Subgroups defined by generating pairs of groups described by presentations

Thesis

SUBGROUPS DEFINED BY GENERATING PAIRS OF GROUPS DESCRIBED BY PRESENTATIONS.

by

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CONTENTS

STATEMENT

In Chapter I the concept of factorizations (which is central to the whole thesis) is original. The rest of Chapter I is expository.

Chapter II is based on my own ideas. Dr. Pride had obtained a description of the subgroups generated by pairs of elements of finite order in small cancellation groups. I was originally interested in describing the two generator subgroups of torsion-free small cancellation groups. The methods I used were then improved to obtain results for small cancellation groups with torsion. The methods were most successful in dealing with subgroups which cannot be generated by a pair (u,v) with one or both u,v of finite order. Dr. Vella and Dr. Pride subsequently refined the techniques to deal adequately with the case that u or v has finite order. Our results are combined in a joint paper [14] In the original proof of the main result of Chapter II the length of a pair of elements was assumed to be minimal. Dr. Vella suggested that we assumed that the length of a factorization was minimal. This has both improved and greatly simplified the work in Chapter II (and Chapter III). Development and use of Dr. Vella's idea are my own. The idea of extending the techniques to Theorem 2.4 is my own.

Chapter III is my own work except where results are quoted and these are acknowledged in the text. Some of the results of Chapter III will appear in [13]. Further results will appear in a joint paper "Commutators, generators and conjugacy equations in groups", with S.J. Pride.

ACKNOWLEDGEMENTS

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ABSTRACT

Two main topics are studied in this thesis.

The first concerns the 2-generator subgroups of small cancellation groups. If G is a finitely presented group satisfying certain small cancellation conditions, we show how to compute a finite set S of pairs of elements of G so that the 2-generator subgroups generated by pairs in S include all the isomorphism types of 2-generator subgroups of G. We describe an algorithm for which given a pair (w,z) of elements of G, finds a pair which belongs to S and generates a subgroup isomorphic to the subgroup generated by (w,z).

The second topic is motivated by a well-known property of the free group F_2 of rank 2. Nielsen has shown that F_2 has the property (called here Property A) that there exists an element c of F_2 such that $u^{-1}v^{-1}uv$ is conjugate to c or c^{-1} if and only if (u,v) is a generating pair. We consider this and related properties for small cancellation groups and one relator groups with torsion. In particular, we show for a 2-generator group $G = \langle a,b;\ R\rangle$ where R satisfies the "right" small cancellation conditions that $u^{-1}v^{-1}uv$ is conjugate to $(a^{-1}b^{-1}ab)^{\pm 1}$ only if u,v is Nielsen equivalent to a,b. Pride has shown that a 2-generator one-relator group, where the relator r is a proper power of form S^n, $n > 3$, has Property A. We provide examples of 2-generator one-relator groups where the relator is a square which do not have Property A.

In addition to these main topics we show that if F_n is the free group of rank n, and G is an m-generator one-relator group, $n \geqslant m \geqslant 3$, then F_n is residually G. This result has a slight connection (which is explained in the thesis) to the second topic.

NOTATIONS AND DEFINITIONS

We adopt the usual notation in set theory:-

$R \cup S$ is the union of sets R, S

$R \cap S$ is the intersection of sets R, S

$R \subset S$ or $S \supset R$ means that R is a subset of S

$r \in R$ means that r is a member of the set R

$|R|$ denotes the cardinal of a set R

$\{x_1, \ldots, x_n\}$ denotes the unordered n-tuple

$(x_1, \ldots, x_n)$ denotes the ordered n-tuple.

When referring to properties of the natural numbers we use the standard notation. Let n, m be integers, then

$n | m$ means n divides m

(n, m) is the HCF of n and m.

$\phi(m)$ denotes the Euler function, $\phi(m) = 1$ if $m = 1$,
$\phi(m)$ = the number of distinct values of k, where
$0 < k < m$, and $(k, m) = 1$, if $m > 1$.

$\varepsilon, \varepsilon'$ (and variations of these) denote the integers +1 or −1.

$\mathbb{Z}$ denotes the integers

$\mathbb{Z}^+$ denotes the positive integers.

If ϕ is a map then if ϕ is an element of a group we write ϕ on the right of the element it acts on. Other functions (e.g. ϕ in "$\phi(m)$") will be written on the left.

The notation used to denote terms in group theory is similar to that of [27].

1 represents the trivial group

$\mathbb{Z}_n$ denotes the cyclic group of order n

$<X>$ is the free group with basis X

Let Y be a set of words in $F = <X>$. Then

$<Y>^F$ denotes the normal closure of Y in F.

sgp$\{Y\}$ is the subgroup of F generated by Y.

$<X;Y>$ is either the presentation with generators $x \in X$, relators $y \in Y$, or the group defined by such a presentation.

Let G,H be groups. Then

$\mathrm{sgp}_G\{g\}$ denotes the subgroups of G generated by g, a subset of G.

$G * H$ is the free product of G and H,

$G \underset{g=h}{*} H$ is the free product of G and H amalgamating $\mathrm{sgp}_G\{g\}$ with $\mathrm{sgp}_H\{h\}$ under the isomorphism $g \mapsto h$,

Aut(G) is the group of automorphisms of G.

F is a free group

F_n is the free group of rank n.

Let a,b be elements of F. Then

a^b denotes $b^{-1}ab$,

$[a,b]$ denotes $a^{-1}b^{-1}ab$

If G is defined by means of a presentation $<X; R>$, then a primitive of G is an element which forms part of a generating $|X|$-tuple of G.

Let X be an n-tuple, and $W(X)$ the set of words in $X \cup X^{-1}$. Let 1 denote the empty word. Then we do not distinguish between the elements of $W(X)$ and the elements of $F = <X>$ they represent. We use the following notation:-

$\equiv$ denotes equality in $W(X)$

$=$ denotes equality in F

$=_G$ or $=_N$ denotes equality in $G = \langle X; R\rangle$, where N is the normal closure of R in F.

$\sim$ denotes conjugacy in F

$\sim_G$ or $\sim_N$ denotes conjugacy in G

If a,b are words in $W(X)$ then we say that a is a subword of b if $b \equiv waz$ for words $w,z \in W(X)$. We say that $w \in W(X)$ is reduced if it contains no subwords xx^{-1} or $x^{-1}x$.

We say that $w \in W(X)$ is cyclically reduced if all cyclic permutations of w are reduced.

We say that $f \in F$ is a proper power if for some $e \in F$ $f = e^n$, where n is an integer > 1.

The following notations and definitions are introduced in the text. The number in brackets refers to the pages where the notations are introduced.

L, L_1	Length functions	(4)
$C(p)$, $C'(\lambda)$, $C'_L(\lambda)$, $T(q)$	Small cancellation conditions	(5)
$N_0, N_1, N_2, N^k, N_G, \bar{N}_G$	Generalised elementary transformations	(7)
$(f,g,h)^{I}, (f,g,h)^{II}$	Factorizations	(8)
S_0^I, ${}^kS^{Il}$, S_G^I, $\bar{S}_G^I$	Factorization transformations	(45)
S^{II}, ${}^kS^{IIl}$, S_G^{II}, $\bar{S}_G^{II}$	Factorization transformations	(46)
$[g_1,\ldots,g_l]$	Higher commutator	(15)
	Subconjugate	(11)
	root-closed	(34)
	Property A	(16)
	Properties B_N, B_T	(16)
	Properties C_N, C_T	(16)

CHAPTER I

INTRODUCTION

SECTION 1 THE PROBLEMS

Let G be a group given in terms of generators and defining relators. This thesis will be mainly concerned with two questions.

I. What can be said about the isomorphism types of two generator groups embeddable in G?

II. Let G be generated by two elements. Then does there exist an element c in G so that the elements (u,v) are a generating pair of G if and only if $u^{-1}v^{-1}uv$ is conjugate to c or c^{-1} ?

Problems I and II are related, for they both involve trying to obtain information about the subgroups of G generated by pairs of words in G.

In relation to I, it follows from the Nielsen-Schreier Theorem [30 p.95] that if G is free, then G has at most three isomorphism types of two-generator subgroups, namely, the free groups of rank 0, 1 and 2. If G is a one-relator group with torsion, then the two-generator subgroups of G have been described in Pride [45]. In this thesis we investigate Problem I in the case that G is a small cancellation group. We show that if G has a finite presentation satisfying a "suitable" small cancellation condition, then G has finitely many isomorphism types of two-generator subgroups. We will give precise statements of our results in Section 2 below.

A general survey of what appears to be known concerning subgroups of small cancellation groups can be found in [46].

When considering II, we use the following definition: A two-generator group has *Property* A if G possesses an element c, so that (u,v) is a generating pair of G if and only if $u^{-1}v^{-1}uv$ is conjugate to c or c^{-1}.
It has been shown by Nielsen [34] that F_2, the free group of rank 2 has Property A. Other groups have also been shown to have Property A:-

(1) The Fuchsian group $\langle a,b,c,d;\ a^2,b^2,c^2,d^q,abcd\rangle$, where $q>1$, $(2,q) = 1$. Then this group can be generated by the two elements ab, ac, [35], [50] and has Property A. (See Kalia and Rosenberger [21]).
(2) $\langle a,b;\ [a,b]^n\rangle$, $n > 1$. Then Rosenberger, [53], has shown that this group has Property A.
(3) $\langle a,b,t;\ R_1(a,b),\ R_2(a,b),\ \ldots, t^{-1}atb\rangle$. Pride (unpublished) has shown that under certain conditions, this group has Property A.
(4) $\langle a,b;\ R^n\rangle$, $(n \geqslant 4)$, R not a power of a primitive. Pride has shown that this group has Property A. (see [14]).

Dicks [6] has verified an analogous property to A for the free algebras of rank 2 over a field. He has shown that if k is a field, then u and v generate $k\langle x, y\rangle$ (as a k-algebra) if and only if $uv-vu$ is a non-zero scalar multiple of $xy-yx$.

In Chapter III of this thesis we study problem II for small cancellation groups. In addition we give examples of groups $<a,b;\ R^2>$ (where R is not a power of a primitive) which do not have Property A, (this should be viewed in the light of (4) above). A more detailed account of the contents of this chapter, and statements of our results are given in Section 2 below.

SECTION 2 SURVEY OF THESIS

We begin with some definitions.

Let X be an alphabet, and let $W(X)$ denote the set of words on X, that is, the set of expressions.

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_m^{\varepsilon_m}, \ m \geqslant 0, \ \varepsilon_i = \pm 1, \ x_i \in X, (i = 1, 2, \ldots, m).$$

Equality in $W(X)$ will be denoted by $\equiv$. An element of $W(X)$ will be called *reduced* if it does not contain an inverse pair xx^{-1}, $x^{-1}x (x \in X)$, and it will be called *cyclically reduced* if all its cyclic permutations are reduced.

A length function on $W(X)$ is a function L: $W(X) \mapsto \mathbb{Z}^+$ satisfying $L(UV) = L(U) + L(V)$, $L(U) = L(U^{-1})$, for all $U, V \in W(X)$. Therefore a length function on $W(X)$ is completely specified by its effect on X. If $j_x : x \in X$ is a set of non-negative integers, we define a length function L, where $L(x) = j_x$, and $L(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_m^{\varepsilon_m}) = \sum_{i=1}^{m} j_{x_i}$. The particular length function obtained by defining $j_x = 1$ for each $x \in X$ will be denoted by L_1.

If F is the free group on X, and w is a word in $W(X)$, then we do not distinguish between w and the element of F that it represents. Equality in F will be denoted by $=$. Conjugacy in F will be denoted by $\sim$.

Let R be a subset of $W(X)$, and N the normal closure of R in F, then equality (mod N) is denoted by $=_N$ or $=_G$, where $G = \langle X;R \rangle$. Similarly conjugacy (mod N) is denoted by $\sim_N$ or $\sim_G$. If $w_1,\ldots,w_l$ is a subset of $W(X)$, then we write $\mathrm{sgp}_G(w_1,\ldots,w_l)$ for the subgroup of G generated by $(w_1N,\ldots,w_lN)$, and we use the expression $(w_1,\ldots,w_l)$ *generates* G when we mean that $(w_1N,\ldots,w_lN)$ generates G.

The *symmetrized closure* of R is the smallest subset of $W(X)$ which contains R and is closed under the taking of inverses and cyclic permutations. If R is the symmetrized closure of itself, then we say that R is *symmetrized*. If ut_1, ut_2 are distinct elements of R, then u is a piece (relative to R).

For R a symmetrized set; p,q positive integers; λ, a positive real number; and L, a length function we define the small cancellation hypothesis as follows:-

C(p): No element of R is the product of fewer than p pieces

T(q): Let $3 \leqslant h < q$. If $r_1,\ldots,r_h \in R$, where $r_1 \neq r_2^{-1}$, $r_2 \neq r_3^{-1},\ldots,r_h \neq r_1^{-1}$, then at least one of the products r_1r_2, $r_2r_3,\ldots,r_hr_1$ is reduced without cancellation.

$C_L'(\lambda)$: If $ut \in R$, and u is a piece, then $L(u) < \lambda L(ut)$.

In common with standard practise we write $C'(\lambda)$ instead of $C'_{L_1}(\lambda)$. If R satisfies $C(p)$ and $G = \langle X;R \rangle$ then we often call G a $C(p)$ *group* (With similar abuses of terminology for other conditions). If R satisfies $C(p)$, $T(q)$ where $1/p + 1/q = 1/2$ (that is $(p,q) = (6,3)$, $(4,4)$, or $(3,6)$, then R is called a *small cancellation set*, and $G = \langle X;R \rangle$ is called a *small cancellation group*.

A word t is called a *p-remnant* (with respect to R) if $tu_1 \ldots u_\rho \in R$, and $u_1, \ldots, u_\rho$ are pieces.

Let $U = (u_1, \ldots, u_n)$ be an n-tuple of words in $W(X)$.
We define two types of transformations on U:

1. replace some u_i by u_i^{-1}
2. replace some u_i by $u_i u_j$ where $j \neq i$ $(1 \leqslant i,j \leqslant n)$.

In both cases it is understood that u_h for $h \neq i$ remains unchanged.

Any finite product of these transformations we call a *Nielsen transformation*. Let $V = (v_1, \ldots, v_n)$ be another n-tuple in $W(X)$. Suppose V is obtained from U by a Nielsen transformation. Then we say U is *Nielsen equivalent to* V, and say that U,V lie in the same *Nielsen equivalence* (NE) *class of n-tuples*. If U is a generating set for a group G, then V is also a generating set for G. (See [30,p.121]).

By [30,p.131] any generating n-tuple of $F_n = \langle x_1,\ldots,x_n \rangle$ is obtainable from $(x_1,\ldots,x_n)$ by a Nielsen transformation. Thus F_n has 1 NE class of generating n-tuples.

Suppose U,V are generating n-tuples of a group G, then we say that they lie in the same T-*system* if $(u_1,\ldots,u_n)$ is Nielsen equivalent to $(v_1\theta,\ldots,v_n\theta)$, for some $\theta \in \mathrm{Aut}\,(G)$.

The main work of the thesis begins in Chapter II. This Chapter is composed of three sections. In the first section we state the main theorems concerning the two-generator subgroups of small cancellation groups, and derive some consequences of these results. We prove the theorems in the second and third sections. The method of proof is described in Section II.2, but a complete list of cases can be found in Section II.3.

Before stating the results of Chapter II, we need to make some further definitions.

Let (u,v) be an ordered pair of words in $W(X)$. Define *elementary transformations* (mod N) as follows.

N_0: $\quad (u,v) \longmapsto (v,u)$

N_1: $\quad (u,v) \longmapsto (u^{-1},v)$

N_2: $\quad (u,v) \longmapsto (uv,v)$

N^k: $\quad (u,v) \longmapsto (k^{-1}uk,k^{-1}vk),\ k \in W(X)$

N_G: $\quad (u,v) \longmapsto (\bar{u},\bar{v}),\ u =_G \bar{u},\ v =_G \bar{v}$

Now if $(u',v') \in \{(v,u), (u^{-1},v), (uv,v), (k^{-1}uk,k^{-1}vk), (\bar{u},\bar{v})\}$ then (u,v) and (u',v') generate conjugate subgroups of G. In particular (u,v), (u',v') belong to the same NE class of generating pairs of G if and only if (u',v') can be obtained from (u,v) by a finite sequence of elementary transformations. (See [30 p.121]). The transformations N_0, N_1, N_2, N^k, N_F we call *free elementary transformations.*

In addition to these elementary transformations, we need a further transformation.

$\bar{N}_G$: $(u,v) \longmapsto (u_1,v)$, where $u \equiv x^{\alpha}$, $u_1 \equiv x^{\alpha_1}$,
$x^{\gamma} \in R$, $\gamma > 1$ and $(\alpha,\gamma) = (\alpha_1,\gamma)$.

We call $\{\bar{N}_G \cup$ elementary transformations$\}$, the *generalised elementary* (GE) *transformations* (mod N). As $\mathrm{sgp}_G\{u,v\}$ and $\mathrm{sgp}_G\{u_1,v\}$ are both generated by the pair $(x^{(\alpha,\gamma)},v)$ these subgroups are equal. Therefore if (u',v') is obtained from (u,v) by a sequence of generalised elementary transformations (mod N), $\mathrm{sgp}_G\{u',v'\}$ is conjugate to $\mathrm{sgp}_G\{u,v\}$.

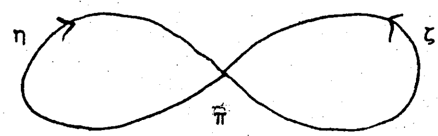
A triple (f,g,h) of elements of $W(X)$ is called a factorization of (u,v) if either

(i) $u \equiv f^{-1}g$, $v \equiv f^{-1}h$, or

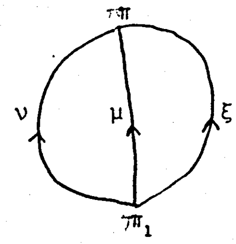
(ii) $u \equiv f^{-1}gf$, $v \equiv h$.

If (i) holds, then we say that the factorization is of Type I while if (ii) holds, then we say that the factorization is of Type II. If in either case, $f \equiv 1$, then the factorization is said to be trivial.

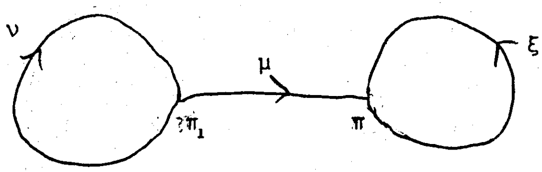
We can describe these two types of factorization geometrically. Any word in (u,v) can be considered as the label of a closed path in the 1-complex K consisting of two closed paths η ζ joined at a single vertex π:



where the label on η is u, ζ is v. If $u \equiv f^{-1}g$, $v \equiv f^{-1}h$, then we can identify the initial paths of η, ζ labelled f^{-1} so that we get K_I:



where the labels on μ, ν, ξ are f, g, h respectively. If $u \equiv f^{-1}gf$, $v \equiv h$, then we can identify the initial and terminal path of η, labelled by $f^{-1}f$ respectively, so that we have K_{II}:



Thus if (f,g,h) is a factorization of Type J, (J = I or II) and if W is a word in (u,v), then W is the label of a closed path ω in K_J whose initial and terminal vertex is π.

Let $W_i(f,g,h)$ be obtained from $W_{i-1}(f,g,h)$ by cancelling adjacent letters f, f^{-1} or f^{-1}, f in W_{i-1} $(0 < i < N)$. Let $W_0 \equiv W(f^{-1}g, f^{-1}h)$ or $W(f^{-1}gf, h)$ (depending on the Type factorization J). Then there is a sequence

$\{W \equiv W_0, W_1, \ldots, W_n \equiv W^*\}$

where W^* is reduced in the letters f, g, h. Now each W_i is the label of a closed path ω_i in K_J whose initial (and terminal) vertex is π. We show this by induction on i, for it is already noted that $W \equiv W_0$ is the label of a closed path in K_J . Consider $W_i \equiv UV = W_{i-1} \equiv Uff^{-1}V$ or $Uf^{-1}fV$. Then the subword ff^{-1} or $f^{-1}f$is the label of a closed path which is a subpath of ω_{i-1}. Deleting this subpath we obtain ω_i whose label is W_i and whose initial and terminal vertex are the same as ω_{i-1}. Thus if ω_{i-1} begins and ends at π, then ω_i also begins and ends at π.

We call the labels of the subpaths μ, ν, ξ of ω^* (the path whose label is W^*), *f-,g- and h-subwords* of W^* respectively. The f-, g- and h-subwords of W^* are called *F-subwords* of W^*. We call the label of any subpath of ω^*, starting at π or π_1 and ending at π or π_1, an *F*-subword* of W^*. Such a label is a word in the elements (f, g, h).

Suppose g, h are non-trivial, and

(a) (f, g, h) is a non-trivial factorization of Type I, and $g^{-1}fh^{-1}gf^{-1}h$ is cyclically reduced, or

(b) (f, g, h) is a non-trivial factorization of Type II, and g, h cyclically reduced, $f^{-1}gf, fhf^{-1}$ reduced, or

(c) (f, g, h) is a trivial factorization, and $g^{-1}h^{-1}gh$, g, h cyclically reduced.

Then we say that the factorization is *reduced*.

By a *subconjugate* of a set Y of words in $W(X)$, we mean a cyclic permutation of a subword of an element of Y.

The main results proved in Chapter II are:-

(THEOREM 2.3) *Let R satisfy $C_L'(1/14)$ or $C_L'(1/10), T(4)$ for some L, and let H be a two-generator subgroup of G. Then either*

(i) H is a free product of cycles, or

(ii) if (w,z) generates H, then (w,z) can be transformed by a finite sequence of GE transformations to (u,v) where $sgp_G\{u,v\} \sim sgp_G\{w,z\}$, and there exists a reduced factorization (f,g,h) of (u,v), so that either, for certain integers $\varepsilon_1,\varepsilon_2,\varepsilon_3$ of modulus 1, the elements of the set $\{f^{\varepsilon_1},g^{\varepsilon_2},h^{\varepsilon_3}\}$ are disjoint subwords of a subconjugate of R, or (f,g,h) is a non-trivial factorization of Type I and $sgp\{u,v\} = \langle u,v;\ u^l,v^m,(u^{-1}v)^n\rangle$ where $l,m,n \neq 0$.

(THEOREM 2.4) *Let R satisfy $C_L'(1/16)$, or $C_L'(1/12)$, $T(4)$ for some L and let H be a two-generator subgroup of G. Then either*

(i) H is a free product of cycles, or

(ii) if (w,z) generates H, then (w,z) can be transformed by a finite sequence of GE transformations to (u,v), where $sgp_G\{u,v\} \sim sgp_G\{w,z\}$, and the elements of a reduced factorization of (u,v) are pieces.

If R is finite and satisfies $C_L'(1/14)$, or $C_L'(1/10)$, $T(4)$ then, by Theorem 2.3, there are only a finite number of triples (f,g,h) whose elements or their inverses are either $\equiv 1$ or subwords of a subconjugate of R. Hence there are only finitely many conjugacy classes of two-generator subgroups of G which are not free products of cycles. However, as there is a bound on the orders of elements of finite order [27 p.281], the number of isomorphism types of two-generator subgroups which are free products of cycles is finite. Therefore we have the following result.

(THEOREM 2.5) *Let* $G = \langle X;R\rangle$, *where* R *is finite and satisfies* $C_L'(1/14)$ *or* $C_L'(1/10)$, $T(4)$. *Then*

(i) G has finitely many conjugacy classes of two-generator subgroups whose members are not free products of cycles.

(ii) G has finitely many isomorphism types of two-generator subgroups.

Let R be finite and satisfy $C_L'(1/14)$ or $C_L'(1/10)$, $T(4)$ (resp $C_L'(1/16)$ or $C_L'(1/12)$, $T(4)$ Let $\{(u_1,v_1),\ldots,(u_k,v_k)\}$ be the smallest set of pairs of elements which contains $(f^{-1}g,f^{-1}h)$, $(f^{-1}gf,h)$ and (g,h), where f,g,h range over every subword of a subconjugate of R, (resp. where f,g,h range over every piece relative to R). Then by Theorem 2.3, (resp 2.4) $\{(u_1,v_1),\ldots,(u_k,v_k)\}$ includes a set of generating pairs for the representatives of the conjugacy classes of two-generator subgroups which are not free products of cycles. (Note that the number of pairs, k, depends on the presentation). In the proof of Theorem 2.3, (resp. 2.4) we will describe an algorithm which takes any pair (w,z) and obtain from it a pair (u,v). If (u,v) is not one of the above list then the $\mathrm{sgp}_G(u,v)$ $(\sim \mathrm{sgp}_G(w,z))$ is a free product of cycles.

Suppose $|X| = 2$ and no element of R is a power of a primitive in G. Then a sequence of GE transformations on a generating pair of G is a sequence of elementary transformations. If, in addition, R satisfies the small cancellation hypothesis $C_L'(1/14)$ or $C_L'(1/10)$, $T(4)$, then by Theorem 2.3, any generating pair (w,z) of G is Nielsen equivalent to (u,v), where (u,v) has a reduced factorization whose elements (or their inverses) are subwords of a subconjugate of R. If R is finite, there are only finitely many such pairs, so that we have established the following:

(THEOREM 2.6) *Let $G = \langle a,b;\ R\rangle$ where*

(i) R is finite

(ii) R satisfies $C_L'(1/14)$ or $C_L'(1/10)$, $T(4)$, and

(iii) no element of R is a power of a primitive in G.

