Modular and Dual-Dedekind Subgroups in Certain Classes of Infinite Groups.

by

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Acknowledgements

I would like to thank my supervisor, Dr. S.E. Stonehewer, without whom, of course, there would have been no thesis at all, and also my colleagues in the Mathematics Department of St. Mary's College,Twickenham, and especially A.G.Vosper Esq. for his encouragement and guidance.

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Summary

The main inspiration of this thesis were the two papers of Schmidt ([I] & [II]) and the paper of Menegazzo ([III]).

Chapter One is concerned with establishing some basic results concerning modular subgroups, and Chapter Two with defining a class of groups $\mathfrak{X}$ (which includes the class of locally finite groups) and extending the theorems in Schmidt ([I]) to groups in this class. Chapter Three, which was the first chapter of the thesis to be written, examines the structure of modular subgroups in locally finite groups with the minimum condition on subgroups (where there is a definitive structure theorem to help us). Chapter Four extends the results of Schmidt ([II]) to locally finite groups. Finally, Chapter Five takes a (by no means exhaustive) look at dual-dedekind subgroups (i.e. subgroups which are dual to modular subgroups). A few theorems in the first section of Chapter Five are simply the dual of theorems in Chapter One; for the sake of clarity, however, their proofs are included.

After the main body of this thesis had been completed, my supervisor, Dr. S.E.Stonehewer, produced a definitive theorem concerning the structure of corefree modular subgroups in locally finite groups analogous to the main theorem of Schmidt ([II]). For the sake of completeness, this theorem is included in an appendix.

A Glossary of some Symbols and Notation used in this Thesis

1. $M \text{ m } G$: M is a modular subgroup of G (see Chapter One).

2. $M \text{ dd } G$: M is a dual-dedekind subgroup of G (see Chapter Five)

3. $M \text{ qn } G$: M is a quasinormal subgroup of G i.e. for all subgroups H of G, $\langle M,H\rangle = MH$

4. M^G: the normal closure of M in G i.e. the smallest normal subgroup of G containing M.

5. M_G: the core of M in G i.e. the largest normal subgroup of G contained in M.

6. $Z(G)$; the centre of G . $Z(G) = \{a \mid ag = ga \text{ for all } g \in G\}$.

7. $N_G(M)$: the normaliser of M in G. $N_G(M) = \{g \in G \mid M^g = M\}$.

8. $C_G(M)$: the centraliser of M in G $= \{g \in G \mid gm = mg \text{ for all } m \in M\}$.

9. $[G/M]$: the lattice of subgroups $\{H \mid M \leq H \leq G\}$.

10. $[G]: = [G/1]$ i.e. the lattice of all subgroups of G.

11. $a \equiv 1 (\text{mod } p)$: $p \mid (a-1)$

12. G a P-group: see 2.2.2.

13. G a generalised P-group: here A (the maximal p-subgroup of G which is elementary abelian and normal in G) is infinite.

14 $[x,y]: = xyx^{-1}y^{-1}$ for any elements x,y of a group G.

15. $P \in Syl_p(G)$: P is a Sylow p-subgroup of G.

16. C_{p^∞}: the quasicyclic group . $G \cong C_{p^\infty} \Rightarrow G = \bigcup_i A_i$

$A_0 = 1$ and $A_{i+1}^p = A_i$ for all i

17. C_q: the cyclic group of order q.

18 $Z_\infty(G)$: the hypercentre of G i.e. the greatest member of the upper central series of G.

19. N m[G/M] : N is a modular element in the lattice [G/M] (see 9 above)

Introduction

We have that any normal subgroup of a group G must be quasinormal in G and any quasinormal subgroup must be modular (for, suppose K qn G. Then, if $V \geqslant U$, $\langle K,U\rangle \cap V = KU \cap V = (K \cap V)U = \langle K \cap V, U\rangle$ (as $K \cap V$ qn V). Similarly, if $V \geqslant K$).

The converse of the latter assertion is not always true: a subgroup may be modular but not quasinormal in G. For example: let $G = S_3$, the symmetric group on the three elements $\{1,2,3\}$. Then $\langle(12)\rangle$ m G, but $(123)(12) = (23) \in \langle(12),(13)\rangle \setminus \langle(12)\rangle\langle(13)\rangle$ so $\langle(12)\rangle$ is not quasinormal in G.

Perhaps the most important property of modular subgroups is that under a (subgroup) lattice isomorphism, modular subgroups must always be mapped onto modular subgroups - such is not the case with normal or even with quasinormal subgroups.

In ([I]), considering finite groups, Schmidt firstly investigates the situation when $[G/M]$ is a chain (M m G), and then considers how a modular subgroup differs from a normal subgroup: he investigates $\frac{M}{M_G}$ (which he finds is nilpotent), $\frac{M^G}{M_G}$ (supersoluble) and $\frac{G}{C_G\left(\frac{M^G}{M_G}\right)}$ (which applying the main theorem of ([II]) may be shown to be supersoluble).

In ([II]), again considering only finite groups, Schmidt firstly investigates some conditions under which a modular subgroup will permute with another subgroup, and then goes on to prove the important theorem concerning the structure of core-free modular subgroups, viz. if M m G and $M_G = 1$, $M = Q_1 \times \ldots \times Q_r \times M \cap K$ and $G = P_1 \times \ldots \times P_r \times K$, where $|Q_i| = q_i$, q_i a prime, $M \cap K$ qn G, P_i is a P-group for all i and $\forall\, x_i \in P_i, x_j \in P_j, k \in K$, $(|x_i|,|x_j|) = (|x_i|,|k|) = 1 \;\forall\, i,j$.

Modular subgroups are referred to by some writers as Dedekind subgroups - hence the use of the term dual-Dedekind by Menegazzo.

Inclusions, intersections and unions are interchanged in the defining axioms of dual-Dedekind subgroups as compared with those of modular subgroups.

Again restricting his attention to finite groups, Menegazzo proves that a simple group can have no non-trivial dual-Dedekind subgroups, and then goes on to investigate those groups all of whose normal subgroups are dual-Dedekind (a normal subgroup of a group G need not necessarily be dual-Dedekind in G, see for example, p.60).

Chapter One

Some basic facts about modular subgroups

A subgroup M of G is said to be modular in G (we write M m G) if

M1. For all X,Y subgroups of G such that $X \leq Y$, we have that $\langle M,X\rangle \cap Y = \langle M \cap Y, X\rangle$.

M2. For all $X,Y \leq G$, such that $M \leq Y$, we have that $\langle M,X\rangle \cap Y = \langle M, X \cap Y\rangle$.

The following propositions 1.1.1. – 1.1.5 are stated but not proved in Schmidt [1].

Proposition 1.1.1

The following statements are equivalent:

(i) M m G

(ii) For all subgroups K of G, the map ϕ_K is a lattice isomorphism where ϕ_K is defined as follows:

$$\phi_K : [\langle M,K\rangle / M] \longrightarrow [K/K \cap M]$$
$$L \longmapsto L \cap K$$

(iii) For all subgroups K of G, the map ψ_K is a lattice isomorphism where ψ_K is defined as follows:

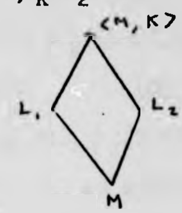
$$\psi_K : [K/K \cap M] \longrightarrow [\langle M,K\rangle/M]$$
$$R \longmapsto \langle R,M\rangle$$

Moreover, in this situation, ϕ_K and ψ_K are mutually inverse.

Proof

(i) $\Rightarrow$ (ii)

Suppose $L_1, L_2 \in [\langle M,K\rangle/M]$ and $\phi_K(L_1) = \phi_K(L_2)$ i.e.

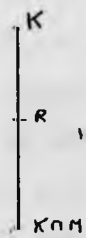


$L_1 \cap K = L_2 \cap K$. Hence $\langle L_1 \cap K, M\rangle = \langle L_2 \cap K, M\rangle$

Hence $\langle M,K\rangle \cap L_1 = \langle M,K\rangle \cap L_2$ by M2

i.e. $L_1 = L_2$. So ϕ_K is injective.

Now let $R \in [K/K\cap M]$ Hence $\langle M,R\rangle \in [\langle M,K\rangle/M]$.



$$\phi_K(\langle M,R\rangle) = \langle M,R\rangle \cap K = \langle M\cap K, R\rangle \text{ by M1}$$
$$= R$$

Hence ϕ_K is surjective.

Also ψ_K is its inverse, as for all $L \in [\langle M,K\rangle/M]$,

$$\psi_K(\phi_K(L)) = \langle L\cap K, M\rangle = \langle M,K\rangle \cap L = L \quad (*)$$

by M2

Clearly ϕ_K preserves intersections. Also as $\psi_K \phi_K =$

$1_{[\langle M,K\rangle/M]}$, $\psi_K(\phi_K(\langle A,B\rangle)) = \langle A,B\rangle$

$$= \langle \psi_K(\phi_K(A)), \psi_K(\phi_K(B))\rangle$$
$$= \psi_K\langle \phi_K(A), \phi_K(B)\rangle$$

(as ψ_K clearly preserves unions) for all $A,B \in [\langle M,K\rangle/M]$

Hence as ψ_K is the inverse of ϕ_K and hence a bijection

$\phi_K(\langle A,B\rangle) = \langle \phi_K(A), \phi_K(B)\rangle$ i.e. ϕ_K preserves

unions.

So ϕ_K is a lattice isomorphism.

(ii) $\Rightarrow$ (iii)

<u>Proof</u>

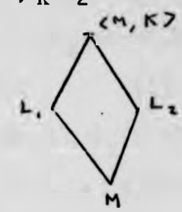
<u>(i) ⇒ (ii)</u>

Suppose $L_1, L_2 \in [\langle M,K\rangle/M]$ and $\phi_K(L_1) = \phi_K(L_2)$ i.e.

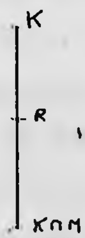
$L_1 \cap K = L_2 \cap K$. Hence $\langle L_1 \cap K, M\rangle = \langle L_2 \cap K, M\rangle$

Hence $\langle M,K\rangle \cap L_1 = \langle M,K\rangle \cap L_2$ by M2

i.e. $L_1 = L_2$. So ϕ_K is injective.



Now let $R \in [K/K\cap M]$ Hence $\langle M,R\rangle \in [\langle M,K\rangle/M]$.



$$\phi_K(\langle M,R\rangle) = \langle M,R\rangle \cap K = \langle M\cap K, R\rangle \text{ by M1}$$
$$= R$$

Hence ϕ_K is surjective.

Also ψ_K is its inverse, as for all $L \in [\langle M,K\rangle/M]$,

$$\psi_K(\phi_K(L)) = \langle L\cap K, M\rangle = \langle M,K\rangle \cap L = L \quad (*)$$

by M2

Clearly ϕ_K preserves intersections. Also as $\psi_K\phi_K =$

$1_{[\langle M,K\rangle/M]}$, $\psi_K(\phi_K(\langle A,B\rangle)) = \langle A,B\rangle$

$$= \langle \psi_K(\phi_K(A)), \psi_K(\phi_K(B))\rangle$$
$$= \psi_K\langle \phi_K(A), \phi_K(B)\rangle$$

(as ψ_K clearly preserves unions) for all $A,B \in [\langle M,K\rangle/M]$

Hence as ψ_K is the inverse of ϕ_K and hence a bijection

$\phi_K(\langle A,B\rangle) = \langle \phi_K(A), \phi_K(B)\rangle$ i.e. ϕ_K preserves unions.

So ϕ_K is a lattice isomorphism.

<u>(ii) ⇒ (iii)</u>

Given that ϕ_K is a lattice isomorphism, we want to prove that ψ_K is. Firstly, we shall prove that $\forall\ S \in [\langle M,K\rangle/M]$, $\langle S\cap K,M\rangle = S$.

$M \le \langle S\cap K,M\rangle \le S \Rightarrow \phi_K(\langle S\cap K,M\rangle) \le \phi_K(S)$. But $\phi_K(S) = S\cap K \le \langle S\cap K,M\rangle\cap K = \phi_K(\langle S\cap K,M\rangle)$ So $\phi_K(S) = \phi_K(\langle S\cap K,M\rangle)$ and as ϕ_K is injective, $S = \langle S\cap K,M\rangle$ as required. (I)

Hence ψ_K is surjective ($\because\ \forall S \in [\langle M,K\rangle/M]$, $\psi_K(S\cap K) = S$)

Now suppose $\psi_K(L_1) = \psi_K(L_2)$ for some $L_1, L_2 \in [K/K\cap M]$. ϕ_K surjective $\Rightarrow \exists\ M_1, M_2 \in [\langle M,K\rangle/M]$ such that $\phi_K(M_1) = L_1$, $\phi_K(M_2) = L_2$ i.e. $M_1\cap K = L_1, L_2 = M_2\cap K$. $\psi_K(L_1) = \psi_K(L_2)$ $\Rightarrow \langle M_1\cap K,M\rangle = \langle M_2\cap K,M\rangle \Rightarrow M_1 = M_2$ by (I). Hence $L_1 = L_2$ and ψ_K is injective.

(I) shows that $\psi_K\phi_K$ is the identity map; ψ_K clearly preserves unions and can be shown to preserve intersections using an argument analagous to the one in the first part of the theorem.

<u>(iii) ⇒ (i)</u>

Firstly we shall prove that $\forall$ K, and $L \in [K|K\cap M]$, $\langle L,M\rangle\cap K = L$.

$L \le \langle L,M\rangle\cap K \Rightarrow \psi_K(L) \le \psi_K(\langle L,M\rangle\cap K)$ i.e. $\langle L,M\rangle \le \langle\langle L,M\rangle\cap K,M\rangle$.

But $\langle\langle L,M\rangle\cap K,M\rangle \le \langle L,M\rangle$. So $\psi_K(L) = \psi_K(\langle L,M\rangle\cap K)$ and hence, as ψ_K is injective, $L = \langle L,M\rangle\cap K$ (**)

We wish to prove M1 i.e. $X\le Y \Rightarrow \langle M,X\rangle\cap Y = \langle M\cap Y,X\rangle$.

$\langle X,Y\cap M\rangle \in [Y/Y\cap M]$ and hence by (**) with $L = \langle X,Y\cap M\rangle$, $K = Y$ we have that $\langle X,Y\cap M\rangle = \langle\langle X,Y\cap M\rangle M\rangle\cap Y = \langle X,M\rangle\cap Y$ as required.

For M2, we want to prove that $M\le Y$ and X any subgroup of G $\Rightarrow \langle M,X\rangle\cap Y = \langle M,X\cap Y\rangle$.

$\langle M,X\rangle\cap Y \in [\langle M,X\rangle/M]$ and as ψ_X is surjective, $\exists\ R \in [X/X\cap M]$ such that $\psi_X(R) = \langle R,M\rangle = \langle M,X\rangle\cap Y$.

By (**) with $L = R$ and $K = X$, we have that $R = \langle R,M\rangle\cap X$

i.e. $\langle R,M\rangle = \langle\langle R,M\rangle \cap X, M\rangle$

i.e. $\langle M,X\rangle \cap Y = \langle\langle M,X\rangle \cap Y \cap X, M\rangle = \langle X \cap Y, M\rangle$ as required.||

<u>Proposition 1.1.2</u>

$M \text{ m } G$ and $U \leq G \Rightarrow M \cap U \text{ m } U$

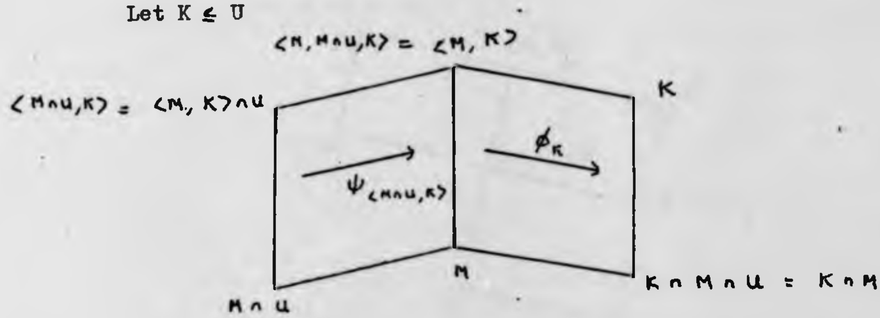
<u>Proof</u>

By 1.1.1., it is sufficient to prove that for all subgroups K of U, the map

$$[\langle M\cap U,K\rangle / M\cap U] \longrightarrow [K/K\cap M\cap U]$$
$$L \longmapsto L\cap K$$

is a lattice isomorphism.

Let $K \leq U$



As $\langle M,K\rangle \cap U = \langle M\cap U,K\rangle$ (by M1) and $\langle M\cap U,K\rangle \cap M = \langle M,K\rangle \cap U \cap M = U\cap M$, we have, as the above diagram indicates, lattice isomorphisms $\phi_K : [\langle M,K\rangle/M] \longrightarrow [K/K\cap M]$

and $\psi_{\langle M\cap U,K\rangle} : [\langle M\cap U,K\rangle/\langle M\cap U,K\rangle \cap M] \rightarrow [\langle M,M\cap U,K\rangle/M]$

(by 1.1.1.) where the notation is as usual.

Thus, their composition $\phi_K \psi_{\langle M\cap U,K\rangle} = \theta$, say, is a lattice isomorphism and $\theta(L) = \phi_K(\langle L,M\rangle) = \langle L,M\rangle \cap K$

$= \langle L,M\rangle \cap U \cap K$ (as $K \leq U$) $= \langle L, M\cap U\rangle \cap K$ (as $L \leq U$ by M1)

$= L\cap K$, for all $L \in [\langle M\cap U,K\rangle / M\cap U]$ as required.||

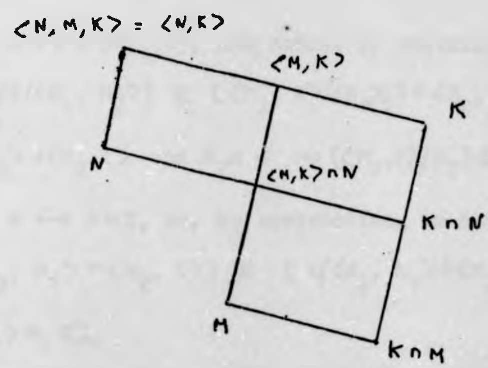
Proposition 1.1.3

$N \text{ m } [G/M]$ and $M \text{ m } G \Rightarrow N \text{ m } G$

Proof

By 1.1.1., again, it is enough to prove that for all subgroups K of G, $\phi_K : [\langle N,K\rangle/N] \longrightarrow [K/K \cap N]$, $L \longmapsto L \cap K$ is a lattice isomorphism.

As $M \leq N$, so $\langle N,K\rangle = \langle N,M,K\rangle$ and $\langle M,K\rangle \in [G/M]$, thus we have that $[\langle N,K\rangle/N] \cong [\langle M,K\rangle/\langle M,K\rangle \cap N]$ via $L \longmapsto L \cap \langle M, K\rangle$ (as $N \text{ m } [G/M]$, using 1.1.1.), and $[\langle M,K\rangle/\langle M,K\rangle \cap N] \cong [K/N \cap K]$ as part of $[\langle M,K\rangle/M] \cong [K/K \cap M]$ via $R \mapsto R \cap K$.



