Hypergraphic matroids

Thesis

HYPERGRAPHIC MATROIDS

BY

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vertex $W_{i-2} \in V(e_{i-2})$, with $W_{i-2} \notin \cup\{V(e_j): 1 \le j \le t, j \neq i-1, i-2\}$.

$W_{i-2} = V_{i-2}$, since V_{i-2} is the only vertex V satisfying

$V \in V(e_{i-2})$ and $V \notin \bigcup_{j=1}^{i-3} V(e_j)$.

Thus, $V_i \in V(e_i) - V(e_{i-1})$,

$V_{i-1} \in V(e_i) \cap V(e_{i-1})$,

$V_{i-2} \in V(e_{i-1}) - V(e_i)$,

and the result follows.

page 138, lines 6 and 13:

In each line, for " $\lambda^{|n'(e)|+|n'(f_1)|-2}$ " read

" $\lambda^{|n'(e)|+|n'(f_1)|-3}$ " .

page 138, line 15:

for " $|n'(e)|+|n'(f_1)|-1$ " read " $|n'(e)|+|n'(f_1)|-2$" .

page 147, line 17:

for " $1 \le i \le m$ " read " $1 \le i \le m_J$ " .

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ABSTRACT OF THESIS

"HYPERGRAPHIC MATROIDS"

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A method of defining a matroid on the edge-set of a k-uniform hypergraph (a k-hypergraph) is defined, which is a generalisation of that used for defining a matroid on the edge-set of a graph; the matroids so defined are called "hypergraphic matroids".

Analogues are found in hypergraphs of the concepts of trees, forests, circuits, cutsets and components; we show that two generalisations are necessary of the concept of a vertex in a graph - a vertex, and a (k-1)-subset of vertices of a k-hypergraph; we call such a subset a node. The class of hypergraphic matroids is not closed under contraction, but may be enlarged to the class of generalised hypergraphic matroids, which is the closure of the class of hypergraphic matroids under the operation of taking minors. These matroids are defined in an analogous way to hypergraphic matroids, but a particular type of submodular function is necessary, instead of the cardinality function used for hypergraphs. We show that no finite set of forbidden minors exists to characterise either hypergraphic or generalised hypergraphic matroids. There is, however, a lattice characterisation of hypergraphic matroids.

Transversal matroids are hypergraphic, and we give a simple method of obtaining a presentation. We also prove that hypergraphic matroids are representable over every characteristic, and that binary generalised hypergraphic matroids are graphic.

The graph-theoretic notion of series-parallel extension is generalised, motivated by hypergraph considerations, to a new operation

ABSTRACT CONTINUED

called generalised series-parallel extension. This operation has many properties similar to series-parallel extension. Generalised series-parallel networks are defined, and characterised by a set of six forbidden minors. An extension of this result characterises ternary base-orderable matroids.

We show that the matroid of a hypergraph can be used to derive weak and strong colourings of the nodes, and that, under obvious necessary conditions, all such colourings arise in this way. Connectedness and paths are investigated, but the results obtained for hypergraphs are less satisfactory than those for graphs, largely because the concepts of "node" and "vertex" do not coincide for general k-hypergraphs.

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CONTENTS

CHAPTER 1

INTRODUCTION, DEFINITIONS AND TERMINOLOGY

INTRODUCTION:

It has long been known that a matroid can be defined on the set of edges of a graph. Indeed, one of the principal reasons for studying matroids is that, in some senses, they are abstractions of graphs - preserving some, though not all, of the structure of graphs. Attempts have been made in the past to extend the definition of a matroid on the edges of a graph to a matroid on the edges of a hypergraph. Such attempts have met with mixed success; some potential definitions may not lead to matroids, while other definitions may fail to reduce to the definition applicable to graphic matroids when the hypergraph is in fact a graph.

Our purpose is two-fold. Firstly, we shall define a matroid on the set of edges of a uniform hypergraph which avoids the above-mentioned difficulties, and we shall examine various properties of the class of matroids so produced. Secondly, we shall investigate what, if any, of the structure of a hypergraph is preserved by the matroid on its edges and, if any structure is preserved, what can be learnt about the hypergraph from the matroid.

A desirable property of any loopless matroid on the edge-set of a graph is that two edges should be parallel in the matroid if and only if they are parallel in the graph. In a loopless graph, two edges are parallel if they have the same vertex-set. We say that two edges of a hypergraph are parallel if and only if they have the same vertex-set - i.e., for most practical purposes they are indistinguishable. A desirable property of any matroid on the set of edges of a hypergraph should be that two edges are parallel in the matroid if and only if they are parallel in the hypergraph. This property is not possessed by many of the definitions of a matroid on the edge-set of a hypergraph. Loopless graphs are in fact uniform hypergraphs. It is not, therefore, unreasonable that any definition of a matroid should be applicable only to uniform hypergraphs. We shall show, however, that our definition can be extended to the case of non-uniform hypergraphs.

Looking ahead briefly to the results of future chapters, we find analogues of many of the concepts of graph theory, such as trees, forests cutsets and components. We also find on several occasions that two generalisations of the concept of a vertex are necessary when we pass from a graph to a k-uniform hypergraph. These are a vertex in the hypergraph, and also the complement, within the vertex-set of an edge, of a vertex of that edge. This latter generalisation we call a node. In the case of a k-uniform hypergraph, the nodes have cardinality (k-1). The use of (k-1)-subsets in the study of simplicial complexes has long been accepted, but appears to be a relatively new idea in hypergraph theory. The nodes of a hypergraph can be used to define paths, are partitioned by the components of the hypergraph, and, under certain circumstances, can be coloured using techniques derived from matroid theory.

The class of hypergraphic matroids is not closed under the operation of contraction, and this leads us to define a larger class of matroids - the generalised hypergraphic matroids - which is closed under the operation of taking minors. It transpires that there is a sense in which an edge contracted in a hypergraph is contracted to a node; this compares favourably with the contraction of an edge of a graph to a vertex.

Various forbidden minor conditions are investigated, and it is shown that no finite set of forbidden minors exists to characterise hypergraphic or generalised hypergraphic matroids.

We generalise the notion of series-parallel extension to an operation called generalised series-parallel extension. This leads to a class of matroids called generalised series-parallel networks, which can be characterised by a set of six forbidden minors. An extension of this result leads to a forbidden-minor characterisation of ternary base-orderable matroids. Finally, a characterisation of hypergraphic matroids is given, in terms of the existence of a family of flats satisfying certain conditions.

TERMINOLOGY AND DEFINITIONS:

SET THEORY:

We begin by defining some set-theoretic notation we shall be using. For typographical convenience, set difference will be denoted by "-". Δ will be used to denote symmetric difference. Thus,

$$A \Delta B = (A-B) \cup (B-A).$$

The notation $\{x_1, x_2, \ldots, x_n\}_{\neq}$ means that $x_i \neq x_j$ for $i \neq j$.

If $(C_i : i \in I)$ is a family of sets, and $J = \{i_1, i_2, \ldots, i_n\}_{\neq} \subseteq I$,

$$\bigcup_J C_i = \cup\{C_i : i \in J\} = C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_n}.$$

$$\bigcap_J C_i = \cap\{C_i : i \in J\} = C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_n}.$$

$$\Delta_J C_i = \Delta\{C_i : i \in J\} = C_{i_1} \Delta C_{i_2} \Delta \ldots \Delta C_{i_n}.$$

If $(\underline{M}_i : i \in I)$ is a family of matroids on disjoint sets, then, with the same notation, $\bigoplus_J \underline{M}_i = \oplus\{\underline{M}_i : i \in J\} = \underline{M}_{i_1} \oplus \underline{M}_{i_2} \oplus \ldots \oplus \underline{M}_{i_n}$.

If E and E' are sets defined to be isomorphic, and $E = \{e_1, \ldots, e_n\}_{\neq}$, then, unless otherwise stated, E' will be defined as $\{e_1', e_2', \ldots, e_n'\}_{\neq}$, and the isomorphism $\theta : E \to E'$ with $\theta(e_i) = e_i'$ will be called the <u>obvious bijection</u> (or <u>obvious isomorphism</u>) between E and E'.

A k-set is a set of cardinality k.

If V is a set, and $W \subseteq V$ with $|W| = k$, W is said to be a k-subset of V.

MATROIDS:

Our matroid terminology will be standard - see, for example, Harary and Welsh [12] or Wilson [30]. However, as we shall need to refer to many of the definitions, axiom systems and properties of matroids, we give a summary here.

A matroid (independence structure) M on the finite set E is an ordered pair $(E, \mathcal{I})$ where $\mathcal{I}$ is a set of subsets of E satisfying the independence axioms:

(I1) $\phi \in \mathcal{I}$;

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$;

(I3) If $X, Y \in \mathcal{I}$ and $|X| = |Y| + 1$, then there exists $x \in X - Y$ such that $(Y \cup \{x\}) \in \mathcal{I}$.

The set $\mathcal{I}$ is called the set of independent sets of M.

A maximal independent set of M is called a base of M.

A set which is not independent is called dependent.

A minimal dependent set is called a circuit of M.

A single element of E which is a circuit of M is called a loop of M.

If $\{x,y\}_{\neq} \subseteq E$ and $\{x,y\}$ is a circuit of M, then x and y are said to be parallel in M.

A simple matroid is a matroid without loops or parallel elements.

It can be shown that all bases of M have the same cardinality. A matroid on the set E is determined uniquely not only by its independent sets, but also by its bases or circuits.

Base Axioms:

A set $\mathbb{B}$ of subsets of E is the set of bases of a matroid on E if and only if

(B1) $\mathbb{B} \neq \phi$;

(B2) If $B_1, B_2 \in \mathbb{B}$ and there exists $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $(B_1 - \{b_1\}) \cup \{b_2\} \in \mathbb{B}$

Circuit Axioms:

A set $\mathbb{C}$ of subsets of E is the set of circuits of a matroid on E if and only if

(C1) If $C_1, C_2 \in \mathbb{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;

(C2) If $C_1, C_2 \in \mathbb{C}$ with $C_1 \neq C_2$, and if $x \in C_1 \cap C_2$, then there exists $C_3 \in \mathbb{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{x\}$.

The rank ρA of a subset A of E is the maximum cardinality of independent sets contained in A. Thus, $\rho A = \max\{|X| : X \subseteq A \text{ and } X \in \mathbb{I}\}$.

The rank of the matroid $\underline{M}$, denoted by rk$\underline{M}$ is equal to ρE.

The closure σA of the set A is the set $\{x: \rho(A \cup \{x\}) = \rho A\}$.

A closed set or flat is the closure of some set. It can be shown that $F \subseteq E$ is a flat of $\underline{M}$ if and only if $\sigma F = F$. A j-flat of $\underline{M}$ is a flat of $\underline{M}$ of rank j.

A point of $\underline{M}$ is a 1-flat of $\underline{M}$.

A line of $\underline{M}$ is a 2-flat of $\underline{M}$.

A plane of $\underline{M}$ is a 3-flat of $\underline{M}$.

A hyperplane of $\underline{M}$ is a flat of $\underline{M}$ of rank rk$\underline{M}$ - 1.

A trivial j-flat of $\underline{M}$ is a j-flat of $\underline{M}$ which is independent.

If $X, Y \subseteq E$ and $Y \subseteq \sigma X$, then X is said to span Y. The span of X is the set σX.

A matroid can be determined uniquely by its hyperplanes, rank function or closure operator.

Hyperplane Axioms:

A set $\mathbf{H}$ of subsets of E is the set of hyperplanes of a matroid on E if and only if

(H1) If $H_1, H_2 \in \mathbf{H}$ and $H_1 \subseteq H_2$, then $H_1 = H_2$;

(H2) If $H_1, H_2 \in \mathbf{H}$, and if $x \notin H_1 \cup H_2$ and $y \in H_1 - H_2$, then there exists $H_3 \in \mathbf{H}$ such that $y \notin H_3$ and $H_3 \supseteq (H_1 \cap H_2) \cup \{x\}$.

Rank Axioms:

A function $\rho: 2^E \to \mathbb{Z}$ is the rank function of a matroid on E if and only if

(R1) $0 \leq \rho X \leq |X|$;

(R2) If $X \subseteq Y$ then $\rho X \leq \rho Y$;

(R3) (Submodularity of the rank function)

For any $X, Y \subseteq E$, $\rho X + \rho Y \geq \rho(X \cup Y) + \rho(X \cap Y)$.

Closure Axioms:

A function $\sigma: 2^E \to 2^E$ is the closure operator of a matroid on E if and only if

(K1) $X \subseteq \sigma X$ for all $X \subseteq E$;

(K2) If $X \subseteq Y$, then $\sigma X \subseteq \sigma Y$;

(K3) $\sigma(\sigma X) = \sigma X$;

(K4) If $x \in \sigma(X \cup \{y\})$ and $x \notin \sigma X$, then $y \in \sigma(X \cup \{x\})$.

Where we are dealing with several matroids, the sets of independent sets, bases, circuits and hyperplanes of $\underline{M}$ will be denoted by $\mathbf{I}(\underline{M})$, $\mathbf{B}(\underline{M})$, $\mathbf{C}(\underline{M})$ and $\mathbf{H}(\underline{M})$ respectively. The rank function of $\underline{M}$ will similarly be denoted by $\rho_{\underline{M}}$.

The dual of the matroid $\underline{M} = (E, \mathbf{B})$ is denoted by $\underline{M}^*$. It is the matroid $(E, \mathbf{B}^*)$, where $\mathbf{B}^* = \{E-B: B \in \mathbf{B}\}$. Clearly $(\underline{M}^*)^* = \underline{M}$.

It can be shown that C is a circuit of $\underline{M}$ if and only if $E-C$ is a hyperplane of $\underline{M}^*$.

A circuit of $\underline{M}^*$ is also called a <u>co-circuit</u> of $\underline{M}$, a base of $\underline{M}^*$ is called a <u>co-base</u> of $\underline{M}$ and so on. In particular, a loop of $\underline{M}^*$ is called a <u>co-loop</u> of $\underline{M}$. A co-loop of $\underline{M}$ is an element of E which is an element of every base of $\underline{M}$ or, equivalently, of no circuit of $\underline{M}$. The rank function of $\underline{M}^*$ is denoted by ρ^*. If $A \subseteq E$, then

$\rho^*A = |A| + \rho(E-A) - \rho E$.

<u>Isomorphism of Matroids</u>:

$\underline{M}_1 = (E_1, \mathbf{I}_1)$ and $\underline{M}_2 = (E_2, \mathbf{I}_2)$ are said to be <u>isomorphic</u>, written $\underline{M}_1 \cong \underline{M}_2$, if and only if there exists a bijection $\phi: E_1 \to E_2$ such that, for $\{x_1, x_2, \ldots, x_m\}_{\neq} \subseteq E_1$,

$\{x_1, x_2, \ldots, x_m\} \in \mathbf{I}_1$ if and only if $\{\phi x_1, \phi x_2, \ldots, \phi x_m\} \in \mathbf{I}_2$.

<u>Operations on Matroids</u>:

RESTRICTION:

Let $T \subseteq E$. Then the <u>restriction</u> of $\underline{M}$ to T, denoted by $\underline{M}\times T$, is the matroid $(T, \mathbf{I}')$ with set of independent sets $\mathbf{I}' = \{X \in \mathbf{I} : X \subseteq T\}$.

If $T = E-X$ for some $X \subseteq E$, we often refer to $\underline{M}\times T$ as the <u>deletion</u> of X from $\underline{M}$. If e is a loop or one of a pair of parallel elements of $\underline{M}$, the deletion of $\{e\}$ is called an <u>elementary simplification</u> of $\underline{M}$. If $\underline{M}$ is a non-simple matroid, the <u>simplification</u> of $\underline{M}$ is the simple matroid obtained from $\underline{M}$ by successive elementary simplifications - i.e. the simple matroid $(..(\underline{M}\times(E-\{e_1\}))\times(E-\{e_1,e_2\}))\times\ldots\times(E-\{e_1,\ldots,e_m\})$ where e_i is a loop or one of a pair of parallel elements of $\underline{M}\times(E-\{e_1,\ldots,e_{i-1}\})$, $(1 \le i \le m)$.

CONTRACTION:

Let $T \subseteq E$. Then the <u>contraction</u> of $\underline{M}$ to T, denoted by $\underline{M}.T$, is the matroid $(T, \mathbf{I}'')$ with set of independent sets $\mathbf{I}''$, where

$\mathbf{I}'' = \{X \in \mathbf{I} : X \cup A \in \mathbf{I}$ for some maximal independent subset $A \subseteq E-T\}$.

If $T = E-X$ for some $X \subseteq E$, we often refer to $\underline{M}.T$ as the <u>contraction</u> or <u>contraction out</u> of X from $\underline{M}$.

It is easy to check that $(\underline{M} \times T)^* = \underline{M}^*.T$.

Let $\underline{M}'' = \underline{M}.(E-X)$. Then, for $A \subseteq E-X$,

$$\rho_{\underline{M}''}(A) = \rho_{\underline{M}}(A \cup X) - \rho_{\underline{M}}(X).$$

A <u>minor</u> of $\underline{M}$ is a matroid obtained from $\underline{M}$ by a sequence of restrictions and contractions.

If X and Y are disjoint subsets of E, then

$$(\underline{M} \times (E-X)).(E-(X \cup Y)) = (\underline{M}.(E-Y)) \times (E-(X \cup Y)).$$

In other words, the matroid obtained from $\underline{M}$ by the deletion of X and the contraction of Y is independent of the order of the operations of deleting X and contracting Y. Thus, every minor of $\underline{M}$ may be written as $(\underline{M} \times A).B$ or as $(\underline{M}.X) \times Y$ for suitable sets $A, B, X, Y \subseteq E$.

The "<u>scum theorem</u>" of Crapo-Rota [6] states the following: If $\underline{M}_1$ is a simple minor of the simple matroid $\underline{M}$, then $\underline{M}_1 \cong (\underline{M}.A) \times B$ where $rk(\underline{M}_1) = rk(\underline{M}.A)$ for some $A, B \subseteq E$.

A set of <u>forbidden minors</u> for a class $\mathfrak{m}$ of matroids is a set $\mathfrak{f}$ of matroids such that $\underline{M} \notin \mathfrak{m}$ if $\underline{M}$ contains a minor isomorphic to a member of $\mathfrak{f}$.

TRUNCATION:

Let $\underline{M}$ be a matroid on E, and let $t \leq rk(\underline{M})$. Then the <u>t-truncation</u> of $\underline{M}$, denoted by $\underline{M}^{(t)}$, is the matroid $(E, \mathfrak{I}^{(t)})$, with set of independent sets $\mathfrak{I}^{(t)} = \{X \in \mathfrak{I} : |X| \leq t\}$.

Our method for obtaining truncations is that used by Piff [21].

DIRECT SUM:

Let $\underline{M}_1 = (E_1, \mathfrak{I}_1)$ and $\underline{M}_2 = (E_2, \mathfrak{I}_2)$, where $E_1 \cap E_2 = \phi$. The <u>direct sum</u> of $\underline{M}_1$ and $\underline{M}_2$, denoted by $\underline{M}_1 \oplus \underline{M}_2$, is the matroid $(E, \mathfrak{I})$, where

$$E = E_1 \cup E_2$$
$$\mathbb{I} = \{I_1 \cup I_2 : I_1 \in \mathbb{I}_1, I_2 \in \mathbb{I}_2\}.$$

A matroid $\underline{M}$ is said to be <u>connected</u> if, for every representation $\underline{M} = \underline{M}_1 \oplus \underline{M}_2$, with $\underline{M}_1$ and $\underline{M}_2$ as above, either $E_1 = \phi$ or $E_2 = \phi$. A <u>connected component</u> of $\underline{M}$ is a maximal subset $E' \subseteq E$ such that $\underline{M} \times E'$ is connected. Equivalently, the matroid $\underline{M}$ on the set E is connected if and only if for every pair of elements $\{x,y\}_{\neq} \subseteq E$, $\{x,y\} \subseteq C$ for some $C \in \mathbb{C}(\underline{M})$. It is easy to show that

$$\rho_{\underline{M}_1 \oplus \underline{M}_2}(A) = \rho_{\underline{M}_1}(A \cap E_1) + \rho_{\underline{M}_2}(A \cap E_2), \text{ for any } A \subseteq E.$$

DILWORTH TRUNCATION:

Let $\underline{M}$ be a matroid on E with rank r, and let $\mathbb{F}^{k+1}$ denote the set of (k+1)-flats of $\underline{M}$ ($0 \le k \le r$, where $\mathbb{F}^{r+1}$ is defined to be ϕ). Then the <u>level-k Dilworth truncation</u> of $\underline{M}$, denoted by $\underline{M}^{d,k}$, is the matroid on the set $\mathbb{F}^{k+1}$, where a set $\mathbb{A} \subseteq \mathbb{F}^{k+1}$ is independent in $\underline{M}^{d,k}$ if and only if either $\mathbb{A} = \phi$ or

$\rho(\vee\mathbb{G}) \ge |\mathbb{G}| + k$ for each non-empty subset $\mathbb{G}$ of $\mathbb{A}$, where $\vee\mathbb{G}$ denotes the supremum in $\underline{M}$ of the flats $G \in \mathbb{G}$ - i.e. that flat of $\underline{M}$ which contains each $G \in \mathbb{G}$ as a subset, and which is minimal with respect to this property. For a proof that the above definition does define a matroid, the reader is referred to Crapo-Rota [6].

<u>Special Types of Matroid</u>:

UNIFORM MATROID:

Let $\underline{M} = (E, \mathbb{I})$ be the matroid on E, with $\mathbb{I} = \{X \subseteq E : |X| \le r\}$, for some $r \le |E|$. Then $\underline{M}$ is called the <u>uniform matroid of rank r</u> on the set E. It is easy to see that all uniform matroids of rank r on sets of n elements are isomorphic; the notation $U_{r,n}$ is used to denote the uniform matroid of rank r on n elements. The uniform matroid of rank r on the n-set X is denoted by $U_{r,n}(X)$. The free matroid on E is the matroid $U_{n,n}(E)$, where $n = |E|$.

FANO MATROID:

This is the matroid on seven elements $\{a,b,c,d,e,f,g\}_{\neq}$ with circuits $\{a,b,f\}$, $\{a,c,e\}$, $\{a,d,g\}$, $\{b,c,d\}$, $\{b,e,g\}$, $\{c,f,g\}$, $\{d,e,f\}$ and all 4-subsets of $\{a,b,c,d,e,f,g\}$ containing none of these.

NON-FANO MATROID:

This is the matroid on seven elements $\{a,b,c,d,e,f,g\}_{\neq}$ with circuits $\{a,b,f\}$, $\{a,c,e\}$, $\{a,d,g\}$, $\{b,c,d\}$, $\{b,e,g\}$, $\{c,f,g\}$ and all 4-subsets of $\{a,b,c,d,e,f,g\}$ containing none of these.

REPRESENTABLE MATROIDS:

A matroid $\underline{M} = (E, \mathbf{I})$ is said to be linearly representable (or representable) over a field F if there exist a vector space V(n,F) of dimension n over F, and a function $\phi : E \to V(n,F)$ such that, for any subset $\{x_1, x_2, \ldots, x_m\}_{\neq} \subseteq E$,
$\{x_1, x_2, \ldots, x_m\} \in \mathbf{I}$ if and only if $\{\phi x_1, \phi x_2, \ldots, \phi x_m\}$ is linearly independent in V(n,F).

$\underline{M}$ is said to be representable over the characteristic q if there exists a field of characteristic q over which $\underline{M}$ is linearly representable. The characteristic set of $\underline{M}$ is the set $\{q:\ \underline{M}$ is representable over characteristic $q\}$.

A binary matroid is one representable over GF(2).

A ternary matroid is one representable over GF(3).

BASE-ORDERABLE MATROIDS:

A matroid $\underline{M} = (E, \mathbf{B})$ is said to be base-orderable if, for each pair of bases $B_1, B_2 \in \mathbf{B}$, there exists a function $\theta : B_1 \to B_2$, depending on B_1 and B_2, such that

(i) θ is 1-1;

(ii) for each $b \in B_1$, both $(B_1 - \{b\}) \cup \{\theta b\}$ and $(B_2 - \{\theta b\}) \cup \{b\}$ are bases of $\underline{M}$.

A matroid $\underline{M} = (E, \mathbb{B})$ is said to be fully (strongly) base-orderable if, for each pair of bases $B_1, B_2 \in \mathbb{B}$, there exists a function $\theta : B_1 \to B_2$, depending on B_1 and B_2, such that

(i) Θ is 1-1;

(ii) for each $X \subseteq B_1$, both $(B_1 - X) \cup \{\Theta x : x \in X\}$ and $(B_2 - \{\Theta x : x \in X\}) \cup X$ are bases of $\underline{M}$.

TRANSVERSAL MATROIDS:

Let E be a finite set, and let $\mathbb{A} = (A_i : i \in I)$ be a finite family of subsets of E. A partial transversal of $\mathbb{A}$ is a set $\{x_i : i \in J \subseteq I\}_{\neq}$ of elements of E such that $x_i \in A_i$ for each $i \in J$.

It can be shown that the set of partial transversals of $\mathbb{A}$ forms the set of independent sets of a matroid on E. Such a matroid is called the transversal matroid on E associated with the family $\mathbb{A}$.

An important theorem (see, for example, Mirsky [20] for a proof of this) states that if r is the rank of the transversal matroid $\underline{M}$ associated with the family $\mathbb{A}$, then there exists a subfamily $\mathbb{A}'$ of $\mathbb{A}$ such that (i) each partial transversal of $\mathbb{A}$ is a partial transversal of $\mathbb{A}'$;

(ii) there are exactly r members of the family $\mathbb{A}'$.

If $X \subseteq E$, we shall use $\mathbb{A}(X)$ to denote the subfamily $(A_i : x \in A_i$ for some $x \in X)$.

Two results we shall be using later (see Mirsky [20] for proofs) are:

If $\underline{M}$ is the transversal matroid on E associated with the family $\mathbb{A}$, and $X \subseteq E$ is independent in $\underline{M}$, then, for each subset $Y \subseteq X$,

$|\mathbb{A}(Y)| \geq |Y|$.

If $X \subseteq E$ is a circuit of $\underline{M}$, then $|\mathbb{A}(X)| < |X|$.

GAMMOIDS:

Let Γ be a directed graph without multiple directed edges on the vertex-set V, and let $B_0 \subseteq V$. A set $B \subseteq V$ is said to be linked into B_0 if there exist pairwise-vertex-disjoint directed paths from B to B_0 such that each element of B is linked by one such path to an element of B_0. It can be shown (e.g. Piff [21]) that the set of subsets of V which can be linked into B_0 forms the set of independent sets of a matroid on V. Such a matroid is called a strict gammoid. A gammoid is a restriction of a strict gammoid. The properties of gammoids that we shall be using are:

(i) a strict gammoid is the dual of a transversal matroid;

(ii) the class of gammoids is closed under the operation of dualising;

(iii) the class of gammoids is closed under the operation of taking minors;

(iv) the class of gammoids is the class of contractions of transversal matroids;

(v) gammoids are representable over every characteristic;

(vi) gammoids are base-orderable;

(vii) gammoids are fully base-orderable.

Proofs of these results can be found in Mason [17], Piff [21] and Ingleton and Piff [15].

WHIRLS:

This class of matroids was introduced by Tutte in [27]. For n ≥ 3, the whirl W_n is defined to be the matroid on the set $E_n = \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}_{\neq}$ with circuits $C_i = \{a_i, a_{i+1}, b_i\}$ (mod n) $(0 \leq i \leq n-1)$, the minimal members of

$\{\Delta_I C_i : I \subsetneq \{0,1,2,\ldots,n-1\}\}$ and the sets $\{b_0,b_1,\ldots,b_{n-1},a_i\}$ $(0 \le i \le n-1)$.

It is easy to check that W_n is a gammoid for $n \ge 3$, that W_n is ternary, and that W_n is not binary.

SUBMODULAR AND SUPERMODULAR FUNCTIONS:

Submodular functions play an important part in matroid theory, and especially so in the theory of generalised hypergraphic matroids.

Let E be a finite set. A function $\mu: 2^E \to \mathbb{Z}$ is said to be submodular (called semi-modular in Crapo-Rota [6]) if, for each pair of sets $A,B \subseteq E$, $\mu A + \mu B \ge \mu(A \cup B) + \mu(A \cap B)$.

A function $\mu: 2^E \to \mathbb{Z}$ is said to be supermodular if, for each pair of sets $A,B \subseteq E$, $\mu A + \mu B \le \mu(A \cup B) + \mu(A \cap B)$.

A function $\mu: 2^E \to \mathbb{Z}$ is said to be modular if, for each pair of sets $A,B \subseteq E$, $\mu A + \mu B = \mu(A \cup B) + \mu(A \cap B)$.

An example of a submodular function is the rank function of a matroid. An example of a supermodular function will be given in Chapter 4 - the ν-function. The cardinality function is an example of a modular function.

GRAPHS AND HYPERGRAPHS:

The terminology of graph theory is fairly standard, and we shall not reproduce it here. Full details can be found in Wilson [29] or Harary [11]. Note, however, that Harary does not allow loops or multiple edges (his graphs are simple graphs); when dealing with matroids, it is preferable to allow these, and so we shall not restrict ourselves to simple graphs. A graph with loops or multiple edges is called a pseudo-graph by Harary.

The terminology of hypergraph theory is far from standard - indeed, each author seems to have his own definitions, and may even change his definitions from paper to paper. We therefore present our basic definitions here. Other definitions, whose motivation will become clearer in later chapters, are postponed until then.

In essence, a hypergraph is a set of vertices, together with a collection of subsets of vertices, called edges. This definition is used by many authors, but will not be entirely suitable for our purposes. We shall need to allow "multiple edges" - i.e. different edges which have the same vertex-set - so we define a hypergraph by means of an incidence relation as follows:

DEFINITION 1.1: A hypergraph H is an ordered triple $(V, \mathcal{E}, \$)$ of sets, where

V is a finite, nonempty set of elements called vertices;

$\mathcal{E}$ is a finite set of elements called edges;

$V \cap \mathcal{E} = \phi$;

and $\$$ is a subset of $V \times \mathcal{E}$ called the incidence relation of the hypergraph.

A vertex $V \in V$ is said to be incident with $e \in \mathcal{E}$ if and only if $(V, e) \in \$$.

Two edges e_1 and e_2 are said to be adjacent if there exists $V \in V$ such that $(V, e_i) \in \$ \quad (i = 1,2)$.

Notation:

Where possible, we shall use the following conventions in connection with hypergraphs, as has been foreshadowed by earlier definitions:

Upper-Case Roman letters will denote vertices - e.g. A, B, A_1, B_n.
Upper-Case Italic letters will denote sets of vertices - e.g. V, W.
Lower-Case Roman letters will denote sets of edges, or elements of a matroid - e.g. e, a, b_1, x_m.

Upper-Case I.B.M. Orator type-face letters will denote sets of edges, or sets of elements of a matroid - e.g. A, X, E.

Those symbols to which special meaning are attached will not be used in the above connection.

Where several hypergraphs are being discussed together, the vertex-set and edge-set of H may be denoted by V(H) and E(H) respectively.

The <u>vertex-set in H</u> of the edge e, denoted by $V_H(e)$ is the set $\{V \in \mathcal{V}: (V,e) \in \$\}$.

If $\mathsf{A} \subseteq \mathsf{E}$, the <u>vertex-set</u> of A in H, denoted by $V_H(\mathsf{A})$ is the set $\cup\{V_H(e): e \in \mathsf{A}\}$.

It is often more convenient to describe H in terms of its vertices, edges and the vertex-sets of its edges. If this is done, \$ is understood to be defined as $\{(V,e): V \in V_H(e), e \in \mathsf{E}\}$.

Where there will be no confusion, the subscript H will be dropped.

The set $\mathsf{A} \subseteq \mathsf{E}$ is said to <u>span</u> $\mathcal{W} \subseteq \mathcal{V}$ if $\mathcal{W} \subseteq V(\mathsf{A})$.

If $|V(e)| = k$ for each $e \in \mathsf{E}$, H is said to be a <u>uniform hypergraph of cardinality k</u>, a <u>k-uniform hypergraph</u>, or simply a <u>k-hypergraph</u>. The value k is called "rank" by Berge in [1], but this term is unsatisfactory when we are also dealing with matroids.

H is said to be <u>simple</u> if $V(e_1) \neq V(e_2)$ for $e_1 \neq e_2$.

If H is a simple hypergraph with $|\mathcal{V}| = p$ and $\{V(e): e \in \mathsf{E}\}$ is equal to the set of all k-subsets of $\mathcal{V}$, then H is said to be the <u>complete k-hypergraph on $\mathcal{V}$</u>, denoted by K_p^k.

If $\mathsf{E}' \subseteq \mathsf{E}$ and $\$' = \{(V,e) \in \$: e \in \mathsf{E}'\}$, then $(\mathcal{V}, \mathsf{E}', \$')$ is called the <u>strict subhypergraph</u> of H induced by E'. With the same notation, $(V(\mathsf{E}'), \mathsf{E}', \$')$ is called the <u>subhypergraph</u> of H induced by E', and is denoted by $H_{\mathsf{E}'}$. Note that Berge calls our "strict subhypergraph" a "partial hypergraph", reserving the term "subhypergraph" for a hypergraph obtained in a particular way from a subset of V. We prefer to use the name inspired by graph theory.

If $V' \subseteq V$, we define $E_{V'}$ to be $\{e \in E: V(e) \subseteq V'\}$, and $\$_{V'}$ to be $\{(V,e) \in \$: e \in E_{V'}\}$. Then $(V', E_{V'}, \$_{V'})$ is called the restriction of H to V', and is denoted by $H|V'$.

A cycle of the hypergraph H is a sequence of edges and vertices of H $(V_0, e_0, V_1, e_1, \ldots, V_{n-1}, e_{n-1}, V_0)$ such that

$V_i \in V(e_i) \cap V(e_{i-1})$ (mod n)

$V_i \neq V_j$ for $i \neq j$

$e_i \neq e_j$ for $i \neq j$.

The edge-set of the cycle is the set $\{e_0, e_1, \ldots, e_{n-1}\}$.

Hypergraph isomorphism has been defined in several ways; we shall be concerned only with the following:

DEFINITION 1.2: The hypergraph $H_1 = (V_1, E_1, \$_1)$ and the hypergraph $H_2 = (V_2, E_2, \$_2)$ are said to be isomorphic if there exist bijections $\Phi: V_1 \to V_2$ and $\Theta: E_1 \to E_2$ such that $V_{H_2}(\Theta e) = \{\Phi V: V \in V_{H_1}(e)\}$ for each $e \in E_1$.

To assist in the presentation of matroids and hypergraphs, we shall often use pictorial representations. For matroids, we shall use Euclidean representation (in which 3 dependent points lie on a line, etc.). For hypergraphs, we shall adopt the method used by Crapo-Rota [6], in which edges are represented as shaded-in faces of a (not necessarily plane) graph.

For example, the hypergraph with edge-set {a,b,c,d}, where V(a) = {A,B,F}, V(b) = {B,C,D}, V(c) = {A,C,E} and V(d) = {D,E,F}, could be shown as the shaded octahedron in Figure 1.

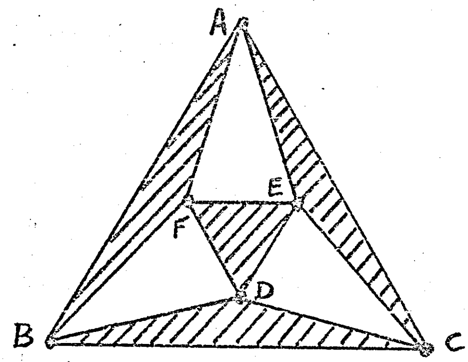


Figure 1.

Before embarking on our construction of hypergraphic matroids, we should mention that Berge [1&2] and Helgason [13] have each produced matroids derived from hypergraphs.

Berge's matroids have as ground-set the vertex-set V of the hypergraph H, and rank function ρ where $\rho W = \max\{|W \cap V(e)| : e \in E\}$ for $W \subseteq V$. In particular, this gives rank 2 to all matroids derived from graphs.

Helgason is concerned with colouring hypergraphs. This is a different approach from ours, and yields a different matroid. This follows immediately from the fact that the class of hypergraphs on which Helgason defines his matroids (those with "geometric colouring closure", which we shall define in Chapter 10) does not contain all uniform hypergraphs. But, even for a uniform hypergraph which does have geometric colouring closure, the matroids need not coincide. For example, consider the hypergraph shown in Figure 1. This has geometric colouring closure, and the matroid produced by Helgason has rank 3. But, anticipating Chapter 2, we can see that the hypergraphic matroid we define has rank 4.

Thus the three methods of obtaining matroids from hypergraphs do not coincide.

CHAPTER 2

CONSTRUCTION OF HYPERGRAPHIC MATROIDS

Throughout this chapter, we shall use H to denote the k-hypergraph $(V, \mathcal{E}, \$)$ $(k \geq 2)$ on the set $V = \{A_1, A_2, \ldots, A_p\}_{\neq}$. We shall not, in general, assume that H is simple.

We know that if H is a graph, then the cycles of H determine the circuits of a matroid on the set of edges of H, called the cycle matroid of H. However, for the purposes of generalising the construction to hypergraphs, it is more satisfactory to consider the forests of H, the set of which is the set of independent sets of the cycle matroid.

Now, a set $\mathcal{A}$ of edges of the graph H is a forest if and only if either $\mathcal{A} = \phi$, or $|V(\mathcal{G})| \geq |\mathcal{G}| + 1$ for each non-empty subset $\mathcal{G}$ of $\mathcal{A}$.

Since a graph is a uniform hypergraph of cardinality 2, we make the obvious generalisation.

DEFINITION 2.1: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph with $k \geq 2$. $\mathcal{A} \subseteq \mathcal{E}$ is called independent if and only if either

(i) $\mathcal{A} = \phi$

or (ii) $|V(\mathcal{G})| \geq |\mathcal{G}| + k - 1$ for each nonempty subset $\mathcal{G}$ of $\mathcal{A}$.

A set which is not independent is called dependent.

There is a very short proof that the independent sets defined in (2.1) do form the independent sets of a matroid on the set $\mathcal{E}$. This method is used by Crapo-Rota [6] and is a frequently-used technique.

Define a set-function $\psi : 2^{\mathcal{E}} \to \mathbb{Z}$ by $\psi(\mathcal{A}) = |V(\mathcal{A})| - (k - 1)$ for $\mathcal{A} \subseteq \mathcal{E}$. Then ψ is an increasing, integer-valued set-function, which takes the value 1 on elements of $\mathcal{E}$. Furthermore, ψ is sub-modular. A theorem proved by Crapo-Rota shows that ψ defines a matroid on E the independent sets of which are those $A \subseteq E$ for which either $A = \phi$ or $\psi(G) \geq |G|$ for each nonempty subset G of A. This is precisely the definition of independent sets given in (2.1).

For a simple hypergraph on V, this is equivalent to proving that the level-(k-1) Dilworth truncation of $U_{p,p}(V)$ is a matroid on the k-subsets of the p-set V, and then restricting this matroid to the subset $\{V(e): e \in E\}$ of this set of k-subsets. However, this method of proof does not bring out any of the similarities to graphic matroids. We shall therefore prove this result again, using an approach derived from graph theory.

DEFINITION 2.2: An independent set A is said to be <u>critical</u> if $|V(A)| = |A| + k - 1$.

LEMMA 2.3: If $C \subseteq E$ is a minimal dependent set, then:

(i) $|V(C)| = |C| + k - 2$;

(ii) $V(C) = V(C-\{e\})$ for any $e \in C$, and hence every vertex $v \in V(C)$ is an element of the vertex-sets of at least two edges of C;

(iii) $C-\{e\}$ is critical for any $e \in C$.

Proof: Let e be any element of C. Then, by minimality of C, $C-\{e\}$ is an independent set, so $|V(C-\{e\})| \geq |C-\{e\}| + k - 1$

$$= |C| + k - 2.$$

Also, since C itself is dependent, there exists a nonempty subset $A \subseteq C$ for which $|V(A)| < |A| + k - 1$. Since this cannot hold for any proper nonempty subset of C, $|V(C)| < |C| + k - 1$.

Combining these two inequalities,

$$|C| + k - 1 > |V(C)| \geq |V(C-\{e\})| \geq |C| + k - 2 \qquad (1).$$

Thus, equality holds throughout, and so $|V(C)| = |C| + k - 2$, $V(C-\{e\}) = V(C)$, and, since this holds for any $e \in C$, there exists no $V \in V(\mathcal{C})$ such that $V \in V(e)$ for only one $e \in \mathcal{C}$.

Also, from (1), $|V(\mathcal{C}-\{e\})| = |\mathcal{C}-\{e\}| + k - 1$, whence, since $\mathcal{C}-\{e\}$ is independent, $\mathcal{C}-\{e\}$ is critical.

LEMMA 2.4: Let $\mathcal{X},\mathcal{Y} \subseteq \mathcal{E}$ be critical sets with $|V(\mathcal{X}) \cap V(\mathcal{Y})| \geq k - 1$.

Then $|V(\mathcal{X} \cup \mathcal{Y})| \leq |\mathcal{X} \cup \mathcal{Y}| + k - 1$.

If, in addition, $\mathcal{X} \cup \mathcal{Y}$ is independent, then $\mathcal{X} \cup \mathcal{Y}$ is critical.

Proof: We have, since $\mathcal{X}$ and $\mathcal{Y}$ are critical, $|V(\mathcal{X})| = |\mathcal{X}| + k - 1$ (1)

$$|V(\mathcal{Y})| = |\mathcal{Y}| + k - 1 \quad (2)$$

CASE I: $\mathcal{X} \cap \mathcal{Y} \neq \phi$.

Then $V(\mathcal{X}) \cap V(\mathcal{Y}) \supseteq V(\mathcal{X} \cap \mathcal{Y})$. Now, $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{X}$ and so, by (2.1), $\mathcal{X} \cap \mathcal{Y}$ is independent. Therefore, $|V(\mathcal{X} \cap \mathcal{Y})| \geq |\mathcal{X} \cap \mathcal{Y}| + k - 1$.

Thus $|V(\mathcal{X}) \cap V(\mathcal{Y})| \geq |\mathcal{X} \cap \mathcal{Y}| + k - 1$ (3)

CASE II: $\mathcal{X} \cap \mathcal{Y} = \phi$.

Then (3) follows from the hypotheses of the lemma.

Now, from (1) and (2), $|V(\mathcal{X})| + |V(\mathcal{Y})| = 2k + |\mathcal{X}| + |\mathcal{Y}| - 2$

$\therefore$ $|V(\mathcal{X}) \cup V(\mathcal{Y})| + |V(\mathcal{X}) \cap V(\mathcal{Y})| = 2k + |\mathcal{X} \cup \mathcal{Y}| + |\mathcal{X} \cap \mathcal{Y}| - 2$.

Thus, from (3), $|V(\mathcal{X}) \cup V(\mathcal{Y})| \leq |\mathcal{X} \cup \mathcal{Y}| + k - 1$.

But, $V(\mathcal{X} \cup \mathcal{Y}) = V(\mathcal{X}) \cup V(\mathcal{Y})$, so $|V(\mathcal{X} \cup \mathcal{Y})| \leq |\mathcal{X} \cup \mathcal{Y}| + k - 1$.

If, in addition, $\mathcal{X} \cup \mathcal{Y}$ is independent, the reverse inequality holds, and so $\mathcal{X} \cup \mathcal{Y}$ is critical.

DEFINITION 2.5: If W is a set of vertices of H such that $W = V(\mathcal{A})$ where $\mathcal{A} \subseteq \mathcal{E}$ and $\mathcal{A}$ is critical, then $(W, \mathcal{E}_W, \$_W)$ is called a <u>fragment</u> of H.

LEMMA 2.6: If $(U, \mathcal{E}_U, \$_U)$ and $(W, \mathcal{E}_W, \$_W)$ are fragments of H with $|U \cap W| \geq k - 1$, then $(U \cup W, \mathcal{E}_{U \cup W}, \$_{U \cup W})$ is a fragment of H.

Proof: Let $U = V(\mathcal{X})$ where $\mathcal{X} \subseteq \mathcal{E}$ and $\mathcal{X}$ is critical, and let $W = V(\mathcal{Y})$ where $\mathcal{Y} \subseteq \mathcal{E}$ and $\mathcal{Y}$ is critical.

Then $|V(X)\cap V(Y)| \geq k - 1$, and so, by (2.4) $|V(X\cup Y)| \leq |X\cup Y| + k - 1$. If $X\cup Y$ is independent, then, by (2.4) it is critical, and we are done, since then $U\cup W = V(X\cup Y)$ where $X\cup Y$ is critical. If $X\cup Y$ is dependent, then there exists a minimal dependent subset of $X\cup Y$, T, say. Then, by (2.3), $|V(T)| = |T| + k - 2$. Pick $e_1 \in T$. Then $V(T-\{e_1\}) = V(T)$. Let $X_1 = X-\{e_1\}$, $Y_1 = Y-\{e_1\}$. Then $V(X_1\cup Y_1) = V(X\cup Y)$.

We claim that $|V(X_1\cup Y_1)| \leq |X_1\cup Y_1| + k - 1$.

For, suppose not. Then $|V(X_1\cup Y_1)| > |X_1\cup Y_1| + k - 1$, and so, from the above, $|X\cup Y| + k - 1 \geq |V(X\cup Y)| = |V(X_1\cup Y_1)| > |X_1\cup Y_1| + k - 1$

$$= |X\cup Y| + k - 2.$$

$$\therefore \quad |V(X\cup Y)| = |X\cup Y| + k - 1. \qquad (1)$$

We shall show that this implies that $X\cup Y$ is independent, contradicting our hypothesis that it contains a dependent subset.

Since $|V(X)| = |X| + k - 1$ and $|V(Y)| = |Y| + k - 1$,

$$|V(X)| + |V(Y)| = |X| + |Y| + 2k - 2$$

$$\therefore \quad |V(X\cup Y)| + |V(X)\cap V(Y)| = |X\cup Y| + |X\cap Y| + 2k - 2$$

$$\therefore \text{ from (1)}, \quad |V(X)\cap V(Y)| = |X\cap Y| + k - 1 \qquad (2)$$

CASE I: $X\cap Y = \phi$. Then $|V(X)\cap V(Y)| = k - 1$.

Let $X' \subseteq X$, $Y' \subseteq Y$. Then, by (2.1), both X' and Y' are independent. Assume $X' \neq \phi$, $Y' \neq \phi$.

$$\text{Then} \quad |V(X')| + |V(Y')| \geq |X'| + |Y'| + 2k - 2$$

$$\therefore \quad |V(X'\cup Y')| + |V(X')\cap V(Y')| \geq |X'\cup Y'| + |X'\cap Y'| + 2k - 2.$$

Now, $X'\cap Y' \subseteq X\cap Y = \phi$, and $V(X')\cap V(Y') \subseteq V(X)\cap V(Y)$.

$$\therefore \quad |V(X'\cup Y')| + k - 1 \geq |X'\cup Y'| + 2k - 2$$

$$\therefore \quad |V(X'\cup Y')| \geq |X'\cup Y'| + k - 1 \qquad (3)$$

If either X' or Y' is empty (but not both), the inequality (3) follows from the independence of the nonempty set. Since (3) is true for every nonempty subset of $X\cup Y$, by (2.1) $X\cup Y$ is independent.

CASE II: $X \cap Y \neq \phi$. Then $X \cap Y$ is independent, since it is a subset of the independent set X. Thus, $|V(X \cap Y)| \geq |X \cap Y| + k - 1$.
Now, $V(X) \cap V(Y) \supseteq V(X \cap Y)$, and, from (2), $|V(X) \cap V(Y)| = |X \cap Y| + k - 1$.
Thus $|V(X \cap Y)| = |X \cap Y| + k - 1$.

Suppose $X \cup Y$ is dependent. Then there exists a minimal dependent subset $X' \cup Y'$ of $X \cup Y$, where $X' \subseteq X$ and $Y' \subseteq Y$. Since X and Y are independent, $X' \neq \phi$ and $Y' \neq \phi$. Now, by (2.3),

$$|V(X' \cup Y')| = |X' \cup Y'| + k - 2.$$

$$\therefore\ |V(X)| + |V(Y)| + |V(X' \cup Y')| = |X| + |Y| + |X' \cup Y'| + 3k - 4$$

$$\therefore\ |V(Y)| + |V(X) \cup V(X' \cup Y')| + |V(X) \cap V(X' \cup Y')|$$
$$= |Y| + |X \cup X' \cup Y'| + |X \cap (X' \cup Y')| + 3k - 4.$$

Now, $X' \subseteq X$, so $V(X') \subseteq V(X)$, and $V(X) \cup V(X' \cup Y') = V(X \cup Y')$.

$$\therefore\ |V(Y)| + |V(X \cup Y')| + |V(X) \cap V(X' \cup Y')|$$
$$= |Y| + |X \cup Y'| + |X \cap (X' \cup Y')| + 3k - 4$$

$$\therefore\ |V(Y) \cup V(X \cup Y')| + |V(Y) \cap V(X \cup Y')| + |V(X) \cap V(X' \cup Y')|$$
$$= |Y \cup X \cup Y'| + |Y \cap (X \cup Y')| + |X \cap (X' \cup Y')| + 3k - 4$$

$$\therefore\ |V(X \cup Y)| + |V(Y) \cap V(X \cup Y')| + |V(X) \cap V(X' \cup Y')|$$
$$= |X \cup Y| + |Y \cap (X \cup Y')| + |X \cap (X' \cup Y')| + 3k - 4.$$

Therefore, from (1),

$$|V(Y) \cap V(X \cup Y')| + |V(X) \cap V(X' \cup Y')|$$
$$= |Y \cap (X \cup Y')| + |X \cap (X' \cup Y')| + 2k - 3 \qquad (4)$$

Now, $|V(Y) \cap V(X \cup Y')| \geq |V(Y \cap (X \cup Y'))|$

$\geq |Y \cap (X \cup Y')| + k - 1$ since $Y \supseteq (X \cup Y') \cap Y \supseteq Y' \neq \phi$.

Similarly, $|V(X) \cap V(X' \cup Y')| \geq |X \cap (X' \cup Y')| + k - 1$.

$$\therefore\ |V(Y) \cap V(X \cup Y')| + |V(X) \cap V(X' \cup Y')|$$
$$\geq |Y \cap (X \cup Y')| + |X \cap (X' \cup Y')| + 2k - 2,$$

which is a contradiction of (4).

Thus, there exist no such sets X' and Y', and so $X \cup Y$ is independent.

But, by hypothesis, $X \cup Y$ is dependent. Thus, $|V(X_1 \cup Y_1)| \leq |X_1 \cup Y_1| + k - 1$. Hence, $|V(X_1 \cup Y_1)| \leq |X_1 \cup Y_1| + k - 1 = |X \cup Y| + k - 2$.
Now, either $X_1 \cup Y_1$ is independent, or it contains a minimal dependent subset T_1, say. In the latter case, let $e_2 \in T_1$, and write $X_2 = X_1 - \{e_2\}$, $Y_2 = Y_1 - \{e_2\}$. We now repeat the above procedure, with X_1, Y_1 in place of X, Y and X_2, Y_2 in place of X_1, Y_1, to show that either $X_2 \cup Y_2$ is independent, or it contains a minimal dependent subset T_2, say. We can thus continue to repeat the procedure, deleting an element from a minimal dependent subset, as long as $X_i \cup Y_i$ is dependent.

Now, at stage i we have $|V(X_i \cup Y_i)| \leq |X \cup Y| + k - (i+1)$ (5)
Since $|V(X_i \cup Y_i)| = |V(X \cup Y)|$, the process must eventually stop, since all sets are finite. Thus, there exists r for which $X_r \cup Y_r$ is independent and so $|V(X_r \cup Y_r)| \geq |X_r \cup Y_r| + k - 1$.
From (5), we have $|V(X_r \cup Y_r)| \leq |X \cup Y| + k - (r+1) = |X_r \cup Y_r| + k - 1$.
Thus, $X_r \cup Y_r$ is critical, and $U \cup W = V(X_r \cup Y_r)$.

Thus $(U \cup W, E_{U \cup W}, \$_{U \cup W})$ is a fragment of H.

LEMMA 2.7: Let A be a subset of the edges of H, and let $H' = H_A$ be the subhypergraph of H induced by A. Then there exists a unique partition of A into $G_1, G_2, \ldots, G_n$ such that H_{G_i} is a fragment of H' for each i, with the property that, if $(W, E_W, \$_W)$ is a fragment of H', then $E_W \subseteq G_i$ for exactly one value of i.

Proof:

(i) Existence of a partition.

Let the fragments of H' be $(U_i, E_{U_i}, \$_{U_i})$ $(i \in I)$.
We may partially order the fragments of H' by inclusion of edge-sets - i.e. $(U_i, E_{U_i}, \$_{U_i}) \preccurlyeq (U_j, E_{U_j}, \$_{U_j})$ if $E_{U_i} \subseteq E_{U_j}$. Let us call this order "containment" of fragments. Let the maximal elements in this partial

order be $H_{G_i} = (V(G_i), G_i, \$_i)$ $(1 \leq i \leq n)$.

By (2.6), if $|V(G_i) \cap V(G_j)| \geq k - 1$ for $i \neq j$, then $(W, E_W, \$_W)$ would be a fragment of H', where $W = V(G_i) \cup V(G_j)$. Thus, since the fragments H_{G_i} are maximal, $|V(G_i) \cup V(G_j)| < k - 1$ for $i \neq j$. Hence, in particular, $G_i \cap G_j = \phi$ for $i \neq j$, and so the G_i partition A. Also, by maximality, every fragment of H' is contained in at least one of the H_{G_i}; furthermore, since $|V(G_i) \cap V(G_j)| < k - 1$ for $i \neq j$, every fragment of H' is contained in at most one of the H_{G_i}. Thus, every fragment of H' is contained in exactly one of the H_{G_i}.

(ii) Uniqueness of the partition.

If $(W, E_W, \$_W)$ is a member of a set $\mathbf{D}$ of fragments of H', then $(W, E_W, \$_W)$ is contained in some maximal fragment H_{G_r}, say. If $\mathbf{D}$ is a partition satisfying the conclusions of (2.7), apart from the uniqueness condition, then H_{G_r} is contained in a fragment $(U, E_U, \$_U)$ which is a member of $\mathbf{D}$. Thus, $(W, E_W, \$_W)$ is contained in $(U, E_U, \$_U)$, and so, by the requirement that every fragment of H' is contained in exactly one member of $\mathbf{D}$, we deduce that $(W, E_W, \$_W) = (U, E_U, \$_U) = H_{G_r}$. Thus, every member of $\mathbf{D}$ is a member of the partition described in (i). Since every member of the partition described in (i) must be contained in some member of $\mathbf{D}$, and is maximal with respect to containment, $\mathbf{D}$ is the set $\{H_{G_i} : 1 \leq i \leq n\}$.

Thus, the partition described in (i) is unique.

DEFINITION 2.8: The fragments $H_{G_i} = (V(G_i), G_i, \$_i)$ constructed in (i) of (2.7) are called the <u>components</u> of H'.

THEOREM 2.9: If X, Y are independent sets of edges of the hypergraph $H = (V, E, \$)$, and if $|X| = |Y| + 1$, then there exists an edge $b \in X - Y$ such that $Y \cup \{b\}$ is independent.

Proof: Suppose first that Y is critical. Then, since $|V(X)| \geq |X| + k - 1 = |Y| + k = |V(Y)| + 1$, there exists $v \in V(X)-V(Y)$. Pick any edge $b \in X$ with $v \in V(b)$. Then $Y \cup \{b\}$ is independent, since

$$|V(G) \cup V(b)| \geq |V(G)| + 1 \geq |G| + k = (|G| + 1) + k - 1$$
$$= |G \cup \{b\}| + k - 1 \text{ for any nonempty}$$

subset G of Y.

Suppose now that Y is not critical. If there exists $v \in V(X)-V(Y)$, then we pick b as in the previous case. Otherwise, consider the subhypergraph H_Y of H induced by Y, and let $G_1, G_2, \ldots, G_n$ be the partition of Y described (for the set A) in (2.7), such that $\{H_{G_i} : 1 \leq i \leq n\}$ is the set of components of H_Y. Let $|G_i| = r_i$ $(1 \leq i \leq n)$. Then $V(G_i) \supseteq V(x)$ for at most r_i edges $x \in X$, since X is independent. Thus, there are at most $\sum_{i=1}^{n}(r_i) = |Y|$ edges x of X satisfying $V(x) \subseteq V(G_i)$ for some i, $1 \leq i \leq n$. Since $|X| = |Y| + 1$, there exists at least one edge $b \in X$ with $V(b) \nsubseteq V(G_i)$ for any i. (1)
Now, if $Y \cup \{b\}$ were dependent, it would contain a minimal dependent subset C. Since Y is independent, $b \in C$. Write $Y' = C - \{b\}$. Then, by (2.3)(iii), Y' is a critical set. Thus, $H_{Y'}$ is a fragment of H_Y. By (2.7), it is therefore contained in a component H_{G_r} for some r. In particular, $V(Y') \subseteq V(G_r)$. But, by (2.3)(ii), $V(b) \subseteq V(Y' \cup \{b\}) = V(Y') \subseteq V(G_r)$, which contradicts (1). Thus, $Y \cup \{b\}$ is independent.

COROLLARY 2.10: The independent sets as defined in (2.1) are the independent sets of a matroid on the edge-set E of H.

Proof: (I1) follows from (2.1)(i);

(I2) Let X be an independent set, and let $Y \subseteq X$. Then, either $Y = \phi$, or, since $|V(G)| \geq |G| + k - 1$ for each nonempty subset G of X,

$|V(G)| \geq |G| + k - 1$ for each nonempty subset G of Y.

Thus, Y is independent.

(I3) follows from (2.9).

DEFINITION 2.11: Any matroid isomorphic to one obtained by the definition (2.1) is called a hypergraphic matroid. The hypergraphic matroid obtained from the hypergraph H is denoted by $\underline{M}(H)$.

PROPOSITION 2.12: If $\underline{M}$ is a hypergraphic matroid on the set S and $S' \subseteq S$, then $\underline{M} \times S'$ is hypergraphic.

Proof: Let $\underline{M} \cong \underline{M}(H)$, where $H = (V, E, \$)$ is a k-hypergraph for some k, and let $\phi: S \to E$ be the bijection which induces the isomorphism. Let $E' = \{e \in E: e = \phi(s) \text{ for } s \in S'\}$. Let $H' = (V, E', \$')$, where $\$' = \{(V,e) \in \$: e \in E'\}$. Then a set $A' \subseteq E'$ is independent in $\underline{M}(H')$ if and only if A' is independent in $\underline{M}(H)$. Thus, $\underline{M}(H') \cong \underline{M}(H) \times E'$. So, since $\underline{M} \times S' \cong \underline{M}(H) \times E'$, $\underline{M} \times S' \cong \underline{M}(H')$, and so $\underline{M} \times S'$ is hypergraphic.

We have already referred to the complete hypergraph K_p^k. The matroid derived from this, $\underline{M}(K_p^k)$, is what Crapo-Rota [6] call the completed k-truncation of the Boolean lattice B_p (which is the lattice of flats of the matroid $U_{p,p}$). This is because the lattice of $\underline{M}(K_p^k)$ contains all the points of B_p from the k-sets upwards, together with the necessary extra points to make the lattice geometric (the completion of the lattice). The lattice point of view, however, brings out none of the similarities to graph theory, and we believe that it is more natural to consider such matroids as arising from hypergraphs in the way we have described.

In the same way that it is sufficient to consider 1-connected graphs in the study of graphic matroids, it is sufficient to consider hypergraphic matroids derived from hypergraphs which are themselves components. This result is the content of the next theorem.

THEOREM 2.13: If $\underline{M}$ is a hypergraphic matroid, there exists a uniform hypergraph $H'' = (V'', E'', \$'')$ such that:

(i) $\underline{M} \cong \underline{M}(H'')$;

(ii) $(V'', E'', \$'')$ is a component of H''.

Proof: Since $\underline{M}$ is hypergraphic, there exists a k-hypergraph $H = (V, E, \$)$ such that $\underline{M} \cong \underline{M}(H)$. As in (2.7), partition E into $G_1, G_2, \ldots, G_n$ such that H_{G_i} is a component for each i $(1 \leq i \leq n)$. Form a new hypergraph H', where H' has components $H'_{G'_i} = (V_{H'}(G'_i), G'_i, \$'_i)$, such that $H' | V_{H'}(G'_i) \cong H | V_H(G_i)$ for each i, and $V_{H'}(G'_i) \cap V_{H'}(G'_j) = \phi$ for $i \neq j$. Extend the isomorphisms to a bijection between the edges of H' and H. Then clearly independence in $\underline{M}(H)$ implies independence in $\underline{M}(H')$. Also, a circuit of $\underline{M}(H)$ is mapped onto a circuit of $\underline{M}(H')$ since, by (2.3), every circuit C of $\underline{M}(H)$ satisfies $C \subseteq G_i$ for some i. Thus, $\underline{M}(H) \cong \underline{M}(H')$.

Now, pick a set W of k-1 vertices in $V_{H'}(G'_1)$, and form a new hypergraph H'' as follows:

Let $V''_1, V''_2, \ldots, V''_n$ be sets of vertices with $|V''_i| = |V_{H'}(G'_i)|$ $(1 \leq i \leq n)$ such that $V''_i \cap V''_j = W$ for $i \neq j$. Let $V'' = V''_1 \cup V''_2 \cup \cdots \cup V''_n$.

For each i $(1 \leq i \leq n)$ define $H''_i = (V''_i, G''_i, \$''_i) \cong H' | V(G'_i)$. (1)

Put $E'' = G''_1 \cup G''_2 \cup \ldots \cup G''_n$ and let $\$'' = \{(V'', e''): (V', e') \in \$'\}$ where V' is the image of V'' under the isomorphism (1), e'' is the image of e' under the isomorphism (1), and $\$' = \$'_1 \cup \$'_2 \cup \ldots \cup \$'_n$.

Define $H'' = (V'', E'', \$'')$, and define a bijection between E'' and E' by the extension of the isomorphisms (1).

Now, since $|V(G''_i) \cap V(G''_j)| = k - 1$ for $i \neq j$, $H''_{E''}$ is a component of H''. Clearly independence in $\underline{M}(H'')$ implies independence in $\underline{M}(H')$. Suppose that X' is an independent set in $\underline{M}(H')$. If $X' = \phi$, then there is nothing to prove. Assume, therefore, that $X' \neq \phi$ and let A' be a nonempty subset of X'. Denote by A'', X'' the images of A', X' respectively under the bijection between E'' and E'.

Let $B_j' = A' \cap G_j'$ $(1 \le j \le n)$, and let $\{A_i' : 1 \le i \le m\}$ be the set of B_j' which are nonempty. Then, since $V_{H'}(G_i') \cap V_{H'}(G_j') = \phi$ for $i \ne j$,

$$|V_{H'}(A')| = |V_{H'}(A_1')| + |V_{H'}(A_2')| + \ldots + |V_{H'}(A_m')|.$$

Thus, (using V( ) to denote vertex-sets in H") we have

$$\begin{aligned}
|V(A'')| &= |V(A_1'' \cup A_2'' \cup \ldots \cup A_m'')| = |V(A_1'' \cup A_2'' \cup \ldots \cup A_m'') - W| + |W \cap V(A_1'' \cup \ldots \cup A_m'')| \\
&= |V(A_1'') - W| + |V(A_2'') - W| + \ldots + |V(A_m'') - W| + |W \cap (\cup A_i'')| \\
&= |V(A_1'')| + |V(A_2'')| + \ldots + |V(A_m'')| - |V(A_1'') \cap W| - |V(A_2'') \cap W| - \ldots \\
&\quad - |V(A_m'') \cap W| + |W \cap V(\cup A_i'')| \\
&= |V_{H'}(A_1')| + \ldots + |V_{H'}(A_m')| - (|V(A_1'') \cap W| + \ldots + |V(A_m'') \cap W|) \\
&\quad + |W \cap V(\cup A_i'')| \\
&\ge |A_1'| + k - 1 + \ldots + |A_m'| + k - 1 \qquad \text{(since } A_i' \text{ is independent} \\
&\quad - (|V(A_1'') \cap W| + \ldots + |V(A_m'') \cap W|) \qquad \text{and nonempty for each } i) \\
&\quad + |W \cap V(\cup A_i'')| \\
&\ge |A_1'| + \ldots + |A_m'| + m(k-1) - m(|V(\cup A_i'') \cap W|) + |V(\cup A_i'') \cap W| \\
&\ge |A_1''| + \ldots + |A_m''| + k - 1.
\end{aligned}$$

Since this is true for each nonempty subset A'' of X'', X'' is independent in $\underline{M}(H'')$. Thus, $\underline{M}(H'') \cong \underline{M}(H') \cong \underline{M}(H) \cong \underline{M}$ as required.

It is therefore sufficient, when dealing with hypergraphic matroids, to restrict our attention to hypergraphs consisting of only one component.

DEFINITION 2.14: A hypergraph $H = (V, E, \$)$ is said to be <u>critical</u> if $(V, E, \$)$ is a component of H.

The reason for using the term "critical" for the hypergraph H itself, is that every base of $\underline{M}(H)$ is a critical subset of E, and that V(H) is spanned by E. From the foregoing results, it is clear that a critical set in a hypergraph is the analogue of a tree in a graph. The motivation leading to (2.1), and the proofs following, show that an independent set in a hypergraph is the analogue of a forest in a graph.

As an example of the construction of (2.13), consider the hypergraph H with vertex-set $V = \{A,B,C,D,E,F,G\}_{\neq}$ and edge-set $E = \{a,b,c,d,e\}_{\neq}$, where $V(a) = \{A,B,C\}$, $V(b) = \{A,B,D\}$, $V(c) = \{A,C,D\}$, $V(d) = \{C,E,F\}$, $V(e) = \{D,F,G\}$. This hypergraph is shown in Figure 2.

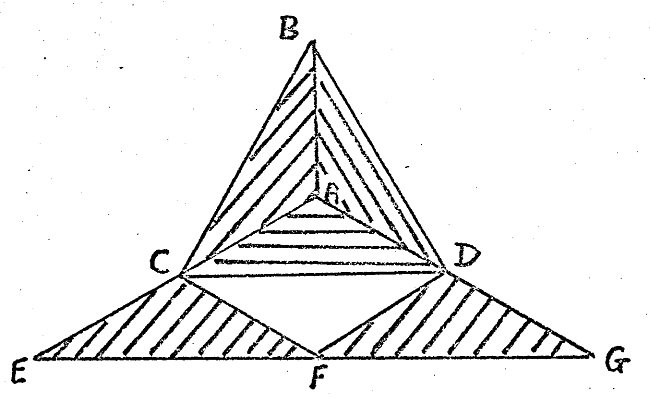


Figure 2.

This has three components, whose edge-sets are {a,b,c}, {d} and {e} respectively. The rank-4 matroid $\underline{M}(H)$ is shown in Euclidean representation in Figure 3.

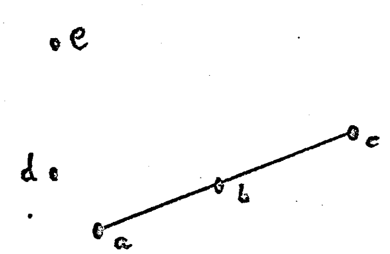


Figure 3

The construction of (2.13) first gives a hypergraph H' with vertex-set $V' = \{A',B',C',D',C',E',F',D^*,F^*,G^*\}_{\neq}$ and edge-set $E' = \{a',b',c',d',e'\}_{\neq}$, where $V_{H'}(a') = \{A',B',C'\}$, $V_{H'}(b') = \{A',B',D'\}$ $V_{H'}(c') = \{A',C',D'\}$ $V_{H'}(d') = \{C',E',F'\}$ and $V_{H'}(e') = \{D^*,F^*,G^*\}$.