Then G has finitely many NE classes of generating pairs.

Using a variation of the method of proof of the main theorems Pride has shown that condition (iii) is not necessary. (See [14]). However the condition (i) that R is finite cannot be lifted. This is illustrated by the following:

Let k be an integer $\geqslant 16$, and R be the symmetrized closure of $\{r_i, i = 1,2,\ldots\}$, where

$$r_i = ab^i a^2 b^i \cdots a^k b^i b, \quad i = 1,2,\ldots$$

and let $G = \langle a,b;\ R\rangle$. Then R satisfies $C'(4/k)$, $T(4)$ and G has an infinite number of NE classes of generating pairs represented by $\{a, b^i\ ;\ i = 1,2\ldots\}$

We have conjectured that if $G = \langle a,b;\ R\rangle$ where each element of R is a proper power, but no element of R is a power of a primitive, and satisfies $C_L'(\lambda)$ for suitably small λ, G will have one NE class. This conjecture is false if the condition that no element of R is a power of a primitive is removed, as the following example shows.

Let k be an odd integer, $k > 18$, and for $i = 1,2,\ldots,n$

$$r_i = (ab^i)^k$$

Let R be the symmetrized closure of $\{r_i;\ i = 1,2,\ldots,n\}$ (n may be ∞), and $G = \langle a,b;\ R\rangle$. Then R satisfies $C'(2/k)$, $T(4)$, and has n NE classes of generating pairs represented by $\{((ab^i)^2, b);\ i = 1,2,\ldots,n.\}$

The question as to whether a group has a finite number of NE classes arises when considering the nature of Aut(G), (see [48], [49]). Let F_2, be the free group of rank 2, R a symmetrized subset of F_2, N the normal closure of R, and $G = F_2/N$. Let $\Pi(N)$ be the group of automorphisms of F_2 such that $N\phi = N$. Each element ϕ of $\Pi(N)$ induces an automorphism $\hat{\phi}$ of G where $wN\hat{\phi} = w\phi N$, ($w \in F_2$)

By Pride [48] if G has l NE classes of generating pairs, then

$$|\text{Aut } G:\ \widehat{\Pi(N)}| \leqslant l.$$

Therefore if G is a two-generator, finitely related, small cancellation group satisfying $C_L'(1/14)$ or $C_L'(1/10)$, $T(4)$, then it follows from Theorem 2.6 and the remark that follows, that Aut(G) is finitely generated (resp, presented) if and only if $\hat{\Pi}(N)$ is finitely generated (resp. presented)

We finish Section 1 of Chapter II with an application of Theorem 2.4. We construct *a two-generator, one-relator group which is not free, but has every proper two-generator subgroup free.*

Chapter III has five sections. The first is mainly expository and we state our results, which are proved in the remaining four sections.

Let G be a group, generated by X, and let $(g_1, \ldots, g_l)$ be a set of elements of G. We say that an element of G is a *higher commutator on* $g_1, \ldots, g_l$ if it is an element of the set $\{[g_1, \ldots, g_l]\}$, where $\{[g_1, \ldots, g_l]\}$ is defined inductively as follows:

$[g_i] = g_i$, and $[g_i, g_j] = g_i^{-1} g_j^{-1} g_i g_j$, $(i, j = 1, 2, \ldots, l)$

If $l > 1$ then $\{[g_1, \ldots, g_l]\} = \{h; h = [h_1, h_2],$

$h_1 \in \{[g_1, \ldots, g_m]\},\ h_2 \in \{[g_{m+1}, \ldots, g_l]\}, l > m > 0\}$

Let Y be a set of $|X|$ elements of G. We investigate those sets Y which satisfy one or more of the following conditions.

a) Y belongs to the same NE class, or T-system of G as X.

b) Y generates G

c) A higher commutator on Y is conjugate to a higher commutator on X or the inverse of such a commutator.

Our main interest is in the case $|X| = 2$. In this case we will write $X = (a,b)$, $Y = (u,v)$. Until further notice we assume that G is a two-generator group. Then a, b, c become:

a) (u,v) and (a,b) belong to the same NE class, or T-system of G.

b) (u,v) generates G.

c) $[u,v] \sim [a,b]^{\pm 1}$.

If G is free ($G = F = \langle a,b \rangle$) then (u,v) satisfies a. if and only if (u,v) satisfies b . Nielsen [34] has shown that (u,v) satisfies c, if and only if (u,v) satisfies b (and so a.)

Thus if G is a free group, it satisfies the following properties:

A: $[u,v] \sim [a,b]^{\pm 1}$ iff (u,v) generates G

B_N, (resp B_T) : $[u,v] \sim [a,b]^{\pm 1}$ iff (u,v) and (a,b) belong to the same NE class (resp. T-system).

C_N, (resp C_T): (u,v) generates G iff (u,v) and (a,b) belong to to the same NE class (resp T-system).

Properties C_N and C_T can be shortened to saying, G has one NE class (resp. T-system) of generating pairs. In general if G satisfies any two of the Properties A, B_N, C_N then G satisfies all three. Similarly if G satisfies two of the Properties A, B_T, C_T, then G satisfies all three.

We discuss the following two questions:

Question 1: *Are there groups which satisfy one of the Properties* A, B_N, *or* C_N *but not all three? Similarly, are there groups which satisfy one of the Properties* A, B_T, *or* C_T, *but not all three?*

Question 2: *Can we show that certain types of groups possess Property* A, B_N, B_T, C_N *or* C_T, *or if possible all five properties?*

In answer to Question 1, we show by means of examples that there are groups which satisfy one of A, B_N, C_N but not all three, and there are groups satisfying one of B_T, C_T but not A. No example has been found of a group which satisfies Property A, but not B_T or C_T.

We consider Question 2 mainly for the case of small cancellation groups. Using methods similar to those of Chapter II we show that if $G = \langle a,b;\ R\rangle$ is a $C_L'(1/16)$-group or a $C_L'(1/12)$, $T(4)$ -group for some L, then G has Property B_N. Property C_N has already been discussed in Chapter II.

It should be pointed out that the results on Property B_N mentioned in the previous paragraph have recently been generalised, in joint work with S.J. Pride [15]. In [15] using geometric techniques, we show that if $G = \langle a,b;\ R\rangle$ is a $C(15)$ -group or a $C(12)$, $T(4)$ -group then G has Property B_N In addition, with regard to Property C_N we have recently established (using the results of Chapter II) that if $G = \langle a,b;\ R\rangle$ is a $C_L'(1/20)$ group, where for all $r \in R$

a) r is not a power of a primitive in G.

b) $r = s^n, n \geqslant 5$,

then G has one NE class.

(Note that we do not require R to be finite for this result.)

Work on Question 2 has been done by Pride for one-relator groups with torsion. In [44] he has shown that if $G = \langle a,b; r^m \rangle, (m > 1)$ then G has Property C_N (unless r is a power of a primitive - in this case, G may not have Property C_N, though it has Property C_T). He has also shown that if $m > 3$ then G has Property B_N, (an account of this work is given in [15]). It is natural to ask whether the condition $m > 3$ (in this latter result) can be replaced by $m > 1$. We give examples with $m = 2$ to show that this is not the case. (The situation for $m = 3$ remains unresolved.)

It is natural to consider ways in which Property A might be generalised. One way would be to choose a fixed but arbitrary pair (a,b) from a group G, (so that (a,b) need not generate G) and thus consider

Property A': $[u,v] \sim [a,b]^{\pm 1}$ iff $\mathrm{sgp}_G\{u,v\}$ is conjugate to $\mathrm{sgp}_G\{a,b\}$

However even if G is free, A' need not hold [2] .

Another possibility is to consider more than two generators. We ask the question:- If G is an n-generator group with fixed generating n-tuple $(a_1,\ldots,a_n)$. Then is there a word $W(x_1,\ldots,x_n)$ on n-variable such that the solutions of

(1) $$W(x_1,\ldots,x_n) \sim W(a_1,\ldots,a_n)^{\pm 1},$$

are precisely the generating n-tuples of G?

Rips [51] has shown that when $G = F_n = \langle a_1,\ldots,a_n\rangle$ and W is a higher commutator, then all solutions to (1) are generating n-tuples of G. However not all generating n-tuples are solutions. In fact it is well known that if $n \geqslant 3$, and $G = F_n$, then there is no word W as described above. For if W were such a word, and $w = W(a_1,\ldots a_n)$ then $w\alpha$ would be conjugate to $w^{\pm 1}$ for each $\alpha \in \mathrm{Aut}\, F_n$. It is well-known that no such non-trivial word w exists [27], [30]. To show how bad things can be, we prove:

(THEOREM 3.4) *Let F be a free group of rank $n \geqslant 3$, and let N be the normal closure of w in F, $w \neq 1$. Then*

$$\bigcap_{\alpha \in \mathrm{Aut}F} N\alpha = 1$$

It follows immediately from Theorem 3.4 that if $G = \langle x_1,x_2,\ldots,x_n;W\rangle$ then F_n is residually G. Since any free group of rank greater than n is residually F_n, [36], we have:

(THEOREM 3.5) *Let G be an n-generator, one-relator group with $n \geqslant 3$, Then any free group of rank greater than or equal to n is residually G.*

Residual properties of free groups have been studied in several papers ([11],[22],[23],[36],[37],[38],[39],[63]). In particular, Pride [39] has shown that if $G = \langle a,b;\ r^k\rangle, k > 1$, then F_2 (and thus F_n for $n \geqslant 2$) is residually G, unless r is conjugate to $[a,b]^l$ for some $l \neq 0$. (If r is conjugate to $[a,b]^l$, then Pride (unpublished) has shown that F_3 is residually G). The following question remains open: If $G = \langle a,b;\ r\rangle$ where r is not a proper power and if G does not satisfy a non-trivial law, then is F_2 residually G?

SECTION 1 PRELIMINARY RESULTS

In this section we introduce some of the results of small cancellation theory which will be used in later chapters. The results in this section may be found in [25], and [54] unless otherwise stated. Wherever possible the notation will be the same as that used in the book [25, Chapter V] by Lyndon and Schupp.

The following theorem is the fundamental result of small cancellation theory.

THEOREM 1.1 *(Greendlinger's Lemma) Let F be a free group. Let R be a symmetrized subset of F, with N the normal closure of R. Assume that R satisfies the hypotheses* C(p) *and* T(q) *where (q,p) is one of the pairs* (6,3),(4.4) *or* (3,6). *If* $w \in N$, $w \neq 1$, *then for some cyclically reduced conjugate* W^* *of* w, $W^* \in R$ *or has the form* $W^* = u_1 s_1 \ldots u_n s_n$ *where each* s_k *is an* $i(s_k)$ *remnant. The number* n *of the* s_k *and the numbers* $i(s_k)$ *satisfy the relation.*

$$\sum_{k=1}^{n} [p/q + 2 - i(s_k)] \geqslant p$$

#

We do not apply Theorem 1.1 directly, but use the following two corollaries.

COROLLARY 1.1 *Let R satisfy* C(6), *and let w be cyclically reduced.*

Then either (1) $w \in R$

or some cyclically reduced conjugate W^* of w contains one of the following:

(2) *two disjoint* 1-*remnants*

(3) *three disjoint* 2-*remnants*

(4) *four disjoint subwords, two* 2-*remnants, and two* 3-*remnants*

(5) *five disjoint subwords, one* 2-*remnant and four* 3-*remnants.*

(6) *six disjoint* 3-*remnants.*

#

COROLLARY 1.2 *Let R satisfy* C(4),T(4) *and let w be cyclically reduced.*

Then either (1) $w \in R$

or some cyclically reduced conjugate w^* of w contains one of the following:

(2) *two disjoint* 1-*remnants*

(3) *three disjoint* 2-*remnants*

#

In particular we observe that if R satisfies C(6) then w contains at least one 3-remnant, and if R satisfies C(4), T(4)w contains at least one 2-remnant.

The next result classifies the torsion elements in small cancellation groups. (see [27]).

THEOREM 1.2 *Let F be a free group, and let R be a symmetrized subset of F satisfying* C(6) *or* C(4),T(4). *Let N be the normal closure of R, and $G = F/N$.*
Then if w has finite order in G, there is an element $r \in R$ which is a proper power in F, say $r \equiv s^n$, $n > 1$, and $w \sim_G s^m$, where m is an integer. Moreover, w has order $n/(m,n)$ in G.

#

The following propositions concerning subwords of R are needed in later chapters. They are originally due to Lipschutz, [24],[25].

PROPOSITION 1.1 *If $r \equiv sxs^{-1}y$, then s is a piece relative to the symmetrized closure of r.*

PROOF: If s is not a piece, then $sxs^{-1}y \equiv sx^{-1}s^{-1}y^{-1}$. Therefore $x \equiv y \equiv 1$ and r is trivial.

#

PROPOSITION 1.2 *If $r \equiv (sx)^m sy$, $m \geqslant 1$, then either*

(1) *$(sx)^{m-1}s$ is a piece relative to the symmetrized closure of r, or*

(2) *$sx, (sx)^{m-1}sy$, r are all powers of a common element, and r is a proper power.*

PROOF: If $(sx)^{m-1}s$ is not a piece, since both $(sx)^m sy$ and $(sx)^{m-1}sysx$ are cyclic permutations of r they must be identical.

Therefore $(sx)^{m-1}sy$ and sx commute. As commuting elements in a free group are powers of a common element, $(sx)^{m-1}sy, sx$ are powers of a common element z, and thus $r \equiv sx(sx)^{m-1}sy$ is a proper power of z.

#

In Chapters II and III we use diagrams to prove results concerning conjugate elements in small cancellation groups. A diagram over a group F is an oriented map M and a function ϕ assigning to each oriented edge e of M as a label, an element $\phi(e)$ of F such that if e is an oriented edge of M, and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$. If α is a path in M, $\alpha = e_1,\ldots,e_k$, then define $\phi(\alpha) = \phi(e_1),\ldots,\phi(e_k)$. If D is a region of M, a label of D is an element $\phi(\alpha)$ for α a boundary cycle of D.
If R is symmetrized subset of F, an R-diagram is a diagram M such that if δ is any boundary cycle of any region D of M, then $\phi(\delta) \in R$.

LEMMA 1.1 *Let N be the normal closure of R in F. Then for any w in F, $w \in N$ if and only if there is a connected, simply connected R-diagram M such that the label on the boundary of M is w.* (see [27, Chapter V]).

#

Thus connected, simply connected diagrams can be used to study membership of normal subgroups.

An *annular map* M is a connected map such that $-M$ has exactly two components. Let M be an annular map. Let K be the unbounded component of $-M$, and let H be the bounded component of $-M$. The intersection of the boundary ∂M of M with the boundary ∂K of K we call the *outer boundary*. Similarly we define the *inner boundary*. A cycle of minimal length (that does not cross itself) which contains all edges in the outer (inner) boundary of M is an *outer (inner) boundary cycle* of M. The next two lemmas show that annular diagrams can be used to study conjugacy in F/N.

LEMMA 1.2 *Let M be an annular R-diagram. If y is a label of an outer boundary cycle of M, and z is a label of an inner boundary cycle of M, then either y and z are conjugate in F/N or y and z^{-1} are conjugate in F/N.*

#

LEMMA 1.3 *Let y and z be two cyclically reduced words of F which are not in N, and which are not conjugate in F. If y and z represent conjugate elements of G, then there is a reduced annular R-diagram M containing at least one region such that:*
If $\sigma = e_1 \ldots e_l$ and $\tau = f_1 \ldots f_k$ are respectively outer and inner boundary cycles of M, then the product $\phi(e_1) \ldots \phi(e_l)$ is reduced without cancellation and is a conjugate of y while the product $\phi(f_1) \ldots \phi(f_k)$ is reduced without cancellation and is a conjugate of $z^{\pm 1}$.

#

We call M a conjugacy diagram for y and z. When R satisfies the small cancellation condition $C'_L(1/6)$, or $C'_L(1/4)$ and $T(4)$ the next theorem describes the geometry of the conjugacy diagrams.

THEOREM 1.3 *(The structure theorem for suitable annular R-diagrams). Let R satisfy either*

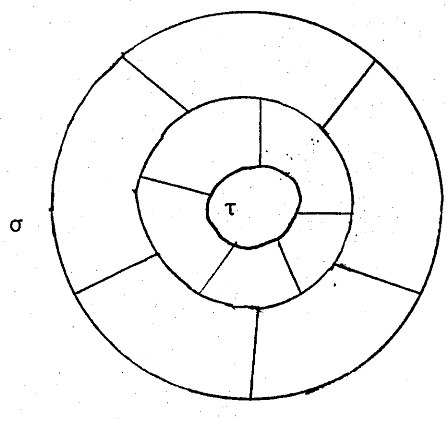
(i) $C'_L(1/6)$

(ii) $C'_L(1/4)$ and $T(4)$.

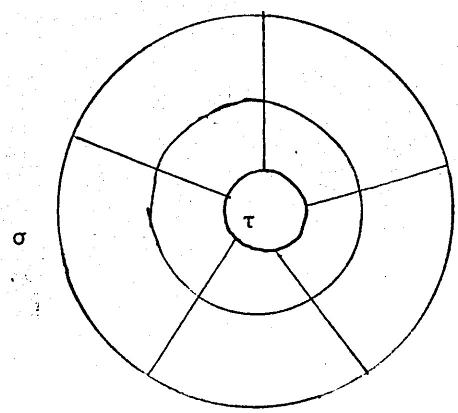
Let M be a reduced annular R-diagram. Let σ, τ be respectively, the outer and inner boundaries of M. Assume that if D is a region of M with $\sigma_1 = \partial D \cap \sigma$ connected then $L(\phi(\sigma_1))$ is not $> 1/2 L(\phi(D))$. Assume the same hypothesis with σ replaced by τ.

(1) *If M does not contain a region D such that ∂D contains an edge of both σ and τ, then M has the form (a) if R satisfies $C'_L(1/6)$, and form (b) if R satisfies $C'_L(1/4)$ and $T(4)$.*

(a)

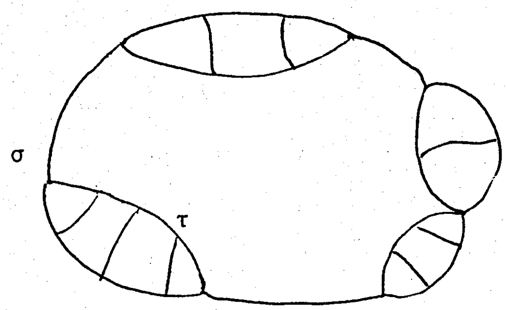


(b)



(the number of layers is always 2, however, the number of regions per layer is variable.)

(2) *If M does contain a region D such that ∂D contains an edge of both σ and τ, then M has the form:*



(the number of 'islands' and the number of regions per 'island' are variable).

#

Note: In [27 pp.252-259] this result is proved when L is the usual length function L_1. However, as observed by Pride, the result can easily be proved for a general length function.

In Chapter III we make use of the strengthened form of the Freiheitssatz [58]. This is a single theorem which strengthens both the Freiheitssatz of Magnus[27 p.104] for one relator groups in general, and Newman's "spelling theorem", [27 p.109], in the torsion case.

THEOREM 1.4 *(The strengthened form of the Freiheitssatz). Let $G = \langle a,b,c\ldots;r\rangle$ where r is cyclically reduced. Write $r = z^n$, $n \geq 1$, where z is not a proper power in the free group on (a,b,c,...). (Elements of the symmetrized set r^* generated by r thus have the form $(z^*)^n$ where z^* is a cyclic permutation of $z^{\pm 1}$). If an equation u = v holds in G, where u and v are freely reduced words, and v omits a generator* *which occurs in both r and u, then u contains a subword t of an element of r* such that* $t \equiv (z^*)^{n-1}s$, *and s contains every generator which occurs in r but not in v.*

#

If $n = 1$, then t is simply s.

In the case $v = 1$, s contains every generator that occurs in r.

CHAPTER II

TWO GENERATOR SUBGROUPS OF SMALL CANCELLATION GROUPS

SECTION 1. SUMMARY

It has been shown that small cancellation groups satisfying C(4) and T(4), or C(6), with certain trivial exceptions, possess a free subgroup of rank 2.
(Collins [4], Al Janabi [20]).
In this Chapter we consider the isomorphism types of non-free 2-generator subgroups of small cancellation groups. However, in order to obtain results, we restrict our attention to small cancellation groups satisfying $C'_L(1/10)$ and T(4), or $C'_L(1/14)$.

There are two similar theorems which we prove in Sections 2 and 3 of this chapter. One of them applies to small cancellation groups satisfying $C'_L(1/10)$ and T(4), or $C'_L(1/14)$, and this is used to derive certain general results concerning these small cancellation groups. The details can be found towards the end of this section. The other, although satisfied by a smaller class of small cancellation groups ($C'_L(1/12)$ and T(4), or $C'_L(1/16)$) is useful, in particular, when dealing with applications of the results to specific groups. An example of this is also given.

The method of proof of the theorems is algorithmic and given a finitely presented small cancellation group satisfying the required small cancellation hypothesis, provides a means of determining the nature of any 2-generator subgroup defined by a given generating pair.

In order to present the theorems, we need to make some definitions.

Let X be an n-tuple , and $W(x)$ be the set of words in X. Let $F(x)$ be the free group freely generated by X; R a symmetrized subset of cyclically reduced words on X; $N = \langle R\rangle^F$, the normal closure in $F(x)$ of R; and $G = F/N$.

We denote equality in $W(x)$ by $\equiv$,

equality in $F(x)$ by $=$,

and equality mod N by $=_G$ or $=_N$.

Similarly we denote conjugacy in $F(x)$ by $\sim$,

and conjugacy mod N by $\sim_G$ or $\sim_N$.

Let (u,v) be a pair of words in $W(x)$. We define the following *elementary transformations* (mod$N$) on (u,v):

$N_0 \;:\; (u,v) \mapsto (v,u)$,

$N_1 \;:\; (u,v) \mapsto (u^{-1},v)$,

$N_2 \;:\; (u,v) \mapsto (uv,v)$,

$N^k \;:\; (u,v) \mapsto (k^{-1}uk, k^{-1}vk)$, where $k \in W(x)$,

$N_G \;:\; (u,v) \mapsto (\bar{u},\bar{v})$ if $u =_G \bar{u}$, $v =_G \bar{v}$.

In addition to these, we need to define a transformation which can only occur when one of the elements of R is a proper power:

$\bar{N}_G \;:\; (u,v) \mapsto (u_1, v)$, where $u \equiv x^{\alpha}$, $u_1 \equiv x^{\alpha_i}$,

$r \equiv x^{\gamma}$ $r \in R, \gamma > 1$, $(\alpha,\gamma) = (\alpha_i, \gamma)$.

We call the transformations $N_0, N_1, N_2, N^k, N_G, \bar{N}_G$, *generalised elementary transformations (or GE transformations)* mod N.

If no element of R is a proper power, then there are no possible transformations of type $\bar{N}_G$. In this case, a sequence of GE transformations (mod N) is the same as a sequence of elementary transformations (mod N). The transformation N_0, N_1, N_2, N^k, N_F are called *free elementary transformations*.

A factorization of (u,v) is a triple (f,g,h) of words in $W(x)$, where either

(i) $u \equiv f^{-1}g$, $v \equiv f^{-1}h$, or

(ii) $u \equiv f^{-1}gf$, $v \equiv h$.

In the first case, we say that the *factorization is of Type I*, and in the second case we say that the *factorization is of Type II*. If $f \equiv 1$, then we say that the *factorization is trivial*.

We assume that $sgp_G\{u,v\}$ *is not cyclic* so that at least two of the elements of a factorization of (u,v) are non-trivial, and if the factorization is of Type II , then g and h are non-trivial.

Suppose g and h are non-trivial, and (f,g,h) is

(i) a non-trivial factorization of Type I and $g^{-1}fh^{-1}gf^{-1}h$ is cyclically reduced, or

(ii) a non-trivial factorization of Type II and $f^{-1}ggfhh$ is cyclically reduced, or

(iii) trivial and $g^{-1}h^{-1}gh$, g and h cyclically reduced,

then we say that (f,g,h) is a reduced factorization of (u,v).

We will show in the next section, (Lemma 2.3) how (u,v) can be mapped by a sequence of free elementary transformations to a pair (u', v') with reduced factorization (f', g', h') where $L(f', g', h') \leq L(f, g, h)$.

Until further notice *let (u,v) be a pair of words with a reduced factorization (f,g,h) of Type J, (J = I or II).*

In the algorithms described in this Chapter, we shall, in each step use a sequence of GE transformations which maps the pair (u,v) to a pair with a reduced factorization of shorter length. This can be repeated until we obtain a pair of words in $W(x)$ with a reduced factorization, and which satisfies one of the three properties defined below.
In order to define two of these properties we shall need to use the following set $\hat{S}$, where each element is conjugate to an element of R, and depends on the factorization (f,g,h) and its Type J.
Let $\hat{S}$ be the set of all words such that
(a) each element of $\hat{S}$ is a proper power, and freely conjugate to an element of R, and
(b) if (f,g,h) is of Type I, then each element of $\hat{S}$ has the form

$$(\dot{f}^{-1}\dot{g}(\dot{f}^{-1}\dot{h})^{\beta})^{\alpha}, \qquad \beta \geq 0,\ \alpha \neq 0$$

and $(\dot{f},\dot{g},\dot{h})$ is a permutation of (f,g,h);
if (f,g,h) is of Type II, then each element of $\hat{S}$ is freely equal to the forms

$$(f^{-1}g^{\varepsilon}fh^{\varepsilon'\beta})^{\alpha} \text{ or } (h^{\varepsilon}f^{-1}g^{\varepsilon'\beta}f)^{\alpha};\ \varepsilon,\ \varepsilon' = \pm 1,$$

$$\beta \geq 0,\ \alpha \neq 0.$$

We say that the pair (u,v) has *Property* 1 if $N \cap \text{sgp}\{u,v\}$ is the normal closure of $\hat{S}$ in $\text{sgp}\{u,v\}$.

