'\| .$$

Hence $orb^+(g,\underline{x}')$ is contained in the ball of radius $2.\|\underline{x}'\|$, centre $\underline{0}$, and eventually $\underline{x}'$ iterates under g into the region X on which g is a quasi-contraction of degree m.

Similarly it can be shown that for x sufficiently small $orb^+(g^{-1},\underline{x}')$ is contained in the ball of radius $2.\|\underline{x}'\|$ centre $\underline{0}$, and that $\underline{x}'$ eventually iterates under g^{-1} into the region Y on which g^{-1} is a quasi-contraction of degree m. Thus g is forced, as required.

<u>Theorem 6.5</u>

For each pair of integers $n,m \geqslant 2$ with m even, there exists a non-empty open set, D, of G_m every member of which is infinitely tangent to a germ, f, of a diffeomorphism $R^n,\underline{0} \to R^n,\underline{0}$ whose centraliser contains only germs of the form f^i for some integer i and germs whose derivatives at $\underline{0}$ is minus the identity.

Moreover every member of D is m+1-tangent to a germ f whose centraliser is simply $\{f^i \mid i \in Z\}$.

<u>Proof</u>.

Let g, X, Y be as in the proof of 6.4. Then every point of R^n which is sufficiently close to $\underline{0}$ is either eventually moved by g into X or eventually moved by g^{-1} into Y. The result now follows from 5.2 and the algebra of A2.3 and A2.9.

The importance of the previous result is highlighted by the following application:

Theorem 6.6

For each pair of integers n, m$\geq$2 with m even, there exists a non-empty open set, D, of G_m, no member of which is flat stable.

Proof

By 6.5 and 6.1 there exists a subset of G_m, every member of which is infinitely tangent to a pair of diffeomorphisms whose centralisers are not isomorphic.

7 Diffeomorphism germs with large centralisers.

Nancy Kopell [6] has shown that the centraliser of a hyperbolic germ is large in the sense that it contains many germs that are infinitely tangent to the identity. The 'partly hyperbolic' germs, introduced below, are a more general type that have large centralisers. Moreover it will be shown that for each positive integer m there exists an open set of G_m, the vector space of germs of diffeomorphisms which are m-tangent to the identity at $\underline{0}$, every member of which is partly hyperbolic. A partly hyperbolic germ may also be a quasi-contraction on some invariant subset, and may even be forced. Nonetheless these results complement those of the previous sections.

Recall the notation 'orb(f,U)' and '$orb^+(f,U)$' introduced in the previous section (6.2).

Definition 7.1

A diffeomorphism $f: R^n,\underline{0} \rightarrow R^n,\underline{0}$ is said to be

partly hyperbolic

if and only if there exists a neighbourhood, U, of $\underline{0}$ and a sequence $\{U_i\}_{i\in N}$ of open subsets of R^n such that one has

a) for each integer i, $U \cap U_i \cap orb^+(f,U_i)$ is empty,

b) the distance between $\underline{0}$ and U_i tends to 0 as i tends to infinity, but is never actually zero, and

c) for each integer i, only finitely many sets of the form $f^j(U_i)$ ($j \in Z$) have a non-empty intersection with U.

Note that each U_i has only wandering points that avoid some fixed neighbourhood of $\underline{0}$. Moreover f has such wandering points arbitrarily close to $\underline{0}$.

Theorem 7.2

Every C^{∞} partly hyperbolic diffeomorphism germ has a large centraliser, in the sense that many germs, infinitely tangent to the identity at $\underline{0}$, commute with the given germ.

Proof.

Firstly, assume without loss of generality, that each $U_i \subset U$, and that the sup. distance from $\underline{0}$ to U_i tends to 0 as $i \to \infty$. Secondly, assume that for each integer j, either $f^j(U_i) \subseteq U$ or $f^j(U_i)$ is disjoint from U.

A C^{∞} partly hyperbolic diffeomorphism germ may be represented by a diffeomorphism, f, say. Every such representation is partly hyperbolic. By taking subsequences and subsets if necessary, assume also without loss of generality that the sets $\mathrm{orb}(f,U_i)$ are disjoint, and that there are no points of accumulation of $\bigcup_i \mathrm{orb}(f,U_i)$ in U other than the origin and those points which are accumulation points of some $\mathrm{orb}(f,U_i)$.

For each positive integer i, let $g_i: R^n,\underline{0} \longrightarrow R^n,\underline{0}$ be an arbitrary C^{∞} map such that

a) $g_i = \underline{0}$ on $R^n - U_i$ and

b) $g_i(x_i) \neq \underline{0}$ for some point $\underline{x}_i \in U$.

Let $\{K_i\}_{i \in N}$ be a sequence of real numbers. Define the map $g: R^n,\underline{0} \longrightarrow R^n,\underline{0}$ by the formula

$$g = id + \sum_i K_i g_i \quad .$$

If the numbers K_i are sufficiently small then g will be a C^{∞} diffeomorphism, and it will also be infinitely tangent to the identity. Define the function $h: R^n,\underline{0} \longrightarrow R^n,\underline{0}$ by

$$h = id \quad \text{outside} \quad \bigcup_i \mathrm{orb}(f,U_i) \cap U$$
$$= f^j g . f^{-j} \quad \text{on the set } f^j(U_i) \cap U \;.$$

The conditions which have been imposed on the sets U_i ensure that the function h is well defined.

Claim 7.3

If the real numbers K_i are sufficiently small then h is C^∞ and infinitely tangent to the identity at $\underline{0}$.

The theorem follows easily from the claim, since h clearly commutes with f.

Proof of claim.

It is obvious that h is C^∞ away from $\underline{0}$.

For $\underline{x} \in f^j(U_i)$ one has the following inequalities:

$$\begin{aligned}\|h(\underline{x})-\underline{x}\| &= \|f^j g f^{-j}(\underline{x})-f^j f^{-j}(\underline{x})\| \\ &\leqslant L_j.\|g f^{-j}(\underline{x})-f^{-j}(\underline{x})\| \\ &\leqslant L_j.K_i.\sup_{\underline{y}\in U_i}\{\|g_i(\underline{y})\|\},\end{aligned}$$

where L_j is a Lipschitz constant for f^j near $\underline{0}$.

Let U be a neighbourhood of $\underline{0}$ such that for each integer i only finitely many sets of the form $f^j(U_i)$ ($j\in Z$) have a non-trivial intersection with U. Then for every $\underline{x}\in U\cap f^j(U_i)$,

$$\|h(\underline{x})-\underline{x}\| \leqslant L.K_i.\sup_{\underline{y}\in U_i}\{\|g_i(\underline{y})\|\},$$

where L is the largest L_j for which $U\cap f^j(U_i)$ is non-empty.

Now, for each integer i the distance between $\underline{0}$ and orb(f,U_i) is non-zero, so for K_i sufficiently small one has that

$$\|h(\underline{x})-\underline{x}\| \leqslant \exp(-\|\underline{x}\|^{-2}) \quad \text{for all} \quad \underline{x}\in U\cap \text{orb}(f,U_i).$$

Now, h is the identity outside $\bigcup_i \text{orb}(f,U_i)$, so

$$\|h(\underline{x})-\underline{x}\| \leqslant \exp(-\|\underline{x}\|^{-2}) \quad \text{for all} \quad \underline{x}\in U.$$

Hence h is C^∞ at $\underline{0}$ and infinitely tangent to the identity at $\underline{0}$, as required. This completes the proof of theorem 7.2.