Composing these two maps, we have $[\langle N,K\rangle/N] \cong [K/N \cap K]$ via $L \longmapsto (L \cap \langle M,K\rangle) \cap K = L \cap K$ as required. ||

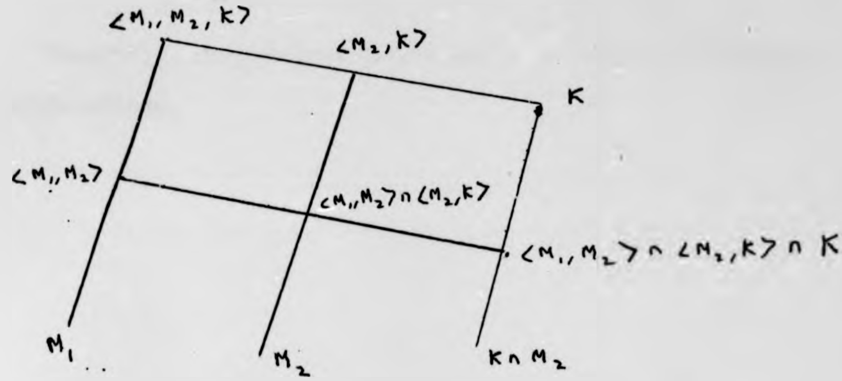
Proposition 1.1.4

$M_1 \text{ m } G$ and $M_2 \text{ m } G \Rightarrow \langle M_1, M_2\rangle \text{ m } G$

Proof

Again by 1.1.1., we wish to prove that for all subgroups K of G the map $\phi_K : [\langle M_1, M_2, K\rangle/\langle M_1, M_2\rangle] \longrightarrow [K/K \cap \langle M_1, M_2\rangle]$

defined by $\phi_K(L) = L \cap K$ is a lattice isomorphism.



As M_1 m G, we have that $[\langle M_1, M_2, K\rangle / M_1] \cong [\langle M_2, K\rangle / M_1 \cap \langle M_2, K\rangle]$ via the map $L \mapsto L \cap \langle M_2, K\rangle$, and hence, by restriction, we have $[\langle M_1, M_2, K\rangle/\langle M_1, M_2\rangle] \cong [\langle M_2, K\rangle/\langle M_1, M_2\rangle \cap \langle M_2, K\rangle]$.

$M_2 \in \langle M_1, M_2\rangle \cap \langle M_2, K\rangle$ and M_2 m G $\Rightarrow [\langle M_2, K\rangle / M_2] \cong [K/K \cap M_2]$ by the map $R \mapsto R \cap K$, so, by restriction, we have that $[\langle M_2, K\rangle/\langle M_2, M_1\rangle \cap \langle M_2, K\rangle] \cong [K/\langle M_2, M_1\rangle \cap \langle M_2, K\rangle \cap K]$ $= [K/\langle M_2, M_1\rangle \cap K]$.

By map composition, we get that $[\langle M_1, M_2, K\rangle / \langle M_1, M_2\rangle] \cong [K / \langle M_1, M_2\rangle \cap K]$ by the map $L \mapsto (L \cap \langle M_2, K\rangle) \cap K$.

So, by 1.1.1., $\langle M_1, M_2\rangle$ m G as required.

Proposition 1.1.5

M m G and $\sigma : [G] \rightarrow [H]$ a lattice isomorphism $\Rightarrow$ $\sigma(M)$ m H.

Proposition 1.1.6

Let $N \leq M$, N normal in G.

Then M m G if and only if $\frac{M}{N}$ m $\frac{G}{N}$

Hopefully, propositions 1.1.5. and 1.1.6 require no further explanations.

Section Two

The following theorem proves that when discussing non-normal subgroups of G, there is no ambiguity involved in speaking of maximal modular subgroups, as every non-normal subgroup maximal in the set of modular subgroups, is a maximal subgroup of G.

Theorem 1.2.1.

(1) Let M be maximal among the modular subgroups of G but not normal in G. Then M is a maximal subgroup of G and for all subgroups H of G, either $H \leq M$ or $H \cap M$ is maximal in H.

(2). Let $M \lessdot G$ be such that for all subgroups H of G, either $H \leq M$ or $H \cap M$ is maximal in H. Then M is a maximal subgroup of G which is modular, and may be normal.

Proof

(1). We suppose M is not a maximal subgroup of G.

For all proper subgroups K of G such that M is contained in K, we have that $M \trianglelefteq K$.

For, suppose not. Suppose there is some subgroup K and an element k of K such that $M^k \neq M$. Then $M < \langle M, M^k \rangle$ and by 1.1.4 and 1.1.5, $\langle M, M^k \rangle$ m G which contradicts the choice of M.

(b). There exists an element x of G such that $\langle M, x \rangle = G$.

For, suppose for each element x of $G \setminus M$, $\langle M, x \rangle < G$. Then M is normal in $\langle M, x \rangle$ by (a), i.e. $M^x = M$ for all x, i.e. M is normal in G, contradicting our choice of M.

(c). Let $M < H$. Then H m $[G/M]$.

For, $[G/M] \cong [\langle x\rangle/\langle x\rangle \cap M]$ which is a modular lattice.
Hence H m $[G/M]$.

Hence by 1.1.3., H m G which contradicts our choice of M.

So we have established that M is a maximal subgroup of G. Now let us consider any subgroup H of G. Then either $H \leq M$ or $\langle H,M\rangle = G$. In the latter case, $[G/M] = [\langle H,M\rangle/M] \cong [H/H \cap M]$ by 1.1.1., so $M \cap H$ is maximal in H.

(2) Taking H = G, we see that M is maximal in G.

Now we wish to prove

i. for all subgroups U,V of G, $U \leq V$, $\langle M,U\rangle \cap V = \langle M \cap V,U\rangle$

ii. for all subgroups U,V of G such that V contains M, $\langle M,U\rangle \cap V = \langle M,U \cap V\rangle$.

i. $U \leq M$ gives $\langle M,U\rangle \cap V = M \cap V = \langle M \cap V,U\rangle$

$U \nleq M$ gives $\langle M,U\rangle \cap V = G \cap V = V = \langle M \cap V,U\rangle$ (as $M \cap V$ is maximal in V and $U \nleq M \cap V$).

ii. $V = M$ gives $\langle M,U\rangle \cap V = M = \langle M,U \cap V\rangle$.

$V = G$ gives $\langle M,U\rangle \cap V = \langle M,U\rangle = \langle M \cap V,U\rangle$.

So the theorem is proved. ||

The following theorem demonstrates that local arguments can be extensively used when examining the properties of modular subgroups in infinite groups.

First a definition:

<u>Definition</u>

M is said to be locally modular in G if for any natural number n and set of n elements $\{x_1, x_2, \ldots, x_n\}$ of G, M is modular in $\langle M, x_1, \ldots, x_n \rangle$.

Theorem 1.2.2.

M is modular in G if and only if M is locally modular in G.

Proof

Only if $M \text{ m } G \Rightarrow M \text{ m } H$ for all H such that $M \leq H \leq G$ (by 1.1.2) $\Rightarrow$ for all natural numbers n and elements $x_1, \cdots, x_n$ of G, $M \text{ m } \langle M, x_1, \ldots, x_n \rangle$.

If Suppose for a contradiction, M is locally modular in G but not modular in G. Then either

(a) there exist subgroups U,V of G such that $U \leq V$ but $\langle M,U \rangle \cap V \neq \langle M \cap V, U \rangle$ or

(b) there exist subgroups U,V of G such that $V \geq M$ but $\langle M,U \rangle \cap V \neq \langle M, U \cap V \rangle$.

In case (a), as $\langle M \cap V, U \rangle \leq \langle M,U \rangle \cap V$, there exists an element y of G such that $y \in \langle M,U \rangle \cap V \setminus \langle M \cap V, U \rangle$. Then $y \in V$ and there exist elements $u_1, \ldots, u_n$ of U such that $y \in \langle M, u_1, \ldots, u_n \rangle$ Let $U_1 = \langle u_1, \ldots, u_n \rangle$ (so $U_1 \leq U$). Let $V_1 = \langle u_1, \ldots, u_n, y \rangle$ (so $V_1 \leq V$ (as $U_1 \leq V$)). Then M locally modular $\Rightarrow M \text{ m } \langle M, V_1 \rangle$ $\Rightarrow \langle M, U_1 \rangle \cap V_1 = \langle M \cap V_1, U_1 \rangle$. So $y \in \langle M, U_1 \rangle \cap V_1 \Rightarrow y \in \langle M \cap V_1, U_1 \rangle$ and $\langle M \cap V_1, U_1 \rangle \leq \langle M \cap V, U \rangle$ contradicting our choice of y. So case (a) cannot hold.

For case (b), there exists some element y of G such that

$y \in \langle M,U\rangle \cap V \setminus \langle M,U \cap V\rangle$. As before, $y \in V$, and there exist elements $u_1, \dots, u_n$ of U such that $y \in \langle M,u_1, \dots, u_n\rangle$

M m $\langle M,u_1, \dots, u_n,y\rangle$. Thus $y \in \langle M,u_1, \dots, u_n\rangle \cap \langle M,y\rangle$

$$= \langle M,\langle u_1, \dots, u_n\rangle \cap \langle M,y\rangle\rangle$$

$$\leq \langle M,U \cap V\rangle (\text{as } M \leq V, y \in V)$$

contradicting our choice of y.

So M locally modular in G implies that M is modular in G as required.||

Similarly:

<u>Theorem 1.2.3.</u>

M m G if and only if for all finite sets of elements $\{a_1, \dots, a_n\}$ of G (n any natural number), $M \cap \langle a_1, \dots, a_n\rangle$ m $\langle a_1, \dots, a_n\rangle$.

<u>Proof</u>

<u>Only if</u> follows from 1.1.2

<u>If</u> Suppose M is not modular in G.

Suppose there are subgroups U,V of G, $U \leq V$ such that $\langle M,U\rangle \cap V \neq \langle M \cap V,U\rangle$ i.e. there exists some element $x \in \langle M,U\rangle \cap V \setminus \langle M \cap V,U\rangle$. Then there exist elements $m_1, \dots, m_n$ of M, $u_1, \dots, u_r$ of U such that $x \in \langle m_1, \dots, m_n, u_1, \dots, u_r\rangle \cap V$.

Let $A = \langle m_1, \dots, m_n, u_1, \dots, u_r\rangle$. Then $M \cap A$ m A.

$x \in \langle M \cap A, U \cap A\rangle \cap V \cap A = \langle M \cap A \cap V \cap A, U \cap A\rangle = \langle M \cap V \cap A, U \cap A\rangle$ $\leq \langle M \cap V,U\rangle$ which is a contradiction to the choice of x.

So there exist subgroups U,V of G such that V contains M and such that $\langle M,U\rangle \cap V \neq \langle M,U \cap V\rangle$ i.e. there exists an element

$z \in \langle M,U\rangle \cap V \smallsetminus \langle M, U\cap V\rangle$. Let $\{m_1, \dots, m_s\} \in M$, $\{u_1, \dots, u_t\} \in U$ be such that $z \in \langle m_1, \dots, m_s, u_1, \dots, u_t\rangle \cap V$. As before, let $A = \langle m_1, \dots, m_s, u_1, \dots, u_t\rangle$. Then $z \in \langle M\cap A, U\cap A\rangle \cap V \cap A = \langle M\cap A, U\cap V\cap A\rangle$ (as $M\cap A$ m A) $\leq \langle M, U\cap V\rangle$, contradicting the choice of z.

So M must be modular in G and the theorem is proved.‖

The next theorem establishes the connection between modular and quasinormal subgroups and generalises a result of Heineken quoted in Schmidt ([II]).

Theorem 1.2.4.

Let M be a subgroup of G.

Then M is quasinormal in G if and only if M is modular and ascendant in G.

Proof

Only if Suppose M qn G and U,V are subgroups of G such that V contains U. Then $\langle M,U\rangle \cap V = MU \cap V = (M\cap V)U = \langle M\cap V, U\rangle$.

Similarly, if U,V are subgroups of G and V contains M, we have that $\langle M,U\rangle \cap V = MU\cap V = (U\cap V)M = \langle U\cap V, M\rangle$. Hence M is modular in G.

M qn G $\Rightarrow$ M ascendant is proved by Stonehewer in ([VI])

if Suppose M is ascendant in G in ρ steps where ρ is some ordinal, i.e. there is a set of subgroups $\{M_\alpha \mid \alpha$ an ordinal, $\alpha \leq \rho\}$ such that $M_0 = M$, $M_\alpha \lhd M_{\alpha+1}$ for all α, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for all limit ordinals β and $M_\rho = G$.

We proceed by induction on ρ. If $\rho = 0$, $M = G$ and there is nothing to prove.

Suppose that ϱ is a limit ordinal, i.e. $G = \bigcup_{\alpha<\varrho} M_\alpha$. Let $g \in G$. Then there exists an α ($\alpha < \varrho$) such that $g \in M_\alpha$ and as by the induction hypothesis, M qn M_α, we have that $M\langle g\rangle = \langle M,g\rangle$ i.e. M qn G as required.

Now suppose that ϱ is not a limit ordinal i.e. $\varrho -1$ exists. $M \text{ m } M_{\varrho-1}$ and by the induction hypothesis, Mqn $M_{\varrho-1}$. Let $K \leq G$ and let $y \in \langle M,K\rangle$. Then $y \in \langle M_{\varrho-1},K\rangle$ (as $M \leq M_{\varrho-1}$)

$$= M_{\varrho-1}K \text{ (as } M_{\varrho-1} \lhd G).$$

Thus there exists some $k \in K$ such that $yk^{-1} \in \langle M,K\rangle \cap M_{\varrho-1}$ $= \langle M, K \cap M_{\varrho-1}\rangle$ (as $M_{\varrho-1} \geq M$, M m G) $= M(K \cap M_{\varrho-1})$ (as M qn$M_{\varrho-1}$) Therefore $y \in MK$. This is true for all $y \in \langle M,K\rangle$ and hence $\langle M,K\rangle \leq MK$, so $\langle M,K\rangle = MK$.

Thus M is quasinormal in G as required.||

Chapter Two

This chapter is concerned with the properties of modular subgroups in a very wide class of groups which we shall call $\mathfrak{X}$.

We define $\mathfrak{X}$ as follows:

Let $\mathfrak{Y} = \{G \mid M$ maximal among the modular subgroups of G and non-normal in $G \Rightarrow |G:M|$ finite$\}$

Let $\mathfrak{X} = \mathfrak{Y}^s$ i.e. the largest subclass of $\mathfrak{Y}$ which is subgroup closed (so $G \in \mathfrak{X}$ and $H \leq G \Rightarrow H \in \mathfrak{X}$).

By defining $\mathfrak{X}$ in this way, we exclude from consideration the Tarski group (in which every proper non-trivial subgroup has order p where p is an odd prime, and the group itself is infinite, ([VII] p.97). It is not known if such a group exists, but if one does, every proper subgroup is modular, maximal and non-normal, and the normal closure of any subgroup is the whole group.).

Theorem 2.1.1.

Let $G \in \mathfrak{Y}$ and let M be a non-normal maximal subgroup which is modular in G. Then $\frac{G}{M_G}$ is nonabelian of order pq where p and q are two primes.

Proof

$G \in \mathfrak{Y} \Rightarrow |G:M|$ finite $\Rightarrow \frac{G}{M_G}$ finite with $\frac{M}{M_G}$ a non-normal maximal modular subgroup of $\frac{G}{M_G}$ (1.1.6). The result follows from lemma 1 of Schmidt ([1]).

Note: if M is a non-normal subgroup of G, then by 1.2.1, there is no ambiguity involved in describing M as a maximal modular subgroup as M is maximal among the modular subgroups of G if and only if M is modular and a maximal subgroup of G.

Theorems 2.1.2. and 2.1.5. were suggested by Dr. S.E. Stonehewer.

Theorem 2.1.2

$$\iota\mathfrak{Y} \subset \mathfrak{Y}$$

Proof

Let $G \in \iota\mathfrak{Y}$ and let M be a non-normal maximal modular subgroup of G. We want to prove that $|G:M|$ is finite.

1. There exists a finitely generated subgroup F of G such that $M \cap F \ntriangleleft F$.

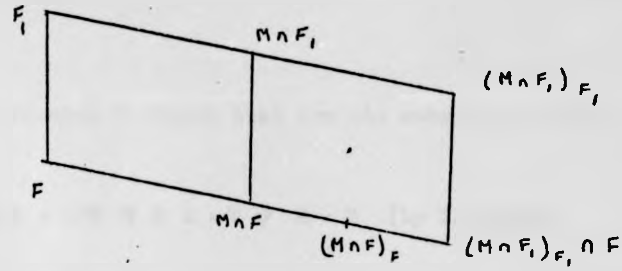
For, $M \ntriangleleft G \Rightarrow$ there exists an element $m \in M$ and an element $x \in G$ such that $m^x \notin M$. Let $F = \langle m,x \rangle$. Then $m \in M \cap F$ and $m^x \notin M \cap F$, so $M \cap F \ntriangleleft F$.

By 2.1.1.,1.2.1. and the facts that $M \cap F$ m F and $G \in \iota\mathfrak{Y}$ we have that $\frac{F}{(M \cap F)_F}$ is non-abelian of order pq where p and q are primes, $p > q$, say. So $|F:M \cap F| = p$

2. If F_1 is any finitely generated subgroup of G such that $F \leq F_1$, then $|F_1 : M \cap F_1| = p$

For $M \cap F_1$ m F_1, and $M \cap F_1 \triangleleft F_1$ would imply $M \cap F_1 \cap F \triangleleft F_1 \cap F$

i.e. $M \cap F \lhd F$, which is not so. So $M \cap F_1$ is not normal in F_1 and is a maximal modular subgroup of F_1 by 1.2.1. So $|F_1 : M \cap F_1|$ is finite as $G \in L\mathfrak{Y}$.



So, by 2.1.1., $\dfrac{F_1}{(M \cap F_1)_{F_1}}$ is non-abelian of order $p'q'$ (p', q' both primes, $p' > q'$).

Consider $\dfrac{F(M \cap F_1)_{F_1}}{(M \cap F_1)_{F_1}} \cong \dfrac{F}{F \cap (M \cap F_1)_{F_1}}$ and $F \cap (M \cap F_1)_{F_1} \leq M \cap F$ and is normal in F, so $F \cap (M \cap F_1)_{F_1} \leq (M \cap F)_F \leq F$

So $\left|\dfrac{F}{(M \cap F)_F}\right|$ (= pq) divides $\left|\dfrac{F(M \cap F_1)_{F_1}}{(M \cap F_1)_{F_1}}\right|$ so $\{p,q\} = \{p',q'\}$

and as $p > q$, $p' > q'$, $p = p'$ and $q = q'$

3. $|G:M| = p$.

For, suppose $|G:M| > p$. Choose $g_1, \ldots, g_{p+1}$ so that each element defines a different element of $\dfrac{G}{M}$ i.e. $g_i g_j^{-1} \notin M$ for all i,j such that $1 \leq i \neq j \leq p+1$. Let $K = \langle F, g_1, \ldots, g_{p+1} \rangle$ where F is the subgroup defined in 1.

Then, by 2, $|K : M \cap K| = p$ i.e. there exist i,j ($1 \leq i \neq j \leq p+1$) such that $g_i g_j^{-1} \in M \cap K \leq M$. This contradiction shows our supposition to be false and hence $|G:M| \leq p$. $|FM : M|$

$= p$ and $\langle F, M\rangle \leq G \Rightarrow |G:M| = p$. So $G \in \mathfrak{Y}$ as required.

Theorem 2.1.3.

$$\mathfrak{X} = L\mathfrak{X}$$

Proof

Let $G \in L\mathfrak{X}$. We wish to prove that for all subgroups H of G, $H \in \mathfrak{Y}$.

$H \leq G \Rightarrow H \in L\mathfrak{X} \Rightarrow H \in L\mathfrak{Y} \Rightarrow H \in \mathfrak{Y}$ (by 2.1.2).||

Theorem 2.1.4.

$$\mathfrak{Y} = Q\mathfrak{Y}$$

Proof

Obvious from 1.1.6.