We say that the pair (u,v) has *Property 2* if either

(i) for certain integers $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$, the non-trivial elements of the set $(f^{\varepsilon_1}, g^{\varepsilon_2}, h^{\varepsilon_3})$ are disjoint subwords of a subconjugate of an element of R, or

(ii) The triple (f,g,h) is a non-trivial factorization of Type I, and $N \cap \text{sgp}\{u,v\}$ is the normal closure of

$$(f^{-1}g)^{\ell},\ (h^{-1}f)^{m},\ (g^{-1}h)^{n}$$

for certain integers $\ell,m,n > 1$.

We say that (u,v) has *Property 2** if the following hold:

(a) There exists an $r \in R$ so that the non-trivial elements of the triple (f,g,h) are pieces relative to the symmetrized closure of r.

(b) For any $r \in R$, $r \notin \hat{S}$, for which there exists a reduced word W^* equal to a word in (u,v), and a subword t of W^*, which is a ρ-remnant of r, the non-trivial elements of the triple (f,g,h) are pieces relative to the symmetrized closure of r.

(where $\rho = 2$ if R satisfies T(4), and $\rho = 3$ otherwise).

The two main Theorems proved in this Chapter are the following:

THEOREM 2.1 *Let R satisfy $C'_L(1/14)$ or $C'_L(1/10), T(4)$. Let (u,v) be a pair of elements of F, where $\text{sgp}_G\{u,v\}$ is not cyclic.*

Then (u,v) can be transformed by a finite sequence of GE *transformations (modN) to a pair which satisfies Property* 1 *or Property* 2.

We say that (u,v) is *root-closed (rel R)* if for all $S^n \in R, n>1$, S^m is not conjugate in G to an element of a generating pair of $\text{sgp}_G\{u,v\}$, for all $m \in \mathbb{Z}$.

THEOREM 2.2 *Let R satisfy* $C_L'(1/16)$ *or* $C_L'(1/12)$, $T(4)$. *Let (u,v) be a pair of elements of F, where* $\text{sgp}_G\{u,v\}$ *is not cyclic. Then*

(a) *(u,v) can be transformed by a finite sequence of* GE *transformations* (modN) *to a pair which satisfies Property* 1 *or Property* 2*.

(b) *if (u,v) is root-closed (rel R), then either (u,v) can be transformed by a finite sequence of elementary transformations* (modN) *to a pair which satisfies Property* 2*, *or* $\text{sgp}_G\{u,v\}$ *is free.*

In order to use these results, it is important to say something further about those pairs that satisfy Property 1.

LEMMA 2.1 *If $(u.v)$ is a pair of elements of F with reduced factorization (f,g,h) and either*

(i) *satisfies Property* 1, *but not Property* 2, *or*

(ii) *satisfies Property* 1, *but f,g,h are not all pieces.*

Then $\text{sgp}_G\{u,v\}$ *is a free product of cycles.*

This Lemma will be proved in Section 2.2.

Now if the pair (w,z) of words in $W(x)$ is obtained from the pair (u,v) by a sequence of elementary transformations N_0, N_1, N_2, N^k, N_G, then $\mathrm{sgp}_G\{u,v\} \sim \mathrm{sgp}_G\{w,z\}$. This can be seen by considering each of the elementary transformations in turn.

In fact this is also true of $\bar{N}_G$, for let $\bar{N}_G(u,v) = (u_1,v)$ where $u \equiv x^{\alpha}$, $u_1 \equiv x^{\alpha_1}$, $r \equiv x^{\gamma}$, $r \in R$, and $(\alpha,\gamma) = (\alpha_1,\gamma)$ Then the pair $(x^{(\alpha,\gamma)},v)$ generates both $\mathrm{sgp}_G\{u,v\}$ and $\mathrm{sgp}_G\{u_1,v\}$.

Using Lemma 2.1 and the definitions of Properties 2 and 2* we have the following:

THEOREM 2.3 *Let R satisfy* $C'_L(1/14)$, *or* $C'_L(1/10)$, T(4) *for some length* L *and let H be a two-generator subgroup of G. Then either*

(i) *H is a free product of cycles, or*

(ii) *if* (w,z) *generates H, then* (w,z) *can be transformed by a finite sequence of* GE *transformations to* (u,v) *where* $\mathrm{sgp}_G\{u,v\} \sim \mathrm{sgp}_G\{w,z\}$, *and there exists a reduced factorization* (f,g,h) *of* (u,v), *so that either for certain integers* ε,ε' *equal to* ± 1, *the elements of the set* $\{f,g^{\varepsilon},h^{\varepsilon'}\}$ *are disjoint subwords of a subconjugate of R, or* (f,g,h) *is a non-trivial factorization of Type* I *and*
$\mathrm{sgp}\{u,v\} = \langle u,v; u^{\ell}, v^{m}, (u^{-1}v)^{n}\rangle$ *where* $\ell,m,n \neq 0$.

THEOREM 2.4 *Let R satisfy* $C'_L(1/16)$, *or* $C'_L(1/12)$, T(4) *for some length* L *and let H be a two-generator subgroup of G. Then either*

(i) *H is a free product of cycles, or*

(ii) *if (w,z) generates H, then (w,z) can be transformed to (u,v) where* $\mathrm{sgp}_G\{u,v\} \sim \mathrm{sgp}_G\{w,z\}$, *and the elements of a reduced factorization of (u,v) are pieces.*

The usefulness of Theorem 2.3 is demonstrated by the following results.

THEOREM 2.5 *Let $G = \langle X;R \rangle$ where R is finite and satisfies* $C_L'(1/14)$ *or* $C_L'(1/10)$, $T(4)$. *Then*

(i) *G has finitely many conjugacy classes of 2-generator subgroups whose members are not free products of cycles.*

(ii) *G has finitely many isomorphism types of 2-generator subgroups.*

In order to see this, note that by Theorem 2.3, for each conjugacy class of two-generator subgroups whose members are not free products of cycles, there exists a factorization where two of the elements are subwords of elements of R, and the third is a product of at most two words, each of which is a subword of an element of R. Since R is finite, there are only finitely many such factorizations, proving part (i).

By (i) there are only finitely many isomorphism types of two-generator subgroups which are not free products of cycles. Also since R is finite, there is a bound on the orders of elements of finite order in G [27,p281], so only finitely many isomorphism types of 2-generator free products can occur as subgroups of G.

#

A result, similar to (i) cannot be obtained for 3-generator subgroups. In [49] Rips shows that given $\lambda>0$, and a finitely presented group A, there is no exact sequence.

$$1 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 1$$

where B is a finitely generated $C'(\lambda)$-group, and C is generated, as a group, by two elements C_1, C_2. In a letter to S.J. Pride, Rips has pointed out that C is not free. Thus if A has an infinite family $\{D_i; i \in I\}$ of pairwise non-conjugate cyclic subgroups, and if b_i is an element of B, mapping onto a generator of D_i, $(i \in I)$, then the subgroups $\mathrm{sgp}\{b_i, C_1, C_2\}$, $(i \in I)$, are non-free and pairwise non-conjugate 3-generator subgroups of B.

Suppose a group G is defined by means of a presentation $\langle X;R\rangle$, then a *primitive* of G is an element which is a member of a generating $|X|$-tuple of G.

THEOREM 2.6 *Let $G = \langle a,b;R\rangle$ where R is finite, and no element of R is a power of a primitive of G, and R satisfies* $C'_L(1/14)$ *or* $C'_L(1/10)$, $T(4)$. *Then G has finitely many NE classes of generating pairs.*

(The result is well-known if G is a free product of cycles). As no element of R is a power of a primitive in G, a sequence of GE transformation (mod N) on a pair of generators of G is a sequence of elementary transformations (mod N). Thus if G is not free, by Theorem 2.3 each generating pair (w,z) of G is in the same NE class as the generating pair (u,v) of G, and a factorization of (u,v) is such that two of the elements are subwords of R, and the third is the product of at most two words, each of which is a subword of R. Since R is finite, there are only finitely many such pairs.

Using a modification of the method used to prove Theorem 2.6, Pride [14] has shown that the restriction that no element of R is a power of a primitive of G in the above theorem is not necessary. However, the following example shows that the condition that R is finite in Theorem 2.1 cannot be omitted.

EXAMPLE 2.1 Let k be an integer, $k \geqslant 16$. For $i = 1,2,3...$ let

$$r_i = ab^i a^2 b^i \dots a^k b^i b$$

Let R be the symmetrized closure in $F = \langle a,b \rangle$ of $\{r_i ; i = 1,2,...\}$ and let $G = \langle a,b;\ R \rangle$. Now R satisfies T(4). In addition R satisfies $C'_-(4/k)$ In order to see this note that the largest pieces contained in a cyclic permutation of r_i are

$$b^i a^k b^{i+1},\ a^{k-1} b^{i+1} a \text{ and } a^{k-2} b^i a^{k-1}$$

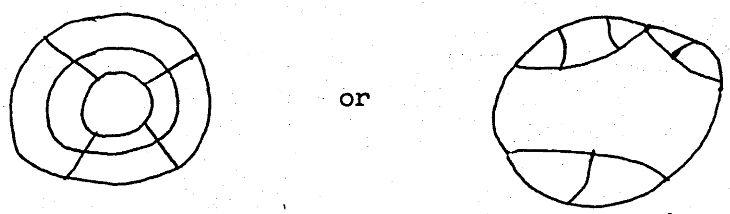
Therefore R satisfies $C'(\lambda)$ where

$\lambda \geqslant 4/k > \max\{2i+1+k,\ k+i+1, \text{ and } 2k+i-3\} / (ki+1+(1+2+...+k)$

In particular, R satisfies $C'(1/4)$.

We will show that G has an infinite number of NE classes of generating pairs, represented by $\{(a,b^i);\ i = 1,2,...\}$.
In G, $b^{-1} = a\, b^i\, a^2\, b^i \dots a^k\, b^i$, so that (a,b^i) generates G. We show that for $i \neq j$, (a,b^i), (a,b^j) are not Nielsen equivalent.

By Nielsen [32] it suffices to show that $[a,b^i]$, $[a,b^j]^\varepsilon$, ($\varepsilon = \pm 1$) are not conjugate in G. However (by Theorem 1.3) if they were conjugate in G, since they are cyclically reduced, but not freely conjugate, there would be a reduced R-diagram D.



with the label on the outer boundary $[a,b^i]$ and the label on the inner boundary $[a,b^j]^\varepsilon$. Now any subword of $[a,b^i]$ or $[a,b^j]$ which is also a subword of an element of R is a piece. Thus all edges of regions of D are labelled by pieces which contradicts the fact that R satisfies $C'(1/4)$.

#

We have conjectured that if $G = \langle a,b;\ R\rangle$ where each element of R is a proper power, and where no element of R is a power of a primitive, then if R satisfies $C'(\lambda)$ for suitably small λ, G will have one Nielsen equivalance class of generating pairs. The following example should be noted in connection with this conjecture.

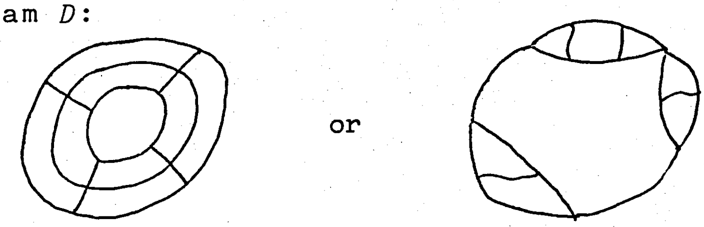
EXAMPLE 2.2 Let k be an odd integer with $k \geqslant 12$, and for $i = 1,2,3\ldots$ let

$$r_i = (ab^i)^k$$

Then each r_i is a power of a primitive in $F = \langle a,b\rangle$. Let R be the symmetrized closure in F of $\{r_i;\ i = 1,2,3\ldots,n\}$ (n may be ∞), and let $G = \langle a,b;\ R\rangle$. Now R satisfies $T(4)$.

Also the largest piece contained in a cyclic permutation of r_i is $b^i a b^i$, and so R satisfies $C'(2/k)$. We show that G has at least n Nielsen equivalence classes, represented by $\{((ab^i)^2, b); i = 1,2,\ldots n\}$.

Let $k = 2l+1$. Then $a^{-1} = b^i(ab^i)^{2l}$, so $((ab^i)^2, b)$ generates G. Now for $i \neq j$, $((ab^i)^2, b)$, $((ab^j)^2, b)$ are not Nielsen equivalent. To show this, it suffices to show that the cyclically reduced forms of $[(ab^i)^2, b]$ and $[(ab^j)^2, b]^\varepsilon$ that is $U = a^{-1}b^{-i}a^{-1}b^{-1}ab^iab$ and $V = (a^{-1}b^{-j}a^{-1}b^{-1}ab^jab)^\varepsilon$ ($\varepsilon = \pm 1$), are not conjugate in G. But if they were conjugate, since they are not freely conjugate, there would be a reduced R-diagram D:



with the label on the outer boundary U and the label on the inner boundary V. (See Theorem 1.3)
Now any subword of U or V which is also a subword of an element of R is no more than 2 pieces. Thus each internal edge of D is labelled by pieces, and each boundary edge is labelled by at most 2 pieces. This contradicts the fact that R satisfies $C'(2/k)$.

#

The question, whether a group has a finite number of Nielsen equivalence classes arises when considering the nature of Aut(G).
Let F_2 be the free group of rank 2, R a symmetrized subset of F_2, N the normal closure of R, and $G = F_2/N$.

Let $\Pi(N)$ be the group of automorphisms ϕ of F_2 such that $N\phi = N$.

Each element ϕ of $\Pi(N)$ induces an automorphism $\hat{\phi}$ of G where

$$wN\hat{\phi} = w\phi N \quad (w \in F_2).$$

By Pride [48] if G has ℓ Nielsen equivalence classes of generating pairs.

$$|\mathrm{Aut}(G) : \hat{\Pi}(N)| \leqslant \ell .$$

Therefore if G is a two-generator, finitely related, small cancellation group, satisfying $C_L'(1/14)$ or $C_L'(1/10)$, $T(4)$, then it follows from Theorem 2.6 and the above remark that $\mathrm{Aut}(G)$ is finitely generated (resp. presented) if and only if $\hat{\Pi}(N)$ is finitely generated (resp. presented). Some results concerning $\Pi(N)$ can be found in [48], [49].

We finish this section with an application of Theorem 2.4.

THEOREM 2.7 *There exist two-generator, one-relator groups which are not free, but which have every proper two-generator subgroup free.*

To show this we construct the following group G:

Let k be an integer, with $k \geqslant 31$. Let

$$r = ab^k\, ab^{k+1} \ldots\ldots ab^{2k}$$

and let $G = \langle a, b;\ R\rangle$, where R is the symmetrized closure of r in F, ($F = \langle a, b\rangle$). Then R satisfies $T(4)$ and $C'(1/12)$ (since the largest piece contained in a cyclic permutation of r is $b^{2k-2}ab^{2k-1}$).

Let (w,z) be a pair of elements of G, where $\text{sgp}_G\{w,z\} = H$ is a non-cyclic, non-free subgroup of G. Then as no element of R is a proper power, all pairs of words in G will be root-closed (rel R). Therefore, by Theorem 2.4(ii), (w,z) is Nielsen equivalent in G to a conjugate of the pair (u,v), and the elements f,g,h of a reduced factorization of (u,v) are pieces.

Since $H^* = \text{sgp}_G\{u,v\}$ is non-free, H^* contains a non-trivial word $w =_G 1$. By small cancellation theory (See Corollary 1.2), w will contain a 2-remnant of R. But any 2-remnant of R must have a subword z where

$$z = ab^{j_1}ab^{j_2}ab^{j_3}ab^{j_4}ab^{j_5}a$$

for distinct j_1, j_2, j_3, j_4, j_5, and where $j_{i+1} = j_i+1$ for some i, $1 \leqslant i < 5$. Note that z is the product of no less than 6 pieces.

We use the result that f,g,h are pieces, and the fact that any piece is of the form $(b^{q_1}ab^{q_2})^{\pm 1}$ or $b^{\pm q}$, $(q,q_1,q_2 \geqslant 0)$ to give specific expressions for f,g,h, namely

$$f \equiv (b^{\ell_1}a^{\theta}b^{\ell_2})^{\varepsilon_1},\quad g \equiv (b^{m_1}a^{\lambda}b^{m_2})^{\varepsilon_2},\quad h \equiv (b^{n_1}a^{\mu}b^{n_2})^{\varepsilon_3}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$; $\theta, \lambda, \mu = 0$ or 1; $\ell_1, \ell_2, m_1, m_2, n_1, n_2 \geqslant 0$. By examining the maximum powers, j, of b in words of form $ab^j a$, where $ab^j a$ is a subword of a word in (u,v), we show that as z must also be a subword of a word in (u,v), $f \equiv 1$, $g \equiv a^{\pm 1}$, $h \equiv b^{\pm 1}$ as required.

We first show that (f,g,h) cannot be a non-trivial factorization of Type II. For if this is the case, as z is a positive word in (a, b), z must be a subword of $g^{\alpha}fh^{\beta}$ or $h^{\beta}f^{-1}g^{\alpha}$, $(\alpha, \beta \geqslant 0)$. We assume z is a subword of $g^{\alpha}fh^{\beta}$, for the other case is similar. Then z is a subword of

$$(b^{m_1}a^{\lambda}b^{m_2})^{\alpha} \; b^{\ell_1}a^{\theta}b^{\ell_2} \; (b^{n_1}a^{\mu}b^{n_2})^{\beta}.$$

However, there can be at most four distinct integers j, such that $ab^{j}a$ is a subword of the above word, and this contradicts our definition of z.

Suppose the triple (f, g, h) is a factorization of Type I. Then as H^* is non-cyclic, at least two elements of the set $\{f, g, h\}$ are non-trivial. It can be seen that the triples $(f^{\varepsilon}, g^{\varepsilon}, h^{\varepsilon})$, $(g^{\varepsilon}, h^{\varepsilon}, f^{\varepsilon})$, $(h^{\varepsilon}, f^{\varepsilon}, g^{\varepsilon})$ are reduced factorizations of Type I of pairs of elements that generate subgroups conjugate to H^*. Thus we may assume that g, h are non-trivial, and if f is non-trivial we can choose (u, v) and its factorization so that $\varepsilon_2 = \varepsilon_3 = 1$. If f is trivial, then as $(1, g^{-1}h)$, $(1, g, h^{-1})$ $(1, g^{-1}, h^{-1})$ are reduced factorizations of (u^{-1}, v), (u, v^{-1}), (u^{-1}, v^{-1}) respectively, we may still choose (u, v) and (f, g, h) so that $\varepsilon_2 = \varepsilon_3 = 1$. If $\lambda = \mu = 0$, then (f, g, h) is not reduced. By the symmetry between g and h we can assume $\lambda = 1$. Thus z is a subword of a positive word in

$$((b^{\ell_1}a^{\theta}b^{\ell_2}b^{m_1}ab^{m_2}), (b^{\ell_1}a^{\theta}b^{\ell_2}b^{n_1}a^{\mu}b^{n_2}))$$

We have four cases to consider, depending on the values of θ and μ.

(a) Suppose $\theta = \mu = 1$. Then there can be no more than four distinct integers j where $ab^{j}a$ is a subword of a word in (u, v), namely ℓ_2+n_1, $n_2+\ell_1$, ℓ_2+m_1, $m_2+\ell_1$, which contradicts our definition of z.

(b) Suppose $\theta = 1$, $\mu = 0$. Then there can be no more than three distinct integers j where $ab^j a$ is a subword of a word in (u,v), namely $\ell_2+n_1+n_2+\ell_1$, $\ell_2+m_1, m_2+\ell_1$, – a contradiction of z.

(c) Suppose $\theta = 0$, $\mu = 1$. Then there can be no more than four distinct integers j where $ab^j a$ is a subword of a word in (u,v) namely $m_2+\ell_1+\ell_2+n_1$, $n_2+\ell_1+\ell_2+m_1$, $m_2+\ell_1+\ell_2+m_1$, $n_2+\ell_1+\ell_2+n_1$ – a contradiction of z.

(d) Suppose $\theta = \mu = 0$. Let $\ell_1+\ell_2 = \ell$, $n_1+n_2 = n$, $(\ell,n > 0)$. If $ab^j a, ab^{j+1}a$ are subwords of a positive word in $(b^\ell b^{m_1} ab^{m_2}, b^\ell b^n)$, then $j = m_2+(\ell+n)\alpha+\ell+m_1$, $j+1 = m_2+(\ell+n)\alpha_1+\ell+m_1$.
Therefore $\ell+n = 1$. However, we assumed h is non-trivial, so that $\ell = 0$ and $n = 1$. Thus

$$f \equiv 1, \; g \equiv b^{m_1}ab^{m_2}, \; h \equiv b.$$

If m_1 or $m_2 \neq 0$, then (f,g,h) is not a reduced factorization. So it follows that $(f,g,h) = (1,a,b)$, and $(u,v) = (a,b)$ as required.

#

SECTION 2 PAIRS OF WORDS AND FACTORIZATIONS

In order to prove Theorem 2.1 and Theorem 2.2 we need to start with an arbitrary pair of words (u,v) in $W(x)$, and describe the transformations which will map (u,v) to a pair of words that satisfy Properties 1,2 or 2*. However, instead of working directly with the pair (u,v) it has been found to be simpler to use a factorization of (u,v) and transform the factorization using the factorization transformations defined below.

So that we can reconstruct a pair (u,v) from a factorization (f,g,h) it is convenient to write $(f,g,h)^{I}$ or $(f,g,h)^{II}$ according to whether (f,g,h) is a factorization of Type I or II respectively. We say that the factorizations $(f,g,h)^{J}$, $(f,g,h)^{J'}$ are equal if and only if either $J = J'$ or $f \equiv 1$, $(J,J' = \text{I or II})$.

Let $(f,g,h)^{I}$ be a factorization of (u,v), then we define the following *factorization transformations:*

$(f,g,h)^{I} \xrightarrow{S_0^I} ((f^{\varepsilon},g^{\varepsilon},h^{\varepsilon})\pi)^{I}$, where π permutes (f,g,h), $\varepsilon = \pm 1$.

$(f,g,h)^{I} \xrightarrow{{}^{k}S^{I\,\ell}} (kf\ell, kg\ell, kh\ell)^{I}$, $k,\ell \in W(x)$.

$(f,g,h)^{I} \xrightarrow{S_G^I} (f',g',h')$, where $f' =_G f$, $g' =_G g$, $h' =_G h$.

$(f,g,h)^{I} \xrightarrow{\bar{S}_G^I} (f_1,g_1,h_1)$, where $f^{-1} \equiv d(cd)^{\mu}$,

$f_1^{-1} \equiv d(cd)^{\mu_1}$, $g \equiv (cd)^{\nu}c$, $g_1 \equiv (cd)^{\nu_1}c$, $h_1 \equiv (cd)^{r-\mu_1}h$;

$r \sim (dc)^{\gamma}$, $\gamma > 1$, $r \in R$; $\mu,\nu,\mu_1,\nu_1 \geqslant 0$;

$(\mu+\nu+1, \gamma) = (\mu_1+\nu_1+1, \gamma)$

In the second transformation ${}^{k}S^{I\ell}$, if k(or ℓ) $\equiv 1$, then we write S_0^I (or ${}^{k}S^I$).

We say that S_0^I, ${}^{k}S^{I\ell}$, S_F^I are *free factorization transformations* (*of Type* I).

For example, let $(f,g,h)^I = (a^{-1}b,b,a)^I$ where $a,b \in X$ so that $u \equiv b^{-1}ab$, $v \equiv b^{-1}a^2$. Then

$(a^{-1}b,b,a)^I \xrightarrow{S_0^I} (a,b,a^{-1}b)^I$, a factorization of $(a^{-1}b,a^{-2}b)$

$(a^{-1}b,b,a)^I \xrightarrow{{}^{a}S^I} (aa^{-1}b,ab,a^2)^I \xrightarrow{S_F^I} (b,ab,a^2)^I$, a factorization of $(b^{-1}ab,b^{-1}a^2)$ and if $a^\gamma \in R$, $\gamma > 1$, and α_1 is an integer such that $(\alpha_1,\gamma) = 1$, $(a^{-1}b,b,a)^I \xrightarrow{S^{Ib^{-1}}, S_F^I}$ $(a^{-1},1,ab^{-1})^I \xrightarrow{\bar{S}_G^I} (a^{-1},a^{\alpha_1-1},ab^{-1})^I$, which is a factorization of (a^{α_1},a^2b^{-1}).