Example 7.4

Suppose $n\geqslant 2$. Then given an integer $m\geqslant 1$, let f be the germ of the C^∞ diffeomorphism $R^n,\underline{0} \to R^n,\underline{0}$ given by the following:

$$(x_1,\ldots x_n) \mapsto (x_1+x_1^{m+1}, x_2-x_2^{m+1}, x_3,\ldots x_n)$$

This clearly lies in G_m, and any germ sufficiently close to f must be partly hyperbolic. To see this, for each strictly positive integer i, let U_i be the ball of radius i^{-2} about

$(2.i^{-1},2.i^{-1},0,\dots 0)$.

Explicit claculation confirms that $\{U_i\}_{i\in N}$ satisfies the condition of 7.1, for U the unit ball. Note that the seperation between iterates of each U_i is at least $i^{-(m+1)}$, and the iterates $f^j(U_i)$ all eventually move away from $\underline{O}$, so (a) and (c) of 7.1 hold for nearby germs with the same U_i and U sufficiently small. Now (b) of 7.1 is trivially satisfied by these, and so they are partly hyperbolic.

Note that, for the case where the number of variables, n, is one, there are no partly hyperbolic germs.

8 Applications.

Theorem 8.1

Let $Y \subset X \subseteq R^n$, with Y a core of X (see 3.2). Then given a C^∞ quasi-contraction $f: R^n, \underline{0} \to R^n, \underline{0}$ of finite degree on X there exists a one-parameter group $\{f^t\}_{t \in R}$ of C^∞ diffeomorphisms $R^n, \underline{0} \to R^n, \underline{0}$ such that on Y the germs of f^1 and f are identical.

Proof.

By 6.1 there exists a one-parameter group $\{g^t\}_{t \in R}$ with g^1 infinitely tangent to f at $\underline{0}$. By 3.4 there exists a diffeomorphism $h: R^n, \underline{0} \to R^n, \underline{0}$, infinitely tangent to the identity at $\underline{0}$, such that for some sufficiently small real number r,

$$g^1 \circ h = h \circ f \qquad \text{on } Y_r .$$

For each real number t let the diffeomorphism f^t be given by $\quad f^t = h^{-1} \circ g^t \circ h$.

Then $\{f^t\}_{t \in R}$ is clearly the one-parameter group required.

It often happens, for example in the standard case (2.10), that f(X) is a core of X. In these cases one may obtain the above equality on the germ of the whole of X.

If one takes a fixed quasi-contraction , f, as above and a given point $\underline{x} \in X_r$ (for r sufficiently small), then the map $p: [0, \infty) \to R^n$ given by the following formula:

$$t \mapsto f^t(\underline{x})$$

is clearly well defined. Since $f^n(\underline{x}) \mapsto \underline{0}$ as $n \mapsto \infty$, the image of the map is a path which goes to $\underline{0}$. What does it look like?

Theorem 8.2

Given an integer m which is both even and at least two, let $f: R^n, \underline{0} \to R^n, \underline{0}$ be a standard quasi-contraction of degree m about the x_1-axis. Suppose that none of $r_2, \ldots r_n$ are integers.

Then there exists a C^∞ diffeomorphism $f^*: R^n, \underline{0} \to R^n, \underline{0}$ which is conjugate to f (in G, the vector space of germs of diffeomorphisms of R^n), m+1-tangent to f, and which leaves the x_1-axis invariant.

Proof.

The x_1-axis is formally invariant under the action of the m+1-jet of f. The Taylor expansion of f is formally conjugate to a Taylor expansion for which the x_1-axis is formally mapped to itself. This is easily proven by induction. The above result is now a straight-forward corollary of 3.4. Note that the linear part of the conjugate constructed here is the identity.

Corollary 8.3

Given a diffeomorphism $f: R^n, \underline{0} \to R^n, \underline{0}$ satisfying the hypothesis of the above theorem, there exist C^∞ functions $k: R, 0 \to R^{n-1}, \underline{0}$ and $g: R, 0 \to R, 0$ such that k is flat at 0 and one has:

$$f(\mathrm{id} \times k) = (\mathrm{id} \times k)_\circ g \quad \text{near } 0.$$

Proof.

The conjugation given in the preceeding theorem gives rise to k and g in the obvious way. The point is that the graph of the C^∞ function k is invariant under f, just as the x_1-axis is invariant under f*.

The following example shows that the Corollary above can not be generalised to arbitrary quasi-contractions.

Example 8.4

Given an integer $m \geq 2$, let $f: R^2, \underline{0} \to R^2, \underline{0}$ be the C^∞ standard quasi-contraction about the x_1-axis given by the following formula:

$$f(x,y) = (x - x^{m+1}, y - 2 . x^m y + x^{m+2}) .$$

Then there do not exist functions $k: R, 0 \to R, 0$ and $g: R, 0 \to R, 0$ with k of class C^∞ which satisfy

$$f(\mathrm{id} \times k) = (\mathrm{id} \times k)_\circ g \quad \text{near } 0 .$$

Proof.

Suppose that on the contrary such functions do exist. Let the Taylor expansion of k be $\sum a_i x^i$. Let j be the least integer such that a_j is non-zero.

Firstly, suppose that $j \leq m+2$. Then the first $m+1$ non-trivial terms in the Taylor expansion of f(idxk) are

$$\sum_{i=j}^{j+m} a_i x^i - 2.a_j x^{j+m} + x^{m+2} \text{ .*}$$

It is clear that $g(x) = x - x^{m+1}$, so the first $m+1$ terms of the Taylor expansion of $k_\circ g$ are as follows:

$$\sum_{i=j}^{j+m} a_i x^i - j.a_j x^{m+j} \text{ .}$$

This situation is insoluble, so one must have $j > m+2$. But now the first term in the Tayor expansion of f(idxk) is x^{m+2}, whereas the first term in the Taylor expansion of $k_\circ g$ is $a_j x_j$. This situation is also insoluble, so such a function k can not possibly exist, even on a formal level.

Thus once again the formal behaviour and the properties of the germ are closely related.

Example 8.5

Given an even integer $m \geq 2$ and a positive real number r, let $f: R^2,\underline{0} \rightarrow R^2,\underline{0}$ be the C^∞ diffeomorphism given by

$$f(x,y) = (\ x - x^{m+1},\ y.(1-x^m)^r\ ) \text{ near } \underline{0}.$$

This is a standard quasi-contraction of degree m about the x_1-axis with $r_2 = r$ and $r_1 = 1$. The x_1-axis is clearly invariant.

For each real number, t, let k_t be the function given by

$$k_t(x) = t.x^r \text{ .}$$

Then for each real number t, one has the following equality:

$$f(idxk_t) = (idxk_t)_\circ g \text{ , where } g(x) = x - x^{m+1}.$$

If r is an integer then each k_t is of class C^∞. Thus one has in this case a one-parameter family of invariant curves.

*Except for j=1, when the last term is omitted.