Theorem 2.1.5.

$$\mathfrak{Y} = P\mathfrak{Y}$$

Proof

Let $G \in P\mathfrak{Y}$. Then there exists a normal subgroup, N, of G such that $N \in \mathfrak{Y}$, and $\frac{G}{N} \in \mathfrak{Y}$. Let M be a maximal non-normal modular subgroup of G. We want to prove that $|G:M|$ is finite.

If $N \leq M$, $\frac{M}{N} \text{ m } \frac{G}{N}$ (1.1.6), $\frac{M}{N}$ is maximal in $\frac{G}{N}$ (as M is maximal in G) and $\frac{M}{N}$ is not normal in $\frac{G}{N}$ (as M is not normal in G).

So, as $\frac{G}{N} \in \mathfrak{Y}$, $\left|\frac{G}{N} : \frac{M}{N}\right| = |G:M|$ is finite.

Suppose now that $N \not\leq M$. So M a maximal subgroup of G implies $NM = G$.

$M \cap N$ m N (1.1.2) and $[N/N \cap M] \cong [NM/M]$ so $N \cap M$ is a maximal subgroup of N.

$N \cap M \lhd N \Rightarrow |N:N \cap M|$ finite, and if $N \cap M \ntriangleleft N$, the same result is true as $N \in \mathcal{Y}$.

Thus, $|N:N \cap M| = |NM:M| = |G:M|$ is finite as required.//

Section Two

Here we investigate the properties of modular subgroups in $\mathfrak{X}$ groups following the pattern of Schmidt ([I]).

Firstly we consider the case when $[G/M]$ is a chain.

Theorem 2.2.1

Let $M \text{ m } G \in \mathfrak{X}$ and let $[G/M]$ be a chain of length n. Then $|G:M| = p^n$ where p is a prime.

Proof

Firstly we note that since $[G/M]$ is a chain, it is certainly a modular lattice. Hence every subgroup in $[G/M]$ is modular in $[G/M]$ and hence in G (by 1.1.3).

Let $M = M_0 < M_1 < \ . \ . \ . \ < M_n = G$ be the chain $[G/M]$.

So, as $G \in \mathfrak{X}$ and M_i is a maximal modular subgroup in M_{i+1} for all i, and $M_{i+1} \in \mathfrak{X}$, thus $|M_{i+1}:M_i|$ is finite for all i, whether M_i is normal in M_{i+1} or not, and hence $|G:M|$ is finite So $\left|\frac{G}{M_G}\right|$ is finite and the result follows from Schmidt lemma 2 ([I])

Definition

We call the group G a P-group (see ([VIII])) if either G is an elementary abelian p-group, or G = AB, where $A = \langle a_1, a_2, , , a_n \rangle$ $B = \langle b \rangle$, $a_i^p = b^q = 1$ for all i, $a_i a_j = a_j a_i$ for all i, j and $b^{-1}ab = a^r$ for all $a \in A$, where $r \not\equiv 1 (\text{mod } p), r^q \equiv 1 (\text{mod } p)$.

It is well known that the lattice of subgroups of a P-group is isomorphic to the lattice of subgroups of an elementary abelian p-group of suitable size.

Theorem 2.2.2

$G \in \mathfrak{X}$ and $[G/M]$ a chain of length n implies that either $\frac{G}{M_G}$ is a p-group or n = 1 and $\frac{G}{M_G}$ is a P-group of order pq.

Proof

This follows from 2.2.1 which shows that with the given hypothesis $|G:M|$ and hence $\frac{G}{M_G}$ is finite, and then from Schmidt ([1]) Lemma 3. As P-groups are lattice isomorphic to element-ary abelian p-groups, if $\frac{G}{M_G}$ is a P-group, then $[G/M]$ must be a chain of length 1 and thus $\frac{G}{M_G}$ has order pq. ||

Theorem 2.2.3

Let $G \in \mathfrak{X}$ and suppose that $[G/M]$ is a chain. Then $\frac{G}{M_G} = \frac{M}{M_G} \cdot \frac{P}{M_G}$ where $\frac{P}{M_G}$ is a cyclic p-group.

Proof

Without loss of generality, we may take $M_G = 1$. By 2.2.2., we have that either G is a P-group of order pq, or G is a p-group.

In the former case, we have that G = AM where A is a subgroup of G of order p $(p > q)$.

In the latter case, let H be a maximal subgroup of G containing M and let $x \in G \setminus H$. Then $\langle M,x\rangle \nleq H$ implies that

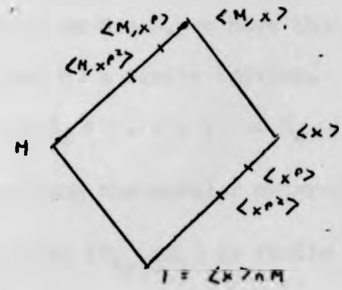
$G = \langle M,x\rangle$. But G a p-group implies that M is subnormal in G and hence quasinormal in G by 1.2.4.. Hence $G = M\langle x\rangle$ and the theorem is proved.||

<u>Theorem 2.2.4</u> Let $G \in \mathfrak{X}$

Let M m G and let x be an element of G such that x is torsion-free and $M \cap \langle x\rangle = 1$. Then $M \lhd \langle M,x\rangle$.

<u>Proof</u>

M modular in G implies that $[\langle M,x\rangle/M] \cong [\langle x\rangle/\langle x\rangle \cap M]$ by 1.1.1. So $[\langle M,x\rangle/M]$ is a modular lattice. $[\langle x\rangle/\langle x^{p^2}\rangle]$ is a chain of length 2, where p is any prime, so by the lattice isomorphism, $[\langle M,x\rangle/\langle M,x^{p^2}\rangle]$ is a chain of length 2.



By 2.2.2., $\dfrac{\langle M,x\rangle}{\underline{\langle M,x^{p^2}\rangle}}$ must be a p-group, where $\underline{\langle M,x^{p^2}\rangle}$ denotes the core of $\langle M,x^{p^2}\rangle$ in $\langle M,x\rangle$. So $\dfrac{\langle M,x^p\rangle}{\underline{\langle M,x^{p^2}\rangle}} \lhd \dfrac{\langle M,x\rangle}{\underline{\langle M,x^{p^2}\rangle}}$ as a maximal subgroup of a p-group, and hence $\langle M,x^p\rangle \lhd \langle M,x\rangle$.

Hence setting $K = \bigcap_p \langle M,x^p\rangle$, p running over all primes, we see that K is normal in $\langle M,x\rangle$. $M \leq K$ and, in fact, $M = K$, as otherwise $[\langle M,x\rangle/K]$ would be finite.

So $M \lhd \langle M,x\rangle$ as required. ||

This gives rise to some more general results:

Theorem 2.2.5.

Let M m $G \in \mathfrak{X}$. Then $\frac{M}{M_G} \in R(\mathfrak{n} \cap \mathfrak{F})$.

Proof

Without loss of generality, we may assume that $M_G = 1$.

Then we shall prove that $\frac{M}{M_{\langle M,x\rangle}} \in (\mathfrak{n} \cap \mathfrak{F})$ for all $x \in G$.

(For, as $M_{\langle M,x\rangle} \leq M \cap M^x$, we have that $\bigcap_{x \in G} M_{\langle M,x\rangle} \leq \bigcap_{x \in G} M^x = 1$.

Hence, $M \in R(\mathfrak{n} \cap \mathfrak{F})$ as required).

We note that as M m G, we have that $[\langle M,x\rangle / M] \cong [\langle x\rangle / \langle x\rangle \cap M]$

case a $[\langle x\rangle / \langle x\rangle \cap M]$ a finite lattice.

Let $M = M_0 < M_1 < \dots < M_n = \langle M,x\rangle$ be a chain so that M_i is maximal among the modular subgroups of M_{i+1} for all i. If $M_i \not\lhd M_{i+1}$, then $|M_{i+1}:M_i|$ is finite as $G \in \mathfrak{X}$. If $M_i \lhd M_{i+1}$ then $|M_{i+1}:M_i|$ is finite as $\frac{M_{i+1}}{M_i}$ is a group with a finite lattice. So $|\langle M,x\rangle : M|$ is finite and hence so is $|\langle M,x\rangle : M_{\langle M,x\rangle}|$ so $\frac{M}{M_{\langle M,x\rangle}} \in (\mathfrak{n} \cap \mathfrak{F})$ by Schmidt Theorem 2 ([I]).

case b $[\langle x\rangle / \langle x\rangle \cap M]$ is an infinite lattice.

Here we have x having infinite order and $\langle x\rangle \cap M = 1$. So, by 2.2.4., we have that $M \lhd \langle M,x\rangle$ i.e. $M_{\langle M,x\rangle} = M$.

Then the theorem is proved as indicated above. ||

Corollary 2.2.6.

$G \in L\mathfrak{F}$ and $M \text{ m } G \Rightarrow \frac{M}{M_G} \in L(\mathfrak{N} \cap \mathfrak{F})$.

Proof

(Note: $G \in L\mathfrak{X}$ and hence $G \in \mathfrak{X}$ by 2.1.3.)

We may assume, as usual, that the core of M in G = 1. Then by 2.2.5., $M \in R(\mathfrak{N} \cap \mathfrak{F})$. Let $\langle x_1, \ldots, x_n \rangle = F \leq M$ (so F is finite). $M \in R(\mathfrak{N} \cap \mathfrak{F}) \Rightarrow$ there exists some index set I such that $\bigcap_{i \in I} \{N_i \mid N_i \lhd M, \frac{M}{N_i} \in \mathfrak{N} \cap \mathfrak{F}\} = 1$. Hence $\bigcap_{i \in I} N_i \cap F = 1$ and

$\frac{F}{N_i \cap F} \cong \frac{FN_i}{N_i} \leq \frac{M}{N_i} \in \mathfrak{N}$, so $F \in R\mathfrak{N}$, and as F is finite, so only finitely many of the subgroups $\{N_i \cap F \mid i \in I\}$ can be distinct, thus we have $F \in R_0\mathfrak{N} = \mathfrak{N}$. (by Fitting, see e.g. ([XII]) p. 49)

So $M \in L(\mathfrak{N} \cap \mathfrak{F})$ as required.||

Theorem 2.2.7.

Let $G \in \mathfrak{X}$, and let M be a corefree modular subgroup of G. Then $M^G \in LRSS$.

Proof

Let n be any natural number, $\{g_1, \ldots, g_n\}$ be any set of elements of G, and $J = \langle M^{g_1}, \ldots, M^{g_n} \rangle$.

1. Then $|J:M^{g_i}|$ is finite for all i.

Without loss of generality, we may take $g_1 = 1$, and $M, M^{g_2} \ldots$ to be distinct conjugates of M. (*)

We write $J_i = \langle M, M^{g_2}, \ldots M^{g_i} \rangle$. Then it is enough to prove that $|J_i : J_{i-1}|$ is finite for all i such that $1 < i \leq n$.

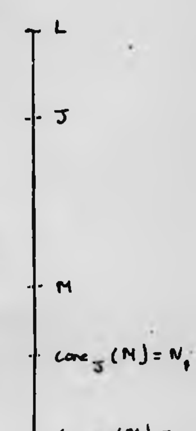
(Without loss of generality, we may take M^{g_i} to be M^{g_1} i.e. M). We note that $J_{i+1} \leq \langle J_i, g_{i+1} \rangle$ (as $M \leq J_i$ and $J_{i+1} = \langle J_i, M^{g_{i+1}} \rangle$), and $[\langle J_i, g_{i+1} \rangle / J_i] \cong [\langle g_{i+1} \rangle / \langle g_{i+1} \rangle \cap J_i]$ (as J_i m G by 1.1.5 and 1.1.4). If this lattice is finite, we have $|\langle J_i, g_{i+1} \rangle : J_i|$ is finite, as $G \in \mathfrak{X}$, and hence so is $|J_{i+1} : J_i|$ as required. If the lattice is infinite, we have that g_{i+1} is torsion-free and $\langle g_{i+1} \rangle \cap J_i = 1$. $M \leq J_i \Rightarrow M \cap \langle g_{i+1} \rangle = 1$, so by 2.2.4., $M \lhd \langle M, g_{i+1} \rangle$ i.e. $M^{g_{i+1}} = M$ which contradicts (*)

So $|J : M^{g_i}|$ is finite for all i.

2. $\dfrac{J}{\text{core}_J(M^{g_i})} \in \mathfrak{F} \cap ss$ for all i.

Let $N_i = \text{core}_J(M^{g_i})$. Without loss of generality we may take $M^{g_i} = M$ and may assume that $n > 2$.

We know from 1, that $\dfrac{J}{N_i}$ is finite.



Let $K = \langle M^{g_3}, \ldots, M^{g_n} \rangle$. Then let $L = \langle M, K, g_2 \rangle = \langle J, g_2 \rangle$. J m G $\Rightarrow [\langle J, g_2 \rangle / J] \cong [\langle g_2 \rangle / \langle g_2 \rangle \cap J]$

and hence as in 1, we have that $|\langle J, g_2 \rangle : J|$ is finite. We know $|J : N_1|$ is finite, hence $|L : N_1|$ must be finite, and, in particular, $|L : M|$ is finite. So $\dfrac{L}{\text{core}_L(M)}$, which we will write $\dfrac{L}{C}$, is finite.

As $\dfrac{M}{C}$ m $\dfrac{L}{C}$, we have that, by Schmidt, corollary to the main theorem of ([II]), that

$$\frac{L}{C} = \frac{P_1}{C} \times \cdot \cdot \times \frac{P_r}{C} \times \frac{K}{C}$$

and $\frac{M}{C} = \frac{Q_1}{C} \times \cdot \cdot \cdot \cdot \times \frac{Q_r}{C} \times \frac{M \cap K}{C}$

where the notation is as in Schmidt ([II]) i.e. $\frac{Q_i}{C}^L = \frac{P_i}{C}$

for all i, $\left|\frac{Q_i}{C}\right| = q_i$ and $\frac{P_i}{C}$ is a P_i-group, $\frac{M \cap K}{C}$ qn $\frac{L}{C}$.

So $\frac{M^L}{C} = \frac{P_1}{C} \times \cdot \cdot \cdot \cdot \times \frac{P_r}{C} \times \frac{(M \cap K)^L}{C}$

$\frac{P_i}{C}$ is supersoluble for all i, and as

any subgroup normal in $\frac{P_i}{C}$ is normal in $\frac{L}{C}$, so $\frac{P_i}{C}$ has an L-

invariant cyclic series for all i. $\frac{(M \cap K)^L}{C} \leq Z_{\sim}\left(\frac{L}{C}\right)$ by

Maier-Schmid ([IX]). Hence $\frac{(M \cap K)^L}{C}$ has an L-invariant cyclic

series. So $\frac{M^L}{C}$ has an L-invariant cyclic series. Thus $\frac{M^L \cap J}{C}$

has a J-invariant cyclic seriesand thus so has $\frac{M^L \cap J}{N_1}$.

($M^{g_2} \leq J \cap M^L$, note).

We repeat this procedure,redefining K to omit M^{g_3} and replace M^{g_2}, and letting $L = \langle M, K, g_3 \rangle$ and then omitting M^{g_4} and replacing M^{g_3} etc. Then $\frac{J}{N_1}$ is a subgroup of the product of the $\frac{M^L \cap J}{N_1}$'s and as this product has a J-invariant cyclic series, so $\frac{J}{N_1}$ is supersoluble as required.

3. Finally, we let F be a finitely generated subgroup of M^G. Then there exist elements $g_1, \ldots, g_n$ of G such that $F \leq \langle M^{g_1}, \ldots, M^{g_n} \rangle$. Let g be any element of G and let $J(g) = \langle M^g, M^{g_1}, \ldots, M^{g_n} \rangle$. Then if $N(g) = \mathrm{core}_{J(g)} M^g$, we have by 2 that $\frac{J(g)}{N(g)} \in \mathfrak{F} \cap \mathfrak{S}\mathfrak{S}$ i.e. $\frac{FN(g)}{N(g)} \in \mathfrak{F} \cap \mathfrak{S}\mathfrak{S}$ i.e. $\frac{F}{F \cap N(g)} \in \mathfrak{F} \cap \mathfrak{S}\mathfrak{S}$.

We repeat this for all $g \in G$ and note that $\bigcap_g F \cap N(g) \leq \bigcap_g N(g) \leq \bigcap_g M^g = 1$. So $\bigcap \{ M : M \triangleleft F, \frac{F}{M} \in \mathfrak{F} \cap \mathfrak{S}\mathfrak{S} \} = 1$ i.e. $F \in R\mathfrak{S}\mathfrak{S}$.

So $M^G \in L(R\mathfrak{S}\mathfrak{S})$ as required.//

<u>Corollary 2.2.8.</u>

Let $G \in L\mathfrak{F}$ and let M be a corefree modular subgroup of G. Then M^G is locally supersoluble.

<u>Proof</u>

By 2.2.7., we have that M^G is locally residually supersoluble. Let F be any finitely generated subgroup of M^G. Then F is finite and F residually supersoluble $\Rightarrow F \in R_0\mathfrak{S}\mathfrak{S} = \mathfrak{S}\mathfrak{S}$ as required.//

<u>Theorem 2.2.9.</u>

Let G be a finite group and let M be a corefree modular subgroup of G. Then $\frac{G}{C_G(M^G)}$ is supersoluble. (This result is well-known but apparently unpublished).

<u>Proof</u>

By the main theorem of Schmidt ([II]), we have that $G = P_1 \times \quad \times P_r \times K$ and $M = Q_1 \times \quad \times Q_r \times M \cap K$ (notation as in Schmidt; $M \cap K$ qn G). $M^G = P_1 \times \quad \times P_r \times (M \cap K)^G$. We consider the automorphism group induced on M^G by conjugation by elements of G. As a P-group induces a supersoluble group of automorphisms on itself by conjugation, and as by Maier-Schmid ([IX]) we have that the group of automorphisms induced by K on $(M \cap K)^K$ ($= (M \cap K)^G$, note) is nilpotent, the theorem is proved. ||

Lemma 2.2.10

Let G be locally finite, M a corefree modular subgroup of G which is finite. Then $\frac{G}{C_G(M^G)}$ is locally supersoluble.

Proof

We write C for $C_G(M^G)$. Then any finitely generated subgroup of $\frac{G}{C}$ is of the form $\frac{FC}{C}$ where F is a finitely generated (and hence finite) subgroup of G.

Let $\mathcal{S} = \{K \mid K \leq G,\ K \text{ finite},\ \langle M,F\rangle \leq K,\ M_K = 1\}$

(as M is finite, $\mathcal{S} \neq \emptyset$ as the intersection of only a finite number of conjugates of M, $M^{g_1}, \ldots, M^{g_n}$, say, is trivial. Hence $\langle M,F,g_1,\ldots,g_n\rangle$ is finite and belongs to $\mathcal{S}$)

Let $K \in \mathcal{S}$. Then, as K is finite, by 2.2.9., we have that $\frac{K}{C_K(M^K)}$ is supersoluble. We write C_K for $C_K(M^K)$. As $C \lhd G$, we have that $\frac{KC}{C_K C}$ is supersoluble.

Let $K_1 \in \mathcal{S}$ be such that $K \leq K_1$ (e.g. $K_1 = \langle K, x \rangle$ for $x \in G \setminus K$)

Then $\frac{KC\, C_{K_1} C}{C_{K_1} C} \leq \frac{K_1 C}{C_{K_1} C}$ which is supersoluble (where $C_{K_1} = C_{K_1}(M^{K_1})$). Hence $\frac{KC}{KC \cap C_{K_1} C}$ is supersoluble.