Then, taking $W = \{C',D'\}$ say, we obtain the critical hypergraph H" with vertex-set $\{A'',B'',C',D',E'',F''\}_{\neq}$ and edge-set $E'' = \{a'',b'',c'',d'',e''\}_{\neq}$ where $V(a'') = \{A'',B'',C'\}$, $V(b') = \{A'',B'',D'\}$, $V(c'') = \{A'',C',D'\}$

$V(d'') = \{C',D',E''\}$ and $V(e'') = \{C',D',F''\}$. The hypergraph H" is shown in Figure 4. It is easy to see that, under the obvious bijection between the edge-sets E and E'', $\underline{M}(H) \cong \underline{M}(H'')$.

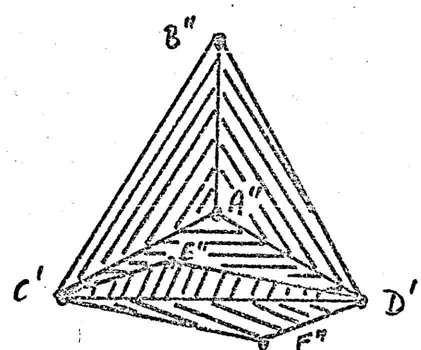


Figure 4

It follows from (2.6) that no subset of V with cardinality k-1 can be contained in the vertex-set of more than one component of a hypergraph. In a sense, therefore, these (k-1)-subsets are partitioned by the components of the hypergraph, in a similar way to the partitioning of the vertex-set of a graph by the vertex-sets of the components of the graph. We shall be using these (k-1)-subsets often in subsequent chapters, and we give those contained in the vertex-set of some edge of the hypergraph the name of "nodes". More formally:

DEFINITION 2.15: Let $H = (V, E, \$)$ be a k-hypergraph with $k \geq 2$. A subset $N \subseteq V$ with $|N| = k-1$, and such that $N \subseteq V(e)$ for some $e \in E$ is called a <u>node</u> of H. If $A \subseteq E$, the set of nodes $\{N: N \subseteq V(e)$ for some $e \in A\}$ is denoted by $n(A)$. We write $n(e)$ for $n(\{e\})$. The set of all nodes of H is denoted by $n(H)$.

With any k-hypergraph H, therefore, we can associate another hypergraph of the same cardinality, whose vertex-set is the set $n(H)$, whose edge-set is the set $E(H)$, and an incidence relation "N is incident with e if and only if $N \subseteq V_H(e)$". We embody this concept in the following definition:

DEFINITION 2.16: Let $H = (V, E, \$)$ be a k-hypergraph, with $k \geq 2$ and $E \neq \phi$. The <u>node-hypergraph</u> of H, denoted by N(H), is the hypergraph $(V_N, E_N, \$_N)$, where $V_N = n(H)$, $E_N = E$ and $\$_N = \{(N, e) : N \subseteq V_H(e)\}$.

EXAMPLE: Let H be the hypergraph K_4^3 on the set $V = \{A, B, C, D\}_{\neq}$. Then n(H) is the set of all 2-subsets of V contained in the vertex-set of some edge of H - i.e., in this case, all 2-subsets of V. N(H) is shown in Figure 5. Some properties of node-hypergraphs will be investigated in Chapter 9.

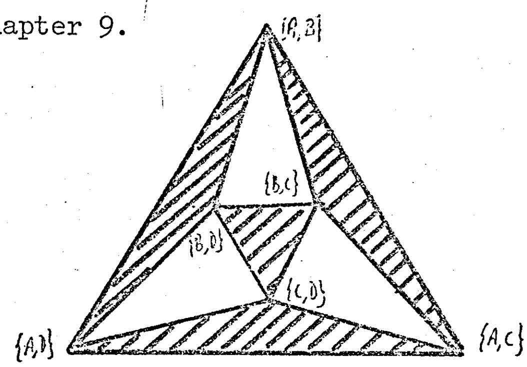


Figure 5

<u>NON-UNIFORM HYPERGRAPHS</u>:

So far in this chapter we have dealt exclusively with uniform hypergraphs. We now consider briefly the non-uniform case, and show that, under one possible definition for the associated matroid, no non-hypergraphic matroids result; under another possible definition matroids are produced which fail to satisfy the desirable property mentioned in Chapter 1; a third possible definition, which permits loops, is one which we shall be using in Chapter 4 - however, every matroid produced by this definition is either hypergraphic, or is such that the matroid formed from it by the deletion of loops is hypergraphic. Essentially, therefore, no greater generality results.

It would be possible to apply (2.1) to non-uniform hypergraphs by dropping the requirement that H be uniform, and taking k to be some integer less than the maximum cardinality of V(e) where $e \in E$. However, if e_1 and e_2 were such that $V(e_1) = V(e_2)$ and $|V(e_1)| = |V(e_2)| > k$, we would have $\{e_1, e_2\}$ independent in the resulting matroid which, as they have identical vertex-sets is rather unnatural. Indeed, that such a set should be dependent was the desirable property that we mentioned in Chapter 1.

An alternative, simpler, approach is to regard all edges with vertex-sets having cardinality less than the maximum of $|V(e)|$ for $e \in E$ as being loops; this is equivalent to dropping the requirement that H be uniform in (2.1) and taking k to be the maximum cardinality of V(e) for $e \in E$. This is the extension we shall make in Chapter 4, in order to allow loops.

The compromise between these two extremes is to vary the value of k, depending on the set of edges under consideration.

DEFINITION 2.17: Let $H'' = (V'', E'', \$'')$ be a hypergraph. A set $A'' \subseteq E''$ is said to be independent if and only if either

(i) $A'' = \phi$;

or (ii) $|V(G'')| \geq |G''| - 1 + \min\{|V(e'')| : e'' \in G''\}$ for each nonempty subset G'' of A''.

It can be proved directly that the set of independent sets defined by (2.17) is the set of independent sets of a matroid on E''. However, we shall prove this in another way, which shows that no matroids result which are not hypergraphic by our earlier definition.

THEOREM 2.18: Let $H'' = (V'', E'', \$'')$ be a hypergraph. Then there exists a uniform hypergraph $H' = (V', E'', \$')$ such that $A'' \subseteq E''$ is independent in $\underline{M}(H')$ if and only if A'' is independent according to (2.17).

Proof: Put $k = \max \{|V_{H''}(e'')|: e'' \in E''\}$. Let $\{A_1, A_2, \ldots, A_k, B_1, B_2\}_{\neq}$ be a set of vertices disjoint from V', and let $V' = V'' \cup \{A_1, A_2, \ldots, A_k, B_1, B_2\}$ Define $H' = (V', E'', \$')$ to be the (k+2)-hypergraph where

$V_{H'}(e'') = V_{H''}(e'') \cup \{A_i: 1 \leq i \leq k - |V_{H''}(e'')|\} \cup \{B_1, B_2\}$

Then $|V_{H''}(G'')| \geq |G''| - 1 + \min\{|V_{H''}(e'')|: e'' \in G''\}$ if and only if

$$|V_{H'}(G'')| \geq |G''| + (k+2) - 1$$

Thus, (2.17) defines the set of independent sets of a matroid $\underline{M}$ on E'', and $\underline{M} \cong \underline{M}(H')$. We therefore obtain no new matroids satisfying the desirable property that two edges with the same vertex-set should be parallel in any matroid obtained from the hypergraph.

Note that (2.17) has been included to show that it is possible to extend the definition of hypergraphic matroids to non-uniform hypergraphs without requiring all edges with less than maximum cardinality to be loops. We shall not be using definition (2.17) in future chapters, because it does not allow loops in the resulting matroid. Our definition of independence will therefore continue to be (2.1), until we modify it in Chapter 4.

We end this chapter by noting the connection between our definitions and those of Berge [1]. Berge defines a hypergraph to be connected if, for any two edges e_1, e_t of the hypergraph, there is a sequence of edges of the hypergraph $(e_1, e_2, \ldots, e_t)$ such that e_i and e_{i+1} are adjacent for each i $(1 \leq i \leq t-1)$.

A critical hypergraph is connected in the sense of Berge, but there are hypergraphs connected in the sense of Berge which are not critical. For example. consider the hypergraph H on the set of 6 vertices $\{A,B,C,D,E,F\}_{\neq}$ with two edges a and b where $V(a) = \{A,B,C,D\}$ and $V(b) = \{C,D,E,F\}$. This hypergraph is shown in Figure 6. It is clearly connected in the sense of Berge, but it has 6 vertices, 2 edges and cardinality 4. So $|V(E)| = 6 > 2 + 4 - 1 = |E| + k - 1$,

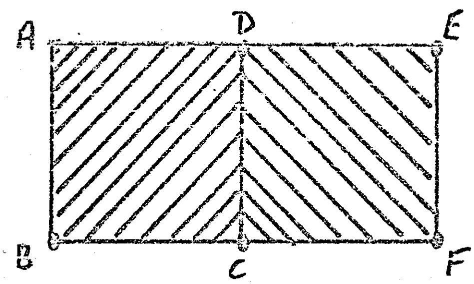


Figure 6

Hence H is not critical, since $\underline{M}(H)$ has rank 2 and $\mathcal{E}$ is a base of $\underline{M}(H)$.

A circuit $\mathcal{C}$ necessarily contains the edge-set of a cycle, since, by (2.3), every vertex in $V(\mathcal{C})$ is contained in the vertex-sets of at least two elements of $\mathcal{C}$. However, not every cycle is such that its edge-set is dependent, and the hypergraph shown in Figure 6 provides a suitable example. This also gives a reason why the cycles of a hypergraph are not appropriate as a starting-point for matroids. We have, in the hypergraph of Figure 6, two edges which together form the edge-set of a cycle. Thus, if any matroid were possible starting from the cycles of the hypergraph, these edges would be parallel. But they do not have the same vertex-set, which is contrary to the desirable property we mentioned in Chapter 1. Our requirement for independence ensures that two edges are parallel if and only if they have the same vertex-set.

Another good reason for not using the cycles of a hypergraph to attempt to define the circuits of a matroid is that there exist uniform and non-uniform hypergraphs, the edge-sets of whose cycles do not satisfy the circuit axioms (C1) and (C2).

CHAPTER 3

ELEMENTARY PROPERTIES OF HYPERGRAPHIC MATROIDS

In this chapter, we shall prove various properties of hypergraphic matroids, and use these to show that the Fano matroid and its dual are not hypergraphic matroids. We also produce a generalisation to hypergraphs of the notion of a cutset in graphs, and show that the set of cutsets of a hypergraph H is the set of circuits of the dual of $\underline{M}(H)$.

PROPOSITION 3.1: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph. If $\mathcal{A} \subseteq \mathcal{E}$, $\mathcal{A} \neq \phi$, then $|V(\mathcal{A})| - (k-1) \geq \rho\mathcal{A}$, where ρ is the rank function of $\underline{M}(H)$.

If $H_{\mathcal{A}}$ is a fragment of H, then $|V(\mathcal{A})| - (k-1) = \rho\mathcal{A}$.

Proof: Let $\mathcal{A} \subseteq \mathcal{E}$ and let $\mathcal{B}$ be a maximal independent subset of $\mathcal{A}$. Then $\rho\mathcal{A} = |\mathcal{B}|$. Also, since $\mathcal{B}$ is independent, by (2.1), $|V(\mathcal{B})| \geq |\mathcal{B}| + k - 1$. Since $V(\mathcal{A}) \supseteq V(\mathcal{B})$,
$|V(\mathcal{A})| \geq |V(\mathcal{B})| \geq |\mathcal{B}| + k - 1 = \rho\mathcal{A} + k - 1$.
If $H_{\mathcal{A}}$ is a fragment, $V(\mathcal{A}) = V(\mathcal{D})$ for some critical subset $\mathcal{D} \subseteq \mathcal{A}$.

Therefore,
$$\begin{aligned} \rho\mathcal{A} + k - 1 &\leq |V(\mathcal{A})| = |V(\mathcal{D})| \\ &= |\mathcal{D}| + k - 1 \quad \text{since } \mathcal{D} \text{ is critical} \\ &\leq |\mathcal{B}| + k - 1 \quad \text{since } \mathcal{B} \text{ is a maximal independent subset of } \mathcal{A} \\ &= \rho\mathcal{A} + k - 1. \end{aligned}$$

Thus, equality holds throughout, and the second half of the proposition now follows.

PROPOSITION 3.2: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph. If $\mathcal{C} = \{e_1, e_2, e_3\}_{\neq}$ is a circuit of cardinality 3 in $\underline{M}(H)$ then, under a suitable labelling of the vertices, $V(e_1) = \{A,C\} \cup W$, $V(e_2) = \{B,C\} \cup W$ and $V(e_3) = \{A,B\} \cup W$, where $\{A,B,C\}_{\neq} \cap W = \phi$ and $\{A,B,C\} \cup W \subseteq V$.

Proof: Since $\mathcal{C}$ is a circuit, by (2.3), $|V(\{e_1,e_2\})| = |V(\mathcal{C})| = k+1$. Since H is a k-hypergraph, $|V(e_i)| = k$ $(1 \leq i \leq 3)$. So, $|V(e_1) \cap V(e_2)| = k - 1$. Let $V(e_1) \cap V(e_2) = U$. Then $V(e_1) = \{A\} \cup U$ and $V(e_2) = \{B\} \cup U$ under a suitable labelling of the vertices. Also by (2.3), $V(e_3) \subseteq V(\mathcal{C}) = V(e_1) \cup V(e_2) = \{A,B\} \cup U$.
Furthermore, since every vertex in $V(\mathcal{C})$ is contained in the vertex-sets of at least two elements of $\mathcal{C}$, $\{A,B\} \subseteq V(e_3)$. Thus $V(e_3) = \{A,B\} \cup W$ for some subset $W \subseteq U$ with $|W| = k - 2$. Let $\{C\} = U - W$. Then $V(e_1) = \{A,C\} \cup W$, $V(e_2) = \{B,C\} \cup W$ and $V(e_3) = \{A,B\} \cup W$.

PROPOSITION 3.3: Let $H = (V, E, \$)$ be a k-hypergraph. If $\mathcal{C}$ is a circuit of cardinality 4 in $\underline{M}(H)$, then, for any $e \in \mathcal{C}$, there exists $e' \in \mathcal{C}$ such that $|V(e) \cap V(e')| = k - 2$, and $V(e) \cup V(e') = V(\mathcal{C})$.

Proof: Let $\mathcal{C} = \{e,e_1,e_2,e_3\}_{\neq}$. Then, since $\mathcal{C}$ is a circuit, by (2.3), $|V(\mathcal{C})| = |V(\{e,e_1,e_2,e_3\})| = k+2$. Since H is a k-hypergraph, $|V(e)| = |V(e_i)| = k$ $(1 \leq i \leq 3)$. Thus, $|V(e) \cap V(e_i)| \geq k-2$ for each i, $1 \leq i \leq 3$. If strict inequality holds for each i, then, since $\mathcal{C}$ is a minimal dependent set, $|V(e) \cap V(e_i)| = k - 1$ for each i.

Suppose this is so. Since $\{e,e_1,e_2\}$ is critical (by (2.3)), $|V(\{e,e_1,e_2\})| = k + 2$. Thus, there exist vertices $V_1 \in V(e_1)$, $V_2 \in V(e_2)$ such that $V_1 \notin V(e) \cup V(e_2)$ and $V_2 \notin V(e) \cup V(e_1)$. But, by (2.3), $\{V_1,V_2\} \subseteq V(e_3)$, so $|V(e_3) \cap V(e)| \leq k-2$, which is a contradiction. Thus, there exists $e' \in \mathcal{C}$ satisfying $|V(e) \cap V(e')| = k-2$. But then $|V(e) \cup V(e')| = k+2$, and so $V(e) \cup V(e') = V(\mathcal{C})$.

(3.2) and (3.3) are not of any great interest in themselves (except that they mirror the behaviour of circuits of similar sizes in graphs), but they are of considerable use in constructing hypergraphic presentations of matroids, if such presentations exist, or in proving that certain matroids are not hypergraphic.

It is routine to check that all matroids on at most six elements are hypergraphic.

THEOREM 3.4: Let $\underline{M}$ be a matroid of rank 3 on the seven-element set $E = \{a,b,c,d,e,f,g\}_{\neq}$. If the set of circuits of $\underline{M}$ contains $\{a,b,c\}$, $\{a,d,e\}$, $\{a,f,g\}$ and no other sets of cardinality 3 containing a, then $\underline{M}$ is not hypergraphic.

Proof: (i) We prove first that $\underline{M}$ is simple.

Since each of the seven elements of E is contained in a circuit of cardinality 3, $\underline{M}$ is loopless. Suppose $\underline{M}$ has a parallel pair of elements, $\{b,d\}$, say. Then, by (C2), there exists a circuit of $\underline{M}$ contained in $(\{a,b,c\} \cup \{b,d\}) - \{b\} = \{a,c,d\}$. This circuit cannot be $\{a,c,d\}$, since this is a set of cardinality 3 containing a. It cannot be $\{a,c\}$ or $\{a,d\}$, since both of these are properly contained in circuits of $\underline{M}$. It must therefore be $\{c,d\}$. But then b and c are parallel, and so $\{b,c\}$ is a circuit, which is impossible, since it is properly contained in the circuit $\{a,b,c\}$.

A similar argument applies to other possible pairs of parallel elements. Thus, $\underline{M}$ is simple.

(ii) We now prove that $\underline{M}$ is not hypergraphic.

Suppose $\underline{M}$ is hypergraphic. Then, by (2.13), there exists a critical hypergraph $H = (V', E', \$')$ of cardinality k, where $|V'| = k+2$, and $\underline{M} \cong \underline{M}(H)$. Let $E' = \{a',b',c',d',e',f',g'\}_{\neq}$, and let the matroid isomorphism be that induced by the obvious bijection between E and E'. Then, since $\{a',b',c'\}$ is a circuit, by (3.2), $V(a') = \{A,C\} \cup W$ $V(b') = \{B,C\} \cup W$ and $V(c') = \{A,B\} \cup W$, for a suitable labelling of the vertex-set V', where $|W| = k-2$. Then $|V(\{a',b',c'\})| = k+1$, so there is exactly one vertex D, say, with $D \in V' - V(\{a',b',c'\})$.
Now, since none of d',e',f',g' forms a circuit with $\{a',b'\}$, D is an element of each of $V(d')$, $V(e')$, $V(f')$ and $V(g')$.

Also, since {a',d',e'} and {a',f',g'} are circuits, we have $|V(\{a',d',e'\})| = k+1$, and so $V(\{a',d',e'\}) = \{A,B,D\} \cup W$. Similarly, $V(\{a',f',g'\}) = \{A,B,D\} \cup W$.

Thus, $V(\{a',d',e',f',g'\}) = \{A,B,D\} \cup W$, so, by (3.1), the rank in M(H) of {a',d',e',f',g'} is at most 2. But, in M, the rank of {a,d,e,f,g} is 3, which is a contradiction. Thus, M is not hypergraphic.

PROPOSITION 3.5: The matroid M on the seven-element set $E = \{a,b,c,d,e,f,g\}_{\neq}$ with circuits {a,b,c}, {a,d,e}, {a,f,g} and all 4-subsets of E containing none of these is the simplest non-hypergraphic matroid, in that it has fewest elements, and, amongst all non-hypergraphic matroids on seven elements, it has fewest circuits of less than full rank.

Proof: That M is not hypergraphic follows from (3.4). That M has fewest elements amongst all non-hypergraphic matroids follows from the fact that every matroid on at most six elements is hypergraphic. That any matroid on seven elements with at most two circuits of less than full rank can be shown by routine check.

PROPOSITION 3.6: The Fano matroid is not hypergraphic.

Proof: (3.4).

THEOREM 3.7: Let M be a matroid of rank 4 on the seven-element set $E = \{a,b,c,d,e,f,g\}_{\neq}$. If the set of circuits of M contains {a,b,c,d}, {a,b,e,f}, {a,c,f,g}and {a,d,e,g}, and no other circuits of cardinality 4 containing a, then M is not hypergraphic.

Proof: (i) We prove first that M is simple.

Since each element of E is properly contained in a circuit, M is loopless. Suppose that M contains a pair of parallel elements. Since no such pair can be a subset of one of the 4-sets given in the hypotheses of the theorem, such a pair can be only $\{b,g\}$, $\{c,e\}$ or $\{d,f\}$. Consider, for example, the pair $\{b,g\}$. If these are parallel, then, by (C2), there exists a circuit contained in $(\{a,b,c,d\}\cup\{b,g\})-\{b\}$, i.e. in $\{a,c,d,g\}$. This circuit cannot be $\{a,c,d,g\}$, since this is a 4-set containing a. It cannot be any of the pairs of elements in the set, since no such pair can be parallel. It cannot be a triple containing a, since any such triple is properly contained in a circuit of $\underline{M}$. It must therefore be $\{c,d,g\}$. But then we have $\rho(\{a,b,c,d\}\cup\{c,d,g\}) = \rho(\{a,b,c,d\})$ and so $\rho(\{a,b,c,d,g\}\cup\{a,d,e,g\}) = \rho(\{a,b,c,d\})$, and $\rho(\{a,b,c,d,e,g\}\cup\{a,b,e,f\}) = \rho(\{a,b,c,d\})$. Thus, $\mathrm{rk}\underline{M} = \rho(\{a,b,c,d\}) = 3$, which is a contradiction of the hypothesis that $\underline{M}$ has rank 4.

A similar argument holds for other possible pairs of parallel elements. Thus, $\underline{M}$ is simple.

(ii) We now prove that $\underline{M}$ is not hypergraphic.

Suppose $\underline{M}$ is hypergraphic. Then by (2.13), there exists a critical k-hypergraph $H = (V', E'\ \$')$ with $|V'| = k+3$, such that $\underline{M} \cong \underline{M}(H)$. Let $E' = \{a',b',c',d',e',f',g'\}_{\neq}$, and let the matroid isomorphism be induced by the obvious bijection from E to E'. We note that the set of circuits given in the hypotheses of the theorem is symmetric in b,c,d in that, for any permutation of b,c,d, there is a corresponding permutation of e,f,g which preserves the circuits listed in the hypotheses of the theorem. Now, by (3.3) applied to the circuit $\{a',b',c',d'\}$, there is an edge x' in $\{b',c',d'\}$ such that $|V(a')\cup V(x')| = k-2$. By the symmetry referred to above, we may assume without loss of generality that $x' = b'$.

Thus, $|V(a')\cup V(b')| = k+2$. Now, by (2.3), $|V(\{a',b',e',f'\})| = k+2$, since $\{a',b',e',f'\}$ is a circuit. Thus, $V(\{a',b',e',f'\}) = V(\{a',b'\})$. Hence $|V(\{a',b',c',d',e',f'\})| = k+2$ and so, by (3.1), in $\underline{M}(H)$, $\{a',b',c',d',e',f'\}$ has rank at most 3. But, in $\underline{M}$, $\{a,b,c,d,e,f\}$ has rank 4, since it spans $\mathcal{E}$ in $\underline{M}$, and $\underline{M}$ has rank 4. Thus, $\underline{M}$ is not hypergraphic.

COROLLARY 3.8: The dual of the Fano matroid is not hypergraphic.

PROPOSITION 3.9: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph. Let $\{a,b,c\}_{\neq} \subseteq \mathcal{E}$. If $\{a,b\}$ and $\{a,c\}$ are critical sets, and if $\{b,c\}$ is independent and not critical, then $V(b) \cup V(c) \supseteq V(a) \supseteq V(b) \cap V(c)$.

Proof: Since $\{a,b\}$ is critical, $|V(a) \cup V(b)| = k+1$. So $V(b) = (V(a)-\{A\}) \cup \{B\}$ for some $\{A,B\}_{\neq} \subseteq V$. Similarly, since $\{a,c\}$ is critical, $V(c) = (V(a)-\{C\}) \cup \{D\}$ for some $\{C,D\}_{\neq} \subseteq V$.

Therefore, $V(b) \cup V(c) = (V(a)-(\{A\} \cap \{C\})) \cup (\{B\} \cup \{D\})$ (1)

$V(b) \cap V(c) = (V(a)-(\{A\} \cup \{C\})) \cup (\{B\} \cap \{D\})$ (2)

Now, $\{b,c\}$ is independent and not critical, so $|V(b) \cup V(c)| \geq k+2$. Thus, since $|V(a)| = k$, from (1), $A \neq C$ and $B \neq D$. Therefore, $V(b) \cup V(c) = V(a) \cup \{B,D\}$ and $V(b) \cap V(c) = V(a)-\{A,C\}$. Hence $V(b) \cup V(c) \supseteq V(a) \supseteq V(b) \cap V(c)$.

PROPOSITION 3.10: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph, and let $\mathcal{A} \subseteq \mathcal{E}$ be a critical set. If $e \in \mathcal{E}-\mathcal{A}$ and $V(e) \subseteq V(\mathcal{A})$, then $\mathcal{A} \cup \{e\}$ is dependent in $\underline{M}(H)$.

Proof: $|V(\mathcal{A} \cup \{e\})| = |V(\mathcal{A}) \cup V(e)| = |V(\mathcal{A})|$ since $V(e) \subseteq V(\mathcal{A})$

$= |\mathcal{A}| + k - 1$ since $\mathcal{A}$ is critical

$< |\mathcal{A} \cup \{e\}| + k - 1$ since $e \notin \mathcal{A}$.

Thus, $\mathcal{A} \cup \{e\}$ is dependent.

PROPOSITION 3.11: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph, such that $\underline{M}(H)$ is simple. Let $\mathcal{E}' = \{a,b,c,d,e,f,g,x,y,z\}_{\neq} \subseteq \mathcal{E}$ be such that the set of circuits of $\underline{M}(H) \times \mathcal{E}'$ contains $\{a,b,c\}$, $\{a,d,e\}$, $\{a,f,g\}$ and no other circuits of cardinality 3 containing a. Suppose further that, in $\underline{M}(H)$, $x \in \sigma(\{b,d\}) \cap \sigma(\{c,e\})$, $y \in \sigma(\{b,f\}) \cap \sigma(\{c,g\})$

and $z \in \sigma(\{d,f\}) \cap \sigma(\{e,g\})$. Then $\{x,y,z\}$ is a circuit of $\underline{M}(H)$.

Proof: We note first that $\underline{M}(H) \times E'$ necessarily has rank ≥ 4, since, by (3.4), $\underline{M}' = \underline{M}(H) \times \{a,b,c,d,e,f,g\}$ cannot have rank 3, and the requirement that $\underline{M}(H) \times E'$ has only the given circuits of cardinality 3 containing a precludes the possibility that $\underline{M}'$ has rank less than 3. Furthermore, by an application of the rank function, we see that $\underline{M}(H) \times E'$ has rank at most 4, since $\rho(\{a,b,c\} \cup \{a,d,e\}) \leq \rho(\{a,b,c\}) + \rho(\{a,d,e\}) - 1 = 3$ (1)

and $\rho(\{a,f,g\} \cup \{a,b,c,d,e\}) \leq \rho(\{a,f,g\}) + \rho(\{a,b,c,d,e\}) - 1 \leq 4$ (2)

and $\rho(\{a,b,c,d,e,f,g,x,y,z\}) = \rho(\{a,b,c,d,e,f,g\})$, since $x \in \sigma(\{b,d\})$, $y \in \sigma(\{b,f\})$ and $z \in \sigma(\{d,f\})$.

Thus, equality must hold in the inequalities (1) and (2). Since $\{a,b,c\}$ is a circuit, by (3.2), $V(a) = W \cup \{A,C\}$, $V(b) = W \cup \{B,C\}$ and $V(c) = W \cup \{A,B\}$ for a suitable labelling of V, where $|W| = k-2$. $\{a,d,e\}$ is a circuit, so, by (2.3), $V(a) \cup V(d) \cup V(e) = W \cup \{A,C\} \cup \{D\}$ for some $D \in V - \{A,C\}$. Since, in $\underline{M}(H)$, $\rho(\{a,b,c,d,e\}) = 3$, by (3.1), $|V(\{a,b,c,d,e\})| \geq k+2$, so $D \notin \{A,B,C\} \cup W$. By (2.3), $D \in V(d) \cap V(e)$.

Since $\{b,d,x\}$ is a circuit, $|V(b) \cup V(d) \cup V(x)| = k+1$. Since $V(b) \cup V(d) \supseteq W \cup \{B,C,D\}$, we must have $V(b) \cup V(d) \cup V(x) = W \cup \{B,C,D\}$. Thus, $V(d) \subseteq (V(a) \cup V(d) \cup V(e)) \cap (V(b) \cup V(d) \cup V(x)) = W \cup \{C,D\}$. Since $|V(d)| = k = |W| + 2$, equality holds, so $V(d) = W \cup \{C,D\}$. Similarly, $V(e) = W \cup \{A,D\}$ and $V(x) = W \cup \{B,D\}$.

Again, using the circuits $\{a,f,g\}$ and $\{b,f,y\}$, $V(f) = W \cup \{C,E\}$, $V(g) = W \cup \{A,E\}$ and $V(y) = W \cup \{B,E\}$ for some $E \notin \{A,B,C,D\} \cup W$. Thus, $V(z) \subseteq (V(d) \cup V(f)) \cap (V(e) \cup V(g))$

$$= (\{C,D,E\} \cup W) \cap (\{A,D,E\} \cup W) = \{D,E\} \cup W.$$

So, since $|V(z)| = k = |W| + 2$, equality holds, and $V(z) = \{D,E\} \cup W$.

Thus, $V(x) = \{B,D\} \cup W$, $V(y) = \{B,E\} \cup W$, $V(z) = \{D,E\} \cup W$ and so $\{x,y,z\}$ is a circuit of $\underline{M}(H)$.

This theorem is a specialisation of Desargues' theorem. Translated into a geometric interpretation in the Euclidean representation of $\underline{M}(H)$, (3.11) states that if bdf and ceg are triangles, such that the lines bc, de and fg are concurrent at a, then the points x,y and z formed by the intersections of corresponding sides of the triangles are collinear.

The proof of (3.11) shows that, up to the common set W of vertices, the allocation of vertices is the same as the allocation of vertices to the edges of a K_5; the matroid $\underline{M}(K_5)$ is, in Euclidean representation, the Desargues configuration without coincidences in three dimensions.

PROPOSITION 3.12 (DIRECT SUM): Let $\underline{M}_1, \underline{M}_2$ be hypergraphic matroids.

Then $\underline{M}_1 \oplus \underline{M}_2$ is hypergraphic.

Proof: Let $H_i' = (V_i', E_i, \$_i')$ $(i = 1,2)$ be a k_i'-hypergraph such that $\underline{M}_i \cong \underline{M}(H_i')$. Without loss of generality, suppose $k_1' \geq k_2'$, and put $k = k_1'$. Let V_1, V_2, W be disjoint sets of vertices such that $|V_i| = |V_i'|$ $(i = 1,2)$ and $|W| = k_1' - k_2'$. Let θ_i denote a bijection between V_i and V_i' $(i = 1,2)$.

Let H_1 be the k-hypergraph $(V_1, E_1, \$_1)$ where

$V_{H_1}(e) = \{V \in V_1 : \theta_1 V \in V_{H_1'}(e)\}$, for each $e \in E_1$.

Let H_2 be the k-hypergraph $(V_2 \cup W, E_2, \$_2)$, where

$V_{H_2}(e) = \{V \in V_2 : \theta_2 V \in V_{H_2'}(e)\} \cup W$, for each $e \in E_2$.

Then $\underline{M}(H_i) \cong \underline{M}_i$ $(i = 1,2)$. Let H be the hypergraph $(V, E, \$)$, where $V = V_1 \cup V_2 \cup W$, $E = E_1 \cup E_2$ and $\$ = \$_1 \cup \$_2$. Then $\underline{M}(H) = \underline{M}(H_1) \oplus \underline{M}(H_2)$. For, let $A_i \subseteq E_i$ be a nonempty set independent in $\underline{M}(H_i)$ $(i = 1,2)$.

Then $|V_{H_i}(G_i)| \geq |G_i| + k - 1$ for each nonempty subset G_i of A_i.

$\therefore$ $|V_H(G_i)| \geq |G_i| + k - 1$ for each nonempty subset G_i of A_i, and so A_i is independent in $\underline{M}(H)$. Write $G = G_1 \cup G_2$. Then, if $G_1, G_2 \neq \phi$,

$$|V_H(G)| = |V_H(G_1)| + |V_H(G_2)| \geq |G_1| + |G_2| + k-1 + k-1$$
$$> |G| + k - 1 \quad \text{since } k > 1.$$

Thus, writing $A = A_1 \cup A_2$, if A_i is independent in $\underline{M}(H_i)$ $(i = 1,2)$, A is independent in $\underline{M}(H)$.

Conversely, suppose A is independent in $\underline{M}(H)$. Write $A_i = A \cap E_i$. Then, if $A_i \neq \phi$, $|V_{H_i}(G_i)| = |V_H(G_i)| \geq |G_i| + k-1$, for each nonempty subset G_i of A_i, and so A_i is independent in $\underline{M}(H_i)$.

Thus, $\underline{M}(H) = \underline{M}(H_1) \oplus \underline{M}(H_2) \cong \underline{M}_1 \oplus \underline{M}_2$, and so $\underline{M}_1 \oplus \underline{M}_2$ is hypergraphic.

PROPOSITION 3.13: Let $\underline{M}$ be a matroid of rank r on the set E and suppose $x \in E$ is such that x is an element of no circuit of $\underline{M}$ of cardinality less than (r+1). Then $\underline{M} \times (E-\{x\})$ is hypergraphic if and only if $\underline{M}$ is hypergraphic.

Proof: Assume $\underline{M}$ is hypergraphic. Then, by (2.12), $\underline{M} \times (E-\{x\})$ is hypergraphic.

Now suppose $\underline{M} \times (E-\{x\})$ is hypergraphic. If x is a coloop of $\underline{M}$, then $\underline{M} = (\underline{M} \times (E-\{x\}) \oplus U_{1,1}(\{x\})$, and hence, by (3.12), $\underline{M}$ is hypergraphic. If x is not a coloop of $\underline{M}$, write $E' = \{e' : e \in E\}$, and let H' be the critical k-hypergraph $(V', E'-\{x'\}, \$')$ with $\underline{M}(H') \cong \underline{M} \times (E-\{x\})$, where the isomorphism is that induced by the obvious bijection between E and E'. Choose a set W' of $|V'|-k$ vertices disjoint from V', and put $V'' = V' \cup W'$, $E'' = \{e'' : e \in E\}$.

Let $H'' = (V'', E'', \$'')$ be the hypergraph with $V_{H''}(e'') = V_{H'}(e') \cup W'$ if $e'' \neq x''$ and $V_{H''}(x'') = V'$. Then H" is a uniform hypergraph of cardinality $|V'|$.

Let $A'' \subseteq E''-\{x''\}$. Then A'' is independent in $\underline{M}(H'')$ if and only if either $A'' = \phi$, or $|V_{H''}(G'')| \geq |G''| + |V'| - 1$ for each nonempty subset G'' of A''.

But this is equivalent to $|V_{H'}(G')| \geq |G'| + k - 1$ for each nonempty subset G' of A'. Thus, A' is independent in $\underline{M}(H')$ if and only if A'' is independent in $\underline{M}(H'')$, where $A'' \subseteq E''-\{x''\}$. Hence $\underline{M}(H'')\times(E''-\{x''\}) \cong \underline{M}(H')$.

Now let $A'' \subseteq E''$, with $x'' \in A''$. If $|A''| \geq 2$, then $V_{H''}(A) = V'$, so there are no circuits of $\underline{M}(H'')$ containing x'' of less than full rank. However, since H' is critical, $r = |V'| - k+1$. By (3.1), $rk\underline{M}(H'') \leq |V(H'')| - |V'| + 1 = |W'| + 1 = r$; also, from the above, $rk\underline{M}(H'') \geq rk\underline{M}(H') = r$. Therefore, $\underline{M}(H'')$ has rank r.
Let B'' be any base of $\underline{M}(H'')$ not containing x''. Then $\{x''\}\cup B''$ is a circuit of $\underline{M}(H'')$. Conversely, if $\{x''\}\cup B''$ is a circuit of $\underline{M}(H'')$, B'' is a base of $\underline{M}(H'')$. Therefore, $\underline{M}(H'') \cong \underline{M}$, the isomorphism being that induced by the obvious bijection between E'' and E. Thus, $\underline{M}$ is hypergraphic.

COROLLARY 3.14: Let $\underline{M}$ be a matroid of rank r on the set E, and suppose that $x_1, x_2, \ldots, x_s \in E$ are such that none of them is contained in a circuit of $\underline{M}$ of less than full rank. Then $\underline{M}\times(E-\{x_1, x_2, \ldots, x_s\})$ is hypergraphic if and only if $\underline{M}$ is hypergraphic.

Proof: Repeated application of (3.13).

DEFINITION 3.15: Let $\underline{M}$ be a matroid of rank r on the set E, and let $x \notin E$. Then the matroid on the set $E\cup\{x\}$, whose set of bases is the set $\mathbf{B}(\underline{M})\cup\{I\cup\{x\}: |I| = r-1 \text{ and } I \in \mathbf{I}(\underline{M})\}$ is called the <u>free, rank-preserving one-point extension</u> of $\underline{M}$ by x.
If $X\cap E = \phi$, the matroid on $E\cup X$ whose set of bases is the set $\{I\cup Y: Y \subseteq X, |Y| \leq r, |I| = r - |Y|, I \in \mathbf{I}(\underline{M})\}$ is called the <u>free, rank preserving $|X|$-point</u> extension of $\underline{M}$ by X.

COROLLARY 3.16: If $\underline{M}$ is a hypergraphic matroid on the set E and $x \notin E$, the free rank-preserving one-point extension of $\underline{M}$ by x is hypergraphic.

Proof: Let $\underline{M}_1$ be the free, rank-preserving one-point extension of $\underline{M}$ by x. Then x is contained in no circuit of $\underline{M}_1$ of less than full rank - i.e. in no circuit of cardinality less than $rk\underline{M}_1 + 1$. Thus, by (3.13), $\underline{M}_1$ is hypergraphic if and only if $\underline{M}_1 \times (E-\{x\})$ is hypergraphic. But $\underline{M}_1 \times (E-\{x\}) = \underline{M}$, and $\underline{M}$ is hypergraphic, so $\underline{M}_1$ is hypergraphic.

Note that the one-point extension of (3.15) is a very particular one-point extension, in which the point x is placed "in general position" in $\underline{M}$. It is not true that placing x elsewhere in $\underline{M}$ will necessarily give a hypergraphic matroid from the hypergraphic matroid $\underline{M}$. For example, consider the matroid $\underline{M} = \underline{M}(K_4)$, shown in Euclidean representation in Figure 7(a). A one-point extension of $\underline{M}$ placing x in the flat $\sigma(\{c,e\})$ gives the matroid shown in Euclidean representation in Figure 7(b). By (3.4), this is not hypergraphic.

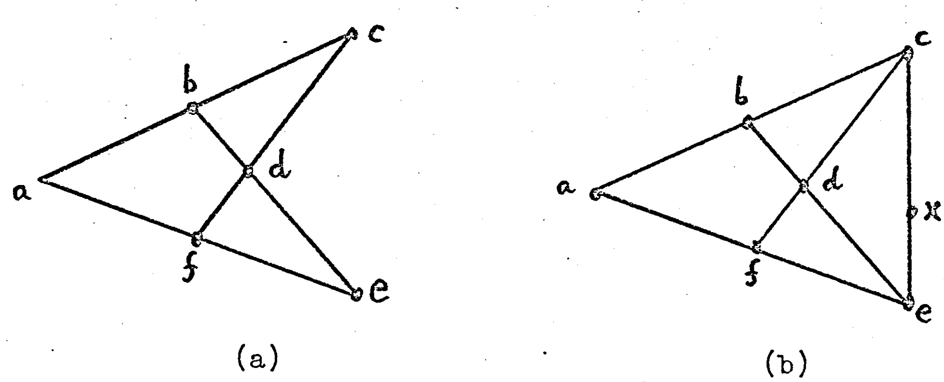


Figure 7

PROPOSITION 3.17: Let $H = (V, E, \$)$ be a k-hypergraph, and let $\{a,b,c,d\}_{\neq} \subseteq E$. If {a,b}, {a,c}, {a,d}, {b,c} and {b,d} are critical sets of edges, and if no three of these edges together form a circuit, then {c,d} is critical.

Proof: Since {a,b} is critical, we have $V(a) = \{A\} \cup W$, $V(b) = \{B\} \cup W$ for a suitable labelling of the elements of V, where $|W| = k-1$. Then $V(a) \cup V(b) = \{A,B\} \cup W$. Since c does not form a circuit with {a,b}, $V(c) \not\subseteq V(a) \cup V(b)$, and so there exists $C \in V$ such that $C \in V(c)$ and

$C \notin V(a) \cup V(b)$. Thus, $V(a) \cup V(c) \supseteq \{A,C\} \cup W$. Since $\{a,c\}$ is critical, equality holds. Similarly, $V(b) \cup V(c) = \{B,C\} \cup W$. Thus, since $V(c) \subseteq (V(a) \cup V(c)) \cap (V(b) \cup V(c)) = \{C\} \cup W$ and $|V(c)| = k$, equality holds here, and $V(c) = \{C\} \cup W$.

Similarly, $V(d) = \{D\} \cup W$, where $D \notin \{A,B,C\} \cup W$.

Thus $V(c) \cup V(d) = \{C,D\} \cup W$, and so $\{c,d\}$ is critical.

PROPOSITION 3.18: Let $H = (V, E, \$)$ be a k-hypergraph, and let $\{(V(G_i), G_i, \$_i): 1 \le i \le m\}$ be the set of components of H. Then, if ρ is the rank function of $\underline{M}(H)$,

$$\rho A = \sum_{i=1}^{m} (\rho(G_i \cap A)) \quad \text{for any } A \subseteq E.$$

Proof: Let $H_i' = (V_i', G_i', \$_i')$ $(1 \le i \le m)$ be k-hypergraphs such that $\underline{M}(H_i') \cong \underline{M}(H) \times G_i$ under the obvious isomorphism, and such that $V_i' \cap V_j' = \phi$ for $i \neq j$. Put $V' = V_1' \cup V_2' \cup \ldots \cup V_m'$, $E' = G_1' \cup G_2' \cup \ldots \cup G_m'$ and $\$' = \$_1' \cup \ldots \cup \$_m'$. Let $H = (V', E', \$')$. Then, as in the proof of (2.13), $\underline{M}(H') \cong \underline{M}(H)$. Also, $\underline{M}(H') = \underline{M}(H_1') \oplus \underline{M}(H_2') \oplus \ldots \oplus \underline{M}(H_m')$.

For, let $A_{i_1}', A_{i_2}', \ldots A_{i_n}'$ be nonempty independent subsets of $G_{i_1}', G_{i_2}', \ldots, G_{i_n}'$ $(n \le m)$.

Then $|V_{H_{i_j}'}(B_{i_j}')| \ge |B_{i_j}'| + k - 1$ for each nonempty subset B_{i_j}' of A_{i_j}'.

$\therefore$ $|V_{H'}(B_{i_1}' \cup B_{i_2}' \cup \ldots \cup B_{i_n}')| \ge |B_{i_1}' \cup B_{i_2}' \cup \ldots \cup B_{i_n}'| + k - 1$, since $V_{i_r}' \cap V_{i_s}' = \phi$ for $r \neq s$, and $k > 1$.

Thus $A' = A_{i_1}' \cup A_{i_2}' \cup \ldots \cup A_{i_n}'$ is independent in $\underline{M}(H')$.

Conversely, if, with the above notation, $A' = A_{i_1}' \cup A_{i_2}' \cup \ldots \cup A_{i_m}'$ is independent in $\underline{M}(H')$ where each $A_{i_j}' \neq \phi$, then, for any nonempty subset B_{i_j}' of A_{i_j}', $|V_{H'}(B_{i_j}')| \ge |B_{i_j}'| + k - 1$, since B_{i_j}' is a nonempty subset of A', and hence

$|V_{H'_{i_j}}(B'_{i_j})| \geq |B'_{i_j}| + k - 1.$

Thus, A'_{i_j} is independent in $\underline{M}(H'_{i_j})$.

Hence, $\underline{M}(H') = \underline{M}(H'_1) \oplus \underline{M}(H'_2) \oplus \ldots \oplus \underline{M}(H'_m)$, and so $\underline{M}(H) \cong \underline{M}(H'_1) \oplus \underline{M}(H'_2) \oplus \ldots \oplus \underline{M}(H'_m)$. Thus, for $A \subseteq E$,

$\rho A = \sum_{i=1}^{m} (\rho(G_i \cap A)).$

We close this chapter by examining the analogue in hypergraphs of cutsets in graphs.

DEFINITION 3.19: Let $(V(G_1), G_1, \$_1)$ be a component of the hypergraph $(V, E, \$)$. A subset $X \subseteq E$ is said to <u>separate</u> $(V(G_1), G_1, \$_1)$ if $(V(G_1), G_1 - X, \$'_1)$ is not a critical hypergraph, where $\$'_1 = \{(V,e) \in \$_1 : e \in G_1 - X\}$; i.e., there exists no critical set $A \subseteq G_1 - X$ such that $V(G_1) = V(A)$.

DEFINITION 3.20: Let $H = (V, E, \$)$ be a k-hypergraph. A <u>cutset</u> of H is a subset $C^* \subseteq E$ such that

(i) C^* separates some component of H;

(ii) no proper subset of C^* separates any component of H.

LEMMA 3.21: Let $H = (V, E, \$)$ be a k-hypergraph with components $(V(G_i), G_i, \$_i)$ $(1 \leq i \leq m)$, and let C^* be a cutset of H. Then $C^* \subseteq G_i$ for some i.

Proof: Suppose $C^* \cap G_i \neq \phi$ and $C^* \cap G_j \neq \phi$ where $i \neq j$. Then $(V(G_i), G_i - C^*, \$')$ is not a critical hypergraph. Let $c \in C^* \cap G_j$. Then $(V(G_i), G_i - (C^* - \{c\}), \$') = (V(G_i), G_i - C^*, \$')$, since $c \in G_j$ and, by (2.7), $G_i \cap G_j = \phi$. Thus, $C^* - \{c\}$ separates $(V(G_i), G_i, \$_i)$, and so C^* is not a cutset, since there exists a proper subset of C^* which separates a component of H. But this contradicts the hypothesis that C^* is a cutset. Thus, $C^* \subseteq G_i$ for some i.

LEMMA 3.22: Let $(V(G_1), G_1, \$_1)$ be a component of the k-hypergraph H. Then $|V(G_1)| = \rho G_1 + k - 1$, where ρ is the rank function of M(H).

Proof: Since $(V(G_1), G_1, \$_1)$ is a component of H, it is a fragment of H. The result now follows from (3.1).

LEMMA 3.23: Let $H = (V, E, \$)$ be a k-hypergraph, and let C^* be a cutset of H. Then $E-C^*$ is a hyperplane of $\underline{M}(H)$.

Proof: Let $\{(V(G_i), G_i, \$_i): 1 \le i \le m\}$ be the set of components of H. Then, by (3.21), $C^* \subseteq G_i$ for some i. Without loss of generality, assume $C^* \subseteq G_1$.

Since C^* separates $(V(G_1), G_1, \$_1)$, $\rho(G_1-C^*) < \rho(G_1)$. For, if not, there exists an independent set $A \subseteq G_1-C^*$ with $\rho A = \rho G_1$. Then $|V(G_1)| \ge |V(A)| \ge |A| + k - 1 = \rho G_1 + k - 1$. But, by (3.22), $|V(G_1)| = |G_1| + k - 1$. So A is critical, and $V(G_1)=V(A)$. But this contradicts the hypothesis that C^* separates $(V(G_1), G_1, \$_1)$. Thus, $\rho(G_1-C^*) < \rho(G_1)$. Furthermore, since C^* is minimal with respect to separation of $(V(G_1), G_1, \$_1)$, for any $c \in C^*$, there exists a critical set $A \subseteq G_1-(C^*-\{c\})$ with $V(A) = V(G_1)$; so $\rho(G_1-(C^*-\{c\})) = \rho(G_1)$ for any $c \in C^*$.

Since the rank function is increasing in unit steps, we must have $\rho(G_1-C^*) = \rho(G_1) - 1$.

Now, by (3.18),

$$\rho E = \sum_{i=1}^{m} \rho(G_i) = \rho(G_1) + \sum_{i=2}^{m} \rho(G_i)$$

$$\rho(E-C^*) = \rho(G_1-C^*) + \sum_{i=2}^{m} \rho(G_i) \quad \text{since } C^* \subseteq G_1$$

and for any $c \in C^*$, $$\rho(E-(C^*-\{c\})) = \rho(G_1-(C^*-\{c\})) + \sum_{i=2}^{m} \rho(G_i).$$

Thus, $$\rho(E-C^*) = \rho(E) - 1$$

and, for any $c \in C^*$, $$\rho(E-(C^*-\{c\})) = \rho(E)$$

Thus $E-C^*$ is a hyperplane of $\underline{M}(H)$.

LEMMA 3.24: Let $H = (V, E, \$)$ be a k-hypergraph. If E' is a hyperplane of $\underline{M}(H)$, then $E-E'$ is a cutset of H.

Proof: Let H have components $(V(G_i), G_i, \$_i)$ $(1 \le i \le m)$.

Then, by (3.18), $$\rho E = \sum_{i=1}^{m} \rho(G_i) \qquad (1)$$

Write $C^* = E-E'$. Then $E-C^* = E'$, so $\rho(E-C^*) = \rho E - 1$. Also, for any $c \in C^*$, $\rho(E-(C^*-\{c\})) = \rho E$.

By (3.18), $$\rho(E-C^*) = \sum_{i=1}^{m} \rho(G_i-C^*) \qquad (2)$$

and, for any $c \in C^*$, $$\rho(E-(C^*-\{c\})) = \sum_{i=1}^{m} \rho(G_i-(C^*-\{c\})) \qquad (3)$$

From (1) and (2), there exists j such that $\rho(G_j-C^*) = \rho(G_j) - 1$, since ρ is an increasing function which increases in unit steps.

Thus, C^* separates $(V(G_j), G_j, \$_j)$, since $|V(G_j)| = \rho G_1 + k - 1$ (by (3.22))

$$= \rho(G_1-C^*) + k - 1 + 1,$$

and so $(V(G_j), G_j-C^*, \$'_j)$ is not a critical hypergraph.

Furthermore, for any $c \in C^*$, $C^*-\{c\}$ does not separate $(V(G_i), G_i, \$_i)$ for any i. For, from (1) and (3),

$$\rho(G_i-(C^*-\{c\})) = \rho(G_i) \quad (1 \le i \le m),$$

so, if A is a maximal independent subset of $G_i-(C^*-\{c\})$, $\rho A = \rho G_i$ and

$$\begin{aligned} \rho G_i + k - 1 &= |V(G_i)| \quad \text{from (3.22)} \\ &\ge |V(A)| \quad \text{since } A \subseteq G_i \\ &\ge |A| + k - 1 \quad \text{since } A \text{ is independent} \\ &= \rho G_i + k - 1. \end{aligned}$$

So equality holds throughout, and $A \subseteq G_i-(C^*-\{c\})$ is a critical set with $V(A) = V(G_i)$, whence $(V(G_i), G_i-(C^*-\{c\}), \$'_i)$ is a critical hypergraph.

Thus, C^* separates some component of H, and $C^*-\{c\}$ separates no component of H, for any $c \in C^*$. Thus, $C^* = E-E'$ is a cutset of H.

THEOREM 3.25: The set of cutsets of the k-hypergraph $H = (V, \mathcal{E}, \$)$ is the set of circuits of a matroid $\underline{M}^*(H)$, where $\underline{M}^*(H) = (\underline{M}(H))^*$.

Proof: From (3.23) and (3.24), $\mathcal{C}^*$ is a cutset of H if and only if $\mathcal{E}-\mathcal{C}^*$ is a hyperplane of $\underline{M}(H)$. Thus, the set of cutsets of H is the set of cocircuits of $\underline{M}(H)$, and hence the set of circuits of $(\underline{M}(H))^*$.

COROLLARY 3.26: Let $H_1 = (V_1, \mathcal{E}_1, \$_1)$ and $H_2 = (V_2, \mathcal{E}_2, \$_2)$ be uniform hypergraphs with $\underline{M}(H_1) \cong \underline{M}(H_2)$. Let the isomorphism be induced by the bijection $\theta: \mathcal{E}_1 \to \mathcal{E}_2$. Then $\mathcal{C}^*$ is a cutset of H_1 if and only if $\{\theta(c): c \in \mathcal{C}^*\}$ is a cutset of H_2.

Proof: $\mathcal{C}^*$ is a cutset of H_1 if and only if $\mathcal{C}^*$ is a cocircuit of $\underline{M}(H_1)$. Since $\underline{M}(H_1) \cong \underline{M}(H_2)$, this is so if and only if $\{\theta(c): c \in \mathcal{C}^*\}$ is a cocircuit of $\underline{M}(H_2)$; i.e. if and only if $\{\theta(c): c \in \mathcal{C}^*\}$ is a cutset of H_2.

EXAMPLE: Let H be the hypergraph shown in Figure 2. The cutsets of H are the sets {d}, {e}, {a,b}, {a,c} and {b,c}.

These are the circuits of the matroid on the distinct elements a,b,c,d,e consisting of the two loops d and e, and the three parallel elements a,b,c – the dual of the matroid $\underline{M}(H)$ shown in Figure 3.

The cutsets of the critical hypergraph shown in Figure 4 are {d"}, {e"}, {a",b"}, {a",c"} and {b",c"}. The isomorphism between the ground-sets of the matroids $\underline{M}(H)$ and $\underline{M}(H")$ thus maps the sets of cutsets of H to the set of cutsets of H".

CHAPTER 4

MINORS OF HYPERGRAPHIC MATROIDS

It is well-known that any minor of a graphic matroid is itself graphic. However, the class of hypergraphic matroids as we have defined it is not closed under the operation of contraction, even if we extend the definition to allow loops in hypergraphic matroids. This observation will lead us to define the class of generalised hypergraphic matroids.

We first modify (2.1) to permit loops in a hypergraphic matroid.

DEFINITION 4.1: Let $H = (V, E, \$)$ be a hypergraph and $k \geq 2$ an integer with $|V|+1 \geq k \geq \max\{|V(e)|: e \in E\}$. A set $A \subseteq E$ is said to be independent if and only if either

(i) $A = \phi$;

or (ii) $|V(G)| \geq |G| + k - 1$ for each nonempty subset G of A.

The set of independent sets thus defined is the set of independent sets of a matroid $\underline{M}(H)$ on E. This follows immediately from the corresponding proof for (2.1). Alternatively, the method used by Crapo-Rota [6] can be applied.

Any matroid isomorphic to $\underline{M}(H)$ for some H is called a hypergraphic matroid.

PROPOSITION 4.2: Let $\underline{M}$ be a matroid on the set E', and let $e' \in E'$ be a loop of $\underline{M}$. Then $\underline{M}$ is hypergraphic if and only if $\underline{M}\times(E'-\{e'\})$ is hypergraphic.

Proof: Let E be an isomorphic copy of E'. Suppose $\underline{M}$ is hypergraphic. Then there exists a hypergraph $H = (V, E, \$)$ such that $\underline{M} \cong \underline{M}(H)$, where the isomorphism is induced by the obvious bijection between E and E'. Let $H' = (V, E-\{e\}, \$')$, where $\$' = \{(V,a) \in \$: a \in E-\{e\}\}$. Then clearly $\underline{M}(H') = \underline{M}(H)\times(E-\{e\}) \cong \underline{M}\times(E'-\{e'\})$.

Conversely, suppose $\underline{M}\times(E'-\{e'\})$ is hypergraphic. Let $H' = (V, E-\{e\}, \$')$ be such that $\underline{M}(H') \cong \underline{M}\times(E'-\{e'\})$, where the isomorphism is induced by the obvious bijection between E and E'. Let $W \subseteq V$ be such that $|W| = k-1$, where $k \geq 2$ defines $\underline{M}(H')$ as in (4.1). Then $|V| \geq |W| > 0$. Define $H = (V, E, \$)$, where $\$ = \$' \cup \{(V,e): V \in W\}$. Define $\underline{M}(H)$ using k as in (4.1). Then clearly $\underline{M}(H) \cong \underline{M}$, and so $\underline{M}$ is hypergraphic.

This result is the justification for using the term "hypergraphic matroid" to describe a matroid obtained either from definition (2.1) or from (4.1). For, a matroid is hypergraphic in accordance with (4.1) if and only if the matroid obtained from it by the deletion of loops is hypergraphic in accordance with (4.1). But, a loopless matroid is hypergraphic in accordance with (4.1) if and only if it is hypergraphic in accordance with (2.11) - i.e. if and only if it is isomorphic to $\underline{M}(H)$, for some uniform hypergraph H, where the independent sets of $\underline{M}(H)$ are as defined in (2.1). Thus, the only difference between the matroids obtained from (4.1) and those obtained from (2.1) is that those from (2.1) are loopless, while those from (4.1) may have loops. The simple underlying matroids are the same. In particular, in order to prove whether a given matroid is or is not hypergraphic, it is sufficient to consider that simplification of the matroid which deletes its loops.

PROPOSITION 4.3: If $\underline{M}$ is a hypergraphic matroid on the set E and $E' \subseteq E$, then $\underline{M}\times E'$ is hypergraphic.

Proof: (4.2) and (2.12).

PROPOSITION 4.4: The class of hypergraphic matroids is not closed under the operation of contraction.

Proof: We shall present a hypergraphic matroid, and a contraction of that matroid which is not hypergraphic.

Let $\underline{M}_1$ be the free, rank-preserving one-point extension by x (see (3.15) for the definition) of the matroid $\underline{M}(H)$, where H = (V, E, \$) is the following 2-hypergraph:

V = {A,B,C,D,E}$_{\neq}$

E = {a,b,c,d,e,f,g}$_{\neq}$

V(a) = {A,B}, V(b) = {A,C}, V(c) = {B,C}, V(d) = {A,D}, V(e) = {B,D}, V(f) = {A,E} and V(g) = {B,E}.

Then, by (3.16), $\underline{M}_1$ is hypergraphic; it has circuits {a,b,c}, {a,d,e}, {a,f,g}, {b,c,d,e}, {b,c,f,g}, {d,e,f,g} and all 5-subsets of $E\cup${x} containing none of these. $\underline{M}_1$ therefore has rank 4.

Contracting x yields the matroid $\underline{M}_1''$ on the set E with rank 3 and circuits {a,b,c}, {a,d,e}, {a,f,g} and all 4-subsets of E containing none of these. By (3.4) $\underline{M}_1''$ is not hypergraphic.

Thus, the class of hypergraphic matroids fails to be closed under the operation of contraction.

Since every minor of a hypergraphic matroid is isomorphic to the contraction of a restriction of a hypergraphic matroid, the class of minors of hypergraphic matroids is the same as the class of contractions of hypergraphic matroids.

DEFINITION 4.5: A matroid isomorphic to the contraction of a hypergraphic matroid is called a generalised hypergraphic matroid.

We shall now derive a method of defining generalised hypergraphic matroids in terms of a submodular function, which is analogous to that described for hypergraphic matroids in Chapter 1. In order to explain the motivation of the method, consider the hypergraph H = (V, E, \$) which consists of K_6^3 on the vertex-set V = {A,B,C,D,E,F}$_{\neq}$, together with an extra edge e with V(e) = {A,E,F}. The matroid is shown in Euclidean representation in Figure 8, with the points a $\in$ E-{e} labelled by V(a).

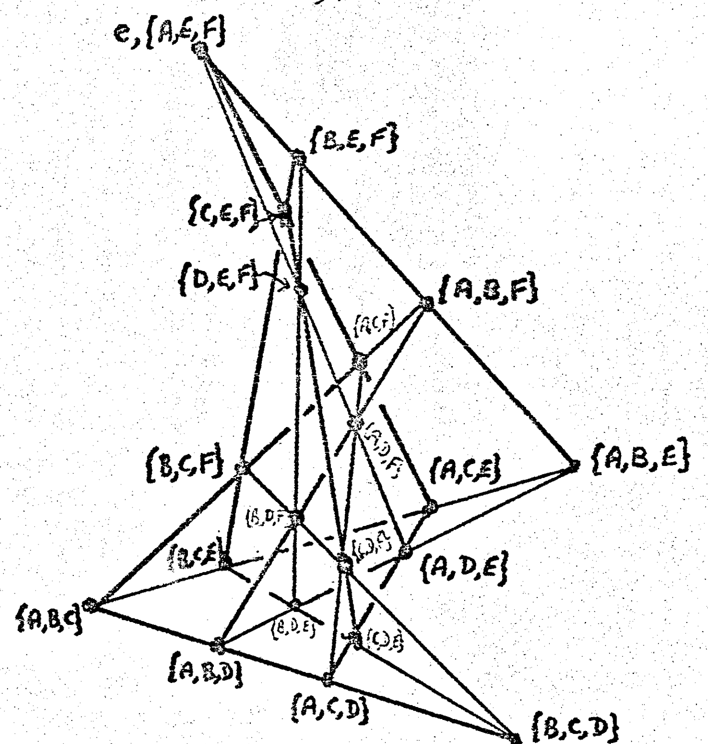


Figure 8

If we now contract e, we obtain the matroid shown in Euclidean representation in Figure 9, with the points $a \in E-\{e\}$ again labelled with V(a). We can see from this that a set of parallel elements of this matroid is a set of edges whose vertex-sets are the 3-subsets of {A,E,F,X}, where $X \in$ {B,C,D}, excluding {A,E,F} itself, and that the edge with vertex-set {A,E,F} has become a loop.

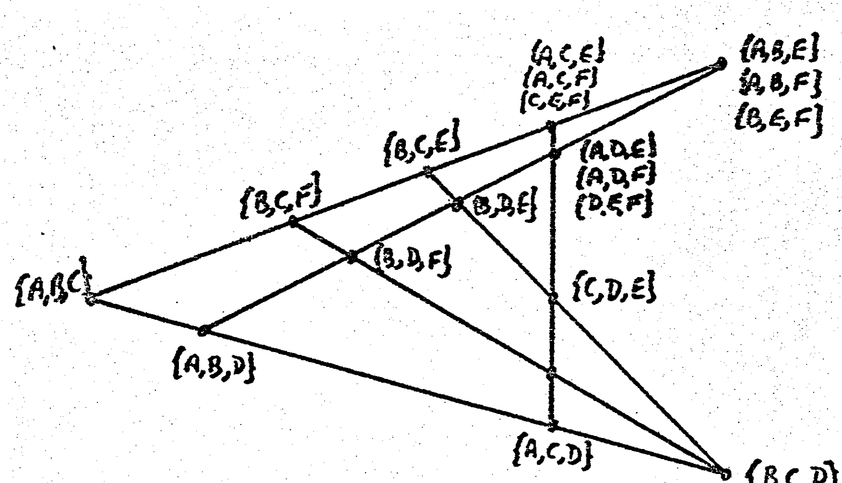


Figure 9

Now, if we are to maintain any sort of hypergraphic structure, we would expect that, in some sense, the "cardinality" of the set {A,E,F} should now be less than 3. Accordingly, we define a function

$\mu: 2^V \to \mathbb{Z}$ such that

$$\mu(W) = \begin{cases} |W| & \text{if } \{A,E,F\} \not\subseteq W \\ |W| - 1 & \text{if } \{A,E,F\} \subseteq W. \end{cases}$$

It is a matter of routine to check that the matroid after the contraction has independent sets $\mathcal{A} \subseteq \mathcal{E}-\{e\}$, where $\mathcal{A}$ is independent if and only if either $\mathcal{A} = \phi$ or $\mu V(\mathcal{G}) \geq |\mathcal{G}| + k - 1$ for each nonempty subset $\mathcal{G}$ of $\mathcal{A}$, where, in this case, $k = 3$.

We now proceed to the case of contraction of a general k-hypergraph.

Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph with $k \geq 2$. Let $\mathcal{K} = \{e_1, e_2, \ldots, e_m\}_{\neq}$ be an independent set of $\underline{M}(H)$, defined by (2.1).

DEFINITION 4.6: For $W \subseteq V$, define $\nu(W) = |\{i: V(e_i) \subseteq W, e_i \in \mathcal{K}\}|$.

PROPOSITION 4.7: ν is increasing and supermodular.

Proof: (i) ν is increasing.

Let $U \subseteq W \subseteq V$. Then
$$\begin{aligned} \nu(U) &= |\{i: V(e_i) \subseteq U, e_i \in \mathcal{K}\}| \\ &\leq |\{i: V(e_i) \subseteq W, e_i \in \mathcal{K}\}| \\ &= \nu(W). \end{aligned}$$

(ii) ν is supermodular. Let $U,W \subseteq V$.

Then
$$\begin{aligned} \nu(U \cup W) &= |\{i: V(e_i) \subseteq U \cup W, e_i \in \mathcal{K}\}| \\ &\geq |\{i: V(e_i) \subseteq U \text{ or } V(e_i) \subseteq W, e_i \in \mathcal{K}\}| \qquad (1) \\ \nu(U \cap W) &= |\{i: V(e_i) \subseteq U \cap W, e_i \in \mathcal{K}\}| \\ &= |\{i: V(e_i) \subseteq U \text{ and } V(e_i) \subseteq W, e_i \in \mathcal{K}\}| \end{aligned}$$

$$\begin{aligned} \therefore \nu(U \cup W) + \nu(U \cap W) &\geq |\{i: V(e_i) \subseteq U \text{ or } V(e_i) \subseteq W, e_i \in \mathcal{K}\}| \\ &\quad + |\{i: V(e_i) \subseteq U \text{ and } V(e_i) \subseteq W, e_i \in \mathcal{K}\}| \\ &= |\{i: V(e_i) \subseteq U, e_i \in \mathcal{K}\}| + |\{i: V(e_i) \subseteq W, e_i \in \mathcal{K}\}| \\ &= \nu(U) + \nu(W) \end{aligned}$$

PROPOSITION 4.8: Let $U,W \subseteq V$. Then $\nu(U \cup W) + \nu(U \cap W) = \nu U + \nu W$ if and only if, for each $e_i \in \mathcal{K}$ with $V(e_i) \subseteq U \cup W$, either $V(e_i) \subseteq U$ or $V(e_i) \subseteq W$.

Proof: (=>). If $\nu(U \cup W) + \nu(U \cap W) = \nu U + \nu W$, equality must hold in (1) of the proof of (4.7). Thus, $V(e_i) \subseteq U$ or $V(e_i) \subseteq W$ for each $e_i \in \mathcal{K}$.

(<=) If $V(e_i) \subseteq U \cup W \Rightarrow V(e_i) \subseteq U$ or $V(e_i) \subseteq W$ for each $e_i \in \mathcal{K}$, equality holds in (1) of the proof of (4.7), and so

$\nu(U \cup W) + \nu(U \cap W) = \nu U + \nu W$.

DEFINITION 4.9: Let $\nu: 2^V \to \mathbb{Z}$ be defined by (4.6). Define

$\mu: 2^V \to \mathbb{Z}$ by $\mu W = \min\{|U| - \nu U: W \subseteq U \subseteq V\}$.