The above situation in which r is an integer is unusual in that for each real number t the diffeomorphism given by the following formula commutes with f:

$$g^t(x,y) = (x , y+k_t(x)) .$$

By A2.3 germs like this are uncommon, in the sense of that section.

Theorem 8.6

Let $f: R^n,\underline{0} \to R^n,\underline{0}$ be a C^∞ standard quasi-contraction of degree m about the x_1-axis with none of $r_2,\dots r_n$ being integers, as in 8.2.

Then there exists exactly one flat C^∞ function $k: R,0 \to R^{n-1},\underline{0}$ whose graph is invariant under f.

Proof.

By 8.2 it suffices to consider the case where the x_1-axis is invariant under f. The zero function is then a solution. Suppose that one had another solution given by

$$k = k_2 x\dots k_n ,$$

where for $i=2\dots n$, $k_i: R,0 \to R,0$ is C^∞ and flat at 0. Then

a) To show that each k_i is infinitely flat:

Suppose otherwise. Choose an integer j such that no function k_i is flatter than k_j. Let k_j have a Taylor expansion given by

$$k_j(x) = a_1 x^p + \dots .$$

The Taylor expansion of the j-th part of f is by hypothesis

$$f(x_1,\dots x_n) = x_j - r_j . x_1^m x_j + \underline{p} + \text{higher order terms},$$

where $\underline{p}$ is a homogenous polynomial of degree m+1 which does not contain any terms of the form $x_1^m x_i$.

It is now easy to show that the Taylor expansion of f(idxk) is

$$f(x,k(x)) = a_1 . x^p + \dots a_m . x^{p+m} - r_j . a_1 . x^{p+m} + \text{higher terms}.$$

(Note that $\underline{p}$ does not contribute any terms of order less than m-1+p and by hypothesis p is greater than one.)

Now the Taylor expansion of f restricted to the x_1-axis is

$$f(x,0,\dots 0) = x-x^{m+1}+\text{higher order terms.}$$

Thus the Taylor expansion of f restricted to the x_1-axis followed by k is given by the following formula:

$$k(f(x,0,\dots 0)_1) = a_1.(x-x^{m+1})^p+a_2.x^{p+1}\dots a_m.x^{p+m}+ \text{terms of order more than } p+m$$

$$= a_1.x^p+\dots a_m.x^{p+m}-p.a_1.x^{p+m}+ \text{higher order terms.}$$

Equating terms of order p+m one has

$$r_j.a_1 = p.a_1 ,$$

which is a contradiction, because a_1 is supposed to be non-zero, but p is an integer while r_j is not an integer.

b) To show that the only infinitely flat solution is zero:

Given a real number, t, let $V(t) \subset R^n$ be defined by

$$V(t) = \{(x_1,\dots x_n) \in X \mid x_1 > 0 \text{ and } \|x_i\| < t.x_1^K \text{ for all } i\}$$

where K is some integer that is much bigger than r_n.

Clearly for any $t > 0$ there exists a real number $d > 0$ such that for all $x \in (0,d)$ one has the following:

$$(x,k(x)) \in V(t) .$$

Let $\underline{x} = (x_1,\dots x_n) \in X$ with $\underline{x} \neq \underline{0}$.

Then for any sufficiently small real number t,

$$\underline{x} \notin V(t) .$$

Now it suffices to show that for any t with $0 < t < 1$,

$$f(X-V(t)) \subset X-V(t) ,$$

for then each k_i must be the zero function to avoid a contradiction.

To show that the above inclusion is in fact true, consider $\underline{x} \in X-V(t)$ as before, and let

$$f(\underline{x}) = (x_1',\dots x_n') .$$

The pyramid, X, of 2.11 is described by constants $l_2, \ldots l_n$, $l_{n+2}, \ldots l_{2n}$ and r. If these are sufficiently small then one has

$$x_1' < x_1 - \tfrac{1}{2}.x_1^{m+1} \quad \text{and}$$

$$x_j' > x_j - (r_j+1).x_1^m x_j ,$$

where j is some integer with the property that for no integer i is $|x_i| > |x_j|$. Now if $K > 2.(r_n+1)$ and r is sufficiently small then one has the following inequalities, as required:

$$x_j' > t.x_1^K(1-(r_n+1).x_1^m) > t.x_1^K.(1-\tfrac{1}{2}.x_1^m)^K > t.(x_1')^K .$$

The above calculations are done for x_j positive, but this does not lose generality as the situation is entirely symmetric about the x_1-axis.

By arguing as above but calculating as in 4.3 it is possible to show that for a general quasi-contraction of degree m and characteristic (a*,c*), there do not exist two curves in X which are more than (m.a*+c*-m-1)-tangent to each other. As a corollary one may deduce a result very similar to that of 5.2, namely that such germs have small centralisers. The only difference is that recently our attention has been restricted to C^∞ quasi-contractions, using the results of section 3, but section 4 can equally well be applied to give analogous results for only finitely differentiable germs.

Notice that if one has a fixed curve in mind, then by restricting one's attention to a possibly smaller set (such as the cores of 3.2) one can make a* equal to 1 without loss of generality. This makes the formulae much simpler, and shows that the arbitrary quasi-contractions aren't very different from the standard ones.

Appendix 1:

Details of Belickiĭ's Theory of Quasi-Contractions, Modified.

The basic concepts underlying Belickiĭ's work are as follows:

Let $M \subset R^n$ be compact. For each $e > 0$, let $V = V(M,e)$ be the closed e-neighbourhood of M. Let $J_e(M)$ be the space of C^∞ maps $h\colon V \to R^n$ with the usual topology given by the norms

$$\|h\|^k = \max_{i \leq k} \max_{x \in V} \|D^i h(x)\| \text{, for non-negative integers, k.}$$

Let $J(M)$ be the indirect limit as e tends to zero of $\{J_e(M)\}_{e>0}$, and let $J_0(M) \subset J(M)$ be the subspace of all germs which vanish on M together with all derivatives.

Given $J^1, J^2 \subseteq J(M)$ and an operator $T\colon J^1 \to J^2$, an operator $T_e\colon J^1_e \to J^2_e$ is a representative of T if and only if

both i) J^i_e is a representative of J^i for $i=1,2$,

and ii) for each $h \in J^1$ and each representative $h_e \in J^1_e$, $T_e(h_e)$ is a representative of the germ $T(h)$.

A subset $J^1 \subseteq J_0$ is suitable if and only if it is complete and closed under the following: subtraction; multiplication by members of J; and the maps induced by composition and the taking of inverses in the space $\{id\} + J^1$.*

Let $T\colon J^1 \to J^2$ be a linear operator. A set $L = \{L_e\}_{e>0}$ of operators $L_e\colon J^2_e \to J^1_e$ is right inverse to T if and only if there exists a set of representatives $\{T_e\}_{e>0}$ of T such that $T_e \circ L_e$ is the identity operator.

Belickiĭ supposes that in addition to the norms $\| \, \|^k$ one has, for each J^i_e and each integer $k \geq 0$, a non-decreasing sequence $\{\| \, \|^k_n\}_{n \in N}$ of norms, with $\| \, \|^k_0 = \| \, \|^k$.