So $\frac{KC}{\bigcap\limits_{\substack{K_1 \in \mathcal{S} \\ K_1 \geq K}} KC \cap C_{K_1} C}$ is residually supersoluble. (*)

But consider $\bigcap\limits_{\substack{K_1 \in \mathcal{S} \\ K_1 \geq K}} C_{K_1} C = D$, say. Obviously, $C \leq D$, and we shall prove that $D \leq C$. For, let $y \in D$, $a \in M^G$. Then there exist elements $y_1, , , y_n$ such that $a \in \langle M^{y_1}, , , M^{y_n} \rangle$ Let $K_2 = \langle K, y_1, , , , y_n \rangle$ $K_2 \in \mathcal{S}$ and $y \in D \Rightarrow y \in C_{K_2} C$ (using our usual notation). But $a \in M^{K_2}$. So y centralises a. This is true for all $a \in M^G$ so $y \in C_G(M^G) = C$ as required. Hence $C = D$.

From (*), we have that $\frac{KC}{KC \cap D}$ is residually supersoluble So $\frac{KC}{KC \cap C} = \frac{KC}{C}$ is residually supersoluble. Hence $\frac{FC}{C}$ $\left(\leq \frac{KC}{C}\right)$ is residually supersoluble , and as $\frac{FC}{C}$ is finite and $ss = R_0 ss$, $\frac{FC}{C}$ is supersoluble as required. ||

<u>Theorem 2.2.11</u>.

Let G be a locally finite group and let M be a corefree modular subgroup of G. Then $\frac{G}{C_G(M^G)}$ is locally supersoluble.

<u>Proof</u>

We write as before $C = C_G(M^G)$.

Let $\frac{A}{C}$ be a finitely generated subgroup of $\frac{G}{C}$ (Then $\frac{A}{C}$ $= \frac{FC}{C}$ where F is a finitely generated, and hence finite, subgroup of G.) We wish to prove that $\frac{A}{C}$ is supersoluble.

Let $X(F) = C\langle M,F\rangle$. Then it is easy to see that $M^{X(F)} = M^{\langle M,F\rangle}$, $M_{X(F)} = M_{\langle M,F\rangle}$. As $[\langle M,F\rangle / M] \cong [F/F \cap M]$ is a finite lattice, and $\langle M,F\rangle$, inparticular, belongs to $\mathfrak{X}$, we have that $|\langle M,F\rangle : M|$ and hence $\frac{\langle M,F\rangle}{M_{\langle M,F\rangle}}$ is finite. So, in particular,

$\frac{M}{M_{\langle M,F\rangle}}$ is finite, and by 2.2.10., we have that $\frac{X(F)}{C_{X(F)}\left(\frac{M^{X(F)}}{M_{X(F)}}\right)}$

is locally supersoluble. Writing $C_{X(F)}$ for $C_{X(F)}\left(\frac{M^{X(F)}}{M_{X(F)}}\right)$

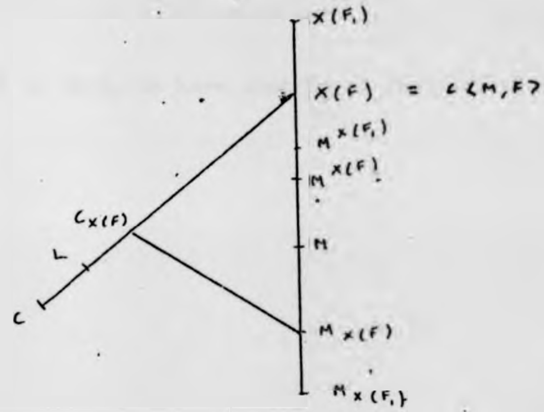
we have that $M_{X(F)} \leq C_{X(F)}$ and $C \leq C_{X(F)}$

Let $\frac{L}{C}$ be the locally supersoluble residual of $\frac{X(F)}{C}$ (such a residual exists because $G \in L\mathfrak{F}$ and $R_{SS} = SS$) We wish to prove that in fact $L = C$. Let $a \in M^G$. Then there will exist elements $y_1, , ,$ y_n of G such that $a \in \langle M^{y_1}, , , , M^{y_n}\rangle$.

Let $X(F_1) = C\langle M,F,y_1, , , , y_n\rangle$. As $F_1 = \langle F,y_1, , y_n\rangle$

is finite, we have, as before, that $\frac{M}{M_{\langle M,F_1\rangle}} = \frac{M}{M_{X(F_1)}}$ is finite

and hence $\frac{X(F_1)}{C_{X(F_1)}\left(\frac{M^{X(F_1)}}{M_{X(F_1)}}\right)}$ is locally supersoluble.



Write $C_{X(F_1)}$ for $C_{X(F_1)}\left(\frac{M^{X(F_1)}}{M_{X(F_1)}}\right)$. Then $\frac{X(F)C_{X(F_1)}}{C_{X(F_1)}} \leq \frac{X(F_1)}{C_{X(F_1)}}$

which is locally supersoluble, so $\frac{X(F)}{X(F)\cap C_{X(F_1)}}$ is locally supersoluble. Hence $L \leq X(F)\cap C_{X(F_1)} \leq C_{X(F_1)}$.

So $a \in M^{X(F_1)} \Rightarrow [L,a] \leq M_{X(F_1)}$. Repeating this argument for all K such that F_1 is contained in K and K is finite, gives us that $\frac{X(K)}{C_{X(K)}}$ is locally supersoluble (where the notation is as usual). Hence $\frac{X(F)\,C_{X(K)}}{C_{X(K)}}$ is locally supersoluble, so

$L \leq X(F) \cap C_{X(K)}$ and $a \in M^{X(K)} \Rightarrow [L,a] \leq M_{X(K)}$. Hence

$[L,a] \leq \bigcap_{K} M_{X(K)} \leq \bigcap_{x \in G} M \cap M^{x} = M_G = 1$ i.e. $L \leq C_G(a)$.

This is true for all elements a of M^G so $L \leq C$ i.e. $L = C$.

Hence we have established that $\frac{X(F)}{C}$ is locally supersoluble so as $F \leq X(F)$, we have that $\frac{FC}{C}$ is supersoluble as required.॥

$L \leq X(F) \cap C_{X(K)}$ and $a \in M^{X(K)} \Rightarrow [L,a] \leq M_{X(K)}$. Hence

$[L,a] \leq \bigcap_{K} M_{X(K)} \leq \bigcap_{x \in G} M \cap M^{x} = M_G = 1$ i.e. $L \leq C_G(a)$.

This is true for all elements a of M^G so $L \leq C$ i.e.

$L = C$.

Hence we have established that $\frac{X(F)}{C}$ is locally supersoluble

so as $F \leq X(F)$, we have that $\frac{FC}{C}$ is supersoluble as required.||

$L \leq X(F) \cap C_{X(K)}$ and $a \in M^{X(K)} \Rightarrow [L,a] \leq M_{X(K)}$. Hence

$[L,a] \leq \bigcap_{K} M_{X(K)} \leq \bigcap_{x \in G} M \cap M^{x} = M_G = 1$ i.e. $L \leq C_G(a)$.

This is true for all elements a of M^G so $L \leq C$ i.e.

$L = C$.

Hence we have established that $\frac{X(F)}{C}$ is locally supersoluble so as $F \leq X(F)$, we have that $\frac{FC}{C}$ is supersoluble as required.||

Chapter Three

I started my investigation into modular subgroups by investigating their properties in locally finite groups with the minimum condition on subgroups (it is well known that such groups are finite extensions of a direct product of a finite number of quasicyclic groups (VII). This chapter is concerned with that theory. More general results will be proved in Chapter four.

Lemma 3.1.1.

Let $G \in (L\mathfrak{N})\mathfrak{F}$. Then $G \in \mathfrak{X}$.

Proof

Let M be a maximal non-normal modular subgroup of a nilpotent group G. But 1.2.1. $\Rightarrow$ M is a maximal subgroup of G and hence normal in G. So $\mathfrak{N} \in \mathfrak{X}$. Then by 2.1.3. $L\mathfrak{N} \in \mathfrak{X}$ and by 2.1.5. $(L\mathfrak{N})\mathfrak{F} \in \mathfrak{X}$ ||

Lemma 3.1.2.

Suppose $G = AM$ where $A \cong C_{p^\infty}$, $M \cong C_q$ where q and p are distinct primes, and $A \lhd G$.

Then $M \ntriangleleft G \Rightarrow M$ not modular in G.

Proof

Consider the set of automorphisms $\alpha_m : A \longrightarrow A$ defined by $\alpha_m(a) = mam^{-1}$ for all elements a of A. $M \ntriangleleft G \Rightarrow M \not\leq C_G(A)$ $\Rightarrow$ there exists some m such that α_m is not the identity automorphism. As $|\langle m \rangle| = |M| = q$, α_m has finite order. So by Robinson,[V], lemma 2.36 page 55, either p is odd and α_m does not fix every element of order p in A or p = 2 and α_m does not fix every element of order 4. (*)

We write $A = \bigcup_i A_i$ where $A_0 = 1$, $A_{i+1}^p = A_i$ and consider $\langle M, A_2 \rangle$. (So $A_2 = \langle a \rangle$ where $a^{p^2} = 1$).

If M m G, then M m MA_2 by 1.1.2 and $\langle M, U \rangle \cap V = \langle M, U \cap V \rangle$ for all $U, V \leq MA_2$ and $M \leq V$ (note: $A_2 = \langle x \in A \mid x^{p^2} = 1 \rangle$ is characteristic in A and hence normal in G).

We choose $V = A_1 M$, $U = \langle am \rangle$.

Then $\langle M, U \rangle \cap V = \langle m, am \rangle \cap A_1 M$.

We now investigate $\langle M, U \cap V \rangle$. The possibilities for $|U|$ are p^2q, pq, q or p.

$|U| = p^2 q$ is ruled out immediately, as then $U = MA_2$ so MA_2 would be cyclic with $M \leq C_G(A_2)$ contradicting (*)

$|U| = p \Rightarrow U \leq A_2 \Rightarrow U = A_1$ which is impossible as $M \cap A = 1$.

$|U| = pq \Rightarrow$ (as U is cyclic) that the subgroup of order p in U (i.e. A_1 as the only subgroup of order p in MA_2) commutes with a subgroup of order q i.e. a conjugate of M which implies that $[M, A_1] = 1$.

If p is odd, this contradicts (*), so the only possibility is $|U| = q$. So $U \cap V = U$ or 1 i.e. $\langle M, U \cap V \rangle = \langle m \rangle$ or $\langle m, am \rangle$ $(= MA_2)$, neither of which equal $A_1 M$ i.e. V $(= \langle U, M \rangle \cap V)$. So $M \ntriangleleft\triangleleft G \Rightarrow M$ not modular in G in the case where p is odd.

If p is even, we consider the map $\alpha : M \longrightarrow \mathrm{Aut}(A_2)$ defined by $\alpha(m) = \alpha_m$ for all $m \in M$. This is a homomorphism and is injective (as $M \nleq C_G(A_2)$ by (*)). So $|M| \mid |\mathrm{Aut}(C_4)|$ i.e. $M = 2$ which contradicts our hypothesis that $p \neq q$. Thus p cannot be even and our result is proved. ||

Theorem 3.1.3.

Let $M \text{ m } G \in \mathfrak{X} \cap \text{min}$. Then $\frac{M}{M_G} \in \mathfrak{N} \cap \mathfrak{F}$

(Here $G \in$ min means that G has the minimal condition on subgroups).

Proof

Without loss of generality, let $M_G = 1$.

Then by 2.2.5, $M \in R(\mathfrak{N} \cap \mathfrak{F})$.

Let $\mathcal{S} = \{ \bigcap_{i=1}^{n} H_i \mid n$ some natural number, $H_i \triangleleft M, \frac{M}{H_i} \in \mathfrak{N} \cap \mathfrak{F} \}$

$\mathcal{S}$ is non-empty. Let K be a minimal subgroup in $\mathcal{S}$. So $K = \bigcap_{i=1}^{m} H_i$. Suppose $K \neq 1$ i.e. there exists an element x such that $1 \neq x \in K$. As $M \in R(\mathfrak{N} \cap \mathfrak{F})$, so $\bigcap \{ H \mid H \triangleleft M, \frac{M}{H} \in \mathfrak{N} \cap \mathfrak{F} \} = 1$, there exists $N_x \triangleleft M$ such that $x \notin N_x$ and $\frac{M}{N_x} \in \mathfrak{N} \cap \mathfrak{F}$. Let $\overline{K} = K \cap N_x$.

$\overline{K} \in \mathfrak{S}$, and $\overline{K} \lneq K$ (as $x \in K \setminus \overline{K}$) which contradicts the minimality of K. So K = 1 and $M \in R_0(\mathfrak{N} \cap \mathfrak{F}) = \mathfrak{N} \cap \mathfrak{F}$ as required. ||

We now come to the main theorem of this section:

<u>Theorem 3.1.4</u>

Let $G \in L\mathfrak{Z} \cap \min$ and let A be the minimal normal subgroup of finite index in G. Let M be a corefree modular subgroup of G. Then $A \leq C_G(M)$.

<u>Proof</u>

Firstly we assume that M is a p-group for some prime p. (Note that by 3.1.3, we know that M is finite).

Let H_q be the direct product of all the quasicyclic q-groups for some prime q i.e. $H_q = \langle x \in A \mid x^{q^n} = 1$ for some $n \rangle$. Then H_q a characteristic subgroup of A $\Rightarrow$ H_q a normal subgroup of G.

<u>case a</u> p = q

It is easy to see that $MH_p \in L\mathfrak{E}^*_p$ i.e. MH_p is locally nilpotent. H_p is countable so we let $\{g_1, g_2, \dots g_n, \dots\}$ be the generators of H_p. $\langle M, g_1, \dots, g_n \rangle$ is nilpotent, so $\langle M, g_1, \dots, g_{n-1} \rangle$ is subnormal in $\langle M, g_1, \dots, g_n \rangle$ for all $n > 1$ i.e. M is ascendent in MH_p. So, by 1.2.4, M is quasinormal in MH_p. M is finite hence M is subnormal in MH_p ([XV]). So there exists a series of subgroups $M = M_0 < M_1 < \dots < M_n = MH_p$ such that $M_i \lhd M_{i+1}$ for all i.

Then M_i is nilpotent for all i.

We prove this claim by induction. When $i = 0$, $M_i = M$ which is nilpotent by 3.1.3. When $i > 0$, $M_i = MH_p \cap M_i = M_{i-1}H_p \cap M_i$ $= (H_p \cap M_i)M_{i-1}$ (as $M \leq M_{i-1} \leq M_i$)

$M_{i-1} \lhd M_i$ and $M_{i-1} \in \mathfrak{N}$ by induction. Also $H_p \cap M_i \lhd M_i$ and $H_p \cap M_i$ is nilpotent as H_p is, so by Fitting's theorem, (VII) p 49, M_i is nilpotent. This is true for all i, so in particular is true when $i = n$ i.e. MH_p is nilpotent. So by Robinson, (LV) lemma 2.32 page 51, we have $[M, H_p] = 1$ i.e. $H_p \leq C_G(M)$ as required.

<u>case b</u> $p \neq q$

Let $H \leq H_q$ be such that $H \cong C_{q^\infty}$. By 4.1.1. and 3.1.1 we have that $MH = HM$ i.e. MH is a subgroup of MH_q.

$MH \in L\mathfrak{F}$ min so has a minimal (normal subgroup of finite index), B, say. As $|BH:H| = |B:B\cap H|$, and $|BH:H| \mid |MH:H|$ and $|MH:H| = |M|$, so $|B:B\cap H|$ is finite and hence so is $|MH:B\cap H|$ and hence $|MH:(B\cap H)_{MH}|$. This contradicts the minimality of B unless $B = (B\cap H)_{MH} = B\cap H$ i.e. $B \leq H$.

Suppose $B \lneq H$. Then $\frac{H}{B}$ is a finite complete group which is impossible, so $H = B$.

Hence $H \lhd MH$. If $|M| = p$, M m $MH \Rightarrow M \lhd MH$ (by 3.1.2.) $[M, H] \leq M \cap H = 1$. So $H \leq C_G(M)$ as required.

If $|M| = p^n$, we prove $H \leq C_G(M)$ by using induction on n.

$n = 1$ is proved above. Suppose $n > 1$. Then there exists a subgroup $\overline{M} \lhd M$ such that $|\overline{M}| = p^{n-1}$. $M \cap H\overline{M} \,\mathrm{m}\, H\overline{M}$ by 1.1.2. but $M \cap H\overline{M} = (H \cap M)\overline{M} = \overline{M}$. It is easy to see that H is the minimal normal subgroup of finite index in $H\overline{M}$ (proof as above). So by the induction hypothesis, $H \leq C_G(\overline{M})$. This implies, in particular, that $\overline{M} \lhd HM$. Now $\frac{M}{\overline{M}} \,\mathrm{m}\, \frac{MH}{\overline{M}}$, we have that

$\left|\frac{M}{\overline{M}}\right| = p$, and it can be seen easily that $\frac{H\overline{M}}{\overline{M}}$ is the minimal normal

subgroup of finite index in $\frac{HM}{\overline{M}}$. So our previous argument gives

us that $\frac{H\overline{M}}{\overline{M}} \leq C_{\frac{HM}{\overline{M}}}\left(\frac{M}{\overline{M}}\right)$. Thus, $[H\overline{M}, M] \leq \overline{M}$ and so $[H, M] \leq H \cap M = 1$

So $H \leq C_G(M)$ as required.

So we have that $A \leq C_G(M)$ whenever M is of prime power order.

We now drop the assumption that M is of prime power order.

Let $x \in M$. By splitting x up into its p-potent and p-prime parts, we may assume that x is of prime power order.

$M \cap A \lhd M$ and $M \cap A \lhd A \Rightarrow M \cap A \lhd AM$. $\frac{M}{M \cap A} \,\mathrm{m}\, \frac{MA}{M \cap A}$ so

$\frac{M}{M \cap A} \cap \frac{\langle x \rangle A}{M \cap A} \,\mathrm{m}\, \frac{\langle x \rangle A}{M \cap A}$ i.e. $\frac{M \cap \langle x \rangle A}{M \cap A} = \frac{(A \cap M)\langle x \rangle}{M \cap A} \,\mathrm{m}\, \frac{\langle x \rangle A}{M \cap A}$ and by

our previous discussion, we have $\frac{A}{M \cap A} \leq C_{\frac{AM}{M \cap A}} \frac{\langle x \rangle (A \cap M)}{A \cap M}$

i.e. $[A, x] \leq M \cap A \leq M$ i.e. $a^{-1}xa \in M$ for all $a \in A$.

This is true for every element x of M, so $A \leq N_G(M)$.

Thus $M \lhd MA$, $A \lhd MA$, A and M both nilpotent $\Rightarrow$ (by Fitting ([VII])) that MA is nilpotent. Hence by Robinson,

$(\text{[V3]}), [M, A] = 1$ i.e. $A \leq C_G(M)$ as required. ||

Section Two

Here we show that using 3.1.4. enables Schmidt's results in the finite case to be carried over easily to locally finite groups with the minimum condition on subgroups.

Theorem 3.2.1.

$M \text{ m } G \in L\mathfrak{F} \cap \min \Rightarrow \frac{M^G}{M_G}$ is supersoluble.

Proof

Without loss of generality, let the core of M in G be 1.

Let A be the minimal normal subgroup of finite index in G. Then $A \leq C_G(M)$ (3.1.4.) $\Rightarrow A \leq N_G(M) \Rightarrow |G:N_G(M)|$, which is the number of conjugates of M in G, is finite.