PROPOSITION 4.10: μ is increasing and submodular, and $0 \leq \mu X \leq |X|$ for each $X \subseteq V$.

Proof: (i) μ is increasing. Let $X \subseteq Y \subseteq V$

$$\begin{aligned}\text{Then } \mu X &= \min\{|U| - \nu U: X \subseteq U \subseteq V\} \\ &\leq \min\{|U| - \nu U: Y \subseteq U \subseteq V\} \\ &= \mu Y\end{aligned}$$

(ii) μ is submodular. Let $X, Y \subseteq V$, and suppose $\mu X = |U| - \nu U$, $\mu Y = |W| - \nu W$ for $X \subseteq U$, $Y \subseteq W$.

$$\begin{aligned}\text{Then } \mu X + \mu Y &= |U| - \nu U + |W| - \nu W \\ &= |U \cup W| + |U \cap W| - (\nu U + \nu W) \\ &\geq |U \cup W| + |U \cap W| - (\nu(U \cup W) + \nu(U \cap W)) \text{ by (4.7)} \\ &= (|U \cup W| - \nu(U \cup W)) + (|U \cap W| - \nu(U \cap W)) \\ &\geq \mu(X \cup Y) + \mu(X \cap Y), \text{ since } X \cup Y \subseteq U \cup W \text{ and } X \cap Y \subseteq U \cap W\end{aligned}$$

(iii) $\mu X \leq |X|$

$\mu X = \min\{|Y| - \nu Y: X \subseteq Y \subseteq V\} \leq |X| - \nu X \leq |X|$

(iv) $\mu X \geq 0$.

Let $Y \supseteq X$, and let $\mathcal{K}' = \{e_i \in \mathcal{K}: V(e_i) \subseteq Y\}$.

Then $|\mathcal{K}'| = \nu Y$ and $V(\mathcal{K}') \subseteq Y$. Now, $\mathcal{K}' \subseteq \mathcal{K}$ and $\mathcal{K}$ is independent in $\underline{M}(H)$, so $|V(\mathcal{K}')| \geq |\mathcal{K}'| + k - 1$, if $\mathcal{K}' \neq \phi$.

$\therefore \quad |Y| \geq \nu Y + k - 1$, if $\nu Y \neq 0$.

$\therefore \quad |Y| - \nu Y \geq k - 1$, if $\nu Y \neq 0$.

If $\nu Y = 0$, $|Y| - \nu Y \geq 0$.

Thus, since $\mu X = \min\{|Y| - \nu Y : X \subseteq Y \subseteq V\}$, $\mu X \geq 0$.

PROPOSITION 4.11: Let $G \subseteq E-K$. Then $\mu V(G) \geq |G| + k - 1$ if and only if $|V(G \cup K')| \geq |G \cup K'| + k - 1$ for each subset K' of K.

Proof: Since K is independent in $\underline{M}(H)$, we have $|V(K')| \geq |K'| + k - 1$ for each nonempty subset K' of K.

(i) Assume $\mu V(G) \geq |G| + k - 1$. Let $K' \subseteq K$. Then

$\nu(V(G \cup K')) \geq |K'|$, so $\mu(V(G \cup K')) \leq |V(G \cup K')| - |K'|$.

$$\begin{aligned} \therefore \quad |V(G \cup K')| &\geq \mu(V(G \cup K')) + |K'| \\ &\geq \mu V(G) + |K'| \quad \text{since } \mu \text{ is increasing} \\ &\geq |G| + k - 1 + |K'| \quad \text{by hypothesis} \\ &= |G \cup K'| + k - 1 \end{aligned}$$

(ii) Assume $|V(G \cup K')| \geq |G \cup K'| + k - 1$ for each subset K' of K.

Let $U \subseteq V$ be such that $V(G) \subseteq U$ and $\mu V(G) = |U| - \nu U$.

Let $K'' = \{e \in K : V(e) \subseteq U\}$. (1)

$$\text{Then} \quad V(G \cup K'') = V(G) \cup V(K'') \subseteq U \qquad (2)$$

$$\begin{aligned} \mu V(G) &= |U| - \nu U \\ &= |U| - |K''| \quad \text{by (1)} \\ &\geq |V(G \cup K'')| - |K''| \quad \text{by (2)} \\ &\geq |G \cup K''| + k - 1 - |K''| \quad \text{by hypothesis} \\ &= |G| + k - 1 \quad \text{since } G \cap K'' = \phi. \end{aligned}$$

Since μ is increasing, integer-valued and submodular, it can be used to define a matroid on $E-K$ in the way indicated in the following definition. For a proof that this does yield a matroid, see Crapo-Rota [6].

DEFINITION 4.12: With the notation of the previous propositions, define $\underline{M}''(H)$ to be the matroid on the set $E-K$ whose independent sets are those $A \subseteq E-K$ for which either $A = \phi$ or $\mu V(G) \geq |G| + k - 1$ for each nonempty subset G of A.

THEOREM 4.13: $\underline{M}''(H) = \underline{M}(H).(E-K)$.

Proof: (i) Let $A \neq \phi$ be an independent set of $\underline{M}''(H)$.

Then $\mu V(G) \geq |G| + k - 1$ for each nonempty subset G of A.
So, by (4.11), $|V(G \cup K')| \geq |G \cup K'| + k - 1$ for each nonempty subset G of A and each subset K' of K. Also, since K is independent in $\underline{M}(H)$, $|V(K'')| \geq |K''| + k - 1$ for each nonempty subset K'' of K.

Thus, $|V(B)| \geq |B| + k - 1$ for each nonempty subset B of $A \cup K$.
Thus $A \cup K$ is independent in $\underline{M}(H)$.

(ii) Suppose $A \cup K$ is independent in $\underline{M}(H)$, with $A \neq \phi$ and $A \subseteq E-K$.
Then $|V(B)| \geq |B| + k - 1$ for each nonempty subset B of $A \cup K$.
So, in particular, $|V(G \cup K')| \geq |G \cup K'| + k - 1$ for each nonempty subset G of A and each subset K' of K. Therefore, by (4.11), $\mu V(G) \geq |G| + k-1$ for each nonempty subset G of A. Thus, A is independent in $\underline{M}''(H)$.

Therefore, A is independent in $\underline{M}''(H)$ if and only if $A \cup K$ is independent in $\underline{M}(H)$. But the set of $A \subseteq E-K$ for which $A \cup K$ is independent in $\underline{M}(H)$ is the set of independent sets of $\underline{M}(H).(E-K)$.

Thus, $\underline{M}''(H) = \underline{M}(H).(E-K)$.

In order to provide a convenient way of describing the contraction $\underline{M}(H).(E-K)$, we now introduce an object called a generalised hypergraph.

DEFINITION 4.14: With the notation of this chapter, given a k-hypergraph $H = (\mathcal{V}, E, \$)$ with $k \geq 2$, and a subset $K \subseteq E$ independent in $\underline{M}(H)$, the <u>generalised hypergraph produced from H by</u> K is defined to be the pair $(H,K) = ((\mathcal{V}, E, \$), K)$. Since the order of the sets in the brackets will remain fixed, we shall also write this as $(\mathcal{V}, E, \$, K)$.

DEFINITION 4.15: If $K = (V, E, \$, K)$ is a generalised hypergraph, the matroid $\underline{M}(K)$ is defined to be $\underline{M}(H).(E-K)$, where $H = (V, E, \$)$.

PROPOSITION 4.16: A matroid $\underline{M}$ is a generalised hypergraphic matroid if and only if $\underline{M} \cong \underline{M}(K)$ where K is a generalised hypergraph.

Proof: If $\underline{M} \cong \underline{M}(K)$, then clearly $\underline{M}$ is a generalised hypergraphic matroid.

Conversely, suppose that $\underline{M}$ is a generalised hypergraphic matroid. Then $\underline{M} \cong \underline{M}(H').(E'-S')$ where $H' = (V', E', \$')$, and $S' \subseteq E'$. Let L' denote the set of loops of $\underline{M}(H').(E'-S')$, and let K' be a maximal subset of S' which is independent in $\underline{M}(H')$. Let $H'' = H_{((E'-S')-L')\cup K'}$. Then H'' is a k-hypergraph for some $k \geq 2$, where, if $E(H'') \neq \phi$, $k = \max\{|V_{H'}(e')|: e' \in E'\}$. Choose $x \notin E'$, and let W be a set of vertices with $W \cap V' = \phi$ and $|W| = k$. Let L be a set of $|L'|$ distinct edges disjoint from $E' \cup \{x\}$ whose vertex-sets are all equal to W. Put $V = V' \cup W$, $E = ((E'-S')-L') \cup K' \cup L \cup \{x\}$, $K = K' \cup \{x\}$ and
$\$ = \{(v,e) \in \$': e \in (E-L)-\{x\}\} \cup \{(v,e): v \in W, e \in L \cup \{x\}\}$.
Then $H = (V, E, \$)$ is a k-hypergraph with $k \geq 2$, such that K is independent in $\underline{M}(H)$, and $\underline{M} \cong \underline{M}(H).(E-K)$. So, writing $K = (V, E, \$, K)$, $\underline{M} \cong \underline{M}(K)$, as required.

PROPOSITION 4.17: If $K = (V, E, \$, K)$ is a generalised hypergraph with $K = \{e_1, e_2, \ldots, e_m\}_{\neq}$, then $\underline{M}(K)$ is the matroid whose independent sets are those $A \subseteq E-K$ for which $A = \phi$ or $\mu V(G) \geq |G| + k - 1$ for each nonempty subset G of A, where $\mu X = \min\{|Y| - |\{i: V(e_i) \subseteq Y\}|: X \subseteq Y \subseteq V\}$

Proof: (4.12), (4.15) and (4.13).

PROPOSITION 4.18: With the notation of this chapter, let $K = (V, E, \$, K)$ be a generalised hypergraph. Then, if C is a circuit of $\underline{M}(K)$, $\mu V(C) = |C| + k - 2$.

Proof: Since $\mathcal{C}$ is dependent in $\underline{M}(K)$, there exists a nonempty subset

$\mathcal{C}' \subseteq \mathcal{C}$ with $\mu V(\mathcal{C}') < |\mathcal{C}'| + k - 1$ (1)

Since (1) cannot hold for any proper nonempty subset $\mathcal{C}'$ of $\mathcal{C}$ (because each such $\mathcal{C}'$ is independent), $\mu V(\mathcal{C}) < |\mathcal{C}| + k - 1$. (2)

Now, if $c \in \mathcal{C}$, $\mathcal{C} - \{c\}$ isindependent, so $\mu V(\mathcal{C}-\{c\}) \geq |\mathcal{C}-\{c\}| + k - 1$

$= |\mathcal{C}| + k - 2$ (3)

Combining (2) and (3), since, by (4.10), μ is increasing,

$\mu V(\mathcal{C}) = |\mathcal{C}| + k - 2$.

PROPOSITION 4.19: With the notation of this chapter, if $X \subseteq V$ and $\mu X < k - 1$, then $\mu X = |X|$.

Proof: Let $Y \subseteq V$ be such that $X \subseteq Y$ and $\mu X = |Y| - \nu Y$.

Then $|Y| - \nu Y < k - 1$ (1)

Let $K' = \{e_i \in K: V(e_i) \subseteq Y\}$. Then $|K'| = \nu Y$ and $V(K') \subseteq Y$.

So, from (1), $|V(K')| - |K'| < k - 1$

$\therefore$ $|V(K')| < |K'| + k - 1$.

Thus, if $K' \neq \phi$, K' is not independent in $\underline{M}(H)$. But $K' \subseteq K$ and K is independent in $\underline{M}(H)$, which is a contradiction. Thus $K' = \phi$ and so

$|Y| = \mu X$. Now, $\mu X \leq |X|$ by (4.10), so

$|Y| = \mu X \leq |X| \leq |Y|$ since $Y \supseteq X$.

Thus, $\mu X = |X|$.

Having defined the class of generalised hypergraphic matroids so that it is closed under the operation of taking minors, it is natural to ask whether it is closed under other matroid operations, such as truncation or the taking of duals. The question of duality will be left to a later chapter. However, the class isclosed under the operation of truncation, as we shall now prove.

LEMMA 4.20: If $\underline{M}$ is a matroid of rank r on the set S, and A is an independent subset of S, then $(\underline{M}.(S-A))^{(t)} = \underline{M}^{(t+|A|)}.(S-A)$ $(0 \leq t \leq r - \rho A)$.

Proof: B is a base of $\underline{M}.(S-A)$ if and only if $B\cup A$ is a base of $\underline{M}$. Thus, $Z \subseteq S-A$ is an independent set of $(\underline{M}.(S-A))^{(t)}$ if and only if $|Z| \leq t$ and $Z \subseteq B$ for some base B of $\underline{M}.(S-A)$; so, $Z \subseteq S-A$ is an independent set of $\underline{M}.(S-A)^{(t)}$ if and only if $|Z\cup A| \leq t + |A|$ and $Z\cup A$ is an independent set of $\underline{M}$. Hence $Z \subseteq S-A$ is an independent set of $(\underline{M}.(S-A))^{(t)}$ if and only if $Z\cup A$ is an independent set of $\underline{M}^{(t+|A|)}$, i.e. if and only if Z is an independent set of $\underline{M}^{(t+|A|)}.(S-A)$.

Thus $(\underline{M}.(S-A))^{(t)} = \underline{M}^{(t+|A|)}.(S-A)$.

Thus, in order to prove that every truncation of a generalised hypergraphic matroid is generalised hypergraphic, it is sufficient to prove that every truncation of a hypergraphic matroid is generalised hypergraphic.

PROPOSITION 4.21: Let $\underline{M}$ be a matroid of rank r on the set S, and let $x_1,x_2,\ldots,x_t \notin S$ $(t \leq r)$ be distinct elements. Define $\underline{M}_t$ to be the free rank-preserving t-point extension of $\underline{M}$ by $\{x_1,x_2,\ldots,x_t\}$. Then $\underline{M}^{(r-t)} = \underline{M}_t.S$.

Proof: The bases of $\underline{M}^{(r-t)}$ are those sets I independent in $\underline{M}$ with $|I| = r-t$ (since $t \leq r$). The bases of $\underline{M}_t.S$ are those $J \subseteq S$ such that $J\cup\{x_1,x_2,\ldots,x_t\}$ is a base of $\underline{M}_t$. Now, by (3.15), $J\cup\{x_1,x_2,\ldots,x_t\}$ is a base of $\underline{M}_t$ if and only if J is independent in $\underline{M}$ and $|J| = r-t$ (since $t \leq r$). Thus, the bases of $\underline{M}^{(r-t)}$ are the same as the bases of $\underline{M}_t.S$ and so $\underline{M}^{(r-t)} = \underline{M}_t.S$.

PROPOSITION 4.22: If $\underline{M}$ is a generalised hypergraphic matroid of rank r and $t \leq r$, $\underline{M}^{(r-t)}$ is generalised hypergraphic.

Proof: Since $\underline{M}$ is generalised hypergraphic, by (4.16) there exists a uniform hypergraph H on the set E such that $\underline{M} \cong \underline{M}(H).(E-K)$, where K is independent in $\underline{M}(H)$. With the notation of (4.21), define $\underline{M}(H)_t$.

Then, by (3.14), $\underline{M}(H)_t$ is hypergraphic.

By (4.21) $\underline{M}(H)^{(rk(\underline{M}(H)-t)} = \underline{M}(H)_t . E$

Now, $rk(\underline{M}(H)) = r + h$, where $h = |K|$, so

$$\underline{M}(H)^{(r+h-t)} = \underline{M}(H)_t . E$$

$$\therefore \quad \underline{M}(H)^{(r+h-t)}.(E-K) = (\underline{M}(H)_t . E).(E-K)$$

$$= \underline{M}(H)_t .(E-K).$$

By (4.20), $\underline{M}(H)^{(r+h-t)}.(E-K) = (\underline{M}(H).(E-K))^{(r-t)}$

$$\cong \underline{M}^{(r-t)}$$

Thus, $\underline{M}^{(r-t)} \cong \underline{M}(H)_t .(E-K)$, and so $\underline{M}^{(r-t)}$ is generalised hypergraphic.

PROPOSITION 4.23: If $\underline{M}_1$ and $\underline{M}_2$ are generalised hypergraphic matroids on disjoint sets, then $\underline{M}_1 \oplus \underline{M}_2$ is generalised hypergraphic.

Proof: follows from (3.12), (4.16) and contraction.

We close this chapter by developing some properties of the functions μ and ν defined earlier.

DEFINITION 4.24: With the notation of this chapter, the (μ-) <u>closure operator</u> $< >$ on V is defined by

$\langle X \rangle = \{v \in V: \mu(X \cup \{v\}) = \mu X\}$.

A set X of vertices for which $\langle X \rangle = X$ will be called <u>closed</u>.

From (4.10) it is clear that μ is the rank function of a matroid on V. We may therefore use the results of matroid theory to prove some results about the closure operator.

PROPOSITION 4.25: With the notation of this chapter, let $U, W \subseteq V$, and let $X, Y \subseteq V$ be closed sets. Then:

(a) $U \subseteq \langle U \rangle$

(b) if $U \subseteq W$, then $\langle U \rangle \subseteq \langle W \rangle$ (continued on next page)

(c) $\mu\langle U\rangle = \mu U$

(d) $\langle U\rangle\cup\langle W\rangle \subseteq \langle U\cup W\rangle$

(e) $\langle U\rangle\cap\langle W\rangle \supseteq \langle U\cap W\rangle$

(f) $\langle X\cap Y\rangle = X\cap Y$.

Proof: These are all elementary results in matroid theory and we omit the details.

PROPOSITION 4.26: With the notation of this chapter, if X is a closed set, then $\mu X = |X| - \nu X$.

Proof: We have $\mu X = \min\{|U| - \nu U: X \subseteq U \subseteq V\}$.
Let Y be a maximal set such that $X \subseteq Y \subseteq V$ and $\mu X = |Y| - \nu Y$.
Then, since $\mu Y \leq |Y| - \nu Y = \mu X$ and μ is increasing, $\mu Y = \mu X$.
Thus, $Y \subseteq \langle X\rangle$. For, if $y \in Y$, $\mu X \leq \mu(X\cup\{y\}) \leq \mu Y = \mu X$. Since X is closed, $X = \langle X\rangle$, so, because $X \subseteq Y \subseteq \langle X\rangle$, $Y = X$, and $\mu X = |X| - \nu X$.

PROPOSITION 4.27: With the notation of this chapter, let $\mathcal{A}$ be a set of edges of the generalised hypergraph $K = (V, \mathcal{E}, \$, \mathcal{K})$. Then

$$\mu V(\mathcal{A}) \geq \rho\mathcal{A} + k - 1, \text{ where } \rho \text{ is the rank function of } \underline{M}(K).$$

Proof: Let $\mathcal{B}$ be a maximal independent subset of $\mathcal{A}$. Then $|\mathcal{B}| = \rho\mathcal{A}$, and $\mu V(\mathcal{B}) \geq |\mathcal{B}| + k - 1$. Since μ is increasing,
$\mu V(\mathcal{A}) \geq \mu V(\mathcal{B}) \geq |\mathcal{B}| + k - 1 = \rho\mathcal{A} + k - 1$.

PROPOSITION 4.28: With the notation of this chapter, let $\mathcal{A},\mathcal{B}$ be sets of edges of the generalised hypergraph $K = (V, \mathcal{E}, \$, \mathcal{K})$, such that there exist integers $t,r > 0$ with $\mu V(\mathcal{A}) = t+r$, $\mu V(\mathcal{A}\cup\mathcal{B}) \geq \mu V(\mathcal{B}) + t$, and such that there exists $\mathcal{C} \subseteq \mathcal{A}\cap\mathcal{B}$ with $\mu V(\mathcal{C}) = r$. Then $\langle V(\mathcal{A})\rangle\cap\langle V(\mathcal{B})\rangle = \langle V(\mathcal{C})\rangle$.

Proof: By (4.25)(c), $\mu\langle V(\mathcal{A})\rangle = t+r$, $\mu\langle V(\mathcal{A}\cup\mathcal{B})\rangle = \mu V(\mathcal{A}\cup\mathcal{B})$ and $\mu\langle V(\mathcal{B})\rangle = \mu V(\mathcal{B})$.
By submodularity of μ, (4.10),

$$\mu\langle V(A)\rangle + \mu\langle V(B)\rangle \geq \mu(\langle V(A)\rangle \cup \langle V(B)\rangle) + \mu(\langle V(A)\rangle \cap \langle V(B)\rangle)$$

$$\geq \mu(V(A) \cup V(B)) + \mu(\langle V(A)\rangle \cap \langle V(B)\rangle)$$ since μ is increasing

$$\geq \mu\langle V(B)\rangle + t + \mu(\langle V(A)\rangle \cap \langle V(B)\rangle)$$

$$= \mu\langle V(A)\rangle + \mu\langle V(B)\rangle - r + \mu(\langle V(A)\rangle \cap \langle V(B)\rangle)$$

$$\therefore \quad r \geq \mu(\langle V(A)\rangle \cap \langle V(B)\rangle) \qquad (1)$$

Now, $\langle V(A)\rangle \cap \langle V(B)\rangle \supseteq V(A) \cap V(B) \supseteq V(C)$, since $C \subseteq A \cap B$.

$\therefore$ by (4.25)(b) $\langle\langle V(A)\rangle \cap \langle V(B)\rangle\rangle \supseteq \langle V(C)\rangle$

$\therefore$ by (4.25)(f) $\langle V(A)\rangle \cap \langle V(B)\rangle \supseteq \langle V(C)\rangle$.

But, from (1), $\mu(\langle V(A)\rangle \cap \langle V(B)\rangle) \leq r$. Thus, since $\mu V(C) = r$,

$\langle V(A)\rangle \cap \langle V(B)\rangle = \langle V(C)\rangle$.

PROPOSITION 4.29: With the notation of this chapter, if $a \in \sigma(A)$, then $\langle V(a)\rangle \subseteq \langle V(A)\rangle$ and $\mu V(A \cup \{a\}) = \mu V(A)$.

Proof: If ~~a~~ $\in A$, there is nothing to prove. So assume $a \notin A$. Let $B \subseteq A$ be such that $\{a\} \cup B$ is a circuit of $\underline{M}(K)$.

Then $\mu V(B \cup \{a\}) = |B \cup \{a\}| + k - 2$ by (4.18) (1)

and, since B is independent $\mu V(B) \geq |B| + k - 1 = |B \cup \{a\}| + k - 2$ (2)

Thus, since μ is increasing, $\mu V(B) = \mu V(B \cup \{a\})$ (3)

By submodularity of μ, $\mu V(B \cup \{a\}) + \mu V(A) \geq \mu V(A \cup \{a\}) + \mu(V(B \cup \{a\}) \cap V(A))$

$\geq \mu V(A) + \mu V(B)$ since μ is increasing

$= \mu V(A) + \mu V(B \cup \{a\})$ by (3)

Thus, equality holds throughout, and so $\mu V(A \cup \{a\}) = \mu V(A)$.

Therefore, $\mu V(A \cup \{a\}) = \mu V(A)$ and so $\langle V(a)\rangle \subseteq \langle V(A)\rangle$.

For convenience, in many of the following chapters we shall restrict our attention to generalised hypergraphs arising from critical hypergraphs. That there is no loss of generality in doing so, follows from the next theorem.

THEOREM 4.30: $\underline{M}$ is a generalised hypergraphic matroid if and only if $\underline{M} \cong \underline{M}(H).(E-K)$ where $H = (V, E, \$)$ is a critical k-hypergraph for some $k \geq 2$, and K is independent in $\underline{M}(H)$.

Proof: Suppose $\underline{M}$ is a generalised hypergraphic matroid. Then $\underline{M} \cong \underline{M}(H').(E'-S')$ for some hypergraph $H' = (V', E', \$')$, $S' \subseteq E'$, and $\underline{M}(H')$ is defined by some $k \geq 2$. Let K' be a maximal independent subset of S', and let L' denote the set of loops of $\underline{M}(H').(E'-S')$. Then $H'_{((E'-S')-L')\cup K'}$ is a k-hypergraph, and so, by (2.13), there exists a critical k-hypergraph $H'' = (V'', E'', \$'')$ with $\underline{M}(H'') \cong \underline{M}(H'_{((E'-S')-L')\cup K'})$, where $E'' = ((E'-S')-L')\cup K'$, and K' is independent in $\underline{M}(H'')$. Choose $x \notin E'$, and define $V_{H''}(x) = \{A\}\cup W$, where $A \notin V''$, and $W \subseteq V''$ with $|W| = k - 1$. Let L be a set of $|L'|$ distinct edges, disjoint from $E'\cup\{x\}$, whose vertex-sets are all equal to $V_{H''}(x)$. Let H be the k-hypergraph $(V, E, \$)$ where $V = V''\cup\{A\}$, $E = E''\cup L\cup\{x\}$ and $\$ = \$''\cup\{(v,e): v \in V_{H''}(x),\ e \in L\cup\{x\}\}$. Put $K = K'\cup\{x\}$. Then clearly $\underline{M}(H).(E-K) \cong \underline{M}$, and K is independent in $\underline{M}(H)$, and H is a critical k-hypergraph, since

$$|V(H)| = |V(B\cup\{x\})| = |V(B)| + 1 = rk\underline{M}(H'') + k$$
$$= rk\underline{M}(H) + k - 1$$

where B is a base of $\underline{M}(H'')$. Since $A \in V(x)-V(B)$, $B\cup\{x\} \in \mathbf{I}(\underline{M}(H))$ so $B\cup\{x\}$ is critical.

The converse is immediate.

THEOREM 4.31: If $K = (V, E, \$, K)$ where $H = (V, E, \$)$ is a critical k-hypergraph, then $\mu V = \mu V(E) = \mu V(E-K) = k + \rho(E-K) - 1$, where ρ is the rank function of $\underline{M}(K)$.

Proof: Since H is critical, $|V| = k + rk\underline{M}(H) - 1$ and $V(E) = V$. Thus,

$$\mu V = |V| - |K| = k - 1 + (rk(\underline{M}(H) - |K|)$$
$$= k - 1 + \rho(E-K).$$

Also, by (4.27), $\mu V(E-K) \geq \rho(E-K) + k - 1$. Since μ is increasing, $\mu V \geq \mu V(E-K)$, and the result now follows.

CHAPTER 5

TRANSVERSAL MATROIDS AND GAMMOIDS

We now turn briefly to the class of transversal matroids and gammoids. We shall prove that transversal matroids are hypergraphic, and use a result of Ingleton and Piff [15] to prove that gammoids are generalised hypergraphic. We also show that whirls are hypergraphic.

Let $\underline{M}$ be a transversal matroid on the set $S = \{x_1, x_2, \ldots, x_n\}_{\neq}$. By a result proved by Mirsky in [20], if r is the rank of $\underline{M}$, there is a presentation of the independent sets of $\underline{M}$ as the partial transversals of a family $\mathbf{A} = (A_i : 1 \leq i \leq r)$, where $A_i \subseteq S$ for each i. Now, if $x \in S$ and $x \notin A_i$ for any i, x is a loop of $\underline{M}$. Since, by (4.2), a matroid is hypergraphic if and only if the minor obtained from it by deletion of its loops is hypergraphic, it is sufficient to prove that $\underline{M} \times (\bigcup_{i=1}^{r} A_i)$ is hypergraphic.

THEOREM 5.1: If $\underline{M}$ is a transversal matroid, then $\underline{M}$ is hypergraphic.

Proof: With the above notation, let $E' = \bigcup_{i=1}^{r} A_i$. We shall show that $\underline{M} \times E'$ is hypergraphic. By re-numbering if necessary, let $E' = \{x_1, x_2, \ldots, x_m\}_{\neq}$. Let E be an isomorphic copy of the set E', under the obvious bijection $\theta : E \to E'$, where $\theta(e_i) = x_i$.
For each i, $1 \leq i \leq m$, define $q_i = |\{j : x_i \in A_j\}|$. Then $q_i \geq 1$ for each i. For $1 \leq i \leq m$, let V_i be a set of vertices such that $|V_i| = q_i - 1$, and $V_i \cap V_j = \phi$ for $i \neq j$. Let W be the set $\{B_0, B_1, \ldots, B_r\}_{\neq}$ of vertices, where $W \cap V_i = \phi$ for each i, $1 \leq i \leq m$. Let $V = V_1 \cup V_2 \cup \cdots \cup V_m$.
Let $H = (V \cup W, E, \$)$ where, for each $e_i \in E$,
$V_H(e_i) = (V - V_i) \cup \{B_j : x_i \in A_j\} \cup \{B_0\}$ $(1 \leq i \leq m)$.

(i) H is a uniform hypergraph of cardinality at least 2.

For, $|V(e_i)| = |\mathcal{V}| - |\mathcal{V}_i| + |\{B_j : x_i \in A_j\}| + |\{B_0\}|$

$$= |\mathcal{V}| - (q_i - 1) + q_i + 1$$

$$= |\mathcal{V}| + 2.$$

Write $k = |\mathcal{V}| + 2$. Then $k \geq 2$.

(ii) $\underline{M}(H) \cong \underline{M} \times E'$, the isomorphism being that induced by the bijection θ.

Suppose $\{x_i : i \in I\}$ is independent in $\underline{M} \times E'$ for some $I \subseteq \{1,2,\ldots,m\}$.

Then, for any nonempty subset $J \subseteq I$,

$$|\{j : x_i \in A_j \text{ for some } i \in J\}| \geq |J| \qquad (1)$$

If $|J| = 1$, then e_i is independent in $\underline{M}(H)$, where $J = \{i\}$. If $|J| \geq 2$,

$$|\bigcup_J V(e_i)| = |\bigcup_J (\mathcal{V} - \mathcal{V}_i)| + |\bigcup_J \{B_j : x_i \in A_j\}| + |\{B_0\}|$$

$$= |\mathcal{V}| + |\{j : x_i \in A_j \text{ for some } i \in J\}| + 1$$

$$\geq |\mathcal{V}| + |J| + 1 \qquad \text{from (1)}$$

$$= |J| + k - 1$$

Thus, $\{e_i : i \in I\}$ is independent in $\underline{M}(H)$.

Conversely, suppose that $\{x_i : i \in I\}$ is a circuit of $\underline{M} \times E'$ for some $I \subseteq \{1,2,3,\ldots,m\}$. Then, since $\underline{M} \times E'$ is loopless, $|I| \geq 2$.

$$|\{j : x_i \in A_j \text{ for some } i \in I\}| < |I|$$

and so $|\bigcup_J V(e_i)| = |\mathcal{V}| + |\{j : x_i \in A_j \text{ for some } i \in I\}| + |\{B_0\}|$

$$< |\mathcal{V}| + |I| + 1$$

$$= |I| + k - 1.$$

Thus, $\{e_i : i \in I\}$ is dependent in $\underline{M}(H)$. Therefore, $\underline{M}(H) \cong \underline{M} \times E'$, and so $\underline{M} \times E'$ is hypergraphic. Thus, by (4.2), $\underline{M}$ is hypergraphic.

We now give two examples of the construction of (5.1).

EXAMPLE 1: Let $\underline{M} = U_{2,4}(\{x_1,x_2,x_3,x_4\})$. Then $\underline{M}$ can be represented as a transversal matroid using the family $\mathbf{A} = (A_1, A_2)$, where $A_1 = A_2 = \{x_1,x_2,x_3,x_4\}$. Then $q_i = 2$ for each i, and so we obtain the 6-hypergraph $H = (\mathcal{V} \cup \mathcal{B}\, E, \$)$, where $V(e_1) = \{V_2,V_3,V_4,B_0,B_1,B_2\}$,

$V(e_2) = \{V_1, V_3, V_4, B_0, B_1, B_2\}$, $\quad V(e_3) = \{V_1, V_2, V_4, B_0, B_1, B_2\}$

$V(e_4) = \{V_1, V_2, V_3, B_0, B_1, B_2\}$, $\quad W = \{B_0, B_1, B_2\}$.

The hypergraphic matroid $\underline{M}(H)$ is clearly isomorphic to $U_{2,4}$.

EXAMPLE 2. Let $\underline{M}$ be the transversal matroid whose presentation is $\mathbb{A} = (A_1, A_2, A_3, A_4)$, where $A_1 = \{x_1, x_2, x_3, x_4\}$, $A_2 = \{x_1, x_2, x_3, x_5\}$, $A_3 = A_4 = \{x_4, x_5\}$.

Then $q_1 = q_2 = q_3 = 2$, $q_4 = q_5 = 3$. Writing $V_4 = \{V_{41}, V_{42}\}$ and $V_5 = \{V_{51}, V_{52}\}$, we obtain the 9-hypergraph $H = (V \cup W, E, \$)$ where

$V(e_1) = \{V_2, V_3, V_{41}, V_{42}, V_{51}, V_{52}, B_0, B_1, B_2\}$

$V(e_2) = \{V_1, V_3, V_{41}, V_{42}, V_{51}, V_{52}, B_0, B_1, B_2\}$

$V(e_3) = \{V_1, V_2, V_{41}, V_{42}, V_{51}, V_{52}, B_0, B_1, B_2\}$

$V(e_4) = \{V_1, V_2, V_3, V_{51}, V_{52}, B_0, B_1, B_3, B_4\}$

$V(e_5) = \{V_1, V_2, V_3, V_{41}, V_{42}, B_0, B_2, B_3, B_4\}$

$W = \{B_0, B_1, B_2, B_3, B_4\}$

It can be checked that $\underline{M}(H) \cong \underline{M}$. Note, however, that although H is a hypergraph with $\underline{M}(H) \cong \underline{M}$, it is by no means the only one. Indeed, in this case, $\underline{M}$ is graphic, being the cycle matroid of the graph shown in Figure 10.

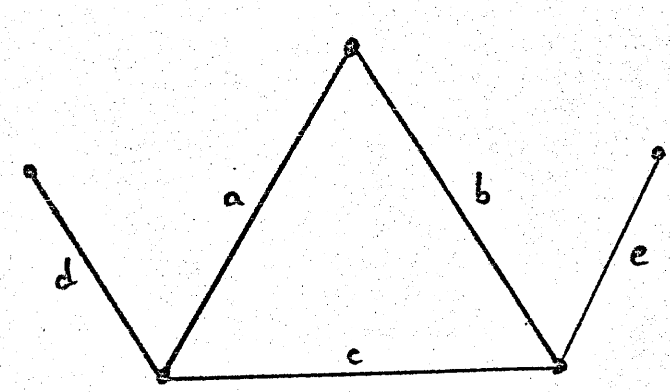


Figure 10

In [15], Ingleton and Piff show that strict gammoids are the duals of transversal matroids, and that every gammoid is the contraction of a transversal matroid. This latter result shows that gammoids are generalised hypergraphic matroids.

PROPOSITION 5.2: If $\underline{M}$ is a gammoid, then $\underline{M}$ is a generalised hypergraphic matroid.

Proof: By (5.1) every transversal matroid is hypergraphic, and so, by (4.5) every contraction of a transversal matroid is generalised hypergraphic.

In his paper [17], Mason gives, as an example of a strict gammoid, the matroid $\underline{M}$ shown in Euclidean representation in Figure 11.

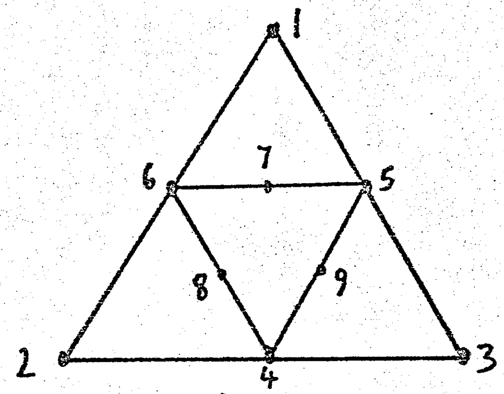


Figure 11

This has rank 3, and has circuits {4,3,2}, {4,6,8} and {4,5,9} (and others) and no circuits of cardinality three containing the element 4 except the three stated. Thus, by (3.4), $\underline{M}\times\{2,3,4,5,6,8,9\}$ is not hypergraphic, and so, by (2.12), $\underline{M}$ is not hypergraphic. Hence

PROPOSITION 5.3: If $\underline{M}$ is a gammoid, then $\underline{M}$ is a generalised hypergraphic matroid but not, in general, a hypergraphic matroid.

In [27], Tutte shows that the only 3-connected matroids in which every element is essential are the wheels and whirls. Wheels are graphic, and hence hypergraphic. It is easy to see that whirls are gammoids, and hence generalised hypergraphic. In fact, they are hypergraphic, as we now prove.

THEOREM 5.4: Let $\underline{M}$ be a whirl. Then $\underline{M}$ is hypergraphic.

Proof: Let $\underline{M}$ be the whirl W_n of order n on the set S defined as

$\{b_0,b_1,\ldots,b_{n-1},a_0,a_1,\ldots,a_{n-1}\}_{\neq}$, where $\underline{M}$ has circuits $\{a_i,a_{i+1},b_i\}$ (mod n) $(1 \leq i \leq n-1)$, the symmetric differences of sets of these given in Chapter 1, and the sets $\{b_0,b_1,\ldots,b_{n-1},a_i\}$ for each i. Let $E = \{e_0,e_1,\ldots,e_{n-1},x_0,x_1,\ldots,x_{n-1}\}_{\neq}$ and let $\theta: E \to S$ be defined by $\theta(e_i) = b_i$, $\theta(x_i) = a_i$ $(0 \leq i \leq n-1)$.

Let $A = \{A_0,A_1,\ldots,A_{n-1}\}_{\neq}$, $B = \{B_0,B_1,\ldots,B_{n-1}\}_{\neq}$ be disjoint sets of vertices, and let $V = A \cup B$. Define $H = (V, E, \$)$ to be the hypergraph with

$$V(x_i) = \{A_i\} \cup B \qquad (0 \leq i \leq n-1)$$

$$V(e_i) = \{A_i,A_{i+1}\} \cup (B-\{B_i\}) \text{ (mod n)} \quad (0 \leq i \leq n-1)$$

It is easy to check that $\underline{M}(H) \cong \underline{M}$.

For example, $V(\{x_i,x_{i+1},e_i\}) = \{A_i,A_{i+1}\} \cup B$ (mod n)

$$= V(\{x_i,x_{i+1}\}) = V(\{x_i,e_i\})$$

$$= V(\{x_{i+1},e_i\})$$

and so $\{x_i,x_{i+1},e_i\}$ is a circuit of $\underline{M}(H)$.

The symmetric differences of these given in Chapter 1 are also circuits, and the proof is of the same form.

$|V(\{e_i: i \in I\})| \geq n + |I|$ for each nonempty set $I \subseteq \{0,1,\ldots,n-1\}$, and so $\{e_0,e_1,\ldots,e_{n-1}\}$ is independent in $\underline{M}(H)$. Also, since $V(\{e_0,e_1,\ldots,e_{n-1}\}) = V$, $\{e_0,e_1,\ldots,e_{n-1}\}$ is spanning in $\underline{M}(H)$.

Now, x_i cannot form a circuit $\{x_i\} \cup \{e_j: j \in J\}$ for any J with $|J| < n$, since, for any such J, there exists $j \in J$ with $V(e_j) \nsubseteq V(\{x_i\} \cup \{e_m: m \in J-\{j\}\})$. Thus, $\{e_0,e_1,\ldots,e_{n-1},x_i\}$ is a circuit for each i, and so $\underline{M}(H) \cong \underline{M}$.

We therefore have the interesting result that the three-connected matroids in which every element is essential are all hypergraphic matroids. This result will be used in Chapter 11.

CHAPTER 6

DUALITY

The title of this chapter is slightly misleading. Our object is to use the results of Chapter 4 to show that neither $(\underline{M}(K_{3,3}))^*$ nor $(\underline{M}(K_5))^*$ is generalised hypergraphic. We use this result to make various deductions.

THEOREM 6.1: $(\underline{M}(K_{3,3}))^*$ is not a generalised hypergraphic matroid.

Proof: The graph $K_{3,3}$ is shown, suitably labelled, in Figure 12.

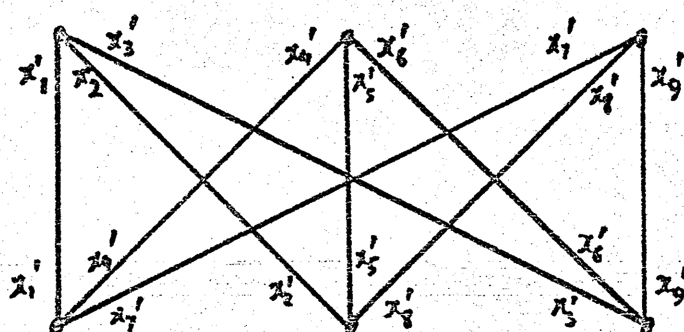


Figure 12

The circuits of $(\underline{M}(K_{3,3}))^*$, with the above labelling, are:

$C_1' = \{x_1', x_2', x_3'\}$ $\quad$ $C_2' = \{x_4', x_5', x_6'\}$ $\quad$ $C_3' = \{x_7', x_8', x_9'\}$

$D_1' = \{x_1', x_4', x_7'\}$ $\quad$ $D_2' = \{x_2', x_5', x_8'\}$ $\quad$ $D_3' = \{x_3', x_6', x_9'\}$

and all minimal symmetric differences of sets of these.

Suppose $(\underline{M}(K_{3,3}))^*$ is generalised hypergraphic. Then, by (4.30), there exists a generalised hypergraph $K = (V, E, \$, K)$ such that $E-K = \{x_1, x_2, \ldots, x_9\}$, $|V(x)| = k$ for each $x \in E$, $H = (V, E, \$)$ is a critical k-hypergraph for some $k \geq 2$, and $(\underline{M}(K_{3,3}))^* \cong \underline{M}(K)$, where the isomorphism is induced by the obvious map between $E-K$ and $\{x_1', x_2', \ldots, x_9'\}$. Denote the images of C_i', D_i' by C_i, D_i respectively.

By (4.18) $\mu V(C_i) = \mu V(D_j) = k+1 \qquad 1 \leq i,j \leq 3 \qquad (1)$

and $\qquad \mu V(C_i \Delta D_j) = k+2 \qquad 1 \leq i,j \leq 3 \qquad (2)$

and $\mu V(C_i \Delta D_j \Delta C_m) = \mu V(D_i \Delta C_j \Delta D_m) = k+3 \quad (1 \le i < m \le 3,\ 1 \le j \le 3)$ (3)

Since H is critical, by (4.31), $\mu V(E-K) = \mu V(E) = k+3$ (4)

Since $C_i \cap D_j \subseteq \sigma(C_i \Delta D_j)$, by (4.29), $\mu V(C_i \cup D_j) = \mu V(C_i \Delta D_j)$. So, from (2),

$\mu V(C_i \cup D_j) = k+2 \quad (1 \le i,j \le 3)$ (5)

Similarly, $\mu V(C_i \cup D_j \cup C_m) = \mu V(D_i \cup C_j \cup D_m) = k+3 \quad (1 \le i < m \le 3,\ 1 \le j \le 3)$ (6)

Write $\mathcal{C}_i = \langle V(C_i)\rangle$, $\mathcal{D}_i = \langle V(D_i)\rangle \quad (1 \le i \le 3)$

and $X_i = \langle V(x_i)\rangle \quad (1 \le i \le 9)$.

From (1) and (5), using (4.25),

$$\begin{aligned} k+1 + k+2 &= \mu C_1 + \mu\langle C_2 \cup D_1\rangle \\ &\ge \mu C_1 + \mu(C_2 \cup D_1) && \text{by (4.25)} \\ &\ge \mu(C_1 \cup C_2 \cup D_1) + \mu(C_1 \cap (C_2 \cup D_1)) && \text{by (4.10)} \\ &\ge \mu(C_1 \cup C_2 \cup D_1) + \mu X_1 && \text{since } \mu \text{ is increasing} \\ &= k+3 + k && \text{by (6).} \end{aligned}$$

Thus, equality holds throughout, so $C_1 \cap (C_2 \cup D_1) = X_1$, (7)

since $\mu(C_1 \cap (C_2 \cup D_1)) = \mu X_1$ and $C_1 \cap (C_2 \cup D_1) \supseteq X_1$.

In particular $C_1 \cap D_1 = X_1$, and similarly for other sets. (8)

Results similar to (7) hold for other sets C_i, D_j and C_m.

Thus, $\langle V(C_1)\rangle = \langle V(x_1)\rangle \cup \langle V(x_2)\rangle \cup \langle V(x_3)\rangle \cup V_1$ (9)

for some set $V_1 \subseteq V$ where $V_1 \cap \langle V(x_i)\rangle = \phi$ for $i = 1,2,3$,

$V_1 \cap \langle V(C_i)\rangle = \phi$ for $i = 2,3$ and $V_1 \cap \langle V(D_j)\rangle = \phi$ for $j = 1,2,3$.

Then (9) becomes $C_1 = X_1 \cup X_2 \cup X_3 \cup V_1$ (10)

Similarly,

$$\left.\begin{aligned} C_i &= \cup\{X_j : x_j \in C_i\} \cup V_i \\ D_i &= \cup\{X_j : x_j \in D_i\} \cup W_i \end{aligned}\right\} \quad (11)$$

where $V_i \cap X_j = V_i \cap W_m = \phi \quad (1 \le i, m \le 3,\ 1 \le j \le 9)$.

and $V_i \cap V_j = W_i \cap W_j = \phi \quad (1 \le i < j \le 3)$

By (4.26) $\nu C_i = |C_i| - \mu C_i$

so, by (11) and (1),

$$\left.\begin{aligned} \nu C_i &= |\cup\{X_j : x_j \in C_i\} \cup V_i| - k - 1 \\ \text{and similarly} \quad \nu D_i &= |\cup\{X_j : x_j \in D_i\} \cup W_i| - k - 1 \end{aligned}\right\} \quad (12)$$

Thus, $\nu C_1 + \nu D_1 = |X_1 \cup X_2 \cup X_3 \cup V_1| + |X_1 \cup X_4 \cup X_7 \cup W_1| - 2k-2$

$$= |X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_7 \cup V_1 \cup W_1| -k-2 + |X_1| - k \quad \text{by (7)}$$

$$\geq \nu(C_1 \cup D_1) + \nu X_1 \quad \text{by (5)}$$

$$\geq \nu C_1 + \nu D_1 \quad \text{by (4.7)}$$

Thus, equality holds throughout, and so $\nu(C_1 \cup D_1) = |C_1 \cup D_1| -k-2$

A similar result holds for other sets - i.e.

$$\nu(C_i \cup D_j) = |C_i \cup D_j| - k - 2 \quad (1 \leq i,j \leq 3) \qquad (13)$$

Now, $(C_1 \cup D_1) \cap (C_2 \cup D_2) = ((C_1 \cup D_1) \cap C_2) \cup ((C_1 \cup D_1) \cap D_2)$

$$= X_2 \cup X_4 \quad \text{by (7)} \qquad (14)$$

Write $X_J = \underset{J}{\cup} X_j$ and $X = X_1 \cup X_2 \cup \ldots \cup X_9$.

$\nu(C_1 \cup D_1) + \nu(C_2 \cup D_2) = |X_{\{1,2,3,4,7\}} \cup V_1 \cup W_1| - k - 2$

$$+ |X_{\{1,2,4,5,8\}} \cup V_2 \cup W_2| - k - 2 \quad \text{from (13)}$$

$$= |X_{\{1,2,3,4,5,6,7,8\}} \cup V_1 \cup V_2 \cup W_1 \cup W_2| - k - 3$$

$$+ |X_2 \cup X_4| - k - 1$$

$$\geq \nu(C_1 \cup C_2 \cup D_1 \cup D_2) + \nu((C_1 \cup D_1) \cap (C_2 \cup D_2)) \quad \text{by (6) and (14)}$$

$$\geq \nu(C_1 \cup D_1) + \nu(C_2 \cup D_2) \quad \text{by (4.7)}$$

Thus equality holds throughout, and so

$$\nu(C_1 \cup C_2 \cup D_1 \cup D_2) = |C_1 \cup C_2 \cup D_1 \cup D_2| - k - 3 \qquad (15)$$

A similar result holds for other sets C_i and D_j.

$\nu(C_1 \cup C_2 \cup D_1 \cup D_2) + \nu C_3 = |X_{\{1,2,3,4,5,6,7,8\}} \cup V_1 \cup V_2 \cup W_1 \cup W_2| - k - 3$

$$+ |X_{\{7,8,9\}} \cup V_3| - k - 1$$

$$= |X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2| -k-3 + |X_7 \cup X_8| - k-1 \quad \text{by (8)}$$

$$\geq \nu(C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2) + \nu(X_7 \cup X_8)$$

$$\geq \nu(C_1 \cup C_2 \cup D_1 \cup D_2) + \nu C_3 \quad \text{by (4.7)}$$

Thus, equality holds throughout, and so

$$\nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2) = |X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2| - k - 3 \qquad (16)$$

Similarly, $\nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_3) = |X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_3| - k - 3$

So, $\nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2) + \nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_3)$

$$= |X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2| - k-3 + |X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_3| - k-3$$

$$= |X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2 \cup W_3| - k-3 + |X \cup V_1 \cup V_2 \cup V_3 \cup W_1| - k-3$$

$$\geq \nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2 \cup W_3) + \nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1) \quad \text{by (6)}$$

$$\geq \nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_2) + \nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1 \cup W_3) \quad \text{by (4.7)}$$

Thus, equality holds throughout, and so

$$\nu(X \cup V_1 \cup V_2 \cup V_3 \cup W_1) = |X \cup V_1 \cup V_2 \cup V_3 \cup W_1| - k - 3$$

$$\text{i.e. } \nu(C_1 \cup C_2 \cup C_3 \cup D_1) = |C_1 \cup C_2 \cup C_3 \cup D_1| - k - 3 \qquad (17)$$

$$\begin{aligned}\nu(C_1 \cup D_1) + \nu(C_2 \cup D_1) &= |X_{\{1,2,3,4,7\}} \cup V_1 \cup W_1| - k - 2 \\ &\quad + |X_{\{1,4,5,6,7\}} \cup V_2 \cup W_1| - k - 2 \quad \text{from (12)} \\ &= |X_{\{1,2,3,4,5,6,7\}} \cup V_1 \cup V_2 \cup W_1| - k - 3 \\ &\quad + |X_{\{1,4,7\}} \cup W_1| - k - 1 \quad \text{from (7)} \\ &\geq \nu(C_1 \cup C_2 \cup D_1) + \nu D_1 \quad \text{from (1) and (4)} \\ &\geq \nu(C_1 \cup D_1) + \nu(C_2 \cup D_1) \quad \text{from (4.7)}\end{aligned}$$

Thus, equality holds throughout, and so

$$\nu(C_1 \cup C_2 \cup D_1) = |C_1 \cup C_2 \cup D_1| - k - 3 \qquad (18)$$

and similar results hold for other choices of C_i, C_j and D_m.

Now, $\nu(C_1 \cup C_2 \cup D_1) + \nu(C_3 \cup D_1)$

$$= |C_1 \cup C_2 \cup D_1| - k-3 + |C_3 \cup D_1| - k-2 \quad \text{by (18) and (13)}$$

$$= |C_1 \cup C_2 \cup C_3 \cup D_1| - k-3 + |D_1| - k-1 - 1 \quad \text{by (7)}$$

$$= |X \cup V_1 \cup V_2 \cup V_3 \cup W_1| - k-3 + |D_1| - k-1 - 1$$

$$= \nu(C_1 \cup C_2 \cup C_3 \cup D_1) + \nu((C_1 \cup C_2 \cup D_1) \cap (C_3 \cup D_1)) - 1 \quad \text{from (17)}$$

Thus, by (4.8), there exists one $e \in K$ with $V(e) \subseteq X \cup V_1 \cup V_2 \cup V_3 \cup W_1$ such that $V(e)$ is not a subset of C_1, C_2, C_3 or D_1. So, since equality holds in the derivation of (16), $V(e)$ must be a subset of D_2, and, by similar reasoning, of D_3.

But, $\quad D_2 \cap D_3 \subseteq D_2 \cap (C_1 \cup D_3) \subseteq C_1 \qquad$ (by (8))

which is a contradiction.

Thus, there exists no such K, and so $(\underline{M}(K_{3,3}))^*$ is not generalised hypergraphic.

The method of proof used here - that of building up a set of vertices from closed sets in different ways, and showing that there are then inconsistencies in the resulting ν-function - is very powerful, and we shall use it often.

It is not possible to show that a matroid is not generalised hypergraphic merely by using the function μ, because all we know about μ (not involving ν) is that it is increasing and submodular, and

$\mu V(A) \geq \rho A + k - 1$ for each nonempty subset A of $E-K$,

$\mu V(x) = k$ for each $x \in E-K$ such that $\{x\}$ is independent,

and $\mu V(C) = |C| + k - 2$ for each circuit $C \subseteq E-K$.

But we can take $\mu V(A) = \rho A + k - 1$ for each $A \subseteq E-K$ with $A \neq \phi$, and $\mu\phi = 0$, and satisfy these conditions.

It is necessary to explore beyond the sets $V(A)$ into the subsets of V and bring in the function ν to produce a contradiction.

We next prove, by a method similar to that of (6.1), that $(\underline{M}(K_5))^*$ is not generalised hypergraphic.

THEOREM 6.2: $(\underline{M}(K_5))^*$ is not a generalised hypergraphic matroid.

Proof: The graph K_5 is shown, suitably labelled, in Figure 13.

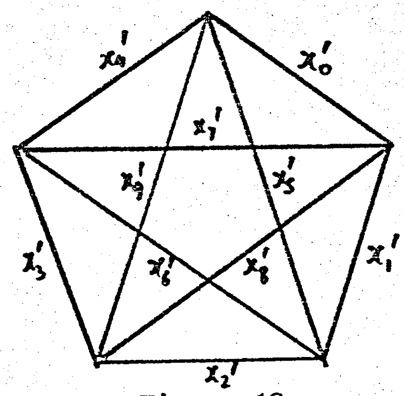


Figure 13

With the above labelling, the circuits of $(\underline{M}(K_5))^*$ are

$C_1' = \{x_0', x_4', x_5', x_9'\}$ $C_2' = \{x_0', x_1', x_7', x_8'\}$ $C_3' = \{x_1', x_2', x_5', x_6'\}$

$C_4' = \{x_2', x_3', x_8', x_9'\}$ $C_5' = \{x_3', x_4', x_6', x_7'\}$

and all symmetric differences (in pairs) of these.

Suppose $(\underline{M}(K_5))^*$ is generalised hypergraphic. Then, by (4.30), there exists a generalised hypergraph $K = (V, E, \$, K)$ such that $E-K = \{x_0, x_1, x_2, \ldots, x_9\}$, $|V(x)| = k$ for each $x \in E$, $H = (V, E, \$)$ is a critical k-hypergraph for some $k \geq 2$, and $(\underline{M}(K_5))^* \cong \underline{M}(H)$, where the isomorphism is induced by the obvious bijection between $E-K$ and $\{x_0', x_1', \ldots, x_9'\}$. Denote the image of C_i' by C_i $(1 \leq i \leq 5)$.

Then, by (4.18), $\mu V(C_i) = k + 2$ $\quad (1 \leq i \leq 5)$ (1)

and $\mu V(C_i \Delta C_j) = k + 4$ $\quad (1 \leq i < j \leq 5)$ (2)

Since H is critical, by (4.31), $\mu V(E-K) = \mu V(E) = k+5$ (3)

From (2), since $C_i \cap C_j \subseteq \sigma(C_i \Delta C_j)$, by (4.29), $\mu V(C_i \cup C_j) = k+4$ (4)

$$\begin{aligned}
\text{Now,}\quad k+2 + k+2 &= \mu\langle V(C_i)\rangle + \mu\langle V(C_j)\rangle \\
&\geq \mu\langle V(C_i \cup C_j)\rangle + \mu(\langle V(C_i)\rangle \cap \langle V(C_j)\rangle) \quad \text{by submodularity of } \mu \\
&\geq \mu\langle V(C_i \cup C_j)\rangle + \mu\langle V(C_i \cap C_j)\rangle \quad \text{since } \mu \text{ is increasing} \\
&= k+4 + k \quad \text{from (4)}
\end{aligned}$$

Thus, equality holds throughout, and so

$$\langle V(C_i)\rangle \cap \langle V(C_j)\rangle = \langle V(C_i \cap C_j)\rangle \qquad (5)$$

Write $\langle V(x_i)\rangle = X_i$ $\quad (0 \leq i \leq 9)$

$X_J = \bigcup_J X_i$ $\quad J \subseteq \{0, 1, \ldots, 9\}$

$X = X_0 \cup X_1 \cup \ldots \cup X_9$

$\langle V(C_i)\rangle = C_i$ $\quad (1 \leq i \leq 5)$

Then, from (5), $C_i = \cup\{X_j : x_j \in C_i\} \cup V_i$ (6)

where $V_i \subseteq V$, $V_i \cap X_j = \phi$ $\quad (1 \leq i \leq 5,\ 0 \leq j \leq 9)$

and $V_i \cap V_j = \phi$ $\quad (1 \leq i < j \leq 5)$

It is easy to check that $\rho(C_i \cup C_j \cup C_m) = 6$ for $1 \leq i < j < m \leq 5$,

$\rho(\{x_i, x_j\}) = 2$ for $0 \leq i < j \leq 9$

$\rho(\{x_i, x_j, x_m\}) = 3$ for $0 \leq i < j < m \leq 9$

and $\rho(\{x_i, x_j, x_m, x_t\}) = 4$ for any subset $\{x_i, x_j, x_m, x_t\}$ not one of the circuits C_s.

Thus, by (4.27), $\mu V(C_i \cup C_j \cup C_m) \geq k+5 \quad (1 \leq i < j < m \leq 5)$ (7)

$\mu V(\{x_i, x_j\}) \geq k+1 \quad (0 \leq i < j \leq 9)$ (8)

$\mu V(\{x_i, x_j, x_m\}) \geq k+2 \quad (0 \leq i < j < m < 9)$ (9)

$\mu V(\{x_i, x_j, x_m, x_t\}) \geq k+3$ for any subset

$\{x_i, x_j, x_m, x_t\}$ not equal to one of the circuits C_s. (10)

From (4), $k+2 + k+4 = \mu V(C_1) + \mu V(C_2 \cup C_3)$

$\geq \mu V(C_1 \cup C_2 \cup C_3) + \mu(V(C_1) \cap V(C_2 \cup C_3))$ since μ is submodular

$\geq \mu V(C_1 \cup C_2 \cup C_3) + \mu V(\{x_0, x_5\})$ since μ is increasing

$\geq k+5 + k+1$ from (7) and (8)

Thus, equality holds throughout, and so

$\mu V(\{x_0, x_5\}) = \mu(X_0 \cup X_5) = k + 1$ (11)

A result similar to (11) holds for any pair $\{x_i, x_j\} \subseteq C_m$ for some m.

Now, $k+4 + k+4 = \mu V(C_1 \cup C_4) + \mu V(C_2 \cup C_3)$

$\geq \mu V(C_1 \cup C_2 \cup C_3 \cup C_4) + \mu((V(C_1 \cup C_4)) \cap (V(C_2 \cup C_3)))$

since μ is submodular

$\geq \mu V(C_1 \cup C_2 \cup C_3 \cup C_4) + \mu V(\{x_0, x_2, x_5, x_8\})$ since μ is increasing

$\geq k+5 + k+3$ from (3) and (10)

Thus, equality holds throughout, and so

$\mu V(\{x_0, x_2, x_5, x_8\}) = k + 3$ (12)

$k+1 + k+1 = \mu V(\{x_0, x_5\}) + \mu V(\{x_2, x_5\})$ from (11)

$\geq \mu V(\{x_0, x_2, x_5\}) + \mu V(x_5)$ since μ is increasing and submodular

$\geq k+2 + k$ from (9)

Thus, equality holds throughout, and so

$\mu V(\{x_0, x_2, x_5\}) = k + 2$ (13)

Similarly, $\mu V(\{x_0, x_2, x_8\}) = k + 2$

Thus, $k+2 + k+2 = \mu V(\{x_0, x_2, x_5\}) + \mu V(\{x_0, x_2, x_8\})$

$\geq \mu V(\{x_0, x_2, x_5, x_8\}) + \mu V(\{x_0, x_2\})$ since μ is increasing and submodular

$\geq k+3 + k + 1$ from (10) and (8)

Thus, equality holds throughout, and so

$$\mu V(\{x_0,x_2\}) = \mu(X_0 \cup X_2) = k + 1 \qquad (14)$$

A result similar to (14) holds for other pairs $\{x_i,x_j\}$ of elements such that $\{x_i,x_j\} \not\subseteq C_m$ for any m.

Thus, from (11) and (14), $\mu V(\{x_i,x_j\}) = \mu(X_i \cup X_j) = k+1$

$$\text{for any } i,j \quad 0 \le i < j \le 9 \qquad (15)$$

So, by (4.28), $\langle V(\{x_i,x_j\})\rangle \cap \langle V(\{x_i,x_m\})\rangle = \langle V(x_i)\rangle$

where $0 \le i \le 9,\ 0 \le j < m \le 9,\ i \ne j,m,$

and so $(X_i \cup X_j) \cap (X_i \cup X_m) = X_i$ (16)

By (4.26), $\nu C_1 + \nu C_2$

$= |X_{\{0,4,5,9\}} \cup V_1| - k-2 + |X_{\{0,1,7,8\}} \cup V_2| - k-2$

$= |X_{\{0,1,4,5,7,8,9\}} \cup V_1 \cup V_2| - k-4 + |X_0| - k$ from (5)

$\ge \nu(C_1 \cup C_2) + \nu X_0$

$\ge \nu C_1 + \nu C_2$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(C_1 \cup C_2) = |X_{\{0,1,4,5,7,8,9\}} \cup V_1 \cup V_2| - k - 4 \qquad (17)$$

Now $\nu(C_1 \cup C_2) + \nu C_3$

$= |X_{\{0,1,4,5,7,8,9\}} \cup V_1 \cup V_2| + |X_{\{1,2,5,6\}} \cup V_3| - 2k - 6$

$= |X_{\{0,1,2,4,5,6,7,8,9\}} \cup V_1 \cup V_2 \cup V_3| - k-5 + |X_{\{1,5\}}| - k-1$ from (5)

$\ge \nu(C_1 \cup C_2 \cup C_3) + \nu((C_1 \cup C_2) \cap C_3)$ from (15) and (5)

$\ge \nu(C_1 \cup C_2) + \nu C_3$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(C_1 \cup C_2 \cup C_3) = |X_{\{0,1,2,4,5,6,7,8,9\}} \cup V_1 \cup V_2 \cup V_3| -k-5 \qquad (18)$$

Also $\nu(X_{\{1,5\}}) = |X_{\{1,5\}}|-k-1$ (19)

Similarly, $\nu(X_{\{2,8\}}) = |X_{\{2,8\}}| - k-1$ and $\nu(X_{\{2,9\}}) = |X_{\{2,9\}}|-k-1$ (20)

$\therefore \quad \nu X_{\{2,8\}} + \nu X_{\{2,9\}} = |X_{\{2,8\}}| + |X_{\{2,9\}}| - 2k-2$

$= |X_{\{2,8,9\}}| - k-2 + |X_2| - k$ from (16)

$\ge \nu X_{\{2,8,9\}} + \nu X_2$ from (9)

$\ge \nu X_{\{2,8\}} + \nu X_{\{2,9\}}$ by (4.7)

Thus, equality holds throughout, and so

$$\nu X_{\{2,8,9\}} = |X_{\{2,8,9\}}| - k-2 \tag{21}$$

So, from (18), $\nu(C_1 \cup C_2 \cup C_3) + \nu C_4$

$= |X_{\{0,1,2,4,5,6,7,8,9\}} \cup V_1 \cup V_3 \cup V_3| - k-5 + |X_{\{2,3,8,9\}} \cup V_4| - k-2$

$= |X \cup V_1 \cup V_2 \cup V_3 \cup V_4| - k-5 + |X_{\{2,8,9\}}| - k-2$ from (16)

$\geq \nu(C_1 \cup C_2 \cup C_3 \cup C_4) + \nu X_{\{2,8,9\}}$

$\geq \nu(C_1 \cup C_2 \cup C_3) + \nu C_4$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(X \cup V_1 \cup V_2 \cup V_3 \cup V_4) = |X \cup V_1 \cup V_2 \cup V_3 \cup V_4| - k - 5 \tag{22}$$

Now, $\nu(C_1 \cup C_2 \cup C_3 \cup C_4) + \nu C_5$

$= |X \cup V_1 \cup V_2 \cup V_3 \cup V_4| - k-5 + |X_{\{3,4,6,7\}} \cup V_5| - k-2$

$= |X \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5| - k-5 + |X_{\{3,4,6,7\}}| - k-2$

$\geq \nu(C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5) + \nu X_{\{3,4,6,7\}}$

$\geq \nu(C_1 \cup C_2 \cup C_3 \cup C_4) + \nu C_5$ by (4.7)

Thus, equality holds throughout, and so

$$\nu X_{\{3,4,6,7\}} = |X_{\{3,4,6,7\}}| - k - 2 \tag{23}$$

From the calculation deriving (22), by (4.8), for each $e \in K$ with $V(e) \subseteq X \cup V_1 \cup V_2 \cup V_3 \cup V_4$, $V(e) \subseteq C_4$ or $V(e) \subseteq C_1 \cup C_2 \cup C_3$.

From the previous equaities, this implies $V(e) \subseteq C_i$ for some i,

$$1 \leq i \leq 4 \tag{24}$$

Now, by an argument similar to that used to derive (20), but applied to the sets $X_{\{3,4\}}$, $X_{\{3,6\}}$ and $X_{\{3,7\}}$,

$\nu X_{\{3,7\}} = |X_{\{3,7\}}| - k-1$, $\nu X_{\{3,4\}} = |X_{\{3,4\}}| - k-1$ and

$\nu X_{\{3,6\}} = |X_{\{3,6\}}| - k-1$. (25)

Then, as in the derivation of (21),

$$\nu X_{\{3,4\}} + \nu X_{\{3,6\}} = \nu X_{\{3,4,6\}} + \nu X_3 \tag{26}$$

So, $\nu X_{\{3,4,6\}} + \nu X_{\{3,7\}} = |X_{\{3,4,6\}}| - k-2 + |X_{\{3,7\}}| - k-1$

$= |X_{\{3,4,6,7\}}| - k-2 + |X_3| - k-1$

$$= \nu X_{\{3,4,6,7\}} + \nu X_3 - 1 \quad \text{from (23)}$$

Thus, by (4.8), there exists $e \in K$ such that $V(e) \subseteq X_{\{3,4,6,7\}}$, but $V(e) \nsubseteq X_{\{3,4,6\}}$ and $V(e) \nsubseteq X_{\{3,7\}}$.

So, applying (4.8) to (26), $V(e) \nsubseteq X_{\{3,4\}}$, $V(e) \nsubseteq X_{\{3,6\}}$ and $V(e) \nsubseteq X_{\{3,7\}}$. (27)

But, from (24), $V(e) \subseteq C_i$ for some i, $1 \le i \le 4$.

Thus, $V(e) \subseteq C_i \cap C_5$ for some i, $1 \le i \le 4$.

Thus, $V(e) \subseteq X_4$ or $V(e) \subseteq X_7$ or $V(e) \subseteq X_6$ or $V(e) \subseteq X_3$, which contradicts (27). Thus, there exists no such K, and so $(\underline{M}(K_5))^*$ is not generalised hypergraphic.

We have thus proved

COROLLARY 6.3: The class of generalised hypergraphic matroids is not closed under the operation of matroid duality.