Note that in each case, if $(f_1,g_1,h_1)^I$ is the new factorization, $\mathrm{sgp}\{f_1^{-1}g_1,f_1^{-1}h_1^{-1}\} \sim \mathrm{sgp}\{f^{-1}g,f^{-1}h\}$, except in the last case when $\mathrm{sgp}\{f_1^{-1}g_1,f_1^{-1}h\} \underset{G}{\sim} \mathrm{sgp}\{f^{-1}g,f^{-1}h\}$.

If $(f,g,h)^{II}$ is a factorization of (u,v) then we define the the following transformations.

$(f,g,h)^{II} \xrightarrow{S_0^{II}} (f,g^{\varepsilon_1},h^{\varepsilon_2})^{II}$ or $(f^{-1},h^{\varepsilon_2},g^{\varepsilon_1})^{II}$ where $\varepsilon_1,\varepsilon_2 = +1$ or -1.

$(f,g,h)^{II} \xrightarrow{{}^{k}S^{II\ell}} (kf\ell,kgk^{-1},\ell^{-1}h\ell)^{II}$, $k,\ell \in W(x)$

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f',g',h')^{II} \text{ where } f =_G f',\ g =_G g',\ h =_G h'$$

$$(f,g,h)^{II} \xrightarrow{\bar{S}_G^{II}} (f,g_1,h)^{II} \text{ where } g \equiv c^{\mu},\ g_1 \equiv c^{\mu_1};$$

$$r \sim c^{\gamma}; r \in R, \gamma > 1; \mu, \mu_1 \geq 0;\ (\mu,\gamma) = (\mu_1,\gamma).$$

In the second transformation, if k(or ℓ) $\equiv 1$, then we write $S^{II\ell}$ (or ${}^{k}S^{II}$).

We say that S_0^{II}, ${}^{k}S^{II\ell}$, S_F^{II} are *free factorization transformations* (*of Type* II).

For example if $(f,g,h)^{II} = (b,a,a^2b)^{II}$ so that $u \equiv b^{-1}ab$, $v \equiv a^2b$, then

$(b,a,a^2b)^{II} \xrightarrow{S_0^{II}} (b^{-1},a^2b,a)^{II}$, a factorization of (ba^2bb^{-1},a)

$(b,a,a^2b)^{II} \xrightarrow{{}^{b}S^{II}} (b^2,bab^{-1},a^2b)^{II}$, a factorization of

$(b^{-2}bab^{-1}b^2,a^2b)$. If $a^{\gamma} \in R$, $\gamma > 1$, then if $(\alpha_1,\gamma) = 1$, we

have $(b,a,a^2b)^{II} \xrightarrow{\bar{S}_G^{II}} (b,a^{\alpha_1},a^2b)$ a factorization of

$(b^{-1}a^{\alpha_1}b,a^2b)$.

Therefore as in the previous set of transformations, if (f_1,g_1,h_1) is the new factorization, $\mathrm{sgp}\{f^{-1}gf,h\} \sim \mathrm{sgp}\{f_1^{-1}g_1f_1,h_1\}$ except in the last case when $\mathrm{sgp}\ \{f^{-1}gf,h\} \sim_G \mathrm{sgp}\ \{f_1^{-1}g_1f_1,h_1\}$.

In general if J = I or II, we only apply factorization transformations S_0^J, ${}^{\ell}S^{Jk}$, S_G^J, $\bar{S}_G^J$ to a factorization of Type J. However, as the trivial factorization can be regarded as a factorization of Type I or Type II, we can apply any of the factorization transformations to it. Any factorization $(f,g,h)^J$ can be mapped by means of ${}^{f^{-1}}S^J$ and S_F^J to a trivial factorization of the same pair (u,v), so that by this device we can change from a Type I factorization to a Type II factorization, and vice versa.

We say that the factorization (f,g,h) , $(f',g',h')^{J'}$ are *weakly related* if there is a finite sequence of factorization transformations mapping (f,g,h) to $(f',g',h')^{J'}$. If this sequence excludes transformations of types $\bar{S}_G^{I}$ and $\bar{S}_G^{II}$, then we say that (f,g,h) , $(f',g',h')^{J'}$ are *related*.

A transformation of type S_0^I will map $(f,g,h)^I$ to $((f^{\varepsilon},g^{\varepsilon},h^{\varepsilon})\pi)^I$, $\varepsilon = \pm 1$, π a permutation of (f,g,h). Thus the transformations of type S_0^I form a group of order 12. (That is, the group is isomorphic to the direct product of $\mathbb{Z}_2$ and S_3 the symmetric groups on 3 elements.) A transformation of type S_0^{II} will map $(f,g,h)^{II}$ to $(f,g^{\varepsilon_1},h^{\varepsilon_2})^{II}$ or $(f^{-1},h^{\varepsilon_2},g^{\varepsilon_1})^{II}$, $\varepsilon_1,\varepsilon_2 = \pm 1$. Thus the transformations of type S_0^{II} form a group of order 8. (That is, the group is isomorphic to the direct product of three copies of $\mathbb{Z}_2$.)

We say that *the factorizations* $(f,g,h)^J$, $(f',g',h')^{J'}$ *are equivalent* if

$$(f,g,h)^J Q = (f',g',h')^{J'}$$

and Q is a sequence of factorization transformations of types S_0^I and S_0^{II} only.

Let S be a factorization transformation, where $(f,g,h)^J S = (f_1,g_1,h_1)^{J_1}$.
Let $(f,g,h)^J$, $(f_1,g_1,h_1)^{J_1}$ be factorizations of (u,v), and (u_1,v_1) respectively.

Suppose S is of type S_0^I. If $(f,g,h)^I\ S = (h,f,g)^I$ then $(u_1,v_1) = (v^{-1}, v^{-1}u)$. However $(u,v)N_1,N_2,N_1,N_0,N_1 = (v^{-1},v^{-1}u)$.
If $(f,g,h)^I\ S = (f^{-1},g^{-1},h^{-1})^I$ then $(u_1,v_1) = (fg^{-1},fh^{-1})$.
However $(u,v)\ N_1,N_0,N_1,N_0,N^{f-1} = ((u^{-1})^{f-1},(v^{-1})^{f-1})$.

Suppose S is of type S_0^{II}. If $(f,g,h)^{II}\ S = (f,g^{-1},h)^{II}$

then $(u_1,v_1) = (f^{-1}g^{-1}f,h)$ However $(u,v)N_1 = (u^{-1},v)$
If $(f,g,h)^{II}\ S = (f^{-1},h,g)^{II}$ then $(u_1,v_1) = (fhf^{-1},g)$
However $(u,v)\ N_0N^{f-1} = (v^{f-1},u)$.
Suppose S is of type ${}^kS^{I\ell}$ or ${}^kS^{II\ell}$. Then $(f,g,h)^I S$ $= (kf\ell,kg\ell,kh\ell)^I$ or $(f,g,h)^{II}\ S = (kf\ell,kgk^{-1},\ell^{-1}h\ell)^{II}$.
In both cases $(u_1,v_1) = (u,v)^{\ell} = (u,v)N^{\ell}$.

Thus if $S = S_0{}^J$ or ${}^kS^{J\ell}$, there exists a sequence N of transformations of types N_0,N_1,N_2,N^{ℓ} so that

$$(u,v)N = (u_1,v_1)\ .$$

If $S = S_G{}^J$ or $\bar{S}_G{}^J$, then it can be seen directly from the definition that there exists an elementary transformation N of type N_G or $\bar{N}_G$ respectively, where

$$(u,v)N = (u_1,v_1)$$

Therefore if $(f,g,h)^J$ is related (or weakly related) to $(f_1,g_1,h_1)^{J_1}$ then there exists a sequence of elementary transformations (or GE transformations) which maps (u,v) to (u_1,v_1). In fact, as we shall show in the next lemma, the reverse situation is also true. That is, if a sequence of elementary transformations (GE transformations) maps (u,v) to (u_1,v_1) then $(f,g,h)^J$ is related (weakly related) to $(f_1,g_1,h_1)^{J_1}$.

LEMMA 2.2 *Let $(f,g,h)^J$ and $(f_1,g_1,h_1)^{J_1}$ be factorizations of (u,v) and (u_1,v_1) respectively. Then $(f,g,h,)^J$ and $(f_1,g_1,h_1)^{J_1}$ are related or weakly related in G if and only if (u,v) can be transformed to (u_1,v_1) by a sequence of elementary transformations in G, or* GE *transformations in G, respectively.*

The sufficiency of the condition has already been noted. In order to see that the condition is necessary, we can assume that (u_1,v_1) is obtained from (u,v) by one of the transformations $N_0, N_1, N_2, N^k, N_G, \overline{N}_G$, we treat each of these in turn.

(1) $(u_1,v_1) = (u,v)N_0$

$$(f,g,h)^J \xrightarrow{f^{-1}S^J,\,S_F^J} (1,u,v)^{II} \xrightarrow{S_0^{II}} (1,v,u)^{J_1}$$

$$\xrightarrow{f_1 S^{J_1},\,S_F^{J_1}} (f_1,g_1,h_1)^{J_1}$$

(2) $(u_1,v_1) = (u,v)N_1$

$$(f,g,h)^J \xrightarrow{f^{-1}S^J,\,S_F^J} (1,u,v)^{II} \xrightarrow{S_0^{II}} (1,u^{-1},v)^{J_1}$$

$$\xrightarrow{f_1 S^{J_1},\,S_F^{J_1}} (f_1,g_1,h_1)^{J_1}$$

(3) $(u_1,v_1) = (u,v)N_2$

$$(f,g,h)^J \xrightarrow{f^{-1}S^J,\,S_F^J} (1,u,v)^{II} \xrightarrow{S_0^{II}} (1,u^{-1},v)^{I}$$

$$\xrightarrow{v^{-1}S^I} (v^{-1},v^{-1}u^{-1},1)^I \xrightarrow{S_0^I} (1,uv,v)^{J_1} \xrightarrow{f_1 S^{J_1},\,S_F^{J_1}} (f_1,g_1,h_1)^{J_1}$$

(4) $(u_1,v_1) = (u,v)N_G$

$$(f,g,h)^J \xrightarrow{f^{-1}S^J,\,S_F^J} (1,u,v)^{J} \xrightarrow{S_G^J} (1,u_1,v_1)^{J_1}$$

$$\xrightarrow{f_1 S^{J_1},\,S_F^{J_1}} (f_1,g_1,h_1)^{J_1}.$$

(5) $(u_1, v_1) = (u,v)\overline{N}_G$

$$(f,g,h)^J \xrightarrow{f^{-1}S^J, S^J_F} (1,u,v)^{II} \xrightarrow{\overline{S}_G^{\,II}} (1,u_1,v_1)^{J_1}$$

$$\xrightarrow{f_1 S^{J_1}, S^{J_1}_F} (f_1, g_1, h_1)^{J_1}$$

where $u = c^{\alpha}$, $u_1 = c^{\alpha_1}$; $r = c^{\gamma}$, $\gamma>1, r \in R; (\alpha,\gamma) = (\alpha_1,\gamma)$

#

We will now prove the main results described in Section 2.1. First we will establish Lemma 2.1, which is the link between Theorems 2.1 and 2.3, and between Theorems 2.2 and 2.4.

PROOF OF LEMMA 2.1

Suppose (u,v) satisfies Property 1. Then if $\mathrm{sgp}_G\{u,v\}$ is not a free product of cycles, since each element of $\hat{S}$ is a power of some element which is one of a generating pair of $\mathrm{sgp}\{u,v\}$, $\hat{S}$ must contain at least two elements S_1 and S_2. If (S_1,S_2) is a generating pair of $\mathrm{sgp}\{u,v\}$, then $\hat{S}$ must contain a third element S_3, where $S_3^{\pm 1}$ is not conjugate to S_1 or S_2.

Suppose $(f,g,h)^{I}$ is a factorization of (u,v).
Then for all $S_i \in \hat{S}$, let $S_i \equiv (f_i^{-1} g_i (f_i^{-1} h_i)^{\beta_i})^{\rho_i}$, where (f_i, g_i, h_i) is a permutation of (f,g,h), $\beta_i \geqslant 0$. We can arrange that $\beta_1 \geqslant \beta_2 \geqslant \beta_i$, $i \neq 1$, and if S_3 exists, we can also arrange that $\beta_3 \geqslant \beta_i$, $i \neq 1,2$. For convenience we drop the suffix 1. We examine the various possibilities for S and S_2.

Case (1) $\beta>0$, $\beta_2>0$.

Then S contains f^{-1}, g and h as subwords, and f^{-1} is a piece since $(f^{-1}g(f^{-1}h)^{\beta})^{\rho} \not\equiv ((f^{-1}h)^{\beta}f^{-1}g)^{\rho}$.

To show g and h are pieces, we consider the six possible values for S_2 separately.

(a) $S_2 \equiv (f^{-1}g(f^{-1}h)^{\beta_2})^{\rho_2}$: Then g and h are pieces, since as $\beta \neq \beta_2$,

$(g(f^{-1}h)^{\beta_2}f^{-1})^{\rho_2} \not\equiv (g(f^{-1}h)^{\beta}f^{-1})^{\rho}$ and

$(hf^{-1})^{\beta_2}gf^{-1})^{\rho_2} \not\equiv ((hf^{-1})^{\beta}gf^{-1})^{\rho}$

(b) $S_2 \equiv (g^{-1}h(g^{-1}f)^{\beta_2})^{\rho_2}$: Then g and h are pieces, since

$(g^{-1}h(g^{-1}f)^{\beta_2})^{\rho_2} \not\equiv (g^{-1}f\ (h^{-1}f)^{\beta})^{\rho}$ and

$(h(g^{-1}f)^{\beta_2}g^{-1})^{\rho_2} \not\equiv ((hf^{-1})^{\beta}gf^{-1})^{\rho}$.

(c) $S_2 \equiv (h^{-1}f(h^{-1}g)^{\beta_2})^{\rho_2}$: Then g and h are pieces, since

$((gh^{-1})^{\beta_2}fh^{-1})^{\rho_2} \not\equiv (gf^{-1}(hf^{-1})^{\beta})^{\rho}$ and

$(h^{-1}f(h^{-1}g)^{\beta_2})^{\rho_2} \not\equiv ((h^{-1}f)^{\beta}g^{-1}f)^{\rho}$

(d) $S_2 \equiv (f^{-1}h(f^{-1}g)^{\beta_2})^{\rho_2}$: Then g and h are pieces since β and β_2 are not both equal to 1, which means that

$(hf^{-1}(gf^{-1})^{\beta_2})^{\rho_2} \not\equiv ((hf^{-1})^{\beta}gf^{-1})^{\rho}$ and

$((gf^{-1})^{\beta_2}hf^{-1})^{\rho_2} \not\equiv (gf^{-1}(hf^{-1})^{\beta})^{\rho}$.

(e) $S_2 \equiv (g^{-1}f(g^{-1}h)^{\beta_2})^{\rho_2}$: Then g and h are pieces since

$(g^{-1}f(g^{-1}h)^{\beta_2})^{\rho_2} \not\equiv g^{-1}f(h^{-1}f)^{\beta})^{\rho}$ and

$((hg^{-1})^{\beta_2}fg^{-1})^{\rho_2} \not\equiv ((hf^{-1})^{\beta}gf^{-1})^{\rho}$

(f) $S_2 \equiv (h^{-1}g(h^{-1}f)^{\beta_2})^{\rho_2}$: Then g and h are pieces since

$(gh^{-1}(fh^{-1})^{\beta_2})^{\rho_2} \not\equiv (gf^{-1}(hf^{-1})^{\beta})^{\rho}$ and

$(h^{-1}g(h^{-1}f)^{\beta_2})^{\rho_2} \not\equiv ((h^{-1}f)^{\beta}g^{-1}f)^{\rho}$

Case (2) $\beta>0$, $\beta_2 = 0$.

Then f^{-1}, g and h are subwords of S. As

$(f^{-1}hf^{-1}g(f^{-1}h)^{\beta-1})^{\rho} \not\equiv ((f^{-1}h)^{\beta}f^{-1}g)^{\rho}$

$f^{-1}h$ is a piece. However $\beta_2 = 0$, so that the only possible values for S_2 are $(f^{-1}h)^{\rho_2}$, $(g^{-1}f)^{\rho_2}$, $(h^{-1}g)^{\rho_2}$, (or their inverses). We consider these separately.

(a) $S_2 \equiv (f^{-1}h)^{\rho_2}$: Then as $(f^{-1}h, f^{-1}g(f^{-1}h)^{\beta})^{\rho}$ is a generating pair of sgp$\{u,v\}$, S_3 exists and $S_3^{\pm 1} \equiv (g^{-1}f)^{\rho_3}$ or $(h^{-1}g)^{\rho_3}$, so that g is a piece.

(b) $S_2 \equiv (g^{-1}f)^{\rho_2}$: Then as $(gf^{-1})^{\rho_2} \not\equiv (gf^{-1}(hf^{-1})^{\beta})^{\rho}$, g is a piece.

(c) $S_2 \equiv (h^{-1}g)^{\rho_2}$: Then as $(gh^{-1})^{\rho_2} \not\equiv (gf^{-1}(hf^{-1})^{\beta})^{\rho}$, g is a piece.

Case (3) $\beta = 1$, $\beta_2 = 0$.

Then f^{-1}, g and h are subwords of S. As $f^{-1}gf^{-1}h \not\equiv f^{-1}hf^{-1}g$, f is a piece. However as $\beta_2 = 0$, the only possible values for S_2 are $(f^{-1}h)^{\rho_2}$, $(g^{-1}f)^{\rho_2}$, $(h^{-1}g)^{\rho_2}$ (or their inverses). We consider these separately.

(a) $S_2 \equiv (f^{-1}h)^{\rho_2}$: Then $(f^{-1}h, f^{-1}gf^{-1}h)$ are a generating pair for sgp$\{u,v\}$. Thus S_3 exists and $S_3 = (g^{-1}f)^{\rho_3}$ or $(h^{-1}g)^{\rho_3}$ (or their inverses) so that g and h are pieces.

(b) $S_2 \equiv (g^{-1}f)^{\rho_2}$: Then as S is symmetric in g and h, the case is similar to (a).

(c) $S_2 \equiv (h^{-1}g)^{\rho_2}$: Then g and h are pieces.

Case (4) $\beta = 0$, $\beta_2 = 0$.

Then we can arrange that $S \equiv (f^{-1}g)^{\rho}$, $S_2 \equiv (f^{-1}h)^{\rho_2}$, and $S_3 \equiv (h^{-1}g)^{\rho_3}$, so that f, g and h are pieces. Since β was assumed to be maximal, these are all the possible values of $\hat{S}$, so that (u,v) satisfies Property 2.

Suppose $(f,g,h)^{II}$ is a factorization of (u,v)

Then for all $S_i \in \hat{S}$, let $S_i = (f^{-1}g^{\varepsilon_i}fh^{\delta_i\beta_i})^{\rho_i}$ or $= (h^{\varepsilon_i}f^{-1}g^{\delta_i\beta_i}f)^{\rho_i}$; $\varepsilon_i, \delta_i = \pm 1$; $\beta \geqslant 0$. We can arrange that $\beta_1 \geqslant \beta_2 \geqslant \beta_i$, $i \neq 1$, and if S_3 exists, we can also arrange that $\beta_3 \geqslant \beta_i$, $i \neq 1,2$. By the symmetry between g and h we can assume $S_1 = (f^{-1}gfh^{\beta_1})^{\rho_1}$.

For convenience we drop the suffix 1. We examine the various possibilities for S and S_2, and where necessary S_3. Note that by replacing S_i by a conjugate of $S_i^{\pm 1}$, we can arrange that $\varepsilon_i = 1$.

Case (1) $\beta>0$, $\beta_2>0$.

Then f, g, and h are subwords of S. As

$(f^{-1}gfh^{\beta})^{\rho} \not\equiv (f^{-1}g^{-1}fh^{-\beta})^{\rho}$

f is a piece.

To show g and h are pieces, we consider the possible values for S_2 separately.

(a) $S_2 \equiv (f^{-1}gfh^{\delta_2\beta_2})^{\rho_2}$: Then g and h are pieces, since $\delta_2\beta_2 \neq \beta$ so that

$(gfh^{\delta_2\beta_2}f^{-1})^{\rho_2} \not\equiv (gfh^{\beta}f^{-1})^{\rho}$, and

$(h^{\delta_2\beta_2}f^{-1}gf)^{\rho_2} \not\equiv (h^{\beta}f^{-1}gf)^{\rho}$

(b) $S_2 \equiv (hf^{-1}g^{\delta_2\beta_2}f)^{\rho_2}$: Then S_2 is in the same form as the S_2 in case (a) with g and h exchanged.

Case (2) $\beta>1$, $\beta_2 = 0$.

Then f, g and h are subwords of S. As

$fh^{\beta}f^{-1}g \not\equiv fh^{-\beta}f^{-1}g^{-1}$, and $hf^{-1}gh^{\beta-1} \not\equiv h^{\beta}f^{-1}gf$,

f and h are pieces. However $\beta_2 = 0$, so that the only possible values for S_2 are $f^{-1}g^{\rho_2}f$ or h^{ρ_2}. We consider them separately.

Suppose $(f,g,h)^{II}$ *is a factorization of* (u,v)

Then for all $S_i \in \hat{S}$, let $S_i = (f^{-1}g^{\epsilon_i}fh^{\delta_i\beta_i})^{\rho_i}$ or $= (h^{\epsilon_i}f^{-1}g^{\delta_i\beta_i}f)^{\rho_i}$; $\epsilon_i, \delta_i = \pm 1$; $\beta \geqslant 0$. We can arrange that $\beta_1 \geqslant \beta_2 \geqslant \beta_i$, $i \neq 1$, and if S_3 exists, we can also arrange that $\beta_3 \geqslant \beta_i$, $i \neq 1,2$. By the symmetry between g and h we can assume $S_1 = (f^{-1}gfh^{\beta_1})^{\rho_1}$.

For convenience we drop the suffix 1. We examine the various possibilities for S and S_2, and where necessary S_3. Note that by replacing S_i by a conjugate of $S_i^{\pm 1}$, we can arrange that $\epsilon_i = 1$.

Case (1) $\beta > 0$, $\beta_2 > 0$.

Then f, g, and h are subwords of S. As

$(f^{-1}gfh^{\beta})^{\rho} \not\equiv (f^{-1}g^{-1}fh^{-\beta})^{\rho}$

f is a piece.

To show g and h are pieces, we consider the possible values for S_2 separately.

(a) $S_2 \equiv (f^{-1}gfh^{\delta_2\beta_2})^{\rho_2}$: Then g and h are pieces, since $\delta_2\beta_2 \neq \beta$ so that

$(gfh^{\delta_2\beta_2}f^{-1})^{\rho_2} \not\equiv (gfh^{\beta}f^{-1})^{\rho}$, and

$(h^{\delta_2\beta_2}f^{-1}gf)^{\rho_2} \not\equiv (h^{\beta}f^{-1}gf)^{\rho}$

(b) $S_2 \equiv (hf^{-1}g^{\delta_2\beta_2}f)^{\rho_2}$: Then S_2 is in the same form as the S_2 in case (a) with g and h exchanged.

Case (2) $\beta > 1$, $\beta_2 = 0$.

Then f, g and h are subwords of S. As

$fh^{\beta}f^{-1}g \not\equiv fh^{-\beta}f^{-1}g^{-1}$, and $hf^{-1}gh^{\beta-1} \not\equiv h^{\beta}f^{-1}gf$,

f and h are pieces. However $\beta_2 = 0$, so that the only possible values for S_2 are $f^{-1}g^{\rho_2}f$ or h^{ρ_2}. We consider them separately.

DUPLICATE

(a) $S_2 \equiv f^{-1}g^{\rho_2}f$: Then as $g^{\rho_2} \not\equiv (gfh^{\beta}f^{-1})^{\rho}$, g is a piece.

(b) $S_2 \equiv h^{\rho_2}$: Then h and $f^{-1}gfh^{\beta}$ are a generating pair for sgp$\{u,v\}$, so that S_3 exists, and as $\beta_3 = 0$ $S_3 \equiv f^{-1}g^{\rho_3}f$. Thus as in case (a), g is a piece.

Case (3) $\beta = 1$, $\beta_2 = 0$.
Then f,g and h are subwords of S, and f is a piece (since $fhf^{-1}g \not\equiv fh^{-1}f^{-1}g^{-1}$). As $\beta_2 = 0$, the only possible values for S_2 are $f^{-1}g^{\rho_2}f$ or h^{ρ_2}. But $(f^{-1}gfh,h)$ and $(f^{-1}gfh,f^{-1}gf)$ are generating pairs for sgp$\{u,v\}$ so that in both cases S_3 exists. As $\beta_3 = 0$, $S_3 \equiv f^{-1}g^{\rho_3}f$ if $S_2 \equiv h^{\rho_2}$, and $S_3 \equiv h^{\rho_3}$ if $S_2 \equiv f^{-1}g^{\rho_2}f$. It can be seen that, in both cases, $S, S_2^{\pm 1}, S_3^{\pm 1}$ are not conjugate, so that g and h are pieces.

Case (4) $\beta = 0$, $\beta_2 = 0$.
Then $S \equiv f^{-1}g^{\rho}f$, $S_2 = h^{\rho_2}$. As β is maximal, sgp$_G\{u,v\}$ is a free product of cycles.

#

In the remainder of this section, we will show that if an arbitrary pair of words (u,v) with factorization $(f,g,h)^{J}_{\sim}$ does not satisfy Properties 1,2 or 2*, then $(f,g,h)^{J}$ is weakly related in G to $(f',g',h')^{J'}$ where $L(f',g',h') < L(f,g,h)$.