Let $M^{x_1}, \ldots . M^{x_n}$ be the distinct conjugates of M in G.

Let $H = \langle M, x_1, \ldots . x_n \rangle$. By 3.1.3. and as G is locally finite, H is finite, so, by Schmidt ([I]), theorem four, $\frac{M^H}{M_H}$ is supersoluble and $M^H = M^G$, $M_H = M_G = 1$, so the theorem is proved. ||

Theorem 3.2.2.

Let $G \in L\mathfrak{F} \cap \min$, $M \text{ m } G \Rightarrow \dfrac{G}{C_G\left(\frac{M^G}{M_G}\right)}$ is supersoluble.

Proof

Without loss of generality, we may take $M_G = 1$. Then $A \leq C_G(M) \Rightarrow |G:N_G(M)|$ i.e. the number of conjugates of M is finite as before (where A is the minimal normal subgroup of finite index as usual). Let $M^{x_1}, \ldots M^{x_n}$ be the distinct conjugates of M. $A \leq C_G(M^{x_i})$ for all i such that $1 \leq i \leq n$, so $A \leq C_G(M^G)$ as $M^G = \langle M^{x_i} \mid 1 \leq i \leq n \rangle$. Thus $\frac{G}{C_G(M^G)}$ is finite and as by 2.2.11, $\frac{G}{C_G(M^G)}$ is locally supersoluble, we have that $\frac{G}{C_G(M^G)}$ is supersoluble as required.||

Theorem 3.2.3.

$$G \in L\mathfrak{F} \cap \min,\ M \text{ m } G,\ \frac{Q}{M_G} \in Syl_q\left(\frac{M}{M_G}\right) \Rightarrow Q \text{ m } G$$

Proof

By 1.1.6., without loss of generality, we may take $M_G = 1$ (so $M \in \mathfrak{N} \cap \mathfrak{F}$ by 3.1.3.). Suppose Q is not modular in G, then by 1.2.2., there exists a finite subgroup A of G such that Q is not modular in $\langle Q, A \rangle$. Let $F = \langle M, A, x_1, \ldots, x_n \rangle$ where $M^{x_1}, \ldots M^{x_n}$ are the distinct conjugates of M as usual. Then F is finite, $M_F = 1$ so Q is modular in F by Schmidt [11] theorem 5. Hence Q m $\langle Q, A \rangle$ by 1.1.2. and this contradiction proves the result.||

Chapter Four

Here we investigate more generally the properties of groups following the pattern of Schmidt [II].

Theorem 4.1.1

Let $G \in \mathfrak{X}$; $M \text{ m } G$, $U \leq G$ be such that M and U are both periodic, and for all $m \in M$, $u \in U$ we have that $(|m|, |u|) = 1$. Then $MU = UM$.

Proof

Suppose that $MU \neq UM$. Then there exists an element $u \in U$ such that $M\langle u\rangle \neq \langle u\rangle M$ i.e. $M\langle u\rangle \neq \langle M,u\rangle$. Consider $\langle M,u\rangle$. $G \in \mathfrak{X} \Rightarrow |\langle M,u\rangle : M|$ is finite (as U is periodic and $[\langle u\rangle / \langle u\rangle \cap M]$ is a finite lattice), so $\frac{\langle M,u\rangle}{M_{\langle M,u\rangle}}$ is finite , $\frac{M}{M_{\langle M,u\rangle}}$ m $\frac{\langle M,u\rangle}{M_{\langle M,u\rangle}}$ by 1.1.6, and $\left(\left| \frac{M}{M_{\langle M,u\rangle}} \right|, \left| \frac{u M_{\langle M,u\rangle}}{M_{\langle M,u\rangle}} \right| \right) = 1$, so by Schmidt [II], theorem one, we have that $\frac{M\langle u\rangle}{M_{\langle M,u\rangle}} = \frac{\langle M,u\rangle}{M_{\langle M,u\rangle}}$ i.e. $M\langle u\rangle = \langle M,u\rangle$ which contradicts our hypothesis. So $MU = UM$ //

Theorem 4.1.2.

Let $M \text{ m } G \in \mathfrak{X}$. Let Q be a locally finite q-subgroup of G (q a prime). Then either $MQ = QM$ or M is maximal in $\langle M,Q\rangle$ and $|\langle M,Q\rangle : M| = p$, p a prime, $p > q$.

Proof

Suppose, for a contradiction, that $MQ \neq QM$ and that M is not maximal in $\langle M,Q\rangle$.

Then there is an element r of Q such that $M\langle r\rangle \neq \langle M,r\rangle$

$\frac{\langle M,r\rangle}{M_{\langle M,r\rangle}}$ is finite, $\frac{M\langle r\rangle}{M_{\langle M,r\rangle}} \neq \frac{\langle M,r\rangle}{M_{\langle M,r\rangle}}$, $\frac{M}{M_{\langle M,r\rangle}}$ m $\frac{\langle M,r\rangle}{M_{\langle M,r\rangle}}$ and

$\frac{\langle r\rangle M_{\langle M,r\rangle}}{M_{\langle M,r\rangle}}$ is a q-group. Hence by Schmidt [II], theorem 2, $\frac{M}{M_{\langle M,r\rangle}}$ is

maximal in $\frac{\langle M,r\rangle}{M_{\langle M,r\rangle}}$ i.e. M is maximal in $\langle M,r\rangle$. As M is not

maximal in $\langle M,Q\rangle$ by hypothesis, this implies, in particular,

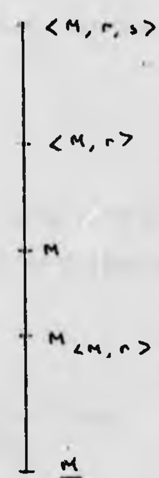
that $\langle r\rangle \lneq Q$. Suppose now that $M\langle r,s\rangle = \langle r,s\rangle M$ for all

$s \in Q \setminus \langle M\cap Q, r\rangle$. Then $MQ = QM$. So there exists an element $s \in Q \setminus \langle M\cap Q, r\rangle$

such that $M\langle r,s\rangle \neq \langle r,s\rangle M$. Q locally finite $\Rightarrow \langle r,s\rangle$ is a

finite q-subgroup and $G \in \mathfrak{X} \Rightarrow |\langle M,r,s\rangle : M|$ is finite. So $\frac{\langle M,r,s\rangle}{M_{\langle M,r,s\rangle}}$

is finite. Write $\underline{M}$ for $M_{\langle M,r,s\rangle}$.



Then $\frac{M}{\underline{M}}$ m $\frac{\langle M,r,s\rangle}{\underline{M}}$, $\frac{\langle r,s\rangle \underline{M}}{\underline{M}}$ is a

q-group and $\frac{M\langle r,s\rangle}{\underline{M}} \neq \frac{\langle M,r,s\rangle}{\underline{M}}$,

so by Schmidt [II] theorem 2, we

have that $\frac{M}{\underline{M}}$ is maximal in $\frac{\langle M,r,s\rangle}{\underline{M}}$

i.e. M is maximal in $\langle M,r,s\rangle$. So

M maximal in $\langle M,r\rangle$ implies that

$\langle M,r\rangle = \langle M,r,s\rangle$. But $\langle r\rangle \lneq \langle r,s\rangle$

implies, as $[\langle M,Q\rangle / M] \cong [Q/Q\cap M]$, that $\langle M,r\rangle \lneq \langle M,r,s\rangle$.

This contradiction proves the required result: viz that either $MQ = QM$ or M is maximal in $\langle M,Q\rangle$.

If the latter case holds, and $MQ \neq QM$ (so $M \ntrianglelefteq \langle M,Q\rangle$), then $|\langle M,Q\rangle : M| = p$ by 2.1.1, and $p > t$ where $t = \left|\frac{M}{M_{\langle M,Q\rangle}}\right|$. $q \mid \left|\frac{\langle M,Q\rangle}{M_{\langle M,Q\rangle}}\right|$ and $q = p \Rightarrow \frac{QM_{\langle M,Q\rangle}}{M_{\langle M,Q\rangle}} \lhd \frac{\langle M,Q\rangle}{M_{\langle M,Q\rangle}}$

i.e. QM is a group, which is a contradiction. So $q = t$. ||

We now investigate the situation when M is a q-group for some prime q.

<u>Theorem 4.1.3.</u>

Let G be locally finite, and let M m G be a q-group for some prime q. If M is not quasinormal in G, then

$$\frac{G}{M_G} = \frac{M^G}{M_G} \times K \text{ where for all } x \in \frac{M^G}{M_G},\ y \in K,\ (\,|x|\,,\,|y|\,) = 1$$

and where $\frac{M^G}{M_G}$ is a P-group, not necessarily finite.

Conversely, if M is a subgroup of G for which $\frac{G}{M_G}$ has the above structure, M m G.

<u>Proof</u>

Let M be a q-subgroup of G such that M is modular in G but not quasinormal in G. Thus there is an element y of G such that $M\langle y\rangle \neq \langle y\rangle M$.

Let $\mathcal{S} = \{F \leq G \mid F \text{ is finite and } \langle y\rangle \leq F\}$

Then for all $F \in \mathcal{S}$, $|\langle M,F\rangle : M|$ and hence $\frac{\langle M,F\rangle}{M_{\langle M,F\rangle}}$ is finite.

$\frac{M}{M_{\langle M,F\rangle}}$ is a modular q-subgroup of $\frac{\langle M,F\rangle}{M_{\langle M,F\rangle}}$ and as $\frac{M\langle y\rangle}{M_{\langle M,F\rangle}} \neq \frac{\langle M,y\rangle}{M_{\langle M,F\rangle}}$,

$\frac{M}{M_{\langle M,F\rangle}}$ is not quasinormal in $\frac{\langle M,F\rangle}{M_{\langle M,F\rangle}}$. So by Schmidt[II], theorem 3,

we have $\frac{\langle M,F\rangle}{M_{\langle M,F\rangle}} = \frac{M^{\langle M,F\rangle}}{M_{\langle M,F\rangle}} \times K_F$ where $\frac{M^{\langle M,F\rangle}}{M_{\langle M,F\rangle}}$ is a P-group of

order $p^n q$ ($p > q$, $n \geq 1$) and for all $x \in \frac{M^{\langle M,F\rangle}}{M_{\langle M,F\rangle}}$, $y \in K_F$, $(\,|x|\,,|y|\,)$

$= 1$. Note that as $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{S} \Rightarrow \langle F_1, F_2\rangle \in \mathcal{S}$, the prime p does not depend on F; note also that $\left|\frac{M}{M_{\langle M,F\rangle}}\right| = q$.

We establish the following facts:

1. Let F be any given subgroup belonging to the set $\mathcal{S}$. Then $M_G = M_{\langle M,F\rangle}$.

For, clearly $M_G \leq M_{\langle M,F\rangle}$. Suppose now that there is an element x of G such that $x \in M_{\langle M,F\rangle}$ but $x \notin M_G$. Thus there exists an element z of G such that $x \notin M^z$. Let $F_1 = \langle F,z\rangle$. Then $G \in L\mathfrak{F} \Rightarrow F_1 \in \mathcal{S}$.

So $\left|\frac{M}{M_{\langle M,F_1\rangle}}\right| = \left|\frac{M}{M_{\langle M,F\rangle}}\right| = q$ and as $M_{\langle M,F_1\rangle} \leq M_{\langle M,F\rangle}$ clearly, we have $M_{\langle M,F_1\rangle} = M_{\langle M,F\rangle}$. So $x \in M_{\langle M,F\rangle} = M_{\langle M,F_1\rangle} \leq M^z$. This contradiction to the choice of x proves that $M_G = M_{\langle M,F\rangle}$ as required, for any choice of F in $\mathcal{S}$.

Now without loss of generality, we take $M_G = 1$ (so $|M| = q$ and $\langle M,F\rangle$ is finite for all $F \in \mathcal{S}$.

2. For any given $F \in \mathcal{S}$, the direct complement K_F of $M^{\langle M,F\rangle}$ in $\langle F,M\rangle$ is unique.

For, suppose $M^{\langle M,F\rangle} \times A = M^{\langle M,F\rangle} \times K_F$. Then $|A| = |K_F|$ which is a $\{p,q\}'$ number. So any element of A is a $\{p,q\}'$ element. Let $a \in A$. Then $a = mk$ for some $m \in M^{\langle M,F\rangle}$, $k \in K_F$. As $[m,k] = 1$, we have $1 = a^{|a|} = m^{|a|}k^{|a|}$ and as $M^{\langle M,F\rangle} \cap K_F = 1$, $m^{|a|} = 1 = k^{|a|}$. But m is a $\{p,q\}$ element and $|a|$ is a $\{p,q\}'$ number so $m = 1$ and $a \in K_F$. This is true for all $a \in A$ so $A = K_F$.

3. If $F_1 \leq F_2$, then $K_{F_1} \leq K_{F_2}$.

For, let $k \in K_{F_1}$. Then $k \in \langle M,F_1\rangle \leq \langle M,F_2\rangle$ and $\langle M,F_2\rangle = M^{\langle M,F_2\rangle} \times K_{F_2}$. So $k = mk'$ where $m \in M^{\langle M,F_2\rangle}$, $k' \in K_{F_2}$, and $m^{|k|} = k'^{|k|} = 1$ as above. But $|k|$ a $\{p,q\}'$ number and m a $\{p,q\}$ element implies that $m = 1$. Hence $K_{F_1} \leq K_{F_2}$.

4. Let $K = \bigcup_{F \in \mathfrak{F}} K_F$. Then $K \lhd G$.

Firstly we show that K is a subgroup of G. Let $x,y \in K$. Then there exist $F_1 \in \mathfrak{F}$, $F_2 \in \mathfrak{F}$ such that $x \in K_{F_1}$, $y \in K_{F_2}$. $F_3 = \langle F_1,F_2\rangle \in \mathfrak{F}$, so by 3, $x \in K_{F_3}$ and $y \in K_{F_3}$ i.e. $xy^{-1} \in K_{F_3} \leq K$.

Now let $x \in K, g \in G$. There exists $F \in \mathfrak{F}$ such that $x \in K_F$. Let $\overline{F} = \langle F,g\rangle$. Then $\overline{F} \in \mathfrak{F}$, and by 3, $x \in K_{\overline{F}}$. $K_{\overline{F}} \lhd \langle M,\overline{F}\rangle \Rightarrow x^g \in K_{\overline{F}} \leq K$ as required. So K is a normal subgroup of G.

5. For all $x_1 \in M^G$, $x_2 \in K$, $(|x_1|, |x_2|) = 1$

For, there exist elements $y_1, \ldots\ldots, y_n$ of G such that $x_1 \in \langle M^{y_1}, \ldots\ldots, M^{y_n}\rangle$. Let $F = \langle y_1, \ldots\ldots y_n, y\rangle$ where y is the defining element of $\mathfrak{F}$. So $F \in \mathfrak{F}$, $x_2 \in K \Rightarrow$ there exists an F_1 such that $x_2 \in K_{F_1}$. Let $F_3 = \langle F,F_1\rangle$. Then $(|M^{\langle M,F_3\rangle}|, |K_{F_3}|) = 1$ by the finite case, and as $x_2 \in K_{F_3}$ (by 3) and $x_1 \in M^{\langle M,F_3\rangle}$, $(|x_1|, |x_2|) = 1$

6. $G = M^G \times K$.

By 4 & 5, $M^G \times K \leq G$. Let $g \in G$, $F \in \mathfrak{F}$. Then $F_1 = \langle F,g\rangle$

$\in \mathcal{S}$. $g \in \langle M,F_1\rangle = M^{\langle M,F_1\rangle} \times K_{F_1}$ i.e. $g \in M^G \times K$ as required.

7. We now show that M^G is a (possibly infinite) P-group.

$M^G = \bigcup_{F \in \mathcal{S}} M^{\langle M,F\rangle}$. Let $M = \langle m\rangle$ (so by 1, $m^q = 1$). Let $A = \langle x \in M^G \mid x^p = 1\rangle$. Then $A \trianglelefteq M^G$ and A is an elementary abelian p-group. (For, let $x,y \in A$. There exists $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{S}$ such that $x \in M^{\langle M,F_1\rangle}$, $y \in M^{\langle M,F_2\rangle}$. Let $F_3 = \langle F_1,F_2\rangle$. Then $F_3 \in \mathcal{S}$ and $x,y \in M^{\langle M,F_3\rangle}$ which is a P-group. Hence $x^p = y^p = 1$ and $[x,y] = 1$). Also $M^G = MA$. (For, clearly $MA \leq M^G$. Let $g \in M^G$ Then there exists $F \in \mathcal{S}$ such that $g \in M^{\langle M,F\rangle} = M\bar{A}$ where $\bar{A} = \langle x \in M^{\langle M,F\rangle} \mid x^p = 1\rangle = A \cap M^{\langle M,F\rangle}$. So $g \in MA$ as required)

Also clearly for all $a \in A$, there exists a positive integer r such that $m^{-1}am = a^r$ where $r \not\equiv 1 (\mathrm{mod}\ p)$ but $r^q \equiv 1 (\mathrm{mod}\ p)$. Hence M^G has the required structure.

We now consider the converse of the theorem, assuming, as before, that $M_G = 1$. We have $G = M^G \times K$ where $M^G = AB$, $A \triangleleft M^G$, $A \in \mathfrak{a}_p$ (possibly infinite), $B \cong C_q$ and for all $b \in B$ $a \in A$ there exists a positive integer r such that $b^{-1}ab = a^r$ where $r \not\equiv 1 (\mathrm{mod}\ p)$, $r^q \equiv 1 (\mathrm{mod}\ p)$.

Firstly we prove that $|M| = q$. For, suppose there is no element of order q in M. Then $M \leq A$ and as all elements of M^G are of the form ba, we have that $M \triangleleft M^G$, and as $[M^G,K] = 1$, it follows that $M \triangleleft G$. So $M = 1$ (which is impossible as $M^G \neq 1$) Now suppose M has an element of order q i.e. there exists $g \in G$ such that $B^g \leq M$. Then $M = (AB)^g \cap M = AB^g \cap M = (A \cap M)B^g$. $A \cap M \triangleleft M^G \Rightarrow A \cap M \triangleleft G \Rightarrow$ (as $M_G = 1$) $A \cap M = 1$. So $M = B^g$ i.e. $|M| = q$ as required.

Suppose now that M is not modular in G. By 1.2.2., and the fact that G is locally finite, we have that there exists a finite subgroup D of G such that M is not modular in $\langle M, D \rangle$ (= $\underline{D}$, say). Then, by the preceding paragraph, $\underline{D}$ is finite and as $G = M^G \times K$, there exists a finite subgroup F ($M \leq F$) such that $\underline{D} \leq M^F(K \cap F) = M^F \times (K \cap F)$ (as $(|M^F|, |K \cap F|) = 1$ and $M^F \lhd F$, $K \cap F \lhd F$). Using the notation of the preceding paragraph, we have $M^F = (AB)^g \cap M^F = AB^g \cap M^F = (A \cap M^F)B^g$ i.e. M^F is a P-group. So, by Schmidt [11], theorem 3, we have $M \text{ m } M^F \times K \cap F$ and hence as $M \leq \underline{D} \leq M^F \times K \cap F$, $M \text{ m } \underline{D}$.

This contradiction proves the result.

(Note: in the above, $\underline{D}$ should not be confused with the core of D in G).

Corollary 4.1.4

Let $M \text{ m } G \in L\mathfrak{F}$, and let M be a q-subgroup for some prime q. If M is not quasinormal in G, then $\left|\frac{M}{M_G}\right| = q$ and M is a maximal q-subgroup of G.

Proof

The last assertion is the only one to require proof.