COROLLARY 6.4: If $\underline{M}$ is a generalised hypergraphic matroid, and $\underline{M}$ is regular (i.e. representable over every field), then $\underline{M}$ is graphic.

Proof: By a proof in [23], $\underline{M}$ is regular if and only if $\underline{M}$ does not contain any minor isomorphic to $U_{2,4}$, the Fano matroid or the dual of the Fano matroid.

By (6.1), (6.2) and (4.5), if $\underline{M}$ is generalised hypergraphic, then $\underline{M}$ contains no minor isomorphic to $(\underline{M}(K_{3,3}))^*$ or $(\underline{M}(K_5))^*$. Thus, by a proof in [24], $\underline{M}$ is graphic.

We shall show in the next chapter that generalised hypergraphic matroids are representable over any characteristic. In particular, this implies that the Fano matroid and its dual are not generalised hypergraphic matroids. (Alternatively, this can be proved using the methods of this chapter). We therefore have the following theorem:

THEOREM 6.5: If M is a binary generalised hypergraphic matroid, then M is graphic.

Proof: $U_{2,4}$ is the forbidden minor for binary matroids ([21] et alibi). The result now follows from (6.4) and the above remarks.

COROLLARY 6.6: If M is transversal and binary, then M is isomorphic to the cycle matroid of a planar graph.

COROLLARY 6.7: If M is a gammoid and M is binary, then M is isomorphic to the cycle matroid of a planar graph.

Proof of (6.7): ((6.6) is a special case).

By (5.2), a gammoid is generalised hypergraphic. The dual of a gammoid is a gammoid ([15], [21]) and is representable over the same fields. Thus, by (6.5), both M and M* are graphic, and so M is isomorphic to the cycle matroid of a planar graph.

Corollary (6.6) is one of the main results of de Souza and Welsh [7].

CHAPTER 7

REPRESENTABILITY OF GENERALISED HYPERGRAPHIC MATROIDS

It is well-known that graphic matroids are representable over every field. However, since the forbidden minor for binary matroids is $U_{2,4}$ (a transversal matroid), this result cannot hold for the class of hypergraphic matroids. Indeed, it is a consequence of (6.4) that a hypergraphic matroid representable over every field is in fact graphic.

In this chapter, we shall use a result of Mason [18] to show that complete hypergraphic matroids (the matroids of the complete hypergraphs) are representable over every characteristic. Since every simple generalised hypergraphic matroid is a minor of a complete hypergraphic matroid, this implies that generalised hypergraphic matroids are representable over every characteristic.

THEOREM 7.1 (Mason [18]): Let $\underline{M}$ be a matroid on the set E with rank at least 4. Then $\underline{M}$ is representable over characteristic q if and only if $\underline{M}^{d,1}$ is representable over characteristic q.

The method of proof is based on an observation by Crapo-Rota [6]. If $\underline{M}$ is embedded in a projective space P, the points of $\underline{M}^{d,1}$ can be identified with the intersections of the lines of $\underline{M}$ in the embedding with a hyperplane of P external to $\underline{M}$ and in general position with respect to the points of $\underline{M}$ in the embedding. Mason's proof in fact uses a vector space rather than a projective space, but the method of proof is similar.

LEMMA 7.2: Let $\mathbf{A} \subseteq \mathbf{F}^{k+2}$, and let ρ_k denote the rank function of $\underline{M}^{d,k}$. Then $\rho_k(\overset{k}{\vee}\mathbf{A}) = \min_{\pi} \left(\sum_{i=1}^{n_\pi} (\rho\mathbf{A}_i - k) \right)$ where $(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n_\pi})$ is a partition of $\mathbf{A}$ into subsets of $\mathbf{F}^{k+2}$, $\overset{k}{\vee}\mathbf{A}$ denotes the supremum in $\underline{M}^{d,k}$ of the elements of $\mathbf{A}$, and $\rho\mathbf{A}_i = \rho(\vee\mathbf{A}_i)$.

Proof: For each $A_i \in \mathbf{A}$, let B_i and $C_i \in \mathbf{F}^{k+1}$ be such that $B_i \vee C_i = A_i$ and $B_i \wedge C_i \in \mathbf{F}^k$. Then $\overset{k}{\vee}\mathbf{A} = \overset{k}{\vee}(\mathbf{B}\cup\mathbf{C})$, where $\mathbf{B} = \{B_i : A_i \in \mathbf{A}\}$, $\mathbf{C} = \{C_i : A_i \in \mathbf{A}\}$.

By a result in Crapo-Rota [6],

$$\rho_k(\overset{k}{\vee}(\mathbf{B}\cup\mathbf{C})) = \min_{\pi'} \left(\sum_{i=1}^{n_{\pi'}} (\rho\mathbf{G}_i - k)\right) \tag{1}$$

where $(\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_{n_{\pi'}})$ is a partition of the set $\mathbf{G} = \mathbf{B}\cup\mathbf{C}$.

Now, suppose that, for some $A_i \in \mathbf{A}$, B_i and C_i are not contained in the same member of the partition $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_m$ at which the minimum in (1) is reached. Suppose $B_i \in \mathbf{G}_1$ and $C_i \in \mathbf{G}_2$, say.

Then $\rho((\vee\mathbf{G}_1)\wedge(\vee\mathbf{G}_2)) \geq \rho(B_i \wedge C_i) = k$.

Now, $\rho\mathbf{G}_1 + \rho\mathbf{G}_2 \geq \rho((\vee\mathbf{G}_1)\vee(\vee\mathbf{G}_2)) + \rho((\vee\mathbf{G}_1)\wedge(\vee\mathbf{G}_2))$

Thus, $(\rho\mathbf{G}_1 - k) + (\rho\mathbf{G}_2 - k) \geq (\rho((\vee\mathbf{G}_1)\vee(\vee\mathbf{G}_2)) - k) + (\rho(\vee\mathbf{G}_1)\wedge(\vee\mathbf{G}_2) - k)$

$$\geq \rho((\vee\mathbf{G}_1)\vee(\vee\mathbf{G}_2)) - k.$$

Thus, if $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_m$ is a partition of $\mathbf{G}$ at which the minimum of (1) is achieved, and $B_i \in \mathbf{G}_1$ and $C_i \in \mathbf{G}_2$, then the minimum is also achieved at $\mathbf{G}_1\cup\mathbf{G}_2, \mathbf{G}_3, \ldots, \mathbf{G}_m$. We can continue thus, combining sets in the partition, until we reach a partition $\mathbf{G}'_1, \mathbf{G}'_2, \ldots, \mathbf{G}'_n$ such that, for any $A_i \in \mathbf{A}$, B_i and C_i are members of $\mathbf{G}'_j$ for some j. Clearly, with each such $\mathbf{G}'_j$ we can associate the subset $\mathbf{A}_j$ of $\mathbf{A}$, where $\mathbf{A}_j = \{A_i : B_i, C_i \in \mathbf{G}'_j\}$. Furthermore, $\vee\mathbf{G}'_j = \vee\mathbf{A}_j$.

Thus, $\rho_k(\overset{k}{\vee}\mathbf{A}) = \sum_{i=1}^{n} (\rho\mathbf{A}_i - k)$.

Now, with any partition $\mathbf{A}'_1, \mathbf{A}'_2, \ldots, \mathbf{A}'_q$ of $\mathbf{A}$, we can associate a partition $\mathbf{G}''_1, \mathbf{G}''_2, \ldots, \mathbf{G}''_q$ where $\mathbf{G}''_j = \{B_i, C_i : A_i \in \mathbf{A}'_j\}$, and each of the partitions is such that $\left(\sum_{j=1}^{q} (\rho\mathbf{G}''_j - k)\right) \geq \left(\sum_{i=1}^{n} (\rho\mathbf{G}'_i - k)\right)$, since the right-hand side of this inequality is the minimum possible.

Thus, $\rho_k(\overset{k}{\vee}\mathbf{A}) = \min_{\pi} \left(\sum_{i=1}^{n_\pi} (\rho\mathbf{A}_i - k)\right)$, where $(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n_\pi})$ is a partition of $\mathbf{A}$.

THEOREM 7.3: $\underline{M}^{d,k+1} = (\underline{M}^{d,k})^{d,1} \times \mathbf{F}^{k+2}$.

Proof: Suppose $\mathbf{A} = \{A_i : i \in I\} \subseteq \mathbf{F}^{k+2}$ is independent in $(\underline{M}^{d,k})^{d,1}$.

For each nonempty subset $J \subseteq I$, let $\mathbf{A}^J = \{A_i : i \in J\}$.

Then $\rho_k(\overset{k}{\vee} \mathbf{A}^J) \geq |J| + 1$ for each nonempty subset $J \subseteq I$.

From (7.2), $\rho_k(\overset{k}{\vee} \mathbf{A}^J) = \min_{\pi} (\sum_{i=1}^{n_\pi}(\rho\mathbf{A}_i^J - k))$, where $(\mathbf{A}_1^J, \mathbf{A}_2^J, \ldots, \mathbf{A}_{n_\pi}^J)$ is a partition of $\mathbf{A}^J$.

$$\therefore \quad \rho_k(\overset{k}{\vee} \mathbf{A}^J) \leq \rho\mathbf{A}^J - k$$

$$\therefore \quad \rho\, \mathbf{A}^J \geq |J| + k + 1 \qquad (1)$$

Since (1) holds for each nonempty subset $J \subseteq I$, $\mathbf{A}$ is independent in $\underline{M}^{d,k+1}$.

Conversely, suppose $\mathbf{A} \subseteq \mathbf{F}^{k+2}$ is a circuit of $(\underline{M}^{d,k})^{d,1}$.

Then $\rho_k(\overset{k}{\underset{I}{\vee}} A_i) < |I| + 1$, where $\mathbf{A} = \{A_i : i \in I\}$.

Now, for some partition $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$ of $\mathbf{A}$,

$$\rho_k(\overset{k}{\underset{I}{\vee}} A_i) = \sum_{j=1}^{m}(\rho\mathbf{A}_j - k) \qquad (2)$$

Suppose $m > 1$. Then each $\mathbf{A}_i$ is independent in $(\underline{M}^{d,k})^{d,1}$ and is nonempty, so, from (1), $\rho\mathbf{A}_j \geq |\mathbf{A}_j| + k + 1$.

$$\therefore \quad \sum_{j=1}^{m}(\rho\, \mathbf{A}_j - k) \geq |I| + m > |I| + 1 > \rho_k(\overset{k}{\underset{I}{\vee}} A_i),$$

which contradicts (2). Therefore, $m = 1$, and so

$\rho_k(\overset{k}{\underset{I}{\vee}} A_i) = \rho\mathbf{A} - k$, and hence $\rho\mathbf{A} < |I| + k + 1$.

Thus, $\mathbf{A}$ is dependent in $\underline{M}^{d,k+1}$.

Since the elements of $\underline{M}^{d,k+1}$ are the elements of $\mathbf{F}^{k+2}$,

$\underline{M}^{d,k+1} = (\underline{M}^{d,k})^{d,1} \times \mathbf{F}^{k+2}$.

THEOREM 7.4: $\underline{M}(K_p^k)$ is representable over every characteristic.

Proof: We have, from earlier remarks (or definition (2.1)), that

$$\underline{M}(K_p^k) \cong (U_{p,p})^{d,k-1}.$$

Thus, by repeated application of (7.3),

$\underline{M}(K_p^k) \cong (\ldots(((U_{p,p})^{d,1} \times \mathbf{F}^2)^{d,1} \times \mathbf{F}^3) \times \ldots\ldots)^{d,1} \times \mathbf{F}^k$,

where $\mathbf{F}^j$ denotes the set of j-flats of $U_{p,p}$ $(0 \le j \le p)$.

If $rk(\underline{M}(K_p^k)) \le 2$, then clearly $\underline{M}(K_p^k)$ is representable over every characteristic. If $rk(\underline{M}(K_p^k) \ge 3$, then, by (7.1) and the above relation, $\underline{M}(K_p^k)$ is representable over the same characteristics as $U_{p,p}$, since representability over a characteristic is preserved under restriction. But $U_{p,p}$ is representable over every field, and hence over every characteristic. Thus, $\underline{M}(K_p^k)$ is representable over every characteristic.

COROLLARY 7.5: If $\underline{M}$ is a generalised hypergraphic matroid, $\underline{M}$ is representable over every characteristic.

Proof: Let $\underline{M}'$ be the simplification of $\underline{M}$. Then, by (4.5), $\underline{M}'$ is isomorphic to a minor of $\underline{M}(K_p^k)$ for some k,p. Thus, since representability is preserved under the operation of taking minors, $\underline{M}'$ is representable over every characteristic. Now, $\underline{M}$ can be represented over the same fields as $\underline{M}'$, so $\underline{M}$ is representable over every characteristic.

COROLLARY 7.6: The Fano matroid and its dual are not generalised hypergraphic matroids.

Proof: Both are representable only over fields of characteristic 2.

The question of which generalised hypergraphic matroids are representable over which fields remains largely unanswered.

We have, of course, Tutte's result on forbidden minors for graphic matroids, which ensures that the class of binary generalised hypergraphic matroids is the class of graphic matroids. A direct proof of this for other than complete hypergraphic matroids is still to be found. Although the condition that there should be only three points on any line is certainly necessary, it is far from sufficient, as the examples in Figures 14 and 15 demonstrate. Each, of course, contains $U_{2,4}$ as a minor, and so cannot be graphic, although each is hypergraphic.

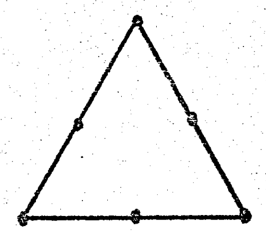


Figure 14

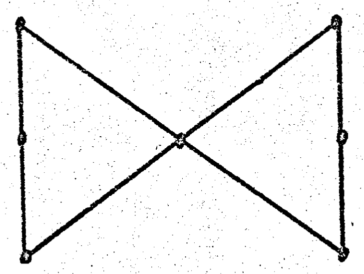


Figure 15

Clearly, a complete ternary hypergraphic matroid is either $U_{2,4}$ or graphic, since $\underline{M}(K_p^k)$ contains $U_{2,5}$ (a forbidden minor for ternary matroids) unless either k = 2, or k = 3 and p = 4.

No necessary and sufficient conditions are known in terms of forbidden minors for representability over GF(q) for q > 3, so this approach cannot be used to examine representability over such fields for generalised hypergraphic matroids. We can, however, put an obvious lower bound on the size of q such that a complete hypergraphic matroid should be representable over GF(q) as follows:

For any point x in $\underline{M}$, where $\underline{M}$ is the complete hypergraphic matroid $\underline{M}(K_p^k)$ of rank r = p-k+1, there are (r-1)k points which are elements of nontrivial lines of $\underline{M}$ containing x. All other points of $\underline{M}$ form trivial lines with x in $\underline{M}$. In PG(n,q) there are $\frac{q^n-1}{q-1}$ lines through any point. Thus, if $\underline{M}$ is to be representable over GF(q), we must have

$$\frac{q^{p-k}-1}{q-1} \geq \binom{p}{k} - (p-k)k - 1 + (p-k)$$

$$= \binom{p}{k} - (p-k)(k-1) - 1$$

$\underline{M}(K_5^3)$ can be represented over GF(4), as we shall see in Chapter 10, (a case in which equality holds in the above), but $\underline{M}(K_6^3)$ cannot be represented over GF(3), although $q = 3$, $p = 6$ and $k = 3$ do satisfy the above. We therefore need at least the extension of this to count the number of $(m+1)$-flats containing a given m-flat of the form $\{e: V(e) \subseteq V' \subseteq V\}$, where $|V'| = k+m$. This leads to the requirement that, for each m with $0 \leq m \leq p-k-1$,

$$\frac{q^{p-k-m} - 1}{q - 1} \geq \binom{p}{k} - \binom{k+m}{m} - (p-k-m)\left(\binom{k+m}{m+1} - 1\right).$$

I do not know whether this condition is also sufficient for complete hypergraphic matroids; even if it is sufficient for deciding the representability of complete hypergraphic matroids, the question of representability for other generalised hypergraphic matroids remains largely unresolved.

We might hope that the obvious analogue of the above - counting the number of 2-flats containing a given point - might be sufficient for such matroids of rank 3, but this is not the case. An example of this is provided by one of the forbidden minors for ternary matroids, $U_{3,5}$. This is uniform, and hence by (5.1) it is hypergraphic. In Figure 16, we show it as a restriction minor of $\underline{M}(K_5^3)$. However, the number of 2-flats containing any point is 4, and $4 = \frac{3^2 - 1}{3 - 1}$. So, since $U_{3,5}$ is not ternary, the condition is not sufficient.

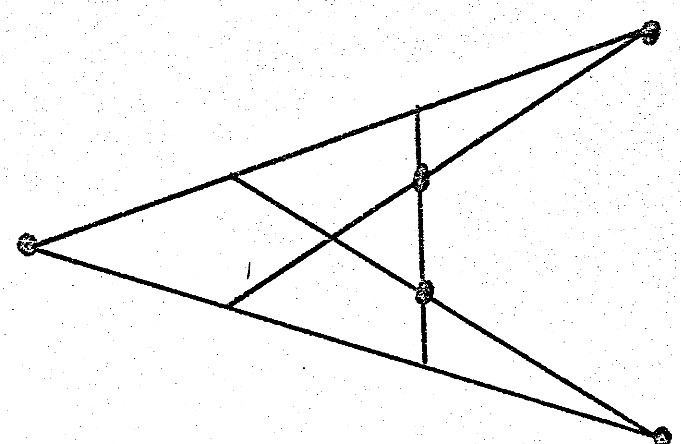


Figure 16

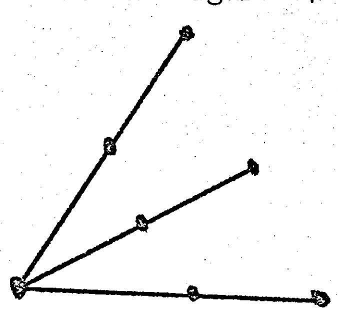
CHAPTER 8

FORBIDDEN MINORS

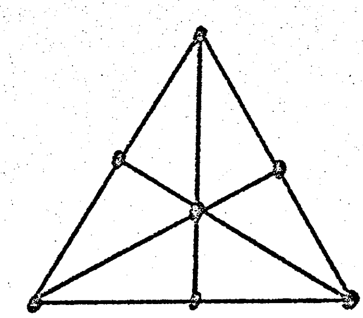
We showed in Chapters 6 and 7 that certain matroids were forbidden minors for the class of generalised hypergraphic matroids. In [23] and [24], Tutte proved that the class of graphic matroids could be defined by a finite set of forbidden minors, viz. $(\underline{M}(K_5))^*$, $(\underline{M}(K_{3,3}))^*$, the Fano matroid and its dual, and $U_{2,4}$. All these matroids are forbidden minors for generalised hypergraphic matroids, with the exception of $U_{2,4}$ (which is transversal, and hence hypergraphic).

Denote the class of generalised hypergraphic matroids by **gh**. Then, if **gh** can be defined by a finite set of forbidden minors, this set must include $(\underline{M}(K_5))^*$, $(\underline{M}(K_{3,3}))^*$, the Fano matroid and its dual, and a finite set of matroids all of which contain $U_{2,4}$ as a minor. In this chapter, we shall first find the smallest matroid which is not a member of **gh**, and then find an infinite family of minimal non-members of **gh**.

Recall that, in Chapter 4, we showed that the matroid of (3.5) was generalised hypergraphic; it follows from (7.4) that the non-Fano matroid is not generalised hypergraphic, since it is not representable over characteristic 2. These two matroids are shown in Euclidean representation in Figure 17.



The matroid of (3.5)



The Non-Fano matroid

Figure 17

The non-Fano matroid is certainly a minimal non-generalised-hypergraphic matroid, and it has fewest elements amongst non-generalised-hypergraphic matroids. However, it may be that there exists a matroid on 7 elements with fewer circuits of cardinality 3 which is also non-generalised-hypergraphic. Any matroid on 7 elements which has at most 4 circuits of cardinality at most 3 can be shown to be generalised hypergraphic. However, a matroid with one fewer circuit of cardinality 3 than non-Fano is non-generalised-hypergraphic, as we now prove.

THEOREM 8.1: Let $S = \{x', a'_1, a'_2, a'_3, b'_1, b'_2, b'_3\}$, and let $\underline{M}$ be the matroid on S with circuits

$C'_i = \{x', a'_i, b'_i\}$ $(1 \leq i \leq 3)$

$D'_1 = \{a'_1, a'_2, b'_3\}$

$D'_2 = \{b'_1, b'_2, b'_3\}$

together with all 4-subsets of S containing none of these. Then $\underline{M}$ is not generalised hypergraphic.

Proof: $\underline{M}$ is shown in Euclidean representation in Figure 18.

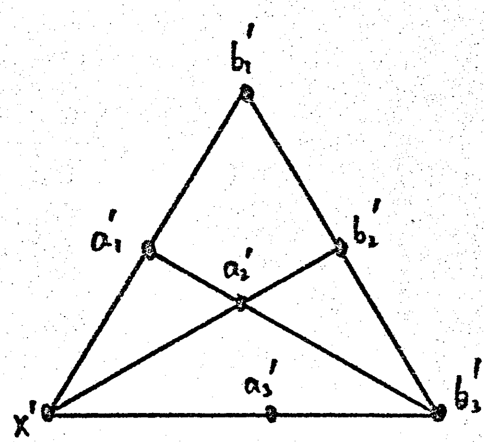


Figure 18

Suppose $\underline{M}$ is generalised hypergraphic. Then, by (4.30), there exists a generalised hypergraph $K = (V, E, \$, K)$, such that $H = (V, E, \$)$ is a critical k-hypergraph for some $k \geq 2$, $E - K = \{x, a_1, a_2, a_3, b_1, b_2, b_3\}$, and $\underline{M} \cong \underline{M}(K)$, where the isomorphism is induced by the obvious bijection between S and $E-K$.

By (4.31), $\mu V(E) = \mu V(E-K) = k+2$ (1)

Denote by C_i, D_i the images of C_i', D_i' respectively. Then, by (4.18),

$$\mu V(C_i) = \mu V(D_j) = k+1 \qquad (1 \le i \le 3,\ j = 1,2) \tag{2}$$

Thus, by (4.28),

$$\left.\begin{array}{ll} \langle V(C_i)\rangle \cap \langle V(C_j)\rangle = \langle V(x)\rangle & (1 \le i < j \le 3) \\ \langle V(D_1)\rangle \cap \langle V(D_2)\rangle = \langle V(b_3)\rangle & \\ \langle V(C_i)\rangle \cap \langle V(D_j)\rangle = \langle V(C_i \cap D_j)\rangle & (1 \le i \le 3,\ j = 1,2) \end{array}\right\} \tag{3}$$

Write $X = \langle V(x)\rangle$, $A_i = \langle V(a_i)\rangle$, $B_i = \langle V(b_i)\rangle$, $C_i = \langle V(C_i)\rangle$ and $D_j = \langle V(D_j)\rangle$. Then

$$\left.\begin{array}{ll} C_i = X \cup A_i \cup B_i \cup V_i & (1 \le i \le 3) \\ D_1 = A_1 \cup A_2 \cup B_3 \cup W_1 & \\ D_2 = B_1 \cup B_2 \cup B_3 \cup W_2 & \end{array}\right\} \tag{4}$$

where $V_i \cap A_i = V_i \cap B_i = W_1 \cap (A_1 \cup A_2 \cup B_3) = W_2 \cap (B_1 \cup B_2 \cup B_3) = \phi$.

Then from (3), $V_i \cap V_j = V_i \cap W_1 = V_i \cap W_2 = \phi$ $(1 \le i \ne j \le 3)$.

By (4.26), $\nu C_i + \nu C_j$

$$\begin{array}{lll} = & |X \cup A_i \cup B_i \cup V_i| + |X \cup A_j \cup B_j \cup V_j| - 2k - 2 & \\ = & |X \cup A_i \cup A_j \cup B_i \cup B_j \cup V_i \cup V_j| - k-2 + |X| - k & (i \ne j) \text{ from } (3) \\ \ge & \nu(X \cup A_i \cup A_j \cup B_i \cup B_j \cup V_i \cup V_j) + \nu X & \text{by } (4.27) \\ \ge & \nu C_i + \nu C_j. & \text{by } (4.7). \end{array}$$

Thus, equality holds throughout, so $\nu(C_i \cup C_j) = |C_i \cup C_j| - k-2$ $(i \ne j)$ (5)

Thus, $\nu(C_1 \cup C_3) + \nu D_2$

$$\begin{array}{lll} = & |X \cup A_1 \cup A_3 \cup B_1 \cup B_3 \cup V_1 \cup V_3| + |B_1 \cup B_2 \cup B_3 \cup W_2| - 2k - 3 & \\ = & |X \cup A_1 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup V_1 \cup V_3 \cup W_2| - k-2 + |B_1 \cup B_3| - k-1 & \text{from } (3) \\ \ge & \nu(C_1 \cup C_3 \cup D_2) + \nu((C_1 \cup C_3) \cap D_2) & \text{by } (4.27) \\ \ge & \nu(C_1 \cup C_3) + \nu D_2 & \text{by } (4.7) \end{array}$$

Thus, equality holds throughout, and so

$$\nu(C_1 \cup C_3 \cup D_2) = |C_1 \cup C_3 \cup D_2| - k - 2 \tag{6}$$

Thus, $\nu(C_1 \cup C_3 \cup D_2) + \nu D_1$

$= |C_1 \cup C_3 \cup D_2| + |D_1| - 2k-3$

$= |C_1 \cup C_3 \cup D_1 \cup D_2| - k-2 + |A_1 \cup B_3| - k-1$ from (3)

$\geq \nu(C_1 \cup C_3 \cup D_1 \cup D_2) + \nu((C_1 \cup C_3 \cup D_2) \cap D_1)$ by (4.27)

$\geq \nu(C_1 \cup C_3 \cup D_2) + \nu D_1$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(C_1 \cup C_3 \cup D_1 \cup D_2) = |C_1 \cup C_3 \cup D_1 \cup D_2| - k - 2 \quad (7)$$

Thus, $\nu(C_1 \cup C_3 \cup D_1 \cup D_2) + \nu C_2$

$= |C_1 \cup C_3 \cup D_1 \cup D_2| + |C_2| - 2k - 3$

$= |C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2| - k-2 + |X \cup A_2 \cup B_2| - k-1$ from (3)

$\geq \nu(C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2) + \nu((C_1 \cup C_3 \cup D_1 \cup D_2) \cap C_2)$ by (4.27)

$\geq \nu(C_1 \cup C_3 \cup D_1 \cup D_2) + \nu C_2$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(X \cup A_2 \cup B_2) = |X \cup A_2 \cup B_2| - k - 1 \quad (8)$$

From (6)&(8), $\nu(C_1 \cup C_3 \cup D_2) + \nu(X \cup A_2 \cup B_2)$

$= |X \cup A_1 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup V_1 \cup V_3 \cup W_2| + |X \cup A_2 \cup B_2| - 2k - 3$

$= |X \cup A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup V_1 \cup V_3 \cup W_2| - k-2 + |X \cup B_2| - k-1$ from (3)

$\geq \nu(X \cup A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup V_1 \cup V_3 \cup W_2) + \nu(X \cup B_2)$ by (4.27)

$\geq \nu(C_1 \cup C_3 \cup D_2) + \nu(X \cup A_2 \cup B_2)$ by (4.7)

Thus, equality holds throughout, and so

$\nu(X \cup B_2) = |X \cup B_2| - k - 1$

Similarly, using the sets C_1, C_3 and D_1, (9)

$\nu(X \cup A_2) = |X \cup A_2| - k - 1$

From (3), $A_2 \cap B_2 \subseteq B_3$. So, $A_2 \cap B_2 \subseteq B_3 \cap B_2 \subseteq X$ (from (3)) (10)

Now, since equality holds in the inequalities used to derive (7), by (4.8) there are no edges $e \in K$ with $V(e) \subseteq C_1 \cup C_3 \cup D_1 \cup D_2$ and $V(e) \not\subseteq C_1$ and $V(e) \not\subseteq C_3$ and $V(e) \not\subseteq D_1$ and $V(e) \not\subseteq D_2$.

Thus, for each $e \in K$ with $V(e) \subseteq X \cup A_2 \cup B_2$,

$V(e) \subseteq (X \cup A_2 \cup B_2) \cap C_1$ or $V(e) \subseteq (X \cup A_2 \cup B_2) \cap C_3$ or $V(e) \subseteq (X \cup A_2 \cup B_2) \cap D_1$

or $V(e) \subseteq (X \cup A_2 \cup B_2) \cap D_2$.

i.e. $V(e) \subseteq X$ or $V(e) \subseteq X$ or $V(e) \subseteq A_2$ or $V(e) \subseteq B_2$.

Thus certainly $V(e) \subseteq X \cup A_2$ or $V(e) \subseteq X \cup B_2$, and so, by (4.8),

$$\nu(X \cup A_2 \cup B_2) = \nu(X \cup A_2) + \nu(X \cup B_2) - \nu X, \qquad (11)$$

since, from (10), $(X \cup A_2) \cap (X \cup B_2) = X$.

Now, from (9), $\nu(X \cup A_2) + \nu(X \cup B_2)$

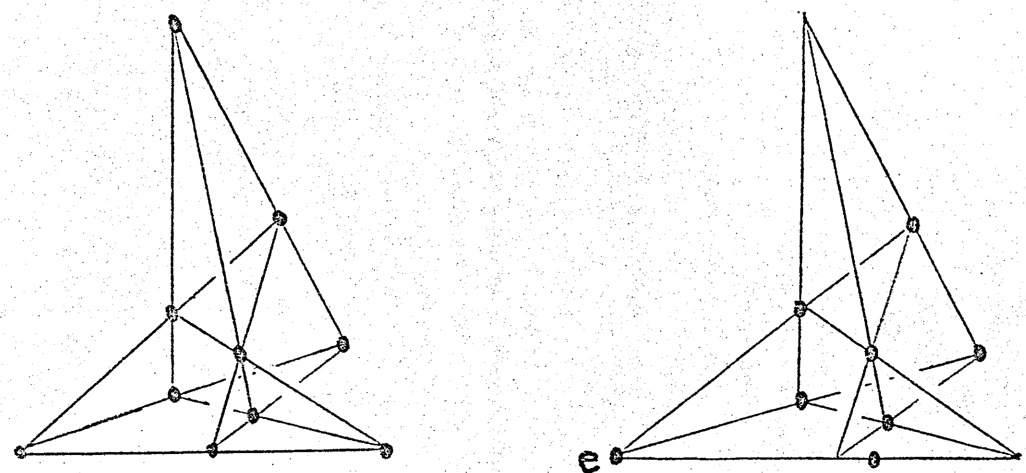
$= |X \cup A_2| + |X \cup B_2| - 2k - 2$

$= |X \cup A_2 \cup B_2| - k-1 + |X| - k - 1$ from (10)

$= \nu(X \cup A_2 \cup B_2) + \nu X - 1$ from (8)

which contradicts (11). Thus, there exists no such K, and so $\underline{M}$ is not generalised hypergraphic.

We now proceed to the second objective of this chapter. Rather than simply pull the family of matroids out of a hat with no apparent reason for choosing them, a little explanation may help. The search was originally for a family of matroids which were generalised hypergraphic, but which had non-generalised-hypergraphic duals. An obvious starting-point was the matroid $\underline{M}(K_5)$, shown in Euclidean representation in Figure 19(a). A slight modification to this produced a matroid which was still hypergraphic, and turned out to have a non-generalised-hypergraphic dual (Figure 19(b)). Attempts to generalise this matroid failed, but a slight modification, including the deletion of the point e and increasing the rank did produce a matroid suitable for generalisation to an infinite family of matroids. The duals of these matroids are defined in (8.2). The matroid $\underline{M}_3$ is shown in Euclidean representation in Figure 20, to assist the reader in following some of the proofs.



(a) (b)

Figure 19

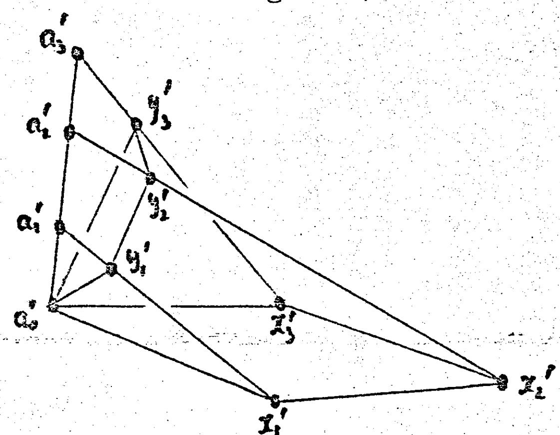


Figure 20

CONSTRUCTION 8.2: Let $E_n' = \{a_0',a_1',\ldots,a_n',x_1',x_2',\ldots,x_n',y_1',y_2',\ldots,y_n'\}_{\neq}$ $(n \geq 3)$. Put

$$C_i' = \{a_i',x_i',y_i'\} \qquad (1 \leq i \leq n)$$

$$D_1' = \{a_0',x_1',x_2',\ldots,x_n'\}$$

$$D_2' = \{a_0',y_1',y_2',\ldots,y_n'\}$$

$$B_{ijm}' = \{a_i',a_j',a_m'\} \qquad (0 \leq i < j < m \leq n)$$

$$X_{ijm}' = \{x_i',y_i',a_j',a_m'\} \quad (1 \leq i \leq n,\ 0 \leq j < m \leq n,\ j,m \neq i)$$

$$Y_{ijm}' = \{x_i',y_i',x_j',y_j',a_m'\} \quad (1 \leq i < j < n,\ 0 \leq m \leq n,\ i,j \neq m)$$

$$Z_{ijm}' = \begin{cases} \phi & (\text{if } n = 3) \\ \{x_i',y_i',x_j',y_j',x_m',y_m'\} & (1 \leq i < j < m \leq n,\ n \geq 4) \end{cases}$$

and let $\mathbf{C}_n'$ be the set of subsets of E_n' defined to be

$\{C_i': 1 \le i \le n\} \cup \{D_1', D_2'\} \cup \{B_{ijm}': 0 \le i < j < m \le n\} \cup$
$\cup \{X_{ijm}': 1 \le i \le n, 0 \le j < m \le n, j,m \ne i\} \cup \{Y_{ijm}': 1 \le i < j \le n, 0 \le m \le n, i,j \ne m\} \cup$
$\cup \{Z_{ijm}': 1 \le i < j < m \le n\}$, together with all $(n+2)$-subsets of E_n' containing none of these.

PROPOSITION 8.3: The set $\mathbf{C}_n'$ $(n \ge 3)$ is the set of circuits of a matroid $\underline{M}_n$ on the set E_n'.

Proof: By construction, $\mathbf{C}_n'$ satisfies circuit axiom (C1). It is therefore sufficient to check axiom (C2). This is routine, and we omit the details.

THEOREM 8.4: $\underline{M}_n$ is not generalised hypergraphic for $n \ge 3$.

Proof: Suppose $\underline{M}_n$ is generalised hypergraphic. Then, by (4.30), there exists a generalised hypergraph $K = (V, E, \$, K)$ such that $H = (V, E, \$)$ is a critical k-hypergraph for some $k \ge 2$, $E-K = \{a_0, a_1, \ldots, a_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and such that $\underline{M}_n \cong \underline{M}(K)$, where the isomorphism is induced by the obvious bijection between E_n' and $E-K$. Denote the images of C_i', D_i', X_{ijm}', Y_{ijm}', Z_{ijm}' and B_{ijm}' by C_i, D_i, X_{ijm}, Y_{ijm}, Z_{ijm} and B_{ijm} respectively, and let $\mathbf{C}_n$ be the set of circuits of $\underline{M}(K)$. By construction, $\underline{M}_n$ has rank $n+1$, and D_1' and D_2' are hyperplanes of $\underline{M}_n$.

By (4.31) $\mu V(E-K) = \mu V(E) = k+n$ (1)

By (4.18) $\mu V(C_i) = \mu V(B_{ijm}) = k+1$ (2)

$\mu V(D_i) = k+n-1$ (3)

$\mu V(X_{ijm}) = k+2$ (4)

$\mu V(Y_{ijm}) = k+3$ (since $n \ge 3$) (5)

$\mu V(Z_{ijm}) = k+4$ if $n \ge 4$ (6)

Write $A_i = \langle V(a_i)\rangle$, $X_i = \langle V(x_i)\rangle$ and $Y_i = \langle V(y_i)\rangle$,

and let $A = \langle V(\{a_0, a_1, \ldots, a_n\})\rangle$

From (3) and (4), $k+2 + k+n-1 = \mu V(X_{i0m}) + \mu V(D_1)$

$\geq \mu V(D_1 \cup X_{i0m}) + \mu V(\{a_0, x_i\})$ since μ is increasing and submodular

$\geq k+n + k+1$ by (4.27)

Thus equality holds throughout, and so $\mu V(\{a_0, x_i\}) = k+1$ (7)

Similarly, $\mu V(\{a_0, y_i\}) = k+1$ (8)

Since $\{a_0, x_i, y_j\}$ is independent for all i,j,

$\mu V(\{a_0, x_i, y_j\}) \geq \mu V(\{a_0, x_i\}) + 1$, so, by (4.28),

$$\left.\begin{aligned} \langle V(a_0)\rangle &= \langle V(\{a_0,x_i\})\rangle \cap \langle V(\{a_0,y_j\})\rangle, \text{ and, similarly,} \\ \langle V(a_0)\rangle &= \langle V(\{a_0,x_i\})\rangle \cap \langle V(\{a_0,x_j\})\rangle \quad (i \neq j) \\ \langle V(a_0)\rangle &= \langle V(\{a_0,y_i\})\rangle \cap \langle V(\{a_0,y_j\})\rangle \quad (i \neq j) \end{aligned}\right\} \quad (9)$$

Write $\langle V(\{a_0,x_i\})\rangle = A_0 \cup X_i \cup U_i$ where $U_i \cap X_i = U_i \cap A_0 = \phi$

$\langle V(\{a_0,y_i\})\rangle = A_0 \cup Y_i \cup V_i$ where $V_i \cap Y_i = V_i \cap A_0 = \phi$

Then, from (9), $U_i \cap U_j \subseteq A_0$ $(i \neq j)$ and $V_i \cap V_j \subseteq A_0$ $(i \neq j)$.

$$\left.\begin{aligned} &\text{Since } U_i \cap A_0 = V_i \cap A_0 = \phi,\ U_i \cap U_j = V_i \cap V_j = \phi \ (i \neq j) \\ &\text{Similarly, } U_i \cap V_j = U_i \cap Y_j = V_j \cap X_i = \phi \\ &\text{and} \quad U_i \cap X_j = V_i \cap Y_j = \phi \\ &\text{Also, from (9), } X_i \cap Y_j \subseteq A_0 \end{aligned}\right\} \quad (10)$$

Write $D_1 = \langle V(D_1)\rangle = A_0 \cup X_1 \cup X_2 \cup \ldots \cup X_n \cup U_1 \cup U_2 \cup \ldots \cup U_n \cup W_1$

$D_2 = \langle V(D_2)\rangle = A_0 \cup Y_1 \cup Y_2 \cup \ldots \cup Y_n \cup V_1 \cup V_2 \cup \ldots \cup V_n \cup W_2$

where $W_1 \cap A_0 = W_1 \cap X_i = W_1 \cap U_i = W_2 \cap A_0 = W_2 \cap Y_i = W_2 \cap V_i = \phi$

From (1), (3) and (8), $\mu V(D_1 \cup \{a_0, y_i\}) \geq \mu V(D_1) + 1$, so, by (4.28),

$\langle V(D_1)\rangle \cap \langle V(\{a_0,y_i\})\rangle = \langle V(a_0)\rangle$.

$$\left.\begin{aligned} &\text{Thus,} \quad V_i \cap W_1 = \phi \\ &\text{Similarly, } U_i \cap W_2 = \phi \end{aligned}\right\} \quad (11)$$

Since $\{a_0, a_1, \ldots, a_n, x_i\}$ has rank 3 for all i, $1 \leq i \leq n$,

$\mu V(\{a_0, a_1, \ldots, a_n, x_i\}) \geq k+2$. By repeated use of (4.29),

$\mu V(\{a_0, a_1, \ldots, a_n\}) = \mu V(\{a_0, a_1, a_2\}) = k+1$ (from (2)).

Thus, $\mu V(\{a_0, a_1, \ldots, a_n, x_i\}) \geq \mu V(\{a_0, a_1, \ldots, a_n\}) + 1$, so, by (4.28),

$\langle V(\{a_0,x_i\})\rangle \cap \langle V(\{a_0,a_1,\ldots,a_n\})\rangle = \langle V(a_0)\rangle$ (12)

Thus, $X_i \cap A \subseteq A_0$ and similarly $Y_i \cap A \subseteq A_0$ (13)

Also, since $\mathbb{D}_i$ is a hyperplane of $\underline{M}(K)$,

$\mu V(\mathbb{D}_i \cup \{a_0,a_1,\dots,a_n\}) = k+n = \mu V(\mathbb{D}_i) + 1$, so, from (4.28),

$$D_i \cap A = A_0 \tag{14}$$

By (4.26), $\nu D_1 + \nu A$

$= |D_1| - k-n+1 + |A| - k-1$

$= |D_1 \cup A| - k-n + |A_0| - k$

$\geq \nu(D_1 \cup A) + \nu A_0$ by (4.27)

$\geq \nu D_1 + \nu A$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(D_1 \cup A) = |D_1 \cup A| - k - n \tag{15}$$

Put $C_i = \langle V(\{a_i,x_i,y_i\})\rangle = A_i \cup X_i \cup Y_i \cup T_i$, where $T_i \cap A_i = T_i \cap X_i = T_i \cap Y_i = \phi$

Then, since $\mathbb{D}_j$ is a hyperplane of $\underline{M}(K)$,

$\mu V(\mathbb{D}_j \cup \{a_i,x_i,y_i\}) = k+n = \mu V(\mathbb{D}_j) + 1$, so, from (4.28),

$$D_1 \cap C_i = X_i \quad \text{and } D_2 \cap C_i = Y_i \tag{16}$$

Thus, $T_i \cap A_0 = T_i \cap X_j = T_i \cap Y_j = T_i \cap W_1 = T_i \cap W_2 = \phi$
and $T_i \cap U_j = T_i \cap V_j = \phi$ $(i \neq j)$. (17)

Since $y_i \in \sigma(\{a_0,a_1,\dots,a_n,x_i\})$, by (4.29),

$\langle V(\{a_0,a_1,\dots,a_n,x_i,y_i\})\rangle = \langle V(\{a_0,a_1,\dots,a_n,x_i\})\rangle$.

$\therefore$ $k+1 + k+1 = \mu\langle V(\{a_0,a_1,\dots,a_n\})\rangle + \mu\langle V(\{a_i,x_i,y_i\})\rangle$

$\geq \mu V(\{a_0,a_1,\dots,a_n,x_i,y_i\}) + \mu V(a_i)$ since μ is increasing and submodular

$\geq \mu V(\{a_0,a_1,x_i,y_i\}) + \mu V(a_i)$ since μ is increasing

$= k+2 + k$ from (4)

Thus, equality holds throughout, and so

$$\mu V(\{a_0,a_1,\dots,a_n,x_i,y_i\}) = k+2 \tag{18}$$

Also, by (4.28), $C_i \cap A = A_i$ (19)

From (18), $k+2 + k+1 = \mu\langle V(\{a_0,a_1,\dots,a_n,x_i,y_i\})\rangle + \mu\langle V(\{a_j,x_j,y_j\})\rangle$

$\geq \mu V(\{a_0,a_1,\dots,a_n,x_i,y_i,x_j,y_j\}) + \mu V(a_j)$ since μ is increasing and submodular

$\geq \mu V(\mathbb{Y}_{ijm}) + \mu V(a_j)$ if $i \neq j$ for $m \neq i,j$ since $n \geq 3$

$= k+3 + k$ from (5)

Thus, equality holds throughout, and, by (4.28),

$$\langle V(\{a_0,a_1,\ldots,a_n,x_i,y_i\})\rangle \cap \langle V(\{a_j,x_j,y_j\})\rangle = \langle V(a_j)\rangle \quad (i \neq j) \tag{20}$$

$$\left.\begin{array}{lll} \text{Thus, in particular,} & C_i \cap C_j \subseteq A_j & (i \neq j) \\ \text{Therefore,} & T_i \cap T_j = \phi & (i \neq j) \end{array}\right\} \tag{21}$$

From (15), $\nu(D_1 \cup A) + \nu C_1$

$= |D_1 \cup A| - k-n + |C_1| - k-1$

$= |D_1 \cup A \cup C_1| - k-n + |X_1 \cup A_1| - n-1$ from (16)&(19)

$\geq \nu(D_1 \cup A \cup C_1) + \nu(X_1 \cup A_1)$ by (4.27)

$\geq \nu(D_1 \cup A) + \nu C_1$ by (4.7)

Thus, equality holds throughout, and so

$$\left.\begin{array}{l} \nu(X_1 \cup A_1) = |X_1 \cup A_1| - k - 1 \\ \nu(D_1 \cup A_1 \cup C_1) = |D_1 \cup A \cup C_1| - k - n \end{array}\right\} \tag{22}$$

Let $P_i = C_1 \cup C_2 \cup \ldots \cup C_i$ $(1 \leq i \leq n)$.

We shall show by induction that $\nu(D_1 \cup A \cup P_i) = |D_1 \cup A \cup P_i| - k-n$

By (22), the induction starts. Assume the result is true for

j, where $1 \leq j < n$. Then $\nu(D_1 \cup A \cup P_j) + \nu C_{j+1}$

$= |D_1 \cup A \cup P_j| - k-n + |C_{j+1}| - k-1$

$= |D_1 \cup A \cup P_j \cup C_{j+1}| - k-n + |X_{j+1} \cup A_{j+1}| - k-1$ from (16),(19)&(21)

$\geq \nu(D_1 \cup A \cup P_j \cup C_{j+1}) + \nu(X_{j+1} \cup A_{j+1})$ by (4.27)

$\geq \nu(D_1 \cup A \cup P_j) + \nu C_{j+1}$ by (4.7)

Thus, equality holds throughout, and so

$$\nu(D_1 \cup A \cup P_j \cup C_{j+1}) = |D_1 \cup A \cup P_j \cup C_{j+1}| - k-n$$

Since $P_{j+1} = P_j \cup C_{j+1}$, $\nu(D_1 \cup A \cup P_{j+1}) = |D_1 \cup A \cup P_{j+1}| - k - n$

Thus, $\nu(D_1 \cup A \cup P_n) = |D_1 \cup A \cup P_n| - k - n$ (23)

Now, $D_1 \cap D_2 = (A_0 \cup X_1 \cup \ldots \cup X_n \cup U_1 \cup \ldots \cup U_n \cup W_1) \cap (A_0 \cup Y_1 \cup \ldots \cup Y_n \cup V_1 \cup \ldots \cup V_n \cup W_2)$

$= A_0 \cup (W_1 \cap W_2)$ from (10) and (11) (24)

Thus, since D_1 and D_2 are closed, $A_0 \cup (W_1 \cap W_2)$ is closed. (25)

Also, $D_2 \cap (D_1 \cup A \cup C_1 \cup \ldots \cup C_n) = A_0 \cup (W_1 \cap W_2) \cup Y_1 \cup \ldots \cup Y_n$.

From (23), $\nu(D_1 \cup A \cup P_n) + \nu D_2$

$$
\begin{aligned}
&= |D_1 \cup A \cup P_n| - k - n + |D_2| - k - n + 1 \\
&= |D_1 \cup D_2 \cup A \cup P_n| - k - n + |A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)| - k - n + 1 \quad \text{from (26)} \\
&\geq \nu(D_1 \cup D_2 \cup A \cup P_n) + \nu(A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)) \quad \text{from (3) and (4.27)} \\
&\geq \nu(D_1 \cup A \cup P_n) + D_2 \quad \text{by (4.7)}
\end{aligned}
$$

Thus, equality holds throughout, and so

$$\nu(A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)) = |A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)| - k - n + 1 \tag{27}$$

Now, since equality holds in those expressions used in the derivation of (23), by (4.8), there exists no $e \in K$ with $V(e) \subseteq D_1 \cup A \cup P_n$, but $V(e) \nsubseteq D_1$, $V(e) \nsubseteq A$ and $V(e) \nsubseteq C_i$ $(1 \leq i \leq n)$.

Thus, there exists no $e \in K$ with $V(e) \subseteq (A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2))$

$$= D_2 \cap (D_1 \cup A \cup P_n)$$

but $V(e) \nsubseteq D_1$, $V(e) \nsubseteq A$ and $V(e) \nsubseteq C_i$ $(1 \leq i \leq n)$.

Thus, there exists no $e \in K$ with $V(e) \subseteq D_2 \cap (D_1 \cup A \cup P_n)$ but $V(e) \nsubseteq D_2 \cap D_1$, $V(e) \nsubseteq D_2 \cap A$ and $V(e) \nsubseteq C_i \cap D_2$ $(1 \leq i \leq n)$; i.e. $V(e) \nsubseteq A_0 \cup (W_1 \cap W_2)$, $V(e) \nsubseteq A_0$ and $V(e) \nsubseteq Y_i$ $(1 \leq i \leq n)$.

So, certainly there exists no $e \in K$ with $V(e) \subseteq D_2 \cap (D_1 \cup A \cup P_n)$ but $V(e) \nsubseteq A_0 \cup (W_1 \cap W_2)$ and $V(e) \nsubseteq A_0 \cup Y_i$ $(1 \leq i \leq n)$.

Therefore, by repeated application of (4.8),

$$
\begin{aligned}
&\nu(A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)) \\
&\quad = \nu(A_0 \cup Y_1) + \ldots + \nu(A_0 \cup Y_n) + \nu(A_0 \cup (W_1 \cap W_2)) - n\nu A_0 \\
&\quad \leq |A_0 \cup Y_1| - k - 1 + \ldots + |A_0 \cup Y_n| - k - 1 + |A_0 \cup (W_1 \cap W_2)| \\
&\qquad - \mu(A_0 \cup (W_1 \cap W_2)) - n|A_0| + nk \quad \text{from (8) and (4.29)} \\
&\quad = |A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)| - n - \mu(A_0 \cup (W_1 \cap W_2))
\end{aligned}
$$

But, from (27),

$$\nu(A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)) = |A_0 \cup Y_1 \cup \ldots \cup Y_n \cup (W_1 \cap W_2)| - n - k + 1$$

$\therefore$ $-\mu(A_0 \cup (W_1 \cap W_2)) \geq -k+1$, so $k-1 \geq \mu(A_0 \cup (W_1 \cap W_2))$.

But $\mu(A_0 \cup (W_1 \cap W_2)) \geq \mu A_0 = k$, which is a contradiction.

Thus, $\underline{M}_n$ is not generalised hypergraphic.

We next prove that $\underline{M}_n^*$ is generalised hypergraphic for each $n \geq 3$. In doing so, we shall demonstrate a technique for identifying the circuits of a generalised hypergraphic matroid which will be much used in subsequent proofs.

THEOREM 8.5: $\underline{M}_n^*$ is generalised hypergraphic for each $n \geq 3$.

Proof: Let $K = (V, E, \$, K)$, where $V = X \cup Y$, $X \cap Y = \phi$, $|X| = n$ and $Y = \{A, B, A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\}_{\neq}$, $k = n+2$, $E = \{a_0, a_1, \ldots, a_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, e\}_{\neq}$, $K = \{e\}$,

and
$$\left.\begin{aligned} V(x_i) &= \{A, A_i\} \cup X \\ V(y_i) &= \{B, A_i\} \cup X \\ V(a_i) &= \{A_i, B_i\} \cup X \end{aligned}\right\} \qquad (1 \leq i \leq n)$$
$$V(a_0) = \{A, B, B_1, B_2, \ldots, B_n\}$$
$$V(e) = \{X, Y, B_1, B_2, \ldots, B_n\}, \text{ where } \{X, Y\}_{\neq} \subseteq X.$$

Denote by x_I the set $\{x_i : i \in I\}$, and use similar notation for other letters. Let $N = \{1, 2, \ldots, n\}$.

Suppose $C = x_I \cup y_J \cup a_T$ is a circuit of $\underline{M}(K)$, where $I, J, T \subseteq N$. Then, provided $I, J \neq \phi$, $V(C) = \{A, B\} \cup A_I \cup A_J \cup A_T \cup B_T \cup X$.

So, provided $T \neq N$, $\mu V(C) = 2 + |I \cup J \cup T| + |T| + n$

$$= k + |I \cup J \cup T| + |T|.$$

Since C is a circuit, by (4.18), $\mu V(C) = |C| + k - 2 = |I| + |J| + |T| + k - 2$.

$$\therefore\ |I| + |J| + |T| + k - 2 = k + |I \cup J \cup T| + |T|$$
$$\therefore\ |I \cup J| + |I \cap J| - 2 = |I \cup J \cup T|$$
$$\therefore\ |I \cap J| - 2 = |I \cup J \cup T| - |I \cup J| \geq 0$$
$$\therefore\ |I \cap J| \geq 2$$

But $C' = \{x_i, y_i, x_j, y_j\}$ is a circuit of $\underline{M}(K)$ for any $1 \leq i < j \leq n$, since

(i) $V(C') = \{A, B, A_i, A_j\} \cup X$ and so $\mu V(C') = k+2 = k + |C'| - 2$, and

(ii) C' is minimal with respect to this property.

Thus, if $T \neq N$, $T = \phi$ and $I = J$, with $|I| = |J| = 2$ (1)

If $T = N$, $\mu V(\mathbb{C}) = k + |I \cup J \cup T| + |T| - 1$

$\therefore \quad |I| + |J| + |T| \quad = \quad |I \cup J \cup T| + |T| + 1.$

So, since $|T| = |I \cup J \cup T| = n$, $|I| + |J| = n+1$.

Therefore $|I \cup J| + |I \cap J| = n+1$. Now, $\mathbb{C}$ is a circuit so, from (1),

$|I \cap J| \leq 1$. Thus, $|I \cap J| = 1$ and $I \cup J = N$ (2)

Suppose now that $J = \phi$. Then, if $\mathbb{C}$ is a circuit, $I \neq \phi$, so

$V(\mathbb{C}) = \{A\} \cup A_I \cup A_T \cup B_T \cup X$

So, if $T \neq N$, $\mu V(\mathbb{C}) = 1 + |I \cup T| + |T| + n = |I \cup J| + |T| + k - 1.$

Since $\mathbb{C}$ is a circuit, $\mu V(\mathbb{C}) = |\mathbb{C}| + k - 2 = |I| + |T| + k - 2.$

$\therefore \quad |I| + |T| + k - 2 \quad = \quad |I \cup T| + |T| + k - 1$

$\therefore \quad |I| \quad = \quad |I \cup T| + 1$, which is impossible.

If $T = N$, $\mu V(\mathbb{C}) = |I \cup T| + |T| + k - 2$, whence $|I| = |I \cup T| = n$.

Thus, $x_N \cup a_N$ is a circuit of $\underline{M}(K)$. (3)

Similarly, $y_N \cup a_N$ is a circuit of $\underline{M}(K)$. (4)

Now suppose $\mathbb{D} = x_I \cup y_J \cup a_T \cup \{a_0\}$ is a circuit of $\underline{M}(K)$, where $I, J, T \subseteq N$. Then $V(\mathbb{D}) = \{A, B\} \cup B_N \cup A_I \cup A_J \cup A_T \cup X$.

So
$$\mu V(\mathbb{D}) = 2 + n + |I \cup J \cup T| + n - 1$$
$$= k + |I \cup J \cup T| + n - 1.$$

Since $\mathbb{D}$ is a circuit, $\mu V(\mathbb{D}) = |\mathbb{D}| + k - 2 = |I| + |J| + |T| + 1 + k - 2.$

$\therefore |I| + |J| + |T| + k - 1 \quad = \quad |I \cup J \cup T| + n + k - 1$

$\therefore |I \cup J| + |I \cap J| + |T| \quad = \quad |I \cup J \cup T| + n$

$\therefore |I \cup J \cup T| + |I \cap J| + |T \cap (I \cup J)| = |I \cup J \cup T| + n$

$\therefore |I \cap J| + |T \cap (I \cup J)| \quad = \quad n.$

Now, since $\mathbb{D}$ is a circuit, by (1), $|I \cap J| < 2$.

If $|I \cap J| = 0$, $|T \cap (I \cup J)| = n$, so $T = N = I \cup J$.

If $|I \cap J| = 1$, $|T \cap (I \cup J)| = n-1$, so, for minimality, $|T| = n-1$ and $I \cup J = T$.

Thus, the circuits of $\underline{M}(K)$ containing a_0 are those of the form

$\{a_0\} \cup x_I \cup y_J \cup a_T$, where either $|T| = n-1$, $I \cup J = T$ and $|I \cap J| = 1$ (5)

or $T = N$, $I \cup J = N$ and $|I \cap J| = 0$. (6)

By minimality in (6), $I,J \neq \phi$, since otherwise $x_N \cup a_N$ or $y_N \cup a_N$ would be properly contained in $\mathcal{D}$.

Thus, the set of circuits of $\underline{M}(K)$ is the set

$\{\{x_i,y_i,x_j,y_j\}: 1 \leq i < j \leq n\} \cup \{x_I \cup y_J \cup a_N: |I \cap J| = 1$ and $I \cup J = N\} \cup$
$\cup \{x_N \cup a_N,\ y_N \cup a_N\} \cup \{\{a_0\} \cup x_I \cup y_J \cup a_T: |T| = n-1,\ I \cup J = T$ and $|I \cap J| = 1\} \cup$
$\cup \{\{a_0\} \cup x_I \cup y_J \cup a_N:\ J = N-I,\ I,J \neq \phi\}$.

Thus, the set of hyperplanes of $(\underline{M}(K))^*$ is the set

$\{\{a_0\} \cup x_I \cup y_I \cup a_N: |I| = n-2\} \cup \{\{a_0\} \cup x_I \cup y_J: |I \cup J| = n-1$ and $I \cap J = \phi\} \cup$
$\cup \{\{a_0\} \cup y_N,\ \{a_0\} \cup x_N\} \cup \{x_I \cup y_J \cup a_T: T = \{t\},\ I \cap J = \{t\}$ and $|I \cup J| = n-1\} \cup$
$\cup \{x_I \cup y_J: J = N-I,\ I,J \neq \phi\}$.

Let $\mathcal{H}$ be a hyperplane of $\underline{M}_n$. Then, from (8.3),
If $\{a_i',a_j'\} \subseteq \mathcal{H}$ for $i \neq j$, then $\{a_0'\} \cup a_N' \subseteq \mathcal{H}$.
If $\{a_i',x_i'\} \subseteq \mathcal{H}$ or $\{a_i',y_i'\} \subseteq \mathcal{H}$ or $\{x_i',y_i'\} \subseteq \mathcal{H}$, then $\{x_i',y_i',a_i'\} \subseteq \mathcal{H}$.
Suppose that $\{a_0'\} \cup a_N' \subseteq \mathcal{H}$. Then $x_i' \in \mathcal{H} \Leftrightarrow y_i' \in \mathcal{H}$.
Since $\mathcal{H}$ is a hyperplane, $\mathcal{H} = \{a_0'\} \cup a_N' \cup x_I' \cup y_I'$ where $|I| = n-2$.
Suppose next that $a_0' \in \mathcal{H}$, but that $a_i' \notin \mathcal{H}$ for $i \neq 0$. Then $x_i' \in \mathcal{H} \Leftrightarrow y_i' \notin \mathcal{H}$. So, since $\mathcal{H}$ is a hyperplane,
$\mathcal{H} \supseteq \{a_0'\} \cup x_I' \cup y_J'$ where $I \cap J = \phi$ and $|I \cup J| = n-1$.
Now, if $J = \phi$, $\mathcal{H} \supseteq \{a_0'\} \cup x_I'$ where $|I| = n-1$, so, from (8.3), since $\mathcal{D}_1'$ is a circuit of $\underline{M}_n$, $\mathcal{H} \supseteq \{a_0'\} \cup x_N'$. Since $\mathcal{D}_1'$ is a hyperplane of $\underline{M}_n$, $\mathcal{H} = \{a_0'\} \cup x_N'$. Similarly, $\{a_0'\} \cup y_N'$ and $\{a_0'\} \cup x_I' \cup y_J'$ ($I \cap J = \phi$, $I,J \neq \phi$ and $|I \cup J| = n-1$) are hyperplanes of $\underline{M}_n$.

Now suppose $a_i' \in \mathcal{H}$ for some $i \neq 0$, but $a_j' \notin \mathcal{H}$ for $i \neq j$.
Then $x_i' \in \mathcal{H} \Leftrightarrow y_i' \in \mathcal{H}$, and $x_j' \in \mathcal{H} \Leftrightarrow y_j' \notin \mathcal{H}$ $(j \neq i)$.
Then $\mathcal{H} = \{a_i'\} \cup x_I' \cup y_J'$ where $|I \cup J| = n-1$ and $I \cap J = \{i\}$, or
$\mathcal{H} = \{a_i'\} \cup x_I' \cup y_J'$ where $I \cap J = \phi$ and $I \cup J = N-\{i\}$.

Now suppose $a_i \notin H$ for all i. Then $x_i \in H \iff y_i \notin H$. Therefore, $H = x_I \cup y_J$ where $|I \cup J| = n$, $I \cap J = \phi$, $I, J \neq \phi$. Thus, the set of hyperplanes of $(\underline{M}(K))^*$ is the set of images of the hyperplanes of $\underline{M}_n$ under the obvious bijection between E and E'. Hence $\underline{M}_n \cong (\underline{M}(K))^*$, and so $\underline{M}_n^* \cong \underline{M}(K)$, and therefore $\underline{M}_n^*$ is generalised hypergraphic.

We next prove all proper minors of $\underline{M}_n$ are generalised hypergraphic. To do this, it is necessary and sufficient to prove that the minors obtained from $\underline{M}_n$ by deleting or contracting one point of E_n' are generalised hypergraphic. It is easy to see that there are three distinct classes of point of $\underline{M}_n$, viz. $\{a_0'\}$, $\{a_i' : 1 \leq i \leq n\}$ and $\{x_i' : 1 \leq i \leq n\} \cup \{y_i' : 1 \leq i \leq n\}$, which are such that there exists an automorphism of $\underline{M}_n$ which maps any point in a particular class to any other point in the same class. We may take as representatives of these classes the points a_0', a_1' and x_1'. Then, in order to prove that all minors of $\underline{M}_n$ are generalised hypergraphic, it is necessary and sufficient to prove that the minors obtained from $\underline{M}_n$ by the deletion or contraction of one of a_0', a_1' or x_1' are all generalised hypergraphic. The proof of this is the content of the next six propositions. In these, sets of circuits of various generalised hypergraphic matroids are given without proof. The technique for each proof is essentially the same as that used in (8.5), and is fairly routine. We therefore omit the details.

PROPOSITION 8.6: $\underline{M}_n \times (E_n' - \{a_0'\})$ is generalised hypergraphic.

Proof: Let $H = (V, E, \$)$ be the hypergraph with

$V = \{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n, X, Y\}_{\neq}$, and

$E = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, a_1, a_2, \ldots, a_n\}_{\neq}$.

Let $A = \{A_1, A_2, \ldots, A_n\}$; then $\$$ is defined by

$$\left.\begin{aligned} V(a_i) &= (\mathcal{A}-\{A_i\})\cup\{X,Y\} \\ V(x_i) &= (\mathcal{A}-\{A_i\})\cup\{X,B_i\} \\ V(y_i) &= (\mathcal{A}-\{A_i\})\cup\{Y,B_i\} \end{aligned}\right\} \quad (1 \le i \le n)$$

Then, with the notation of (8.4), $\underline{M}(H)$ has as circuits

$\{C_i: 1 \le i \le n\} \cup \{B_{ijm}: 1 \le i < j < m \le n\} \cup$
$\cup \{X_{ijm}: 1 \le i \le n, 1 \le j < m \le n, j,m \ne i\} \cup$
$\cup \{Y_{ijm}: 1 \le i < j \le n, 1 \le m \le n, m \ne i,j\} \cup \{Z_{ijm}: 1 \le i < j < m \le n\}$,

and $\underline{M}(H)$ has rank $(n+2)$. Thus, the $(n+1)$-truncation of $\underline{M}(H)$ has as circuits the images of $\underline{M}_n \times (E'_n - \{a'_0\})$ under the obvious bijection between $E'_n - \{a'_0\}$ and E. Thus, $\underline{M}_n \times (E'_n - \{a'_0\}) \cong \underline{M}(H)^{(n+1)}$ and so, by (4.22), $\underline{M}_n \times (E'_n - \{a'_0\})$ is generalised hypergraphic.

PROPOSITION 8.7: $\underline{M}_n \times (E'_n - \{a'_1\})$ is generalised hypergraphic.

Proof: Let $K = (V, E, \$, K)$ where

$$V = \{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n, X, Y, Z\}_{\ne},$$
$$E = \{a_0, a_2, a_3, \ldots, a_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, e_0, e_1, \ldots, e_n\}_{\ne}$$
$$K = \{e_0, e_1, \ldots, e_n\}.$$

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. Then \$ is defined by

$$V(a_0) = \{B_1, C_1\} \cup \mathcal{A}$$
$$V(x_1) = \{B_1, B_2, \ldots, B_n, Y, Z\}$$
$$V(y_1) = \{C_1, C_2, \ldots, C_n, Y, Z\}$$

$$\left.\begin{aligned} V(a_i) &= (\mathcal{A}-\{A_i\})\cup\{X,Y,Z\} \\ V(x_i) &= (\mathcal{A}-\{A_i\})\cup\{B_i,Y,Z\} \\ V(y_i) &= (\mathcal{A}-\{A_i\})\cup\{C_i,Y,Z\} \\ V(e_i) &= (\mathcal{A}-\{A_i\})\cup\{B_i,C_i,X\} \end{aligned}\right\} \quad (2 \le i \le n)$$

$$V(e_0) = (\mathcal{A}-\{A_1\})\cup\{B_1,Y,Z\}$$
$$V(e_1) = (\mathcal{A}-\{A_1\})\cup\{C_1,Y,Z\}$$

Then, with the notation of (8.4), the circuits of $\underline{M}(K)$ are

$\{C_i: 2 \le i \le n\} \cup \{B_{ijm}: 0 \le i < j < m \le n, i,j,m \ne 1\} \cup$

$\cup\{X_{ijm}: 2 \le i \le n,\ 0 \le j < m \le n,\ j,m \ne 1,\ j,m \ne i\} \cup$

$\cup\{Y_{ijm}: 2 \le i < j \le n,\ 0 \le m \le n,\ m \ne 1,\ m \ne i,j\} \cup$

$\cup\{Z_{ijm}: 2 \le i < j < m \le n,\ n \ge 4\} \cup \{D_1, D_2\}$, together with all (n+2)-subsets of $E-K$ containing none of these. But these are the images of the circuits of $\underline{M}_n \times (E_n' - \{a_1'\})$ under the obvious bijection between $E-K$ and $E_n' - \{a_1'\}$. Thus, $\underline{M}_n \times (E_n' - \{a_1'\})$ is generalised hypergraphic.

PROPOSITION 8.8: $\underline{M}_n \times (E_n' - \{x_1'\})$ is generalised hypergraphic.