First we consider transformations in F only.
If, in a word in u and v, cancellation occurs, apart from the obvious cancellation between f and f^{-1}, then we will show in the next lemma how $(f,g,h)^{J}$ can be mapped to the factorization

$(f',g',h')^{J'}$ with shorter length.

LEMMA 2.3 *Let $(f,g,h)^J$ be a factorization of (u,v) where $[u,v] \neq 1$ and where either*

(i) *J = I and $g^{-1}fh^{-1}gf^{-1}h$ is not cyclically reduced or*

(ii) *J = II and $f^{-1}gf, fhf^{-1}$, gg or hh is not reduced or*

(iii) *$f \equiv 1$ and $g^{-1}h^{-1}gh, g$ or h is not cyclically reduced.*

Then there is a sequence of free factorization transformations mapping $(f,g,h)^J$ to $(f',g',h')^{J'}$ where $L(f',g',h')<L(f,g,h)$ and either $J = J'$ or $f \equiv 1$.

(i) Suppose J = I.

Then at least 2 of the elements of (f,g,h) are non-trivial. Suppose f and g are non-trivial, and $f^{-1}g$ is not reduced. Then either f(or g) is itself not reduced, in which case using S^I_F we can replace f(or g) by a shorter word, or there exists a word $k \not\equiv 1$, where

$g \equiv kg_2$, $f \equiv kf_2$.

Thus we have the following sequence of transformations.

$$(f,g,h)^I \xrightarrow{k^{-1}S^I} (k^{-1}kf_2, k^{-1}kg_2, k^{-1}h)^I \xrightarrow{S^I_F} (f_2, g_2, k^{-1}h)^I$$

$$\text{But } L(f_2,g_2,k^{-1}h) = L(f,g,h) - L(k) < L(f,g,h)$$

Transformations of type S^I_0 do not alter the length of a factorization. Thus as the pairs $(g,f^{-1}),(h^{-1},f),(f,h^{-1})$, (g^{-1},h), (h,g^{-1}) can be obtained from (f^{-1},g) by means of a transformation of type S^I_0 acting on $(f,g,h)^I$, part (i) of the lemma is proved unless f,g or $h \equiv 1$.

Suppose J = I but $f \equiv 1, g$ and h reduced, gh not reduced. Then there exists $k \not\equiv 1$ where

$g \equiv g_1 k^{-1}$, $h \equiv kh_2$

so that $(1,g,h)^{\mathrm{I}} = (1,g,h)^{\mathrm{II}} \xrightarrow{S_0^{\mathrm{II}}} (1,g^{-1},h)^{\mathrm{II}}$

$$= (1,g^{-1},h)^{\mathrm{I}} \xrightarrow{{}^{k^{-1}}S^{\mathrm{I}}} (k^{-1},k^{-1}kg_1^{-1},k^{-1}kh_2)^{\mathrm{I}}$$

$$\xrightarrow{S_F^{\mathrm{I}}} (k^{-1},g_1^{-1},h_2)^{\mathrm{I}}.$$

But $L(k^{-1},g_1^{-1},h_2) = L(1,g,h) - L(k)$

$$< L(1,g,h).$$

As the pairs (h,g), (f,g), (h,f) can be obtained from the pair (g,h) by means of a permutation of (f,g,h), part (i) of the lemma is proved.

(ii) Suppose J = II.

Then g and h are non-trivial. If f,g or h is not reduced, then using S_F^{II}, we can transform $(f,g,h)^{\mathrm{II}}$ to a shorter factorization.

If g is reduced but not cyclically reduced, then for some $k \not\equiv 1$,

$g \equiv kg_0 k^{-1}$.

Therefore we have the following transformations:

$$(f,g,h)^{\mathrm{II}} \xrightarrow{{}^{k^{-1}}S^{\mathrm{II}}} (k^{-1}f,k^{-1}kg_0k^{-1}k,h)^{\mathrm{II}} \xrightarrow{S_F^{\mathrm{II}}} (k^{-1}f,g_0,h)^{\mathrm{II}}$$

But $L(k^{-1}f,g_0,h) = L(f,g,h) - L(k)$

$$< L(f,g,h).$$

If f^{-1} and g are reduced, but $f^{-1}g$ is not reduced, then there exists $k \not\equiv 1$ where

$g \equiv kg_2$, $f \equiv kf_2$.

Therefore we have the following transformations:

$$(f,g,h)^{II} \xrightarrow{k^{-1}S^{II}} (k^{-1}kf_2, k^{-1}kg_2k, h)^{II} \xrightarrow{S_F^{II}} (f_2, g_2k, h)^{II}.$$

But $L(f_2, g_2k, h) = L(f,g,h) - L(k)$

$< L(f,g,h)$.

Transformations of type S_0^{II} do not alter the length of a factorization. Therefore as the pairs (f^{-1}, g^{-1}), (f,h), (f,h^{-1}), (h,h) can be obtained from (f^{-1},g) or (g,g) by means of a transformation of type S_0^{II} acting on $(f,g,h)^{II}$, part (ii) of the lemma is proved.

(iii) Part (iii) follows from parts (i) and (ii). For if $f \equiv 1$ we can regard the factorization to be of Type I or Type II, so that if $g^{-1}h^{-1}gh$ is not cyclically reduced, by part (i), we can map $(1,g,h)^{I}$ to a shorter factorization, and if g or h is not cyclically reduced, by part (ii), we can map $(1,g,h)^{II}$ to a shorter factorization.

#

Thus a factorization $(f,g,h)^{J}$, which is not reduced can be mapped to a reduced factorization $(f',g',h')^{J'}$ where $L(f,g,h) \geqslant L(f',g',h')$ by means of a sequence of free factorizations. For any given factorization, the sequence can be constructed using the method described in Lemma 2.3.

Therefore we only need to consider a pair (u,v) which has a reduced factorization $(f,g,h)^{J}$. We are interested in the $sgp_G\{u,v\}$. To obtain the structure of this group we need to analyze $sgp\{u,v\} \cap N$.

We will assume

(a) $\text{sgp}_G\{u,v\}$ is not cyclic, so that two of the triple (f,g,h) are non-empty, and if the factorization is of Type II, then g and h are non-empty.

(b) $\text{sgp}_G\{u,v\}$ is not free so that $\text{sgp}\{u,v\} \cap N \neq 1$.

Note that if $\text{sgp}_G\{u,v\}$ is free, then (u,v) satisfies Property 1 and the Theorems 2.1, 2.2 are proved.

Let $W(x,y)$ be a word in two variables. Let W^* be the word obtained from $W(u,v)$ by freely reducing in terms of the generators of F. Note that W^* is obtained by cancelling the letters from adjacent subwords f,f^{-1} and f^{-1},f. Thus W^* can be partitioned into subwords $f^{\pm1}$, $g^{\pm1}$ or $h^{\pm1}$. We call these f-, g-, *or* h-*subwords* of W^* respectively. We call an f-, g-, or h-subword of W^* an *F-subword* of W^*.

If $W^* \not\equiv 1$ defines an element of N, then by small cancellation theory, W^* contains a ρ-remnant ($\rho = 2$ if R satisfies T(4), and $\rho = 3$ otherwise). Let t be any ρ-remnant which is a subword of a freely reduced word W^* equal to a word in (u,v), (W^* does not necessarily belong to N). We define below 6 properties ({i},{ii},{iiia},{iiib},{iva},{ivb}) on the set (u,v,W,t). By analyzing the position of t in relation to the f-, g-, h-subwords of W^*, we will show that the set (u,v,W,t)

(a) possesses at least one of the properties {i},{ii},{iiia},{iva}, and

(b) possesses at least one of the properties {i},{ii},{iiib},{ivb}.

The properties {i},{ii},{iiia},{iiib},{iva} and {ivb} are defined as follows:

{i} There is another word $W_1(x,y)$ so that $W_1(u,v) =_{<\hat{S}>} W(u,v)$, (where $\hat{S}$ is as defined in Section 2.1), and if W_1^* is the freely reduced word equal to W_1, then $L(W_1^*) < L(W^*)$.

{ii} There is another pair (u',v') of elements of F with factorization $(f',g',h')^{J'}$, where $L(f',g',h')<L(f,g,h)$ and $(f',g',h')^{J'}$ is weakly related to $(f,g,h)^J$. If (u,v) is root-closed (rel r), then $(f',g',h')^{J'}$ is related to $(f,g,h)^J$.

{iiia} A subconjugate of t contains $f^{\varepsilon_1}, g^{\varepsilon_2}, h^{\varepsilon_3}$ disjointly, for certain integers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of modulus 1.

{iiib} The element t contains, disjointly, the F-subwords $f^{\varepsilon_1}, f^{\varepsilon_2}, g^{\varepsilon_3}, g^{\varepsilon_4}, h^{\varepsilon_5}, h^{\varepsilon_6}$ ($\varepsilon_i = \pm 1$, $1 \leq i \leq 6$), and f,g,h are pieces relative to the symmetrized closure of r, where t is a subword of $r \in R$.

{iva} $L(t) < \max\{(1/2,(2\rho+4)\lambda,(3\rho+2)\lambda)L(r)\}$

{ivb} $L(t) < \max\{(1/2,(\rho+8)\lambda,(2\rho+6)\lambda,(3\rho+4)\lambda)L(r)\}$.

Suppose (u,v,W,t) has property {i}. Then (u,v) is not root-closed (rel R) and $W^* =_{<\hat{S}>} W_1^*$, where $L(W_1^*)<L(W^*)$, $W_1^* = W_1$, and W_1 is a word in (u,v). We can repeat this process until we have a word which cannot be reduced any further by this method. We say that such a word is $\hat{S}$-minimal. Thus if W^* is $\hat{S}$-minimal, then (u,v,W,t) does not possess property {i}.

Suppose (u,v,W,t) has property {ii}. Then $(f,g,h)^J$ can be replaced by a weakly related factorization $(f',g',h')^{J'}$, where $L(f',g',h')<L(f,g,h)$. If the factorization $(f,g,h)^J$

has minimal length, that is for any factorization $(f',g',h')^{J'}$ weakly related to $(f,g,h)^{J}$, $L(f,g,h)<L(f',g',h')$, then (u,v,W,t) cannot possess property {ii}.
Note that by the definition of weakly related factorization, if (u,v) is root-closed (rel R), and $(f,g,h)^{J}$, $(f',g',h')^{J'}$ are weakly related factorizations, then $(f,g,h)^{J}$, $(f',g',h')^{J'}$ are related factorizations.

If (u,v,W,t) has property {iiia}, then G has Property 2.

If G satisfies the hypothesis of Theorem 2.1 then (u,v,W,t) cannot possess property {iva}. In order to see this suppose R satisfies $C'_L(1/14)$. Then $\lambda = 1/14$, $\rho = 3$, so that

$L(t) > 11/14\ L(r)$.

If we substitute for ρ,λ and $L(t)$ in the inequality in {iva} we get

$11/14 < \max\{1/2, 10/14, 11/14\}$

which is not possible.

If on the other hand R satisfies $C'_L(1/10)$ and $T(4)$, then $\lambda = 1/10$, $\rho = 2$ so that

$L(t) > 8/10\ L(r)$

If we substitute for ρ,λ and $L(t)$ in the inequality in {iva} we get

$8/10 < \max\{1/2, 8/10, 8/10\}$

which also is not possible.

Therefore, in order to prove Theorem 2.1, it remains to show that (u,v,W,t) possesses one of the properties {i},{ii},{iiia}, {iva}.

Suppose (u,v,W,t) possesses property {iiib} then G has Property 2*.

If G satisfies the hypothesis of Theorem 2.2, then the inequality in {ivb} is not possible. In order to see this, suppose R satisfies $C'_L(1/16)$. Then $\lambda = 1/16$, $\rho = 3$ so that

$L(t) > 13/16\ L(r)$

If we substitute for ρ,λ and $L(t)$ in the inequality in {ivb} we get

$13/16 < \max\{1/2,\ 11/16,\ 12/16,\ 13/16\}$

which is not possible.

If on the other hand R satisfies $C'_L(1/12)$ and $T(4)$, then

$\lambda = 1/12$, $\rho = 2$ so that

$L(t) > 10/12\ L(r)$

If we substitute for ρ,λ and $L(t)$ in the inequality in {ivb} we get

$10/12 < \max\{1/2,\ 10/12,\ 10/12,\ 10/12\}$

which is not possible.

Therefore, in order to prove Theorem 2.2, it remains to show that (u,v,W,t) possesses one of the properties {i}, {ii}, {iiib} or {ivb}.

The inequalities in {iva} and {ivb} are obtained by considering t as the product of a number of subwords. We will show that if (u,v,W,t) do not possess one of the properties {i}, {ii}, {iiia} (or {iiib} for Theorem 2.2), then the length of each of these subwords of t is bounded in size by $\lambda L(r)$ or $\rho\lambda L(r)$, depending on the subword and the method used. In order to obtain the inequalities for the length of t, we add the bounds on the lengths of the subwords.

There are four possible methods to show that these subwords are bounded. We call these methods A,B,C and D, and according to the method used, we refer to each of these subwords as of types A,B.C or D respectively. (This notation is only needed in Theorem 2.2).

We will show that unless there is another factorization $(f',g',h')^{J'}$ of (u',v'), where $(f',g',h')^{J'}$ is related to $(f,g,h)^{J}$ and $L(f',g',h')^{J'} < L(f,g,h)^{J}$, then a subword of Type A is bounded by $\rho\lambda L(r)$.

Subwords of types B and D are shown to be pieces.

A subword of type C is shown to be a piece unless (u,v) is not root-closed (rel R), and either there is another factorization $(f',g',h)^{J'}$ of (u',v') where $(f',g',h)^{J'}$ is weakly related to $(f,g,h)^{J}$ and $L(f',g',h')^{J'} < L(f,g,h)$, or W^* is not S-minimal.

In assuming that either t is a subword of $\hat{S}$ or $f^{\varepsilon_1},g^{\varepsilon_2},h^{\varepsilon_3}$ are not distinct subconjugates of t, (or $f^{\varepsilon_1},f^{\varepsilon_2},g^{\varepsilon_3},g^{\varepsilon_4},h^{\varepsilon_5}$, h^{ε_6}, are not distinct subwords of t), we restrict the possible positions that t can have in relation to the F-subwords of W^*.

Each position, relative to these F-subwords of W^* provides a possible case. In addition to these cases we will consider the case when $f^{\varepsilon_1},f^{\varepsilon_2},g^{\varepsilon_3},g^{\varepsilon_4},h^{\varepsilon_5},h^{\varepsilon_6}$ are all distinct F-subwords of t, and show that unless f,g and h are pieces (relative to the symmetrized closure of r) the set (u,v,W,t) possesses properties {i} or {ii}.

However because of the number of cases, (even after factoring out similar cases) needed for Theorem 2.2, we provide only a guide to the proof for each. This is done by writing t as the product of A-, B- and C-type subwords, and indicating the type of subword, and therefore the corresponding method by writing the letter (A, B, or C) above the subword.

A complete list of cases, and a shortened proof for each case in Theorem 2.1 is given in Section 2.3. A complete list of cases and a guide to the proof for each case in Theorem 2.2 is also given in Section 2.3.

The following examples provide a detailed description of the methods used, and how the inequalities are obtained. In these examples, and in all the detailed consideration of cases in Section 2.3 we let

$t\mathfrak{p}^{-1} \equiv r,\ r \in R.$

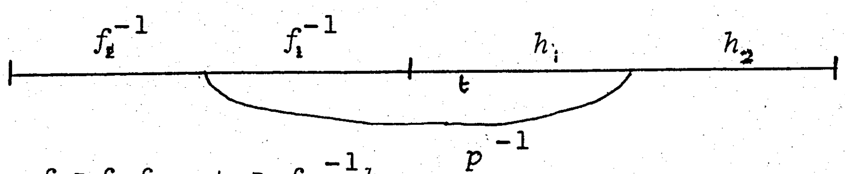
Then as t is a ρ-remnant, $L(\mathfrak{p}) < \rho\lambda\ L(r)$.

In each example we begin with a diagram which shows the position of t relative to the f-, g- and h-subwords of W^*.

We say that "{i} occurs" to mean that (u, v, W, t) has property {i}, with similar abuses of terminology for the other properties.

In the first 6 examples, the factorization is of Type I.

Example 1



$h \equiv h_1 h_2$, $f \equiv f_1 f_2$, $t \equiv f_1^{-1} h_1$.

Then $h_1 =_G f_1 p$, and $f_1 =_G h_1 p^{-1}$. Using the first of these, the transformation S_G^I maps $(f,g,h)^I$ to $(f,g,f_1 p h_2)^I$ Then the transformation $f_1^{-1} S^I$ maps this to $(f_1^{-1} f, f_1^{-1} g, f_1^{-1} f_1 p h_2)^I$ and then S_F^I will map this to $(f_2, f_1^{-1} g, p h_2)^I$. In a similar way, using the expression for f_1, we obtain

$$(f,g,h)^I \xrightarrow{S_G^I} (h_1 p^{-1} f_2, g, h)^I$$

$$\xrightarrow{h_1^{-1} S^I} (h_1^{-1} h_1 p^{-1} f_2, h_1^{-1} g, h_1^{-1} h)^I \xrightarrow{S_F^I} (p^{-1} f_2, h_1^{-1} g, h_2)^I .$$

Therefore $(f,g,h)^I$ is related to $(f_2, f_1^{-1} g, p h_2)^I$ and $(p^{-1} f_2, h_1^{-1} g, h_2)^I$.

But $L(f,g,h) = L(f_2, f^{-1} g, p h_2) - L(p) + L(h\ )$

$= L(p^{-1} f_2, h_1^{-1} g, h_2) - L(p) + L(f_1)$

Thus if $(f,g,h)^I$ is not related in G to a factorization with shorter length $L(p) \geqslant L(f_1)$ and $L(h_1)$, and as a consequence

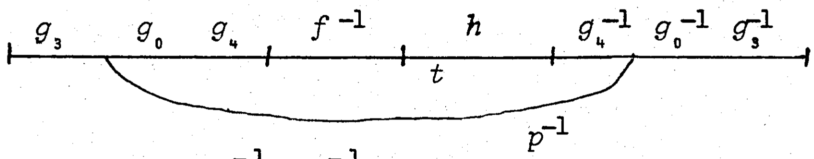
$L(t) = L(f_1) + L(h_1) \leqslant 2L(p) < 2\rho\lambda L(r)$.

#

This example used 'Method A', and shows that unless $(f,g,h)^I$ is related in G to a shorter factorization, then the length of f_1, and the length of h_1 are less than $\rho\lambda L(r)$. We call f_1 and h_1, A-subwords of t.

For the remainder of the examples, the transformations S^I_F, S^{II}_F (that is cancellation within the elements of a factorization) will be used without specific reference to them.

Example 2



$g \equiv g_3 g_0 g_4$, $t \equiv g_0 g_4 f^{-1} h g_4^{-1}$.

Then $f =_G h g_4^{-1} p^{-1} g_0 g_4$, $h =_G f g_4^{-1} g_0^{-1} p g_4$, $g_0 g_4 =_G p g_4 h^{-1} f$

Using the expression for f, we have the following

$$(f,g,h)^I \xrightarrow{S^I_G} (h g_4^{-1} p^{-1} g_0 g_4, g, h)^I$$

$$\xrightarrow{g_4 h^{-1} S^I g_4^{-1} g_0^{-1}} (p^{-1}, g_4 h^{-1} g_3, g_0^{-1})^I$$

Using the expressions for h, we have

$$(f,g,h)^I \xrightarrow{S^I_G} (f, g, f g_4^{-1} g_0^{-1} p g_4)^I$$

$$\xrightarrow{g_4 f^{-1} S^I g_4^{-1}} (1, g_4 f^{-1} g_3 g_0, g_0^{-1} p)^I$$

$$= (1, g_4 f^{-1} g_3 g_0, g_0^{-1} p)^{II}$$

$$\xrightarrow{S^{II}_0} (1, g_4 f^{-1} g_3 g_0, p^{-1} g_0)^{II} = (1, g_4 f^{-1} g_3 g_0, p^{-1} g_0)^I$$

$$\xrightarrow{S^I g_0^{-1}} (g_0^{-1}, g_4 f^{-1} g_3, p^{-1})^I$$

Using the expression for $g_0 g_4$ we have

$$(f,g,h)^I \xrightarrow{S^I_G} (f, g_3 p g_4 h^{-1} f, h)^I \xrightarrow{S^I f^{-1}} (1, g_3 p g_4 h^{-1}, h f^{-1})^I$$

$$= (1, g_3 p g_4 h^{-1}, h f^{-1})^{II} \xrightarrow{S^{II}_0} (1, g_3 p g_4 h^{-1}, f h^{-1})^{II}$$

$$= (1, g_3 p g_4 h^{-1}, f h^{-1})^I \xrightarrow{S^I h} (h, g_3 p g_4, f)^I$$

Therefore it can be seen that if $(f,g,h)^I$ is not related to a shorter factorization, then

$L(p) \geq L(f)$, $L(h)$, and $L(g_0)$.

If g_4 is non-trivial and is not a piece, then as t is a subword of r,

$$g_4 f^{-1} h g_4^{-1} p^{-1} g_0 \equiv g_4 h^{-1} f \, g_4^{-1} g_0^{-1} p$$

which is not possible. Thus either $(f,g,h)^{I}$ is related to a shorter factorization, or

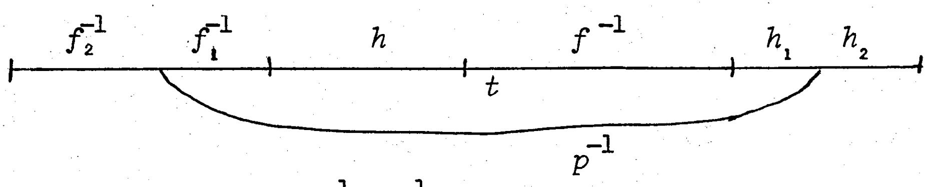
$$L(t) = 2L(g_4)+L(g_0)+L(h)+L(f)$$

$$<(2+3\rho)\lambda L(r).$$

#

The words, f, g_0, h are A-subwords of t. However we call g_4 a B-subword of t, and the above argument used to show that g_4 is a piece is called Method-B.

Example 3



$f \equiv f_1 f_2$, $h \equiv h_1 h_2$, $t \equiv f_1^{-1} h f^{-1} h_1$.

Then $h =_G f_1 p h_1^{-1} f$, $f =_G h_1 p^{-1} f_1^{-1} h$. Using the first of these, we have

$$(f,g,h)^{I} \xrightarrow{S_G^{I}} (f,g,f_1 p h_1^{-1} f)^{I} \xrightarrow{S^{I f-1}} (1, gf^{-1}, f_1 p h_1^{-1})^{I}$$

$$= (1, gf^{-1}, f_1 p h_1^{-1})^{II} \xrightarrow{S_0^{II}} (1, fg^{-1}, f_1 p h_1^{-1})^{II}$$

$$= (1, fg^{-1}, f_1 p h_1^{-1})^{I} \xrightarrow{f_1 -1 S^{I}} (f_1^{-1}, f_2 g^{-1}, p h_1^{-1})^{I}$$

and using the second of the expressions, for f, we get

$$(f,g,h)^{I} \xrightarrow{S_G^{I}} (h_1 p^{-1} f_1^{-1} h, g, h)^{I} \xrightarrow{S^{I h-1}} (h_1 p^{-1} f_1^{-1}, gh^{-1}, 1)^{I}$$

$$\xrightarrow{S_0^{I}} (1, h_1 p^{-1} f_1^{-1}, gh^{-1})^{I} = (1, h_1 p^{-1} f_1^{-1}, gh^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1, f_1 p h_1^{-1}, g h^{-1})^{II} = (1, f_1 p h_1^{-1}, g h^{-1})^{I}$$
$$\xrightarrow{S^{I h_1}} (h_1, f_1 p, g h_2^{-1})^{I}.$$

Therefore if $(f,g,h)^I$ is not related in G to a factorization with shorter length, $L(p) \geqslant L(h_2)$ and $L(f_2)$.

However we cannot place a bound on $L(t)$ until we know the maximum permitted length of $f_1^{-1}h_1$.

Suppose $f_1^{-1}h_1$ is not a piece. Then by definition of a piece $f_1^{-1}hf^{-1}h_1p^{-1} \equiv f_1^{-1}h_1p^{-1}f_1^{-1}hf_2^{-1} \equiv r \in R$.
This implies that $f_1^{-1}h_1p^{-1}$ and $f_1^{-1}hf_2^{-1}$ commute. But commuting elements in a free group are power of a common element, so that r is a proper power, and (u,v) is not root-closed (rel R).

If r is a power of $f_1^{-1}hf_2^{-1}$, then as $L(t) > 3L(p)$, $p^{-1} \equiv h_2f_2^{-1}$, so that $(f^{-1}h)^2 \in R$, and therefore $(f^{-1}h)^2 \in \hat{S}$. From the diagram it can be seen that $f^{-1}hf^{-1}h$ is an F^*-subword of W^*; and therefore

$W^* = Uf^{-1}hf^{-1}hV$, where U,V are words in $(u\ v)$

$\quad \equiv_{\langle\hat{S}\rangle} UV \equiv W_1 = W_1^*$

and UV is a word in (u,v), $L(W_1^*) < L(W_1)$,

so that {i} occurs.