A set L of linear operators $L_e\colon J^1_e \to J^2_e$ is a set of order r if and only if there exists a constant c such that:

for each integer $k \geq 0$ and each sufficiently large integer m (depending on k) there exists a positive integer m' such that:

for each $e>0$ there exists d such that for all $h \in J^1_e$,

$$\|L_e h\|^k_m \leq c.\|h\|^k_{m+r} + k.d.\|h\|^{k-1}_{m'} .$$

*Only such sets will be considered from now on.

A linear operator $T: J^1 \to J^2$ is an <u>operator of order r</u> if there exists a set of representatives of T which are of order r.

The infimum of the numbers c above is denoted $n_r(L)$ (or $n_r(T)$). The function n_r is a semi-norm on the space of linear operators of order r.

Belickiǐ gives the following example:

<u>Lemma A1.1</u>

Suppose the norms $\| \|_m^k$ are defined by the formula

$$\|h\|_m^k = \sup_{i \leq k} \sup_{x \in V} \{\|D^i h(x)\| / \|x\|^m\},$$

and that $f \in J(M)$ has order of at most r (that is,

$$\inf_{\underline{x} \in V} \{\|f(\underline{x})\| / \|\underline{x}\|^r\} \text{ is strictly positive }),$$

then the operator $L: J_0(M) \to J_0(M)$ defined by the formula

$$h \mapsto h/f$$

has order r .

The proof is straightforward: the constant c is simply the inverse of the above infimum, and each d is zero.

Operators of finite order combine as follows:

<u>Lemma A1.2</u>

The composition of an operator of order r with an operator of order s is an operator of order r+s.

The proof is trivial. Note also that an operator of order r is also an operator of order s for each integer s which is greater than r.

The next result is based on Belickiǐ's Proposition 1.2 [3], but is significantly more general:

<u>Theorem A1.3</u>

Let $T: J^1 \to J^1$ be a linear operator. Suppose that there exists a constant c such that:

for each non-negative integer k and for each sufficiently large integer m there exists a positive integer m' such that:

for each sufficiently small strictly positive real number e there exists a real number d such that for every integer M and every $h \in J^1_e$ one has the following inequality:

$$\sum_{i=0}^{M} \|T^i h\|^k_m \leqslant c.\|h\|^k_{m+r} + k.d.\|h\|^{k-1}_{m'} .$$

(Belickii's hypothesis is that r is 0 and $n_0(T)$ is less than 1.)

Then the linear operator I+T has an inverse set of order r. (Here I denotes the identity operator.)

Proof:

The series $\sum_{i=0}^{\infty} T^i h$ is absolutely convergent. Denote its limit by Lh. Clearly -L is a linear operator of order r and $-L \circ T = I - L = -T \circ L$, and so $-L \circ (I-T) = I = (I-T) \circ L$. Thus I-T has an inverse. Replacing T by -T proves that I+T has an inverse of order r, as required.

The next important concept is as follows:

Definition A1.4

An operator $H: J^1 \to J^2$ is small if and only if for each non-negative integer s there exists an integer i such that:

for every pair of strictly positive real numbers e and K there exists a set of continuous representatives $\{H_e\}_{e>0}$ of H such that:

for each non-negative integer k one has:

i) there exist non-decreasing functions $c: R,0 \to R,0$, and

ii) for each sufficiently large integer m there exists an integer m' such that for all $h \in J^1_e$ with $\|h\|^0_i \leqslant K$, $\|H_e h\|^k_{m+s} \leqslant e.\|h\|^k_m + k.c(\|h\|^{k-1}_{m'})$.

Note that the sum of two small operators is also small. The composition of a small operator and an operator of finite order (either way round) is small. Belickii gives the following results:

Lemma A1.5 [3]

For each function $f \in J_0$ the operator $H: J_0 \to J_0$ defined by

$h \mapsto f \circ (id+h) - f$ is small.

Lemma A1.6 [3]

Let $T:R^n \times R^n \to R^n$ be C^∞. Then the operator $H:J_0 \to J_0$ given by

$$h \mapsto T(id,h) - T(id,\underline{0}) - D_2T(id,\underline{0}).h$$

is small, where D_2 denotes partial differentiation with respect to the second variable.

Belickiǐ goes on to introduce the following notion:

Definition A1.7

An operator $B:J^1 \to J^2$ is differentiable at $\underline{0} \in J^1$ if and only if there exists a linear operator $T:J^1 \to J^2$ and a small operator $H:J^1 \to J^2$ such that one has

$$B = T + H + B(\underline{0}) .$$

In general the derivative is not uniquely defined.

Theorem A1.8

Suppose that an operator $B:J^1 \to J^2$ is differentiable at $\underline{0}$, and that one of its derivatives has a right inverse set of finite order. Then B is surjective. Moreover,

$$B^{-1}(B(\underline{0})) = \{\underline{0}\} .$$

Proof

Belickiǐ [3] proves the first part. The second part is new, and is vital to the application of Belickiǐ's work.

Suppose $Bh = B(\underline{0})$, then $Th = -Hh$. Let L be the right inverse of T. Then $HLh = -h$.

However L is of finite order, say of order r, so there exists a constant c such that for m sufficiently large

$$\|Lh\|_m^0 \leqslant c\|h\|_{m+r}^0 .$$

But H is a small operator, so choosing $e = 1/2c$,

$$\|h\|_{m+r}^0 \leqslant \|HLh\|_{m+r}^0 \leqslant e\|Lh\|_m^0 \leqslant \tfrac{1}{2}\|h\|_{m+r}^0$$

for m sufficiently large.

Hence $\|h\|_m^0 = 0$, and so h must be the zero function, as required.

This completes the preliminary results. These relate to concepts of stability as follows:

Definition A1.9

A map $f:R^n,\underline{0} \longrightarrow R^n,\underline{0}$ is J^1-stable if and only if for each $h \in J^1$ there exists a germ $k \in J^1$ such that

$$(id+k)_\circ f = (f+h)_\circ (id+k) \; .$$

The map f is said to be uniquely J^1-stable if and only if the germ k above is unique.

Definition A1.10

A C^∞ map $f:R^n,\underline{0} \longrightarrow R^n,\underline{0}$ is a J^1-quasi-contraction of degree m if and only if the linear operator on J^1 defined by

$$T(f):h \longmapsto Df.h - h_\circ f$$

has a right inverse set of order m.

(Note that $T(f) = Df.T(f^{-1})_\circ f$, so the inverse of a Belickiĭ-type quasi-contraction is also a quasi-contraction!)

The main result may now be stated:

Theorem A1.11

Every J^1-quasi-contraction is J^1-stable.

Proof (after Belickiĭ).

Let $f:R^n,\underline{0} \longrightarrow R^n,\underline{0}$ be a quasi-contraction. Let $h \in J^1$. Define the operator $A:J^1 \longrightarrow J^1$ by the formula

$$A(k) = (f+h)_\circ (id+k) - (id+k)_\circ f \; .$$

It suffices to show that A maps onto $\underline{0}$. Let T be the linear operator defined in A1.10 above. Let H be the operator given by

$$H(k) = (f+h)_\circ (id+k) - (f+h) - Df.k \; .$$

Then $A = T + H + A(\underline{0})$, so by Theorem A1.8 it suffices to show that the operator H is small in the sense of definition A1.4 . This is, however, a simple application of Lemmas A1.5 and A1.6 : simply take $T(x,y) = f(x+y)$ in A1.6 .