Proof: Let $K = (V, E, \$, K)$, where

$$V = \{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n, X, Y\}_{\ne},$$

$$E = \{a_0, a_1, a_2, \ldots, a_n, x_2, x_3, \ldots, x_n, y_1, y_2, \ldots, y_n, e\}_{\ne},\quad K = \{e\}.$$

Let $A = \{A_1, A_2, \ldots, A_n\}$. Then \$ is defined by

$$V(a_0) = A \cup \{Y\}$$

$$V(e) = A \cup \{B_1\}$$

$$V(a_1) = (A - \{A_1\}) \cup \{X, Y\}$$

$$V(y_1) = \{B_1, B_2, \ldots, B_n, Y\}$$

$$\left.\begin{aligned} V(a_i) &= (A - \{A_i\}) \cup \{X, Y\} \\ V(x_i) &= (A - \{A_i\}) \cup \{X, B_i\} \\ V(y_i) &= (A - \{A_i\}) \cup \{Y, B_i\} \end{aligned}\right\} \qquad (2 \le i \le n)$$

Then, with the notation of (8.4), $\underline{M}(K)$ has as circuits

$\{C_i: 2 \le i \le n\} \cup \{B_{ijm}: 0 \le i < j < m \le n\} \cup$

$\cup\{X_{ijm}: 2 \le i \le n,\ 0 \le j < m \le n,\ j,m \ne i\} \cup$

$\cup\{Y_{ijm}: 2 \le i < j \le n,\ 0 \le m \le n,\ m \ne i,j\} \cup \{Z_{ijm}: 2 \le i < j < m \le n\} \cup \{D_2\}$,

together with all (n+2)-subsets of $E-K$ containing none of these. But these are the images of the circuits of $\underline{M}_n \times (E_n' - \{x_1'\})$ under the obvious bijection between $E_n' - \{x_1'\}$ and $E-K$. Thus, $\underline{M}_n \times (E_n' - \{x_1'\})$ is generalised hypergraphic.

PROPOSITION 8.9: $\underline{M}_n \cdot (E_n' - a_0')$ is generalised hypergraphic.

Proof: Let $K = (V, E, \$, K)$ where

$V = \{X,Y,B_1,B_2,\ldots,B_n\}_{\neq} \cup A$, $A \cap \{X,Y,B_1,\ldots,B_n\} = \phi$ and $|A| = n-2$,

$E = \{a_1,a_2,\ldots,a_n,x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n,e\}$, $K = \{e\}$ and

$$\left.\begin{aligned} V(a_i) &= A \cup \{X,Y\} \\ V(x_i) &= A \cup \{X,B_i\} \\ V(y_i) &= A \cup \{Y,B_i\} \end{aligned}\right\} \quad (1 \le i \le n)$$

$V(e) = \{B_1,B_2,\ldots,B_n\}$.

Then, with the notation of (8.4), $\underline{M}(K)$ has the set of circuits $\{\{a_i,x_j,y_j\}: 1 \le i,j \le n\} \cup \{\{a_i,a_j\}: 1 \le i < j \le n\} \cup$ $\cup\{D_1-\{a_0'\}, D_2-\{a_0\}\} \cup \{\{x_i,y_i,x_j,y_j\}: 1 \le i < j \le n\}$ together with all $(n+1)$-subsets of $E-K$ containing none of these. But these are the images of the circuits of $\underline{M}_n \cdot (E_n'-\{a_0'\})$ under the obvious bijection between $E_n'-\{a_0'\}$ and $E-K$. Thus, $\underline{M}_n \cdot (E_n'-\{a_0'\})$ is generalised hypergraphic.

PROPOSITION 8.10: $\underline{M}_n \cdot (E_n'-\{a_1'\})$ is hypergraphic.

Proof: Let $G = (V, E, \$)$, where $V = \{X,Y,A_2,A_3,\ldots,A_n\}_{\neq}$,

$E = \{a_0,a_2,\ldots,a_n,x_2,x_3,\ldots,x_n,y_2,y_3,\ldots,y_n\}_{\neq}$,

$V(a_0) = \{X,Y\}$

$$\left.\begin{aligned} V(a_i) &= \{X,Y\} \\ V(x_i) &= \{X,A_i\} \\ V(y_i) &= \{Y,A_i\} \end{aligned}\right\} \quad (2 \le i \le n)$$

Then G is the graph consisting of n-1 triangles A_iXY with common base XY, together with a further n-1 edges parallel to XY. Then the circuits of $\underline{M}(G)$ are the sets

$\{\{a_i,x_j,y_j\}: 0 \le i,j \le n,\ i,j \ne 1\} \cup \{\{a_i,a_j\}: 0 \le i < j \le n,\ i,j \ne 1\} \cup$
$\cup \{\{x_i,y_i,x_j,y_j\}: 2 \le i < j \le n\}$.

Let $\underline{M}$ be the free, rank-preserving, one-point extension of $\underline{M}(G)$

by x_1. Then, by (3.13), $\underline{M}$ is hypergraphic. Let $\underline{M} = \underline{M}(H)$, where H is thehypergraph obtained from G by the construction of (3.13). Since G is critical, H is defined. Let $\underline{M}' = \underline{M}(H')$, where H' is the hypergraph $(V', E', \$')$, such that $V' = V(H)$, $E' = E(H) \cup \{y_1\}$, and $\$' = \$(H) \cup \{(V, y_1) : V \in V_H'(x_1)\}$.

Then $\underline{M}(H')$ has as circuits the circuits of $\underline{M}(G)$, together with $\{x_1, y_1\}$ and all (n+1)-subsets of E' containing none of these. But these are the images of the circuits of $\underline{M}_n . (E_n' - \{a_1'\})$ under the obvious bijection between $E_n' - \{a_1'\}$ and E'. Thus, $\underline{M}_n . (E_n' - \{a_1'\})$ is hypergraphic.

PROPOSITION 8.11: $\underline{M}_n . (E_n' - \{x_1'\})$ is generalised hypergraphic.

Proof: Let $K = (V, E, \$, K)$, where

$$V = \{A_1, A_2, \ldots, A_n, B_2, B_3, \ldots, B_n, X, Y\}_{\neq},$$

$$E = \{a_0, a_1, \ldots, a_n, x_2, x_3, \ldots, x_n, y_1, y_2, \ldots, y_n, e\}_{\neq}, K = \{e\}.$$

Denote $\{A_1, A_2, \ldots, A_n\}$ by A. Then \$ is defined by

$$V(a_0) = A \cup \{X\}$$

$$V(a_1) = V(y_1) = (A - \{A_1\}) \cup \{X, Y\}$$

$$V(e) = \{A_1, A_2, B_2, \ldots, B_n\}$$

$$\left.\begin{aligned} V(a_i) &= (A - \{A_i\}) \cup \{X, Y\} \\ V(x_i) &= (A - \{A_i\}) \cup \{X, B_i\} \\ V(y_i) &= (A - \{A_i\}) \cup \{Y, B_i\} \end{aligned}\right\} \quad (2 \le i \le n)$$

Then, with the notation of (8.4), $\underline{M}(K)$ has the set of circuits

$\{C_i : 2 \le i \le n\} \cup \{B_{ijm} : 0 \le i < j < m \le n\} \cup$

$\cup \{\{y_1, a_j, a_m\} : 0 \le j < m \le n,\ j, m \ne 1\} \cup \{a_1, y_1\} \cup$

$\cup \{X_{ijm} : 2 \le i \le n,\ 0 \le j < m \le n,\ j, m \ne i\} \cup$

$\cup \{(X_{i1m} - \{a_1\}) \cup \{y_1\} : 2 \le i \le n,\ 0 \le m \le n,\ m \ne 1\} \cup$

$\cup \{Y_{ijm} : 2 \le i < j \le n,\ 0 \le m \le n,\ i, j \ne m\} \cup$

$\cup \{(Y_{ij1}-\{a_1\}) \cup \{y_1\}: 2 \leq i < j \leq n\} \cup \{Z_{ijm}: 2 \leq i < j < m \leq n, n \geq 4\}$, together with all (n+1)-subsets of $E-K$ containing none of these. But these are the images of the circuits of $\underline{M}_n.(E_n'-\{x_1'\})$ under the obvious bijection between $E_n'-\{x_1'\}$ and $E-K$. Thus, $\underline{M}_n.(E_n'-\{x_n'\})$ is generalised hypergraphic.

We have therefore proved

THEOREM 8.12: For each $n \geq 3$, $\underline{M}_n$ is not generalised hypergraphic, but every proper minor of $\underline{M}_n$ is generalised hypergraphic.

COROLLARY 8.13: The set $\{\underline{M}_n : n \geq 3\}$ is an infinite set of forbidden minors for the class **gh**.

COROLLARY 8.14: The class **gh** cannot be characterised by a finite set of forbidden minors.

Denote the class of matroids with generalised hypergraphic duals by **gch**. Then

COROLLARY 8.15: The class **gch** cannot be characterised by a finite set of forbidden minors.

Proof: $\{\underline{M}_n^*: n \geq 3\}$ is an infinite set of forbidden minors for **gch**.

By (8.5), each $\underline{M}_n^*$ is generalised hypergraphic. Consider, for each n, the matroid $\underline{N}_n = \underline{M}_n \oplus (\underline{M}_n')^*$, where $\underline{M}_n'$ is a matroid isomorphic to $\underline{M}_n$ on a set disjoint from E_n'. Then, for any proper minor $\underline{N}_n'$ of $\underline{N}_n$, either $\underline{N}_n'$ is generalised hypergraphic, or $(\underline{N}_n')^*$ is generalised hypergraphic (or both). But $\underline{N}_n$ is not generalised hypergraphic, nor is $(\underline{N}_n)^*$. Hence

THEOREM 8.16: The class of matroids which are either generalised hypergraphic or have generalised hypergraphic duals, cannot be characterised by a finite set of forbidden minors.

We shall make further use of the set $\{\underline{M}_n : n \geq 3\}$ in Chapter 12.

CHAPTER 9

CONNECTION IN HYPERGRAPHS AND NODE HYPERGRAPHS

In this chapter, we change the emphasis from hypergraphic matroids to uniform hypergraphs and their associated node hypergraphs. Connectivity in graphs has been studied extensively by Tutte [25]. Connectedness in hypergraphs is defined by Berge [1], but his definition of connected component does not coincide, in general, with our definition of component given in (2.8), which is also used by Crapo-Rota [6]. We therefore begin by seeking an alternative definition of connectedness.

In a graph G, two vertices V_1 and V_2 are connected if and only if $\{V_1,V_2\}$ is a subset of the vertex-set of a tree (i.e. of a critical subset of $E(G)$). However, we may also speak of a walk (cf. Harary [11]) from V_1 to V_2 in G as a sequence of edges $(e_1,e_2,\ldots,e_t)$ such that $V_1 \in V(e_1)$, $V_2 \in V(e_t)$ and e_i is adjacent to e_{i+1} (i.e. $\{e_i,e_{i+1}\}$ is critical) for each i, $1 \leq i \leq t-1$. V_1 and V_2 are then connected if there exists a walk from V_1 to V_2.

Each of these definitions may be generalised from a graph G to a hypergraph H, but, unfortunately, the two definitions of connectedness no longer, in general, coincide. We must therefore define two types of connectedness in hypergraphs.

Throughout this chapter, H will denote the k-hypergraph $(V, E, \$)$ where $k \geq 2$.

DEFINITION 9.1: Two subsets $V_1,V_2 \subseteq V$ are said to be weakly connected if there exist edges $e_1,e_2 \in E$ such that $V_1 \subseteq V(e_1)$, $V_2 \subseteq V(e_2)$ and either $e_1 = e_2$, or $\{e_1,e_2\}$ is a subset of a critical subset of E.

DEFINITION 9.2: Two subsets $V_1, V_2 \subseteq V$ are said to be strongly connected if there exists a sequence of edges $(e_1, e_2, \ldots, e_t)$ of H such that $V_1 \subseteq V(e_1)$, $V_2 \subseteq V(e_t)$ and either $t = 1$, or $\{e_i, e_{i+1}\}$ is a critical set for each i, $1 \leq i \leq t-1$.

We have already remarked (in Chapter 2) that there are two analogues in a hypergraph of the concept of a vertex in a graph - a vertex and a node. We shall see that these share the roles played by vertices in graphs, but that some properties of vertices in graphs are not completely generalisable to the hypergraphic case, because, for $k > 2$, a vertex is not a node.

The types of sets of vertex V_1, V_2 in (9.1) and (9.2) that we shall consider are single vertices, nodes and the vertex-sets of edges. We shall then refer to the connectedness as being vertex-, node- or edge-connectedness respectively. It is easy to see that vertex-connectedness is not an equivalence relation on the vertex-set of H for a general k-hypergraph H (consider, for instance, the hypergraph shown in Figure 2). However, by (2.6), node-connectedness and edge-connectedness are equivalence relations on the set of nodes and the set $\{V(e): e \in E\}$ respectively.

DEFINITION 9.3: If $e_1, e_2 \in E$ are such that $V(e_1)$ and $V(e_2)$ are weakly (strongly) connected, then e_1 and e_2 are said to be weakly (strongly) connected.

Edge-connectedness is thus an equivalence relation on the set of edges of H.

DEFINITION 9.4: If H is such that every pair of nodes (edges) is weakly connected, then H is said to be weakly node- (edge-) connected.

DEFINITION 9.4 (CONTD): If every pair of nodes (edges) is strongly connected, then H is said to be strongly node- (edge-) connected.

LEMMA 9.5: H is weakly (strongly) edge-connected if and only if H is weakly (strongly) node-connected.

Proof: Suppose H is weakly edge-connected. Let N_1, N_2 be two nodes of H. Then there exist $e_1, e_2 \in E$ such that $N_1 \subseteq V(e_1)$ and $N_2 \subseteq V(e_2)$. If $V(e_1) = V(e_2)$, there is nothing to prove. Otherwise, since e_1 and e_2 are weakly connected, $\{e_1, e_2\}$ is a subset of a critical set. If $\{e_1, e_2\}_{\neq}$ is a subset of a critical set, then N_1 and N_2 are weakly connected by (9.1). Since this holds for any two nodes N_1, N_2 of H, H is weakly node connected.

Suppose H is weakly node connected. Let $\{e_1, e_2\}_{\neq} \subseteq E$. If $V(e_1) = V(e_2)$, there is nothing to prove. Otherwise, let $N_1 \subseteq V(e_1)$ and $N_2 \subseteq V(e_2)$ be two nodes of H. Then there exist edges e_3, e_4 of H with $N_1 \subseteq V(e_3)$ and $N_2 \subseteq V(e_4)$ such that either (a) $\{e_3, e_4\}_{\neq} \subseteq A$ for some critical set A, or (b) $e_3 = e_4$.
If case (a) holds, since $\{e_1\}$, $\{e_2\}$ and A are critical, by (2.6), $\{e_1, e_2\} \cup A$ is a subset of the edge-set of a fragment H_G of H. Thus, since $V(e_1) \neq V(e_2)$, $\{e_1, e_2\}$ can be extended to a maximal independent subset of G, which must therefore be critical.
If case (b) holds, then, since $\{e_1\}$, $\{e_2\}$ and $\{e_3\}$ are critical, by (2.6), $\{e_1, e_2, e_3\}$ is a subset of the edge-set of a fragment H_G of H. Thus, since $\{e_1, e_2\}$ is independent, $\{e_1, e_2\}$ can be extended to a maximal independent subset of G which must therefore be critical, and so $\{e_1, e_2\}$ is a subset of a critical set.
Therefore, in either case, e_1 and e_2 are weakly connected. Since this holds for any two edges of H, H is weakly edge-connected.

Suppose H is strongly edge-connected. Let N_1, N_2 be two nodes of H. Then there exist edges $e_1, e_2 \in E$ such that $N_1 \subseteq V(e_1)$ and $N_2 \subseteq V(e_2)$. If $V(e_1) = V(e_2)$, there is nothing to prove. Otherwise, there exists a sequence $(e_1=f_1, f_2, \ldots, f_{t-1}, f_t=e_2)$ of edges of H such that $\{f_i, f_{i+1}\}$ is critical for each i, $1 \leq i \leq t-1$. Thus, N_1 and N_2 are strongly connected. Since this holds for any two nodes of H, H is strongly node-connected.

Suppose H is strongly node-connected. Let $\{e_1, e_2\}_{\neq} \subseteq E$. If $V(e_1) = V(e_2)$, there is nothing to prove. Otherwise, let N_1 and N_2 be nodes of H with $N_1 \subseteq V(e_1)$ and $N_2 \subseteq V(e_2)$. Then there exists a sequence $(f_1, f_2, \ldots, f_t)$ of edges of H such that $N_1 \subseteq V(f_1)$, $N_2 \subseteq V(f_t)$ and either $t = 1$, or $\{f_i, f_{i+1}\}$ is critical for each i, $1 \leq i \leq t-1$. Now, either $V(e_1) = V(f_1)$, or, by (2.4), $\{e_1, f_1\}$ is critical, since $V(e_1) \cap V(f_1) \supseteq N_1$. Similarly, $V(e_2) = V(f_t)$ or $\{e_2, f_t\}$ is critical. If $V(e_1) = V(f_1)$ and $V(e_2) = V(f_t)$, then the sequence $(e_1, f_2, \ldots, f_{t-1}, e_2)$ shows that e_1 and e_2 are strongly connected. If $V(e_1) = V(f_1)$ and $V(e_2) \neq V(f_t)$, the sequence $(e_1, f_2, \ldots, f_t, e_2)$ shows that e_1 and e_2 are strongly connected.
If $V(e_1) \neq V(f_1)$ and $V(e_2) = V(f_t)$, then the sequence $(e_1, f_1, \ldots, f_{t-1}, e_2)$ shows that e_1 and e_2 are strongly connected.
If $V(e_1) \neq V(f_1)$ and $V(e_2) \neq V(f_t)$, the sequence $(e_1, f_1, \ldots, f_t, e_2)$ shows that e_1 and e_2 are strongly connected. Since this holds for any two edges of H, H is strongly edge-connected.

This result is the analogue of that for vertex- and edge-connectedness in graphs without isolated vertices. The proviso about isolated vertices arises because a node is, by definition, a subset of $V(e)$ for some $e \in E$, whereas a vertex V need not satisfy $V \in V(e)$ for any $e \in E$.

We shall, in future, refer to strong or weak connectedness without qualification, meaning node- or edge-connectedness, since, by (9.5), these are equivalent.

PROPOSITION 9.6: Let $H = (V, E, \$)$ be a k-hypergraph with $V = V(E)$. Then H is critical if and only if H is weakly connected.

Proof: Suppose H is critical. Then, for any maximal independent subset $A \subseteq E$, A is critical, and $V(A) = V(E)$. Let $e_1, e_2 \in E$. If $V(e_1) = V(e_2)$, then e_1 and e_2 are weakly connected. Otherwise, $\{e_1, e_2\}$ is independent in $\underline{M}(H)$. Thus, there exists a maximal independent set $A \subseteq E$ with $\{e_1, e_2\} \subseteq A$. But A is critical, so e_1 and e_2 are weakly connected.

Conversely, suppose H is weakly connected, and suppose H has components $(V_i, G_i, \$_i)$ $(1 \leq i \leq m)$. If $m = 1$, then $H = (V_1, G_1, \$_1)$, since $V = V(E)$. If $m > 1$, let $e_1 \in G_1$, $e_2 \in G_2$. Then, since H is weakly connected, there exists a critical set $A \supseteq \{e_1, e_2\}$. Consider $(V(A), E_{V(A)}, \$_{V(A)})$. Since A is critical, this is a fragment of H, so, by (2.7), there exists a unique i for which $A \subseteq G_i$. Since the G_i partition E, and since $A \cap G_1 \neq \phi$ and $A \cap G_2 \neq \phi$, this is a contradiction. Thus, $m = 1$, and so H is critical.

PROPOSITION 9.7: If $V_1, V_2 \subseteq V$ are strongly connected, then V_1 and V_2 are weakly connected.

Proof: Since V_1 and V_2 are strongly connected, there exists a sequence $(e_1, e_2, \ldots, e_t)$ of edges of H such that $V_1 \subseteq V(e_1)$, $V_2 \subseteq V(e_t)$, and either $t = 1$ or $\{e_i, e_{i+1}\}$ is critical for each i, $1 \leq i \leq t-1$. If $t = 1$, there is nothing to prove. Otherwise, by repeated application of (2.6), $(W, E_W, \$_W)$ is a fragment of H, where $W = V(e_1) \cup V(e_2) \cup \ldots \cup V(e_t)$. If $\{e_1, e_t\}$ is dependent, then $V(e_1) = V(e_t)$, and there is nothing to prove. Otherwise, $\{e_1, e_t\}$ is independent, and so it can be extended to a maximal independent subset A of E_W. A is critical since $(W, E_W, \$_W)$ is a fragment, so V_1 and V_2 are weakly connected.

COROLLARY 9.8: If H is strongly connected, then H is weakly connected.

The converses to (9.7) and (9.8) are false, as can be seen from the hypergraph shown in Figure 1. In this case, $|V| = 6$, $rk\underline{M}(H) = 4$ and $k = 3$, so H is critical. Since $V = V(E)$, by (9.6), H is weakly connected. But, taking $V_1 = \{A,B\}$ and $V_2 = \{D,E\}$, for example, we see that V_1 and V_2 are not strongly connected.

It is clear that, if a graph G contains a connected spanning subgraph, then G is itself connected. The analogous results for hypergraphs are given in (9.9) and (9.10).

PROPOSITION 9.9: If there exists a strict subhypergraph $H' = (V, E', \$')$ of H such that H' is weakly connected, and $V(E') = V(E)$, then H is weakly connected.

Proof: Let $V' = V(E)$. Then, since H' is weakly connected, $(V', E', \$')$ is weakly connected, and so, by (9.6), $(V', E', \$')$ is critical. Thus, there exists a critical set $A \subseteq E'$ with $V(A) = V'$. Now, $E' \subseteq E$ and $V(E') = V(E)$, so $(V', E, \$)$ is critical, since A is a critical subset of E. Thus, by (9.6), $(V', E, \$)$ is weakly connected, and so H is weakly connected.

PROPOSITION 9.10: If there exists a strict subhypergraph $H' = (V, E', \$')$ of H such that H' is strongly connected and $n(H) = n(H')$, then H is strongly connected.

Proof: Let $N_1, N_2 \in n(H)$. Then, since $n(H) = n(H')$, $N_1, N_2 \in n(H')$. If $N_1 = N_2$, there is nothing to prove. Otherwise, since N_1 and N_2 are strongly connected in H', there exists a sequence $(e_1, \ldots, e_t)$ of edges of H' such that $N_1 \subseteq V(e_1)$, $N_2 \subseteq V(e_t)$ and either $t = 1$

or $\{e_i, e_{i+1}\}$ is critical for each i, $1 \le i \le t-1$. Since $(e_1, \ldots, e_t)$ is also a sequence of edges in H, N_1 and N_2 are strongly connected in H. Since this holds for any two nodes $N_1, N_2 \in n(H)$, H is strongly connected.

From (9.2) we have a natural definition of a walk in a hypergraph. We can therefore define a path in a hypergraph in a way analogous to that used for graphs.

DEFINITION 9.11: Let $V_1, V_2 \subseteq V$. A <u>path</u> from V_1 to V_2 is a sequence $(e_1, e_2, \ldots, e_t)$ of edges of H such that $V_1 \subseteq V(e_1)$, $V_2 \subseteq V(e_t)$ and either t = 1, or t > 1 and, for each i,j,m with $1 \le i,j,m \le t$,

$$|n(e_i) \cap n(e_j)| = \begin{cases} 0 & \text{if } i-j \neq 0, \pm 1; \\ k & \text{if } i-j = 0; \\ 1 & \text{if } i-j = \pm 1; \end{cases}$$

and $n(e_i) \cap n(e_j) \cap n(e_m) = \phi$ for $i \neq j \neq m \neq i$.

This definition ensures that the edges of a path are all distinct, and that a node N is a subset of $V(e_i)$ for at most two values of i, this occuring only when the values are consecutive, when N is the intersection of the vertex-sets of the consecutive edges. The requirement that $n(e_i) \cap n(e_j) \cap n(e_m) = \phi$ for $i \neq j \neq m \neq i$ cannot be relaxed, as can be seen by consideration of the star graph $K_{1,3}$.

We next wish to define an analogue of a cycle of a graph, following on from the above definition of a path. The words "cycle" and "circuit" already have special meaning, so, since our definition of path is related to strong connectedness, we call the analogue a "strong cycle".

DEFINITION 9.12: A <u>strong cycle</u> of H is a sequence $(e_0,e_1,\dots,e_{t-1})$ of edges of H such that either t = 2 and $|n(e_0)\cap n(e_1)| = k$, or $t > 2$ and, for each i,j,m with $0 \le i,j,m \le t-1$,

$$|n(e_i)\cap n(e_j)| = \begin{cases} 0 & \text{if } i-j \ne 0, \pm 1 \pmod t; \\ k & \text{if } i-j = 0; \\ 1 & \text{if } i-j = \pm 1 \pmod t; \end{cases}$$

and $n(e_i)\cap n(e_j)\cap n(e_m) = \phi$ for $i \ne j \ne m \ne i$.

PROPOSITION 9.13: Let $(e_0,e_1,\dots,e_{t-1})$ be a strong cycle of H. Then $\underline{M}(H)\times(\{e_0,e_1,\dots,e_{t-1}\})$ is connected.

Proof: Let the connected components of $\underline{M}(H)\times(\{e_0,e_1,\dots,e_{t-1}\})$ be $G_1, G_2,\dots, G_m$. If m = 1, then $\underline{M}(H)\times(\{e_0,e_1,\dots,e_{t-1}\})$ is connected. Suppose therefore that m > 1. Since any $H' = (V', E', \$')$ with $V'=V(E')$ which is not critical cannot have $\underline{M}(H')$ connected, for each i, $H_{G_i} = (V(G_i),G_i,\$_i)$ is critical. For each i, let $A_i \subseteq G_i$ be a critical set with $V(A_i) = V(G_i)$. Suppose there exist i,j with $i \ne j$ and $|V(G_i)\cap V(G_j)| \ge k$. Then $|V(A_i)\cap V(A_j)| \ge k$. But $|V(A_s)| = k + |A_s| - 1$ for each s, since A_s is critical. Thus, $|V(A_i\cup A_j)| \le k + |A_i\cup A_j| - 2$, since $A_i\cap A_j = \phi$ for $i \ne j$. But then $A_i\cup A_j$ is dependent and so contains a circuit C of $\underline{M}(H)$. This circuit cannot be wholly contained in either A_i or A_j since these are independent, so $C \subseteq G_i\cup G_j$ with $C \not\subseteq G_i$ and $C \not\subseteq G_j$, which contradicts the definition of the G_s.

Consider now a set $G_{j_0} = \{e_i: i \in I_{j_0} \subseteq \{0,1,\dots,t-1\}\}$. Since $m > 1$, $|I_{j_0}| \ne t$, so there exists $e_{n_1} \in \{e_0,e_1,\dots,e_{t-1}\}$ such that $e_{n_1} \notin G_{j_0}$, but that e_{n_1} is an immediate predecessor or successor in the sequence $(e_0,e_1,\dots,e_{t-1})$ of an element of G_{j_0}. Let $e_{n_1} \in G_{j_1}$. Then $|V(G_{j_0})\cap V(G_{j_1})| \ge k-1$. Since strict inequality cannot hold (by the previous part), $|V(G_{j_0})\cap V(G_{j_1})| = k-1$. Thus, by the definition of a strong cycle, no other element of G_{j_1} is an immediate predecessor or successor of an element of G_{j_0}. If $|G_{j_0} \cup G_{j_1}| \neq t$, then there exists an element $e_{n_2} \notin G_{j_0} \cup G_{j_1}$ which is an immediate predecessor or successor of an element of G_{j_1} and $e_{n_2} \in G_{j_2}$ (say), where $|V(G_{j_1}) \cap V(G_{j_2})| = k-1$. Clearly, we may continue thus, until we have a sequence $G_{j_0}, G_{j_1}, \ldots, G_{j_r}$ which exhausts $\{G_1, G_2, \ldots, G_m\}$.

But, consider G_{j_r}. By the argument, $|V(G_{j_r}) \cap V(G_{j_{r-1}})| = k-1$.

But, since there exists no element $e_{n_{r+1}} \in \{e_0, \ldots, e_{t-1}\} - (G_{j_0} \cup \ldots \cup G_{j_r})$, some element of G_{j_r} is also an immediate predecessor or successor of some element $e \in G_{j_s}$ for some $s < r-1$. But then $|V(G_{j_r}) \cap V(G_{j_s})| = k-1$, and, since $(e_0, e_1, \ldots, e_{t-1})$ is a strong cycle, $V(G_{j_r}) \cap V(G_{j_s}) \neq V(G_{j_r}) \cap V(G_{j_{r-1}})$. Consider the sequence $G_{j_s}, G_{j_{s+1}}, \ldots, G_{j_{r-1}}, G_{j_r}$. We have $V(G_{j_i}) = V(A_{j_i})$ for some critical subset $A_{j_i} \subseteq G_{j_i}$.

$$\therefore \quad |V(A_{j_i})| = |A_{j_i}| + k - 1$$

$$\therefore |V(A_{j_s} \cup A_{j_{s+1}} \cup \ldots \cup A_{j_{r-1}})| \leq |A_{j_s} \cup \ldots \cup A_{j_{r-1}}| + (r-s)(k-1) - (r-s-1)(k-1)$$

$$= |A_{j_s} \cup \ldots \cup A_{j_{r-1}}| + k - 1.$$

$$\begin{aligned}\therefore |V(A_{j_s} \cup \ldots \cup A_{j_{r-1}} \cup A_{j_r})| &= |V(A_{j_s} \cup \ldots \cup A_{j_{r-1}})| + |V(A_{j_r})| \\ &\quad - |V(A_{j_r}) \cap V(A_{j_s} \cup \ldots \cup A_{j_{r-1}})| \\ &\leq |A_{j_s} \cup \ldots \cup A_{j_{r-1}}| + k-1 + |A_{j_r}| + k-1 - k \\ &= |A_{j_s} \cup \ldots \cup A_{j_r}| + k - 2.\end{aligned}$$

Thus, $(A_{j_s} \cup \ldots \cup A_{j_r})$ is dependent. Since each A_{j_i} is independent, this contradicts the definition of the G_i as the components of $\underline{M}(H) \times (\{e_0, e_1, \ldots, e_{t-1}\})$. Thus, $m = 1$, and $\underline{M}(H) \times (\{e_0, e_1, \ldots, e_{t-1}\})$ is connected.

The converse to (9.13) is false. Consider, for example, the hypergraph $H = (V, \mathcal{E}, \$)$, where $V = \{A,B,C,D,E\}_{\neq}$, $\mathcal{E} = \{a,b,c,d\}_{\neq}$ and $V(a) = \{A,B,C\}$, $V(b) = \{B,C,D\}$, $V(c) = \{C,D,E\}$ and $V(d) = \{A,D,E\}$. Then $\{a,b,c,d\}$ is a circuit of $\underline{M}(H)$, whence $\underline{M}(H)\times(\{a,b,c,d\})$ is connected, but no sequence of elements of $\mathcal{E}$ forms a strong cycle.

It is easy to see that the element-sets of the strong cycles of a hypergraph do not, in general, satisfy the circuit axioms for a matroid. The axiom (C1) is satisfied, but (C2) need not be, as can be seen from the following example.

Let $H = (V, \mathcal{E}, \$)$, where $V = \{A,B,C,D,E\}_{\neq}$, $\mathcal{E} = \{a,b,c,d,e\}_{\neq}$ and $V(a) = \{A,C,D\}$, $V(b) = \{A,B,C\}$, $V(c) = \{A,B,D\}$, $V(d) = \{B,C,E\}$ and $V(e) = \{B,D,E\}$. Then (a,b,c) and (b,c,e,d) are strong cycles of H. However, there is no strong cycle whose elements are all contained in $\{a,c,d,e\} = (\{a,b,c\}\cup\{b,c,e,d\})-\{b\}$.

The nodes of a hypergraph have several of the properties which are possessed by the vertices of a graph. The following results are easy to prove, and we omit the details.

PROPOSITION 9.14: Let $H = (V, \mathcal{E}, \$)$ be a simple k-hypergraph with $k \geq 2$. Then:

(i) If $e_i, e_j \in \mathcal{E}$ and $e_i \neq e_j$, $V(e_i) \cap V(e_j) \supseteq N$ for at most one $N \in n(H)$;

(ii) If $N_1, N_2 \in n(H)$ and $N_1 \subseteq V(e)$, $N_2 \subseteq V(e)$, then, for any e' with $V(e') \supseteq N_1$ and $V(e') \supseteq N_2$, $e' = e$;

(iii) If $(V_i, \mathcal{G}_i, \$_i)$ is a component of H, and $e \in \mathcal{E}$ is such that $V(e) \cap \mathcal{G}_i \supseteq N$ for some $N \in n(H)$, then $e \in \mathcal{G}_i$;

(iv) If $(V_1, \mathcal{G}_1, \$_1)$ and $(V_2, \mathcal{G}_2, \$_2)$ are distinct components of H, then $n(\mathcal{G}_1) \cap n(\mathcal{G}_2) = \phi$.

For reasons which will become apparent shortly, we shall use the form of vertex-connectedness defined by Berge [1] for connectedness in node-hypergraphs. We shall use $n(A)$ to denote the set of nodes in H of the set $A \subseteq E$, even when A is being regarded as a set of edges of N(H).

DEFINITION 9.15: Let $H = (V, E, \$)$, and let N(H) be the node-hypergraph of H. Then two vertices N_1, N_2 of N(H) are said to be <u>V-connected</u> if there exists a sequence of edges $(e_1, \ldots, e_t)$ of N(H) such that $N_1 \in n(e_1)$, $N_2 \in n(e_t)$, and $n(e_i) \cap n(e_{i+1}) \neq \phi$ $(1 \leq i \leq t-1)$. If each pair of vertices N_1, N_2 of N(H) is V-connected, N(H) is said to be <u>V-connected</u>.

DEFINITION 9.16: A <u>V-cycle</u> of N(H) is a sequence $(e_0, e_1, \ldots, e_{t-1})$ of edges of N(H) such that either $t = 2$ and $|n(e_0) \cap n(e_1)| = k$, or $t > 2$, and for each i, j, m with $0 \leq i, j, m \leq t-1$,

$$|n(e_i) \cap n(e_j)| = \begin{cases} 0 & \text{if } i-j \neq 0, \pm 1 \pmod t; \\ k & \text{if } i-j = 0; \\ 1 & \text{if } i-j = \pm 1 \pmod t; \end{cases}$$

and $n(e_i) \cap n(e_j) \cap n(e_m) = \phi$ for $i \neq j \neq m \neq i$.

This definition ensures that the V-cycles of N(H) form a clutter - i.e. if C_1, C_2 are the element-sets of two V-cycles of N(H) such that $C_1 \subseteq C_2$, then $C_1 = C_2$.

The similarity between (9.16) and (9.12) is clear, and prompts the following proposition:

PROPOSITION 9.17: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph with $k \geq 2$.

Then:

(i) N(H) is V-connected if and only if H is strongly connected;

(ii) $\mathcal{C}$ is a V-cycle of N(H) if and only if $\mathcal{C}$ is a strong cycle of H;

(iii) If $\mathcal{C}$ is a circuit of $\underline{M}$(H) that does not contain the element-set of a strong cycle of H, then $\mathcal{C}$ contains the element-sets of no V-cycles of N(H).

The proofs are routine, and we omit the details.

We call a circuit of $\underline{M}$(H) that does not contain the element-set of any strong cycle of H as a subset, a <u>weak circuit of H</u>. A circuit of $\underline{M}$(H) which is the set of elements of a strong cycle of H is called a <u>strong circuit of H</u>. Since the set of elements of a strong cycle is necessarily dependent, every circuit of $\underline{M}$(H) is either a strong circuit of H or a weak circuit of H.

DEFINITION 9.18: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph with $k \geq 2$.

A pair of edges $e_1, e_2 \in \mathcal{E}$ with $e_1 \neq e_2$ is said to be <u>V-cycle-connected</u> if there exists a V-cycle of N(H), $(f_0, f_1, \ldots, f_{t-1})$, such that $e_1 = f_0$ and $e_2 = f_i$ for some i, $1 \leq i \leq t-1$.

N(H) is said to be <u>V-cycle-connected</u> if every pair of distinct edges is V-cycle-connected. By convention, a node-hypergraph with a single edge is V-cycle-connected.

DEFINITION 9.19: Let $H = (V, \mathcal{E}, \$)$ be a k-hypergraph with $k \geq 2$.

A pair of edges $e_1, e_2 \in \mathcal{E}$ with $e_1 \neq e_2$ is said to be <u>strongly 2-connected</u> if there exists a strong cycle $(f_0, f_1, \ldots, f_{t-1})$ of H such that $e_1 = f_0$ and $e_2 = f_i$ for some i, $1 \leq i \leq t-1$.

H is said to be <u>strongly 2-connected</u> if every pair of distinct elements of $\mathcal{E}$ is strongly 2-connected. By convention, a hypergraph with a single edge is strongly 2-connected.

The similarity between (9.18) and (9.19) is to be expected in the light of previous remarks.

DEFINITION 9.20: Let $H = (V, E, \$)$ be a k-hypergraph with $k \geq 2$.

A strongly-connected component of H is a hypergraph $H' = H_{E'}$ such that $E' \subseteq E$, H' is strongly connected, and, for any $E'' \subseteq E$ with $E' \subsetneqq E''$, $H_{E''}$ is not strongly connected.

A V-connected component of N(H) is N(H') where H' is a strongly connected component of H.

A strongly-2-connected component of H is a hypergraph $H' = H_{E'}$ such that $E' \subseteq E$, H' is strongly 2-connected, and, for any $E'' \subseteq E$ with $E' \subsetneqq E''$, $H_{E''}$ is not strongly 2-connected.

A 2-V-connected component of N(H) is N(H') where H' is a strongly 2-connected component of H.

We shall later need to distinguish between two types of node – those contained in the vertex-set of only one edge, and those contained in the vertex-sets of two or more edges. We therefore make the following definition:

DEFINITION 9.21: Let $H = (V, E, \$)$ be a k-hypergraph with $k \geq 2$.

If $N \in n(H)$ is such that $N \subseteq V(e_i)$ for some $e_i \in E$, and there exists $e_j \in E$ with $e_j \neq e_i$ such that $N \subseteq V(e_j)$, then N is called a valency node of e_i. A valency node of N(H) is a valency node of e_i for some $e_i \in E$.

A loopless graph has the property that each edge has at most two valency nodes. This prompts the following definition, which will be used in the next chapter:

DEFINITION 9.22: Let H = (V, $\mathcal{E}$, \$) be a k-hypergraph with $k \geq 2$.

If, for each $e \in \mathcal{E}$, e has at most two valency nodes in n(H), then H is said to be <u>strictly pseudo-graphic</u>.

If, for each strongly 2-connected component H' of H, H' is strictly pseudo-graphic, then H is said to be <u>pseudo-graphic</u>.

With the definition of strong 2-connectedness in (9.19), we might hope to derive analogues of results in graph theory on the existence of cut-vertices or cut-nodes. We define a cut-node in the obvious way:

DEFINITION 9.23: Let H = (V, $\mathcal{E}$, \$) be a strongly-connected k-hypergraph. Then $N \in$ n(H) is said to be a <u>cut-node</u> of H if either

(i) N is the unique valency node in n(H) of some $e \in \mathcal{E}$; or

(ii) H' = (V($\mathcal{E}'$), $\mathcal{E}'$, \$') is not strongly connected, where $\mathcal{E}' = \{e \in \mathcal{E}: N \not\subseteq V(e)\}$, and \$' = $\{(V,e) \in$ \$: $e \in \mathcal{E}'\}$.

A cut-node of H is thus either a cut-vertex of N(H), or an articulation vertex of N(H) - i.e. a vertex N of N(H) such that the removal from N(H) of N and its incident edges leaves a node-hypergraph which is not V-connected, or a vertex N of N(H) such that N is the unique valency node of some $e \in \mathcal{E}$.

PROPOSITION 9.24: If H is strongly 2-connected, H has no cut-nodes.

Proof: Immediate from (9.19) and (9.23).

The converse to (9.24) is false. This is most easily seen by consideration of the node-hypergraph shown in Figure 21. This has no cut-nodes, but there is no V-cycle containing e_1 and e_2.

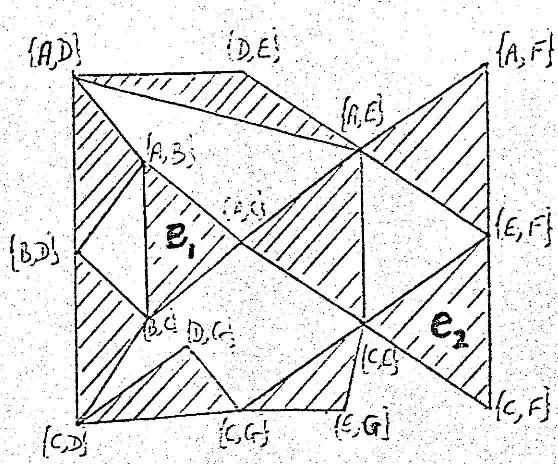


Figure 21

Recall from (4.9) that if K = (V, E, \$,K) is a generalised hypergraph, and e $\in$ E then $\mu V(e) \leq k-1$. From (4.19) it follows that $\mu V(e) = k-1$. There is thus a sense in which the contraction of the edge e contracts it to a node. In the node-hypergraph, this could reasonably be described as identification of the nodes of e. This certainly is the definition of contraction in graphs.

Consider the following simple hypergraph H = (V, E, \$) where V = $\{A,B,C,D,E\}_{\neq}$, E = $\{a,b,c,d\}_{\neq}$ and V(a) = {A,B,C}, V(b) = {A,C,D}, V(c) = {A,D,E} and V(d) = {A,B,E}. N(H) is shown in Figure 22. N(H) is a V-cycle of cardinality 4, and E is a circuit of cardinality 4 in M(H). The generalised hypergraph K = (V, E, \$, {d}) is such that E-{d} is a circuit of cardinality 3. The identification of the nodes of d in N(H) would produce a V-cycle of cardinality 3, as can be seen from Figure 22.

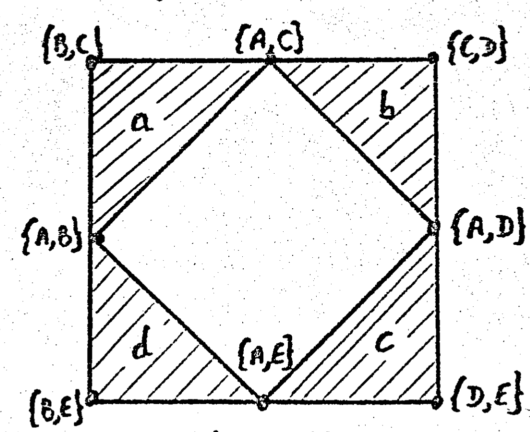


Figure 22

With this as motivation, we define contraction in a node-hypergraph as follows:

DEFINITION 9.25: Let N = (V, E, \$) be a k-hypergraph, and let e $\in$ E. Then the contraction of e in N is defined to be the identification of the vertices in N of e.

DEFINITION 9.26: Let K = (V, E, \$, K) be a generalised hypergraph. Then N(K) is defined to be that hypergraph obtained from N((V, E, \$)) by the contraction of the elements of K.

It is easy to see that N(K) is well-defined.

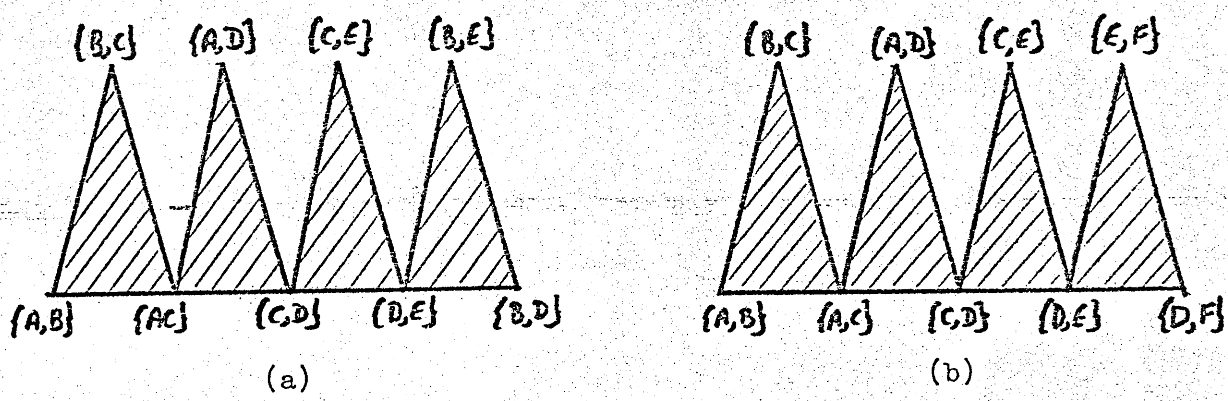


Figure 23

It might be thought that $\underline{M}$(H) could be derived directly from N(H). That this is not so can be seen from the node-hypergraphs shown in Figures 23(a) and 23(b). Although the node-hypergraphs are isomorphic, the matroid corresponding to that in 23(a) has rank 3, whereas the matroid corresponding to that in Figure 23(b) has rank 4. A possible way of avoiding this type of anomaly would be to define a canonical method of obtaining a matroid from a node-hypergraph.

We could, for instance, require that the V-cycles of N(H) should be circuits of the matroid. However, not only do these cycles not, in general, satisfy the circuit axiom (C2), but also there are node-hypergraphs (e.g. those of the complete hypergraphs) which uniquely determine their node-hypergraphs and fail to satisfy this condition.

An approach via independent sets would yield a condition of the form: "A is independent in $\underline{M}$ if and only if either $A = \phi$, or $|V(G)| \geq (k-1)|G| + 1$ for each nonempty subset G of A".

However, if $k > 2$, this does not, in general, give the independent sets of a matroid (see, for instance, Crapo-Rota [6], Chapter 7). We are therefore forced back to our original hypergraph H and its vertices to define $\underline{M}(H)$, and to derive N(H) from H.

CONNECTIVITY

In [27], Tutte defines connectivity for matroids in terms of a function called the ξ-function. The motivation for this comes from graph theory in the following way:

DEFINITION 9.27: A graph G is said to be κ-separated, where κ is a positive integer, if there exist complementary sets of edges E_1, E_2 of G with $|E_i| \geq \kappa$, and such that $|V(E_1) \cap V(E_2)| = \kappa$.

G is said to be κ-connected if there exists a least positive integer κ for which G is κ-separated.

Tutte then defines an analogous concept for matroids:

DEFINITION 9.28: Let $\underline{M}$ be a matroid on the set E. For every subset $T \subseteq E$, define $\xi(\underline{M}; T) = \rho(\underline{M} \times T) + \rho(\underline{M} \times (E-T)) - \rho E + 1$.

Then $\underline{M}$ is said to be κ-separated, where κ is a positive integer, if there exists $T \subseteq E$ with $|T| \geq \kappa$, $|E-T| \geq \kappa$ and such that $\xi(\underline{M}; T) = \kappa$.

$\underline{M}$ is said to be κ-connected if there exists a least positive integer κ for which $\underline{M}$ is κ-separated.

The relationship between $\xi(\underline{M};T)$ and $|V(E_1)\cap V(E_2)|$ can be seen as follows: for a connected graph G, $|V(G)| = rk\underline{M}(G) + 1$. Thus, provided E, T and E-T are all connected,

$$\begin{aligned}|V(T)\cap V(E-T)| &= |V(T)| + |V(E-T)| - |V(E)| \\ &= \rho T + \rho(E-T) - \rho E + 1 \\ &= \xi(\underline{M}(G);T).\end{aligned}$$

It is easy to see that, in the case of graphs, the minimum value of κ always occurs when T, E-T and E are connected.

In the case of a critical k-hypergraph $H = (V, E, \$)$, a critical subhypergraph H_G satisfies $|V(G)| = \rho G + k - 1$. Thus, if H, H_G and H_{E-G} are all critical,

$$\begin{aligned}|V(G)\cap V(E-G)| &= |V(G)| + |V(E-G)| - |V(E)| \\ &= \rho G + \rho(E-G) - \rho E + k - 1 \\ &= \xi(\underline{M}(H); G) + k - 2.\end{aligned}$$

$\xi(\underline{M}(H); G)$ thus attempts to give a measure of the vertex-connectivity of a hypergraph in that, provided H, H_G and H_{E-G} are all critical, $\xi(\underline{M}(H); G) + k - 2$ is the number of vertices common to H_G and H_{E-G}.

The analogue of (9.27) for hypergraphs would therefore be

DEFINITION 9.29: A k-hypergraph $H = (V, E, \$)$ with $k \geq 2$ is said to be $\underline{\kappa\text{-separated}}$, where κ is a positive integer, if there exists $T \subseteq E$ such that $|T| \geq \kappa$, $|E-T| \geq \kappa$ and
$|V(T)\cap V(E-T)| = \kappa + k - 2$.
H is said to be $\underline{\kappa\text{-connected}}$ if there exists a least positive integer κ for which H is κ-separated.

Consider the critical hypergraph $H = (V, E, \$)$, where $V = \{A,B,C,D,E,F,G\}_{\neq}$, $E = \{a,b,c,d,e\}_{\neq}$, $V(a) = \{A,B,C,D\}$, $V(b) = \{B,C,F,G\}$, $V(c) = \{A,B,E,F\}$, $V(d) = \{A,C,E,G\}$ and $V(e) = \{A,D,F,G\}$. Then the only H_G with $G \subseteq E$ which are critical are those with $|G| = 1$, $|G| = 4$ and $|G| = 5$. It is easy to check that $|V(G) \cap V(E-G)| = 4$ if $|G| = 1$, and $|V(G) \cap V(E-G)| \geq 5$ if $|G| = 2$. Thus, there exists no κ for which H is κ-connected. However, $\underline{M}(H)$ is 2-connected. Thus, the relationship between connectivity in H and connectivity in $\underline{M}(H)$ is less close than the corresponding relationship in the case of graphs.

CHAPTER 10

COLOURING HYPERGRAPHS

In this chapter, we examine the ways in which the matroid of a hypergraph gives rise to colourings of the hypergraph, which are generalisations of the vertex-colourings of graphs. We shall be concerned with only two of the possible generalisations, because these arise naturally from matroid considerations. These are the weak and strong colourings defined below. In fact, we shall find that the matroid gives colourings of the node-hypergraph rather than of the hypergraph itself.

The two types of colouring we shall be using are defined as follows:

DEFINITION 10.1: A hypergraph $H = (V, E, \$)$ is said to be strongly (vertex-) colourable with q colours, if there exists a partition $V = V_1 \cup \ldots \cup V_q$ of V such that $|V(e) \cap V_i| \leq 1$ for each $e \in E$ and $1 \leq i \leq q$.

Any such partition is called a strong (vertex-) q-colouring of H, and H is said to be strongly (vertex-) coloured with q colours if such a partition is given.

The strong chromatic polynomial $P_s(H; \lambda)$ of H is that polynomial whose value for each integer $\lambda \geq 0$ is the number of strong vertex colourings of H with λ colours.

DEFINITION 10.2: A hypergraph $H = (V, E, \$)$ is said to be weakly (vertex-) colourable with q colours, if there exists a partition $V = V_1 \cup \ldots \cup V_q$ of V such that, for each $e \in E$ there exist distinct integers $i_1(e)$ and $i_2(e)$, with $1 \leq i_j(e) \leq q$, such that $|V(e) \cap V_{i_j(e)}| \geq 1$, $j = 1,2$.

Any such partition is called a weak (vertex-) q-colouring of H, and

H is said to be weakly (vertex-) coloured with q colours if such a partition is given.

The weak chromatic polynomial $P(H; \lambda)$ of H is that polynomial whose value for each integer $\lambda \geq 0$ is the number of weak vertex colourings of H with λ colours.

In either case, the set $\{i: 1 \leq i \leq q\}$ is referred to as the set of colours of the vertices. The vertex V is said to be coloured with colour i if $V \in V_i$.

A strong colouring of H is thus a proper colouring of the underlying graph of H - i.e. the simplification of the graph $(V, E', \$')$, where $E' = \cup\{\{e_{i1}, e_{i2}, \ldots, e_{im}\} : e_i \in E$ and $m = \binom{k}{2}\}$, $e_{ij} \neq e_{rs}$ unless $i = r$ and $j = s$, and $\{V(e_{i1}), V(e_{i2}), \ldots, V(e_{im})\}_{\neq} = \{\{X,Y\} \subseteq V(e_i)\}_{\neq}$.

In general, many different hypergraphs will have the same underlying graph, and so it may be expected that the hypergraphic matroid will be of little assistance in answering questions about strong colourings. For example, both K_4^3 and K_4^3 less one edge have K_4 as their underlying graph, but have hypergraphic matroids $U_{2,4}$ and $U_{2,3}$ respectively.

Weak colourings have been studied by many authors, including Berge [1] and Helgason [13 & 14]. Helgason has shown that integer polymatroids (see Dunstan [8] for a treatment of these), which he calls "hypermatroids", are the appropriate concept for calculating weak colouring polynomials in general hypergraphs. The reader is referred to his paper [14] for further details. We shall be using Helgason's results to show the relationship between the matroid of a k-hypergraph and the colouring polynomial of its node-hypergraph.

We begin by examining ways of colouring graphs, to see which methods are suitable for generalisation.

It has been shown (e.g. by Crapo-Rota [6]) that the problem of vertex-colouring a graph is equivalent to the critical problem for its matroid - i.e. the problem of finding a minimal set of hyperplanes of PG(n,q) whose intersection with an embedding of the matroid is null. That this should be the case is quite remarkable; it depends on the facts that:

(i) graphic matroids are binary;

(ii) there is a 1-1 correspondence between the hyperplanes of PG(n,2) and the hyperplanes of $\underline{M}(K_{n+2})$;

(iii) there is a 1-1 correspondence between the hyperplanes of $\underline{M}(K_{n+2})$ and the partitions of the vertex-set of K_{n+2} into two nonempty sets.

For a general k-hypergraph H = (V, $\mathcal{E}$, \$) where k ≥ 2, we have:

(i) $\underline{M}(H)$ is not, in general, binary. Indeed, it follows from (6.5) that $\underline{M}(H)$ is binary if and only if $\underline{M}(H)$ is graphic.

(ii) It may be possible to embed $\underline{M}(H)$ in a minimal projective geometry P such that the embedding is affine - i.e. there exists a hyperplane $\mathcal{J}$ of P such that $\mathcal{J} \cap \mathcal{E} = \phi$. As an example of this, Figure 24 shows an embedding of $\underline{M}(K_5^3)$ in PG(2,4). PG(2,4) is the minimal projective geometry in which $\underline{M}(K_5^3)$ can be embedded, since, as we have seen, $U_{3,5}$ is a minor of $\underline{M}(K_5^3)$, from which it follows that $\underline{M}(K_5^3)$ is not binary or ternary. The numbering of the points in Figure 24 is that used in the table of Hall [10], and it is clear that the embedding is affine, because the hyperplane {1,11,14,15,20} has null intersection with $\mathcal{E}(K_5^3)$.

(iii) While it is true that a partition of $V(K_p^k)$ into two sets W_1 and W_2 with $|W_i| \geq k$ and $|W_1 \cap W_2| = k-2$ does correspond to the

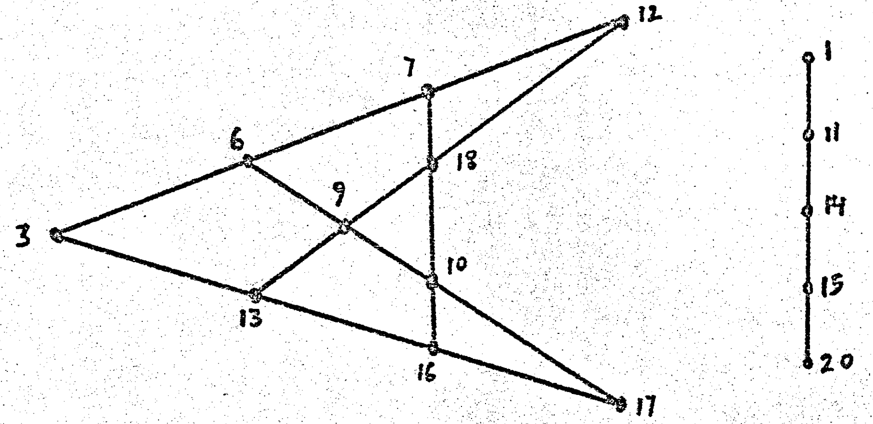


Figure 24

hyperplane $\{e \in E(K_p^k): V(e) \subseteq W_1$ or $V(e) \subseteq W_2\}$ of $\underline{M}(K_p^k)$, and that a subset $W \subseteq V$ with $|W| = p-1$ does correspond to the hyperplane $\{e \in E(K_p^k): V(e) \subseteq W\}$ of $\underline{M}(K_p^k)$, for many values of k and p there exist hyperplanes of $\underline{M}(K_p^k)$ with more than two components. For example, consider K_6^3 on the vertex-set $\{A,B,C,D,E,F\}_{\neq}$. Then it is easy to check that if a,b,c are edges of K_6^3, with $V(a) = \{A,C,E\}$, $V(b) = \{B,C,D\}$ and $V(c) = \{A,B,F\}$, then $\{a,b,c\}$ is a hyperplane of $\underline{M}(K_6^3)$, but $(K_6^3)_{\{a,b,c\}}$ has three components.

It is, therefore, not surprising that the results on graph colourings obtained from the method of intersecting hyperplanes do not extend to the case of vertex-colourings of hypergraphs. It is possible to use the method to colour the vertices of a hypergraph, but the colourings are not particularly well-defined. However, for completeness, we give the construction here.

PROPOSITION 10.3: Let $H = (V, E, \$)$ be a simple k-hypergraph with $k \geq 2$ and $|V| = p$. Embed H in the hypergraph K_p^k on V, and let $\{J_1, J_2, \ldots, J_c\}_{\neq}$ be a set of hyperplanes of $\underline{M}(K_p^k)$ whose intersection with the embedding of H in K_p^k is null - i.e. $J_1 \cap J_2 \cap \cdots \cap J_c \cap E = \phi$. Then H can be coloured in the following way, so that H is weakly vertex coloured:

For each hyperplane J of $\underline{M}(K_p^k)$, let $V_1(J), V_2(J),\ldots,V_{i_J}(J)$ denote the vertex-sets of the components of $(K_p^k)_J$. If $i_J > 1$, then $V_1(J)\cup\ldots\cup V_{i_J}(J) = V$. If $i_J = 1$, set $V_2(J) = V - V_1(J)$. With each $V \in V$, associate the vector $\underline{V} = (a_1,a_2,\ldots,a_c)$, where $V \in V_{a_i}(J_i)$. If, for a particular value of i, $V \in V_j(J_i)$ for more than one value of j, set a_i to be any one of these values. Now associate a different colour with each distinct vector obtained in this way and, for each $V \in V$, colour V with the colour associated with $\underline{V}$. Partition V into sets of vertices coloured with the same colour. Then this partition is a weak colouring of H.

The proof is straightforward, and we omit the details.

If H is a k-hypergraph that is not simple, we can use (10.3) to colour the hypergraph H' which is the maximal simple strict subhypergraph of H. Then this colouring is clearly a weak colouring of H.

As an example of the method of (10.3), consider the hypergraph $H = (V, E, \$)$, where $V = \{A,B,C,D,E\}_{\neq}$, and $H \cong K_5^3$. Let $\{a,b,c,d,e,f\}_{\neq} \subseteq E$ where $V(a) = \{A,B,C\}$, $V(b) = \{C,D,E\}$, $V(c) = \{A,B,D\}$, $V(d) = \{A,B,E\}$, $V(e) = \{A,D,E\}$ and $V(f) = \{B,D,E\}$. Then $J_1 = \{a,b\}$ and $J_2 = \{c,d,e,f\}$ are both hyperplanes of $\underline{M}(K_5^3)$ with $J_1 \cap J_2 \cap E = \phi$.

We can take $V_1(J_1) = \{A,B,C\}$ $\quad V_2(J_1) = \{C,D,E\}$

$V_1(J_2) = \{A,B,D,E\}$ $\quad V_2(J_2) = \{C\}$

So $\underline{A} = (1,1)$, $\underline{B} = (1,1)$, $\underline{C} = (1,2)$, $\underline{D} = (2,1)$ and $\underline{E} = (2,1)$ (say). So, associating the colour 1 with (1,1), 2 with (1,2) and 3 with (2,1), we obtain the colouring $V_1 \cup V_2 \cup V_3$ of H, where $V_1 = \{A,B\}$, $V_2 = \{C\}$ and $V_3 = \{D,E\}$.

Because of the lack of uniqueness in the colouring produced by (10.3), and the fact referred to earlier that, for many values of k and $|V|$, $\underline{M}(H)$ is affine, the method of intersecting hyperplanes cannot be used to obtain a formula for the chromatic polynomial for the weak vertex-colourings of the hypergraph. However, a "chromatic polynomial" can, under certain conditions, be derived from $\underline{M}(H)$ which is meaningful in terms of vertex-colourings, but of $N(H)$ rather than of H.

We recall that, for a graph $G = (V, E, \$)$, the chromatic polynomial $P(G; \lambda)$ satisfies:

$P(G; \lambda) = P(G'; \lambda) - P(G''; \lambda)$, where G', G'' denote respectively the graphs obtained from G by deleting and contracting an edge e which is not a loop or coloop of $\underline{M}(G)$;

$P(G_1 \cup G_2; \lambda) = P(G_1; \lambda)P(G_2; \lambda)$ if G_1 and G_2 are disjoint;

$P(G_1 . G_2; \lambda) = \frac{1}{\lambda}P(G_1; \lambda)P(G_2; \lambda)$ where $V(G_1) \cap V(G_2) = \{V\}$ and "." denotes union with identification of the common vertex.

The chromatic polynomial of G is thus defined uniquely by $P(L; \lambda)$ and $P(C; \lambda)$ where L and C denote the graphs with a single edge which is respectively a loop and coloop.

The chromatic polynomial of a graph is a special case of a more general polynomial in graph theory and matroid theory. This polynomial is due to Tutte [26], is denoted by $T(\underline{M}; x,y)$ and is called the <u>Tutte polynomial</u>. We use it in the form

$$T(\underline{M}; x,y) = \sum_{A \subseteq E} (x-1)^{\rho E - \rho A}(y-1)^{|A| - \rho A}$$

where $\underline{M}$ is a matroid on E with rank function ρ.

We then have $T(\text{loop}; x,y) = y$ and $T(\text{coloop}; x,y) = x$. Furthermore, $P(G; \lambda) = (-1)^{\rho E}\lambda T(\underline{M}(G); 1-\lambda, 0)$ for a connected graph G.

The Tutte polynomial has properties similar to those of the graphic chromatic polynomial. For reference purposes, we embody them in a proposition, the proof of which can be found in [26].

PROPOSITION 10.4: Let $\underline{M}$ be a matroid on E, and let $\underline{M}_1$ and $\underline{M}_2$ be matroids on disjoint sets. Then:

(i) $T(\underline{M}; x,y) = T(\underline{M}'; x,y) + T(\underline{M}''; x,y)$ where $\underline{M}' = \underline{M} \times (E - \{e\})$

and $\underline{M}'' = \underline{M}.(E-\{e\})$ for any element $e \in E$ not a loop or coloop of $\underline{M}$;

(ii) $T(\underline{M}^*; x,y) = T(\underline{M}; y,x)$;

(iii) $T(\underline{M}_1 \oplus \underline{M}_2; x,y) = T(\underline{M}_1; x,y)T(\underline{M}_2; x,y)$.

We seek a chromatic polynomial for a hypergraph H which is derived from $T(\underline{M}(H); x,y)$ in a way similar to that in which $P(G;\lambda)$ is derived from $T(\underline{M}(G); x,y)$ - i.e. such that the chromatic polynomial we obtain is of the form $(-1)^{\alpha(H)}\lambda^{\beta(H)}T(\underline{M}(H); x,y)$ where x and y are polynomials in λ depending only on k, and $\alpha(H)$ and $\beta(H)$ are integers depending on H, to be determined.

Before continuing, we need to establish what is to be coloured in the case of hypergraphs. The Tutte polynomial is defined in terms of deletions and contractions, so contraction needs to be defined for the hypergraph. As we have already said, contraction of an edge e of a hypergraph is meaningful only in N(H), since the identification of the nodes of e in H, implied by the definition of μ, cannot be achieved in a satisfactory way for $k > 2$. Thus, the chromatic polynomial of H will, if it has any meaning at all in terms of colouring, be a measure of the number of ways of (vertex) colouring N(H).

Consider a single edge $e \in E$ not a loop or coloop of $\underline{M}(H)$. Then the number of ways of colouring n(e) in N(H) with λ colours is equal to the number of ways of colouring n(e) with λ colours without restriction, less the number of ways of colouring n(e) with λ colours after the identification of n(e) caused by the contraction of e. Thus, the number of ways of colouring n(e) in N(H) is $\lambda^k-\lambda$. But, this is the number of *weak λ-colourings* of n(e). Thus, the chromatic polynomial will, if it has any meaning at all in terms of colouring N(H), be a measure of the number of *weak colourings* of N(H).

Now consider a hypergraph H which has a strong circuit $A \subseteq E$.

Let $B = A - \{a\}$ for some $a \in A$, and let $K = (H_A, B)$. Then $\underline{M}(K)$ is a loop, and $N(K)$ is seen to be an edge with $(k-1)$ vertices. The chromatic polynomial of $N(K)$ is this $\lambda^{k-1} - \lambda$. This compares favourably with the graphical case. We therefore define the <u>chromatic polynomial of a loop</u> to be $\lambda^{k-1} - \lambda$.

From the preceding comments, therefore, $P(N(\text{loop}); \lambda) = \lambda^{k-1} - \lambda$ and $P(N(\text{coloop}); \lambda) = \lambda^{k} - \lambda$. Thus,

$T(\text{loop}; x,y) = (-1)^{\alpha_0}\lambda^{\beta_0}(\lambda^{k-1} - \lambda)$ and

$T(\text{coloop}; x,y) = (-1)^{\alpha_1}\lambda^{\beta_1}(\lambda^{k} - \lambda)$, whence

$x = (-1)^{\alpha_1}\lambda^{\beta_1}(\lambda^{k} - \lambda)$ and

$y = (-1)^{\alpha_0}\lambda^{\beta_0}(\lambda^{k-1} - \lambda)$.

We shall use $P(N(K); \lambda)$ to denote the weak chromatic polynomial of $N(K)$.

We shall assume that $N(H)$ is V-connected (if not, we can consider each V-connected component of $N(H)$ separately). We than have the result of Helgason [14] that

$P(N(H); \lambda) = \lambda^{\gamma(H)} \sum_{A \subseteq E} (-1)^{|A|} \lambda^{rE - rA}$ where $\gamma(H)$ is the number of V-components of $N(H)$, which in this case is equal to 1, and r is the chromatic hyperrank function defined by

$r(A) = |n(A)| - \gamma(H_A)$.

It is easy to show that

$r(A) = (k-1)|A|$ if A is independent in $\underline{M}(H)$,

$r(A) = (k-1)|A| - 1$ if A is a strong circuit of H, and

$r(A) = (k-1)|A|$ if A is a weak circuit of H.