If on the other hand, r is not a power of $f_1^{-1}hf_2^{-1}$, then $\alpha > (\alpha,\gamma)$, where $f^{-1}h \equiv x^{\alpha}$, $r \sim x^{\gamma}$.
Thus $f^{-1} \equiv d(cd)^{\mu}$, $h \equiv (cd)^{\nu}c$, where $dc = x^{(\alpha,\gamma)}$ and $\mu+\nu+1 = \alpha$. Therefore we have the following transformations:

$$(f,g,h)^I \xrightarrow{S_0^I} (f,h,g)^I \xrightarrow{S_G^I} (d^{-1},c,g)^I$$

But $\alpha > (\alpha,\gamma) \geqslant 1$, so that μ or $\nu > 1$ and $L(f,g,h) > L(d^{-1},c,g)$.
Thus $(f,g,h)^I$ is weakly related in G to the factorization $(d^{-1}c,g)^I$ which has shorter length.

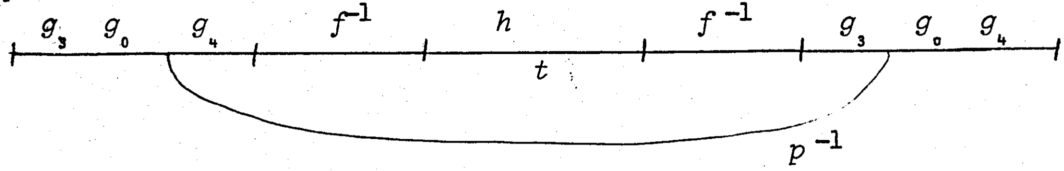
As a consequence if {i} and {ii} do not occur,

$$L(t) = 2L(f_1^{-1}h_1) + L(f_2) + L(h_2)$$
$$< (2+2\rho)\lambda L(r)$$

#

In this example, f_2 and h_2 are A-subwords of t, but we used a different argument to establish a bound on $L(f_1^{-1}h_1)$. We call this Method-C, and say that $f_1^{-1}h_1$ is a C-subword of t.

Example 4



$g \equiv g_3g_0g_4$, $t \equiv g_4f^{-1}hf^{-1}g_3$, $g_4 \not\equiv 1$.

Then $g_4 \equiv pg_3^{-1}fh^{-1}f$, $g_3 \equiv fh^{-1}fg_4^{-1}p$, $h \equiv fg_4^{-1}pg_3^{-1}f$

Using the first expression (for g_4) we have

$$(f,g,h)^{I} \xrightarrow{S^{I}_{G}} (f, g_3g_0pg_3^{-1}fh^{-1}f, h)^{I}$$

$$\xrightarrow{S^{If-1}} (1, g_3g_0pg_3^{-1}fh^{-1}, hf^{-1})^{I}$$

$$= (1, g_3g_0pg_3^{-1}fh^{-1}, hf^{-1})^{II} \xrightarrow{S^{II}_{0}} (1, g_3g_0pg_3^{-1}fh^{-1}, fh^{-1})^{II}$$

$$= (1, g_3g_0pg_3^{-1}fh^{-1}, fh^{-1})^{I} \xrightarrow{S^{Ihf-1}} (hf^{-1}, g_3g_0pg_3^{-1}, 1)^{I}$$

$$\xrightarrow{S^{I}_{0}} (1, g_3g_0pg_3^{-1}, hf^{-1})^{I} = (1, g_3g_0pg_3^{-1}, hf^{-1})^{II}$$

$$\xrightarrow{g_3^{-1}S^{II}} (g_3^{-1}, g_0p, hf^{-1})^{II}$$

Note that we finish here with a different Type of factorization to the Type with which we started. Using the expression for g_3, a similar set of transformations will map $(f,g,h)^{I}$ to $(g_4^{-1}, pg_0, hf^{-1})^{II}$

Finally, using the expression for h, we have

$$(f,g,h)^{I} \xrightarrow{S_G^I} (f,g,fg_4^{-1}pg_3^{-1}f)^{I}$$

$$\xrightarrow{S^{If-1}} (1,gf^{-1},fg_4^{-1}pg_3^{-1})^{I} = (1,gf^{-1},fg_4^{-1}pg_3^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1,gf^{-1},g_3p^{-1}g_4f^{-1})^{II} = (1,gf^{-1},g_3p^{-1}g_4f^{-1})^{I}$$

$$\xrightarrow{g_3{}^{-1}S^{Ifg_4-1}} (g_3^{-1}fg_4^{-1},g_0,p^{-1})^{I}.$$

Thus either $(f,g,h)^{I}$ is related to a factorization with shorter length or $L(p) \geqslant L(g_4)$, $L(g_3)$ or $L(h)$.

Suppose f^{-1} is not a piece, then by definition of a piece $f^{-1}hf^{-1}g_3p^{-1}g_4 \neq f^{-1}g_3p^{-1}g_4f^{-1}h \in R$.

It can be shown using Method C that in this case (u,v) is not root-closed (rel R), and {i} or {ii} must occur. However in this case there is a simpler argument, that provides a stronger result. For by the above identity, g_4 and $f^{-1}h$ have a common terminal subword which is non-trivial provided g_4 and $f^{-1}h$ are non-trivial. However we assumed that g_4 and h are non-trivial, so that $gh^{-1}f$, and therefore $(f,g,h)^{I}$ is not reduced, a contradiction. Thus f^{-1} must be a piece.

Therefore if $(f,g,h)^{I}$ is not related in G to a smaller factorization

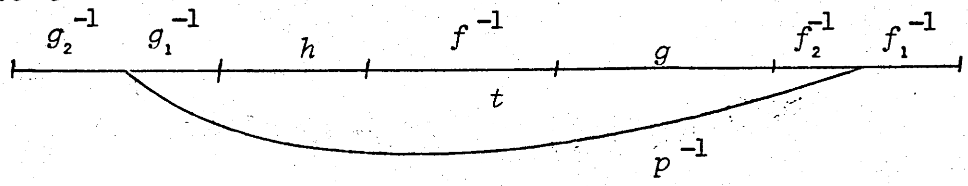
$$L(t) = L(g_4) + 2L(f) + L(h) + L(g_3)$$

$$< (2+3\rho)\ L(r).$$

#

The argument used by this example to show that f^{-1} must be a piece is called Method D, and f^{-1} is called a D-subword of t.

Example 5



$g \equiv g_1 g_2$, $t \equiv g_1^{-1} h f^{-1} \; g f_2^{-1}$, $f \equiv f_1 f_2$

Then $f =_G g f_2^{-1} p^{-1} g_1^{-1} h$, $g =_G f h^{-1} g_1 p f_2$, $h =_G g_1 p f_2 g^{-1} f$,

and using the first of these we have

$$(f, g, h)^{I} \xrightarrow{S_G^{I}} (g f_2^{-1} p^{-1} g_1^{-1} h, g, h)^{I} \xrightarrow{{}^{g^{-1}}S^{I h^{-1} g_1}} (f_2^{-1} p^{-1}, h^{-1} g_1, g_2^{-1})^{I}$$

Similarly using the expression for g, we have

$$(f, g, h)^{I} \xrightarrow{S_G^{I}} (f, f h^{-1} g_1 p f_2, h)^{I}$$

$$\xrightarrow{{}^{f^{-1}}S^{I}} (1, h^{-1} g_1 p f_2, f^{-1} h) = (1, h^{-1} g_1 p f_2, f^{-1} h)^{II}$$

$$\xrightarrow{S_0^{II}} (1, h^{-1} g_1 p f_2, h^{-1} f)^{II} = (1, h^{-1} g_1 p f_2, h^{-1} f)^{I}$$

$$\xrightarrow{{}^{h}S^{I f_2^{-1}}} (h f_2^{-1}, g_1 p, f_1)^{I}.$$

Similarly, using the expression for h, we have

$$(f, g, h)^{I} \xrightarrow{S_G^{I}} (f, g, g_1 p f_2 g^{-1} f)^{I}$$

$$\xrightarrow{S^{I f^{-1}}} (1, g f^{-1}, g_1 p f_2 g^{-1})^{I} = (1, g f^{-1}, g_1 p f_2 g^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1, f g^{-1}, g_1 p f_2 g^{-1})^{II} = (1, f g^{-1}, g_1 p f_2 g^{-1})^{I}$$

$$\xrightarrow{{}^{g_1^{-1}}S^{I g f_2^{-1}}} (g_2 f_2^{-1}, g_1^{-1} f_1, p)^{I}$$

Thus either $(f, g, h)^{I}$ is related to a smaller factorization, (and therefore {ii} occurs), or $L(p) \geqslant L(g_2)$, $L(f_1)$ or $L(h)$.

Now g_2 and g_2^{-1} are both subwords of r, and therefore g_2 is a piece.

Suppose f_2 is not a piece, then

$f_2^{-1} f_1^{-1} g f_2 p g_1^{-1} h \equiv f_2^{-1} p g_1^{-1} h f^{-1} g$,

which is not possible since g, h are non-trivial, and

gh^{-1} is reduced.

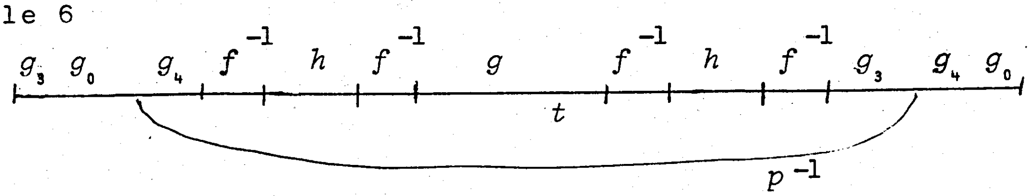
Therefore either $(f,g,h)^{I}$ is related to a shorter factorization, or

$$L(t) = 2L(g_1) + L(h) + 2\,(f_2) + L(f_1) + L(g_2)$$
$$< (4+3\rho)\lambda L(r)$$

#

In this example h, f_1^{-1}, g_2^{-1} are A-subwords of t, g_2 is a B-subword of t, and f_2 is a D-subword of t.

Example 6



$g \equiv g_3g_0g_4$, $t \equiv g_4f^{-1}hf^{-1}gf^{-1}hf^{-1}g_3$.

Then $g =_G fh^{-1}fg_4^{-1}pg_3^{-1}fh^{-1}f$, and using this we have

$$(f,g,h)^{I} \xrightarrow{S_G^{I}} (f,fh^{-1}fg_4^{-1}pg_3^{-1}fh^{-1}f,h)^{I}$$

$$\xrightarrow{S^{If-1}} (1,fh^{-1}fg_4^{-1}pg_3^{-1}fh^{-1},hf^{-1})^{I}$$

$$= (1,fh^{-1}fg_4^{-1}pg_3^{-1}fh^{-1},hf^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1,fh^{-1}fg_4^{-1}pg_3^{-1}fh^{-1},fh^{-1})^{II}$$

$$= (1,fh^{-1}fg_4^{-1}pg_3^{-1}fh^{-1},fh^{-1})^{I}$$

$$\xrightarrow{S^{Ihf-1}} (hf^{-1},fh^{-1}fg_4^{-1}pg_3^{-1},1)^{I}$$

$$\xrightarrow{S_0^{I}} (1,fh^{-1}fg_4^{-1}pg_3^{-1},hf^{-1})^{I} = (1,fh^{-1}fg_4^{-1}pg_3^{-1},hf^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1,fh^{-1}fg_4^{-1}pg_3^{-1},fh^{-1})^{II} = (1,fh^{-1}fg_4^{-1}pg_3^{-1},fh^{-1})^{I}$$

$$\xrightarrow{{}^{hf-1}S^{I}} (hf^{-1},fg_4^{-1}pg_3^{-1},1)^{I} = (1,fg_4^{-1}pg_3^{-1},hf^{-1})^{I}$$

$$= (1, fg_4^{-1}\rho g_3^{-1}, hf^{-1})^{II} \xrightarrow{S_0^{II}} (1, fg_4^{-1}\rho g_3^{-1}, fh^{-1})^{II}$$

$$= (1, fg_4^{-1}\rho g_3^{-1}, fh^{-1})^{I} \xrightarrow{f^{-1}S^{I}} (f^{-1}, g_4^{-1}\rho g_3^{-1}, h^{-1})^{I}$$

Thus unless $(f,g,h)^I$ is related in G to a factorization with shorter length, $L(\rho) \geq L(g_0)$.

If $g_4f^{-1}hf^{-1}g_3$ is not a piece, then $g_4f^{-1}hf^{-1}gf^{-1}hf^{-1}g_3\rho^{-1} \equiv g_4f^{-1}hf^{-1}g_3\rho^{-1}g_4f^{-1}hf^{-1}g_3g_0 \equiv r$ and $g_4f^{-1}hf^{-1}g_3g_0$ commutes with $g_4f^{-1}hf^{-1}g_3\rho^{-1}$. But commuting elements in a free group are powers of a common element, so that r is a proper power, and (u,v) is not root-closed (rel R).

If r is a power of $g_4f^{-1}hf^{-1}g_3g_0$, then as $L(t) > 3L(\rho)$, $\rho^{-1} \equiv g_0g_4f^{-1}hf^{-1}g_3$, so that $(f^{-1}hf^{-1}g)^2 \in R$ and therefore $(f^{-1}hf^{-1}g)^2 \in \hat{S}$. From the diagram it can be seen that $gf^{-1}hf^{-1}g$ is an F^* subword of W^*, and therefore $W^* \equiv Ugf^{-1}hf^{-1}gV$, where Uf, V are words in (u,v)

$$=_{\langle\hat{S}\rangle} Ufh^{-1}fV \equiv W_1(u,v) = W_1^*,$$

Where W_1^* is freely reduced, and $L(W_1^*) < L(W^*)$, so that {i} occurs.

If, on the other hand, r is not a power of $g_4f^{-1}hf^{-1}g_3g_0$, then $\alpha > (\alpha,\gamma)$ where $f^{-1}hf^{-1}g \equiv x^\alpha$, $r \sim x^\gamma$. Thus $f^{-1} \equiv d(cd)^\mu, hf^{-1}g \equiv (cd)^\nu c$, where $dc = x^{(\alpha,\gamma)}$ and $\mu+\gamma+1 = \alpha$. Now $L(g) \neq L(h)$, or gh^{-1}, and therefore $(f,g,h)^I$ are not reduced. Thus we can assume $L(g) > L(h)$, and so $L(g) > L(c)$. However we have:

$$(f,g,h)^{\mathrm{I}} \xrightarrow{S^{\mathrm{I}f-1}} (1,gf^{-1},hf^{-1})^{\mathrm{I}} = (1,gf^{-1},hf^{-1})^{\mathrm{II}}$$

$$\xrightarrow{S_0^{\mathrm{II}}} (1,gf^{-1},fh^{-1})^{\mathrm{II}} = (1,gf^{-1},fh^{-1})^{\mathrm{I}}$$

$$\xrightarrow{S^{\mathrm{I}h}} (h,gf^{-1}h,f)^{\mathrm{I}} \xrightarrow{S_0^{\mathrm{I}}} (f,gf^{-1}h,h)^{\mathrm{I}}$$

$$\xrightarrow{\overline{S}_G^{\mathrm{I}}} (d^{-1},c,h)^{\mathrm{I}}$$

But as $L(f) \geqslant L(d)$ and $L(g) > L(c)$, $L(d^{-1},c,h) < L(f,g,h)$. Thus $(f,g,h)^{\mathrm{I}}$ is weakly related in G to a shorter factorization.

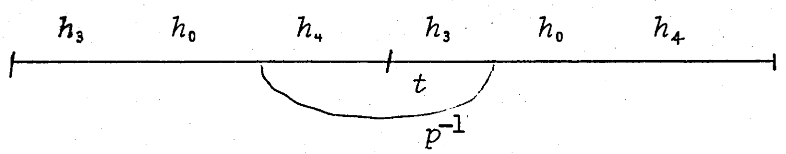
As a consequence if {i} and {ii} do not occur,
$L(t) = 2L(g_4 f^{-1} h f^{-1} g_3) + L(g_0) < (2+\rho)\lambda L(r)$.

#

These last examples, 5 and 6 are needed for Theorem 2.2. They illustrate the fact that although the method is essentially the same, because of the greater complexity of the expressions for t, the details are lengthier.

In the next 6 examples, the factorization is of Type II.

Example 7



$h \equiv h_3 h_0 h_4$, $t \equiv h_4 h_3$.
Then $h_3 =_G h_4^{-1} p$, $h_4 =_G p h_3^{-1}$. Using the expression for h_3, we have:

$$(f,g,h)^{\mathrm{II}} \xrightarrow{S_G^{\mathrm{II}}} (f,g,h_4^{-1} p\, h_0 h_4)^{\mathrm{II}}$$

$$\xrightarrow{S^{\mathrm{II}h_4^{-1}}} (fh_4^{-1},g,ph_0)^{\mathrm{II}}$$

In a similar way, using the expression for h_4, we have:

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,h_3h_0ph_3^{-1})^{II} \xrightarrow{S^{IIh_3}} (fh_3,g,h_0p)^{II}$$

Therefore $(f,g,h)^{II}$ is related to $(fh_4^{-1},g,ph_0)^{II}$ and $(fh_3,g,h_0p)^{II}$. But

$L(f,g,h) = L(fh_4^{-1},g,ph_0)-L(p)+L(h_3)$, and

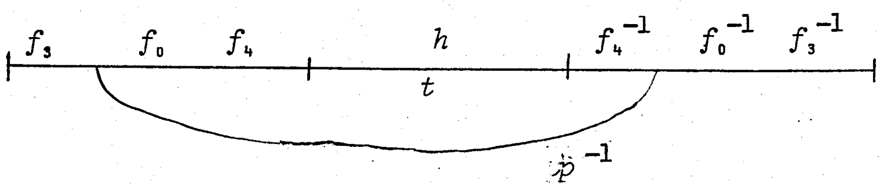
$L(f,g,h) = L(fh_3,g,h_0p)-L(p)+L(h_4)$.

Thus if $(f,g,h)^{II}$ is not related in G to a factorization with shorter length, $L(p)\geqslant L(h_3)$ and $L(h_4)$, and as a consequence $L(t) = L(h_3)+L(h_4)\leqslant 2L(p)<2\rho\lambda L(r)$.

#

The argument which is used to show that h_3 and h_4 are bounded in length by $\rho\lambda L(r)$ is called Method-A and h_3,h_4 are A-subwords of t.

Example 8



$f \equiv f_3f_0f_4$, $t \equiv f_0f_4hf_4^{-1}$.

Then $f_0f_4 =_G pf_4h^{-1}$, $h =_G f_4^{-1}f_0^{-1}pf_4$.

Using the expression for f_0f_4, we have:

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f_3pf_4h^{-1},g,h)^{II} \xrightarrow{S^{IIh}} (f_3pf_4,g,h)^{II}$$

and using the expression for h, we have:

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,f_4^{-1}f_0^{-1}pf_4)^{II} \xrightarrow{S^{IIf_4^{-1}f_0^{-1}}} (f_3,g,pf_0^{-1})^{II}$$

Thus it can be seen that if $(f,g,h)^{II}$ is not related to a shorter factorization, then

$L(p) \geq L(f_0)$ and $L(f_4, h)$.

As f_4 and f_4^{-1} are subwords of r, if they are not pieces $f_4 h f_4^{-1} p^{-1} f_0 \equiv f_4 h^{-1} f_4^{-1} f_0^{-1} p$ which is clearly not possible.

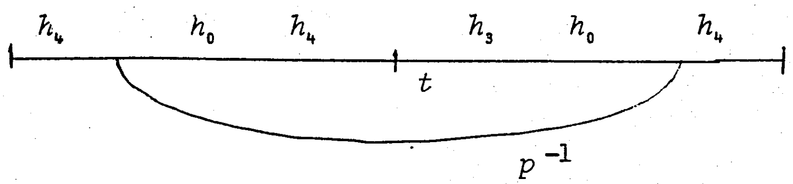
Therefore either $(f,g,h)^{II}$ is related in G to a shorter factorization or

$L(t) = L(f_0) + L(f_4 h) + L(f_4^{-1}) < (1+2\rho)\lambda L(r)$.

#

In this example f_0 and h are A-subwords of t, and f_4 is a B-subword of t.

Example 9



$h \equiv h_3 h_0 h_4$, $t \equiv h_0 h_4 h_3 h_0$.

Then $h_0 h_4 =_G p h_0^{-1} h_3^{-1}$, $h_3 h_0 =_G h_4^{-1} h_0^{-1} p$. Using the first of these we have:

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,h_3 p h_0^{-1} h_3^{-1})^{II} \xrightarrow{S^{II h_3}} (fh_3, g, p h_0^{-1})^{II}$$

and similarly using the expression for $h_3 h_0$ we have:

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,h_4^{-1} h_0^{-1} p h_4)^{II} \xrightarrow{S^{II h_4 -1}} (fh_4^{-1}, g, h_0^{-1} p)^{II}$$

Therefore if $(f,g,h)^{II}$ is not related in G to a shorter factorization, $L(p) \geq L(h_3)$ and $L(h_4)$.

Before we can place a bound on the $L(t)$, we need to determine the maximum permitted length of h_0. Suppose h_0 is not a piece, then

$h_0h_4h_3h_0p^{-1} \equiv h_0p^{-1}h_0h_4h_3 \equiv r$.

This implies that $h_0h_4h_3$ and h_0p^{-1} commute. But commuting elements in a free group are powers of a common element, so that r is a proper power, and (u,v) is not root-closed (rel R).

If r is a power of $h_0h_4h_3$ then as $L(t)>3L(p)$, $p^{-1} \equiv h_4h_3$ so that $h^2 \in R$ and therefore $h^2 \in \hat{S}$. From the diagram it can be seen that h^2 is an F^*-subword of W^*, and therefore

$W^* \equiv Uh^2V$ where U,V are words in (u,v)

$=_G UV \equiv W_1 = W_1{}^*$,

and UV is a word in (u,v), $L(W_1{}^*)<L(W^*)$, so that {i} occurs.

If on the other hand, r is not a power of $h_0h_4h_3$ then $\alpha>(\alpha,\gamma)$, where $h \equiv x^\alpha$, $r \sim x^\gamma$. Therefore we have the following transformation.

$$(f,g,h)^{II} \xrightarrow{\bar{S}_G^{II}} (f,g,x^{(\alpha,\gamma)})^{II},$$

But $\alpha>(\alpha,\gamma)$, so that $L(f,g,h)>L(f,g,x^{(\alpha,\gamma)})$. Thus $(f,g,h)^{II}$ is weakly related in G to the factorization $(f,g,x^{(\alpha,\gamma)})^{II}$, which has shorter length.

As a consequence if {i} and {ii} do not occur

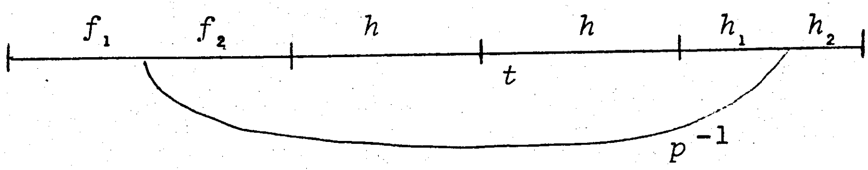
$L(t) = 2L(h_0)+L(h_3)+L(h_4)$.

$< (2+2\rho)\lambda L(r)$

#

In this example, h_3 and h_4 are A-subwords of t, and h_0 is a C-subword of t.

Example 10



$f \equiv f_1 f_2$, $h \equiv h_1 h_2$, $t \equiv f_2 hhh_1$, $f_2 \not\equiv 1$.

Then $f_2 =_G \rho h_1{}^{-1} h^{-2}$, and we have

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f_1 \rho h_1{}^{-1} h^{-2}, g, h)^{II} \xrightarrow{S^{II} h^2 h_1} (f_1 \rho, g, h_2 h_1)^{II}$$

Therefore unless $(f,g,h)^{II}$ is related in G to a factorization with shorter length, $L(\rho) \geqslant L(f_2)$.

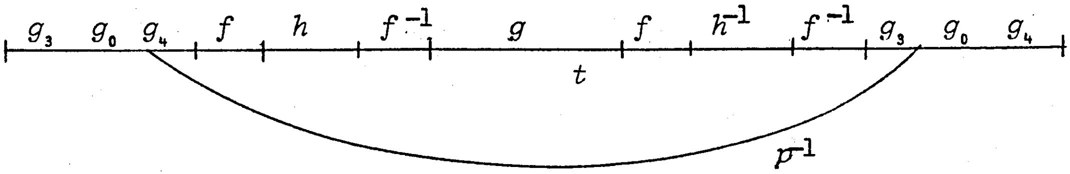
Suppose hh_1 is not a piece, then
$hh_1 h_2 h_1 \rho^{-1} f_2 \equiv hh_1 \rho^{-1} f_2 h \in R$.
Therefore as f_2 and h are non-trivial, the elements f_2 and h have a common terminal subword, and hf^{-1}, and therefore $(f,g,h)^{II}$, is not reduced.

Therefore if $(f,g,h)^{II}$ is not related in G to a shorter factorization,
$L(t) = L(f_2) + L(hh_1) + L(h) < (2+\rho)\lambda L(r)$.

#

In this example f_2 is an A-subword of t, and h, hh_1 are both referred to as D-subwords of t.

Example 11



$g \equiv g_3 g_0 g_4$, $t \equiv g_4 fhf^{-1}\ gfh^{-1} f^{-1} g_3$.