Without loss of generality, we may take $M_G = 1$. Suppose for a contradiction, that M is not a maximal q-subgroup of G i.e. there exists a q-subgroup Q such that $Q \gneq M$ i.e. there exists a q-element x, say, such that $x \in Q \setminus M$. By 4.1.3. $G = M^G \times K$ where $|M| = q$. Thus there exists some finite subgroup F of G such that $x \in M^{\langle M, F \rangle} \times K_F$ where the notation is as in the previous theorem. $M \in Syl_q(M^{\langle M, F \rangle} \times K_F)$, but $M \underset{\neq}{<} \langle M,x\rangle \leq M^{\langle M,F\rangle} \times K_F$ and $\langle M,x\rangle$ is a q-subgroup. This contradiction proves the result. ||

Corollary 4.1.5.

Let $M \text{ m } G \in L\mathfrak{F}$ be such that M is a q-subgroup of G for some prime q and M is not quasinormal in G. Then M^G is a $\{p,q\}$ group, and $N_G(M)$ contains all $\{p,q\}'$ elements.

Proof

Without loss of generality we take M_G to be 1. Then $G = M^G \times K$ (by 4.1.3.). Let y be a $\{p,q\}'$ element of G. Then $y = mk$ for some $m \in M^G$, $k \in K$. i.e. $y^{|y|} = m^{|y|} = k^{|y|} = 1$. But m a $\{p,q\}$ element implies that $m = 1$, so $y \in K$. $[M^G,K] = 1$ implies that $K \leq N_G(M)$ so the corollary is proved. (In fact, $N_G(M) = M^G K \cap N_G(M) = (M^G \cap N_G(M))K = MK$ as M^G is a P-group.) ||

Having dispensed with modular subgroups which are q-groups for some prime q, we now consider locally nilpotent modular subgroups.

Theorem 4.1.6

Let $M \text{ m } G \in L\mathfrak{F}$, $M \in L\mathfrak{N}$. Let Q be a maximal q-subgroup of M. Then either Q qn G or Q is a maximal q-subgroup of G and $\left|\frac{Q}{Q_G}\right| = q$.

Proof

Suppose that Q is not quasinormal in G i.e. there exists an element y of G such that $Q\langle y\rangle \neq \langle Q,y\rangle$ i.e. there exists an element z of G such that $z \in \langle Q,y\rangle \setminus Q\langle y\rangle$. Let $q_1, \dots q_n$ be elements of Q such that $z \in \langle q_1, \dots, q_n, y\rangle$. Suppose also that Q is not a maximal q-subgroup of G i.e. there exists a $t \in G$ such that $Q \lneq \langle Q,t\rangle$ and $\langle Q,t\rangle$ is a q-subgroup. Let $F = \langle q_1, \dots, q_n, t, y\rangle$. Then $G \in L\mathfrak{F} \Rightarrow$ F is finite. $M \cap F$ m F and $M \cap F \in \mathfrak{N}$. As $M = Q \times A$ where A is a q'-group, $Q \cap F \in Syl_q(M \cap F)$. By Schmidt [II], lemma 4, either $Q \cap F$ is quasinormal in F or $Q \cap F \in Syl_q(F)$.

But $z \in \langle Q \cap F, y\rangle \setminus (Q \cap F)\langle y\rangle$ so $Q \cap F$ is not quasinormal in F and $Q \cap F \lneq \langle Q \cap F, t\rangle$ which is a q-subgroup of F. So this contradiction proves the result that either Q qn G or Q is a maximal q-subgroup of G.

Suppose the former case does not hold i.e. Q is not quasinormal in G., and suppose that $\left|\frac{Q}{Q_G}\right| > q$ i.e. there exists elements $y_1, y_2, \dots, y_{q+1}$ of Q such that $y_i y_j^{-1} \notin Q_G$ for all i,j such that $1 \le i \ne j \le q+1$ i.e. there exist elements x_{ij} of G such that $y_i y_j^{-1} \notin Q^{x_{ij}}$. Let $F = \langle y, q_1, \dots, q_n, y_1, \dots y_{q+1}, x_{ij} \mid 1 \le i \ne j \le q+1\rangle$ Then F is finite, $M \cap F$ m F, $Q \cap F \in Syl_q(M \cap F)$, $M \cap F \in \mathfrak{N}$ and $(Q \cap F)\langle y\rangle \ne \langle Q \cap F, y\rangle$ (where the notation is as above). By Schmidt, [II], lemma 4, $\left|\frac{Q \cap F}{(Q \cap F)_F}\right| = q$. Elements $y_1, \dots, y_{q+1} \in Q \cap F$ so there exist i,j where $1 \le i \ne j \le q+1$, such that $y_i y_j^{-1} \in (Q \cap F)_F$. In particular, $y_i y_j^{-1} \in (Q \cap F)^{x_{ij}}$. This contradicts our choice of the elements $\{x_{ij}\}$.

So $\left|\frac{Q}{Q_G}\right| \leq q$, and as Q is a q-group and is not quasinormal in G, $\left|\frac{Q}{Q_G}\right| = q$ as required. ||

Chapter Five

Here we turn our attention to the investigation of dual-Dedekind subgroups following the pattern of Menegazzo ([III]).

Definition

A subgroup H of G is said to be dual-Dedekind in G (written H dd G) if it obeys the following two properties:

D1. For all subgroups X and Y of G such that $X \leq Y$, we have that $\langle H,X\rangle \cap Y = \langle H \cap Y, X\rangle$.

D2. For all subgroups X and Y of G such that $Y \leq H$, we have that $\langle X,Y\rangle \cap H = \langle X \cap H, Y\rangle$.

(Note: D1 is a property shared by both dual-Dedekind and modular subgroups.)

Section One

In this section, we consider some elementary properties of dual-Dedekind subgroups of which 5.1.2.-5.1.6. are stated but not proved in Menegazzo ([III]).

Theorem 5.1.1. (cf 1.1.1.)

The following statements are equivalent:

i. H dd G

ii. For all subgroups K of G the map ϕ_K defined as follows

$$\phi_K: [\langle K,H\rangle/K] \longrightarrow [H/H \cap K]$$

$\phi_K(L) = L \cap H$, is a lattice isomorphism.

iii. For all subgroups K of G, the map ψ_K defined as follows:

$$\psi_K : [H/H \cap K] \longrightarrow [\langle H,K\rangle / K]$$
$$R \longrightarrow \langle R,K\rangle$$

is a lattice isomorphism.

Moreover, in this situation, ϕ_K and ψ_K are mutually inverse.

<u>Proof</u>

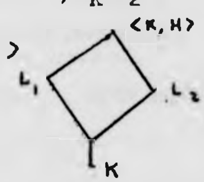
<u>$i \Rightarrow ii$</u>

Let $L_1, L_2 \in [\langle K,H\rangle/K]$ be such that $\phi_K(L_1) = \phi_K(L_2)$.

So $L_1 \cap H = L_2 \cap H$ i.e. $\langle L_1 \cap H, K\rangle = \langle L_2 \cap H, K\rangle$

Hence $\langle H,K\rangle \cap L_1 = \langle H,K\rangle \cap L_2$ by D1

and so $L_1 = L_2$. So ϕ_K is injective.



Now let $R \in [H/H \cap K]$. So $K \leq \langle R,K\rangle \leq \langle H,K\rangle$

Thus $\phi_K(\langle R,K\rangle) = \langle R,K\rangle \cap H = \langle R, K \cap H\rangle$ by D2

$= R$

Hence ϕ_K is surjective.

Thus ϕ_K is a bijection and ψ_K is its inverse as $\psi_K \phi_K(L)$ $= \psi_K(L \cap H) = \langle L \cap H, K\rangle$, $\forall\ L \in [\langle K,H\rangle/K]$. But $\langle L \cap H, K\rangle = \langle H,K\rangle \cap L$ (by D1) and $\langle H,K\rangle \cap L = L$. So $\psi_K \phi_K$ is the identity map on $[\langle K,H\rangle / K]$ *

ϕ_K preserves intersections clearly, and also preserves unions as for all subgroups $L,N \in [\langle K,H\rangle/K]$, $\psi_K(\phi_K(\langle L,N\rangle)) =$

$\langle L,N\rangle = \langle \psi_K(\phi_K(L)), \psi_K(\phi_K(N)) = \psi_K(\langle \phi_K(L), \phi_K(N)\rangle)$

(as ψ_K clearly preserves unions). Hence $\phi_K(\langle L,N\rangle) =$

$\langle \phi_K(L), \phi_K(N)\rangle$ as ψ_K being the inverse of ϕ_K must be injective.

Thus ϕ_K is a lattice isomorphism.

<u>ii ⇒ iii</u>

Given that ϕ_K is a lattice isomorphism, we wish to show that so is ψ_K. Let $R \in [\langle H,K\rangle / K]$. Firstly we shall prove that $\langle R \cap H, K\rangle = R$ (*) . $\langle R\cap H,K\rangle \leq R \Rightarrow \phi_K(\langle R\cap H,K\rangle) \leq \phi_K(R)$ but $\langle R\cap H,K\rangle \cap H \geq \langle R\cap H\rangle$, so $\phi_K(\langle R\cap H,K\rangle) = \phi_K(R)$ and as ϕ_K is injective, (*) is proved. Hence ψ_K is surjective as $\psi_K(R\cap H) = \langle R\cap H,K\rangle = R \quad \forall\ R \in [\langle H,K\rangle / K]$.

Also ψ_K is injective. For suppose $\psi_K(L_1) = \psi_K(L_2)$ for $L_1, L_2 \in [H | H\cap K]$. ϕ_K surjective $\Rightarrow \exists\ M_1, M_2 \in [\langle H,K\rangle / K]$ such that $\phi_K(M_1) = L_1$, $\phi_K(M_2) = L_2$ i.e. $M_1 \cap H = L_1$, $M_2 \cap H = L_2$. So $\langle M_1 \cap H,K\rangle = \langle M_2\cap H,K\rangle$,hence by (*) $M_1 = M_2$ and hence $L_1 = L_2$.

ψ_K preserves unions and as by (*) we have $\psi_K \phi_K$ is the identity map, so ψ_K is the inverse of ϕ_K, and as before we can prove easily that ψ_K preserves intersections.

<u>iii ⇒ i</u>

Firstly we shall prove that for all subgroups K of G, $L \in [H/K\cap H] \Rightarrow \langle L,K\rangle \cap H = L$ (**). $L \leq \langle L,K\rangle \cap H \Rightarrow \psi_K(L) \leq \psi_K(\langle L,K\rangle\cap H)$ i.e. $\langle L,K\rangle \leq \langle\langle L,K\rangle \cap H,K\rangle$.But $\langle\langle L,K\rangle\cap H,K\rangle \leq \langle L,K\rangle$ so $\psi_K(L) = \psi_K(\langle L,K\rangle \cap H)$, and as ψ_K is injective, (**) is proved.

We wish to prove D1 i.e. $X \leq Y \Rightarrow \langle H,X\rangle \cap Y = \langle H\cap Y, X\rangle$. $X \leq \langle H,X\rangle \cap Y \leq \langle H,X\rangle$. Hence as ψ_X is surjective, $\exists$ a subgroup $R \in [H/H\cap X]$ such that $\psi_X(R) = \langle R,X\rangle = \langle H,X\rangle \cap Y$ (***). By (**) with L = R and K = X, we have that $R = \langle R,X\rangle \cap H = \langle H,X\rangle \cap Y \cap H = Y \cap H$. So by(***), we have that $\langle Y\cap H,X\rangle = \langle H,X\rangle \cap Y$

which was to prove.

For D2, consider subgroups X,Y of G such that $X \leq H$. We wish to prove that $\langle X,Y\rangle \cap H = \langle X, Y\cap H\rangle$. $Y \cap H \leq \langle X, Y\cap H\rangle \leq H$ so by ** with $K = Y$, $L = \langle X, Y \cap H\rangle$, we have that $\langle X, Y\cap H\rangle = \langle X, Y\cap H, Y\rangle \cap H = \langle X,Y\rangle \cap H$ as required. ||

Note The fact that $[\langle H,K\rangle / K] \cong [H/K\cap H]$ does not necessarily imply that the map $L \mapsto L \cap H$ for all subgroups L belonging to $[\langle H,K\rangle / K]$ is a lattice isomorphism.

For example, let $G = S_5$ i.e. the permutation group on five elements. Let $K = \langle (1234),(13)\rangle$, $H = \langle (2345)\rangle$.

As $(13)(1234)(13) = (1432) = (1234)^3$, we have that K is isomorphic to the dihedral group of order eight.

Let L be the symmetric group on the four elements $\{1,2,3,4\}$. Then $K < L$ and I claim that L is the only proper subgroup containing K.

For, let $K < J$. $|G:K| = 15$ and G having no subgroup of index 3 implies that $|J| = 24$. So J must have an element of order 3.

Suppose J contains (abc) where $a,b,c \in \{1,2,3,4\}$. (We use here the properties that S_4 may be generated by a3-cycle and a 2-cycle whose product is a 4-cycle, and by (abcd) and (ab) (see e.g. ([X]) p.253 andp.320)). Also recall $J > K$.

As (abc)(abc) = (bac), we need only consider four possibilities,viz:

i. (124) But(124)(13) = (1243) and hence (124) and (13) generate S_4 so J = L.

ii.(234) But (234)(13) = (1342),so J = L

iii.(123) But (123)(13) = (12) and (1234) and(12) generate S_4,so J = L

iv. (134) But (134)(1234) = (1423) and (13)(134) = (14) so J = L.

So J must have an element (ab5). As $(ab5)^2 = (ba5)$, there are only six possibilities:

i.(125) So J contains (1234)(125) = (15)(234). As$((15)(234))^2$ = (324) and (324)(13) = (1324) so J = L.

ii. (135) J contains (1234)(135) = (125)(34) and as$((125)(34))^2$ = (215), this situation is covered by i.

iii. (145). J contains (24)(145)(24) = (125), the situation already covered by i (note: (24) $=(1234)^2(13)$ is in K).

iv. (235) J contains (13)(235)(13) = (152) covered by i.

v.(245) J contains (1234)(245) = (14)(235). Squaring, we see that (325) belongs to J, a situation covered by iv.

vi.(345) Here J contains (14)(23)(345)(23)(14) = (152) covered by i. (Note that (14)(23) = (13)(1234) is in K)

So we have that the only subgroup of G containing K is L. As $K < \langle K,H \rangle$ and $\langle K,H \rangle \nleq L$, we have that $\langle K,H \rangle = G$ As H is cyclic of order 4, it is lattice isomorphic with $[G/K]$ and $H \cap K = 1$. But $L \cap H' = 1$, and hence this example does exhibit the required property, viz that$[\langle H,K \rangle / K] \cong [H/H \cap K]$does not necessarily imply that the map $L \mapsto L \cap H$ is a lattice isomorphism.

<u>Theorem 5.1.2.</u> (cf 1.1.2.)

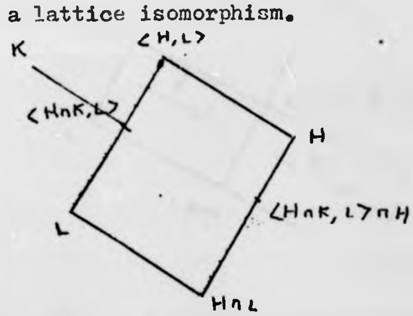
$H \text{ dd } G$ and $K \leq G \Rightarrow H \cap K \text{ dd } K$

<u>Proof</u>

By 5.1.1., it is sufficient to prove that for all subgroups L of K, the map $[\langle H \cap K, L\rangle / L] \longrightarrow [H \cap K / H \cap K \cap L]$

$$R \longmapsto R \cap H \cap K$$

is a lattice isomorphism.



$\langle H \cap K, L\rangle \leq \langle H, L\rangle$ and the map $[\langle H,L\rangle / L] \longrightarrow [H / H \cap L]$ via $R \longmapsto R \cap H$ is a lattice isomorphism as H dd G. So the restriction of this map to $[\langle H \cap K, L\rangle / L]$ is an isomorphism i.e.

$$[\langle H \cap K, L\rangle / L] \longrightarrow [\langle H \cap K, L\rangle \cap H / H \cap L]$$

$$R \longmapsto R \cap H \quad (= R \cap H \cap K \text{ as } R \leq K)$$

and this is the required map as $\langle H \cap K, L\rangle \cap H = \langle H \cap K, H \cap L\rangle$ by D2 and $\langle H \cap K, H \cap L\rangle = H \cap K$ (as $L \leq K$) and $H \cap L = K \cap H \cap L$.||

<u>Theorem 5.1.3.</u> (cf 1.1.3)

$H \text{ dd } K$ and $K \text{ dd } G \Rightarrow H \text{ dd } G$.

<u>Proof</u>

By 5.1.1., it is sufficient to prove that for all subgroups L of G, $[\langle H,L\rangle / L] \longrightarrow [H / H \cap L]$ given by $R \longmapsto R \cap H$ is a

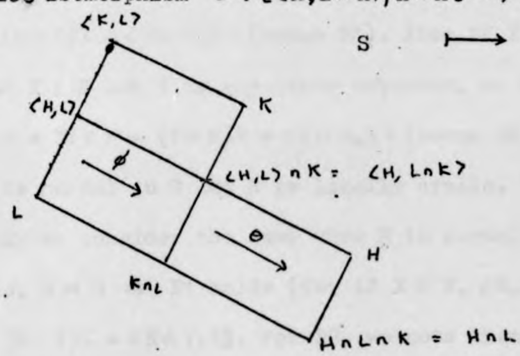
lattice isomorphism.

Let ϕ be the restriction of the isomorphism $[\langle K,L\rangle/L] \longrightarrow [K/K\cap L]$ (given by $T \mapsto T\cap K$) (which we know exists by 5.1.1. as K dd G) to $[\langle H,L\rangle/L]$ i.e. $\phi : [\langle H,L\rangle/L] \longrightarrow [\langle H,L\rangle\cap K / L\cap K]$

$$R \longmapsto R\cap K$$

As $\langle H,L\rangle \cap K = \langle H, L\cap K\rangle$ (by D2 as $H \leq K$) and H dd K, we have a lattice isomorphism $\theta : [\langle H, L\cap K\rangle / L\cap K] \longrightarrow [H/H\cap L\cap K]$

$$S \longmapsto S\cap H$$



So $\theta\phi$ is a lattice isomorphism defined by

$$\theta\phi : [\langle H,L\rangle/L] \longrightarrow [H/H\cap L] \quad (\text{as } H \leq K, H\cap L\cap K = H\cap L)$$

$$R \longmapsto R\cap K\cap H \ (=R\cap H)$$

Hence H dd G which was to prove.\\

<u>Theorem 5.1.4.</u> (cf 1.1.4)

H dd G and K dd G $\Rightarrow$ $H\cap K$ dd G.

<u>Proof</u>

By 5.1.2., $H\cap K$ dd K and by 5.1.3., $H\cap K$ dd G.

<u>Theorem 5.1.5.</u>

If $\phi : [G] \longrightarrow [G']$ is a lattice isomorphism, then H dd G if and only if $\phi(H)$ dd G'.

Proof : obvious.||

Examples of dual-Dedekind subgroups

1. It is clear that both the identity element and the whole group G are dual-Dedekind in G.

2. If $H \leq Z(G)$, then H dd G.

For if X,Y are subgroups of G such that $X \leq Y$, $\langle H,X\rangle \cap Y = HX \cap Y = (H \cap Y)X = \langle H \cap Y, X\rangle$ (hence D1). Also if X is a subgroup such that $X \leq H$ and Y is any other subgroup, we have that $\langle X,Y\rangle \cap H = XY \cap H = (Y \cap H)X = \langle Y \cap H, X\rangle$ (hence D2).