In fact Theorem A1.8 has a stronger consequence:

Theorem A1.12

Every J^1-quasi-contraction is uniquely J^1-stable.

Proof

Otherwise there exists a J^1-quasi-contraction, f, together with germs h, k and $l \in J^1$ (with k and l distinct) such that

$$(id+k)_\circ f = (f+h)_\circ (id+k) \quad \text{and}$$

$$(id+l)_\circ f = (f+h)_\circ (id+l) \; .$$

Let $g \in J^1$ be defined by the following formula:

$$g = (id+k)^{-1}{}_\circ (id+l) - id \; .$$

Define the operator $A: J^1 \longrightarrow J^1$ by the equality

$$A(j) = f_\circ (id+j) - (id+j)_\circ f \; .$$

Then $(id+k)_\circ A(g) = \underline{0}$,

so $A(g) = \underline{0} = A(\underline{0})$,

hence by Theorem A1.8 ,

$$g = \underline{0} \; .$$

Thus the germs k and l are equal, as required.

Besides quasi-contractions, Belickiĭ introduces the following:

Definition A1.13

Let $f: R^n, \underline{0} \longrightarrow R^n, \underline{0}$. Let $J^1 = \sum_{i=1}^{k} J_i \subseteq J_0$.

Suppose that for each integer i, f is a J_i-quasi-contraction of degree $m_i \leq m$.

Then f is J^1-quasi-hyperbolic of degree m.

Recall that quasi-contractions need not be topological contractions. This definition corresponds to the hyperbolic functions of Kopell [6]. Belickiĭ [3] proves the following:

Theorem A1.14

Every J^1-quasi-hyperbolic diffeomorphism is J^1-stable.

However the stability is not always unique: see Theorem 7.2.

Appendix 2:

Taylor Series of Diffeomorphisms.

For each non-negative integer m, denote by T_m the set of n-tuples of formal power series in n variables whose m-jet is the identity map. By a well known theorem of E.Borel any such power series is the Taylor series of some C^∞ diffeomorphism that is m-tangent to the identity at $\underline{0}$. Let T_m have the group structure induced by composition of germs of diffeo-morphisms. This composition is by substitution. For each integer m the quotient space T_m/T_{m+1} is a finite dimensional Euclidean vector space. Give each T_m the topology induced by the usual topology on Euclidean vector space. In other words two power series in T_m are said to be close if and only if the coefficients of their m+1-linear parts are close.

Given any integer m and any $t \in T_m$ the group generated by t is a subgroup of the centraliser of t, and so there is an obvious homomorphism of the integers into the centraliser of t which sends 1 to t. For each integer r this homomorphism gives rise to a map from the integers into the space of r-linear maps $R^n \rightarrow R^n$. Suppose now that m is not zero and r is at most twice m. Denote by $\underline{a}$ the r-linear part of t. Then the homomorphism described above takes an integer i to the r-linear map $i.\underline{a}$. This homomorphism extends uniquely to a polynomial map of the reals into the space of r-linear maps, namely $i \mapsto i.\underline{a}$, where now i is an arbitrary real number. Putting the maps for individual values of r together produces a canonical homomorphism from the reals into T_m/T_{2m+1}. It will now be shown that this lifts canonically to a map into the whole of the topological vector space T_m.

As a first step, consider an arbitrary monomial of order 2m+1. Let c(n) be the coefficient of that monomial in t^n. Since $t^{n+1}=t(t^n)$, c(n+1) can be expressed in terms of the coefficients of t and t^n. Let g=c(2)-2.c(1). Then since the terms of order m+1 of t^n are✦ just n times the terms of order m+1 of t, one has that

$$c(n+1) = c(n)+c(1)+n.g \ .$$

Hence

$$c(n) = n.c(1)+g.\sum_{r=1}^{n-1} r \ ,$$
$$= n.(c(1)-\tfrac{1}{2}g)+n^2.\tfrac{1}{2}g \ ,$$

a polynomial of degree two in n.

Now, suppose that for r at most some r_0 the coefficients of terms of order 2m+r are polynomials of degree r+1 in n. Then, as above, for $r=r_0+1$ one has the following:

$$c(n+1) = c(n)+c(1)+ \text{ a sum of polynomials of degree } r_0+1.$$

As is well known* the sum from 1 to m of polynomials in n of degree at most r_0+1 is a polynomial of degree r_0+2 in m, hence by induction on r_0 all the coefficients of all the terms of the Taylor expansion of t^n are polynomials in n. Thus one has a canonical map, z, from the reals into T_m extending the original map from the integers $n \mapsto t^n$. Let $X:R^2 \to T_m$ be given by

$$X:(s,t) \mapsto z(s+t)-z(s)-z(t) \ .$$

Then X is a polynomial map which is zero on the integer lattice, and so is identically zero. Thus z is a homomorphism, and the following result has been proven:

<u>Theorem A2.1</u>

Given any strictly positive integer m and any $t \in T_m$ there exists a canonical homomorphism of the reals under addition into T_m which maps 1 to t.

* As can be deduced from the following school-book formula:

$$\sum_{r=1}^{n} r(r+1)\ldots(r+d-1) = n(n+1)\ldots(n+d)/(d+1) \ .$$

✦The 'cross terms' are $\sum_I a_I x_I(n\sum_J a_J x_J)$.

When $t \in T_m$ is the identity map this homomorphism maps everything to the identity. For every other map, $t \in T_k$ for some least integer k, and the canonical map takes the real number s to an element of T_k whose k+1_st order term is just s times the k+1_st order term of t, and so the canonical map is injective. Note also that the 'component maps' into the spaces of multi-linear maps are continuous, and the canonical map is the only homomorphism from R into T_m which takes 1 to t and has this property.

Some power series have a centraliser which contains many more elements than those in the image of the canonical map, as here:

<u>Example A2.2</u>

Let $t \in T_1$ be a power series in two variables given by

$$t = (p(x), p(y)), \text{ for some } p \in T_1, \text{ in one variable.}$$

Then any power series q of the form

$$q = (p^a(x), p^b(y)) \quad \text{for some real numbers a, b,}$$

lies in T_1 and commutes with t.

It is tempting to suppose that the centraliser of a map $R^n,0 \to R^n,0$ will 'often' contain an image of R^n, as above. This is not the case however, as the following important result shows:

<u>Theorem A2.3</u>

For each strictly positive integer m there exists a generic subset of T_m for which the image of the canonical map given by A2.1 is the intersection of the centraliser with T_1.

Moreover for each strictly positive integer k there exists an open dense subset $U_k \subset T_m$ such that for every $t \in U_k$ the image in T_k/T_{k+1} of the canonical map is the centraliser of t in T_k/T_{k+1}.