PROPOSITION 10.5: Let $H = (V, E, \$)$ be a k-hypergraph with a weak circuit. Then there exist $E' \subseteq E$ and $K' \subseteq E'$ such that, if $K = (H_{E'}, K')$, then

$P(N(K); \lambda) \neq (-1)^{\alpha(K)}\lambda^{\beta(K)} T(\underline{M}(K); x,y)$ where x and y are as defined above, for any integers $\alpha(K)$ and $\beta(K)$.

Proof: Consider the weak circuit $A \subseteq E$, and let $B = A - \{a\}$ for some

$a \in \mathcal{A}$. Let $K = (H_{\mathcal{A}}, \mathcal{B})$. Then $M(K)$ is a loop, and $N(K)$ is the hypergraph consisting of an edge with k vertices and m (≥ 0) isolated vertices. Thus, the chromatic polynomial of $N(K)$ is $\lambda^m(\lambda^k - \lambda)$.

$\therefore \lambda^m(\lambda^k - \lambda) = (-1)^\alpha \lambda^\beta y$ for some integers α and β.

But we know that $y = (-1)^{\alpha_0} \lambda^{\beta_0}(\lambda^{k-1} - \lambda)$ for some integers α_0 and β_0.

Thus, $\lambda^m(\lambda^k - \lambda) = (-1)^{\alpha+\alpha_0} \lambda^{\beta+\beta_0}(\lambda^{k-1} - \lambda)$. But this must be an identity in λ, which is impossible. Thus, if H contains a weak circuit $\mathcal{A}$, and $K = (H_{\mathcal{A}}, \mathcal{A} - \{a\})$ $(a \in \mathcal{A})$, then $P(N(K); \lambda)$ is not equal to $(-1)^{\alpha(K)} \lambda^{\beta(K)} T(\underline{M}(K); x, y)$ for any integers α and β, where x and y are as defined earlier.

In order for the chromatic polynomial $P(N(H); \lambda)$ to be such that, for any $K = (H_{E'}, K')$ where $E' \subseteq E$ and K' is independent in $\underline{M}(H)$, $P(N(K); \lambda) = (-1)^{\alpha(K)} \lambda^{\beta(K)} T(\underline{M}(K); x, y)$ where x and y are as defined earlier, we must therefore restrict our attention to those hypergraphs H in which every circuit of H is a strong circuit.

LEMMA 10.6: Let H be a simple, strongly 2-connected k-hypergraph, in which every circuit of $\underline{M}(H)$ is a strong circuit. Then $H = (V, E, \$)$ is strictly pseudographic if and only if there exists a graph $G = (V', E, \$')$ such that $V_H(e) = V_G(e) \cup (V - V')$ for each $e \in E$.

Proof: If such a graph G exists, then clearly H is strictly pseudographic, since every edge of H has at most 2 valency nodes.

Conversely, suppose that H is strictly pseudographic. Consider a strong cycle $(e_1, e_2, \ldots, e_t)$ of H, and let $C = \{e_1, e_2, \ldots, e_t\}$. Then C is a circuit of $\underline{M}(H)$, and therefore $|V(C)| = k + |C| - 2$, and $|V(C')| \geq k + |C'| - 1$ for each nonempty proper subset C' of C.

Since C is a strong cycle, $|V(\{e_1, \ldots, e_r\})| \leq k + r - 1$ for each $r < t$.

Thus, $|V(\{e_1, \ldots, e_r\})| = k + r - 1$ for each $r < t$. Since H is simple, $t \geq 3$, so, for each i, $2 \leq i \leq t-1$, $|V(e_i) - V(\{e_1, \ldots, e_{i-1}\})| = 1$.

Let $v_i = V(e_i) - V(\{e_1, \ldots, e_{i-1}\})$ for $2 \leq i \leq t-1$. Then $v_i \in V(e_i)$.

If $V_i \notin V(e_{i+1})$, then $V(e_i) \cap V(e_{i+1}) = V(e_i) - \{V_i\} = V(e_i) \cap V(e_{i-1})$, so there exists a node common to three edges of $\mathcal{C}$, which is a contradiction. Thus, $V_i \in V(e_i) \cap V(e_{i+1})$, and so, for $3 \le i \le t-1$, $\{V_{i-1}, V_i\} \subseteq V(e_i)$. Now, $|(V(e_i) - \{V_{i-1}, V_i\}) \cap (V(e_{i-1}) - \{V_{i-2}, V_{i-1}\})| = k-2$, so $V(e_i) - \{V_{i-1}, V_i\} = V(e_{i-1}) - \{V_{i-2}, V_{i-1}\}$. Thus, for $3 \le i \le t-1$, $V(e_i) = X \cup \{V_{i-1}, V_i\}$ where $|X| = k-2$.

Now, $X \cup \{V_2\} \subseteq V(e_2)$, since $|(X \cup \{V_2, V_3\}) \cap V(e_2)| = k-1$. Therefore, $V(e_2) = X \cup \{V_1, V_2\}$, say, where $V_1 \notin V(\{e_3, e_4, \ldots, e_{t-1}\})$. Similarly, $V(e_1) = X \cup \{V_0, V_1\}$, where $V_0 \notin V(\{e_2, e_3, \ldots, e_{t-1}\})$. Now, $|V(e_t) \cap (X \cup \{V_{t-2}, V_{t-1}\})| = |V(e_t) \cap (X \cup \{V_0, V_1\})| = k-1$, and so $V_0 \in V(e_t)$, and $V(e_t) = X \cup \{V_{t-1}, V_0\}$.

A similar result holds for any other strong cycle of H. Since H is strongly 2-connected, and each edge has at most 2 valency nodes, $X \subseteq V(e)$ for each $e \in \mathcal{E}$. Put $V_G(e) = V(e) - X$. Then $G = (\mathcal{V} - X, \mathcal{E}, \$')$, where $\$' = \{(V,e) : V \in V_G(e), e \in \mathcal{E}\}$, is a graph satisfying the conclusions of the lemma.

PROPOSITION 10.7: Let $H = (\mathcal{V}, \mathcal{E}, \$)$ be a simple, strongly 2-connected hypergraph in which every circuit of $\underline{M}(H)$ is a strong circuit. Then, if H is not strictly pseudographic, there exists a sub-hypergraph $H_{E'}$ of H, and a generalised hypergraph $K = (H_{E'}, \mathcal{K}')$ such that $P(N(K); \lambda) \neq (-1)^{\alpha(K)} \lambda^{\beta(K)} T(\underline{M}(K); x, y)$ where x and y are as defined earlier, for any integers $\alpha(\underline{K})$ and $\beta(K)$.

Proof: Suppose H is not strictly pseudographic. Then there exists an edge $e \in \mathcal{E}$ with at least three valency nodes $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$. Since H is strongly 2-connected and simple, there is a V-cycle $(e, e_1, e_2, \ldots, e_t)$ $(t \ge 2)$ of N(H) such that $\mathcal{N}_1 = V(e_t) \cap V(e)$ and $\mathcal{N}_2 = V(e) \cap V(e_1)$. Consider a path $(e, f_1, f_2, \ldots, f_m)$ in H such that $V(e) \cap V(f_1) = \mathcal{N}_3$, and $n(f_m) \cap n(\{e_1, e_2, \ldots, e_t\}) \neq \phi$. Such a path exists, since H is strongly 2-connected. Now, there exists a least i for which

$n(f_i) \cap n(\{e_1,\ldots,e_t\}) \neq \phi$. Let $E' = \{e,e_1,\ldots,e_t,f_1,\ldots,f_i\}$, $K' = \{f_2,\ldots,f_i,e_1,\ldots,e_t\}$, and let $K = (H_{E'}, K')$. Then $\underline{M}(K)$ is a matroid on the set $\{e,f_1\}$ consisting of two loops. $N(K)$ is a hypergraph with two edges, each with at least two vertices, and with two vertices in common. From Helgason [14], we have that the chromatic polynomial $P(N(K);\lambda) = \lambda(\lambda^{|n'(e)|+|n'(f_1)|-2} - \lambda^{|n'(e)|-2} - \lambda^{|n'(f_1)|-2} + 1)$, where $n'(e)$ and $n'(f_1)$ are the vertex-sets in $N(K)$ of e and f_1 respectively. Suppose $P(N(K);\lambda) = (-1)^{\alpha(K)}\lambda^{\beta(K)}T(\underline{M}(K);x,y)$ where x and y are as defined earlier. Since $\underline{M}(K)$ is a matroid with two components, each of which is a loop, by (10.4)(iii),

$$T(\underline{M}(K);x,y) = T(\text{loop};x,y)T(\text{loop};x,y) = y^2.$$

Thus, $(-1)^{\alpha(K)}\lambda^{\beta(K)}\lambda^{2\beta_0}(\lambda^{k-1}-\lambda)^2$

$$= \lambda(\lambda^{|n'(e)|+|n'(f_1)|-2} - \lambda^{|n'(e)|-2} - \lambda^{|n'(f_1)|-2} + 1).$$

So, since this is an identity in λ, $(-1)^{\alpha(K)} = 1$; $\beta(K)+2\beta_0+2 = 1$, so $\beta(K) + 2\beta_0 = -1$; $-1 + 2(k-1) = |n'(e)|+|n'(f_1)|-1$; $-1 + (k-1) + 1 = |n'(e)|-1 = |n'(f_1)|-1$.

But this is a contradiction. Thus,

$P(N(K);\lambda) \neq (-1)^{\alpha(K)}\lambda^{\beta(K)}T(\underline{M}(K);x,y)$ where x and y are as defined earlier.

PROPOSITION 10.8: Let $H = (V, E, \$)$ be a simple, strongly 2-connected, strictly pseudo-graphic k-hypergraph such that every circuit of $\underline{M}(H)$ is a strong circuit. Then, for each $E' \subseteq E$, and each $K' \subseteq E'$ such that K' is independent in $\underline{M}(H_{E'})$, the generalised hypergraph $K = (H_{E'}, K')$ is such that $P(N(K);\lambda) = (-1)^{\alpha(K)}\lambda^{\beta(K)}\lambda^{\gamma(K)}T(\underline{M}(K);x,y)$ where $\alpha(K) = rk\underline{M}(K)$, $\beta(K) = (k-2)(|E'-K'|-rk\underline{M}(K))$, $\gamma(K)$ is the number of V-connected components of $N(K)$, $x = 1-\lambda^{k-1}$ and $y = 1-\lambda^{-(k-2)}$.

Proof: Suppose H satisfies the hypotheses of the proposition. Then, by (10.6), there exists a graph $G = (V', E', \$')$ such that $V_H(e) = V_G(e) \cup (V-V')$ for each $e \in E$. Now, to each generalised hypergraph $K = (H_{E'}, K')$, there corresponds a generalised hypergraph $(G_{E'}, K')$. But $(G_{E'}, K')$ is a graph G', and $N(K) \cong N(G'')$ where $V_{G''}(e) = V_{G'}(e) \cup (V-V')$ for each $e \in E''$. Furthermore, $\underline{M}(K) \cong M(G') \cong M(G'')$. For a strictly pseudographic hypergraph H in which every circuit of $\underline{M}(H)$ is a strong circuit. it is not difficult to check that $rA = \rho A + (k-2)|A|$ for any $A \subseteq E$, where r and ρ are the chromatic hyperrank function of N(H) and the rank function of $\underline{M}(H)$ respectively. In addition, $r''A = \rho''A + (k-2)|A|$ for any $A \subseteq E''$, where r'' and ρ'' denote the corresponding functions for the hypergraph G". From Helgason's result [14],

$$P(N(K);\lambda) = P(N(G'');\lambda) = \lambda^{\gamma(K)} \sum_{A \subseteq E''} (-1)^{|A|} \lambda^{r''E''-r''A}$$

$$= \lambda^{\gamma(K)} \sum_{A \subseteq E''} (-1)^{|A|} \lambda^{\rho''E''+(k-2)|E''|-\rho''A-(k-2)|A|}$$

$$= \lambda^{\gamma(K)} (-1)^{\rho''E''} \sum_{A \subseteq E''} (-1)^{\rho''E''-|A|} \lambda^{\rho''E''+(k-2)|E''|} \lambda^{-\rho''A-(k-2)|A|}$$

$$= \lambda^{\beta(K)} \lambda^{\gamma(K)} (-1)^{\alpha(K)} \sum_{A \subseteq E''} (-1)^{\rho''E''-\rho''A} \lambda^{(k-1)\rho''E''} (-1)^{\rho''A-|A|} \lambda^{-\rho''A-(k-2)|A|}$$

$$= (-1)^{\alpha(K)} \lambda^{\beta(K)} \lambda^{\gamma(K)} \sum_{A \subseteq E''} (-\lambda^{k-1})^{(\rho''E''-\rho''A)} (-\lambda^{-(k-2)})^{(|A|-\rho''A)}$$

$$= (-1)^{\alpha(K)} \lambda^{\beta(K)} \lambda^{\gamma(K)} T(\underline{M}(K);x,y) \text{ where } x = 1-\lambda^{k-1} \text{ and } y = 1-\lambda^{-(k-2)}.$$

THEOREM 10.9: Let $H = (V, E, \$)$ be a simple, strongly 2-connected k-hypergraph. Then:

(a) for any generalised hypergraph $K = (H_{E'}, K')$, $P(N(K);\lambda) = (-1)^{\alpha(K)} \lambda^{\beta(K)} \lambda^{\gamma(K)} T(\underline{M}(K);x,y)$ where $\alpha(K)$, $\beta(K)$, $\gamma(K)$, x and y are as defined in (10.8), if and only if

(b) H is strictly pseudographic, and every circuit of $\underline{M}(H)$ is a strong circuit.

Proof: (b) $\Rightarrow$ (a). If H is strictly pseudographic, and every circuit of $\underline{M}(H)$ is a strong circuit, then, by (10.8), every generalised hypergraph

$K = (H_{E'}, \mathcal{K}')$ satisfies (a).

(a) $\Rightarrow$ (b). We shall show that, if (b) is not satisfied, then (a) does not hold.

(i) If H has a weak circuit, then, by (10.5), there exists a generalised hypergraph $(H_{E'}, \mathcal{K}')$ for which (a) does not hold.

(ii) If every circuit of $\underline{M}(H)$ is a strong circuit, but H is not strictly pseudographic, then, by (10.7), there exists a generalised hypergraph $(H_{E'}, \mathcal{K}')$ for which (a) does not hold.

Since, if (b) is not satisfied, (i) or (ii) (or both) must hold, we have that, if (b) is not satisfied, then (a) is not satisfied. This completes the proof.

COROLLARY 10.10: Let $H = (V, E, \$)$ be a simple k-hypergraph (not necessarily strongly 2-connected). Then:

(a) for any generalised hypergraph $K = (H_{E'}, \mathcal{K}')$, $P(N(K); \lambda) = (-1)^{\alpha(K)} \lambda^{\beta(K)} \lambda^{\gamma(K)} T(\underline{M}(K); x,y)$ where $\alpha(K)$, $\beta(K)$, $\gamma(K)$, x and y are as defined in (10.8), if and only if

(b) H is pseudographic, and every circuit of $\underline{M}(H)$ is a strong circuit.

Proof: (b) $\Rightarrow$ (a). If every circuit of $\underline{M}(H)$ is a strong circuit, and H is pseudographic, then the edge-sets of the 2-V-connected components of N(H) are the connected components of $\underline{M}(H)$. Thus, each generalised hypergraph $K = (H_{E'}, \mathcal{K}')$ corresponds in the node-hypergraph to a set of subsets of the edge-sets of the 2-V-connected components of N(H). Let these subsets be $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$ (say) where each $\mathcal{K}_i \subseteq G_i$ and G_i is the edge-set of a 2-V-connected component of $\underline{M}(H)$. Denote $N(K)_{\mathcal{K}_i}$ by $N(K_i)$.

Then $P(N(K); \lambda) = \lambda^{-\gamma'(K)} \prod_{i=1}^{m} P(N(K_i); \lambda)$, where

$\gamma'(K) = \sum_{i=1}^{m} |n(\mathcal{K}_i)| - |n(\mathcal{K}_1 \cup \ldots \cup \mathcal{K}_m)|$.

Now, by (10.9), $P(N(K_i); \lambda) = (-1)^{\rho'' K_i} \lambda^{\beta(K_i)} \lambda^{\gamma(K_i)} T(\underline{M}(K_i); x,y)$

$\therefore P(N(K);\lambda) = \lambda^{-\gamma'(K)}(-1)^{\Sigma\rho'' K_i}\lambda^{\Sigma\beta(K_i)}\lambda^{\Sigma\gamma(K_i)}\prod_{i=1}^{m} T(\underline{M}(K_i);x,y)$

Since $\underline{M}(K) = (\underline{M}(K)\times K_1) \oplus \ldots \oplus (\underline{M}(K)\times K_m)$, by (10.4),

$$P(N(K);\lambda) = \lambda^{-\gamma'(K)}(-1)^{\rho''E'}\lambda^{|E'-K'|-\rho''E'}\lambda^{\Sigma\gamma(K_i)}T(M(K);x,y)$$
$$= (-1)^{\alpha(K)}\lambda^{\beta(K)}\lambda^{\gamma(K)}T(\underline{M}(K);x,y).$$

For the converse, we shall show that if (b) does not hold, then (a) cannot be satisfied.

(i) If H has a weak circuit, then, by (10.5), there exists a generalised hypergraph $(H_{E'}, K')$ for which (a) does not hold.

(ii) If every circuit of $\underline{M}(H)$ is a strong circuit, but H is not pseudographic, then there exists a 2-V-connected component of N(H) which is not strictly pseudographic, with edge-set E'', say. But then, by (10.7), there exists a generalised hypergraph $(H_{E'}, K')$ with $E' \subseteq E''$ for which (a) does not hold.

Since, if (b) is not satisfied, (i) or (ii) (or both) must hold, we have that, if (b) is not satisfied, then (a) cannot hold. This completes the proof.

THEOREM 10.11: Let $H = (V, E, \$)$ be a k-hypergraph (not necessarily simple, not necessarily strongly 2-connected). Then:

(a) for any generalised hypergraph $K = (H_{E'}, K')$, $P(N(K);\lambda) = (-1)^{\alpha(K)}\lambda^{\beta(K)}\lambda^{\gamma(K)}T(\underline{M}(K);x,y)$, where $\alpha(K)$, $\beta(K)$, $\gamma(K)$, x and y are as defined in (10.8), if and only if

(b) H is pseudographic and every circuit of $\underline{M}(H)$ is a strong circuit.

Proof: (b) ⇒ (a). If H is simple, this follows from (10.10). Suppose H is not simple. Since H is pseudographic, every edge of H has at most 2 valency nodes. But, a pair of parallel edges $\{e_1,e_2\}$ of H has $n(e_1) = n(e_2)$, so we must have $|n(e_1)| = |n(e_2)| = 2$. Thus, $k = 2$, and (a) follows, since H (and hence K) is a graph.

(a) $\Rightarrow$ (b). We shall show that if (b) is not satisfied, then (a) cannot hold.

(i) If H is simple, this follows from (10.10).

(ii) If H is not simple, and $k = 2$, then (b) is always true, and there is nothing to prove.

(iii) If H is not simple, and $k \geq 3$, consider a pair $\{e_1, e_2\}$ of edges of H, with $V(e_1) = V(e_2)$. Let $K = (H_{\{e_1,e_2\}}, \{e_2\})$. Then M(K) is a loop, and N(K) is an edge with a single vertex. Thus, $P(N(K); \lambda) = 0$ for any λ. Now, $T(\underline{M}(K); x,y) = y = 1-\lambda^{-(k-2)}$. Thus, since $k \neq 2$, (a) does not hold for $K = (H_{\{e_1,e_2\}}, \{e_2\})$.

Since, if (b) is not satisfied, at least one of (i), (ii) and (iii) must hold, we have that, if (b) is not satisfied, then (a) does not hold. This completes the proof.

It can be shown that, apart from strictly pseudographic hypergraphs, the only strongly-connected hypergraphs in which every circuit is strong are the subhypergraphs of K^k_{k+1} for $k \geq 3$, with, possibly, isolated vertices. In this case, $\underline{M}(H) \cong U_{2,n}$ for some n. It is easy to calculate $P(N(H); \lambda)$ for such a hypergraph, but, if $n \geq 4$, it does not bear much relation to $T(\underline{M}(H); x,y)$.

We now turn from an investigation of the weak chromatic polynomial of the node-hypergraph to a study of the weak chromatic number of the node-hypergraph. Helgason, in his paper "Geometric Hypergraphs" [13], proves that the members of a restricted class of hypergraphs all have weak chromatic number 2. To demonstrate just how restrictive this class is, we state his result, and then show which node-hypergraphs satisfy his conditions. We shall then consider the weak chromatic number of more general node-hypergraphs.

DEFINITION 10.12: Let $H = (V, E, \$)$ be a k-hypergraph. Then the <u>covering closure</u> of a set $A \subseteq E$ is the set

$\hat{A} = \{e \in E : V(e) \subseteq V(A)\}$.

The operator ^ is said to be <u>geometric</u> if it satisfies the matroid closure axioms (K1) - (K4).

A hypergraph is said to be <u>covering-geometric</u> if its covering closure operator is geometric.

THEOREM 10.13 (Helgason [13]): Let $H = (V, E, \$)$ be a connected (in the sense of Berge [1]), covering-geometric hypergraph such that $\cap\{V(e) : V \in V(e)\} = \{V\}$ for each $V \in V$. Then, either H is a (one-connected) graph, or H is weakly 2-colourable.

Proof: see [13].

THEOREM 10.14: Let $H = (V, E, \$)$ be a simple, strongly-connected, k-hypergraph. Then

(a) N(H) is covering-geometric if and only if

(b) either (i) for each $e \in E$, there exists a node of e which is not a valency node of e;

or (ii) $H \cong K^k_{k+1}$.

Proof: By (9.17), H is strongly connected if and only if N(H) is V-connected (i.e. connected in the sense of Berge [1]).

(b) => (a) is easy to prove, and we omit the details.

(a) => (b). Suppose N(H) is covering-geometric, and that (b)(i) does not hold. Then there exists $e \in E$ such that every node of e is a valency node. For each node $N_i \subseteq V(e)$ $(1 \leq i \leq k)$, let $a_i \in E - \{e\}$ be such that $N_i \subseteq V(a_i)$. Since each N_i is a valency node of e, there must exist at least one such a_i for each i. Furthermore, since H is simple, and N(H) is its node-hypergraph, $V(a_i) \neq V(a_j)$ for $i \neq j$, and $V(a_i) \neq V(e)$ for any i. Let $A = \{a_i : 1 \leq i \leq k\}$.

Then $e \in \hat{A}$, and so, since $N(H)$ is covering-geometric, for each i $(1 \le i \le k)$, $a_i \in \widehat{(A-\{a_i\})\cup\{e\}}$. Thus, each node $\mathcal{N} \in n(A\cup\{e\})$ is a subset of the vertex-sets of at least two elements of $A\cup\{e\}$. Therefore, $|n(A\cup\{e\})|\times 2 \le (k+1)k$. However, since H is simple, each edge can have at most one node in common with each other edge, so $|n(A\cup\{e\})| \ge \frac{1}{2}k(k+1)$. Thus, equality holds, and so each node is a subset of the vertex-sets of exactly two edges of $H_{A\cup\{e\}}$. Thus, $H_{A\cup\{e\}} \cong K^k_{k+1}$. We note that, therefore, $H_{A\cup\{e\}}$ is symmetric in the elements of $A\cup\{e\}$.

Now suppose there exists $e'' \notin A\cup\{e\}$. Then, since $N(H)$ is V-connected, there exists $e' \notin A\cup\{e\}$ such that $n(e')\cap n(A\cup\{e\}) \ne \phi$. By the symmetry referred to above, there is no loss of generality in assuming that $n(e')\cap n(e) = \mathcal{N}_1$. But, applying the above argument to the set $(A-\{a_1\})\cup\{e,e'\}$, we obtain $H_{(A-\{a_1\})\cup\{e,e'\}} \cong K^k_{k+1}$, and so $V(e') = V(a_1)$ which, since H is simple, is a contradiction. Thus, there exists no such e', and $H \cong K^k_{k+1}$.

For case (b)(i) of (10.14), the condition $\cap\{n(e)\colon \mathcal{N} \subseteq V(e)\} = \mathcal{N}$ for each $\mathcal{N} \in n(H)$ is not satisfied, and so the conditions of (10.13) are not satisfied for $N(H)$. Thus, the only node-hypergraphs which satisfy (10.13) are $N(K^k_{k+1})$ for $k \ge 2$.

For general node-hypergraphs, therefore, we do not have the conditions of (10.13) satisfied. It is thus not surprising to find that the conclusions of (10.13) do not hold for general node-hypergraphs. In order to demonstrate this, we first observe that there is a convenient representation of $N(K^k_p)$ which simplifies the proofs somewhat.

PROPOSITION 10.15: Let $H = (V, E, \$) \cong K_p^{k-1}$. Then there is a 1-1 correspondence θ between E and $n(K_p^k)$ such that θ gives rise to an isomorphism between the elements of the non-trivial 2-flats of $\underline{M}(H)$ and the edges of K_p^k.

Proof: Let $H' = (V', E', \$') \cong K_p^k$; let $V = \{V_i : 1 \le i \le p\}_{\neq}$, $V' = \{V_i' : 1 \le i \le p\}_{\neq}$, and let ψ be the obvious bijection between V and V'.

Then, for each $e \in E$, $\{\psi(V): V \in V(e)\} \in n(H')$, and, for each $N' \in n(H')$, $\{\psi^{-1}(V'): V' \in N'\} \in E$. Thus, ψ gives rise to an isomorphism θ between E and $n(H')$. The non-trivial 2-flats of $\underline{M}(H)$ are the sets $\{e \in E: V(e) \subseteq W, |W| = k\}$. Thus, $\{e: e \in F\}$ is a non-trivial 2-flat of $\underline{M}(H)$ if and only if $\{\psi(V): V \in V(e)$ for some $e \in F\} = V(e')$ for some $e' \in E'$. This completes the proof.

Let us distinguish by a capital letter the non-trivial lines of $\underline{M}(H)$, and call them "Lines" of H. Then, by (10.15), a weak q-colouring of $N(K_p^k)$ is equivalent to a partition of the edges of K_p^{k-1} into q classes so that no Line is contained wholly in one class. In the particular case $k = 3$, we require a colouring of the edges of K_p with q colours so that K_p has no monochromatic triangles. Berge [1] calls this a good q-colouring. The problem of what values of q will give a good q-colouring of K_p is essentially a Ramsey-type problem; K_5 has a good 2-colouring, but K_6 has not.

DEFINITION 10.16: Let $Q = \{E_1, E_2, \ldots, E_q\}$ be a partition of $E(K_p^{k-1})$ into q classes. Then Q is said to be a k-good q-colouring of K_p^{k-1} if no Line of K_p^{k-1} is a subset of E_i for any i.

We shall prove that, for any $q \ge 2$, and any $k > 2$, there exists p such that K_p^{k-1} has a k-good q-colouring, but not a k-good (q-1)-colouring. From (10.15) and the discussion following it, this is equivalent to the statement that $N(K_p^k)$ has weak chromatic number q. The generalised Ramsey theorem that we require is

THEOREM 10.17 (Ramsey): Given an integer h and a set $\{p_1,\ldots,p_q\}$ of integers, none less than h, there exists a finite integer $R_h(p_1,\ldots,p_q)$ such that, if X is any set with $|X| > R_h(p_1,\ldots,p_q)$, then, for any partition $\{\mathbf{T}_1, \mathbf{T}_2,\ldots, \mathbf{T}_q\}$ of the set $\mathbf{P}_h(X)$ of h-subsets of X, there exists a class $\mathbf{T}_i$ and a set $A_i \subseteq X$ such that $|A_i| = p_i$ and $\mathbf{P}_h(A_i) \subseteq \mathbf{T}_i$.

We define $n_k(q) = R_{k-1}(\underbrace{k,k,\ldots,k}_{q \text{ terms}})-1$; then $n_k(q)$ is the maximum value of p for which K_p^{k-1} has a k-good q-colouring.

Berge [1] proves that $K_{n_2(q)+1}$ has a 3-good (q+1)-colouring, but no 3-good q-colouring. An analogous proof holds for $k > 2$.

THEOREM 10.18: $K^{k-1}_{n_k(q)+1}$ has a k-good (q+1)-colouring, but no k-good q-colouring.

Proof: By definition of $n_k(q)$, $K^{k-1}_{n_k(q)+1}$ has no k-good q-colouring, but $K^{k-1}_{n_k(q)}$ has a k-good q-colouring. Let Q be a k-good q-colouring of $H = K^{k-1}_{n_k(q)}$, and let $V \in V(H)$, $W \notin V(H)$ be two vertices. We shall now form a hypergraph $K^{k-1}_{n_k(q)+1}$ on $V(H)\cup\{W\}$, and produce a k-good (q+1)-colouring of it. Let H' be such a hypergraph, and identify $e \in E(H)$ with that $e'' \in E(H')$ with $V_H(e) = V_{H'}(e'')$. For each $e \in E(H')$ with $V_{H'}(e) \subseteq V(H)$, let $e' \in E(H')$ be the unique edge with $V_{H'}(e') = (V_H(e)-\{V\})\cup\{W\}$. Colour the edges e and e' with the colour of Q used to colour e in H. For each $f \in E(H')$ with $\{V,W\} \subseteq V_{H'}(f)$, colour f with a new colour (q+1). Then it is easy to see that the resulting colouring is a k-good (q+1)-colouring of H'.

COROLLARY 10.19: $N(K^k_{n_k(q)+1})$ can be weakly coloured with (q+1) colours, but not with q colours.

COROLLARY 10.20: $N(N^k_{n_k(q)})$ has weak chromatic number (q+1) for $q \geq 2$.

Proof: By (10.18), $n_k(q) \geq n_k(q-1)+1$, so $N(K^k_{n_k(q)})$ can be weakly coloured with q colours, but not with (q-1) colours.

COROLLARY 10.21: Given integers $q \geq 2$, $k \geq 3$, there exists an integer p such that $N(K^k_p)$ has weak chromatic number q.

We have remarked several times that the components of a hypergraph partition the nodes, in the sense that, if $\{H_{G_i} : 1 \leq i \leq m\}$ is the set of components of H, and $\mathcal{N} \in n(H)$, then $\mathcal{N} \subseteq V(G_i)$ for exactly one value of i. It is therefore possible that an adaptation of (10.3) to the node-hypergraph might yield a well-defined colouring.

THEOREM 10.22: Let $H = (V, E, \$)$ be a simple k-hypergraph with $|V| = p$. Embed H in the hypergraph K^k_p on the set V of vertices, and let $J_1, J_2, \ldots, J_c$ be a set of hyperplanes of $\underline{M}(K^k_p)$ such that $J_1 \cap J_2 \cap \ldots \cap J_c \cap E = \phi$.

For each hyperplane J, let $(K^k_p)_{G_i(J)}$ $(1 \leq i \leq m)$ denote the components of $(K^k_p)_J$, and let $\mathcal{N}_i(J) = n(G_i(J))$ $(1 \leq i \leq m_J)$.

Let $\{\mathcal{N}_i(J) : m_J+1 \leq i \leq i_J\}_{\neq}$ denote the set $\{\mathcal{N} \in n(K^k_p) : \mathcal{N} \not\subseteq V(G_j),\ 1 \leq j \leq m_J\}_{\neq}$.

With each $\mathcal{N} \in n(H)$ associate the vector $\underline{\mathcal{N}} = (a_1, a_2, \ldots, a_c)$, where $\mathcal{N} \in \mathcal{N}_{a_i}(J_i)$. Now associate a different colour with each distinct vector so produced, and, for each $\mathcal{N} \in n(H)$, colour $\mathcal{N}$ with the colour associated with $\underline{\mathcal{N}}$. Partition n(H) into sets of nodes coloured with the same colour. Then this partition is a strong colouring of N(H).

Proof: Let $e \in \mathcal{E}$. Then, since $J_1 \cap J_2 \cap \ldots \cap J_c \cap \mathcal{E} = \phi$, there exists i for which $e \notin J_i$. For definiteness, suppose i = 1. Now, if $|n(e) \cap N_j(J_1)| \geq 2$ for any j, $1 \leq j \leq m_{J_1}$, by (9.14) $V(e) \subseteq V(G_j(J_1))$. But $(K^k_p)_{G_j(J_1)}$ is a complete k-hypergraph, so this implies that $e \in G_j(J_1) \subseteq J_1$, which is a contradiction. Thus, $|n(e) \cap N_j(J_1)| \leq 1$ for each j, $1 \leq j \leq m_{J_1}$. Since $|N_j(J_1)| = 1$ for each j with $m_{J_1}+1 \leq j \leq i_{J_1}$, we have $|n(e) \cap N_j(J_1)| \leq 1$ for each j, $1 \leq j \leq i_{J_1}$. Let the k nodes of e be $N_1, N_2, \ldots, N_k$. Then the first components of the vectors $\underline{N}_j$ $(1 \leq j \leq k)$ associated with the k nodes of e are all different, and so each vector is associated with a different colour. Thus, the k nodes of e are all in different colour classes, and so e is strongly coloured. Since this holds for each $e \in \mathcal{E}$, N(H) is strongly coloured.

It is not possible to obtain every strong colouring of N(H) by the method of (10.22). For example, let $H = (V, \mathcal{E}, \$)$, where $V = \{A,B,C,D,E\}_{\neq}$, $\mathcal{E} = \{a,b\}_{\neq}$, $V(a) = \{A,B,C,D\}$ and $V(b) = \{A,B,C,E\}$. Let Q be the colouring which colours the nodes {A,B,D} and {B,C,E} with the same colour, and colours all other nodes of H with different colours. Then, if this colouring arises from (10.22), {A,B,D} and {B,C,E} are subsets of the vertex-set of the same component of each of the hyperplanes $J_1, J_2, \ldots, J_c$ used to obtain the colouring. But this is impossible, because any complete subhypergraph of K^3_5 on V whose vertex-set contains {A,B,D} and {B,C,E} as subsets must contain every vertex of K^3_5 on V, and hence must be K^3_5 on V.

Again, consider $H = (V, \mathcal{E}, \$)$, where $V = \{A,B,C,D\}_{\neq}$, $\mathcal{E} = \{a,b,c\}_{\neq}$, $V(a) = \{A,B,C\}$, $V(b) = \{A,C,D\}$ and $V(c) = \{A,B,D\}$. Let Q be the colouring which colours {B,C} and {B,D} with the same colour, and colours all the other nodes of H with different colours. Then this colouring cannot arise from (10.22), since every component of a hyperplane of $\underline{M}(K_4^3)$ whose vertex-set contains {B,C} and {B,D} necessarily contains {C,D} as well, and so {C,D} would have to be coloured with the same colour as {B,C} and {B,D}.

We next prove a necessary and sufficient condition on the strong colouring Q of N(H) for Q to be derived from a set of hyperplanes by the construction of (10.22).

THEOREM 10.23: Let $H = (V, E, \$)$ be a simple k-hypergraph with $|E| \geq 1$, and let $Q = \{\Delta_1, \Delta_2, \ldots, \Delta_q\}$ be a strong colouring of N(H). Let $|V| = p$. Then

(a) there exist hyperplanes $J_1, J_2, \ldots, J_c$ of $\underline{M}(K_p^k)$ such that Q is derived from $J_1, J_2, \ldots, J_c$ by the method of (10.22)

if and only if

(b) for each $I \subseteq \{1,2,\ldots,q\}$ with $|I| \geq 2$,

$\sum_I |V(\Delta_i)| \leq |\bigcup_I V(\Delta_i)| + (k-1)(|I|-1)-1$, where

$V(\Delta_i) \equiv \cup\{N \in n(H): N \in \Delta_i\}$.

Proof: Without loss of generality, assume that Q is such that $|\Delta_i| > 1$ for $1 \leq i \leq q'$, and that $|\Delta_i| = 1$ for $q'+1 \leq i \leq q$. Let H be embedded in K_p^k on V (this is possible, since H is simple), and define $Y_i = \{e \in E(K_p^k): V(e) \subseteq V(\Delta_i)\}$ $(1 \leq i \leq q')$. Let $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_{q'}$.

(a) $\Rightarrow$ (b). Suppose Q is derived from the hyperplanes $J_1, J_2, \ldots, J_c$ by the method of (10.22). Then $Y = J_1 \cap J_2 \cap \ldots \cap J_c$. For, if $e \in Y$, $V(e) \subseteq V(Y_i)$ for some i, and so $e \in J_j$ for each j $(1 \leq j \leq c)$. If $e \notin Y$, $V(e) \nsubseteq V(Y_i)$ for each i, so there exists j for which $e \notin J_j$. Thus, Y is a flat of $\underline{M}(K_p^k)$. Since $|E| \geq 1$, $Y \neq E(K_p^k)$. By the construction of (10.22), each Y_i is the intersection of a component of each of the J_j, and so, since each H_{Y_i} is complete, the H_{Y_i} are the components of H_Y. If $I \subseteq \{1,2,\ldots,q\}$ with $|I| \geq 2$, let $I_1 = I \cap \{1,2,\ldots,q'\}$ and $I_2 = I \cap \{q'+1,\ldots,q\}$. Then, if $|I_1| \geq 2$,

$$\sum_I |V(\Delta_i)| = \sum_{I_1} |V(Y_i)| + \sum_{I_2} (k-1) = \sum_{I_1} (\rho(Y_i) + k - 1) + |I_2|(k-1)$$

$=\rho(\bigcup_{I_1} Y_i) + |I|(k-1)$

$<|V(\bigcup_{I_1} Y_i)| - (k-1) + |I|(k-1)$ since $|I_1| \geq 2$, and so $H_{\bigcup_{I_1} Y_i}$ is not critical.

$=|\bigcup_{I_1} V(\Delta_i)| + (k-1)(|I|-1)$

$\leq |\bigcup_{I} V(\Delta_i)| + (k-1)(|I|-1)$.

If $I_1 = \{i_1\}$, say, let $i_2 \in I_2$. Then

$\sum_I |V(\Delta_i)| = |V(\Delta_{i_1})| + |V(\Delta_{i_2})| + (k-1)(|I|-2)$

$\leq |V(\Delta_{i_1} \cup \Delta_{i_2})| + (k-2) + (k-1)(|I|-2)$, since, by the construction of (10.22), if $|V(\Delta_{i_1}) \cap V(\Delta_{i_2})| \geq k-1$, $\Delta_{i_2} \subseteq \Delta_{i_1}$.

If $I_1 = \phi$, then, since $|V(\Delta_i)| = k-1$ for each $i \in I_2$, and $V(\Delta_i) \neq V(\Delta_j)$ for $i,j \in I_2$, $i \neq j$, (b) follows immediately.

Thus, in each case, if $I \subseteq \{1,2,\ldots,q\}$ with $|I| \geq 2$,

$\sum_I |V(\Delta_i)| \leq |\bigcup_I V(\Delta_i)| + (k-1)(|I|-1)-1$.

(b) $\Rightarrow$ (a). For each i, $1 \leq i \leq q'$, let $A_i \subseteq Y_i$ be a critical set such that $V(A_i) = V(Y_i)$. Such a set exists since H_{Y_i} is complete, and hence critical. Now, for each i, let $B_i \subseteq A_i$ with $B_i \neq \phi$. Then, for any $I \subseteq \{1,2,\ldots,q'\}$ with $|I| \geq 2$,

$0 \leq \sum_I |V(B_i)| - |\bigcup_I V(B_i)| \leq \sum_I |V(A_i)| - |\bigcup_I V(A_i)|$ since $B_i \subseteq A_i$ for each $i \in I$,

$\leq (k-1)(|I|-1)-1$ by hypothesis.

$\therefore \quad |\bigcup_I V(B_i)| \geq \sum_I |V(B_i)| - (k-1)(|I|-1)+1$

$\geq \sum_I (|B_i|+k-1) - (k-1)(|I|-1)+1$ since each B_i is nonempty and independent

$\geq \sum_I |B_i| + k$.

Since this holds for each $I \subseteq \{1,2,\ldots,q'\}$ with $|I| \geq 2$, and for each nonempty subset $B_i \subseteq A_i$, and since each $B_i \subseteq A_i$ is independent in $\underline{M}(K_p^k)$, $\bigcup_I A_i$ is independent in $\underline{M}(K_p^k)$. Conversely, if $\bigcup_I A_i$ is independent in $\underline{M}(K_p^k)$, each A_i is independent. Thus, $\underline{M}(K_p^k)\times Y = \underline{M}(K_p^k)\times Y_1 \oplus \ldots \oplus \underline{M}(K_p^k)\times Y_{q'}$, and so the Y_i are the components of $\underline{M}(K_p^k)\times Y$ (1).

Now let $e \in \sigma(A_1 \cup A_2 \cup \ldots \cup A_{q'}) - (A_1 \cup A_2 \cup \ldots \cup A_{q'})$, and suppose that $\{e\} \cup (\cup\{B_i : i \in J\})$ is a circuit, where $B_i \neq \phi$ for each $i \in J$, the B_i being as defined above. Then, since each H_{Y_i} is critical, $H_{\bigcup_J Y_i}$ is critical, since, by (2.3), if C is a circuit, $V(C) \subseteq V(X)$ for some critical set X, and, by (2.6), if two fragments $(U, E_U, \$_U)$ and $(W, E_W, \$_W)$ have a common edge, then $(U \cup W, E_{U\cup W}, \$_{U\cup W})$ is a fragment. Thus, $|\underset{J}{\cup} V(\Delta_i)| = |V(\underset{J}{\cup} Y_i)| = |V(\underset{J}{\cup} A_i)| = k + |\underset{J}{\cup} A_i| - 1$

$$= k - 1 + \sum_J |A_i|$$

$$= k - 1 + \sum_J |V(A_i)| - |J|(k-1)$$

$$= \sum_J |V(\Delta_i)| - (k-1)(|J| - 1),$$

which is a contradiction of (b) if $|J| \geq 2$. Thus, $|J| = 1$, and so $e \in \sigma(A_1 \cup \ldots \cup A_{q'}) \Rightarrow e \in \sigma(A_i) = Y_i$ for some i. Thus, Y is a flat of $\underline{M}(K_p^k)$. Now, Y cannot be a flat of $\underline{M}(K_p^k)$ of full rank, since, if it were, from (1) we would have $V(\Delta_1) = V$. Now, $q > 1$, since $|E| \geq 1$ and Q is a strong colouring of N(H). Thus, there exists $\Delta_2 \neq \phi$, $\Delta_2 \neq \Delta_1$. Then, by (b), $|V(\Delta_1)| + |V(\Delta_2)| \leq |V(\Delta_1 \cup \Delta_2)| + (k-2)$, whence $|V(\Delta_2)| \leq (k-2)$, which is impossible. Thus Y is a flat of $\underline{M}(K_p^k)$ of less than full rank, and so there exist hyperplanes $J_1, J_2, \ldots, J_c$ of $\underline{M}(K_p^k)$ such that $Y = J_1 \cap J_2 \cap \ldots \cap J_c$. It is easy to see that Q is derived from $J_1, J_2, \ldots, J_c$ by the method of (10.22).

It is clear from the results in this chapter that we are dealing with two essentially different methods of colouring N(H); one which gives weak colourings, and one which gives strong colourings. From the matroid point of view, therefore, these are the natural generalisations of the vertex-colourings of graphs.

Since weak and strong colourings coincide in graphs, it is not surprising that there is a relationship between the methods in this case. As has been shown by Crapo-Rota [6], the connection is *via* the critical problem for representable matroids. As we noted at the beginning of this chapter, there are several facts which are crucial to this result, and these facts are, in general, not true for hypergraphic matroids.

Even if it were possible to find a formula for the number of hyperplanes of $\underline{M}(K_p^k)$ whose common intersection with the edge-set of an embedding of a hypergraph H is empty, this still would not give a strong chromatic number or strong chromatic polynomial for N(H) because, for many values of k and p, the number of components of $(K_p^k)_J$ for a hyperplane J of $\underline{M}(K_p^k)$ is not constant.

As a final result on colouring node-hypergraphs, we give a result due to Grünbaum [9]. Although his result is given in terms of simplicial complexes, he uses no properties of these not possessed by the edges and nodes of uniform hypergraphs. We first define the notion of embedding a hypergraph in Euclidean space.

DEFINITION 10.24: Let $H = (V, E, \$)$ be a simple k-hypergraph. Then H is said to be <u>embeddable in Euclidean d-space E^d</u> if there exist distinct points a_V $(V \in V)$ of E^d such that, for each $e \in E$, and each $W \subseteq V(e)$, $\dim(\mathrm{co}W) \geq |W| - 1$, and, for any subset $\{e_i : i \in I\} \subseteq E$, $\bigcap_I(\mathrm{co}(\underline{V}(e_i)) = \mathrm{co}(\bigcap_I(V(e_i)))$, where $\mathrm{co}(X)$ for $X \subseteq V$ is the convex hull of $\{a_V : V \in X\}$.

THEOREM 10.25: (Grünbaum [9]). If H is a simple k-hypergraph that is embeddable in E^k, then there exists a strong colouring of N(H) in $6(k-1)$ colours.

CHAPTER 11

A GENERALISATION OF SERIES-PARALLEL EXTENSION

In this chapter, we shall use ideas derived from hypergraphs to generalise the notion of a series-parallel extension. Since the definition of series-parallel extension is motivated by graph-theoretic considerations, this is a reasonable approach. The chapter falls into three main sections. In the first section, we shall introduce an operation of matroid union which we term "pointed union". Dr. J.H. Mason has pointed out that this operation has been described previously by other authors, including Bixby [3]. In the second section, we shall define generalised series-parallel extension, and compare the properties of this operation with those of series-parallel extension. In the final section, we define a generalised series-parallel network, and characterise the class of generalised series-parallel networks by a set of six forbidden minors. An extension of this leads to a characterisation of ternary base-orderable matroids.

POINTED UNIONS:

It is clear that, given two graphs G_1 and G_2, we can form a new graph by identifying one edge of G_1 with an edge of G_2, provided neither edge is a loop, but otherwise keeping the vertex-sets and edge-sets disjoint. The pointed union of two matroids is the translation of this operation into matroid theory.

DEFINITION 11.1: Let $\underline{M}_1$, $\underline{M}_2$ be matroids on the sets $E_1 \cup \{x\}$, $E_2 \cup \{x\}$ respectively, where $E_1 \cap E_2 = \phi$, $x \notin E_1 \cup E_2$ and x is not a loop or coloop of $\underline{M}_1$ or $\underline{M}_2$. Then the <u>pointed union</u> of $\underline{M}_1$ and $\underline{M}_2$, denoted by $\underline{M}_1 \dot{\cup} \underline{M}_2$ is the matroid $\underline{M} = (E_1 \cup E_2 \cup \{x\}, \mathbf{B})$ with set of bases

$\mathbf{B} = \{B_1 \cup I_2 : B_1 \in \mathbf{B}(\underline{M}_1),\ I_2 \cup \{x\} \in \mathbf{B}(\underline{M}_2)\} \cup$

$\cup \{B_2 \cup I_1 : B_2 \in \mathbf{B}(\underline{M}_2),\ I_1 \cup \{x\} \in \mathbf{B}(\underline{M}_1)\}.$

The deleted pointed union of $\underline{M}_1$ and $\underline{M}_2$, written $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is the matroid $(\underline{M}_1 \dot{\cup} \underline{M}_2) \times (E_1 \cup E_2)$.

PROPOSITION 11.2: With the above notation, $\underline{M}_1 \dot{\cup} \underline{M}_2$ is a matroid on the set $E_1 \cup E_2 \cup \{x\}$.

Proof: Routine verification of the base axioms.

PROPOSITION 11.3: With the above notation, the set of circuits of $\underline{M}_1 \dot{\cup} \underline{M}_2$ is given by

$\mathbf{C}(\underline{M}_1 \dot{\cup} \underline{M}_2) = \mathbf{C}(\underline{M}_1) \cup \mathbf{C}(\underline{M}_2) \cup \{(C_1 \Delta C_2) : C_i \in \mathbf{C}(\underline{M}_i) \text{ and } x \in C_i,\ i = 1,2\}.$

Proof: Routine verification.

Although there is a superficial similarity between the definition of pointed union and the "series connection" defined by Brylawski [4], in fact the concepts are quite different; for example, the "series connection" of two matroids $\underline{M}_1$ and $\underline{M}_2$ has rank $rk\underline{M}_1 + rk\underline{M}_2$, whereas the pointed union of $\underline{M}_1$ and $\underline{M}_2$ has rank $rk\underline{M}_1 + rk\underline{M}_2 - 1$.

PROPOSITION 11.4: PROPERTIES OF THE POINTED UNION.

Let $\underline{M}_1$, $\underline{M}_2$ be matroids on $E_1 \cup \{x\}$, $E_2 \cup \{x\}$, where $E_1 \cap E_2 = \phi$, $x \notin E_1 \cup E_2$ and x is not a loop or coloop of $\underline{M}_1$ or $\underline{M}_2$. Then:

(a) $\underline{M}_1 \dot{\cup} \underline{M}_2$ is representable over the field F if and only if $\underline{M}_1$ and $\underline{M}_2$ are representable over F;

(b) $\underline{M}_1 \dot{\cup} \underline{M}_2$ is base-orderable if and only if $\underline{M}_1$ and $\underline{M}_2$ are base-orderable;

(c) $\underline{M}_1 \dot{\cup} \underline{M}_2$ is fully base-orderable if and only if $\underline{M}_1$ and $\underline{M}_2$ are fully base-orderable;

(d) $\underline{M}_1 \dot{\cup} \underline{M}_2$ is hypergraphic if and only if $\underline{M}_1$ and $\underline{M}_2$ are hypergraphic.

Proof: Since each of the properties mentioned is preserved under the operation of matroid restriction, and since $\underline{M}_i = (\underline{M}_1 \dot{\cup} \underline{M}_2) \times (E_i \cup \{x\})$ $(i = 1,2)$, one half of each equivalence is trivial.

The converses are proved as follows:

(a) Assume $\underline{M}_1$ and $\underline{M}_2$ are representable over F. Let $I_i \cup \{x\}$ be a base of $\underline{M}_i$ (i = 1,2). Let V be a vector space of dimension $|I_1|+|I_2|+1$ over F, and let $\{\underline{V}(e): e \in I_1 \cup I_2 \cup \{x\}\}$ be a basis of V. Let V_i be the subspace of V spanned by $\{\underline{V}(e): e \in I_i \cup \{x\}\}$ (i = 1,2). Then, since $\underline{M}_i$ is representable over F, there exists a map $\theta_i : E_i \cup \{x\} \to V_i$ such that for any $\{x_1, x_2, \ldots, x_m\} \underset{\neq}{\subseteq} E_i \cup \{x\}$, $\{x_1, x_2, \ldots, x_m\} \in \mathbf{I}(\underline{M}_i)$ if and only if $\{\theta_i x_1, \theta_i x_2, \ldots, \theta_i x_m\}$ is linearly independent in V_i, and such that $\theta_i e = \underline{V}(e)$ for $e \in I_i$, and $\theta_i x = \underline{V}(x)$. Define

$$\theta(s) = \begin{cases} \theta_1(s) & \text{if } s \in E_1 \cup \{x\} \\ \theta_2(s) & \text{if } s \in E_2 \cup \{x\} \end{cases}$$

Then it is easy to check that θ is a representation of $\underline{M}_1 \dot{\cup} \underline{M}_2$.

(b) and (c) follow from routine checking of the various possibilities for two bases in a base-ordering or full base-ordering.

(d) Suppose that $\underline{M}_i \cong \underline{M}(H_i)$ (i = 1,2), where $\underline{H}_i = (V_i, E_i' \cup \{x'\}, \$_i)$ is a k_i-hypergraph and the isomorphism is induced by the obvious bijection between $E_i \cup \{x\}$ and $E_i' \cup \{x'\}$. Suppose, without loss of generality, that $k_1 \geq k_2$; let V_2' be a set of cardinality $k_1 - k_2$ disjoint from V_1 and V_2, and define $E_2' = (V_2 \cup V_2', E_2' \cup \{x'\}, \$_2')$ by $V_{H_2'}(e') = V_{H_2}(e) \cup V_2'$ for each $e' \in E_2' \cup \{x'\}$. Then $\underline{M}(H_2) = \underline{M}(H_2')$.

Let $V_{H_1}(x) = X$, and let $H_2'' = (V_2'', E_2' \cup \{x'\}, \$_2'')$ be a hypergraph isomorphic to H_2', with $V_2'' \cap V_1 = X$, and $X = V_{H_2''}(x')$. Define $H = (V, E', \$)$, where $V = V_1 \cup V_2''$, $E' = E_1' \cup E_2' \cup \{x'\}$, and

$$V_H(e') = \begin{cases} V_{H_1}(e') & \text{if } e' \in E_1' \cup \{x'\} \\ V_{H_2''}(e') & \text{if } e' \in E_2' \cup \{x'\} \end{cases}$$

Then, clearly, $C' \subseteq E_i' \cup \{x'\}$ is a circuit of $\underline{M}(H)$ if and only if C' is a circuit of $\underline{M}(H_1) \dot{\cup} \underline{M}(H_2)$. Now suppose that $C' \subseteq E_1' \cup E_2' \cup \{x'\}$ but $C' \not\subseteq E_1' \cup \{x'\}$ and $C' \not\subseteq E_2' \cup \{x'\}$, and that C' is a circuit of $\underline{M}(H)$. Let $C_i' = C' \cap (E_i' \cup \{x'\})$ (i = 1,2). Then C_i' is independent in $\underline{M}(H)$.

Therefore, $|V(C_i')| \geq |C_i'| + k_1 - 1 \quad (i = 1,2)$ (1)

Thus, $|V(C_1')| + |V(C_2')| \geq |C_1' \cup C_2'| + 2k_1 - 2$

$\therefore \quad |V(C_1' \cup C_2')| + |V(C_1') \cap V(C_2')| \geq |C_1' \cup C_2'| + 2k_1 - 2$

Since C' is a circuit, $|V(C_1' \cup C_2')| = |V(C')| = |C'| + k_1 - 2$,

so $|V(C_1') \cap V(C_2')| \geq k_1$. Since $|V_2'' \cap V_1| = k_1$, equality must hold, and

therefore $V(C_1') \cap V(C_2') = X$. (2)

Now, $x \notin C_1'$ and $x \notin C_2'$ since, from (1) and (2)

$|V(C_i')| = |C_i'| + k_1 - 1$, and hence $C_i' \cup \{x'\}$ is dependent $(i = 1,2)$.

Also, since C' is a circuit, for no proper subset C_i'' of C_i' is $X \subseteq V(C_i'')$,

so $C_i' \cup \{x'\}$ is a circuit for $i = 1,2$, and so $C' = (C_1' \cup \{x'\}) \Delta (C_2' \cup \{x'\})$.

Thus, by (11.3), $C' \in \mathbf{C}(\underline{M}(H_1) \dot{\cup} \underline{M}(H_2))$.

The proof of the converse to this, that if C' is a circuit of $\underline{M}(H_1) \dot{\cup} \underline{M}(H_2)$, and $C' \not\subseteq E_i \cup \{x'\}$ for $i = 1,2$, then C' is a circuit of $\underline{M}(H)$, is trivial, and we omit the details.

Thus, $\underline{M}_1 \dot{\cup} \underline{M}_2 \cong \underline{M}(H_1) \dot{\cup} \underline{M}(H_2) \cong \underline{M}(H)$, and so $\underline{M}_1 \dot{\cup} \underline{M}_2$ is hypergraphic.

PROPOSITION 11.5: Let $\underline{M}_1$, $\underline{M}_2$ be matroids on $E_1 \cup \{x\}$ and $E_2 \cup \{x\}$ respectively, where $E_1 \cap E_2 = \phi$, $x \notin E_1 \cup E_2$ and x is not a loop or coloop of $\underline{M}_1$ or $\underline{M}_2$. Let $y \in E_1$, where y is not a coloop of $\underline{M}_1$. Then:

(a) $(\underline{M}_1 \dot{\cup}' \underline{M}_2)^* = \underline{M}_1^* \dot{\cup}' \underline{M}_2^*$;

(b) $$(\underline{M}_1 \dot{\cup}' \underline{M}_2) \times (E_1 \cup E_2 - \{y\}) = \begin{cases} (\underline{M}_1 \times (E_1 - \{y\}) \dot{\cup}' \underline{M}_2 & \text{if } x \text{ is not a coloop of } \underline{M}_1 \times (E_1 - \{y\}) \\ (\underline{M}_1 \times (E_1 - \{y\})) \oplus (\underline{M}_2 \times E_2) & \text{otherwise.} \end{cases}$$

Proof:

(a) $$\mathbf{B}((\underline{M}_1 \dot{\cup}' \underline{M}_2)) = \{B_1 \cup I_2 : x \notin B_1 \in \mathbf{B}(\underline{M}_1),\ I_2 \cup \{x\} \in \mathbf{B}(\underline{M}_2)\} \cup \{B_2 \cup I_1 : x \notin B_2 \in \mathbf{B}(\underline{M}_2),\ I_1 \cup \{x\} \in \mathbf{B}(\underline{M}_1)\}.$$

$$\therefore \quad \mathbf{B}((\underline{M}_1 \dot{\cup}' \underline{M}_2)^*) = \{(E_1 - B_1) \cup (E_2 - I_2) : x \notin B_1 \in \mathbf{B}(\underline{M}_1),\ I_2 \cup \{x\} \in \mathbf{B}(\underline{M}_2)\} \cup \{(E_2 - B_2) \cup (E_1 - I_1) : x \notin B_2 \in \mathbf{B}(\underline{M}_2),\ I_1 \cup \{x\} \in \mathbf{B}(\underline{M}_1)\}.$$

Write $E_i - B_i = I_i^*$, $E_i - I_i = B_i^*$. Then $I_i^* \cup \{x\} \in \mathbf{B}(\underline{M}_i^*)$, $B_i^* \in \mathbf{B}(\underline{M}_i^*)$ and $x \notin B_i^*$. Thus,

$$\mathbf{B}((\underline{M}_1\dot{\cup}'\underline{M}_2)^*) = \{B_2^*\cup I_1^*:\ x \notin B_2^* \in \mathbf{B}(\underline{M}_2^*),\ I_1^*\cup\{x\} \in \mathbf{B}(\underline{M}_1^*)\}\cup$$

$$\cup\ \{B_1^*\cup I_2^*:\ x \notin B_1^* \in \mathbf{B}(\underline{M}_1^*),\ I_2^*\cup\{x\} \in \mathbf{B}(\underline{M}_2^*)\}$$

$$= \mathbf{B}(\underline{M}_1^*\ \dot{\cup}'\ \underline{M}_2^*)$$ since x is not a loop or coloop of $\underline{M}_1^*$ or $\underline{M}_2^*$.

(b) $\mathbf{B}((\underline{M}_1\ \dot{\cup}'\ \underline{M}_2)\times(E_1\cup E_2-\{y\}))$

$$= \{B_1\cup I_2:\ x,y \notin B_1\in\ \mathbf{B}(\underline{M}_1),\ I_2\cup\{x\} \in \mathbf{B}(\underline{M}_2)\}\ \cup$$

$$\cup\{B_2\cup I_1:\ x \notin B_2 \in \mathbf{B}(\underline{M}_2),\ y \notin I_1\cup\{x\} \in \mathbf{B}(\underline{M}_1)\}$$

$$= \{B_1\cup I_2:\ x \notin B_1 \in \mathbf{B}(\underline{M}_1\times((E_1\cup\{x\})-\{y\})),\ I_2\cup\{x\}\in\mathbf{B}(\underline{M}_2)\}$$ since y is not a coloop of $\underline{M}_1$

$$\cup\{B_2\cup I_1:\ x \notin B_2 \in \mathbf{B}(\underline{M}_2),\ I_1\cup\{x\} \in \mathbf{B}(\underline{M}_1\times((E_1\cup\{x\})-\{y\}))\}$$

$$= \begin{cases} \mathbf{B}((\underline{M}_1\times(E_1-\{y\})\dot{\cup}'\underline{M}_2) & \text{if } x \text{ is not a coloop of } \underline{M}_1\times(E_1-\{y\}) \\ \{B_2\cup I_1:\ B_2 \in \mathbf{B}(\underline{M}_2\times E_2),\ I_1 \in \mathbf{B}(\underline{M}_1\times(E_1-\{y\}))\} & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathbf{B}((\underline{M}_1\times(E_1-\{y\})\dot{\cup}'\underline{M}_2) & \text{if } x \text{ is not a coloop of } \underline{M}_1\times(E_1-\{y\}) \\ \mathbf{B}((\underline{M}_1\times(E_1-\{y\}))\oplus(\underline{M}_2\times E_2)) & \text{otherwise.} \end{cases}$$

PROPOSITION 11.6: PROPERTIES OF THE DELETED POINTED-UNION.

With the notation of (11.4),

(a) If $\underline{M}_1$ and $\underline{M}_2$ are representable over the field F, then $\underline{M}_1\ \dot{\cup}'\ \underline{M}_2$ is representable over F;

(b) If $\underline{M}_1$ and $\underline{M}_2$ are base-orderable, then $\underline{M}_1\ \dot{\cup}'\ \underline{M}_2$ is base-orderable;

(c) If $\underline{M}_1$ and $\underline{M}_2$ are fully base-orderable, then $\underline{M}_1\ \dot{\cup}'\ \underline{M}_2$ is fully base-orderable;

(d) If $\underline{M}_1$ and $\underline{M}_2$ are hypergraphic, then $\underline{M}_1\ \dot{\cup}'\ \underline{M}_2$ is hypergraphic.

Proof: The properties of representability over a field, base-orderability, full base-orderability and hypergraphicity are all preserved under the operation of restriction. The result now follows from (11.4).

PROPOSITION 11.7: If $\underline{M} = \underline{M}_1\dot{\cup}\underline{M}_2$, there exist matroids $\underline{M}_1'$ and $\underline{M}_2'$ such that $\underline{M} = \underline{M}_1'\ \dot{\cup}'\ \underline{M}_2'$.

Proof: Let $\underline{M}_1$ be a matroid on $E_1\cup\{x\}$, $\underline{M}_2$ a matroid on $E_2\cup\{x\}$, where $x \notin E_1\cup E_2$, $E_1\cap E_2 = \phi$ and x is not a loop or coloop of $\underline{M}_1$ or $\underline{M}_2$, and let $y \notin E_1\cup E_2\cup\{x\}$. Define $\underline{M}_1'$ to be the matroid on $E_1\cup\{y\}$ isomorphic to $\underline{M}_1$

under the map which is the identity on E_1 and maps x to y; define $\underline{M}_2''$ to be the matroid on $E_2 \cup \{y\}$ isomorphic to $\underline{M}_2$ under the map which is the identity on E_2 and maps x to y. Put $\underline{M}_2' = \underline{M}_2'' \dot{\cup} U_{1,2}(\{x,y\})$. Then clearly $\underline{M} = \underline{M}_1' \dot{\cup}' \underline{M}_2'$.

DEFINITION 11.8: Let $\underline{M}$ be a matroid on the set E. If there exist matroids $\underline{M}_1$, $\underline{M}_2$ such that $\underline{M}_i$ is a matroid on the set $E_i \cup \{x\}$, where $|E_i| \geq 2$ $(i = 1,2)$ such that $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2$, then $\underline{M}$ is said to be pointed reducible. A matroid which is not pointed reducible is said to be pointed irreducible.

PROPOSITION 11.9: A connected matroid $\underline{M}$ on the set E with rank function ρ is pointed reducible if and only if there exists a partition T, $E-T$ of E such that $|T| \geq 2$, $|E-T| \geq 2$ and $\rho T + \rho(E-T) = \rho E + 1$.

Proof: Since $\underline{M}$ is connected, for no partition E_1, $E-E_1$ of E does $\rho E_1 + \rho(E-E_1) = \rho E$. Thus, for every partition E_1, $E-E_1$ of E,

$$\rho E_1 + \rho(E-E_1) \geq \rho E + 1. \qquad (1)$$

Suppose $\underline{M}$ is pointed reducible. Then there exist matroids $\underline{M}_1$, $\underline{M}_2$ on sets $E_1 \cup \{x\}$, $E_2 \cup \{x\}$ respectively, such that $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2$, and $|E_i| \geq 2$ $(i = 1,2)$. Now, a base of $\underline{M}$ is of the form $B_i \cup I_j$, where $x \notin B_i \in \mathbf{B}(\underline{M}_i)$ and $I_j \cup \{x\} \in \mathbf{B}(\underline{M}_j)$ $(i \neq j)$, so $rk\underline{M} = rk\underline{M}_1 + rk\underline{M}_2 - 1$.
Now, $rk\,\underline{M}_i = \rho E_i$, since if, say, $rk\underline{M}_1 = \rho E_1 + 1$ and $rk\underline{M}_2 = \rho E_2$, we would have $\rho E = \rho E_1 + \rho E_2$, which contradicts (1), and if $rk\underline{M}_i = \rho E_i + 1$ $(i = 1,2)$, we would have $\rho E = \rho E_1 + \rho E_2 + 1$, which is impossible. Thus, taking $T = E_1$ and $E-T = E_2$, the result follows.
Conversely, suppose there exists $E_1 \subseteq E$ such that $\rho E_1 + \rho(E-E_1) = \rho E + 1$, where $|E_1| \geq 2$ and $|E-E_1| \geq 2$.
Since $\rho E_1 + \rho(E-E_1) = \rho E + 1$, $\rho((\sigma E_1) \cap (\sigma(E-E_1))) \leq 1$.

A special case is where $\underline{M}$ is not simple. Since $\underline{M}$ is connected, there exists a parallel pair $\{a,b\} \subseteq E$. Choose $x \notin E$, and put $\underline{M}_1' = \underline{M}\times(E-\{a\})$, $\underline{M}_2 = U_{1,3}(\{a,b,x\})$. Let $\underline{M}_1$ be a matroid on $(E-\{a,b\})\cup\{x\}$, isomorphic to $\underline{M}_1'$ under the map which is the identity on $E-\{a,b\}$, and maps b to x. Consider $\underline{M}' = \underline{M}_1 \dot{\cup}' \underline{M}_2$. Then clearly $\underline{M}' = \underline{M}$. Put $T = \{a,b\}$;. then, since $|E| = |E-E_1|+|E_1| \geq 4$, $|E-T| \geq 2$, and $\rho T + \rho(E-T) = \rho E + 1$.

For the general case (where $\underline{M}$ may be simple), write $E_2 = E-E_1$. Let $G_1 \in \mathbf{B}(\underline{M}\times E_1)$, and let J_2 and x_2 be such that $G_1\cup J_2 \in \mathbf{B}(\underline{M})$ and $J_2\cup\{x_2\} \in \mathbf{B}(\underline{M}\times E_2)$. Define $\underline{M}_1' = \underline{M}\times(E_1\cup J_2\cup\{x_2\}).(E_1\cup\{x_2\})$. Then, for $X \subseteq E_1$, $\rho_{\underline{M}_1'}X = \rho(X\cup J_2) - \rho J_2 = \rho X$, since $G_1\cup J_2 \in \mathbf{B}(\underline{M})$ and $G_1 \in \mathbf{B}(\underline{M}\times E_1)$. Thus, $\underline{M}_1'\times E_1 = \underline{M}\times E_1$. Also, from the definition, $rk\underline{M}_1' = rk(\underline{M}\times E_1)$. Similarly, let $G_2 \in \mathbf{B}(\underline{M}\times E_2)$, and let J_1 and x_1 be such that $G_2\cup J_1 \in \mathbf{B}(\underline{M})$ and $J_1\cup\{x_1\} \in \mathbf{B}(\underline{M}\times E_1)$. Define $\underline{M}_2' = \underline{M}\times(E_2\cup J_1\cup\{x_1\}).(E_2\cup\{x_1\})$. Then $\underline{M}_2'\times E_2 = \underline{M}\times E_2$ and $rk\underline{M}_2' = rk(\underline{M}\times E_2)$. Choose $x \notin E_1\cup E_2$, and let $\underline{M}_i$ be a matroid on $E_i\cup\{x\}$ isomorphic to $\underline{M}_i'$ under the map which is the identity on E_i and maps x_i to x. We claim $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2$.