3. If N is normal in G and N is locally cyclic, then N dd G.

Firstly we consider the case when N is normal in G and N is cyclic. $N \lhd G \Rightarrow$ D1 holds (for if $X \leq Y$, $\langle N,X\rangle \cap Y = NX \cap Y = (N \cap Y)X = \langle N \cap Y, X\rangle$). For D2, we note that $X \leq N$ implies X a characteristic subgroup of N, and so X is normal in G. So, for any other subgroup Y of G, we have that $\langle X,Y\rangle \cap N = XY \cap N = (Y \cap N)X = \langle Y \cap N, X\rangle$ as required.

Now suppose that N is a locally cyclic normal subgroup of G and that N is not dual-Dedekind in G. So D2 must be the axiom that cannot hold as D1 is always true for a normal subgroup. Hence there exist subgroups X and Y such that X is contained in N and $\langle X,Y\rangle \cap N \neq \langle X, Y \cap N\rangle$ i.e. there is an element $z \in \langle X,Y\rangle \cap N \setminus \langle X, Y \cap N\rangle$. Hence there are elements $\{x_1, \ldots, x_r\}$ of X, $\{y_1, \ldots, y_s\}$ of Y such that $z \in \langle x_1, \ldots, x_r, y_1, \ldots, y_s\rangle \cap N$. Let $F = \langle x_1, \ldots, x_r, y_1, \ldots, y_s\rangle$. Then $z \in \langle X \cap F, Y \cap F\rangle$

$\cap (N \cap F)$ and as N is locally cyclic, $N \cap F$ is cyclic and $N \cap F$ is normal in F. So, by the previous argument, $N \cap F$ dd F and as $X \cap F \leq N \cap F$, $z \in \langle X \cap F, Y \cap N \cap F \rangle$ i.c. $z \in \langle X, Y \cap N \rangle$ which contradicts the choice of z. So D2 must hold, and N dd G.

(Note: a normal subgroup of a group need not be dual-dedekind in the group. For example, let $G = S_4$, the symmetric group on the four elements $\{1,2,3,4\}$. Then the alternating group A_4 is normal in S_4 but not dual-dedekind in S_4. For, $(123) \in A_4$. $\langle (123),(34) \cap A_4 \rangle = \langle (123) \rangle$, but $((123)(34))^2 = (1243)(1243) = (14)(23) \in \langle (123),(34) \rangle \cap A_4 \setminus \langle (123), (34) \cap A_4 \rangle$).

4. Any subgroup of the kernel ($= \cap \{ N_G(X) \mid X \leq G \}$) is dual-dedekind in G.

Section Two

Here we investigate some slightly more complex properties of dual-Dedekind subgroups.

Definition

H is locally dual-Dedekind in G if and only if for all natural numbers n and for all sets of elements $\{x_1, , , , x_n\}$ of G , H dd $\langle H, x_1, , , , x_n \rangle$.

Theorem 5.2.1.

H dd G if and only if H is locally dual-Dedekind in G.

Proof

only if: follows from 5.1.2.

if : Suppose H is locally dual-Dedekind in G but not dual-Dedekind in G i.e. either A. there exist subgroups K,L of G such that $K \leq L$ and $\langle H,K\rangle \cap L \neq \langle H \cap L, K\rangle$.

or B. there exist subgroups K and L of G such that $K \leq H$ and $\langle K,L\rangle \cap H \neq \langle K, L \cap H\rangle$.

Suppose A. holds. Then $\langle H,K\rangle \cap L \not\leq \langle H \cap L, K\rangle$ i.e. there is some element $y \in \langle H,K\rangle \cap L \setminus \langle H \cap L, K\rangle$ i.e. there are elements $h_1, , , h_n$ of H and $k_1, , , k_r$ of K such that $y \in \langle h_1, , h_n, k_1, , k_r\rangle \cap L \setminus \langle H \cap L, K\rangle$. H locally dual-Dedekind implies that H dd $\langle H, k_1, , , k_r\rangle = K_1$, say. So $y \in \langle H, K \cap K_1\rangle \cap L \cap K_1 = \langle H \cap L \cap K_1, K \cap K_1\rangle$ (as $K \cap K_1 \leq L \cap K_1$) i.e. $y \in \langle H \cap L, K\rangle$

which contradicts our choice of y. So case A. cannot hold.

Hence case B. holds. So there exists some element y such that $y \in \langle K,L\rangle \cap H \setminus \langle K,L\cap H\rangle$ i.e. there exist elements $k_1, , , k_n$ of K, $l_1, , , l_r$ of L such that $y \in \langle k_1, , k_n, l_1, , l_r\rangle \cap H$

H dd $\langle H,k_1, , , , k_n, l_1, , l_r\rangle = L_1$, say. So $y \in \langle K\cap L_1, L\cap L_1\rangle$ $\cap H$ implies that $y \in \langle K\cap L_1, L\cap H\cap L_1\rangle$ (as $K\cap L_1 \leq H$)

$\Rightarrow y \in \langle K,L\cap H\rangle$ contradicting our choice of y.

So H locally dual-Dedekind in G does imply H dual-Dedekind in G as required//

Note: this does not appear to be anything like as useful a result as 1.2.2.

We now investigate the relationship between dual-Dedekind and quasinormal subgroups in locally nilpotent groups.

The result: H dd G $\Rightarrow$ H qn G if G is a finite nilpotent group is due to Napolitani; the proof reproduced below is due to me.

Theorem 5.2.2.

Let G be nilpotent (not necessarily finite). Then H dd G implies that G is quasinormal in G.

Proof

We wish to prove that for all subgroups K of G, $HK = \langle H,K\rangle$ As K is a subgroup of a nilpotent group, K must be subnormal in G, in n steps, say. If n = 1, K is normal in G, so HK =

$\langle H,K\rangle$ obviously.

We consider the following induction hypothesis: H permutes with all subgroups subnormal in G in n-1 steps, $n \geq 2$.

Let K be subnormal in G in n steps. Then $K = K_0 \lhd K_1 \lhd \ldots \lhd K_n = G$.

We wish to prove that $\langle H,K\rangle = HK$. Let $y \in \langle H,K\rangle$. We wish to prove that $y \in HK$. $y \in \langle H,K\rangle \Rightarrow y \in \langle H,K_1\rangle$

$\Rightarrow y \in HK_1$ (by the induction hypothesis)

$\Rightarrow y = hk_1$ (for some $h \in H, k_1 \in K_1$)

$\Rightarrow h^{-1}y = k_1$

$\Rightarrow h^{-1}y \in \langle H,K\rangle \cap K_1$

$\Rightarrow h^{-1}y \in \langle H \cap K_1, K\rangle$ (as H dd G)

$\Rightarrow h^{-1}y \in (H \cap K_1)K$ (as $H \cap K_1 \leq N_G(K)$)

$\Rightarrow y \in HK$ as required. ||

Theorem 5.2.3.

Let G be a locally nilpotent group. Then H ddG $\Rightarrow$ H qn G.

Proof

Suppose H dd G and H is not quasinormal in G. Then there is an element x of G such that $H\langle x\rangle \neq \langle H,x\rangle$ i.e. there is some element $y \in \langle H,x\rangle \setminus H\langle x\rangle$ i.e. there are elements $h_1, \ldots, h_n$ of H such that $y \in \langle h_1, \ldots, h_n, x\rangle = F$, say. Then F is nilpotent, $H \cap F$ dd F and hence $H \cap F$ qn F by 5.2.2. So $y \in \langle H \cap F, x\rangle = (H \cap F)\langle x\rangle$. Hence $y \in H\langle x\rangle$ as required. ||

Section Three

Here, following the pattern of Menegazzo, ([III]), section three, we investigate the structure of subgroups which are non-trivial and minimal in the set of dual-dedekind subgroups of a locally finite group, and establish that a locally finite group which has such a minimal dual-dedekind subgroup cannot be simple.

Theorem 5.3.1.(cf Schmidt ([I]) lemma 1)

Let H be minimal and non-trivial among the dual-dedekind subgroups of a locally finite group G, Then either H is normal in G or the order of H is p, for some prime p.

In the former case, if $|H| \neq p$ we have in addition that

1. if H has an element of order p, then all elements of G having order p lie in H and
2. $C_G(H) = \{g \mid g \in G, (|g|, |h|) = 1$ for all $h \in H$.

Proof

(Note: this proof follows very closely that of Menegazzo in the finite case).

Suppose $|H| \neq p$, for any prime p. Then we wish to prove that H is normal in G.

i. Suppose $1 \neq A \leq H$. Then $N_G(A) \leq N_G(H)$

For, for all $g \in G, H^g$ dd G (5.1.5.) and hence $H \cap H^g$ dd G (5.1.4.) Thus, by the minimality of H, either $H \cap H^g = 1$ or $H \cap H^g = H$. So, for all $g \in G$, $H \cap H^g \neq 1 \Rightarrow g \in N_G(H)$.

Now $g \in N_G(A)$ (where A is a non-trivial subgroup of H) $\Rightarrow$

$1 \neq A = A \cap A^g \leq H \cap H^g$ i.e. $g \in N_G(H)$ as required.

ii. Suppose there exists an element x of H such that the order of x is p, for some prime p. Then all elements of G having order p belong to H.

For, suppose $\exists\, g \in G$ such that $|g| = p$ and g does not belong to H. Then $\langle x\rangle = \langle x,g\rangle \cap H$ (by D2 as $\langle g\rangle \cap H = 1$) dd $\langle x,g\rangle$ (by 5.1.2.).

G locally finite $\Rightarrow$ ($R =$) $\langle x,g\rangle$ is finite, so by Menegazzo ([III]) lemma 2.1., we have $|R| = p^2$ or $|R| = pq$ (q a prime, $q > p$).

In the former case, $[x,g] = 1$; in particular, $g \in N_G(\langle x\rangle)$ and hence $g \in N_G(H)$ by i. Let y be any element of H. Then $\langle y\rangle = \langle y,g\rangle \cap H$ (D2, as $\langle g\rangle \cap H = 1$) and $\langle y,g\rangle \cap H \lhd \langle y,g\rangle$, so g normalises every element of H i.e. g belongs to the kernel of H. Similarly, $[x,g] = 1 \Rightarrow |xg| = p$ and $g \notin H \Rightarrow xg \notin H$ so by the above argument, xg belongs to the kernel of H. Hence x belongs to the kernel of H and so $\langle x\rangle$ dd H. Hence $\langle x\rangle$ dd G by 5.1.3. which contradicts the minimality of H.

So we must have $|R| = pq$ and as $\langle x\rangle$ and $\langle g\rangle$ are both Sylow p-subgroups of R, they must be conjugate in R. As $x \in H$, this means there exists some element r of R such that $g \in H^r$. As $H^r \neq H$, we have by i., that $H \cap H^r = 1$. $\langle g\rangle = \langle g,H\rangle \cap H^r$ (by D2 as H^r dd G) and $\langle g,H\rangle \cap H^r$ dd $\langle g,H\rangle$ (5.1.2.) so $\langle g\rangle$ dd $\langle g,H\rangle$, and as $R \leq \langle g,H\rangle$, so x is conjugate to g in $\langle g,H\rangle$, $\langle x\rangle$ dd $\langle g,H\rangle$ and hence by 5.1.2., $\langle x\rangle$ dd H, contradicting the minimality of H.

So if there is an element x of H having order p there cannot be an element g having order p such that g does not belong to H Hence ii (and i) is proved.

iii. Here we prove that H is normal in G.

Let $g \in G$. We may assume that $|g| = p^n$ for some prime p, $n \geq 1$.

There are two possible cases:

a. there is some element x in H of order p. Then,by ii, $g^{p^{n-1}} \in H$, and $g \in N_G(g^{p^{n-1}}) \Rightarrow g \in N_G(H)$ byi.

b. There is no element in H of order p. Let $y \in H$ have prime order. Then $g^{-1}yg \in \langle y,g\rangle \cap H$ (by ii) $= \langle y\rangle$(by D2 as $\langle g\rangle \cap H = 1$). So $g \in N_G(\langle y\rangle)$ and hence $g \in N_G(H)$ as required.

We now turn our attention to assertion 2.

Let $g \in G$ be such that $|g| = q^m$, and H has no element of order q. Let $x \in H$, and let us assume for the moment that $|x| = p$ (p some prime, $p \neq q$,obviously). Then$\langle x\rangle = \langle x,g\rangle \cap H$ (by D2) $\lhd \langle x,g\rangle$. So $|\langle x,g\rangle| = pq^m$. Consider $|\langle xg\rangle|$. If p divides $|\langle xg\rangle|$, then $\langle x\rangle \leq \langle xg\rangle$(as $\langle x\rangle$ is the unique Sylow p-subgroup of $\langle x,g\rangle$), so $\langle xg\rangle = \langle xg,x\rangle = \langle x,g\rangle$ and $[g,x] = 1$.

Now suppose p does not divide $|\langle xg\rangle|$. So $\langle xg\rangle$ is a q element and hence like g, normalises every subgroup of H. So $\langle xg\rangle \in$ Ker H ,$\langle g\rangle \in$ Ker H $\Rightarrow \langle x\rangle \in$ Ker H $\Rightarrow \langle x\rangle$ dd H and hence $\langle x\rangle$ dd G by 5.1.2., which contradicts the minimality of H. So p must divide $|\langle xg\rangle|$ and $[g,x] = 1$.

We now take $|x| = p^n$ $(n > 1)$ and prove $[x,g] = 1$.

by induction on n.

$\langle x\rangle \triangleleft \langle x,g\rangle \Rightarrow x^g = x^k$, say. $[x^p,g] = 1$ by the induction hypothesis, and $(x^p)^g = x^{kp} = x^p$ i.e. as $|x| = p^n$ $p^n | p(k-1)$ so $p^{n-1} | (k-1)$ i.e. there exists some integer s such that $k = 1 + sp^{n-1}$.

$$\text{Consider } x^{g^p} = x^{k^p} = x^{(1+sp^{n-1})^p}$$
$$= x^{1 + psp^{n-1} + p^2s^2p^{2n-2} + \dots}$$
$$= x$$

So $[x,g^p] = 1$ and as $|g| = q^m$, g^p generates $\langle g\rangle$, and hence $g \in C_G(\langle x\rangle)$.

So $\{g | g \in G, (|g|,|h|) = 1 \text{ for all } h \in H\} \subset C_G(H)$

Conversely suppose $y \in C_G(H)$ and $|y| = p^n$, where H has some element of order p. Then, as above, we can show that $\langle y\rangle \cap H \neq 1$, and hence $Z(H) \neq 1$. But Z(H) dd H and hence Z(H) dd G which contradicts the minimality of H.

Hence, $\{g \in G | (|g|,|h|) = 1 \text{ for all } h \in H\} = C_G(H)$ ‖

Of course we now consider the situation when $|H| = p$.

<u>Theorem 5.3.2.</u>

Let H dd $G \in \iota 3$, and let $|H| = p$. Then either

i. H^G is an elementary abelian p-group or

ii. $G = S(N \times K)$ where N is a maximal q-subgroup (q a prime) which is elementary abelian and normal in G, $K = C_G(H^G)$ is a maximal $\{p,q\}'$ subgroup of G, and S is a maximal p subgroup of G which is either locally cyclic or locally generalised quaternion, and $H \leq S$. $H^G = HN$ is a Q-group.

Proof

Suppose H^G is not elementary abelian. Then there exists some finite subgroup of G, F, say, such that H^F is not elementary abelian. Let $\mathcal{S} = \{F_1 \leq G \mid F_1$ finite and $F_1 \geq F\}$. Then for all $F_1 \in \mathcal{S}$, H^{F_1} is not elementary abelian. So, by Menegazzo ([III]) theorem 3.2., $F_1 = S_{F_1}(N_{F_1} \times K_{F_1})$ for all $F_1 \in \mathcal{S}$ where $S_{F_1} \in Syl_p(F_1)$ is either cyclic or generalised quaternion, $N_{F_1} \in Syl_q(F_1)$ is elementary abelian, K_{F_1} is is a $\{p,q\}'$ subgroup such that $K_{F_1} = C_{F_1}(H^{F_1})$

1. Let $N = \bigcup_{F_1 \in \mathcal{S}} N_{F_1}$. Then $N \lhd G$, and is an elementary abelian q-group.

Firstly, we note that $F_1 \leq F_2 \Rightarrow N_{F_1} \leq N_{F_2}$ ($\because H^{F_2} = HN_{F_2}$ and N_{F_2} is the unique Sylow q-subgroup of F_2). Thus N is a subgroup of G which is an elementary abelian q-subgroup (for, let $x \in N, y \in N$. Then there exist subgroups F_1, F_2 of $\mathcal{S}$ such that $x \in N_{F_1}, y \in N_{F_2}$. Let $F_3 = \langle F_1, F_2 \rangle$. Then $F_3 \in \mathcal{S}$ and hence $xy^{-1} \in N_{F_3}$, by the note above. Also, x and y are both q elements and $[x,y] = 1$) Also $N \lhd G$. For, let $n \in N$, $g \in G$. Thus there exists a subgroup $F_4 \in \mathcal{S}$ such that $n \in N_{F_4}$. Let $F_5 = \langle F_4, g \rangle$. $F_4 \leq F_5 \Rightarrow n \in N_{F_5} \Rightarrow n^g \in N_{F_5}$

(as $N_{F_s} \lhd F_s) \leq N$.

2. Let $K = \bigcup_{F_i \in \mathcal{S}} K_{F_i}$. Then K is a maximal $\{p,q\}'$ subgroup of G and $K = C_G(H^G)$

We note that $F_1 \leq F_2 \Rightarrow K_{F_1} \leq K_{F_2}$. For, let $k \in K_{F_1}$

Then $k \in F_2 = S_{F_2}(N_{F_2} \times K_{F_2})$ and hence there exists elements s of S_{F_2}, $n \in N_{F_2}$, $k' \in K_{F_2}$ such that $k = snk'$.

Then $k^{|k|} = (sn)^{|k|}k''$ for some $k'' \in K_{F_2}$. $|K|$ $\{p,q\}'$,

sn a $\{p,q\}$ element and $(S_{F_2}N_{F_2}) \cap K_{F_2} = 1 \Rightarrow sn = 1$

i.e. $k = k'$ so $k \in K_{F_2}$ as required.

Hence K is a $\{p,q\}'$ subgroup of G and is clearly a maximal such subgroup (for, let y be any $\{p,q\}'$ element. Then $L = \langle F,y \rangle \in \mathcal{S}$, and hence $y \in K_L \leq K$)

Also let $k \in K$, (so $k \in K_{F_i}$, say) and let $h \in H^G$ (so there exists $F_j \in \mathcal{S}$ such that $h \in H^{F_j}$). Let $R = \langle F_i, F_j \rangle$ Then $k \in K_R \Rightarrow k \in C_R(H^R) \Rightarrow [k,h] = 1$. This is true for all $h \in H^G$ so $K \leq C_G(H^G)$. Conversely, if $y \in C_G(H^G)$, there exists $F_l \in \mathcal{S}$ such that $y \in F_l$, and hence $y \in C_{F_l}(H^{F_l})$ i.e. $y \in K_{F_l} \leq K$. So $K = C_G(H^G)$.

3. $\frac{G}{NK}$ is a p-group. For let $NKy \in \frac{G}{NK}$. Let T be a subgroup $\in \mathcal{S}$ such that $y \in T$. Then $\frac{TNK}{NK} \cong \frac{T}{T \cap NK} = \frac{T}{S_T N_T K_T \cap NK}$

$= \frac{T}{(S_T \cap NK)\, N_T K_T} = \frac{T}{N_T K_T} \cong S_T$ which is a p-group and is either cyclic or generalised quaternion.