Thus if one restricts one's attention to terms of finite order it is open dense for the image of the canonical map (a one-parameter group) to contain the intersection of the centraliser with T_1. The T_0 part of the centraliser is studied later. The proof of A2.3 utilises the following:

<u>Lemma A2.4</u>

Let $M^* : (T_m/T_{m+1}) x (T_k/T_{k+1}) \rightarrow (T_{m+k}/T_{m+k+1})$ be the quotient map of $M : (T_m) x (T_k) \rightarrow T_{m+k}$ defined by the following:

$$(r,t) \mapsto rtr^{-1}t^{-1}.$$

Then M^* is a continuous bi-linear map between finite dimensional Euclidean vector spaces.

<u>Proof</u>.

Given any $r \in T_m$ and $t \in T_k$, $id + rt - tr \in T_{m+k}$, so $rt(tr)^{-1} \in T_{m+k}$. Moreover the term of order m+k+1 of rt-tr depends bi-linearly on the leading terms of r and t. This property is preserved by multiplication by $(tr)^{-1}$, so M^* is bi-linear, as required.

<u>Proof of A2.3</u>.

It suffices to show that for some $t \in T_m$ the map M^* restricted to $(\{t\}/T_{m+1}) x (T_k/T_{k+1})$ has nullity at most one or zero, depending on whether k=m or k≠m , for then* this property will be open dense for T_m. Thus it suffices to find a single $t \in T_m$ such that the centraliser of t in T_k/T_{k+1} is equal to the image of the canonical map of A2.1 in T_k/T_{k+1}.

<u>Example A2.5</u>.

Consider first the case where n, the number of variables, is just one. Given a strictly positive integer m and a non-zero real number a, let $t \in T_m$ be

$$x + ax^{m+1} + \text{higher order terms.}$$

Suppose that $q \in T_1$ commutes with t up to terms of order m+2. Let b be the coefficient of the linear part of q and let c be

*By A3.4.

the coefficient of the m+1-st order term of q. Then comparing terms of order m+1 of tq and qt one has

$$c+ab^{m+1} = ba+c,$$

and so b is either plus or minus one: it cannot be 0 because q is supposed to be invertible. Moreover if m is odd then b can only be plus one.

Suppose now that $r \in T_l$ commutes with t up to terms of order m+l+1, for some non-zero positive integer l. Let b be the l-th coefficient of r, let c be the m+l+1-st coefficient of r, and let d be the m+l+1-st coefficient of t. Then comparing terms of order m+l+1 of tr and rt one has

$$c+a.(m+1).b+d = d+b.(l+1)a+c .$$

Hence either b=0 or l=m . Thus for any element of T_1 other than the identity the canonical map given by A2.1 maps onto the centraliser of t in T_1. For the cases where there is more than one variable it was shown in example A2.2 that this is not the case.

Example A2.6.

More generally now, given a strictly positive integer m and a set of real numbers $\{1,e_1,\dots e_n\}$ which are linearly independent over the reals, define a power series $p \in T_m$ to be the identity plus the m+1-linear map $\underline{a}$ given by the following:

$$\underline{a}_i(\underline{x}) = e_i x_1^m x_i + x_i^{m+1} \text{ for } i=1\dots n,$$

where $\underline{x}=(x_1,\dots x_n)$, $\underline{a}(\underline{x})=(\underline{a}_1(\underline{x}),\dots \underline{a}_n(\underline{x}))$ and $e_1=0$.

Let l be a positive integer. Let $q \in T_l$ be the identity plus an l+1-linear map $\underline{b}$. Suppose that p and q commute up to terms of order m+l+1. Then to complete the proof of A2.3 it suffices to show that either b is the zero map or l=m and $\underline{b}$ is a scalar multiple of $\underline{a}$.

As in the case n=1, compare terms of order m+l+1 appearing in the first co-ordinates of the power series pq and qp. Then

$$(m+1).x_1^m b_1(\underline{x}) = \text{terms of order } m+l+1 \text{ in } \underline{b}_1(p(\underline{x})) .$$

Suppose that the lowest power of x_1 occuring in $\underline{b}_1$ is x_1^c. Let $\underline{b}_1(\underline{x}) = x_1^c.(u(\underline{x})+x_1 v(\underline{x}))$, where x_1 does not appear in $u(\underline{x})$. Then substitution in the above equality gives rise to

$$(m+1).u(\underline{x}).x_1^{c+m}+(m+1).v(\underline{x}).x_1^{c+m+1}$$
$$= (x_1+x_1^{m+1})^c.(u(p(\underline{x}))+(x_1+x_1^{m+1}).v(p(\underline{x})))$$

to order m+l+1. Now let $u(\underline{x})=\sum_I u_I x_I$. Then

$$(m+1).x_1^{c+m}\sum_I u_I x_I = x_1^c \sum_I u_I \sum_{i\in I} e_i x_1^m x_I + c x_1^{c+m}\sum_I u_I x_I .$$

Hence $m+1 = \sum_{i\in I} e_i + c$, for each index I. But the e_i's are linearly independent, so u must be constant and c=l+1. In other words one has $\underline{b}_1(\underline{x}) = b.x_1^{l+1}$ for some constant b. The equality is now

$$(m+1).b = b.(l+1) .$$

Hence either l=m or b=0, as required.

If l=m then the power series q may be composed with the canonical power series corresponding to the constant -b to give a power series q' commuting with p such that $\underline{b}_1$ is zero.

Now compare the i-th co-ordinates of pq and qp. One has:

$$e_i.x_1^m.\underline{b}_i(\underline{x})+(m+1).x_1^m \underline{b}_i(\underline{x})$$
$$= \text{terms of order } m+l+1 \text{ in } \underline{b}_i(p(\underline{x})).$$

Again, let c be the highest power of x_1 occuring in $\underline{b}_i$, and let $\underline{b}_i(\underline{x}) = x_1^c.\sum_I u_I x_I + \sum_J v_J x_J$.

Then $$e_i x_1^{c+m}\sum_I u_I x_I = c.x_1^{c+m}\sum_I u_I x_I + x_1^{c+m}\sum_I\sum_{j\in I} e_j u_I x_I ,$$

and so $e_i = c+\sum_{j\in I} e_j$ for each index I. But $\{1,e_1,\ldots e_n\}$ are linearly independent over the integers, so $\underline{b}_i$ is zero.

This completes the proof of A2.3.

<u>Example A2.7.</u>

Let $p\in T_0$ contain only terms of odd order. Then the map $x \rightarrow -x$ commutes with p.

It is easy to construct further examples where the centraliser contains elements whose linear term is not the identity. However:

Theorem A2.8.

Consider first the case where the number of variables, n, is one. Let m be a strictly positive integer, and let $t \in T_m$ with $t \notin T_{m+1}$. Then one has the following statements:

a) if m is odd then the linear part of every member of the centraliser of t is the identity.

b) if m is even then there exists a power series which commutes with t and has -identity as its linear part. Moreover the m-jet of each member of the centraliser of t is either +identity or -identity.

Theorem A2.9.

Consider the case where the number of variables, n, is more than one. Let m be a strictly positive integer. Then one has:

a) if m is odd then it is generic for T_m that the linear part of every member of the centraliser of a member of T_m is the identity.

b) if m is even then it is generic for $t \in T_m$ that

i) the linear part of every member of the centraliser of t is either the identity or -identity, and

ii) it is open dense for power series $u \in T_{m+1}$ that the centraliser of t+u contains only power series whose linear part is the identity.