For, let $B_1\cup I_2$ (say) be a base of $\underline{M}_1 \dot{\cup}' \underline{M}_2$, where $x \notin B_1 \in \mathbf{B}(\underline{M}_1)$ and $I_2\cup\{x\} \in \mathbf{B}(\underline{M}_2)$. Then $B_1 \in \mathbf{B}(\underline{M}\times E_1)$. Now, $I_2\cup\{x_1\} \in \mathbf{B}(\underline{M}_2')$, so $I_2\cup\{x_1\}\cup J_1$ is independent in $\underline{M}$. By hypothesis,

$$\rho E = \rho E_1 + \rho E_2 - 1 = |J_1\cup\{x_1\}| + |I_2\cup\{x_1\}| - 1 = |J_1\cup I_2\cup\{x_1\}|.$$

Thus, $J_1\cup I_2\cup\{x_1\} \in \mathbf{B}(\underline{M})$. Thus, $B_1\cup J_1\cup I_2\cup\{x_1\}$ spans E in $\underline{M}$. Now, $J_1\cup\{x_1\} \subseteq E_1 \subseteq \sigma(B_1)$, so $B_1\cup I_2$ spans E in $\underline{M}$. But $|B_1\cup I_2| = \rho E$, so $B_1\cup I_2 \in \mathbf{B}(\underline{M})$. A similar result holds for a base of $\underline{M}_1\dot{\cup}'\underline{M}_2$ of the form $B_2\cup I_1$.

Conversely, suppose $B \in \mathbf{B}(\underline{M})$. Let $I_i = B\cap E_i$ $(i = 1,2)$. Then $|I_1| + |I_2| = |B| = \rho E = \rho E_1 + \rho E_2 - 1$, and $\rho E_i \geq |I_i|$ $(i = 1,2)$. Thus, either (i) $\rho E_1 = |I_1|$ and $\rho E_2 = |I_2| + 1$

or (ii) $\rho E_2 = |I_2|$ and $\rho E_1 = |I_1| + 1$.

Suppose that (i) holds. Then $I_1 \in \mathbb{B}(\underline{M}\times E_1)$, so $I_1 \in \mathbb{B}(\underline{M}_1)$.
Now, for any base $B_1' \in \mathbb{B}(\underline{M}\times E_1)$, $B_1' \cup I_2$ is independent in $\underline{M}$, since $I_1 \cup I_2$ and hence $E_1 \cup I_2$ spans E in $\underline{M}$. Thus, in particular, $I_2 \cup J_1 \cup \{x_1\} \in \mathbb{B}(\underline{M})$, so $I_2 \cup \{x_1\} \in \mathbb{B}(\underline{M}_2')$, and so $I_2 \cup \{x\} \in \mathbb{B}(\underline{M}_2)$.
Thus, $I_1 \cup I_2 \in \mathbb{B}(\underline{M}_1 \dot{\cup}' \underline{M}_2)$. A similar result holds for (ii).
Thus $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2$, and, by hypothesis, $|E_1| \geq 2$ and $|E-E_1| \geq 2$.

PROPOSITION 11.10: Let $\underline{M}$ be a matroid on the set E. Then

(a) $\underline{M}$ is 2-connected (in the sense of Tutte (9.28)) if and only if

(b) $\underline{M}$ is connected and pointed reducible.

Proof: (a) $\Rightarrow$ (b).

Suppose $\underline{M}$ is 2-connected. Then $\underline{M}$ is not 1-separated, so, for no $T \subseteq E$ with $|T| \geq 1$ and $|E-T| \geq 1$ do we have $\rho T + \rho(E-T) = \rho E$. Thus, $\underline{M}$ is connected. Since $\underline{M}$ is 2-connected, $\underline{M}$ is 2-separated, so there exists $T \subseteq E$ such that $|T| \geq 2$, $|E-T| \geq 2$ and $\rho T + \rho(E-T) = \rho E + 1$. Thus, by (11.9), $\underline{M}$ is pointed reducible.

(b) $\Rightarrow$ (a).

Suppose $\underline{M}$ is connected and pointed reducible. Then

(i) for no $T \subseteq E$ with $|T| \geq 1$ and $|E-T| \geq 1$ do we have $\rho T + \rho(E-T) = \rho E$;

(ii) by (11.9) there exists $T \subseteq E$ with $|T| \geq 2$, $|E-T| \geq 2$ such that

$$\rho T + \rho(E-T) = \rho E + 1.$$

Thus $\underline{M}$ is 2-separated but not 1-separated, and so, by (9.28), $\underline{M}$ is 2-connected.

PROPOSITION 11.11: Let $\underline{M}$ be a matroid on the set E. Then

(a) If $\underline{M}$ is κ-connected, $\underline{M}^*$ is κ-connected;

(b) If $\underline{M}$ is κ-connected, and $\underline{M}\times(E-\{x\})$ is κ'-connected for some $x \in E$, then $\kappa' \geq \kappa-1$;

(c) If $\underline{M}$ is κ-connected, and $\underline{M}.(E-\{x\})$ is κ''-connected for some $x \in E$, then $\kappa'' \geq \kappa-1$;

(continued overleaf)

(d) If $\underline{M}$ is not κ-connected for any $\kappa > 0$, and $\underline{M}\times(E-\{x\})$ is κ'-connected for some $x \in E$ and some $\kappa' \geq 1$, then $|E| = 2\kappa' + 1$.

Proof:

(i) $\rho T + \rho(E-T) = \rho E + \lambda - 1$ for some positive integer λ if and only if $\rho^*(E-T) - |E-T| + \rho E + \rho^* T - |T| + \rho E = \rho E + \lambda - 1$

i.e. $\rho^*(T) + \rho^*(E-T) = \rho^*(E) + \lambda - 1$.

Thus, $\underline{M}$ is λ-separated if and only if $\underline{M}^*$ is λ-separated, and so $\underline{M}$ is κ-connected if and only if $\underline{M}^*$ is κ-connected.

(ii) Suppose that $\underline{M}\times(E-\{x\})$ is λ-separated. Then there exists $T \subseteq (E-\{x\})$ such that $|T| \geq \lambda, |(E-\{x\})-T| \geq \lambda$ and

$\rho T + \rho((E-\{x\})-T) = \rho(E-\{x\}) + \lambda - 1.$

Suppose $|E| > 2\lambda + 1$. We have

$$\rho(E-T) - 1 \leq \rho((E-\{x\})-T) \leq \rho(E-T) \tag{1}$$

and
$$\rho E - 1 \leq \rho(E-\{x\}) \leq \rho E \tag{2}$$

If the upper equality holds in (1), then the upper equality holds in (2). Since $|E-\{x\}| \geq 2\lambda + 1$, we may assume, without loss of generality, that $|T| \geq \lambda+1$ and $|(E-\{x\})-T| \geq \lambda$. From (1) and (2),

$$\rho E + \lambda - 1 = \begin{cases} \rho T + \rho(E-T) - 1 & \text{if the lower equality holds in (1), and the upper equality in (2)} \\ \rho T + \rho(E-T) & \text{otherwise.} \end{cases}$$

$$\text{Thus, } \rho T + \rho(E-T) = \begin{cases} \rho E + (\lambda+1) - 1 & (3) \\ \rho E + \lambda - 1 & (4) \end{cases}$$

where (3) holds if the upper equality holds in (2) and the lower in (1), and (4) holds otherwise.

Now, since $|T| \geq \lambda+1$ and $|E-T| \geq \lambda+1$, $\underline{M}$ is either λ- or $(\lambda+1)$-separated, depending on which of (3) and (4) holds.

Suppose $|E| = 2\lambda+1$; then, without loss of generality we may assume that $|T| = \lambda+1$ and $|E-T| = \lambda$. If (4) holds for some T with $|T| = \lambda+1$ then $\underline{M}$ is λ-separated. If (3) holds for every T with $|T| = \lambda+1$, then $\underline{M}$ is not λ-separated. Since $|E| = 2\lambda+1$, $\underline{M}$ is not λ'-separated for any $\lambda' \geq \lambda$. Clearly, if $\underline{M}\times(E-\{x\})$ is λ-separated, $|E| \geq 2\lambda+1$.

If $\underline{M}$ is κ-connected, $\underline{M}$ is κ-separated; if $\underline{M}\times(E-\{x\})$ is κ'-connected, $\underline{M}\times(E-\{x\})$ is κ'-separated, but not $(\kappa'-1)$-separated. Thus, $\kappa' \geq \kappa-1$.

(iii) follows by the application of (i) and (ii) to $\underline{M}^*$.

(iv) If $\underline{M}\times(E-\{x\})$ is κ'-connected, and $\underline{M}$ is not κ-connected for any $\kappa > 0$, then, from the proof of (ii), $|E| = 2\kappa' + 1$.

DEFINITION 11.12: Let $\underline{M}$ be a matroid on E, which is κ-connected. Then $x \in E$ is said to be essential if $\underline{M}\times(E-\{x\})$ and $\underline{M}.(E-\{x\})$ are both $(\kappa'-1)$-connected.

THEOREM 11.13 (Tutte [27]): Let $\underline{M}$ be a 3-connected matroid on the set E such that every $x \in E$ is essential. Then either

$\underline{M} \cong W_n$ $(n \geq 3)$, the whirl of order n, or

$\underline{M} \cong \underline{W}_n$ $(n \geq 3)$, the wheel of order n.

(11.13) will be used in the proofs of the characterisation of generalised series-parallel networks by a set of six forbidden minors.

GENERALISED SERIES-PARALLEL EXTENSION:

A series extension at $e \in E$ of the graph $G = (V, E, \$)$ is effected by placing a new vertex at the mid-point of e, thus dividing it into two new edges. Conventionally, one of these edges is still labelled e. We believe this to be somewhat unsatisfactory, and so we shall label the new edges with two new labels (e_1 and e_2, say). It is easy to see that the series extension at e of G is then obtained from G and a triangle e, e_1, e_2 by identifying the edge e in each graph, and then deleting it.

The series extension at $e \in E$ of a matroid $\underline{M}$ on E is commonly defined in terms of the bases of the resulting matroid - the set of bases being $\{B \cup \{e\}: e \notin B \in \mathbf{B}(\underline{M})\} \cup \{B \cup \{e'\}: B \in \mathbf{B}(\underline{M})\}$, where $e' \notin E$. Again, we believe this to be unsatisfactory, since the element e is used as an element of the new matroid as well as of $\underline{M}$. It is, however, clear, that provided e is not a loop or coloop of $\underline{M}$, the series extension at e of $\underline{M}$ is isomorphic to $\underline{M}\,\dot{\cup}'\,U_{2,3}(\{e,e_1,e_2\})$, where $e_1,e_2 \notin E$. With this as motivation, we make the following definition:

DEFINITION 11.14: SERIES-PARALLEL EXTENSIONS.

(a) If $\underline{M}$ is a matroid on the single element x, then the <u>elementary series extension of M at x</u>, written $\underline{M}s(x;\, x_1,x_2)$, is defined by

$$\underline{M}s(x;\, x_1,x_2) = \begin{cases} U_{1,2}(\{x_1,x_2\}) \text{ if } x \text{ is a loop;} \\ U_{2,2}(\{x_1,x_2\}) \text{ if } x \text{ is a coloop.} \end{cases}$$

If $\underline{M}$ is a connected matroid on the set $E\cup\{x\}$ where $x \notin E \neq \phi$, the <u>elementary series extension of M at x</u> is defined to be

$\underline{M}s(x;\, x_1,x_2) = \underline{M}\,\dot{\cup}'\,U_{2,3}(\{x,x_1,x_2\})$ where $x_1,x_2 \notin E$.

If $\underline{M} = \underline{M}_1 \oplus \underline{M}_2 \oplus \ldots \oplus \underline{M}_n$ where each $\underline{M}_i$ is a connected matroid on the set E_i, then the <u>elementary series extension of M at x</u> is defined to be

$\underline{M}s(x;\, x_1,x_2) = \bigoplus_{N-\{j\}}(\underline{M}_i) \oplus (\underline{M}_j s(x;\, x_1,x_2))$ where $x \in E_j$, $x_1,x_2 \notin \bigcup_N E_i$ and $N = \{1,2,\ldots,n\}$.

The <u>elementary parallel extension of M at x</u>, written $\underline{M}p(x;\, x_1,x_2)$, is defined to be $(\underline{M}^*s(x;\, x_1,x_2))^*$.

(b) A matroid of the form

$(\ldots(\underline{M}s(x_{10};\, x_{11},x_{12}))s(x_{20};\, x_{21},x_{22})\ldots)s(x_{m0};\, x_{m1},x_{m2})$ where $\underline{M}$ is a matroid on E and $x_{i0} \notin E$ for each i, $2 \le i \le m$ is called a <u>series extension of M at x_{10}</u>, written $\underline{M}s(x_{10})$.

A matroid of the form $(\underline{M}^*s(x))^*$ is called a <u>parallel extension of M at x</u>, written $\underline{M}p(x)$.

(c) A <u>series extension of M</u>, written $s(\underline{M})$, is a matroid of the form $(\ldots(\underline{M}s(x_1))s(x_2)\ldots)s(x_m)$ where $\underline{M}$ is a matroid on E and $x_i \in E$, $(1 \le i \le m)$.

A <u>parallel extension of M</u>, written $p(\underline{M})$, is a matroid of the form $(\ldots(\underline{M}p(x_1))p(x_2)\ldots)p(x_m)$ where $\underline{M}$ is a matroid on E and $x_i \in E$, $(1 \le i \le m)$.

(d) A <u>series-parallel extension</u> of $\underline{M}$, written sp($\underline{M}$), is a matroid obtained from $\underline{M}$ by a sequence of series and parallel extensions.

(e) A <u>connected series-parallel network</u> is a connected matroid which is a series-parallel extension of a matroid on a single element.

(f) A <u>series-parallel network</u> (<u>sp network</u>) is a direct sum of connected series-parallel networks.

If we consider an operation on a k-hypergraph $H = (V, E, \$)$, similar to that outlined at the beginning of this section for the graph G, we see that two generalisations of the triangle e, e_1, e_2 are possible. One is a hypergraph with edges e and e_i $(1 \leq i \leq k)$, where $V(e_i) = N \cup \{V_i\}$, such that $|N| = k-1$, $N \cap V = \phi$ and $V(e) = \{V_1, V_2, \ldots, V_k\}_{\neq}$. Using this as a definition of generalised series extension, and mimicking (11.14) accordingly, leads to a definition of "generalised series-parallel network" which defines a subclass of the class of series-parallel networks. The alternative is to generalise the triangle to a hypergraph H' with edges e and e_i $(1 \leq i \leq k)$ where $V_{H'}(e_i) = N_i \cup \{V\}$ such that $\{N_i : 1 \leq i \leq k\}$ is the set of nodes of e and $V \notin V$. This definition does give rise to a new class of matroids, since $\underline{M}(H') \cong U_{2,k+1}$ and, for $k \geq 3$, this is not graphic. Mimicking (11.14) would then lead to a class of "generalised series-parallel networks" that was not a subclass of the class of series-parallel networks. In general, we shall not wish to restrict our definition of generalised series extension to a particular value of k, but only require that $k \geq 2$. We therefore make the following definition, analogous to (11.14), of generalised series and generalised parallel extension.

DEFINITION 11.15: GENERALISED SERIES-PARALLEL EXTENSIONS.

Let $\underline{M}$ be a matroid on the set $E \cup \{x\}$ where $x \notin E$.

(continued overleaf)

(a) If $\underline{M}$ is a matroid on the single element x, then the elementary generalised series extension of M at x by X is defined to be the matroid $\underline{M}gs(x; X) = \begin{cases} U_{1,k}(X) & \text{if } x \text{ is a loop} \\ U_{2,k}(X) & \text{if } x \text{ is a coloop} \end{cases}$

where $x \notin X$ and $|X| = k \geq 2$.

If $\underline{M}$ is a connected matroid and $E \neq \phi$, then the elementary generalised series extension of M at x by X is the matroid

$\underline{M}gs(x; X) = \underline{M} \,\dot{\cup}'\, U_{2,k+1}(X \cup \{x\})$, where $X \cap (E \cup \{x\}) = \phi$ and $|X| = k \geq 2$.

If $\underline{M} = \bigoplus_I M_i$ where each $\underline{M}_i$ is a connected matroid on the set E_i, then the elementary generalised series extension of M at x by X is defined to be the matroid

$\underline{M}gs(x; X) = (\bigoplus_{I-\{j\}} \underline{M}_i) \oplus (\underline{M}_j gs(x; X))$ where $x \in E_j$, and $X \cap E = \phi$.

The elementary generalised parallel extension of M at x by X is defined to be the matroid $\underline{M}gp(x; X) = (\underline{M}^* gs(x; X))^*$.

(b) A matroid of the form $(\ldots(\underline{M}gs(x_1; X_1))gs(x_2; X_2)\ldots)gs(x_m; X_m)$ where $x_i \notin E$ $(2 \leq i \leq m)$ is called a generalised series extension of M at x, written $\underline{M}gs(x)$.

A matroid of the form $(\underline{M}^* gs(x))^*$ is called a generalised parallel extension of M at x, written $\underline{M}gp(x)$.

(c) A generalised series extension of M, written $gs(\underline{M})$, is a matroid of the form $(\ldots(\underline{M}gs(x_1))gs(x_2))\ldots)gs(x_m)$, where $x_i \in E$, $1 \leq i \leq m$.

A generalised parallel extension of M, written $gp(\underline{M})$, is a matroid of the form $(\ldots(\underline{M}gp(x_1))gp(x_2))\ldots)gp(x_m)$, where $x_i \in E$, $1 \leq i \leq m$.

(d) A generalised series-parallel extension of $\underline{M}$, written $gsp(\underline{M})$, is a matroid obtained from $\underline{M}$ by a sequence of series and parallel extensions.

(e) A connected generalised series-parallel network is a connected matroid which is a generalised series-parallel extension of a matroid on a single element.

(f) A <u>generalised series-parallel network</u> (<u>gsp network</u>) is a direct sum of connected generalised series-parallel networks.

LEMMA 11.16:

(A) Let $\underline{M}$ be a series-parallel network. Then $\underline{M}^*$ is a series-parallel network, and $\underline{M}$ is graphic.

(B) Let $\underline{M}$ be a generalised series-parallel network. Then $\underline{M}^*$ is a generalised series-parallel network, and $\underline{M}$ is hypergraphic.

Proof: The first parts of (A) and (B) follow from the definitions (11.14) and (11.15), and the duality result (11.5).

The second part of (A) follows from the forbidden minor conditions (11.25) proved in [4].

The second part of (B) follows from (11.15) and (11.4), since, by (5.1), uniform matroids are hypergraphic, and, by (3.12) and (4.2), a direct sum of hypergraphic matroids is hypergraphic.

LEMMA 11.17: Let $\underline{M}_i$ be a matroid on E_i ($i = 1,2,3,4$). Then

(a) $\underline{M}_1 \dot{\cup}' (\underline{M}_2 \dot{\cup}' \underline{M}_3) = (\underline{M}_1 \dot{\cup}' \underline{M}_2) \dot{\cup}' \underline{M}_3$;

(b) $\underline{M}_1 \dot{\cup}' (\underline{M}_2 \oplus \underline{M}_4) = (\underline{M}_1 \dot{\cup}' \underline{M}_2) \oplus \underline{M}_4$;

whenever the operations indicated are defined.

Proof: Routine verification.

(11.17)(a) shows that the use of brackets in strings of deleted pointed unions is unnecessary, and we shall therefore omit them in future.

LEMMA 11.18: If $\underline{M}$ is a connected gsp network with more than one element, then $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2 \dot{\cup}' \ldots \dot{\cup}' \underline{M}_n$ for some n, where $\underline{M}_1$ is isomorphic to U_{1,r_1} ($r_1 \geq 2$) or U_{2,r_1} ($r_1 \geq 3$) or U_{r_1-2,r_1} ($r_1 \geq 3$) or U_{r_1-1,r_1} ($r_1 \geq 2$), and each $\underline{M}_i$ ($i \geq 2$) is isomorphic to U_{2,r_i} or U_{r_i-2,r_i} ($r_i \geq 3$).

Proof: From the definition of generalised series and generalised parallel extension (11.15) and the duality result (11.5).

LEMMA 11.19:

(A) If $\underline{M}_1$ and $\underline{M}_2$ are series-parallel networks on $E_1 \cup \{x\}$ and $E_2 \cup \{x\}$ respectively, where $x \notin E_1 \cup E_2$, $E_1 \cap E_2 = \phi$ and x is not a loop or coloop of $\underline{M}_1$ or $\underline{M}_2$, then $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is a series-parallel network.

(B) If $\underline{M}_1$ and $\underline{M}_2$ are gsp networks on $E_1 \cup \{x\}$ and $E_2 \cup \{x\}$ respectively, where $x \notin E_1 \cup E_2$, $E_1 \cap E_2 = \phi$ and x is not a loop or coloop of $\underline{M}_1$ or $\underline{M}_2$, then $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is a gsp network.

Proof: (A) follows from the forbidden minor conditions (11.25) proved in [4]; alternatively, a proof similar to that used for (B) can be used.

(B) By virtue of (11.17) and (11.18), it is sufficient to prove this result for the case where $\underline{M}_1$ is connected, and $\underline{M}_2$ is isomorphic to (a) $U_{1,r}$ $(r \geq 2)$; or (b) $U_{2,r}$ $(r \geq 3)$; or (c) $U_{r-2,r}$ $(r \geq 3)$; or (d) $U_{r-1,r}$ $(r \geq 2)$.

(a) $\underline{M}_1 \dot{\cup}' \underline{M}_2 = \underline{M}_1 \dot{\cup}' U_{1,r}(E_2 \cup \{x\}) \qquad (|E_2| \geq 1)$

$$= \begin{cases} \underline{M}_1 p(x;\, x_1, x_1')p(x_1';\, x_2, x_2') \ldots p(x_{n-3}';\, x_{n-2}, x_{n-2}') \ldots \\ p(x_{n-2}';\, x_{n-1}, x_n), \text{ where } E_2 = \{x_1, \ldots, x_n\}_{\neq} \text{ and} \\ \{x_1', \ldots, x_{n-2}'\}_{\neq} \cap (E_1 \cup E_2 \cup \{x\}) = \phi, \quad \text{if } |E_2| \geq 2; \\ \\ \underline{M}_1', \text{ where } \underline{M}_1' \text{ is the matroid on } (E_1 - \{x\}) \cup \{x_1\}, \text{ isomorphic} \\ \text{to } \underline{M}_1 \text{ under the map which is the identity on } E_1 - \{x\}, \\ \text{and which maps } x \text{ to } x_1, \quad \text{if } E_2 = \{x_1\}. \end{cases}$$

Thus, $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is a gsp network.

(b) $\underline{M}_1 \dot{\cup}' \underline{M}_2 = \underline{M}_1 \dot{\cup}' U_{2,r}(E_2 \cup \{x\}) \qquad (|E_2| \geq 2)$

$= \underline{M}_1 gs(x;\, E_2)$, and so $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is a gsp network.

(c) $\underline{M}_1 \dot{\cup}' \underline{M}_2 = \underline{M}_1 \dot{\cup}' U_{r-2,r}(E_2 \cup \{x\}) \qquad (|E_2| \geq 2)$

$= (\underline{M}_1^* \dot{\cup}' U_{2,r}(E_2 \cup \{x\}))^* \qquad$ (by (11.5))

and so, by (b) and (11.16), $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is a gsp network.

(d) $\underline{M}_1 \dot{\cup}' \underline{M}_2 = \underline{M}_1 \dot{\cup}' U_{r-1,r}(E_2 \cup \{x\}) \qquad (|E_2| \geq 1)$

$= (\underline{M}_1^* \dot{\cup}' U_{1,r}(E_2 \cup \{x\}))^* \qquad$ (by (11.5))

and so, by (a) and (11.16), $\underline{M}_1 \dot{\cup}' \underline{M}_2$ is a gsp network.

LEMMA 11.20:

(A) If $\underline{M}$ is a series-parallel network, and $\underline{M}'$ is a minor of $\underline{M}$, then $\underline{M}'$ is a series-parallel network.

(B) If $\underline{M}$ is a gsp network and $\underline{M}'$ is a minor of $\underline{M}$, then $\underline{M}'$ is a gsp network.

Proof: (A) is proved in [4]; alternatively, a proof similar to that for (B) can be used.

(B) It is sufficient to prove this result for $\underline{M}$ connected, and for a minor of $\underline{M}$ formed by the deletion or contraction of one point. Furthermore, since, by (11.16), $\underline{M}^*$ is also a gsp network, and $\underline{M}^*$ is connected if $\underline{M}$ is connected, it is sufficient to prove the result for $\underline{M}$ connected, and $\underline{M}'$ formed from $\underline{M}$ by the deletion of one point.

From (11.18), we have $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2 \dot{\cup}' \ldots \dot{\cup}' \underline{M}_m$, where $\underline{M}_1$ is an elementary generalised series or generalised parallel extension of a matroid on a single element, and each $\underline{M}_i$ $(i \geq 2)$ is isomorphic to U_{2,r_i} or U_{r_i-2,r_i} $(r_i \geq 3)$. Let $\underline{M}_i$ be a matroid on E_i and let $\underline{M}$ be a matroid on E. We shall show that, for any $x \in E$, $\underline{M}\times(E-\{x\})$ is a gsp network. The proof is by induction on m and $|E|$. Clearly the result holds if $m = 1$ or 2, for any $|E|$. Suppose the result holds for $m \leq n-1$, and for all matroids on at most $|E|-1$ elements. Let $\underline{M} = \underline{M}_1 \dot{\cup}' \ldots \dot{\cup}' \underline{M}_n$ where the $\underline{M}_i$ are as above. Assume without loss of generality that $E_i \cap E_1 \neq \phi$ for $1 \leq i \leq t$, and $E_i \cap E_1 = \phi$ for $t < i \leq n$. Let $E_i \cap E_1 = \{x_i\}$ $(2 \leq i \leq t)$, and put $\underline{N}_i = \underline{M}_i \dot{\cup}' \underline{M}_{i_1} \dot{\cup}' \ldots \dot{\cup}' \underline{M}_{i_j}$, where $E_{i_r} \cap (E_i \cup E_{i_1} \cup E_{i_2} \cup \ldots \cup E_{i_{r-1}}) \neq \phi$, $i_r \geq 2$ for each i_r, and $\underline{N}_i$ is maximal with respect to this property. Then each $\underline{N}_i$ is a connected gsp network on a set F_i, where $F_i \cap E_1 = \{x_i\}$, and $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{N}_1 \dot{\cup}' \ldots \dot{\cup}' \underline{N}_t$.

CASE I: Suppose $x \in E_1$. We shall show that $\underline{M}\times(E-\{x\})$ is a gsp network. $\underline{M}_1$ is isomorphic to (a) $U_{1,r}$ $(r \geq 2)$; or (b) $U_{2,r}$ $(r \geq 3)$; or (c) $U_{r-2,r}$ $(r \geq 3)$; or (d) $U_{r-1,r}$ $(r \geq 2)$.

Therefore, $\underline{M}_1\times(E_1-\{x\})$ is isomorphic to

(a) $U_{1,r-1}$ $(r \geq 2)$; or (b) $U_{2,r-1}$ $(r \geq 3)$; or (c) $U_{r-2,r-1}$ $(r \geq 3)$;
or (d) $U_{r-1,r-1}$ $(r \geq 2)$.

(a) By (11.5), $\underline{M}\times(E-\{x\}) = (\underline{M}_1\times(E_1-\{x\}))\ \dot{\cup}'\ \underline{N}_2\ \dot{\cup}'\ \ldots\ \dot{\cup}'\ \underline{N}_t$ if $r \geq 3$,

and hence $\underline{M}\times(E-\{x\})$ is a gsp network;

If $r = 2$, then clearly

$\underline{M} \cong \underline{N}_2$, and the result follows by applying (b) or (d) to
$\underline{N}_2$. This also applies to the case $r = 2$ in (d).

(b) By (11.5), if $r = 3$,

$\underline{M}\times(E-\{x\}) = U_{0,0}(\phi) \oplus (\underline{N}_2\times(F_2-\{x_2\})) \oplus (\underline{N}_3\times(F_3-\{x_3\}))$. By the inductive hypothesis, each term is a gsp network, so $\underline{M}\times(E-\{x\})$ is a gsp network.

If $r > 3$,

$\underline{M}\times(E-\{x\}) = (\underline{M}_1\times(E_1-\{x\})\ \dot{\cup}'\ \underline{N}_2\ \dot{\cup}'\ \ldots\ \dot{\cup}'\ \underline{N}_t$, and so is a gsp network.

(c) By (11.5),

$\underline{M}\times(E-\{x\}) = (\underline{M}_1\times(E_1-\{x\})\ \dot{\cup}'\ \underline{N}_2\ \dot{\cup}'\ \ldots\ \dot{\cup}'\ \underline{N}_t$, and so is a gsp network.

(d) By (11.5),

$\underline{M}\times(E-\{x\}) = (\underline{N}_2\times(F_2-\{x_2\})) \oplus (\underline{N}_3\times(F_3-\{x_3\})) \oplus \ldots \oplus (\underline{N}_t\times(F_t-\{x_t\})) \oplus \underline{N}$,

where $\underline{N} := U_{r-t,r-t}(E_1-\{x,x_2,\ldots,x_t\})$. So, by the inductive hypothesis,

$\underline{M}\times(E-\{x\})$ is a gsp network, if $r > 2$. For the case $r = 2$, see (a).

CASE II Suppose $x \notin E_1$. Then $x \in E_i$ for some $i \geq 2$. So, if $\underline{M}_i \not\cong U_{2,3}$,
by (11.5), $\underline{M}\times(E-\{x\}) = \underline{M}_1\ \dot{\cup}'\ \ldots\ \dot{\cup}'\ \underline{M}_{i-1}\ \dot{\cup}'(\underline{M}_i\times(E_i-\{x\}))\dot{\cup}'\ldots\dot{\cup}'\ \underline{M}_n$.

Now, $\underline{M}_i\times(E_i-\{x\})$ is isomorphic to (a) U_{2,r_i-1} $(r_i \geq 3)$; or
(b) U_{r_i-2,r_i-1} $(r_i \geq 3)$.

If $r_i > 3$, then $\underline{M}_i\times(E_i-\{x\})$ is a gsp network in which x_i is not a loop or coloop of $\underline{M}_i$, and so the result follows from (11.19).

If $r_i = 3$, then, for possibility (b), since $\underline{M}_i\times(E_i-\{x\})$ is a gsp network in which x_i is not a loop or coloop, the result follows from (11.19). If $\underline{M}_i \cong U_{2,3}$ ((a) with r_i=3), then the result follows from (11.5) in the same way as for Case I(b) above.

Thus, $\underline{M}\times(E-\{x\})$ is a gsp network for any $x \in E$.

Before continuing with our investigation of gsp networks, we shall prove some properties of generalised series-parallel extension, and compare them with the corresponding properties of series-parallel extension.

PROPOSITION 11.21:

(A) Properties of Series-Parallel Extension.

1. If $\underline{M}$ is base-orderable, sp($\underline{M}$) is base-orderable.
2. If $\underline{M}$ is fully base-orderable, sp($\underline{M}$) is fully base-orderable.
3. If $\underline{M}$ is representable over the field F, sp($\underline{M}$) is representable over the field F.
4. If $\underline{M}$ is a gammoid, sp($\underline{M}$) is a gammoid.

(B) Properties of Generalised Series-Parallel Extension.

1. If $\underline{M}$ is base-orderable, gsp($\underline{M}$) is base-orderable.
2. If $\underline{M}$ is fully base-orderable, gsp($\underline{M}$) is fully base-orderable.
3. If $\underline{M}$ is representable over characteristic p, gsp($\underline{M}$) is representable over characteristic p.
4. If $\underline{M}$ is a gammoid, gsp($\underline{M}$) is a gammoid.

Proofs: (A)1, (A)2, (A)3, (B)1, (B)2 and (B)3 follow from the definitions (11.14) and (11.15), and the properties of deleted pointed unions (11.6).

Proof of (B)4: ((A)4 is a special case.). Let $\underline{M}$ be a matroid on the set E. By the duality of the definitions for elementary generalised series and elementary parallel extensions, and the fact that the dual of a gammoid is a gammoid, it is sufficient to prove that if $\underline{M}$ is a gammoid, $\underline{M}gs(x_0; X)$ is a gammoid. Let $B \ni x_0$ be a base of $\underline{M}$ and let Γ be a directed graph inducing $\underline{M}$ from B. (with restriction of the vertex-set if $\underline{M}$ is not a strict gammoid). Form a new directed graph Γ' on the vertex-set $V(\Gamma) \cup X$, with $X = \{x_1, x_2, \ldots, x_k\}_{\neq}$ and $X \cap V(\Gamma) = \phi$, where the set of directed edges of Γ' is the set

$\{(v_1,v_2): (v_1,v_2)$ is a directed edge of $\Gamma\} \cup \{(x_i,x_j): i-j = \pm 1, 0 \leq i,j \leq k\}$.

Let $B \cup \{x_k\}$ be the distinguished set of vertices of Γ', and let $\underline{M}''$ be the strict gammoid induced from $B \cup \{x_k\}$ by Γ'. Let $\underline{M}' = \underline{M}'' \times (E \cup X)$.

Then the bases of $\underline{M}'$ are the sets $S \cup \{x_i\}$ ($S \in \mathbf{B}(\underline{M})$, $1 \leq i \leq k$), together with the sets $I \cup \{x_i, x_j\}$ ($I \cup \{x_0\} \in \mathbf{B}(\underline{M})$, $1 \leq i < j \leq k$).

Thus, clearly, $\underline{M}' = \underline{M} \dot{\cup} U_{2,k+1}(X \cup \{x_0\})$. Therefore, $\underline{M} \dot{\cup}' U_{2,k+1}(X \cup \{x_0\})$ is a gammoid. Since $\underline{Mgs}(x_0;X) = \underline{M} \dot{\cup}' U_{2,k+1}(X \cup \{x_0\})$, $\underline{Mgs}(x_0;X)$ is a gammoid, which completes the proof.

In [5], Crapo investigates the properties of a matroid function, called the β-function, defined as follows:

DEFINITION 11.22: Let $\underline{M}$ be a matroid on the set E with rank function ρ.

Then the function β defined by

$$\beta(\underline{M}) = (-1)^{\rho E} \sum_{A \subseteq E} (-1)^{|A|} \rho A$$

is called the __β-invariant of $\underline{M}$__.

PROPOSITION 11.23: __Properties of the β-invariant__. With the above notation:

1. $\beta(\underline{M}) = \beta(\underline{M}.(E-\{e\})) + \beta(\underline{M} \times (E-\{e\}))$ provided e is not a loop or coloop of $\underline{M}$.
2. $\beta(\underline{M}) = 0$ if and only if $\underline{M}$ is not connected.
3. $\beta(U_{k,n}) = \binom{n-2}{k-1}$.

The proofs are routine, and we omit the details; alternatively, see Crapo's paper [5].

PROPOSITION 11.24:

(A) The β-invariant and series-parallel extension.

1. $$\beta(\underline{Ms}(x;\, x_1,x_2)) = \begin{cases} \beta(\underline{M}) & \text{if } \underline{M} \text{ is not a coloop} \\ 0 & \text{if } \underline{M} \text{ is a coloop.} \end{cases}$$

2. $$\beta(\underline{Mp}(x;\, x_1,x_2)) = \begin{cases} \beta(\underline{M}) & \text{if } \underline{M} \text{ is not a loop} \\ 0 & \text{if } \underline{M} \text{ is a loop.} \end{cases}$$

PROPOSITION 11.24 (CONTD):

(B) The β-invariant and generalised series-parallel extension.

Let $|X| = k \geq 2$. Then:

$$1. \quad \beta(\underline{M}gs(x;\, X)) = \begin{cases} (k-1)\beta(\underline{M}) & \text{if } \underline{M} \text{ is not a loop or coloop} \\ k-2 & \text{if } \underline{M} \text{ is a coloop} \\ 1 & \text{if } \underline{M} \text{ is a loop.} \end{cases}$$

$$2. \quad \beta(\underline{M}gp(x;\, X)) = \begin{cases} (k-1)\beta(\underline{M}) & \text{if } \underline{M} \text{ is not a loop or coloop} \\ k-2 & \text{if } \underline{M} \text{ is a loop} \\ 1 & \text{if } \underline{M} \text{ is a coloop.} \end{cases}$$

Proof: The results in (A) are proved by Crapo [5]. However, some of his proofs do not apply to the case where $\underline{M}$ is a matroid on a single element; the proofs in this case are easy, and we omit the details.

(B)1. If $\underline{M}$ is a coloop, $\underline{M}gs(x;\, X) \cong U_{2,k}$, and, by (11.23),

$\beta(U_{2,k}) = k-2$.

If $\underline{M}$ is a loop, $\underline{M}gs(x;\, X) \cong U_{1,k}$, and, by (11.23), $\beta(U_{1,k}) = 1$.

If $\underline{M}$ is neither a loop nor a coloop, by (11.23), for $y \in X$,

$$\begin{aligned}
\beta(\underline{M}gs(x;\, X)) &= \beta(\underline{M}gs(x;\, X).(E \cup X - \{y\})) + \beta(\underline{M}gs(x;\, X) \times (E \cup X - \{y\})) \\
&= \beta(\underline{M}p(x; x_1, x_2)p(x_2; x_3, x_4)\ldots.) + \beta(\underline{M}gs(x;\, X - \{y\})) \\
&= \beta(\underline{M}) + \beta(\underline{M}gs(x;\, X - \{y\})) \qquad \text{from (A)(2)} \\
\text{(by iteration)} &= (k-2)\beta(\underline{M}) + (\underline{M}gs(x;\, X - Y)) \text{ where } Y \subseteq X \text{ and } |X - Y| = 2 \\
&= (k-2)\beta(\underline{M}) + \beta(\underline{M}) \quad \text{from (A)(1)} \\
&= (k-1)\beta(\underline{M}).
\end{aligned}$$

(B)2. The proof follows the same lines as (B)1, and we omit the details.

The operation of generalised series extension on a hypergraph H, mentioned earlier, has an easy geometric interpretation in N(H). Essentially, the edge e is replaced by k edges $e_1, e_2, \ldots, e_k$, where $V(e_i) = N_i \cup \{V\}$, and $\{N_i : 1 \leq i \leq k\} = n(e)$. The nodes of the new hypergraph are then $n(H) \cup \{(N_i \cap N_j) \cup \{V\} : 1 \leq i < j \leq k\}$. Now, in N(H), consider $\{N_i: 1 \leq i \leq k\}$ as the set of vertices of a K_k. Place the new vertex $(N_i \cap N_j) \cup \{V\}$ at the mid-point of the edge joining N_i and N_j for each j, and join up the vertices produced as necessary. We give two examples of this operation in Figure 25.

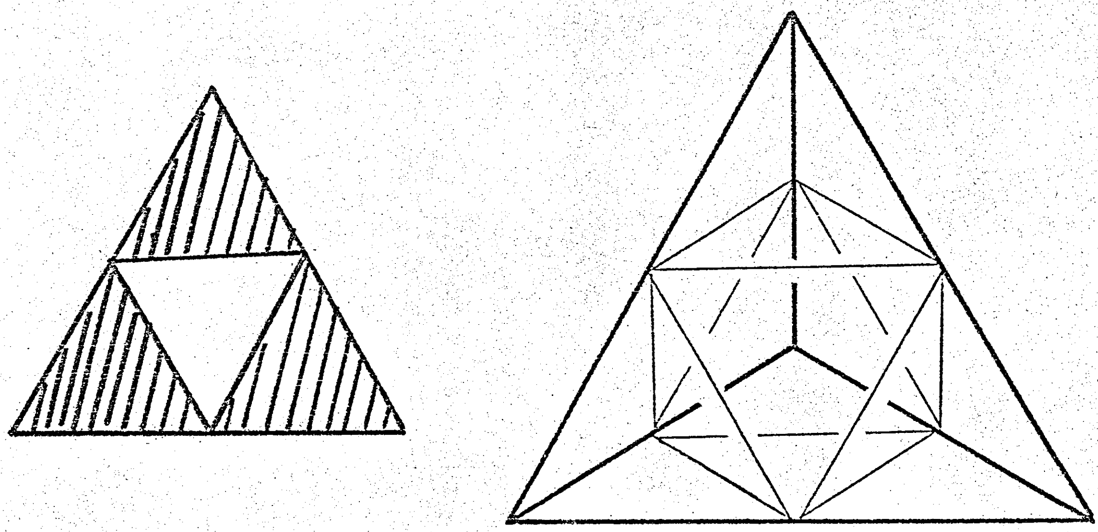


Figure 25

The operation can be thought of as replacing e in N(H) with k "half-size" edges; this is consistent with the usual idea of a series extension, in which a new vertex is placed at the mid-point of the edge e, thus dividing it into two "half-size" edges.

GENERALISED SERIES-PARALLEL NETWORKS:

It can be shown that the class of series-parallel networks can be characterised by a finite set of forbidden minors.

THEOREM 11.25: $\underline{M}$ is a series-parallel network if and only if $\underline{M}$ does not contain any minor isomorphic to $U_{2,4}$ or $\underline{M}(K_4)$.

Proof: see Brylawski [4].

Our purpose in this section is to prove an analogous result for generalised series-parallel networks. Since $U_{2,4}$ is a gsp network, but $\underline{M}(K_4)$ is not, the set of forbidden minors contains $\underline{M}(K_4)$ and certain other matroids, all of which contain $U_{2,4}$ as a proper minor.

LEMMA 11.26: Let $\underline{M}$ be a matroid of rank 3 on the set E. Then

(a) $\underline{M}$ is a gsp network if and only if

(b) $\underline{M}$ does not contain a minor isomorphic to any member of

$\mathbf{g} = \{U_{3,6}, \underline{M}(K_4), L_6, V_6, T_6\}$, where $U_{3,6}$, $\underline{M}(K_4)$, L_6, V_6 and T_6 are the simple matroids shown in Euclidean representation in Figure 26.

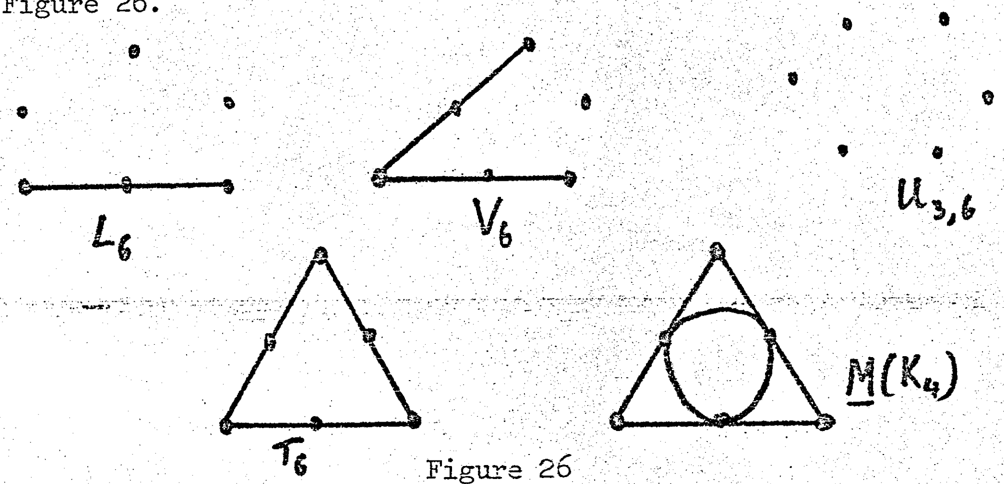


Figure 26

Proof: (a) $\Rightarrow$ (b). It is easy to check that no member of $\mathbf{g}$ is a gsp network. Therefore, by (11.20), if $\underline{M}$ is a gsp network, $\underline{M}$ contains no minor isomorphic to any member of $\mathbf{g}$.

(b) $\Rightarrow$(a): It is easy to check that any matroid of rank at most 2 is a gsp network. If $\underline{M}$ is a matroid of rank 3 that is not connected, or is connected and pointed reducible, then, by the above remarks, (11.15) and (11.19), $\underline{M}$ is a gsp network.

If $\underline{M}$ has at most 5 elements, then $\underline{M}^*$ has rank at most 2, so, by the above remarks and (11.16), $\underline{M}$ is a gsp network.

Suppose that $\underline{M}$ is of rank 3, pointed irreducible and connected, contains no minor isomorphic to a member of $\mathbf{g}$, has at least 6 elements and is not a gsp network. We shall prove that this is impossible.

Since $\underline{M}$ is pointed irreducible, $\underline{M}$ is simple. If $\underline{M}$ has no non-trivial 2-flats, then $\underline{M}$ contains $U_{3,6}$ as a restriction minor, which is a contradiction. Therefore, $\underline{M}$ has at least one non-trivial 2-flat, L, say. Since $\underline{M}$ is connected and pointed irreducible, $|E-L| \geq 3$. Consider $\rho(E-L)$. If $\rho(E-L) = 2$, then $\rho L + \rho(E-L) = \rho E + 1$, so, by (11.9), $\underline{M}$ is pointed reducible, which is a contradiction. So, since $\underline{M}$ is simple, $\rho(E-L) = 3$. Thus, there exist three distinct points, $x,y,z \in E-L$ such that $\rho(\{x,y,z\}) = 3$. Let $\{a,b,c\} \underset{\neq}{\subseteq} L$. Then $\underline{M}\times(\{a,b,c,x,y,z\})$ is simple, and so is isomorphic to L_6, T_6, V_6 or $\underline{M}(K_4)$, which is a contradiction.

Thus, no such $\underline{M}$ exists, which completes the proof.

THEOREM 11.27: Let $\underline{M}$ be a matroid on the set E. Then

(a) $\underline{M}$ is a gsp network if and only if

(b) $\underline{M}$ contains no minor isomorphic to a member of $\mathbf{g}$, where

$$\mathbf{g} = \{U_{3,6}, L_6, V_6, T_6, \underline{M}(K_4)\}.$$

Proof: (a) ⇒ (b). Suppose $\underline{M}$ is a gsp network. Then, by (11.20), every minor of $\underline{M}$ is a gsp network. Since, by (11.26), no member of is a gsp network, $\underline{M}$ contains no minor isomorphic to a member of $\mathbf{g}$.

(b) ⇒ (a). Suppose $\underline{N}$ is a minimal non-gsp-network on the set F. Then $\underline{N}$ is not a gsp network, but, for any $x \in F$, both $\underline{N}^{\times}(F-\{x\})$ and $\underline{N}.(F-\{x\})$ are gsp networks. Since $\underline{N}$ is not a gsp network and is minimal, $\underline{N}$ is connected and pointed irreducible. Thus, $\underline{N}$ is not 1- or 2-connected. Therefore, by (11.11), if $\underline{N}$ is κ-connected, we must have $\kappa \geq 3$, and so, by (11.11), $\underline{N}\times(F-\{x\})$ and $\underline{N}.(F-\{x\})$ are connected for any $x \in F$. If $\underline{N}$ is not κ-connected for any $\kappa > 0$, but $\underline{N}\times(F-\{x\})$ or $\underline{N}.(F-\{x\})$ is 1-connected for some $x \in F$, then, by (11.11), $|F| = 3$, which is a contradiction, since any matroid on 3 elements is a gsp network. Therefore, either

(I) $\underline{N}\times(F-\{x\})$ and $\underline{N}.(F-\{x\})$ are 2-connected gsp networks for any $x \in F$,

or

(II) there exists $x \in F$ such that $\underline{N}\times(F-\{x\})$ or $\underline{N}.(F-\{x\})$ is a pointed irreducible gsp network.

(I). If $\underline{N}\times(F-\{x\})$ and $\underline{N}.(F-\{x\})$ are 2-connected for each $x \in F$, and $\underline{N}$ is κ-connected, by (11.11) we must have $\kappa = 3$, so, by (11.13) $\underline{N}$ is a whirl W_n $(n \geq 3)$, or a wheel $\underline{W}_n$ $(n \geq 3)$. But W_n contains T_6 as a minor, and $\underline{W}_n$ contains $\underline{M}(K_4)$ as a minor, so $\underline{N}$ contains a minor isomorphic to a member of **g**.

If $\underline{N}$ is not κ-connected for any $\kappa > 0$, then, by (11.11), $|F| = 5$, which is a contradiction, since any matroid with 5 elements is a gsp network.

(II). Suppose there exists $x \in F$ such that $\underline{N}\times(F-\{x\})$ or $\underline{N}.(F-\{x\})$ is a pointed-irreducible gsp network. A matroid is a pointed-irreducible gsp network if and only if it is isomorphic to $U_{r-2,r}$ or $U_{2,r}$ for some $r \geq 3$, since both $U_{1,r}$ and $U_{r-1,r}$ are pointed reducible for $r \geq 4$. Now, since $\underline{N}$ is not a gsp network, $|F| \geq 6$, so, since, by hypothesis, $\underline{N}\times(F-\{x\})$ or $\underline{N}.(F-\{x\})$ is a pointed irreducible network, $\underline{N}\times(F-\{x\})$ or $\underline{N}.(F-\{x\})$ is isomorphic to $U_{r-2,r}$ or $U_{2,r}$ for some $r \geq 5$.

(A) Suppose $\underline{N}_1 = \underline{N}\times(F-\{x\}) \cong U_{2,r}$ for some $r \geq 5$.

Then, if $rk\underline{N} = 3$, $\underline{N}$ is 1-connected, which is a contradiction. Therefore, $rk\underline{N} = 2$, so, since $\underline{N}$ is simple (because it is not 1- or 2-connected) $\underline{N} \cong U_{2,r+1}$, and so $\underline{N}$ is a gsp network, which is a contradiction.

(B) Suppose $\underline{N}_1 = \underline{N}\times(F-\{x\}) \cong U_{r-2,r}$ for some $r \geq 5$.

Then $\underline{N}_1^* = \underline{N}^*.(F-\{x\}) \cong U_{2,r}$ for some $r \geq 5$.

Since $\underline{N}$ is not 1-connected, $rk\underline{N}^* = 3$. If $\underline{N}^*$ has no non-trivial 2-flats, then, since $r \geq 5$, $\underline{N}^*$ contains $U_{3,6}$ as a minor, and so $\underline{N}$ contains $U_{3,6}$ as a minor. If $\underline{N}^*$ has a non-trivial 2-flat, L, say, then, since $\underline{N}^*$ is pointed irreducible and connected, $\rho^*(F-L) = 3$. Since $U_{2,r}$ is simple, $x \notin L$. Let $\{a,b,c\} \subsetneq L$ and $x,y,z \in F-L$ be such that $\rho^*(\{x,y,z\}) = 3$. Then $\underline{N}^* \times (\{a,b,c,x,y,z\})$ is simple, and hence it is isomorphic to L_6 or V_6. Therefore, $\underline{N}^*$ contains a minor isomorphic to L_6 or V_6, and so $\underline{N}$ contains a minor isomorphic to L_6 or V_6 (since these matroids are both self-dual).

(C) Suppose $\underline{N}_2 = \underline{N}.(F-\{x\}) \cong U_{2,r}$ for some $r \geq 5$. Then, by the argument of (B) applied to $\underline{N}_2$ instead of $\underline{N}_1^*$, and to $\underline{N}$ instead of $\underline{N}^*$, $\underline{N}$ contains $U_{3,6}$, L_6 or V_6 as a minor.

(D) Suppose $\underline{N}_2 = \underline{N}.(F-\{x\}) \cong U_{r-2,r}$ for some $r \geq 5$. Then, by the argument of (A) applied to $\underline{N}_2^*$ instead of $\underline{N}_1$, and to $\underline{N}^*$ instead of $\underline{N}$, we obtain a contradiction.

Thus, if $\underline{N}$ is a minimal matroid not a gsp network, $\underline{N}$ contains a minor isomorphic to a member of $\mathfrak{g}$, and so, since no member of $\mathfrak{g}$ is a gsp network, $\underline{N} \in \mathfrak{g}$, which completes the proof.

We can use a similar argument to characterise ternary base-orderable matroids by a finite set of forbidden minors. We shall need a few preliminary results.

PROPOSITION 11.28: Let $\underline{M}$ be a connected matroid on the set E, which is pointed irreducible, contains T_6 as a minor, and contains no minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$. Then, either $\underline{M}$ is a whirl, or there exists $x \in E$ such that $\underline{M} \times (E-\{x\})$ or $\underline{M}.(E-\{x\})$ is connected, pointed irreducible, and contains T_6 as a minor.

Proof:

(I) Suppose $\underline{M}$ is κ-connected for some $\kappa > 0$. Then, since $\underline{M}$ is pointed irreducible, $\kappa \geq 3$. Thus, by (11.11) $\underline{M} \times (E-\{x\})$ and $\underline{M}.(E-\{x\})$ are connected for any $x \in E$.

(A) Suppose that $\kappa > 3$. Then, by (11.10) and (11.11), for any $x \in E$,

$\underline{M}\times(E-\{x\})$ and $\underline{M}.(E-\{x\})$ are pointed irreducible and connected. Thus, since $\underline{M} \not\cong T_6$ (because T_6 is 3-connected), there exists $x \in E$ such that $\underline{M}\times(E-\{x\})$ or $\underline{M}.(E-\{x\})$ contains T_6 as a minor.

(B) If $\kappa = 3$, and, for every $x \in E$, $\underline{M}\times(E-\{x\})$ and $\underline{M}.(E-\{x\})$ are pointed reducible, then, by (11.13), $\underline{M} \cong W_n$ or $\underline{M} \cong \underline{W}_n$ for some $n \geq 3$. Since $\underline{M}$ does not contain $\underline{M}(K_4)$ as a minor, $\underline{M} \not\cong \underline{W}_n$, and so $\underline{M} \cong W_n$ for some $n \geq 3$.

(C) Suppose that $\kappa = 3$, and that, for some $x \in E$, $\underline{M}\times(E-\{x\})$ or $\underline{M}.(E-\{x\})$ is pointed irreducible. If $\underline{M}\times(E-\{x\})$ is pointed irreducible and contains no minor isomorphic to T_6, then by hypothesis, $\underline{M}\times(E-\{x\})$ contains no minor isomorphic to a member of $\mathbf{g}$ and so $\underline{M}\times(E-\{x\})$ is a pointed-irreducible gsp network. Thus, $\underline{M}\times(E-\{x\})$ is isomorphic to $U_{2,r}$ or $U_{r-2,r}$ for some $r \geq 3$. Since $\underline{M}$ contains T_6 as a minor, $|E| \geq 6$, so $r \geq 5$. But $\underline{M}$ contains no minor isomorphic to $U_{2,5}$ or $U_{3,5}$, which is a contradiction.

A similar argument holds for the case where $\underline{M}.(E-\{x\})$ is pointed irreducible and contains no minor isomorphic to T_6.

(II) Suppose $\underline{M}$ is not κ-connected for any $\kappa > 0$. Now, for any $x \in E$, $\underline{M}\times(E-\{x\})$ or $\underline{M}.(E-\{x\})$ is pointed reducible if and only if it is 2-connected, which, by (11.11), implies $|E| = 5$, which is a contradiction. So, for any $x \in E$, $\underline{M}\times(E-\{x\})$ and $\underline{M}.(E-\{x\})$ are pointed-irreducible and (by (11.11)) connected. Thus, since $\underline{M} \not\cong W_n$ (since W_n is 3-connected), there exists $x \in E$ such that $\underline{M}\times(E-\{x\})$ or $\underline{M}.(E-\{x\})$ is pointed-irreducible, connected, and contains T_6 as a minor.

Thus, either $\underline{M} \cong W_n$ for some $n \geq 3$, or there exists $x \in E$ such that $\underline{M}\times(E-\{x\})$ or $\underline{M}.(E-\{x\})$ is connected and pointed irreducible, and contains T_6 as a minor.

PROPOSITION 11.29: Let $\underline{M}$ be a connected matroid on the set E which is pointed irreducible, contains T_6 as a minor and contains no minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$. Then $\underline{M} \cong W_n$ for some $n \geq 3$.

Proof: By (11.28), if $\underline{M}$ satisfies the hypotheses of the theorem, either $\underline{M}$ is a whirl, or there exists $x_1 \in E$ such that $\underline{M}\times(E-\{x_1\})$ or $\underline{M}.(E-\{x_1\})$ satisfies the hypotheses of the theorem. Let $\underline{M}_1$ denote one of $\underline{M}\times(E-\{x_1\})$ and $\underline{M}.(E-\{x_1\})$ which satisfies these hypotheses, and write $E_1 = E-\{x_1\}$. Then, applying (11.28) to $\underline{M}_1$, either $\underline{M}_1$ is a whirl, or there exists $x_2 \in E_1$ such that $\underline{M}_1\times(E_1-\{x_2\})$ or $\underline{M}_1.(E_1-\{x_2\})$ satisfies the hypotheses of the theorem. Let $\underline{M}_2$ denote one of $\underline{M}_1\times(E_1-\{x_2\})$ and $\underline{M}_1.(E_1-\{x_2\})$ which satisfies these hypotheses, and write $E_2 = E_1-\{x_2\}$. Proceeding thus, we obtain a sequence of matroids $\underline{M}_i$ on sets E_i which either terminates with some $\underline{M}_n \cong W_r$ for some $r \geq 3$, or continues indefinitely with each $\underline{M}_i$ satisfying the hypotheses of the theorem. Since E is a finite set, and $|E_i| = |E_{i-1}|-1$ for each i, there must come a time when $|E_i| < 6$. But then the hypotheses of the theorem cannot be satisfied, since T_6 has 6 elements, which implies that each $\underline{M}_i$ is a matroid on at least 6 elements. Thus, the sequence terminates with some $\underline{M}_n \cong W_r$ for some $r \geq 3$. Consider $\underline{M}_{n-1}$. Then either (I) $\underline{M}_{n-1}\times(E_{n-1}-\{x_n\}) \cong W_r$ for some $r \geq 3$, or

(II) $\underline{M}_{n-1}.(E_{n-1}-\{x_n\}) \cong W_r$ for some $r \geq 3$.

(I) Suppose $\underline{M}_{n-1}\times(E_{n-1}-\{x_n\}) \cong W_r$ for some $r \geq 3$. By hypothesis, $\underline{M}_{n-1}$ is pointed irreducible, connected, and contains no minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$. We shall show that this is impossible.

For, suppose that $\underline{N}$ is a connected matroid on the set F containing no minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$, such that $\underline{N}\times(F-\{x\}) \cong W_r$ for some $r \geq 3$ and some $x \in F$. We shall show that $\underline{N}$ is not simple. Clearly the result is true for $r = 3$. For $r > 3$, the proof is by induction. Suppose the result holds for all matroids of rank $\le r-1$. $\underline{N}$ has rank r, since $\underline{N}$ is connected. Let a_i', b_i' be the elements of F which are mapped into a_i, b_i of W_r under the isomorphism between $\underline{N}\times(F-\{x\})$ and W_r. Let $C'\cup\{x\}$ be the fundamental circuit of x with the base $\{a_0',a_1',\ldots,a_{r-1}'\}$ of $\underline{N}$.

(A). Suppose there exists some $a_i' \notin C'$. Without loss of generality, suppose $a_{r-1}' \notin C'$. Then $\underline{N}.(F-\{b_{r-1}'\})\times(F-\{a_{r-1}',b_{r-1}',x\}) \cong W_{r-1}$, and $\underline{N}' = \underline{N}.(F-\{b_{r-1}'\})\times(F-\{a_{r-1}',b_{r-1}'\})$ is connected. So, by the inductive hypothesis, $\underline{N}'$ is not simple, and so x is parallel in $\underline{N}'$ to some a_i' or b_i' $(0 \le i \le r-2)$. If $0 < i < r-2$, then x is parallel to a_i' or b_i' in $\underline{N}$. If $i = r-2$, then, either

x is parallel in $\underline{N}$ to a_{r-2}' or b_{r-2}',

or

$\{x,a_{r-2}',b_{r-1}'\}$ or $\{x,b_{r-2}',b_{r-1}'\}$ is a circuit of $\underline{N}$.

If either of the latter cases holds, contract out $\{a_1',a_2',\ldots,a_{r-3}'\}$ from $\underline{N}$, to leave a rank-3 matroid with elements $\{a_0',a_{r-2}',a_{r-1}',b_0',\ldots,b_{r-1}',x\}$. Then clearly this matroid has a minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$. A similar argument holds for the case $i = 0$.

(B). Suppose that $C' = \{a_0',a_1',\ldots,a_{r-1}'\}$. Contract out $\{a_3',a_4',\ldots,a_{r-1}'\}$ and simplify, to give a rank-3 matroid $\underline{N}''$ on the seven elements $\{a_0',a_1',a_2',b_0',b_1',b_2',x\}$, in which $\{a_0',a_1',a_2',x\}$ is a circuit. But $\underline{N}''\times(\{a_0',a_1',a_2',b_0',b_1',b_2'\}) \cong T_6$, and so, since $\{a_0',a_1',a_2',x\}$ is a circuit of $\underline{N}''$, $\underline{N}''$ contains a minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$, which is a contradiction.

Thus, x is parallel to a_i' or b_i' for some i, $0 \le i \le r-1$. Now, by hypothesis, $\underline{M}_{n-1}$ is pointed irreducible, and so, in particular, simple. There there can exist no $x_n \in E_{n-1}$ such that $\underline{M}_{n-1}\times(E_{n-1}-\{x_n\}) \cong W_r$ for some $r \ge 3$.

(II) Suppose that $\underline{M}_{n-1} \cdot (E_{n-1} - \{x_n\}) \cong W_r$ for some $r \geq 3$. Then, since $W_r \cong W_r^*$, we may apply the argument of (I) to the duals to show that, in $\underline{N}^*_{n-1}$, x is parallel to a_i' or b_i' for some i, $0 \leq i \leq r-1$. But, by hypothesis, $\underline{M}_{n-1}$ is pointed irreducible, so $\underline{M}^*_{n-1}$ is pointed irreducible, and hence, in particular, simple, so there can exist no $x_n \in E_{n-1}$ such that $\underline{M}_{n-1} \cdot (E_{n-1} - \{x_n\}) \cong W_r$ for some $r \geq 3$.

Thus, the sequence of matroids $\underline{M}_i$ cannot exist, and so $\underline{M} \cong W_n$ for some $n \geq 3$.

THEOREM 11.30: Let $\underline{M}$ be a matroid on the set E. Then

(a) $\underline{M}$ is ternary and base-orderable if and only if

(b) $\underline{M}$ contains no minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$.

Proof: (a) ⇒ (b) is immediate, since $U_{2,5}$ and $U_{3,5}$ are not ternary, $\underline{M}(K_4)$ is not base-orderable, and both properties are preserved under the operation of taking minors.

(b) ⇒ (a). Suppose $\underline{N}$ is a matroid which is not ternary-and-base-orderable, all of whose proper minors are ternary and base-orderable, and suppose that $\underline{N}$ contains no minor isomorphic to $U_{2,5}$, $U_{3,5}$ or $\underline{M}(K_4)$.

Since, by (11.6), the properties of being ternary and of being base-orderable are preserved under deleted pointed unions, $\underline{N}$ is pointed irreducible. Since these properties are also preserved under direct sums, $\underline{N}$ is connected.

(I). Suppose $\underline{N}$ does not contain a minor isomorphic to T_6. Then $\underline{N}$ contains no minor isomorphic to any member of $\mathfrak{g}$ and so, by (11.27), $\underline{N}$ is a gsp network. Thus, by (11.6), $\underline{N}$ is base-orderable. Since $\underline{N}$ is pointed irreducible, and contains no minor isomorphic to $U_{2,5}$ or $U_{3,5}$, $\underline{N}$ is isomorphic to $U_{2,r}$ or $U_{r-2,r}$ for some $r \leq 4$, and so $\underline{N}$ is ternary, which is a contradiction.

(II). Suppose $\underline{N}$ contains a minor isomorphic to T_6. Then, by (11.29), $\underline{N} \cong W_n$ for some $n \geq 3$. But W_n is ternary and base-orderable, which is a contradiction.

Thus, there exists no such $\underline{N}$, which completes the proof.

COROLLARY 11.31: A connected matroid is ternary and base-orderable if and only if $\underline{M}$ is a deleted-pointed-union of whirls and ternary gsp networks.

Proof: $\Rightarrow$. Suppose $\underline{M}$ is ternary and base-orderable. Let $\underline{M} = \underline{M}_1 \dot{\cup}' \underline{M}_2 \dot{\cup}' \ldots \dot{\cup}' \underline{M}_r$ where $r \geq 1$, and each $\underline{M}_i$ is pointed irreducible. By (11.29), if $\underline{M}_i$ contains T_6 as a minor, then $\underline{M}_i \cong \mathcal{W}_{n_i}$ for some $n_i \geq 3$. If $\underline{M}_i$ does not contain T_6 as a minor, then, by (11.27), $\underline{M}_i$ is a pointed irreducible gsp network. Thus, $\underline{M}$ is a deleted-pointed-union of whirls and ternary gsp networks.

$\Leftarrow$. This is immediate, since the properties of being ternary and being base-orderable are preserved under deleted pointed unions.

COROLLARY 11.32: A matroid $\underline{M}$ is ternary and base-orderable if and only if $\underline{M}$ is the direct sum of deleted-pointed-unions of whirls and ternary gsp networks.

Proof: Immediate from (11.31).

COROLLARY 11.33: If $\underline{M}$ is ternary and base-orderable, then $\underline{M}$ is ternary and fully base-orderable.

Proof: Since a whirl is a gammoid, a whirl is fully base orderable. By (11.6), a gsp network is fully base orderable. Thus, since by (11.6) the property of being fully base-orderable is preserved under deleted pointed unions, by (11.31) a connected ternary base-orderable matroid is fully base orderable. Since being fully base orderable is preserved under direct sums, by (11.32) a ternary base-orderable matroid is ternary and fully base-orderable.

COROLLARY 11.34: If $\underline{M}$ is ternary and base-orderable, $\underline{M}$ is hypergraphic.

Proof: By (5.4), whirls are hypergraphic, and by (11.6) gsp networks are hypergraphic. Thus, by (11.6), deleted-pointed-unions of whirls and gsp networks are hypergraphic, and so, by (11.31), a connected ternary base-orderable matroid is hypergraphic. Since, by (3.12)&(4.2), a direct sum of hypergraphic matroids is hypergraphic, by (11.32) a ternary base-orderable matroid is hypergraphic.

This result should be compared with the corresponding result for binary base-orderable matroids. A matroid is binary if and only if it contains no minor isomorphic to $U_{2,4}$. If such a matroid is also base-orderable, it contains no minor isomorphic to $\underline{M}(K_4)$, and so it is a series-parallel network. By (11.21), a series-parallel network is base-orderable and fully base-orderable. Since series-parallel networks are graphic, we have the following result:

PROPOSITION 11.35:

(a) If $\underline{M}$ is binary and base-orderable, then $\underline{M}$ is fully base-orderable.

(b) If $\underline{M}$ is binary and base-orderable, then $\underline{M}$ is graphic.

(c) $\underline{M}$ is binary and base-orderable if and only if $\underline{M}$ is a series-parallel network.