Now let S be a maximal p-subgroup of G. Let F be a finite subgroup of $\mathcal{S}$. Then there exists a subgroup L of $\mathcal{S}$ such that $F \leq L$ and $\frac{LNK}{NK} \cong \frac{L}{L \cap NK} \cong S_L$, as before, which is either cyclic or generalised quaternion. So F ($\cong \frac{FNK}{NK} \leq \frac{LNK}{NK}$) is either cyclic or generalised quaternion.

Suppose S is locally cyclic. Then $\frac{G}{NK}$ is locally cyclic i.e. $\frac{G}{NK} \cong C_{p^\infty}$, i.e. there is a chain of finite subgroups $\{F_i\}$ such that $F_i NK < F_{i+1} NK$ for all i, and $\bigcup_i (F_i NK) = G$. Without loss of generality, we may take the $\{F_i\}$ to belong to $\mathcal{S}$. $\frac{F_i NK}{NK} \cong S_{F_i} \Rightarrow$ we may take Sylow p-subgroups $\{S_{F_i}\}$ of the $\{F_i\}$ such that $S_{F_i} < S_{F_{i+1}}$ for all i. Let $S = \bigcup_i S_{F_i}$. Then as $\bigcup_i (F_i NK) = \bigcup_i (S_{F_i} NK)$, we have that $G = S(N \times K)$

Suppose now that there exists a subgroup M belonging to $\mathcal{S}$ such that S_M is generalised quaternion. We redefine $\mathcal{S}$ so that $\mathcal{S} = \{F_i \mid F_i \text{ finite}, F_i \geq M\}$. Then S_F is generalised quaternion for all $F \in \mathcal{S}$ and $\frac{G}{NK}$ is locally generalised quaternion. So, by for example ([XI]) p.191, there are, as above, finite subgroups $\{F_i\}$ such that $F_i NK < F_{i+1} NK$ and $\bigcup_i (F_i NK) = G$. Then we proceed as above and the theorem is proved.||

Section Four

Finally, I consider a theorem in finite group theory. That there is a wealth of theorems yet to be proved concerning dual-dedekind subgroups in finite groups (e.g. dualizing Schmidt's results) I have no doubt; but their proof will have to wait until some future date.

Theorem 5.4.1.

Let H dd G be such that H is a maximal subgroup of G where G is finite. Then $|G:H| = q$ and either

i. $H \lhd G$ or

ii. H is modular in G and $\frac{G}{H_G} = \frac{H}{H_G}\frac{N}{H_G}$ where $\left|\frac{N}{H_G}\right| = q$ and $\left|\frac{H}{H_G}\right| = p$

Proof

Suppose that H is not normal in G and without loss of generality, take the core of H in G to be 1.

Let H_1 be a minimum dual-dedekind subgroup of G contained in H. By Menegazzo, ([III]), theorems 3.1. & 3.2., either H_1 is normal in G (which cannot be as the core of H in G is trivial) or $|H_1| = p$, where p is some prime, and either H_1^G is an elementary abelian p-group, or H_1^G ($= H_1N$, $N \neq 1$) is a Q-group.

In the former case, we have that $H \cap H_1^G$ is normal in H and also in H_1^G (as the latter is an abelian group), so, as by the maximality of H and the fact that its core is trivial, $HH_1^G = G$, we have that $H \cap H_1^G$ is normal in G and hence $H \cap H_1^G = 1$. But $1 \neq H_1 \leq H \cap H_1^G$, so this case cannot hold.

In the latter case, we have that as $N \trianglelefteq H_1^G \triangleleft G$, then $N \triangleleft G$ so $HN = G$, by the maximality of H and the fact that H has a trivial core in G. $H \cap N \triangleleft N$ (as N is an elementary abelian q-group (q some prime)) and $H \cap N \triangleleft H$, so $H \cap N = 1$.

Thus $|G:H| = |HN:H| = |N:N \cap H| = |N|$.

Let L be such that $L \leq N, |L| = q$. As $H \cap L = 1$, $G = \langle H,L \rangle$

$N = \langle H,L \rangle \cap N = \langle H \cap N, L \rangle$ (by D1) $= L$ so $|N| = q$ and $|G:H| = q$ (*)

Now let $K = C_G(H_1^G)$. By ([III]), theorem 3.2., K is a $\{p,q\}'$ subgroup. $K \triangleleft G \Rightarrow G = HK$, so $|G| = \frac{|K||H|}{|H \cap K|} = |H| \times \{p,q\}'$ number

But (*) gives $|G| = |H| \times q$, so $K = 1$. Hence by ([III]), 3.2., $G = SN$ where S is a Sylow p-subgroup of G which is either cyclic or generalised quaternion. Choosing S so that $H \leq S$, we find by (*), that $H = S$

Let $C = C_G(N)$. Then, as $N \cong C_q$, we have that $N \leq C$. Suppose $N \lneq C$. Then $C = HN \cap C = (H \cap C)N$ and $N < C \Rightarrow H \cap C$ is non-trivial. But $H \cap C \triangleleft H$ and hence $H \cap C \triangleleft G$ which is impossible as $H_G = 1$. So $C = N$, and $\frac{G}{C} = \frac{G}{N}$ is isomorphic to a subgroup of Aut(N). Thus $\frac{G}{N}$ is abelian and hence so is $\frac{H}{H \cap N}$. As $H \cap N = 1$, we have that H is abelian and so is cyclic.

Now let $N = \langle x \rangle$, and consider H^x. $H \neq H^x$ and so $G = \langle H,H^x \rangle$. $[\langle H,H^x \rangle / H^x] \simeq [H / H \cap H^x]$ gives us that $H \cap H^x$ is a maximal subgroup in H. But H and H^x both abelian implies that $H \cap H^x \triangleleft G$ and hence $H \cap H^x = 1$. So, $|H| = p$ as required.

Thus G is a Q-group, and hence has a modular lattice.

Hence H is modular in G ∥

APPENDIX

The following major theorem is due to Dr. S.E.Stonehewer:

Theorem

Let $M \text{ m } G \in \iota\mathfrak{F}, M_G = 1$. Then there exist subgroups $K, P_1, P_2 \ldots$ of G such that

(i) $G = K \times P_1 \times P_2 \times \ldots$

(ii) P_i is a generalised P_i-group for all i

(iii) For all $x_i \in P_i, x_j \in P_j, k \in K$, we have $(|x_i|, |x_j|) = (|x_i|, |k|) = 1$ for all i, j, $i \neq j$.

(iv) $M = M \cap K \times Q_1 \times Q_2 \ldots\ldots$ where Q_i is a maximal q_i-subgroup of P_i for all i, and $M \cap K$ qn G.

Proof

We may suppose that M is not quasinormal in G (for otherwise we may take $G = K$).

By 2.2.6., we have that $M \in \iota\mathfrak{N}$. So M is a direct product of its maximal p-subgroups (p a prime). Let $Q_1, Q_2 \ldots$ be the maximal q_1-, q_2- . . . subgroups of M which are not quasinormal in G and let R be the product of all those maximal p-subgroups which are quasinormal in G.

So $M = R \times Q_1 \times Q_2 \times \ldots$

a). For all i, there exists an X_i such that $|X_i : M|$ is finite and $Q_i M_{X_i}$ is not quasinormal in X_i.

Suppose not. Suppose $\exists$ i such that $\forall X, |X:M|$ finite $\Rightarrow$ $Q_i M_X$ qn X. Let $g \in G$. Then we shall prove that $Q_i\langle g\rangle$ is a subgroup which cannot be, as Q_i is not quasinormal in G.

Let $A = \{X \mid |X:M| \text{ is finite and } X \geq \langle M, g\rangle\}$.

So, $\forall\ X \in A, Q_i M_X\langle g\rangle$ is a subgroup of X.

Let $B = \bigcap (Q_i M_X \langle g\rangle)\ \forall X \in A$.

Let $b \in B$. Then $b = q_{i_X} m_X g^{n_X}$ where $q_{i_X} \in Q_i$, $m_X \in M_X$ and n_X is such that $1 \leq n_X \leq |g|$, $\forall$ X.

Let $A_n = \{X \in A \mid n_X = n\}$ for $n = 1,2, \ldots |g|$. Then there exists an n such that $\bigcup_{X \in A_n} X = G$ (I)

For, suppose not. $\exists\ g_i \in G \setminus \bigcup_{X \in A_i} X$ for $i = 1,2,\ldots |g|$

Let $X = \langle M, g, g_1, \ldots, g_{|g|}\rangle$. Then $X \in A$ and hence $X \in A_n$ for some n . So $g_n \in \bigcup_{X \in A_n} X$, which contradicts our choice of g_n. Hence our supposition was incorrect and $\exists$ n such that $\bigcup_{X \in A_n} X = G$

So $b \in \bigcap_{X \in A_n} Q_i M_X\langle g\rangle$. Let T be the q_i-complement of M. So $M = Q_i \times T$, $M_X = (Q_i \cap M_X) \times (T \cap M_X)$.

$$\text{So } \bigcap_{X \in A_n} Q_i M_X = \bigcap_{X \in A_n} Q_i((Q_i \cap M_X) \times (T \cap M_X)) = \bigcap_{X \in A_n} Q_i \times (T \cap M_X)$$

$$= Q_i \times \bigcap_{X \in A_n} T \cap M_X$$

But $\bigcap_{X \in A_n} T \cap M_X \leq \bigcap_{g \in G} T \cap M^g = 1$ by (I). So $\bigcap_{X \in A_n} Q_i M_X = Q_i$.

Now, by the definition of A_n, we have that $\bigcap_{X \in A_n} Q_i M_X\langle g\rangle = (\bigcap_{X \in A_n} Q_i M_X)\langle g\rangle$. So $b \in \bigcap_{X \in A_n} Q_i M_X\langle g\rangle \Rightarrow b \in (\bigcap_{X \in A_n} Q_i M_X)\langle g\rangle \Rightarrow$ $b \in Q_i\langle g\rangle$. This is true for all $b \in B$. So $B \leq Q_i\langle g\rangle$. But $Q_i\langle g\rangle \leq B$ obviously. So $B = Q_i\langle g\rangle$ i.e. $Q_i\langle g\rangle$ is a subgroup which was to prove.

Thus, our original supposition is incorrect . Consequently $\forall i$ $\exists\ X_i$ such that $|X_i : M|$ is finite (and hence so is $|X_i : M_{X_i}|$) and $Q_i M_{X_i}$ is not quasinormal in X_i.

Let $S = \{X \mid X \geq X_i,\ (X:M) \text{ is finite}\}$

b). $\forall\, X \in S, Q_iM_x$ is not quasinormal in X.

$$\begin{array}{l} X \\ X_i \\ M \\ M_{x_i} \\ M_x \end{array}$$

For, suppose otherwise. Then Q_iM_x is quasinormal in $X \Rightarrow Q_iM_x$ quasinormal X_i. Let $g \in X_i$. Then $Q_iM_{x_i} \langle g\rangle = Q_i\langle g\rangle M_{x_i}$ (as $M_{x_i} \lhd X_i$)

$$\leq \langle g\rangle Q_iM_xM_{x_i} = \langle g\rangle Q_iM_{x_i}$$

i.e. $Q_iM_{x_i}$ qn X_i which cannot be so.

We now investigate the structure of Q_i for all i and then the structure of Q_i^G.

c). $|Q_i| = q$.

For suppose $|Q_i| > q$. Then $\exists$ $X \in S$ such that $\left|\frac{Q_i}{Q_i \cap M_x}\right| > q$ (as $M_G = 1$) i.e. $\left|\frac{Q_iM_x}{M_x}\right| > q$. $M = Q_i \times T$ where T is the q_i-complement of $M \Rightarrow \frac{M}{M_x} = \frac{Q_iM_x}{M_x}\cdot\frac{TM_x}{M_x}$. hence $\frac{Q_iM_x}{M_x} \in \mathrm{Syl}_q \frac{M}{M_x}$. As $X \in S$, Q_iM_x is not quasinormal in X (by b).) so $\frac{Q_iM_x}{M_x}$ is not quasinormal in $\frac{X}{M_x}$.

Also $M \in L\mathfrak{N}$ and $\frac{M}{M_x} \in \mathfrak{F} \Rightarrow \frac{M}{M_x} \in \mathfrak{N}$.

Hence by Schmidt [17] lemma 4, applied to $\frac{X}{M_x}$, we have that $\left|\frac{QM_x}{M_x}\right| = q$. This contradiction proves the result.

d). Q_i is a maximal q_i-subgroup of G.

For, suppose not. Suppose $\exists$ a finite q_i-subgroup A such that

$Q_i \not\leq A$. Then as Q_i is not quasinormal in G, $\exists\, g \in G$ such that $Q_i\langle g\rangle \neq \langle Q_i, g\rangle$. Let $F = \langle A, g\rangle$. $M \cap F$ m F and $Q_i \in Syl_{q_i}(M \cap F)$. $M \cap F \in \mathfrak{M}$ and Q_i not quasinormal in F implies (by Schmidt [Ⅱ] lemma 4) that $Q_i \in Syl_{q_i}(F)$. This contradiction proves the result.

So all the maximal q_i-subgroups of G are of order q_i and lie in Q_i^G.

We now investigate the structure of Q_i^G.

e). Q_i^G is a $\{p_i, q_i\}$ group.

Let F be a finite subgroup of G in which Q_i is not quasinormal ($Q_i \leq F$). Then by lemma 5 Schmidt [Ⅱ], $\frac{Q_i^F}{Q_{i_F}}$ is a $\{p_i, q_i\}$ subgroup

but $|Q_i| = q$ and Q_i not quasinormal in F $\Rightarrow Q_{i_F} = 1$ i.e. Q_i^F is a $\{p_i, q_i\}$ group. Hence so is Q_i^G.

f). M has no elements of order p_i.

For, suppose $x \in M$ has order p_i. $M_G = 1 \Rightarrow \exists\, h \in G$ such that $x \notin M^h$. Let $X = \langle X_i, h\rangle$. Then $X \in S$ and $x \notin M_X$. So $p_i \mid |M:M_X|$. To clarify notation, let us, for the moment, take $M_X = 1$. By the corollary to the Main Theorem of Schmidt, we have

$$X = P_1 \times \qquad \times P_r \times K \text{ and}$$

$$M = Q_1 \times \qquad \times Q_r \times M \cap K, \text{ where } Q_i^x = P_i.$$ As $(|P_i|, |P_j|) = (|P_i|, |K|) = 1 \ \forall\, i,j,\ 1 \leq i \neq j \leq r$, $p_i \nmid |K|$ and $p_i \nmid |Q_j|$ for $j \neq i$ (as, if so, Q_j a maximal q_j- subgroup of G $\Rightarrow Q_i^X = Q_i Q_j^x$ and hence Q_j^x is normal (and hence of course quasinormal) in X).

So $p_i \nmid |M|$. Hence M has no elements of order p_i.

g). $p_i \neq p_j$ if $Q_i \neq Q_j$

For, let X be such that $M \leq X$, neither Q_i nor Q_j are quasinormal in X and $|X:M|$ is finite. Then, by the Main Theorem of Schmidt the result follows.

h). Q_i^G (= P_i, say) is a generalised P-group.

For. let $X \in S$. Then by the Main Theorem of Schmidt [II], $\frac{Q_i^x M_x}{M_x}$ is a P-group of order $p^n q$ $(p > q)$. But $Q_i^x \cap M_x \leq Q_i$ by f). and as $|Q_i| = q$, $Q_i^x \cap M_x = Q_i$ or $Q_i^x \cap M_x = 1$. $Q_i^x \cap M_x = Q_i \Rightarrow$ $Q_i \leq M_x \Rightarrow M_x = Q_i M_x$ which is normal in X. So, as $Q_i M_x$ is not quasinormal in X, we have that $Q_i^x \cap M_x = 1$ i.e. $Q_i^x = \frac{Q_i^x}{Q_i^x \cap M_x}$ $\cong \frac{Q_i^x M_x}{M_x}$ is a P-group.

Hence, as S is a local system of G, Q_i^G is a generalised P-group.

Let A_i be the (unique) maximal p_i-subgroup of P_i.

i). A_i is the unique maximal p_i-subgroup of G.

For. let x_i be a p_i-element of G such that $x_i \notin A_i$. Let $X \in S$ be such that $x_i \in X$. By the Main Theorem of Schmidt [II], $\frac{Q_i^x M_x}{M_x}$ contains the unique Sylow p_i-subgroup of $\frac{X}{M_x}$. So $x_i \in Q_i^x M_x$ $= Q_i(A_i \cap X)M_x \leq Q_i A_i M_x$. But $\frac{Q_i M_x A_i}{A_i} \cong \frac{Q_i M_x}{Q_i M_x \cap A_i}$ which is a p_i'-group by f). So $x_i \in A_i$ as required.

So $(Q_1 \times Q_2 \times \quad\quad)^G = P_1 \times P_2 \times$

Let $\pi = \{p_i, q_i \mid i = 1, 2 \; \cdots \}$

j). The π elements of G form a subgroup of G.

Suppose not. Suppose y is a π element and $x_1, \cdots, x_n$ are π' elements of G such that $y \in \langle x_1, \ldots\ldots, x_n \rangle$.

By i). and d). we may suppose that y has prime power order and that $y \in P_i$ for some i.

Let X be such that $X \geq \langle x_1, \ldots\ldots, x_n \rangle$, $|X:M|$ is finite and

$Q_1 M_X$ is not quasinormal in X.

Then, by the main theorem of Schmidt ([II]), we have that if $\frac{X}{M_X} = \frac{Q^X M_X}{M_X} \times K$ (where the notation is as in Schmidt ([II])),

$x_1, \ldots, x_n$ and hence $y \in K$. This contradiction proves j).

$$\text{So } G = K \times P_1 \times P_2 \times \cdots$$

$$\text{and} \quad M = M \cap K \times Q_1 \times Q_2 \times \cdots$$

where $M \cap K = R$ as required.

Considering the main body of the thesis (which, of course, was completed before the above theorem was proved), we see that the first part of theorem 4.1.3. and theorem 4.1.6. both follow directly from Dr.Stonehewer's theorem.

<u>Corollary 1.</u>(cf theorem 5 of Schmidt ([II]))

Let $M \text{ m } G \in L\mathfrak{F}$, $M_G = 1$. Let Q be a maximal q-subgroup of M. Then $Q \text{ m } G$.

<u>Proof</u>

By 1.1.2., it is sufficient to prove that Q is locally modular in G.

Using the notation of the main theorem, we have either that $Q \leq M \cap K$ and $Q \text{ qn } G$, in which case there is nothing to prove, or $Q = Q_1$, say, and $|Q| = q_1$. Let F be any finite subgroup of G containing Q. Then it is enough to prove that $Q \text{ m } F$.

Using the same notation as in the main theorem, we have that

$G = K \times P_1 \times P_2 \times \ldots$ i.e. $G = P_1 \times H$ (where $H = P_2 \times \ldots \times K$). Thus $F = (F \cap P_1) \times (F \cap H)$ and $(|F \cap P_1|, |F \cap H|) = 1$

Now $F \cap P_1$ is a P_1-group containing Q and thus, as any P-group has a modular lattice of subgroups, $Q \text{ m } F \cap P$. So, by Schmidt ([II]), lemma 3, $Q \text{ m } F$ as required.||

From this follows immediately:

<u>Corollary 2</u>

If $M \text{ m } G \in \mathfrak{L}$, $M_G = 1$, then M a minimum modular subgroup of $G \Rightarrow$ M a q-group.||

<u>Corollary 3.</u>

If $G \in \mathfrak{L}$ is simple, G can have no non-trivial modular subgroups.||

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