$g =_G fh^{-1}f^{-1}g_4^{-1}pg_3^{-1}fhf^{-1}$, and we have:

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, fh^{-1}f^{-1}g_4^{-1}pg_3^{-1}fhf^{-1}, h)^{II}$$

$$\xrightarrow{fhf^{-1}S^{II}h^{-1}} (f, g_4^{-1}pg_3^{-1}, h)^{II}$$

Therefore unless $(f,g,h)^{II}$ is related to a factorization with shorter length, $L(p) \geq L(g_0)$.

Now fh and $h^{-1}f^{-1}$ are both subwords of r, and therefore by Method-B, fh is a piece. Similarly fh^{-1} is a piece.

Suppose g_3 is not a piece, then

$gfh^{-1}f^{-1}g_3p^{-1}g_4fhf^{-1} \equiv g_3p^{-1}g_4fhf^{-1}gfh^{-1}f^{-1}$

which is not possible or $h^2 = 1$ and therefore $h = 1$.

Similarly g_4 is a piece.

Thus either $(f,g,h)^{II}$ is related in G to a shorter factorization, or

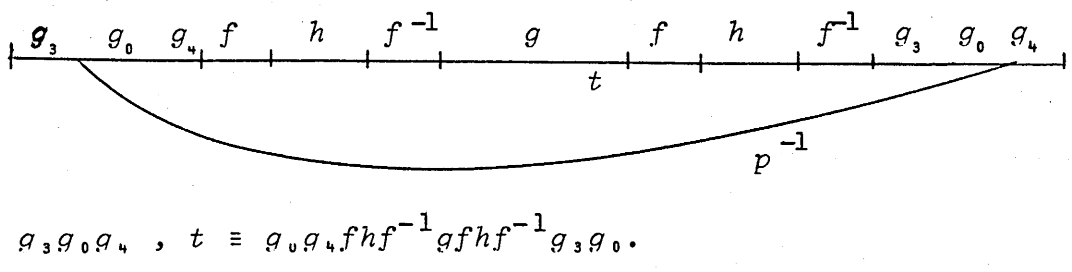
$L(t) = 2L(g_3)+2L(g_4)+2L(fh)+2L(f)+L(g_0)$

$\quad < (8+\rho)\lambda L(r)$

#

In this example g_0 is an A-subword of t, fh, fh^{-1} are B-subwords of t, and g_3, g_4 are D-subwords of t.

Example 12



$g \equiv g_3g_0g_4$, $t \equiv g_0g_4fhf^{-1}gfhf^{-1}g_3g_0$.

Suppose $g_0g_4fhf^{-1}g_3g_0$ is not a piece, then

$$r \equiv g_0 g_4 f h f^{-1} g f h f^{-1} g_3 g_0 \rho^{-1}$$
$$\equiv g_0 g_4 f h f^{-1} g_3 g_0 \rho^{-1} \; g_0 g_4 f h f^{-1} g_3$$

This implies that $g_0 g_4 f h f^{-1} g_3 g_0 \rho^{-1}$ and $g_0 g_4 f h f^{-1} g_3$ commute. But commuting elements, in a free group, are powers of a common element, so that r is a proper power, and (u,v) is not root-closed (rel R).

If r is a power of $g_0 g_4 f h f^{-1} g_3$, then as $L(t) > 3L(\rho)$, $\rho^{-1} \equiv g_4 f h f^{-1} g_3$, so that $(hf^{-1}gf)^3 \in R$ and therefore $(hf^{-1}gf)^3 \in \hat{S}$. From the diagram it can be seen that $gfhf^{-1}gfhf^{-1}g$ is an F*-subword of W*, and therefore $W^* \equiv Ugfhf^{-1}gfhf^{-1}gV$, (where $Uf, f^{-1}V$ are words in (u,v))

$$\equiv_{<\hat{S}>} Ufh^{-1}f^{-1}V = W_1(u,v) = W_1^*$$

and $L(W_1^*) < L(W^*)$, so that {i} occurs.

If, on the other hand r is not a power of $g_0 g_4 f h f^{-1} g_3$, then $\alpha > (\alpha,\gamma)$, where $f^{-1}gfh \equiv x^{\alpha}$, $r \sim x^{\gamma}$. Now either $L(f^{-1}g)$ or $L(fh) > L(x^{(\alpha,\gamma)})$. For if $L(f^{-1}g) < L(x^{(\alpha,\gamma)})$ then $L(fh) > L(x^{(\alpha,\gamma)})$ and vice versa.
However if $L(f^{-1}g) = L(fh) = L(x^{(\alpha,\gamma)})$, then $f^{-1}g \equiv x^{(\alpha,\gamma)}$, $fh \equiv x^{(\alpha,\gamma)}$, so that $f \equiv 1$ and $g^{-1}h$ is not reduced, and therefore the factorization $(f,g,h)^{II}$ is not reduced.
If $L(f^{-1}g) > L(x^{(\alpha,\gamma)})$, then consider the following sequence of factorization transformations:

$$(f,g,h)^{II} \xrightarrow{S_0^{II}} (f,g,h)^{II} \xrightarrow{f^{-1}S^{II}} (1,f^{-1}gf,h^{-1})^{II}$$
$$= (1,f^{-1}gf,h^{-1})^{I} \xrightarrow{S^{Ih}} (h,f^{-1}gfh,1)^{I}$$
$$\xrightarrow{S_0^{I}} (1,f^{-1}gfh,h)^{I} = (1,f^{-1}gfh,h)^{II}$$
$$\xrightarrow{\bar{S}_G^{II}} (1,x^{(\alpha,\gamma)},h)^{II}.$$

Then the factorization $(1,x^{(\alpha,\gamma)},h)^{II}$ is weakly related to $(f,g,h)^{II}$ and has shorter length.

Similarly if $L(fh)>L(x^{(\alpha,\gamma)})$

As a consequence, if {i} or {ii} do not occur,

$$L(t) = L(g_0g_4fhf^{-1}g_3g_0)+L(g_4fhf^{-1}g_3g_0)$$
$$< 2\lambda L(r)$$

#

In this example $g_0g_4fhf^{-1}g_3g_0$ and $g_4fhf^{-1}g_3g_0$ are both C-subwords of t.

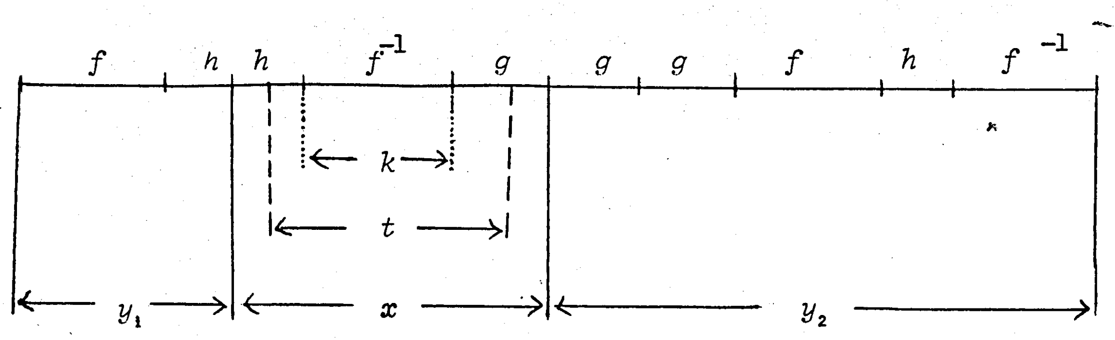
SECTION 3. DETAILED SURVEY OF CASES

INTRODUCTION:

In this section we will list all the cases described in Section 2.2. The method of proof we shall use, depends, not only on the size of t, but also on its position in W^*. Cases are defined so as to take account of this.

In order to define the position of t in W^*, and so that its position and size (relative to the f-, g-, h-subwords of W^*) are limited for a particular set of cases, we work with *F^*-subwords of W^**, which are products of F-subwords of W^*. For example if $W^* = fh^{\beta_1}f^{-1}g^{\alpha}fh^{\beta_2}f^{-1}$; $\alpha, \beta_1, \beta_2 \neq 0$, then $hf^{-1}g, fh, g^{\alpha}fh$ are F^*-subwords of W^*. If k is an F-subword of W^*, then $W^* \equiv UkV$, where U,V are F^*-subwords of W^*. If t is a subword of W^*, and $t \equiv t_1kt_2$; $W^* \equiv U_1t_1kt_2V_2$; U_1t_1, t_2V_2 F^*subwords of W^*, then we say that k is an *F-subword of t* (relative to W). If k is an F-subword of t or t^{-1}, then we say that k is an *F-subword of tvt^{-1}*.

For example, if $W^* = fh^2f^{-1}g^3fhf^{-1}$ then we can illustrate these definitions with the following diagram.



Then y_1, x, y_2 are F^*-subwords of W^*, $k \equiv f^{-1}$ is an F-subword of W^* and of t. However g and h are F-subwords of W^* but not of t.

Let E be an F^*-subword of W^*, and let Q be a sequence of free factorization transformations of types S_0^{I} and S_0^{II} which act on $(f,g,h)^J$. Then Q acts on the F-subwords of E. Let $EQ \equiv E_1$, then we say that E, E_1 are *similar*. By showing that the factorizations $(f,g,h)^J$ and $(f',g',h')^{J'}$ are related, (that is

$$(f,g,h)^J\ T = (f',g',h')^{J'}$$

where T is a finite sequence of factorization transformations of types $S_0{}^J$, ${}^kS^{J\ell}$, and S_G), then the factorizations $(f_1,g_1,h_1)^{J_1}$ and $(f',g',h')^{J'}$ are also related. Similarly if $(f,g,h)^J$ and $(f',g',h')^{J'}$ are weakly related, then $(f_1,g_1,h_1)^{J_1}$ and $(f',g',h')^{J'}$ are also weakly related.
In both cases, as $L(f_1,g_1,h_1) = L(f,g,h)$ if $L(f',g',h')<L(f,g,h)$, then $L(f',g',h')<L(f_1,g_1,h_1)$.
Therefore by proving the result for those cases which arise when considering t as a subword of E, we have also proved the result for all similar cases, obtained by considering t as a subword of $E_1{}^{\pm 1}$.

In most cases, the Types of factorization J and J' are equal. However, in certain cases, the Type of factorization can change. These cases are marked with an asterisk. This is necessary, because in the next chapter, we will make use of some of these results and we need to make certain that in these cases the Type of factorization is unaltered.

Where, in the details of a case, the Type of factorization changes, then instead of writing

$$\ldots\ldots \xrightarrow{S^{J}} (1,x,y)^{J} = (1,x,y)^{J'} \xrightarrow{S^{J'}} \ldots\ldots$$

for certain elementary factorization transformations S^{J} and $S^{J'}$, we shorten this as follows:

$$\ldots\ldots \xrightarrow{S^{J}} (1,x,y)^{J'} \xrightarrow{S^{J'}} \ldots\ldots$$

Note that we can only do this where the factorization is trivial.

The transformations S_F^{I}, S_F^{II} (that is cancellation within the elements of a factorization) will be used without specific reference to them.

LIST OF CASES FOR THEOREM 2.1

The cases are considered in two sections, according to the Type of factorization of (f,g,h).

A. *Let $(f,g,h)^{J}$ be a factorization of Type J = I.*

We assume that no cycle of a subword of t contains the subword $g^{\pm 1}$, for the cases that arise by assuming that no cycle of t contains the subwords $f^{\pm 1}$ or $h^{\pm 1}$ are similar. Therefore t or t^{-1} is a subword of either

$g_2(f^{-1}h)^{\alpha}f^{-1}g_1, \alpha \geqslant 0$, or $g(f^{-1}h)^{\alpha}g^{-1}, \alpha > 0$

where $g \equiv g_1g_2$. In general let

$f \equiv f_1f_2 \equiv f_3f_0f_4;\ f_2, f_3, f_4 \not\equiv 1$

$g \equiv g_1g_2 \equiv g_3g_0g_4;\ g_2, g_3, g_4 \not\equiv 1$

$h \equiv h_1h_2 \equiv h_3h_0h_4;\ h_2, h_3, h_4 \not\equiv 1$

Suppose, first, that f, g, and h are not F-subwords of tut^{-1}.
If t is a subword of $f^{-1}h$, either

1. t is a subword of f^{-1} or h, or
2. $t \subset f^{-1}h$, $t \equiv f_1^{-1}h_1$.

If $t^{\pm 1} \subset gf^{-1}$, $f^{-1}g, hf^{-1}$ or hg^{-1}, then the cases are similar.

Secondly let f^{-1} be an F-subword of t, but h and g not F-subwords of tut^{-1}. Then we have the following three cases

3. $t \subset hf^{-1}h$, $t \equiv h_4 f^{-1} h_3$
4. as for 3, but $t \equiv h_0 h_4 f^{-1} h_3 h_0$
5. $t \subset hf^{-1}g$, $t \equiv h_2 f^{-1} g_1$

If $t \subset gf^{-1}g$, then the cases are similar to 3 and 4.
If $f, g^{\pm 1}$ or $h^{\pm 1}$ is the only F-subword of t, then the cases are similar.

Lastly, let h and f^{-1} be F-subwords of t, but g not an F-subword of t.

(i) Let t begin in f^{-1} and end in h,
then $t \subset f^{-1}(hf^{-1})^{\alpha_1}h$, $\alpha \geqslant 1$ and we have the following cases:

6. $\alpha_1 = 1$, $t \equiv f_i^{-1}hf^{-1}h_1$
7. $\alpha_1 > 1$, $t \equiv f_1^{-1}(hf^{-1})^{\alpha_1}h_1$

If t begins in h and ends f^{-1}, then the cases are similar.

(ii) Let t begin in f^{-1} and end in f^{-1},
then $t \subset f^{-1}(hf^{-1})^{\alpha_1}hf^{-1}$, $\alpha_1 \geqslant 1$ and we have the following cases:

8. $\alpha_1 = 1$, $t \equiv f_3^{-1}hf^{-1}hf_4^{-1}$
9. $\alpha_1 > 1$, $t \equiv f_3^{-1}(hf^{-1})^{\alpha_1}hf_4^{-1}$
10. $\alpha_1 \geqslant 1$, $t \equiv f_0^{-1} f_3^{-1} (hf^{-1})^{\alpha_1}hf_4^{-1} f_0^{-1}$

If t begins in h and ends in h, then the cases are similar.

(iii) Let t begin in g and end in h,

then $t \subset g(f^{-1}h)^{\alpha_1}f^{-1}h$, $\alpha_1 \geq 1$ and we have two cases:

11. $\alpha_1 = 1$, $t \equiv g_2 f^{-1}hf^{-1}h_1$

12. $\alpha_1 > 1$, $t \equiv g_2(f^{-1}h)^{\alpha_1}f^{-1}h_1$

If t begins in h and ends in g, or begins in f^{-1} and ends in g^{-1}, then the cases are similar.

(iv) Let t begin in g and end in f^{-1},

then $t \subset g(f^{-1}h)^{\alpha_1}f^{-1}$, $\alpha_1 \geq 1$ and we have two cases:

13. $\alpha_1 = 1$, $t \equiv g_2 f^{-1}hf_2^{-1}$

14. $\alpha_1 > 1$, $t \equiv g_2(f^{-1}h)^{\alpha_1}f_2^{-1}$.

If t begins f^{-1} and ends in g, or begins in h and ends in g^{-1}, then the cases are similar.

(v) Let t begin in g and end in g^{-1},

then $t \subset g(f^{-1}h)^{\alpha}g^{-1}$ $\alpha \geq 1$ and there are two cases:

15. $\alpha = 1$, $t \equiv g_0 g_4 f^{-1}hg_4^{-1}$

16. $\alpha > 1$, $t \equiv g_0 g_4(f^{-1}h)^{\alpha}g_4^{-1}$.

If t begins with g_4 and ends with $g_4^{-1}g_0^{-1}$, then the cases are similar.

(vi) Let t begin in g and end in g,

then $t \subset g(f^{-1}h)^{\alpha}f^{-1}g$, $\alpha \geq 1$ and as no cycle of a subword of t contains the subword g we have the following two cases only:

17. $\alpha = 1$, $t \equiv g_4 f^{-1}hf^{-1}g_3$, $g_4 \not\equiv 1$

18. $\alpha > 1$, $t \equiv g_4(f^{-1}h)^{\alpha}f^{-1}g_3$, $g_4 \not\equiv 1$.

B. *Let* $(f,g,h)^J$ *be a factorization of Type* J = II.

We assume that no cycle of a subword of t contains the subword $f^{\pm 1}$ or $g^{\pm 1}$, for the cases that arise by assuming that no cycle of t contains the subword $h^{\pm 1}$ are similar.

Therefore t (or t^{-1}) is a subword of either

g_2g_1, $g_2fh^{\alpha}f^{-1}g_1$, or $gfh^{\alpha}f^{-1}g^{-1}$,

where $\alpha>0, g \equiv g_1g_2$.

In general let

$f \equiv f_1f_2 \equiv f_3f_0f_4$; $f_2, f_3, f_4 \not\equiv 1$

$g \equiv g_1g_2 \equiv g_3g_0g_4$; $g_2, g_3, g_4 \not\equiv 1$

$h \equiv h_1h_2 \equiv h_3h_0h_4$; $h_2, h_3, h_4 \not\equiv 1$.

Suppose, first, that f, g and h are not F-subwords of $t \cup t^{-1}$. If t is a subword of fh or hh we have one of the following cases:

1. t is a subword of f or h
2. $t \subset fh$, $t \equiv f_2h_1$
3. $t \subset hh$, $t \equiv h_4h_3$
4. As for 3, but $t \equiv h_0h_4h_3h_0$.

If $t^{\pm 1} \subset hf^{-1}, gf, f^{-1}g$ or gg then the cases are similar.

Secondly let h be an F-subword of t, but f not an F-subword of $t \cup t^{-1}$.

(i) Let t begin in h and end in h,

then $t \subset h^{\alpha}$, $\alpha>2$ and we have the following cases.

5. $t \subset hhh$, $t \equiv h_4h\,h_3$
6. $t \subset hh^{\alpha_1}h, \alpha_1>1, t \equiv h_4h^{\alpha_1}h_3$
7. $t \subset hh^{\alpha_1}h, \alpha_1\geq 1, t \equiv h_0h_4h^{\alpha_1}h_3h_0$

(ii) Let t begin in f and end in h,

then $t \subset fh^{\alpha_1}h$, $\alpha_1\geq 1$, and we have two cases:

8. $\alpha_1 = 1$, $t \equiv f_2hh_1$
9. $\alpha_1 > 1$, $t \equiv f_2h^{\alpha_1}h_1$

If t begins in h and ends in f^{-1}, then the cases are similar.

(iii) Let t begin in f and end in f^{-1},

then $t \subset fh^{\alpha}f^{-1}$, $\alpha \geq 1$, and we have two cases:

10. $\alpha = 1$, $t \equiv f_0 f_4 h f_4^{-1}$

11. $\alpha > 1$, $t \equiv f_0 f_4 h^{\alpha} f_4^{-1}$.

If t begins with f_4 and ends with $f_4^{-1} f_0^{-1}$, then the cases are similar.

Suppose, f is an F-subword of t, but g and h not F-subwords of tvt^{-1}. Then there is only one case:

12. $t \subset gfh$, $t \equiv g_2 f h_1$

If f^{-1} is an F-subword of t, but g and h not F-subwords of tvt^{-1}, then the cases are similar.

Lastly suppose f and h are F-subwords of t, but g is not an F-subword of tvt^{-1}. Then t must begin or end in g or g^{-1}.

(i) Let t begin in g and end in h,

then $t \subset gfh^{\alpha_1}h$, $\alpha_1 > 0$, and we have the following cases:

13. $\alpha_1 = 1$, $t \equiv g_2 f h h_1$

14 $\alpha_1 > 1$, $t \equiv g_2 f h^{\alpha_1} h_1$

If t begins in h and ends in g or g^{-1}, then the cases are similar.

(ii) Let t begin in g and end in f^{-1},

then $t \subset gfh^{\alpha}f^{-1}$, $\alpha \geq 1$ and we have the following two cases:

15. $\alpha = 1$, $t \equiv g_2 f h f_2^{-1}$

16. $\alpha > 1$, $t \equiv g_2 f h^{\alpha} f_2^{-1}$.

If t begins in f and ends in $g^{\pm 1}$,

then the cases are similar.

(iii) Let t begin in g and end in g^{-1},

then $t \subset gfh^{\alpha}f^{-1}g^{-1}$, $\alpha \geq 1$ and we have the following two cases:

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17. $\alpha = 1,$ $t \equiv g_0 g_4 f h f^{-1} g_4^{-1}$

18. $\alpha > 1$ $t \equiv g_0 g_4 f h^{\alpha} f^{-1} g_4^{-1}.$

If t begins with g_4 and ends with $g_4^{-1} g_0^{-1}$, then the cases are similar.

(iv) Let t begin in g and end in g, then $t \subset g f h^{\alpha} f^{-1} g$ $\alpha \geqslant 1$ and as no cycle of a subword of t contains the subword g, we have the following two cases only:

19. $\alpha = 1,$ $t \equiv g_4 f h f^{-1} g_3$

20. $\alpha > 1$ $t \equiv g_4 f h^{\alpha} f^{-1} g_3.$

THE PROOF OF THE CASES REQUIRED FOR THEOREM 2.1

A. *Let $(f,g,h)^J$ be a factorization with $J = I$*

1. $t \subset f^{-1}$ or h. If $h \equiv h_1 t h_4$ then $(f,g,h)^I$ is related to $(f,g,h_1 \rho h_4)^I$, which has shorter length, unless $L(t) < 1/2L(r)$. If $t \subset f^{-1}$, then the case is similar.

2. $t \equiv f_1^{-1} h_1 =_G \rho$. Then this case is described in Example 1 of Section 2.2. Thus, unless {ii} occurs $L(t) = L(f_1, h_1) < 2L(\rho) < 2\rho\lambda L(r)$.

3*. $t \equiv h_4 f^{-1} h_3 =_G \rho$. Then

$$(f,g,h)^I \xrightarrow{S_G^I} (h_3 \rho^{-1} h_4, g, h)^I \xrightarrow{h_3^{-1} S^I h_4^{-1}} (\rho, h_3^{-1} g h_4^{-1}, h_0)^I.$$

But unless $L(f) \leqslant L(\rho)$, $L(f,g,h) > L(\rho, h_3^{-1} g h_4^{-1}, h_0)$.

In addition,

$$(f,g,h)^I \xrightarrow{S_G^I} (f, g, h_3 h_0 \rho h_3^{-1} f)^I \xrightarrow{S^{I f^{-1}}} (1, g f^{-1}, h_3 h_0 \rho h_3^{-1})^{II}$$

$$\xrightarrow{S^{II h_3}} (h_3, g f^{-1}, h_0 \rho)^{II}.$$

(Note that the Type of factorization changes). But unless $L(h_4) \leqslant L(p)$, $L(f,g,h) > L(h_3, gf^{-1}, h_0 p)$. Similarly for h_3.
Thus, unless {ii} occurs,
$L(t) = L(h_4, f^{-1}, h_3) \leqslant 3L(p) < 3\rho\lambda L(r)$.

4*. $t \equiv h_0 h_4 f^{-1} h_3 h_0 =_G p$. Then

$$(f,g,h)^{I} \xrightarrow{S^{I}_{G}} (h_3 h_0 p^{-1} h_0 h_4, g, h)^{I}$$
$$\xrightarrow{h_0{-1}h_3{-1}S^{I}h_4{-1}} (p^{-1}h_0, h_0{}^{-1}h_3{}^{-1}gh_4{}^{-1}, 1)^{I}$$
$$\xrightarrow{S^{I}_{0}} (1, p^{-1}h_0, h_0{}^{-1}h_3{}^{-1}gh_4{}^{-1})^{II}$$
$$\xrightarrow{S^{II}_{0}} (1, p^{-1}h_0, h_4 g^{-1} h_3 h_0)^{I}$$
$$\xrightarrow{S^{I}h_0{-1}} (h_0{}^{-1}, p^{-1}, h_4 g^{-1} h_3)^{I}.$$

But unless $L(f) \leqslant L(p)$, $L(f,g,h) > L(h_0{}^{-1}, p^{-1}, h_4 g^{-1} h_3)$
In addition

$$(f,g,h)^{I} \xrightarrow{S^{I}_{G}} (f, g, h_3 p h_0{}^{-1} h_3{}^{-1} f)^{I}$$
$$\xrightarrow{S^{I}f{-1}} (1, gf^{-1}, h_3 p h_0{}^{-1} h_3{}^{-1})^{II}$$
$$\xrightarrow{S^{II}h_3} (h_3, gf^{-1}, p h_0{}^{-1})^{II}.$$

(Note that the Type of factorization changes). But unless $L(h_4) \leqslant L(p)$, $L(f,g,h) > L(h_3, gf^{-1}, ph_0{}^{-1})$. Similarly for h_3.
Suppose f_0 is not a piece. Then
$r \equiv h_0 h_4 f^{-1} h_3 h_0 p^{-1} \equiv h_0 p^{-1} h_0 h_4 f^{-1} h_3$
so that $h_0 p^{-1}$ and $h_0 h_4 f^{-1} h_3$ commute. Thus r and $h_0 h_4 f^{-1} h_3$ are powers of a common element, and r is a proper power.
Thus (u,v) is not root-closed (rel R).
If r is a power of $h_0 h_4 f^{-1} h_3$ then as $L(t) > 3L(p)$,
$p^{-1} \equiv h_4 f^{-1} h_3$. Thus $(f^{-1}h)^2 \in R$, which implies $(f^{-1}h)^2 \in \hat{S}$.