Proof of A2.8.

Part (a) was proven in A2.5. To prove part (b), let t be

$$x+a_1.x^{m+1}+a_2.x^{m+2}+\dots,$$

where $a_1, a_2, \dots$ are arbitrary real numbers, save that a_1 is not zero. Let p be an arbitrary power series with linear part -identity. Let b_i be the coefficient of x^i in p for each integer i, at least two. It suffices to solve inductively for the b_i's so that t and p commute, given that b_1 is minus one.

Suppose that $b_2 \ldots b_{r-1}$ have been found in terms of $a_1 \ldots a_{r-1}$ so that t and p commute up to terms of order m+r-1. Compare terms of order m+r of tp and pt. One has the following:

$$(-1)^r . a_r + b_{m+r} + (m+1) . a_1 . b_r = r . b_r . a_1 - a_r + c_r + b_{m+r},$$

where c_r depends only on $a_1 \ldots a_{r-1}$. Simplifying, one has

$$((m+1)-r) . a_1 . b_r = c_r - a_r . (1+(-1)^r).$$

When $r \neq m+1$ this has a unique solution for b_r. Since -identity commutes with t up to terms of order 2m, $b_2, \ldots b_m = 0$ are the solutions for r less than m+1. The coeffient b_{m+1} may be chosen arbitrarily, the choice determining the higher order terms. This completes the proof of A2.8.

<u>Proof of A2.9</u>.

Let $p \in T_m$ be as in A2.6. Let $q \in T_0$ be given by the formula

$$q(\underline{x}) = (b_1^1 x_1 + \ldots b_n^1 x_n , \ldots , b_1^n x_1 + \ldots b_n^n x_n),$$

where as usual $\underline{x} = (x_1, \ldots, x_n)$.

Comparing the first co-ordinates of pq and qp up to terms of order m+2 gives rise to the following equality:

$$(b_1^1 x_1 + \ldots b_n^1 x_n)^{m+1} = b_1^1 x_1^{m+1} + b_2^1 (e_2 x_1^m x_2 + x_2^{m+1}) + \ldots b_n^1 (e_n x_1^m x_n + x_n^{m+1}).$$

Comparing the coefficients of x_i^{m+1} in the above renders

$$(b_i^1)^{m+1} = b_i^1 \quad \text{for } i=1 \ldots n.$$

Similarly, comparing coefficients of $x_1^m x_i$ for $i \neq 1$ gives rise to

$$(m+1) . (b_1^1)^m . b_i^1 = b_i^1 e_i \quad \text{for } i=2 \ldots n.$$

Since $e_2 \ldots e_n$ are not integers and q is invertible the above two equalities imply that b_1^1 is either plus or minus one and the other b_i^1's are all zero.

Now comparing the i-th co-ordinates of pq and qp up to terms of order m+2 gives rise to the following equality: for i=2...n,

$$e_i . x_1^m . (b_1^i x_1 + \ldots b_n^i x_n) + (b_1^i x_1 + \ldots b_n^i x_n)^{m+1}$$
$$= b_1^i x_1^{m+1} + b_2^i (e_2 x_1^m x_2 + x_2^{m+1}) + \ldots b_n^i (e_n x_1^m x_n + x_n^{m+1}) .$$

Comparing coefficients of x_1^{m+1} in this gives rise to

$$e_i.b_1^i+(b_1^i)^{m+1} = b_1^i.$$

For j=2...n comparing coefficients of x_j^{m+1} renders

$$(b_j^i)^{m+1} = b_j^i \ .$$

Similarly the coefficients of $x_1^m x_j$ lead to the following:

$$e_i.b_j^i+(m+1).(b_1^i)^m.b_j^i = b_j^i.e_i \quad \text{for } j=1...n \ .$$

As with the first co-ordinate the only solution is for i=2...n

$$b_i^i = +/- \ 1 \text{ and } b_j^i = 0 \text{ for } j=1...n,\ j\neq i \ .$$

To complete the proof of A2.9 (a) note that if m is odd then $(-1)^m \neq 1$, so q can only be the identity. The required property is generic by A2.4.

For the case where m is even it has been shown that the linear part of any power series which commutes with t is diagonal and that each entry on the diagonal is either plus or minus one. If arbitrarily small coefficients of the terms of the form $x_1^{m-1}x_i^2$ are introduced into the first co-ordinate of t then the maps $(x_1,...x_i,...x_n) \mapsto (x_1,...-x_i,...x_n)$ no longer commute with t. By A2.4 no new elements have been introduced into the centraliser of t. Clearly -identity commutes with any power series of the form the identity plus a homogenous polynomial of odd degree, so one cannot hope to do any better than A2.9 (b), of which part (i) has now been proven. In order to prove part (ii), consider the higher order terms:

As in A2.4 each (m+1)-linear map $\underline{a}$ induces a linear map

$$M^*: (T_2/T_3) \rightarrow (T_{m+2}/T_{m+3}) \ ,$$

and if $-id+b_2$ commutes with $id+\underline{a}+\underline{a}'$ up to terms of order m+4 (where b_2 is a 2-linear map and $\underline{a}'$ is an m+2-linear map) then

$$M^*(b_2) = 2.\underline{a}+c(\underline{a}'),$$

where $c(\underline{a}')$ is an m+2-linear map depending only on $\underline{a}'$.

Now M* is a linear map, and the dimension of (T_2/T_3) is less than the dimension of (T_{m+2}/T_{m+3}) for n greater than one and m positive. Thus M* cannot be onto for any **a**, and so for any fixed **a** it is generic for **a**' that 2.**a**+c(**a**') is not in the range of M*.

This completes the proof of A2.9.

To summarise then:

For n=1 and m odd the centraliser of any element of $T_m - T_{m+1}$ is a one-parameter group.

For n=1 and m even the centraliser of any element of $T_m - T_{m+1}$ is of the form $Z_2 x R$.

For $n \neq 1$ and m odd it is generic that the centraliser of a member of T_m is a one-parameter group.

For $n \neq 1$ and m even it is generic that the centraliser of a member of T_m is either a one-parameter group or of the form $Z_2 x R$.

Moreover it is generic for $t \in T_m$ that it is generic for $r \in T_{m+1}$ that the centraliser of t+r is a one-parameter group.

Appendix 3:

Miscellaneous Analysis Results.

Some well known results, vital to this thesis, are given.

Theorem A3.1 (A special case of Leibniz's Formula.)

Let $f,g\colon R^n,\underline{0} \longrightarrow R^n,\underline{0}$ be m times continuously differentiable, and let $\cdot : R^n \times R^n \longrightarrow R^n$ be a continuous bilinear map. Then the product function $f.g\colon R^n,\underline{0} \longrightarrow R^n,\underline{0}$ given by

$$(f.g)(\underline{x}) = f(\underline{x}).g(\underline{x})$$

is m times continuously differentiable.

The derivatives are given by the formula

$$(f.g)^{(m)} = \sum_{p=0}^{m} C_p^m f^{(m-p)}.g^{(p)} ,$$

where C_p^m are the ... Binomial Coefficients, $m!/p!/(m-p)!$.