(11.33) is the analogue of (11.35)(a), and (11.34) is the analogue of (11.35)(b). However, because T_6 is not a gsp network, there is no direct analogue of (11.35)(c); we do, however, have the result of (11.32), and it is clear from (11.32) that a ternary gsp network is base-orderable, and, by (11.33), that it is fully base-orderable.

CHAPTER 12

DUALISM

The purpose of this chapter is to try to find an analogue in hypergraph theory of the concept of planarity in graph theory. We shall be seeking a purely matroidal definition, in terms of the hypergraphicity of the dual matroid. In fact, we shall produce two definitions, one involving hypergraphs, the other involving generalised hypergraphs.

To generalise our notion of planarity, we introduce the concept of dualism.

DEFINITION 12.1: Let $\mathbf{m}$ be a class of matroids with property P. Then a matroid $\underline{M} \in \mathbf{m}$ is said to be <u>P-dualistic</u> if and only if $\underline{M}^* \in \mathbf{m}$. Where the property P is embodied in the definition of $\mathbf{m}$, we shall say simply that $\underline{M}$ is <u>dualistic</u>.

Thus, a graphic matroid is dualistic if and only if it is the matroid of a planar graph. Gammoids are dualistic, uniform matroids are dualistic, but transversal matroids in general are not.

THEOREM 12.2: Let $\mathbf{m}$ be the class of dualistic graphic matroids. Then a graphic matroid $\underline{M}$ is a member of $\mathbf{m}$ if and only if $\underline{M}$ contains no minor isomorphic to $\underline{M}(K_5)$ or $\underline{M}(K_{3,3})$.

Proof: This is a standard result in graph theory. Proofs can be found in Harary [11] or (in matroid form) in Tutte [24].

Since a minor of a hypergraphic matroid is a generalised hypergraphic matroid, but not, in general, a hypergraphic matroid, we consider first dualism in generalised hypergraphic matroids; in other words, we consider the class $\mathbf{m}$ of matroids $\underline{M}$ such that both $\underline{M}$ and $\underline{M}^*$ are generalised hypergraphic.

(12.2) shows that the class of dualistic graphic matroids can be characterised within the class of graphic matroids by a finite set of forbidden minors. Our next result shows that the analogue of this for generalised hypergraphic matroids is false.

THEOREM 12.3: Let $\mathbf{m}$ be the class of dualistic generalised hypergraphic matroids. Then there exists an infinite family $\{\underline{M}_i : i \geq 3\}$ of generalised hypergraphic matroids, such that $\underline{M}_i^* \notin \mathbf{m}$ for $i \geq 3$, but that every proper minor of $\underline{M}_i^*$ is a member of $\mathbf{m}$.

Proof: Let $\underline{M}_i$ $(i \geq 3)$ be the matroid defined in (8.2) and (8.3). Then, by (8.12), $\underline{M}_i$ is not generalised hypergraphic, but every proper minor of $\underline{M}_i$ is generalised hypergraphic. By (8.5), $\underline{M}_i^*$ is generalised hypergraphic, and hence every minor of $\underline{M}_i^*$ is generalised hypergraphic. Thus $\{\underline{M}_i^* : i \geq 3\}$ is an infinite set of matroids such that $\underline{M}_i^* \notin \mathbf{m}$ for each $i \geq 3$, but every proper minor of each $\underline{M}_i^*$ is a member of $\mathbf{m}$.

We now consider dualism in hypergraphic matroids. Clearly, a desirable property of dualistic hypergraphic matroids is that a restriction of a dualistic hypergraphic matroid should be dualistic. By the duality relationship between restriction and contraction, this is equivalent to the requirement that any contraction of a dualistic hypergraphic matroid should be dualistic, and hence that every minor of a dualistic hypergraphic matroid should be dualistic. We therefore define a subclass of the class of hypergraphic matroids as follows:

DEFINITION 12.4: Let $\underline{M}$ be a matroid on the set E. Then $\underline{M}$ is said to be <u>strongly hypergraphic</u> if and only if every minor of $\underline{M}$ is hypergraphic.

The class of strongly hypergraphic matroids is thus, in a sense, the "opposite" of that of generalised hypergraphic matroids - the latter is the smallest superclass of the class of hypergraphic matroids which is closed under the operation of taking minors, the former is the largest subclass of the class of hypergraphic matroids which is similarly closed.

We are therefore considering dualism in strongly hypergraphic matroids. We shall prove the analogue of (12.3) - that the class of dualistic strongly hypergraphic matroids cannot be characterised within the class of strongly hypergraphic matroids by a finite set of forbidden minors. In order to do this, we shall find a family $\{M_n : n \geq 3\}$ of matroids such that, for each $n \geq 3$, $\underline{M}_n$ is not strongly hypergraphic, every proper minor of $\underline{M}_n$ is strongly hypergraphic, and $\underline{M}_n^*$ is strongly hypergraphic. Thus result will also show that the class of strongly hypergraphic matroids cannot be characterised by a finite set of forbidden minors.

PROPOSITION 12.5: For $n \geq 3$, let $E_n' = (\{e'\} \cup \{a_{ij}' : 0 \leq i < j \leq n\})_{\neq}$,

$C_m' = \{e'\} \cup \{a_{ij}' \in E_n' : i = m \text{ or } j = m\}$ $(0 \leq m \leq n)$,

and let $\mathbf{C}_n''$ denote the set of all $(n+2)$-subsets of E_n' containing no C_m' $(0 \leq m \leq n)$.

Then $\mathbf{C}_n' = \mathbf{C}_n'' \cup \{C_m' : 0 \leq m \leq n\}$ is the set of circuits of a matroid $\underline{M}_n$ of rank $(n+1)$ on the set E_n'.

Proof: This is routine verification of the circuit axioms (C1) and (C2), and we omit the details.

THEOREM 12.6: For each $n \geq 3$, $\underline{M}_n$ is not hypergraphic.

Proof: Suppose $\underline{M}_n$ is hypergraphic. Then, by (2.13) there exists a critical hypergraph $H = (V, E, \$)$ such that $\underline{M}_n \cong \underline{M}(H)$. Let $E = \{e\} \cup \{a_{ij} : 0 \leq i < j \leq n\}_{\neq}$ and let the isomorphism between $\underline{M}_n$ and $\underline{M}(H)$ be induced by the obvious bijection between E and E_n'. Denote by C_m the image of C_m'.

Then, for each i, it is easy to see that C_i is a circuit and a hyperplane of $\underline{M}(H)$. (1)

Since H is critical, and $\underline{M}(H)$ has rank n+1, by (2.2) and (2.14),

$|V| = k+n$ (2)

Since $\mathcal{C}_i$ is a hyperplane, $\rho\mathcal{C}_i = n$; since $\mathcal{C}_i$ is also a circuit,

by (2.3), $|V(\mathcal{C}_i)| = k+n-1$ (3)

Since $\mathcal{C}_i$ is a hyperplane, $\rho(\mathcal{C}_i \cup \{x\}) = n+1$ for any $x \notin \mathcal{C}_i$.

Thus, from (1) and (3.1), $|V(\mathcal{C}_i \cup \{x\})| = k+n$ for each $x \notin \mathcal{C}_i$ (4)

Let $\{A_i\} = V - V(\mathcal{C}_i)$ $\quad 0 \le i \le n$.

Then, by (3) and (4), $A_i \neq A_j$ for $i \neq j$, since $A_j \in V(\mathcal{C}_i)$ for $i \neq j$.

Thus, $\{A_0, A_1, \ldots, A_{n-1}\} \subseteq V(\mathcal{C}_n)$.

Now, $V(e) \subseteq V(\mathcal{C}_i)$ for each i $(0 \le i \le n)$, so $A_i \notin V(e)$ $(0 \le i \le n)$.

Thus, $V(\mathcal{C}_n) \supseteq V(e) \cup \{A_0, A_1, \ldots, A_{n-1}\}$.

Thus, $|V(\mathcal{C}_n)| \ge k+n$, which contradicts (2). Thus, $\underline{M}_n$ is not hypergraphic.

There is a high degree of symmetry in the definition of $\underline{M}_n$. In fact, there are exactly two isomorphism classes of elements - $\{e'\}$ and $E_n' - \{e'\}$. If a' and b' are members of the same class, then there exists an automorphism of $\underline{M}_n$ which maps a' into b'. We next consider the one-point deletions and contractions of $\underline{M}_n$, and show that these are all hypergraphic. By the above remarks, it is sufficient for this to prove that $\underline{M}_n \times (E_n' - \{e'\})$, $\underline{M}_n \times (E_n' - \{a'_{n-1,n}\})$, $\underline{M}_n \cdot (E_n' - \{e'\})$ and $\underline{M}_n \cdot (E_n' - \{a'_{n-1,n}\})$ are all hypergraphic. These results form the content of the next four propositions.

PROPOSITION 12.7: $\underline{M}_n \times (E_n' - \{e'\})$ is strongly hypergraphic.

Proof: From (12.5), $\underline{M}_n \times (E_n' - \{e'\})$ is uniform of rank $n+1$ on the set $E_n' - \{e'\}$. Since every minor of a uniform matroid is uniform, and, by (5.1), every uniform matroid is hypergraphic, $\underline{M}_n \times (E_n' - \{e'\})$ is strongly hypergraphic.

PROPOSITION 12.8: $\underline{M}_n \times (E_n' - \{a_{n-1,n}'\})$ is hypergraphic.

Proof: Let $H = (V, E, \$)$ be the hypergraph with

$V = A^0 \cup A^1 \cup \ldots \cup A^{n-3} \cup B$ where these sets are all disjoint, and

$A^0 = \{A_0, A_1, \ldots, A_n\}_{\neq}$

$A^m = \{A_{ij}^m : 0 \le i < j \le n,\ (i,j) \neq (n-1,n)\}_{\neq}$

$B = \{B_{ij} : 0 \le i < j \le n,\ (i,j) \neq (n-1,n)\}_{\neq} \cup \{B_{00}\}$

$E = E_n' - \{a_{n-1,n}'\}$

$V(e') = \{A_{n-1}, A_n\} \cup A^1 \cup \ldots \cup A^{n-3} \cup (B - \{B_{00}\})$

$V(a_{ij}') = (A - \{A_i, A_j\}) \cup (\bigcup_{m=1}^{n-3} (A^m - \{A_{ij}^m\})) \cup (B - \{B_{ij}\})$

It is routine to check that the circuits of $\underline{M}(H)$ are the sets C_i' $(0 \le i \le n-2)$ and all $(n+2)$-subsets of E containing none of these. But, from (12.5), these are the circuits of $\underline{M}_n \times (E_n' - \{a_{n-1,n}'\})$, and so $\underline{M}_n \times (E_n' - \{a_{n-1,n}'\})$ is hypergraphic.

PROPOSITION 12.9: $\underline{M}_n \cdot (E_n' - \{e'\})$ is hypergraphic.

Proof: Let $H = (V, E, \$)$ be the hypergraph with

$V = A \cup A^1 \cup \ldots \cup A^{n-3}$ where these sets are all disjoint, and

$A = \{A_0, A_1, \ldots, A_n\}_{\neq}$

$A^m = \{A_{ij}^m : 0 \le i < j \le n\}_{\neq}$

$E = E_n' - \{e'\}$

$V(a_{ij}') = (A - \{A_i, A_j\}) \cup (\bigcup_{m=1}^{n-3} (A^m - \{A_{ij}^m\}))$

It is routine to check that the circuits of $\underline{M}(H)$ are the sets $C_i' - \{e'\}$ $(0 \le i \le n)$ and all $(n+1)$-subsets of E containing none of these. But, from (12.5), these are the circuits of $\underline{M}_n \cdot (E_n' - \{e'\})$, and so $\underline{M}_n \cdot (E_n' - \{e'\})$ is hypergraphic.

PROPOSITION 12.10: $\underline{M}_n \cdot (E_n' - \{a_{n-1,n}'\})$ is hypergraphic.

Proof: We shall prove, for notational convenience, that $\underline{M}_n \cdot (E_n' - \{a_{01}'\})$ is hypergraphic. From the remarks following (12.6), this is equivalent to the statement of (12.10).

From (12.5), the circuits of $\underline{M}_n \cdot (E_n' - \{a_{01}'\})$ are the sets $C_0' - \{a_{01}'\}$, $C_1' - \{a_{01}'\}$ and all (n+1)-subsets of $E_n' - \{a_{01}'\}$ containing neither of these. By (3.14), therefore, it is sufficient to prove that

$\underline{M}_n \cdot (E_n' - \{a_{01}'\}) \times ((C_0' \cup C_1') - \{a_{01}'\})$ is hypergraphic.

Let $H = (V, E, \$)$ be the hypergraph with

$$V = A_1 \cup A_2 \cup B_1 \cup \ldots \cup B_{n-3}$$
$$A_1 = \{X, Y\}_{\neq}$$
$$A_2 = \{A_1, A_2, \ldots, A_{2n-1}\}_{\neq}$$
$$B_i = \{B_{i0}, B_{i1}, \ldots, B_{i,2n-1}\}_{\neq}, \quad (1 \le i \le n-3).$$
$$E = (C_0' \cup C_1') - \{a_{01}'\}$$
$$V(e) = A_2 \cup (\bigcup_{m=1}^{n-3} (B_m - \{B_{m0}\}))$$
$$V(a_{0i}') = \{X\} \cup (A_2 - \{A_i\}) \cup (\bigcup_{m=1}^{n-3} (B_m - \{B_{m,i}\})) \qquad (2 \le i \le n).$$
$$V(a_{1i}') = \{Y\} \cup (A_2 - \{A_{n+i-1}\}) \cup (\bigcup_{m=1}^{n-3} (B_m - \{B_{m,n+i-1}\})) \quad (2 \le i \le n).$$

It is easy to check that the circuits of $\underline{M}(H)$ are the sets $C_0' - \{a_{01}'\}$, $C_1' - \{a_{01}'\}$ and all (n+1)-subsets of E containing neither of these. Thus, $\underline{M}_n \cdot (E_n' - \{a_{01}'\}) \times ((C_0' \cup C_1') - \{a_{01}'\})$ is hypergraphic, and this, by the remarks at the beginning of the proof, is sufficient to prove the result.

PROPOSITION 12.11: Let $\underline{M}$ be a matroid of rank r on the set $E = A \cup B \cup C$ where these sets are disjoint, $|A| = |B| = r$, and such that $\underline{M}$ has as circuits the sets A, B and all (r+1)-subsets of E containing neither of these. Then $\underline{M}$ is hypergraphic.

Proof: If r = 1, the result is trivial. Otherwise, let $A = \{a_0, \ldots, a_{r-1}\}_{\neq}$, $B = \{b_0, \ldots, b_{r-1}\}_{\neq}$. Let $H = (V, A \cup B, \$)$ be the hypergraph with

$V = A \cup B \cup \{X, Y\}_{\neq}$, where these sets are disjoint and

$$A = \{A_0, A_1, \ldots, A_{r-1}\}_{\neq}$$
$$B = \{B_0, B_1, \ldots, B_{r-1}\}_{\neq}$$
$$V(a_i) = \{A_i, A_{i+1}\} \cup B \cup \{X\} \qquad (\text{mod } r)$$
$$V(b_i) = \{B_i, B_{i+1}\} \cup A \cup \{Y\} \qquad (\text{mod } r).$$

Then A and B are circuits of $\underline{M}(H)$, and the other circuits of $\underline{M}(H)$ are those $(r+1)$-subsets of E containing neither A nor B. Let $\underline{M}'$ be the matroid obtained from $\underline{M}(H)$ by the free, rank-preserving, $|C|$-point extension of $\underline{M}(H)$ by C. Then, by (3.14), $\underline{M}'$ is hypergraphic. Clearly, $\underline{M} \cong \underline{M}'$, so $\underline{M}$ is hypergraphic.

PROPOSITION 12.12: Let $\underline{M}$ be a matroid of rank $r \geq 2$ on the set $E = A \cup C$ where $A \cap C = \phi$ and $|A| = r$, such that $\underline{M}$ has as circuits the set A together with all $(r+1)$-subsets of E not containing A. Then $\underline{M}$ is hypergraphic.

Proof: If $|C| < 2$, $\underline{M}$ is not connected, and the result follows from (3.12). Suppose, therefore, that $|C| \geq 2$. Let $c_1, c_2 \in C$, and let $A = \{a_0, a_1, \ldots, a_{r-1}\}$. Let $H = (V, A\{c_1, c_2\}, \$)$ be the hypergraph with

$V \;\; = A \cup B_1 \cup B_2 \cup \{X\}$ where these sets are all disjoint, and

$A \;\; = \{A_0, A_1, \ldots, A_{r-1}\}_{\neq}$ and $|B_i| = r-1$ $(i = 1,2)$.

$V(a_i) = \{A_i, A_{i+1}\} \cup B_1 \cup B_2 \pmod r$

$V(c_1) = A \cup \{X\} \cup B_2$

$V(c_2) = A \cup \{X\} \cup B_1$

Then $\underline{M}(H)$ has as circuits the set A together with all $(r+1)$-subsets of $A \cup \{c_1, c_2\}$ not containing A. The matroid obtained from $\underline{M}(H)$ by the free, rank-preserving $(|C|-2)$-point extension of $\underline{M}(H)$ by $C - \{c_1, c_2\}$ is hypergraphic, and, from the above, has circuits A together with all $(r+1)$-subsets of E not containing A. This matroid is therefore isomorphic to $\underline{M}$, and so $\underline{M}$ is hypergraphic.

PROPOSITION 12.13: Let $\underline{M}$ be a proper minor of $\underline{M}_n$ $(n \geq 3)$. Then $\underline{M}$ is hypergraphic.

Proof: Let $\underline{M} = \underline{M}_n.(E_n'-X')\times((E_n'-X')-Y') = \underline{M}_n\times(E_n'-Y').((E_n'-Y')-X')$, where X' is independent in $\underline{M}_n$ and $X' \cap Y' = \phi$. If $e' \in Y'$, the result follows from (12.7). If $|X'| \leq 1$, the result follows by restriction from (12.7)-(12.10).

Suppose that $|X'| \geq 2$.

(A) Suppose $e' \in X'$. Then, by (12.9), the circuits of $\underline{M}_n \cdot (E_n' - \{e'\})$ are the sets $C_i' - \{e'\}$ $(0 \leq i \leq n)$ and all $(n+1)$-subsets of $E_n' - \{e'\}$ containing none of these. Since $|X'| \geq 2$, there exists $x' \in X'$, $x' \neq e'$. Then $x' \in C_i'$ for two values of i, i_1 and i_2, say. Therefore, $\underline{M}_n \cdot (E_n' - \{e', x'\})$ has as circuits the sets $C_{i_1}' - \{e', x'\}$, $C_{i_2}' - \{e', x'\}$ and all n-subsets of $E_n' - \{e', x'\}$ containing neither of these. Since $C_{i_1}' \cap C_{i_2}' = \{e', x'\}$, this matroid is hypergraphic by (12.11). If there exists $y' \in X' - \{e', x'\}$, with $y' \notin C_{i_1}' \cup C_{i_2}'$, then $\underline{M}_n \cdot (E_n' - \{e', x', y'\})$ is uniform, and hence strongly hypergraphic. If, for each $y' \in X' - \{e', x'\}$, $y' \in C_{i_1}' \cup C_{i_2}'$, then $\underline{M}_n \cdot (E_n' - \{e', x', y'\})$ has circuits $C_{i_1}' - \{e', x', y'\}$ (say), together with all $(n-1)$-subsets of $E_n' - \{e', x', y'\}$ not containing this set. By (12.12), this is hypergraphic. If there exists $z' \in X' - \{e', x', y'\}$ with $z' \in C_{i_1}' - C_{i_2}'$, then $\underline{M}_n \cdot (E_n' - \{e', x', y', z'\})$ is uniform, and hence strongly hypergraphic. If, for every $z' \in X' - \{e', x', y'\}$, $z' \in C_{i_1}'$, then $\underline{M}_n \cdot (E' - X')$ has circuits $C_{i_1}' - X'$, together with all $((n+2) - |X'|)$-subsets of $E_n' - X'$ not containing this set. If $|C_{i_1}' - X'| \geq 2$, this matroid is hypergraphic, by (12.12). If $|C_{i_1}' - X'| \leq 1$, then $\underline{M}_n \cdot (E_n' - X')$ has rank at most 2, and is therefore hypergraphic.

Thus, if $e' \in X'$, $\underline{M}$ is hypergraphic.

(B) Suppose $e' \notin X'$. Let $x' \in X'$. Then $x' = a_{i_1, i_2}'$ for some i_1, i_2. Therefore, $\underline{M}_n \cdot (E_n' - \{x'\})$ has circuits $C_{i_1}' - \{x'\}$, $C_{i_2}' - \{x'\}$ and all $(n+1)$-subsets of $E_n' - \{x'\}$ containing neither of these. By (12.11), this matroid is hypergraphic. If there exists $y' \in X' - \{x'\}$ with $y' \notin C_{i_1}' \cup C_{i_2}'$, then $\underline{M}_n \cdot (E_n' - \{x', y'\})$ is uniform, and hence strongly hypergraphic. If $y' \in C_{i_1}'$ (say), then $\underline{M}_n \cdot (E_n' - \{x', y'\})$ has as circuits the set $C_{i_1}' - \{x', y'\}$ together with all n-subsets of $C_n' - \{x', y'\}$ not containing this set. By (12.12), this matroid is hypergraphic. If there exists $z' \in X' - \{x', y'\}$ such that $z' \notin C_{i_1}'$, then $\underline{M}_n \cdot (E_n' - \{x', y', z'\})$ is uniform and hence strongly hypergraphic. If, for all $z' \in X' - \{x', y'\}$,

$z' \in C'_{i_1}$, then $\underline{M}_n\cdot(E'_n-X')$ has as circuits the set $C'_{i_1}-X'$, together with all $(n+2-|X'|)$-subsets of E'_n-X' not containing this set. If $|C'_{i_1}-X'| \geq 2$, then, by (12.12), this matroid is hypergraphic. If $|C'_{i_1}-X'| \leq 1$, then $\underline{M}_n\cdot(E'_n-X')$ has rank at most 2, and hence is hypergraphic.

So, for any nonempty independent set $X' \subseteq E'_n$, $\underline{M}_n\cdot(E'_n-X')$ is hypergraphic. Therefore, for any disjoint sets $X', Y' \subseteq E'_n$, not both empty, and with X' independent in $\underline{M}_n$, $\underline{M}_n\cdot(E'_n-X')\times((E'_n-X')-Y')$ is hypergraphic. Thus, every proper minor of $\underline{M}_n$ is hypergraphic.

COROLLARY 12.14: (i) The class of hypergraphic matroids cannot be characterised by a finite set of forbidden minors.

(ii) The class of strongly hypergraphic matroids cannot be characterised by a finite set of forbidden minors.

Proof: $\underline{M}_n$ is a minimal non-member of each class, for each $n \geq 3$.

We shall now prove that, for each n, $\underline{M}^*_n$ is hypergraphic; we shall then show that every contraction of $\underline{M}^*_n$ is hypergraphic, and deduce from this that $\underline{M}^*_n$ is strongly hypergraphic for each $n \geq 3$.

PROPOSITION 12.15: $\underline{M}^*_n$ is hypergraphic for each $n \geq 3$.

Proof: Let $H = (V, E, \$)$ be the hypergraph with

$$V = A_0 \cup A_1 \cup \ldots \cup A_{\frac{1}{2}n(n-3)} \quad \text{where these sets are all disjoint, and}$$

$$A_0 = \{A_0, A_1, \ldots, A_n\}_{\neq}$$

$$A_m = \{A^m_{ij} : 0 \leq i < j \leq n\}_{\neq} \qquad (m \geq 1)$$

$$E = E'_n - \{e'\}$$

$$V(a'_{ij}) = \{A_i, A_j\} \cup \left(\bigcup_{m=1}^{\frac{1}{2}n(n-3)} (A_m - \{A^m_{ij}\}) \right)$$

Then the circuits of $\underline{M}(H)$ are the sets $E-C'_i$ $(0 \leq i \leq n)$, together with all $(\frac{1}{2}n(n-1)+1)$-subsets of E containing none of these. Now, let $\underline{M}'$ be the free, rank-preserving, one-point extension of $\underline{M}(H)$ by e'. Then, by (3.13), $\underline{M}'$ is hypergraphic, and clearly $\underline{M}' \cong \underline{M}^*_n$, whence $\underline{M}^*_n$ is hypergraphic.

PROPOSITION 12.16: If $X' \subseteq E_n'$ is a nonempty independent set in $\underline{M}_n^*$ $(n \geq 3)$, then $\underline{M}_n^*.(E_n'-X')$ is hypergraphic.

Proof:

(A) The case $n = 3$ is trivial, since $\underline{M}_n^*.(E_n'-X')$ then has at most six elements, and is therefore hypergraphic.

(B) Assume $n \geq 4$, and suppose that $e' \in X'$. Then, by (12.7), $\underline{M}_n \times (E_n'-\{e'\})$ is uniform. Thus, $\underline{M}_n^*.(E_n'-\{e'\})$ is uniform, and hence strongly hypergraphic. Thus, $\underline{M}_n^*.(E_n'-X')$ is hypergraphic.

(C) Assume $n \geq 4$, and suppose that $e' \notin X'$.

Let $I = \{i: X' \cap C_i' = \phi\}$. Then, if $I = \phi$, $\underline{M}_n \times (E_n'-X')$ is uniform, and so $\underline{M}_n^*.(E_n'-X')$ is hypergraphic. If $I = \{i_1\}$, then the set of circuits of $\underline{M}_n \times (E_n'-C')$ is the set C_{i_1}', together with all (n+2)-subsets of $E_n'-X'$ not containing this set. Thus, the circuits of $\underline{M}_n^*.(E_n'-X')$ are the set $(E_n'-X')-C_{i_1}'$, together with all $(|E_n'-X'|-n)$-subsets of $E_n'-X'$ not containing this set. If $|(E_n'-X')-C_{i_1}'| \geq 2$, then, by (12.12), this matroid is hypergraphic. If $|(E_n'-X')-C_{i_1}'| \leq 1$, then $\underline{M}_n^*.(E_n'-X')$ has rank at most 2, and is therefore hypergraphic.

Since $C_0' \cup \ldots \cup C_n' = E_n'$, $|I| \neq n+1$, because $X' \neq \phi$. Since also, for each $a_{ij}' \in E_n'$, $a_{ij}' \in C_i' \cap C_j'$, $|I| \neq n$, because $X' \neq \phi$.

Assume, therefore, that $n \geq 4$, and that $2 \leq |I| \leq n-1$. Then, the set of circuits of $\underline{M}_n \times (E_n'-X')$ is the set $\{C_i': i \in I\}$, together with all (n+2)-subsets of $E_n'-X'$ containing none of these. Thus, the set of hyperplanes of $\underline{M}_n \times (E_n'-X')$ is the set $\{C_i': i \in I\}$, together with all n-subsets of $E_n'-X'$ contained in none of these. Therefore, the set of circuits of $\underline{M}_n^*.(E_n'-X')$ is the set $\{(E_n'-X')-C_i': i \in I\}$, together with all $(|E_n'-X'|-n)$-subsets of $E_n'-X'$ containing none of these.

Put $t = rk(\underline{M}_n^*.(E_n'-X'))$. Then $t+1 = |E_n'-X'|-n$.

Let $H = (V, E, \$)$ be the hypergraph with

$V = A_0 \cup A_1 \cup \ldots \cup A_{t'} \cup B_0 \cup B_1$ where these sets are all disjoint, and $t' = t - |I| - 1$,

$\mathcal{A}_0 = \{A_i : i \in I\}_{\neq}$

$\mathcal{A}_m = \{A^m_{ij} : 0 \le i < j \le n\}_{\neq}$ $(m \ge 1)$

$\mathcal{B}_0 = \{B^0_{ij} : i,j \in I,\ i < j\}_{\neq}$

$\mathcal{B}_1 = \{B^1_{ij} : 0 \le i < j \le n,\ \{i,j\} \cap \bar{I} \neq \phi\}_{\neq}$

$\mathsf{E} = (\mathsf{E}'_n - \mathsf{X}') - \{e'\}$

$V(a'_{ij}) = (\{A_i, A_j\} \cap \mathcal{A}_0) \cup (\mathcal{B}_0 - \{B^0_{ij}\}) \cup (\mathcal{B}_1 - \{B^1_{ij}\}) \cup (\bigcup_{m=1}^{t'} (\mathcal{A}_m - \{A^m_{ij}\}))$

We first need to show that $t' \ge 1$.

Now, $\mathsf{X}' \cap \mathsf{C}'_i = \phi$ for $i \in I$.

$\therefore$ $\mathsf{X}' \cap (\bigcup_I \mathsf{C}'_i) = \phi$ and so $\mathsf{X}' \subseteq \mathsf{E}'_n - (\bigcup_I \mathsf{C}'_i)$

$\therefore$ $\mathsf{E}'_n - \mathsf{X}' \supseteq \bigcup_I \mathsf{C}'_i$, and therefore

$|\mathsf{E}'_n - \mathsf{X}'| \ge |\bigcup_I \mathsf{C}'_i| = 1 + \frac{1}{2}|I|(2n+1-|I|)$

$\therefore$ $t \ge \frac{1}{2}|I|(2n+1-|I|)-n$ $(2 \le |I| \le n-1)$

Now, for $|I|$ in the range indicated, and $n \ge 4$,

$\frac{1}{2}|I|(2n+1-|I|)-n \ge |I|+2$. Therefore, $t \ge |I|+2$, and so $t' \ge 1$.

From the definition above,

$|\mathcal{V}| = |I| + t'(\frac{1}{2}n(n+1)) + |\mathcal{B}_0| + |\mathcal{B}_1|$ and, using k to denote the cardinality of H, $k = t'(\frac{1}{2}n(n+1)-1) + |\mathcal{B}_0| + |\mathcal{B}_1|$.

$\therefore$ $|\mathcal{V}|-k+1 = |I|+t'+1 = t$.

Now, for $\mathcal{L} \subseteq \{(i,j) : a'_{ij} \in \mathsf{E}\}$ with $|\mathcal{L}| \ge 2$,

$|\cup\{V(a'_{ij}) : (i,j) \in \mathcal{L}\}| = |(\cup(\{A_i, A_j\} : (i,j) \in \mathcal{L})) \cap \mathcal{A}_0| + k + t'$.

Suppose D is a circuit of $\underline{M}(H)$, with $|V(\mathsf{D})| \le |\mathcal{V}| - 2$.

Then there exist A_i, A_j $(i,j \in I)$ such that $A_i \notin V(\mathsf{D})$ and $A_j \notin V(\mathsf{D})$.

Thus, $\mathsf{D} \subseteq \{x \in \mathsf{E} : A_i \notin V(x)\} \cap \{x \in \mathsf{E} : A_j \notin V(x)\}$

$= ((\mathsf{E}'_n - \mathsf{X}') - \mathsf{C}'_i) \cap ((\mathsf{E}'_n - \mathsf{X}') - \mathsf{C}'_j)$

$= (\mathsf{E}'_n - \mathsf{X}') - (\mathsf{C}'_i \cup \mathsf{C}'_j)$.

$\therefore$ $|\mathsf{D}| \le \frac{1}{2}n(n+1) + 1 - |\mathsf{X}| - 2n = \frac{1}{2}n(n-1) - |\mathsf{X}| + 1 - n$

$= t-n+1$.

$\therefore$ since D is a circuit, $|V(\mathsf{D})| \le t-n+1+k-2$.

Thus, if $\mathcal{L} = \{(i,j) : a'_{ij} \in \mathsf{D}\}$,

$|\cup(\{A_i,A_j\}: (i,j) \in \pounds)| + k + t' \leq t-n+1+k-2$

$\therefore \ |\cup(\{A_i,A_j\}: (i,j) \in \pounds)| \leq \ |I| - n,$

which is impossible, since $|I| \leq n-1$. Thus, for any circuit D of $\underline{M}(H)$ with $V(D) \neq V$, $|V(D)| = |V| - 1$. Therefore, $V(D) = V-\{A_i\}$ for some $i \in I$, and so, since D is a circuit, $|D| = t$. Also, since $A_i \notin V(D)$, $D \subseteq (E_n'-X')-C_i'$. Thus, since $|(E_n'-X')-C_i'| = t$, $D = (E_n'-X')-C_i'$. Thus, the circuits of $\underline{M}(H)$ are the sets $(E_n'-X')-C_i'$ $(i \in I)$, together with all $(t+1)$-subsets of $(E_n'-X')-\{e'\}$ containing none of these. Let $\underline{M}'$ be the matroid obtained by the free, rank-preserving one-point extension of $\underline{M}(H)$ by e'. Then, by (3.13), $\underline{M}'$ is hypergraphic, and, since $\underline{M}' \cong \underline{M}_n^*.(E_n'-X')$, $\underline{M}_n^*.(E_n'-X')$ is hypergraphic.

COROLLARY 12.17: $\underline{M}_n^*$ is strongly hypergraphic for each $n \geq 3$.

Proof: By (12.15) and (12.16), $\underline{M}_n^*.(E_n'-X')$ is hypergraphic, where X' is independent in $\underline{M}_n^*$. Since every minor of $\underline{M}_n^*$ is isomorphic to $(\underline{M}_n^*.(E_n'-X'))\times((E_n'-X')-Y')$ for some disjoint sets $X',Y' \subseteq E_n'$, with X' independent in $\underline{M}_n^*$, every minor of $\underline{M}_n^*$ is hypergraphic, and so $\underline{M}_n^*$ is strongly hypergraphic.

COROLLARY 12.18: Let $\mathfrak{m}$ be the class of dualistic strongly hypergraphic matroids. Then $\{\underline{M}_n^*: n \geq 3\}$ is an infinite family of matroids with the property that $\underline{M}_n^* \notin \mathfrak{m}$ for each $n \geq 3$, but every proper minor of $\underline{M}_n^*$ is a member of $\mathfrak{m}$.

COROLLARY 12.19: The class of dualistic, strongly hypergraphic matroids cannot be characterised by a finite set of forbidden minors.

COROLLARY 12.20: The class of dualistic, strongly hypergraphic matroids cannot be characterised within the class of hypergraphic matroids by a finite set of forbidden minors.

CHAPTER 13

A CHARACTERISATION OF HYPERGRAPHIC MATROIDS

We have already remarked that graphic matroids can be characterised by a finite set of forbidden minors. It is clear that hypergraphic matroids cannot be so characterised, not only because the class of hypergraphic matroids is not closed under contraction, but also because, in Chapter 12, we found an infinite set of forbidden minors for strongly hypergraphic matroids.

There are other characterisations of graphic matroids in the literature. Most are in terms of forbidden minors and a representability condition, or its equivalent. These are not particularly appropriate as starting-points for generalisation to hypergraphic matroids, since (7.5) shows that hypergraphic matroids are representable over every characteristic, and (12.14) shows that there is no finite set of forbidden minors for hypergraphic matroids.

We shall use, as our motivation in this chapter, the characterisation of graphic matroids due to Sachs [22], which is a lattice-theoretic version of the result of MacLane [16].

THEOREM 13.1: An irreducible lattice $\mathbf{L}$ is isomorphic to the lattice of a non-separable graphic matroid if and only if there exists a family $\mathbf{F} = \{H_i : i \in I\}$ of hyperplanes of $\mathbf{L}$ satisfying

(i) every atom of $\mathbf{L}$ has exactly two complements in $\mathbf{F}$, and no two atoms have the same pair of complements;

(ii) If $J \subseteq I$, then $\rho(\bigcap_J H_i) \leq |I-J|-1$, whenever this is non-negative, where ρ is the rank function of $\mathbf{L}$.

We now re-state this is a modified form, suitable for generalisation.

THEOREM 13.1': A connected matroid $\underline{M}$ on the set E is graphic if and only if there exists a family $\mathbf{F} = \{F_i: i \in I\}$ of hyperplanes of $\underline{M}$ satisfying

(i) for each $e \in E$, $e \notin F_i$ for exactly 2 values of $i \in I$;

(ii) if $J \subseteq I$, then $\rho(\bigcap_J F_i) \leq |I-J|-1$, whenever this is non-negative;

(iii) every 1-flat R of $\underline{M}$ is the intersection of a set $\{F_i: i \in I(R) \subseteq I\}$ of hyperplanes with $|I(R)| = |I|-2$.

The proof of this theorem is by construction. For hypergraphic matroids, an analogous result holds.

THEOREM 13.2: A loopless matroid $\underline{M}$ on the set E is isomorphic to $\underline{M}(H)$, where H is a k-hypergraph, if and only if there exists a family $\mathbf{F} = \{F_i: i \in I\}$ of flats of $\underline{M}$ satisfying

(i) for each $e \in E$, $e \notin F_i$ for exactly k values of $i \in I$;

(ii) if $J \subseteq I$, then $\rho(\bigcap_J F_i) \leq |I-J|-(k-1)$, whenever this is non-negative;

(iii)(a) for every circuit C of $\underline{M}$, $\sigma(C)$ is the intersection of a set $\{F_i: i \in I(C) \subseteq I\}$ of flats, with $|I(C)| = |I|-(k-1)-\rho C$

(b) every 1-flat R of $\underline{M}$ is the intersection of a set $\{F_i: i \in I(R) \subseteq I\}$ of flats with $|I(R)| = |I|-k$.

Proof: (A) Suppose $\underline{M}$ satisfies (i) - (iii).

Let $|I| = p$, and let $V = \{A_1, A_2, \ldots, A_p\}_{\neq}$ be a set of p vertices.

Let $E = \{e_i: 1 \leq i \leq n\}_{\neq}$, and let $E' = \{e_i': 1 \leq i \leq n\}$ be an isomorphic copy of E. Let $H = (V, E', \$)$ be the hypergraph with

$$V(e') = \{A_i: e \notin F_i, i \in I\}, \text{ where } e' \text{ is the image of } e$$

under the obvious bijection between E and E'.

Then, by (i), H is uniform, of cardinality k.

Let C' be a circuit of $\underline{M}(H)$, and let C denote the image of C' under the obvious bijection between E and E'.

Since C' is a circuit of $\underline{M}(H)$, $|V(C')| = k+|C'|-2$.

Thus, there exist $p-(k+|C'|-2)$ vertices of H not elements of $V(C')$, and so $C \subseteq F_i$ for at least $p-(k+|C|-2)$ values of $i \in I$.

Therefore, by (ii),

$$\rho C \leq \rho(\cap\{F_i: C \subseteq F_i\}) \leq |I| - |\{i \in I: C \subseteq F_i\}| - (k-1)$$
$$\leq p - p + (k+|C|-2) - k + 1$$
$$= |C| - 1$$

Thus, C is dependent in $\underline{M}$.

Conversely, suppose C is a circuit of $\underline{M}$. Then, by (iii),

$$\sigma(C) = \bigcap_{I(C)} F_i, \text{ where } |I(C)| = |I|-(k-1)-\rho C.$$

By (ii), for $J \subseteq I$, $\rho \bigcap_J F_i \leq |I-J|-(k-1)$, whenever this is non-negative.

Therefore, if $J \supsetneq I(C)$,

$$\rho \bigcap_J F_i \leq |I|-|I(C)|-1-(k-1) \quad \text{whenever this is nonnegative}$$
$$= \rho C - 1.$$

Thus, C is the intersection of exactly $|I(C)|$ members of **F**.

Thus, $|V(C')| = p - p + (k-1) + \rho C = k + |C'| - 2$.

So C' is dependent in $\underline{M}(H)$.

Thus, $\underline{M} \cong \underline{M}(H)$, and so $\underline{M}$ is hypergraphic.

(B) Suppose now that $\underline{M} \cong \underline{M}(H)$, where $H = (V, E', \$)$ is a critical k-hypergraph, and the isomorphism is induced by the obvious bijection between E and E'. Let $V = \{A_1, A_2, \ldots, A_p\}_{\neq}$, $I = \{1,2,\ldots,p\}$. For each $i \in I$, define $F_i = \{e \in E: V(e') \subseteq V-\{A_i\}\}$, and let F_i' denote the image of F_i under the obvious bijection between E and E'.

Then each F_i' is a flat of $\underline{M}(H)$, since, for any $x' \notin F_i'$, $V(x') \nsubseteq V(F_i')$, and $\underline{M}(H)$ is loopless.

(i) $e \notin F_i$ if and only if $A_i \in V(e')$, which is true for exactly k values of i $(1 \leq i \leq p)$.

(ii) For any $J \subseteq I$, $|V(\bigcap_J F_i)| \leq p-|J|$; By (3.1),

$$\rho(\bigcap_J F_i) \leq |V(\bigcap_J F_i)|-(k-1) \quad \text{whenever this is non-negative, so}$$
$$\rho(\bigcap_J F_i) \leq p - |J| - (k-1) \quad \text{whenever this is non-negative}$$
$$= |I-J|-(k-1).$$

(iii)(a) Let C' be a circuit of $\underline{M}(H)$. Then, by (2.3),

$|V(C')| = k-1+\rho C'$.

By (4.29), $V(\sigma C') = V(C')$, so $|V(\sigma C')| = k-1+\rho C'$.

Thus, $C' \subseteq F_i'$ for exactly $p-(k-1+\rho C')$ values of i,

so $C \subseteq F_i$ for exactly $|I|-(k-1)-\rho C$ values of i. Let $I(C)$ denote this set of values. Then, from (ii),

$\rho C \leq \rho \bigcap_{I(C)} F_i \leq \rho C$ so equality holds, and, since σC is a flat of $\underline{M}$,

$\sigma C = \bigcap_{I(C)} F_i$.

(iii)(b) Let R' be a 1-flat of $\underline{M}(H)$. Then $|V(R')| = k$, since $\underline{M}(H)$ is loopless. Thus, $R' \subseteq F_i'$ for exactly p-k values of i $(1 \leq i \leq p)$. Therefore, $R \subseteq F_i$ for exactly p-k values of i. Let $I(R)$ denote this set of values. From (ii), if $J \subseteq I$ with $|J| \geq p-k+1$,

$\rho \bigcap_J F_i = 0$, so $R = \cap\{F_i : i \in I(R) \subseteq I\}$, where $|I(R)| = |I|-k$.

COROLLARY 13.3: A loopless matroid $\underline{M}$ of rank r on the set E is hypergraphic if and only if there exist an integer $k \geq 2$, and a family $\mathbf{F} = \{F_i : i \in I\}$ of flats of $\underline{M}$ such that:

(i) for each $e \in E$, $x \notin F_i$ for exactly k values of $i \in I$;

(ii) if $J \subseteq I$, $\rho(\bigcap_J F_i) \leq |I-J|-(k-1)$ whenever this is non-negative;

(iii)(a) if C is a circuit of $\underline{M}$, $\sigma C = \bigcap_{I(C)} F_i$, where $|I(C)| = |I|-(k-1)-\rho C$;

(b) if R is a 1-flat of $\underline{M}$, $R = \bigcap_{I(R)} F_i$, where $|I(R)| = |I|-k$.

Proof: (13.2).

Note that, in (13.1'), $\mathbf{F}$ is a family of hyperplanes, but that, in (13.2) it is a family of flats. This is because the construction of the proof of (13.2) for graphs always yields hyperplanes when $\underline{M}$ is connected. However, in hypergraphs, this is no longer the case, as can be seen from the following example.

Consider the hypergraphic matroid M(H) shown in Euclidean representation in Figure 27. For ease of explanation, we have labelled each point e with the vertex-set V(e). M(H) is connected, but the flat F_2 (with the notation of the proof of (13.2)(A)) has rank 2, and is, therefore, not a hyperplane, since M(H) has rank 4.

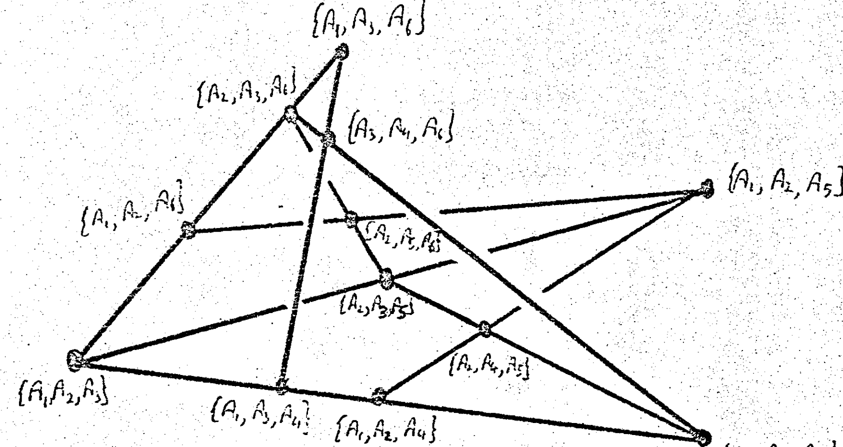


Figure 27

The strengthening of (13.1')(iii) to (13.2)(iii)(a)&(b) is necessary, as can be seen from the matroid of (3.5). Denoting this matroid by M, with the notation of (3.5), let the distinguished family **F** of flats of M be {{a,b,c}, {a,d,e}, {b,f}, {c,g}, {d,f}, {e,g}}. Then **F** satisfies the conditions (13.3)(i) and (13.3)(ii) and (13.3)(iii)(b), with k = 4. However, by (3.5), M is not hypergraphic. This is because (13.3)(iii)(a) is not satisfied for the circuit {a,f,g} of M.

The situation for generalised hypergraphic matroids is more complicated, because of the existence of μ and its associated closure operator. The example we have just given will illustrate the problem. The matroid of (3.5) is generalised hypergraphic, as was proved in (4.4). In fact, it can be obtained as M(K), where K = (V, E, \$, K) with V = {A,B,C,E,E,F}$_{\neq}$, E = {a,b,c,d,e,f,g,x}$_{\neq}$, K = {x}, and V(a) = {A,B,C}, V(b) = {A,B,D}, V(c) = {A,C,D}, V(d) = {B,C,E}, V(e) = {A,C,E}, V(f) = {A,B,F}, V(g) = {B,C,F} and V(x) = {D,E,F}. The sets E(V) ⊆ E−K with V(E(V)) ⊆ V−{V} for each V ∈ V are shown below in Table 1.

V	E(V)	V	E(V)
A	{d,g}	D	{a,d,e,f,g}
B	{c,e}	E	{a,b,c,f,g}
C	{b,f}	F	{a,b,c,d,e}

Table 1

E(D), E(E) and E(F) are not flats of M(K), but unions of flats; indeed, ρE(D) = rkM(K). Such a set of unions will, in general, occur for each contracted element of a hypergraphic matroid. The only possible analogue of (13.2) or (13.3) would therefore involve such unions of flats, which, we believe, is rather an unnatural approach.

It should be noted that the family **F** defined in (13.3) is far from unique, and, indeed, different choices of **F** can lead to different values of k. An easy example is provided by the rank-3 matroid M on the set $E = \{a,b,c,d,e\}_{\neq}$, with circuits {a,b,c}, {a,d,e} and all 4-subsets of E containing neither of these. Two possible choices of **F** are:

{{a,b,c}, {a,d,e}, {b,d}, {c,e}} yielding k = 2;

{{a,b,c}, {a,d,e}, {b},{d},{c},{e}} yielding k = 3.

Although the conditions of (13.2) and (13.3) can be modified to give a characterisation of strongly hypergraphic matroids, such a characterisation is of little value, because of the difficulty of obtaining the circuits of a contraction in a suitable form. However, we can modify the formulation of (13.2), to replace (iii) with a condition on independent sets.

THEOREM 13.4: A loopless matroid $\underline{M}$ of rank r on the set E is hypergraphic if and only if there exist an integer $k \geq 2$, and a family $\mathbf{F} = \{F_i : i \in I\}$ of flats of $\underline{M}$ such that

(i) for each $e \in E$, $e \notin F_i$ for exactly k values of $i \in I$;

(ii) if $J \subseteq I$, $\rho(\bigcap_J F_i) \leq |I-J| - (k-1)$ whenever this is nonnegative;

(iii) X is independent in $\underline{M}$ if and only if, for each nonempty subset Y of X, $Y \nsubseteq F_i$ for at least $|Y| + (k-1)$ values of $i \in I$.

Proof: (A) Suppose $\underline{M}$ satisfies (i)-(iii). Let $|I| = p$, and let $V = \{A_1, A_2, \ldots, A_p\}_{\neq}$ be a set of p vertices. Let $E = \{e_i : 1 \leq i \leq n\}_{\neq}$, and let $E' = \{e_i' : 1 \leq i \leq n\}$ be an isomorphic copy of E.

Let $H = (V, E', \$)$ be the hypergraph with

$V(e') = \{A_i : e \notin F_i, i \in I\}$, where e' is the image of e under the obvious bijection between E and E'. Then, by (i), $\underline{M}$ is k-uniform. Let C' be a circuit of $\underline{M}(H)$, and let C be its image under the obvious bijection between E and E'.

Since C' is a circuit of $\underline{M}(H)$, $|V(C')| = k+|C'|-2$. Thus, there exist $p-(k+|C'|-2)$ vertices of H not elements of $V(C')$, and so $C \subseteq F_i$ for at least $p-(k+|C'|-2)$ values of $i \in I$.

By (ii),

$$\begin{aligned}\rho(\cap(F_i : C \subseteq F_i)) &\leq |I| - |\{i \in I : C \subseteq F_i\}| - (k-1)\\ &\leq p - p + (k+|C'|-2) - (k-1)\\ &= |C'| - 1\\ &= |C| - 1\end{aligned}$$

Thus, C is dependent in $\underline{M}$.

Conversely, suppose that C is a circuit of $\underline{M}$; let C' be the image of C under the obvious bijection between E and E'. For any $c \in C$, $C-\{c\} \nsubseteq F_i$ for at least $|C-\{c\}|+k-1$ values of $i \in I$. Thus,

$C \nsubseteq F_i$ for at least $|C|+k-2$ values of $i \in I$.

Since C is a circuit, $C \nsubseteq F_i$ for at most $|C|+k-2$ values of $i \in I$, since otherwise, by (iii), C would be independent. Therefore, $C \nsubseteq F_i$ for exactly $|C|+k-2$ values of $i \in I$. Thus, $C \subseteq F_i$ for exactly $p-(k+|C|-2)$ values of $i \in I$, and therefore

$$|V(C')| \leq |\cap\{V(F_i): C \subseteq F_i, i \in I\}| \leq p-(p-(k+|C|-2))$$
$$= k + |C'| - 2.$$

thus, C' is dependent in $\underline{M}(H)$. Thus, $\underline{M} \cong \underline{M}(H)$, and so $\underline{M}$ is hypergraphic.

(B) Suppose now that $\underline{M} \cong \underline{M}(H)$, where $H = (V, E', \$)$, and the isomorphism is induced by the obvious bijection between E and E'. Define V and $\mathbf{F}$ as in (13.2). Then (i) and (ii) follow from (13.2)(i) and (ii).

(iii) Let X be independent in $\underline{M}$, let Y be a nonempty subset of X, and denote by X',Y' respectively, the images of X and Y under the obvious bijection between E and E'.

Then $|V(Y')| \geq k+|Y'|-1$. Thus, there exist at most $p-(k+|Y'|-1)$ flats $F_i \in \mathbf{F}$ for which $Y' \subseteq F_i'$. Thus, $Y \not\subseteq F_i$ for at least $|Y|+k-1$ values of $i \in I$.

Conversely, suppose C is a circuit of $\underline{M}$; let C' be the image of C under the obvious bijection between E and E'.

Then $|V(C')| = k+|C'|-2$, and so $A_i \notin V(C')$ for $p-(k+|C'|-2)$ values of $i \in I$, and no more.

$\therefore$ $C' \subseteq F_i'$ for at least $p-(k+|C'|-2)$ values of $i \in I$,

$\therefore$ $C' \not\subseteq F_i'$ for at most $|C'|+k-2$ values of $i \in I$,

$\therefore$ $C' \not\subseteq F_i'$ for less than $|C'|+k-1$ values of $i \in I$.

Thus, $C \not\subseteq F_i$ for less than $|C|+k-1$ values of $i \in I$.

Therefore, X is independent in $\underline{M}$ if and only if, for each nonempty subset Y of X, $Y \not\subseteq F_i$ for at least $|Y|+(k-1)$ values of $i \in I$.

This formulation leads us to a characterisation of strongly hypergraphic matroids.

THEOREM 13.5: A matroid of rank r on the set E is strongly hypergraphic if and only if, for each independent set $B \subseteq E$, there exist an integer $k_B \geq 2$, and a family $\mathbf{F}_B = \{F_i : i \in I_B\}$ of flats of $\underline{M}$, with $B \subseteq F_i$ for each $i \in I_B$, such that

(i) for each $e \in E - \sigma B$, $e \notin F_i$ for exactly k_B values of $i \in I_B$;

(ii) if $J \subseteq I_B$, $\rho(\bigcap_J F_i) - |B| \leq |I_B - J| - (k_B - 1)$, whenever this is non-negative;

(iii) $X \supseteq B$ is independent in $\underline{M}$ if and only if, for each $Y \subseteq X$, with $Y \supsetneq B$, $Y \nsubseteq F_i$ for at least $|Y| - |B| + (k_B - 1)$ values of $i \in I_B$.

Proof: (A) Suppose $\underline{M}$ is strongly hypergraphic. Then, for each independent set $B \subseteq E$, $\underline{M}.(E-B)$ is hypergraphic. Denote by L_B the set of loops of $\underline{M}.(E-B)$. Let $\underline{M}_B = \underline{M}.(E-B) \times ((E-B) - L_B)$. Then $\underline{M}_B$ is loopless and hypergraphic, and so, by (13.4), there exists $k_B \geq 2$, and a family $\mathbf{F}''_B = \{F''_i : i \in I_B\}$ of flats of $\underline{M}_B$ such that

(i)' if $e \in (E-B) - L_B$, $e \notin F''_i$ for exactly k_B values of $i \in I_B$;

(ii)' if $J \subseteq I_B$, $\rho_{\underline{M}_B}(\bigcap_J F''_i) \leq |I_B - J| - (k_B - 1)$, whenever this is non-negative;

(iii)' X'' is independent in $\underline{M}_B$ if and only if, for each nonempty subset Y'' of X'', $Y'' \nsubseteq F''_i$ for at least $|Y''| + (k_B - 1)$ values of $i \in I_B$.

Now, let $\mathbf{F} = \{F_i = F''_i \cup B \cup L_B : i \in I_B\}$. Then each F_i is a flat of $\underline{M}$. Furthermore, $\sigma B = B \cup L_B \subseteq F_i$ for each $i \in I_B$.

Write $X = X'' \cup B$, $Y = Y'' \cup B$. Then (i)' - (iii)' become:

(i) for each $e \in E - \sigma B$, $e \notin F_i$ for exactly k_B values of $i \in I_B$;

(ii) if $J \subseteq I_B$, $\rho(\bigcap_J F_i) - |B| \leq |I_B - J| - (k_B - 1)$, whenever this is non-negative;

(iii) $X \supseteq B$ is independent in $\underline{M}$ if and only if, for each $Y \subseteq X$, with $Y \supsetneq B$, $Y \nsubseteq F_i$ for at least $|Y| - |B| + (k_B - 1)$ values of $i \in I_B$.

(B) Conversely, suppose $\underline{M}$ is such that (i) - (iii) hold. Let $B \subseteq E$ be independent in $\underline{M}$. We shall show that $\underline{M}.(E-B)$ is hypergraphic.

Let L_B denote the set of loops of $\underline{M}.(E-B)$, and let $\underline{M}_B = \underline{M}.(E-B)\times((E-B)-L_B)$. Then $\underline{M}$ is loopless. We note that $\sigma B = B\cup L_B$. Let $\mathbf{F}''_i = \{F''_i = F_i-(B\cup L_B): i \in I_B\}$. Then each F''_i is a flat of $\underline{M}_B$. Furthermore,

from (i), for each $e \in (E-B)-L_B$, $e \notin F_i$ for exactly k_B values of $i \in I_B$ so $e \notin F''_i$ for exactly k_B values of $i \in I_B$;

from (ii), if $J \subseteq I_B$, $\rho_{\underline{M}_B}(\bigcap_J F''_i) = \rho(\bigcap_J F_i) - |B|$

$\leq |I_B-J|-(k_B-1)$, whenever this is non-negative;

$X\supseteq B$ is independent in $\underline{M}$ if and only if $X'' = X-B$ is independent in $\underline{M}_B$; therefore, from (iii), X'' is independent in $\underline{M}_B$ if and only if, for each nonempty subset $Y'' \subseteq X''$, $Y'' \not\subseteq F''_i$ for at least $|Y''\cup B|-|B|+(k_B-1)$ values of $i \in I_B$.

Thus, by (13.4), $\underline{M}_B$ is hypergraphic. Therefore, by (4.2), $\underline{M}.(E-B)$ is hypergraphic.

Now, if $\underline{M}$ is strongly hypergraphic, then, by (A), (i)-(iii) hold. Conversely, if $\underline{M}$ is such that (i)-(iii) hold, then $\underline{M}.(E-B)$ is hypergraphic for each independent subset $B \subseteq E$. Since every minor of $\underline{M}$ is isomorphic to $\underline{M}.(E-B)\times((E-B)-D)$ for some sets B, D, where $B \subseteq E$ is independent in $\underline{M}$, and $B\cap D = \phi$, every minor of $\underline{M}$ is hypergraphic. Thus, $\underline{M}$ is strongly hypergraphic.

(13.4) can be used to derive a characterisation of co-hypergraphic matroids, although the resulting expressions are rather less wieldy. Some preliminary definitions will be needed to simplify the notation.

DEFINITION 13.6: Let $\underline{M}$ be a matroid on the set E.

(i) A <u>fully dependent set</u> of $\underline{M}$ is a set G which is a (possibly empty) union of circuits of $\underline{M}$.

(ii) The <u>nullity</u> of a set $A \subseteq E$, denoted by δA is equal to $|A| - \rho A$.

THEOREM 13.7: A coloop-free matroid $\underline{M}$ of rank r on the set E is co-hypergraphic if and only if there exist an integer $k \geq 2$, and a family $\mathbf{G}=\{G_i : i \in I\}$ of fully dependent sets of $\underline{M}$ such that:

(i) for each $e \in E$, $e \notin G_i$ for exactly k values of $i \in I$;

(ii) if $J \subseteq I$, $\delta E - \delta(\underset{J}{\cup} G_i) \leq k+1-|J|$ whenever this is non-negative;

(iii) $X \subseteq E$ is spanning in $\underline{M}$ if and only if, for each $Y \supseteq X$ with $Y \neq E$, $Y \supseteq G_i$ for at most $k+1-|E-Y|$ values of $i \in I$.

Proof: $\underline{M}$ is coloop-free and co-hypergraphic if and only if $\underline{M}^*$ is loopless and hypergraphic.

By (13.4), the loopless matroid $\underline{M}^*$ is hypergraphic if and only if there exist an integer $k^* \geq 2$, and a family $\mathbf{F}^* = \{F_i^* : i \in I\}$ of flats of $\underline{M}^*$ such that

(i)* for each $e \in E$, $e \notin F_i^*$ for exactly k^* values of $i \in I$;

(ii)* if $J \subseteq I$, $\rho^*(\underset{J}{\cap} F_i^*) \leq |I-J|-(k^*-1)$, whenever this is non-negative;

(iii)* $X^* \subseteq E$ is independent in $\underline{M}^*$ if and only if, for each non-empty subset $Y^* \subseteq X^*$, $Y^* \not\subseteq F_i^*$ for at least $|Y^*|+(k^*-1)$ values of $i \in I$.

From (13.6), it is easy to see that G is fully dependent in $\underline{M}$ if and only if $E-G$ is a flat of $\underline{M}^*$.

Write $G_i = E-F_i^*$ $(i \in I)$, $k = |I|-k^*$ and $X = E-X^*$. Then

(i)* holds if and only if, for each $e \in E$, $e \notin G_i$ for exactly k values of $i \in I$;

(ii)* holds if and only if, for each $J \subseteq I$,

$\rho^*(\underset{J}{\cap}(E-G_i)) \leq |I|-|J|-(|I|-k-1)$ whenever this is non-negative,

i.e. if and only if $\rho^*(E-(\underset{J}{\cup}G_i)) \leq k-|J|+1$ whenever this is nonnegative,

i.e. if and only if $\rho(\underset{J}{\cup}G_i)+|E|-|\underset{J}{\cup}G_i|-\rho E \leq k-|J|+1$, whenever this is non-negative, i.e. if and only if $\delta E-\delta(\underset{J}{\cup}G_i) \leq k-|J|+1$ whenever this is non-negative.

(iii)* holds if and only if:

$(E-X^*)$ is spanning in $\underline{M}$ if and only if, for each $Y \subsetneq E$ with $Y \supseteq (E-X^*)$

$Y \not\supseteq G_i$ for at least $|E-Y|+|I|-k-1$ values of $i \in I$; i.e. if and only if X is spanning in $\underline{M}$ if and only if, for each $Y \subsetneq E$ with $Y \supseteq X$, $Y \supseteq G_i$ for at most $|I|-(|E-Y|+|I|-k-1) = (k+1)-|E-Y|$ values of $i \in I$. Thus, (i)-(iii) hold if and only if (i)*-(iii)* hold, i.e. if and only if the loopless matroid $\underline{M}^*$ is hypergraphic, i.e. if and only if the coloop-free matroid $\underline{M}$ is co-hypergraphic.

Note that, in (13.7), the value k is not the cardinality of the hypergraph H with $\underline{M}(H) \cong \underline{M}^*$, if this were derived by the method of (13.2). The cardinality of this hypergraph is k*; k has been used in (13.7) to bring the formulation into line with earlier theorems.

CHAPTER 14

CONCLUSION

It is probably helpful to tabulate some of the main results of the previous chapters, and to compare, where appropriate, the corresponding properties of graphic, hypergraphic and generalised hypergraphic matroids. This we do in Table 2, which also references the main result for each entry. In Table 3, we compare various graphical and hypergraphical concepts, and reference the hypergraphic definitions where appropriate.

It can be seen from the proofs of (8.5)-(8.11) and (12.7)-(12.13) that the value of k used to give a presentation of the one-point minors of the forbidden minors grows with the value of n. It may be the case that, for a fixed value of k, the set of matroids isomorphic to a minor of $\underline{M}(H)$, where H is a k-hypergraph, can be characterised by a finite set of forbidden minors. Clearly this is true if k = 2. For a value of k greater than 2, it is likely to be a difficult problem to prove whether such a finite set exists. For example, taking k = 3, we have the following necessary forbidden minors:

Fano, the dual of Fano, $(\underline{M}(K_5))^*$, $(\underline{M}(K_{3,3}))^*$ (because all proper minors of these are graphic);

Non-Fano, and the dual of Non-Fano (because all proper minors of these are isomorphic to $\underline{M}(H)$ for some 3-hypergraph H);

The matroid of (8.1) (for the same reason).

I conjecture that $\underline{M}(K_6^4)$ is not isomorphic to a minor of $\underline{M}(H)$ where H is a 3-hypergraph, but can see no way of proving this. It is clear that, for other than very small values of k, the forbidden-minor classes are going to be very large, even if they are finite.

It is well-known that the lattice of $\underline{M}(K_n)$ is isomorphic to the lattice of partitions of a set of n elements. There is a partial analogue

MATROID CONCEPT	TYPE OF MATROID		
	GRAPHIC	HYPERGRAPHIC	GENERALISED HYPERGRAPHIC
Restrictions are	Graphic	Hypergraphic (2.12)	Generalised Hypergraphic (4.5)
Contractions are	Graphic	Generalised Hypergraphic (4.5)	Generalised Hypergraphic (4.5)
Representability	Over every field	Over every characteristic (7.4)	Over every characteristic (7.4)
Special Subclasses	Series-Parallel Networks Binary Base-Orderable Matroids	Series-Parallel Networks (11.16) Binary Base-Orderable Matroids (11.35) GSP Networks (11.16) Ternary Base-Orderable Matroids (11.34) Transversal Matroids (5.1)	Series-Parallel Networks (11.16) Binary Base-Orderable Matroids (11.35) GSP Networks (11.16) Ternary Base-Orderable Matroids (11.34) Transversal Matroids (5.1) Gammoids (5.2)
Characterisation	5 forbidden minors Special family of hyper-planes (13.1)	No finite set of forbidden minors (12.14) Special family of flats (13.2)	No finite set of forbidden minors (8.13)
Colourings	Valuation of the Tutte polynomial and the method of intersecting hyper-planes give vertex colourings	Valuation of the Tutte poly-nomial gives weak colouring of the node-hypergraph if pseudo-graphic (10.11) Method of intersecting hyper-planes gives strong colouring of the node-hypergraph (10.23	

TABLE 2

GRAPHIC CONCEPT	HYPERGRAPHIC CONCEPT	
Tree	Critical Set	(2.2)
Forest	Independent Set	(2.1)
Cycle	Circuit	(2.3)
Component	Component	(2.8)
Cutset	Cutset	(3.20)
Series-Parallel Extension	Generalised Series-Parallel Extension	(11.15)
Vertex	{ Vertex Node	(2.15)
Edge	Edge	

TABLE 3

of this in hypergraph theory, as has recently been pointed out by Matthews [19]. The lattice of $\underline{M}(K_p^k)$ is isomorphic to the lattice of (k-1)-paving matroids which satisfy the conditions of (10.23).

It must be admitted that matroids are not the whole answer to the problem of finding a satisfactory abstraction of hypergraph theory. Nevertheless, as we have shown in the previous chapters, the application of matroid theory does give information about the structure of hypergraphs, and, in particular, demonstrates the double role of vertex and node played by a vertex in a graph. Other authors have used other techniques for investigating hypergraph structure; none is complete in itself, but most give insights into the structure of hypergraphs not given by other techniques. In this context, we must mention particularly the authors whose results have been used or referred to by us in previous chapters - Berge [1 & 2] and Helgason [14]. However, as we have seen, the theory of hypergraphic matroids does throw light on hypergraph structure in a way not achieved by either of these authors.

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INDEX OF SYMBOLS

See also:

Pages 4-8 for matroid symbols;

Pages 15-16 for hypergraph symbols;

Pages 14-15 for use of particular type-faces.

Symbol	Description	Reference
$\beta(\underline{M})$	β-invariant of $\underline{M}$.	(11.22)
H	Hypergraph	(1.1)
H_G	Subhypergraph of H induced by G.	p15
$H\|V'$	Restriction of H to V'.	p16
(H, K)	Generalised hypergraph.	(4.14)
κ	Connectivity	pp125-127
K	Generalised hypergraph.	(4.14)
K_p^k	Complete k-hypergraph on p vertices.	p15
μ	μ-function of generalised hypergraph.	(4.9)
$\underline{M}^{d,k}$	Level-k Dilworth truncation of $\underline{M}$.	p9
$\underline{M}(H)$	Matroid of the hypergraph H.	(2.11)
$\underline{M}(K)$	Matroid of the generalised hypergraph K.	(4.15)
ν	ν-function of generalised hypergraph.	(4.6)
N(H)	Node-hypergraph of hypergraph H.	(2.16)
N(K)	Node-hypergraph of generalised hypergraph K.	(9.26)
$T(\underline{M}; x,y)$	Tutte polynomial of matroid $\underline{M}$.	p133
$(V, E, \$)$	Hypergraph.	(1.1)
$(V, E, \$, K)$	Generalised hypergraph.	(4.14)
$\underline{W}_n$	Wheel of order n (the matroid of the wheel on n+1 vertices).	
W_n	Whirl of order n.	p12
$< >$	μ-closure operator.	(4.24)
$\dot{\cup}$	Pointed union.	(11.1)
$\dot{\cup}'$	Deleted pointed union.	(11.1)

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