No proof of this result is given, but the formulation of the next result is sufficiently novel to warrant a proof.

Theorem A3.2 (A re-formulation of the Chain Rule.)

Let $f,g\colon R^n,\underline{0} \longrightarrow R^n,\underline{0}$ be m times continuously differentiable. Then the composite function, $g_\circ f$, is m times continuously differentiable.

In order to formulate the derivatives, the following notation is used:

$\underline{u}$ is a q-tuple $(u_1,\ldots u_q)$ of non-negative integers.

I_q^m is the set of all q-tuples, $\underline{u}$, with $u_1+\ldots u_q = m$.

$f^{(\underline{u})}$ is the product of derivatives (i.e. the composition of linear maps) $f^{(u_1)}.f^{(u_2)}\ldots f^{(u_q)}$.

$g^{(q)}f(\underline{x})$ is the q-th derivative of g at the point $f(\underline{x})$.

The derivatives (for $m>0$) may now be given by the formula

$$(g_\circ f)^{(m)} = \sum_{q=1}^{m} g^{(q)}f.\sum_{\underline{u}\in I_q^m} c_{\underline{u}} f^{(\underline{u})} ,$$

for some real numbers $c_{\underline{u}}$ with $c_{(1,\ldots,1)} = 1$ and

$c_{\underline{u}} = 0$ whenever any u_i is zero.

Proof

The case where m is one is the ordinary Chain Rule,

$$D(g_o f) = (Dg)_o f . Df \quad .$$

Proceeding by induction on m, one has

$$(g_o f)^{(m+1)} = D((g_o f)^{(m)})$$

$$= D\left(\sum_{q=1}^{m} g^{(q)} f . \sum_{\underline{u} \in I_q^m} c_{\underline{u}} f^{(\underline{u})}\right)$$

$$= \sum_{q=1}^{m} \left(g^{(q+1)} f . Df . \sum_{\underline{u} \in I_q^m} c_{\underline{u}} f^{(\underline{u})} + g^{(q)} f . D\left(\sum_{\underline{u} \in I_q^m} c_{\underline{u}} f^{(\underline{u})}\right)\right)$$

(Applying the ordinary Chain Rule above and Leibniz's Formula)

$$= \sum_{q=1}^{m+1} g^{(q)} f . \sum_{\underline{v} \in I_q^{m+1}} c_{\underline{v}} f^{(\underline{v})} \quad .$$

The first term in the above expansion contributes only to

$$\underline{v} = (1, \underline{u}) \quad .$$

The second term contributes only to terms of the form

$$\underline{v} = (u_1, \ldots u_r, u_r + 1, u_{r+1}, \ldots u_q) \quad .$$

Thus

$c_{(1,\ldots 1)} (\in I_{q+1}^{q+1}) = c_{(1,\ldots 1)} (\in I_q^q) = 1$, and

for each $\underline{v}$ with some v_i zero, the corresponding u_j was zero, and so

$$c_{\underline{v}} = 0 \; ,$$

as required.

Corollary A3.3

Let $f, g \colon R^p, \underline{0} \rightarrow R^n, \underline{0}$ be m times continuously differentiable. Suppose that for each integer $i = 2 \ldots m-1$,

$$f^{(i)}(g(\underline{0})) = \underline{0} \; .$$

Then $D(f_o g)(\underline{0}) = Df(g(\underline{0})) . Dg(\underline{0})$,

$$(f_o g)^{(m)}(\underline{0}) = f^{(m)}(g(\underline{0})) . Dg(\underline{0})^m + Df(g(\underline{0})) . g^{(m)}(\underline{0}) \; ,$$

and intermediate derivatives vanish.

Moreover, if g is invertible, and if for each integer $i = 2 \ldots m-1$,

$$(g^{-1})^{(i)}(f_{\circ}g(\underline{O})) = g^{(i)}(\underline{O}) = \underline{O} ,$$

then

$$(g^{-1}{}_{\circ}f_{\circ}g)^{(m)}(\underline{O}) = (g^{-1})^{(m)}(f_{\circ}g(\underline{O})).f^{(m)}(g(\underline{O})).Dg(\underline{O})^{m},$$

$$D(g^{-1}{}_{\circ}f_{\circ}g)(\underline{O}) = D(g^{-1})(f_{\circ}g(\underline{O})).Df(g(\underline{O})).Dg(\underline{O}) ,$$

and intermediate derivatives vanish .

<u>Theorem A3.4</u> .

Let U,V,W be finite dimensional Euclidean Vector Spaces.
Let $B: U\times V \to W$ be a continuous bilinear map.
Suppose that for some $\underline{u} \in U$,

$B|\{\underline{u}\}\times V$ has rank r.

Then it is open dense for $\underline{u} \in U$ that

$B|\{\underline{u}\}\times V$ has rank at least r.

In other words some restriction maps can have smaller ranks than the majority, but never greater rank.

<u>Proof</u>

Choose an arbitrary basis for U,V,W.
Then for each basis element $\underline{w}_i \in W$, the map $B_i: U\times V \to W$,

$B_i(\underline{u},\underline{v})$ = the i-th component of $B(\underline{u},\underline{v})$,

is a continuous bilinear map into a one-dimensional Euclidean Vector Space. Hence there exists a matrix representation M_i such that

$$B_i(\underline{u},\underline{v}) = \underline{u}M_i\underline{v} ,$$

where the elements of U,V are treated as row and column vectors respectively.

Suppose that $B_i|\{\underline{u}\}\times V$ has rank r. Then, re-ordering the basis of W if necessary, for i = 1...r, $B_i|\{\underline{u}\}\times V$ has rank 1, which implies that $\underline{u}M_i$ is non-zero, hence M_i is non-zero, and so kernel(M_i) is a proper subspace of U.

Now, for $\underline{u} \in U \setminus \bigcup_{i=1}^{r}$ kernel(U_i), for $i = 1 \ldots r$, $\underline{u}M_i$ is non-zero so $B_i | \{\underline{u}\} \times V$ has rank 1.

Hence for such $\underline{u}$, $B | \{\underline{u}\} \times V$ has rank at least r, as required.

Theorem A3.5

Let $f: R^n, \Omega \to R^n, \Omega$ be C^1, m-tangent to the identity. Let a^*, c^* be the characteristic constants for f (as in 4.2). Then $c^* \geqslant a^*(m+1)$.

Proof

One has

$$f(\underline{x}) = \int_0^1 Df(t\underline{x})dt.\underline{x} .$$

Let $h(\underline{x}) = Df(\underline{x}) - id$.

Then $f(\underline{x}) = \underline{x} + \int_0^1 h(t\underline{x})dt.\underline{x}$.

If $\|h(\underline{x})\| \leqslant d\|\underline{x}\|^m$

then
$$\|f(\underline{x}) - \underline{x}\| \leqslant \int_0^1 \|h(t\underline{x})\| dt.\|\underline{x}\| \leqslant d\|\underline{x}\|^{m+1}/(m+1) .$$

Hence $a^* \leqslant d/(m+1)$.

But c^* is a lower bound for the admissable d's. (See the note following the proof of 2.18). Hence

$$c^* \geqslant a^*(m+1) ,$$

as required.

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* Sometimes spelt Belitskii.

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