But $hf^{-1}h$ is an F^*-subword of W^*, so that

$W^* \equiv Uhf^{-1}hV$ where U and hV are words in (u,v)

$\equiv_{<\hat{S}>} UfV = W_1{}^*$.

However $UfV = Ufh^{-1}hV$ is a word in (u,v), and $L(W_1{}^*) < L(W^*)$.

Therefore {i} must occur.

If, on the other hand, r is not a power of $h_0h_4f^{-1}h_3$, then $\alpha>(\alpha,\gamma)$, where $f^{-1}h \equiv x^{\alpha}$, $r \sim x^{\gamma}$. Thus $f^{-1} \equiv d(cd)^{\mu}$, $h \equiv (cd)^{\nu}c$, where $dc \equiv x^{(\alpha,\gamma)}$, and $\mu+\nu+1 = \alpha$. Therefore we have the transformation

$$(f,g,h)^{I} \xrightarrow{S_0^I} (f,h,g)^{I} \xrightarrow{\bar{S}_G^I} (d^{-1},c,g)^{I},$$

But $\alpha>(\alpha,\gamma) \geqslant 1$, so that μ or $\nu > 1$, and $L(f,g,h) > L(d^{-1},c,g)$. Thus $(f,g,h)^{I}$ is weakly related in G to the factorization $(d^{-1},c,g)^{I}$ which has shorter length .

As a consequence, if {i} and{ii} do not occur,

$L(t) = L(h_0,h_4,f^{-1},h_3,h_0) < (2+3\rho)\lambda L(r)$.

5. $t \equiv h_2f^{-1}g_1 =_G p$. Then

$$(f,g,h)^{I} \xrightarrow{S_G^I} (g_1p^{-1}h_2,g,h)^{I} \xrightarrow{g_1^{-1}S^{I}h_2^{-1}} (p^{-1},g_2h_2{}^{-1},g_1{}^{-1}h_1)^{I}$$

But unless $L(f) \leqslant L(p)$, $L(f,g,h) > L(p^{-1},g_2h_2{}^{-1},g_1{}^{-1}h_1)$.

In addition

$$(f,g,h)^{I} \xrightarrow{S_G^I} (f,g,h_1pg_1{}^{-1}f)^{I} \xrightarrow{S^{I}f^{-1}} (1,gf^{-1},h_1pg_1{}^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1,gf^{-1},g_1p^{-1}h_1{}^{-1})^{I}$$

$$\xrightarrow{g_1^{-1}S^{I}} (g_1{}^{-1},g_2f^{-1},p^{-1}h_1{}^{-1})^{I}.$$

But unless $L(h_2) \leqslant L(p)$, $L(f,g,h) > L(g_1{}^{-1},g_2f^{-1},p^{-1}h_1{}^{-1})$

Similarly for g_1.

Thus unless {ii} occurs,

$L(t) = L(h_2, f^{-1}, g_1) < 3\rho\lambda L(r)$.

6. $t \equiv f_1^{-1}hf^{-1}h_1$. Then this case is described in Example 3 of Section 2.2. We have shown, that unless {i} or {ii} occurs

$L(t) = L(f_1^{-1}h_1, f_1^{-1}h_1, h_2) < (2+2\rho)\lambda L(r)$.

7. $t \equiv f_1^{-1}(hf^{-1})^{\alpha_1}h_1$, $\alpha>1$. Then if $(f_1^{-1}hf_2^{-1})^{\alpha_1-1}f_1^{-1}h_1$ is not a piece, using the same method as that used in the previous case (6.), it can be seen that {i} or {ii} occurs.

If $(f_1^{-1}hf_2^{-1})^{\alpha_1-1}f_1^{-1}h_1$ is a piece, then $f_1^{-1}hf_2^{-1}$ is also a piece, and

$L(t) = L((f_1^{-1}hf_2^{-1})^{\alpha_1-1}f_1^{-1}h_1, f_1^{-1}hf_2^{-1}) < 2\lambda L(r)$.

8. $t \equiv f_3^{-1}hf^{-1}hf_4^{-1} =_G \rho$. Then

$$(f,g,h)^I \xrightarrow{S_G^I} (hf_4^{-1}\rho^{-1}f_3^{-1}h, g, h)^I \xrightarrow{h^{-1}S^I} (f_4^{-1}\rho^{-1}f_3^{-1}h, h^{-1}g, 1)$$
$$\xrightarrow{S_0^I} (1, f_4^{-1}\rho^{-1}f_3^{-1}h, h^{-1}g)^{II}$$
$$\xrightarrow{S_0^{II}} (1, h^{-1}f_3\rho f_4^{-1}, h^{-1}g)^I$$
$$\xrightarrow{{}^hS^I} (h, f_4^{-1}\rho^{-1}f_3^{-1}, g)^I .$$

But unless $L(f_0) \leqslant L(\rho)$, $L(f,g,h) > L(h, f_4^{-1}\rho^{-1}f_3^{-1}, g)$

If $f_3^{-1}hf_4^{-1}$ is not a piece, then

$r \equiv f_3^{-1}hf_4^{-1}f_0^{-1}f_3^{-1}hf_4^{-1}\rho^{-1} \equiv f_3^{-1}hf_4^{-1}\rho^{-1}f_3^{-1}hf_4^{-1}f_0^{-1}$

so that $f_3^{-1}hf_4^{-1}\rho^{-1}$ and $f_3^{-1}hf_4^{-1}f_0^{-1}$ commute. Therefore r and $f_3^{-1}hf_4^{-1}f_0^{-1}$ are powers of a common element, and

(u,v) is not root-closed (rel R). Therefore, as in Case 4 unless {i} or {ii} occurs, $f_3^{-1}hf_4^{-1}$ is a piece.

As a consequence if {i} or {ii} do not occur,

$L(t) = L(f_3^{-1}hf_4^{-1}, f_0, f_3^{-1}hf_4^{-1}) < (2+\rho)\lambda L(r)$.

9. $t \equiv f_3^{-1}(hf^{-1})^{\alpha_1}hf_4^{-1}$, $\alpha_1 > 1$. Then using the same method as in the previous case (8), it can be seen that if {i} and {ii} do not occur,

$L(t) < 2\lambda L(r)$.

10. $t \equiv f_0^{-1}f_3^{-1}h(f^{-1}h)^{\alpha_1}f_4^{-1}f_0^{-1} =_G \rho$, $\alpha_1 > 0$. Then using the same method as in case (8), it can be seen if {i} and {ii} do not occur,

$L(t) < 2\lambda L(r)$.

11. $t \equiv g_2f^{-1}hf^{-1}h_1 =_G \rho$. Then

$$(f,g,h)^{I} \xrightarrow{S^{I}_{G}} (f, g_1\rho h_1^{-1}fh^{-1}f, h)^{I} \xrightarrow{S^{If-1}} (1, g_1\rho h_1^{-1}fh^{-1}, hf^{-1})^{II}$$

$$\xrightarrow{S^{II}_{0}} (1, g_1\rho h_1^{-1}fh^{-1}, fh^{-1})^{I}$$

$$\xrightarrow{S^{Ihf-1}} (hf^{-1}, g_1\rho h_1^{-1}, 1)^{I} \xrightarrow{S^{I}_{0}} (1, g_1\rho h^{-1}, hf^{-1})^{II}$$

$$\xrightarrow{S^{II}_{0}} (1, g_1\rho h_1^{-1}, fh^{-1})^{I} \xrightarrow{S^{Ih}_{0}} (h_1, g_1\rho, fh_2^{-1})^{I}.$$

But unless $L(g_2) \leqslant L(\rho)$, $L(f,g,h) > L(h_1^{-1}, g_1\rho, fh_2^{-1})$.

In addition

$$(f,g,h)^{I} \xrightarrow{S^{I}_{G}} (f, g, fg_2^{-1}\rho h_1^{-1}f)^{I}$$

$$\xrightarrow{S^{If-1}} (1, gf^{-1}, fg_2^{-1}\rho h_1^{-1})^{II}$$

$$\xrightarrow{S^{II}_{0}} (1, fg^{-1}, fg_2^{-1}\rho h_1^{-1})^{I}$$

$$\xrightarrow{{}^{g_2f-1}S^{I}} (g_2f^{-1}, g_1^{-1}, \rho h_1^{-1})^{I}.$$

But unless $L(h_2) \leq L(\rho)$, $L(f,g,h) > L(g_2 f^{-1}, g_1^{-1}, \rho h_1^{-1})$

If $f^{-1}h_1$ is not a piece, then

$f^{-1}hf^{-1}h_1\rho^{-1}g_2 \equiv f^{-1}h_1\rho^{-1}g_2 f^{-1}h$,

so that as $f^{-1}h$ and g_2 are non-trivial, $f^{-1}hg_2^{-1}$ is not reduced, and $(f,g,h)^{\mathrm{I}}$ is not reduced.

As a consequence, unless (ii) occurs,

$L(t) = L(g_2, f^{-1}h_1, h_2, f^{-1}h_1) < (2+2\rho)\lambda L(r)$.

12. $t \equiv g_2(f^{-1}h)^{\alpha_1}f^{-1}h_1 =_G \rho$, $\alpha>1$. Then

$$(f,g,h)^{\mathrm{I}} \xrightarrow{S_G^{\mathrm{I}}} (f, g_1\rho h_1{}^{-1}f(h^{-1}f)^{\alpha_1}, h)^{\mathrm{I}}$$

$$\xrightarrow{S^{\mathrm{I}f-1}} (1, g_1\rho h_1{}^{-1}(fh^{-1})^{\alpha_1}, hf^{-1})^{\mathrm{II}}$$

$$\xrightarrow{S_0^{\mathrm{II}}} (1, g_1\rho h_1{}^{-1}(fh^{-1})^{\alpha_1}, fh^{-1})^{\mathrm{I}}$$

$$\xrightarrow{S^{\mathrm{I}hf-1}} (hf^{-1}, g_1\rho h_1{}^{-1}(fh^{-1})^{\alpha_1-1}, 1)$$

$$\xrightarrow{S_0^{\mathrm{I}}} (1, g_1\rho h_1{}^{-1}(fh^{-1})^{\alpha_1-1}, hf^{-1})^{\mathrm{I}}.$$

The last mappings are repeated $\alpha-1$ times until:

$$(1, g_1\rho h_1{}^{-1}, hf^{-1}) \xrightarrow{S_0^{\mathrm{II}}} (1, g_1\rho h_1{}^{-1}, fh^{-1})^{\mathrm{I}}$$

$$\xrightarrow{S^{\mathrm{I}h_1}} (h_1, g_1\rho, fh_2{}^{-1})$$

As in the previous case (11), either $(f^{-1}h)^{\alpha_1-1}f^{-1}h_1$ is a piece or $(f,g,h)^{\mathrm{I}}$ is not reduced.

As a consequence, unless (ii) occurs,

$L(t) = L(g_2, (f^{-1}h)^{\alpha_1-1}f^{-1}h_1, f^{-1}h) < (2+\rho)\lambda L(r)$.

13. $t \equiv g_2 f^{-1}hf_2{}^{-1} =_G \rho$. Then

$$(f,g,h)^{\mathrm{I}} \xrightarrow{S_G^{\mathrm{I}}} (f, g_1\rho f_2 h^{-1}f, h)^{\mathrm{I}}$$

$$\xrightarrow{S^{\mathrm{I}f-1}} (1, g_1\rho f_2 h^{-1}, hf^{-1})^{\mathrm{II}}$$

$$\xrightarrow{S_0^{II}} (1, g_1 p f_2 h^{-1}, f h^{-1})^I$$

$$\xrightarrow{S^I h f_2^{-1}} (h f_2^{-1}, g_1 p, f_1)^I$$

But unless $L(g_2) \leq L(p)$, $L(f,g,h) > L(hf_2^{-1}, g_1 p, f_1)$

In addition

$$(f,g,h)^I \xrightarrow{S_G^I} (hf_2^{-1} p^{-1} g_2, g, h)^I$$

$$\xrightarrow{h^{-1} S^I g_2^{-1}} (f_2^{-1} p^{-1}, h^{-1} g_1, g_2^{-1})^I$$

But unless $L(f_1) \leq L(p)$, $L(f,g,h) > L(f_2^{-1} p^{-1}, h^{-1} g_1, g_2^{-1})$

In addition

$$(f,g,h)^I \xrightarrow{S_G^I} (f, g, f g_2^{-1} p f_2)^I$$

$$\xrightarrow{f^{-1} S^I} (1, f^{-1} g, g_2^{-1} p f_2)^{II}$$

$$\xrightarrow{S_0^{II}} (1, g^{-1} f, g_2^{-1} p f_2)^I$$

$$\xrightarrow{g_2 S^I f^{-1}} (g_2 f_2^{-1}, g_1^{-1} f_1, p)^I$$

But unless $L(h) \leq L(p)$, $L(f,g,h) > L(g_2 f_2^{-1}, g_1^{-1} f, p)$

If f_2 is not a piece, then

$f_2^{-1} f_1^{-1} h f_2^{-1} p^{-1} g_2 \equiv f_2^{-1} p^{-1} g_2 f_2^{-1} f_1^{-1} h$,

so that as $f^{-1}h$ and g_2 are non-trivial, $f^{-1} h g_2^{-1}$ is not reduced, and thus $(f,g,h)^I$ is not reduced.

As a consequence, unless {ii} occurs

$L(t) = L(g_2, f_2^{-1}, f_1^{-1}, h, f_2^{-1}) < (2+3\rho)\lambda L(r)$.

14. $t \equiv g_2 (f^{-1} h)^{\alpha_1} f_2^{-1} =_G p$, $\alpha > 1$. Then

$$(f,g,h)^I \xrightarrow{S_G^I} (f, g_1 p f_2 (h^{-1} f)^{\alpha_1}, h)^I$$

$$\xrightarrow{S^I f^{-1}} (1, g_1 p f_2 h^{-1} (f h^{-1})^{\alpha_1 - 1}, h f^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1, g_1 p f_2 h^{-1}(fh^{-1})^{\alpha_1 - 1}, fh^{-1})^{I}$$

$$\xrightarrow{S^{Ih}} (h, g_1 p f_2 (h^{-1}f)^{\alpha_1 - 1}, f)^{I}$$

$$\xrightarrow{S_0^{I}} (f, g_1 p f_2 (h^{-1}f)^{\alpha_1 - 1}, h)^{I}.$$

These last four mappings are repeated α-times, until:

$$(f, g_1 p f_2, h)^I \xrightarrow{S^{If_2 - 1}} (f_1, g_1 p, hf_2^{-1})^I$$

But unless $L(g_2) \leqslant L(p)$, $L(f,g,h) > L(f_1, g_1 p, hf_2^{-1})$

If $(f^{-1}h)^{\alpha_1 - 1} f_2^{-1}$ is not a piece, then using the same method as in Case (11), we find that $(f,g,h)^I$ is not reduced - a contradiction.

As a consequence, unless {ii} occurs,

$L(t) = L(g_2, (f^{-1}h)^{\alpha_1} f_2^{-1})) < (2+\rho)\ L(r)$.

15. $t \equiv g_0 g_4 f^{-1} h g_4^{-1} =_G p$. Then this case is described in Example 2 of Section 2.2. Thus unless {ii} occurs,

$L(t) = L(g_0 g_4 f^{-1} h g_4^{-1}) < (2+3\rho)\lambda L(r)$.

16. $t \equiv g_0 g_4 (f^{-1}h)^{\alpha} g_4^{-1} =_G p$. $\alpha > 1$. Then

$$(f,g,h)^I \xrightarrow{S_G^{I}} (f, g_3 p g_4 (h^{-1}f)^{\alpha}, h)^I$$

$$\xrightarrow{S^{If-1}} (1, g_3 p g_4 (h^{-1}f)^{\alpha - 1} h^{-1}, hf^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1, g_3 p g_4 (h^{-1}f)^{\alpha - 1} h^{-1}, fh^{-1})^{I}$$

$$\xrightarrow{S^{Ih}} (h, g_3 p g_4 (h^{-1}f)^{\alpha - 1}, f)^{I}$$

$$\xrightarrow{S_0^{I}} (f, g_3 p g_4 (h^{-1}f)^{\alpha - 1}, h)^{I}.$$

These last four mappings are repeated α-1 times, when we have $(f, g_3 p g_4, h)^I$

But unless $L(g_0) \leqslant L(p)$, $L(f,g,h) > L(f, g_3 p g_4, h)$

If $(f^{-1}h)^{\alpha-1}$ is not a piece, then

$(f^{-1}h)^{\alpha}g_4{}^{-1}\text{p}^{-1}g_0g_4 \equiv (f^{-1}h)^{\alpha-1}g_4{}^{-1}\text{p}^{-1}g_0g_4f^{-1}h,$

so that as $f^{-1}h$ and g_4 are non-trivial, $g_4f^{-1}h$ is not reduced, and thus $(f,g,h)^{\text{I}}$ is not reduced.

If g_4 is not a piece, then $g_4f^{-1}h \equiv g_4h^{-1}f$, which is not possible.

As a consequence, if {ii} does not occur,

$\text{L}(t) = \text{L}(g_0,g_4,(f^{-1}h)^{\alpha},g_4) < (4+\rho)\ \text{L}(r).$

17*. $t \equiv g_4f^{-1}hf^{-1}g_3 =_G \text{p}$. Then this case is described in Example 4 of Section 2.2. Thus unless {i} or {ii} occurs,

$\text{L}(t) = \text{L}(g_4,f,h,f,g_3) < (2+3\rho)\lambda\text{L}(r).$

18*. $t \equiv g_4(f^{-1}h)^{\alpha}f^{-1}g_3 =_G \text{p}$. Then

$$(f,g,h)^{\text{I}} \xrightarrow{S_G^{\text{I}}} (f,g_3g_0\text{p}g_3{}^{-1}(fh^{-1})\ f,h)^{\text{I}}$$

$$\xrightarrow{S^{\text{I}f-1}} (1,g_3g_0\text{p}g_3{}^{-1}(fh^{-1})^{\alpha},hf^{-1})^{\text{II}}$$

$$\xrightarrow{S_0^{\text{II}}} (1,g_3g_0\text{p}g_3{}^{-1}(fh^{-1})^{\alpha},fh^{-1})^{\text{I}}$$

$$\xrightarrow{S^{\text{I}hf-1}} (hf^{-1},g_3g_0\text{p}g_3{}^{-1}(fh^{-1})^{\alpha-1},1)^{\text{I}}$$

$$\xrightarrow{S^{\text{I}}} (1,g_3g_0\text{p}g_3{}^{-1}(fh^{-1})^{\alpha-1},hf^{-1})^{\text{II}}.$$

These last mappings are repeated $\alpha-1$ times until

$$(1,g_3g_0\text{p}g_3{}^{-1},hf^{-1})^{\text{II}} \xrightarrow{g_3-1S^{\text{II}}} (g_3{}^{-1},g_0\text{p},hf^{-1})^{\text{II}}.$$

Thus in this case, the Type of factorization changes.

But unless $\text{L}(g_4) \leqslant \text{L}(\text{p})$, $\text{L}(f,g,h) > \text{L}(g_3{}^{-1},g_0\text{p},hf^{-1})$.

Similarly for g_3.

By the same method as in case (17), as $(f,g,h)^{\text{I}}$ is assumed to be reduced, $(f^{-1}h)^{\alpha-1}f^{-1}$ is a piece. As $\alpha>1$, $f^{-1}h$ is also a piece.

As a consequence, unless {ii} occurs,

$L(t) = L(g_4,(f^{-1}h)^{\alpha}f^{-1},g_3) < (2+2\rho)\lambda L(r)$.

B. *Let $(f,g,h)^J$ be a factorization with J = II.*

1. $t \subset f$ or h, then $h \equiv h_3th_4 =_G h_3ph_4$ or $f \equiv f_3tf_4 =_G f_3ph_4$. In both cases, $(f,g,h)^{II}$ is related to a smaller factorization, unless $L(t) < 1/2L(r)$.

2. $t \equiv f_2h_1 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f_1ph_1^{-1},g,h)^{II} \xrightarrow{S^{IIh_1}} (f_1p,g,h_2h_1)^{II}$$

But unless $L(f_2) \leqslant L(p)$, $L(f,g,h) > L(f_1p,g,h_2h_1)$

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,f_2^{-1}ph_2)^{II} \xrightarrow{S^{IIf_2^{-1}}} (f_1,g,ph_2f_2^{-1})^{II}$$

But unless $L(h_1) \leqslant L(p)$, $L(f,g,h) > L(f_1,g,ph_2f_2^{-1})$

As a consequence, unless {ii} occurs,

$L(t) = L(f_2,h_1) < 2\rho\lambda L(r)$.

3. $t \equiv h_4h_3 =_G p$. Then this case is described in Example 7 of Section 2.2. Therefore, unless {ii} occurs,

$L(t) = L(h_4,h_3) < 2\rho\lambda L(r)$.

4. $t \equiv h_0h_4h_3h_0 =_G p$. Then this case is described in Example 9 of Section 2.2. Therefore unless {i} or {ii} occurs,

$L(t) = L(h_0,h_4,h_3,h_0) < (2+2\rho)\lambda L(r)$.

5. $t \equiv h_4 h h_3 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,h_4^{-1} p h_3^{-1})^{II}$$

But unless $L(h_0) \leq L(p)$, $L(f,g,h) > L(f,g,h_4^{-1} p h_3^{-1})$

If $h_4 h_3$ is not a piece, then

$r \equiv h_4 h_3 h_0 h_4 h_3 \equiv h_4 h_3 h_4 h_3 h_0$,

so that $h_4 h_3$ and $h_4 h_3 h_0$ commute. Thus r and $h_4 h_3 h_0$ are powers of a common element, and r is a proper power. Thus (u,v) is not root-closed (rel R). Therefore, as in the previous case (4), {i} or {ii} occurs. As a consequence, unless {i} or {ii} occurs.

$L(t) = L(h_4 h_3, h_0, h_4 h_3) < (2+\rho)\lambda L(r)$.

6. $t \equiv h_4 h^{\alpha_1} h_3$, $\alpha_1 > 1$. Then as in the previous case (5), if $(h_4 h_3 h_0)^{\alpha_1 - 1} h_4 h_3$ is not a piece, {i} or {ii} occurs, Thus unless {i} or {ii} occurs,

$L(t) = L((h_4 h_3 h_0)^{\alpha_1 - 1} h_4 h_3, h_4 h_3 h_0) < 2\lambda L(r)$

7. $t \equiv h_0 h_4 h^{\alpha_1} h_3 h_0$, $\alpha \geq 1$. Then as in case (5), if $(h_0 h_4 h_3)^{\alpha_1} h_0$ is not a piece, {i} or {ii} occurs. Thus, unless {i} or {ii} occurs,

$L(t) = L((h_0 h_4 h_3)^{\alpha_1} h_0, h_0 h_4 h_3) < 2\lambda L(r)$.

8. $t \equiv f_2 h h_1 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f_1 p h_1^{-1} h^{-1}, g, h)^{II} \xrightarrow{S^{II} h h_1} (f_1 p, g, h_2 h_1)^{II}.$$

But unless $L(f_2) \leq L(p)$, $L(f,g,h) > L(f_1 p, g, h_2 h_1)$.

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,f_2^{-1} p h_1^{-1})^{II} \xrightarrow{S^{II} f_2^{-1}} (f_1, g, p h_1^{-1} f_2^{-1})^{II}.$$

But unless $L(h_2) \leqslant L(p)$, $L(f,g,h) > L(f_1,g,ph_1^{-1}f_2^{-1})$

if h_1 is not a piece,

$h_1h_2h_1p^{-1}f_2 \equiv h_1p^{-1}f_2h_1h_2$,

so that h and f_2 are non-trivial, hf_2^{-1} is not reduced,

and therefore $(f,g,h)^{II}$ is not reduced.

As a consequence, unless {ii} occurs,

$L(t) = L(f_2,h_1,h_2,h_1) < (2+2\rho)\lambda L(r)$.

9. $t \equiv f_2h^{\alpha_1}h_1 =_G p$, $\alpha_1>1$. Then this case is described in Example 10, with $\alpha_1 = 2$. Thus unless {i} or {ii} occurs,

$L(t) = L(f_2,h^{\alpha_1-1}h_1,h_2h_1) < (2+\rho)\lambda L(r)$.

10. $t \equiv f_0f_4hf_4^{-1} =_G p$. Then this case is described in Example 8 of Section 2.2. Therefore unless {ii} occurs,

$L(t) = L(f_0,f_4,h,f_4) < (2+2\rho)\lambda L(r)$.

11. $t \equiv f_0f_4h^{\alpha}f_4^{-1} =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f_3pf_4h^{-\alpha},g,h)^{II} \xrightarrow{S^{II}h^{\alpha}} (f_3pf_4,g,h)^{II}$$

But unless $L(f_0) \leqslant L(p)$, $L(f,g,h) > L(f_3pf_4,g,h)$

If f_4 is not a piece, then $h \equiv h^{-1}$ which is not possible.

If $h^{\alpha-1}$ is not a piece, then as in Case 8, $(f,g,h)^{II}$ is not reduced.

As a consequence, unless {ii} occurs,

$L(t) = L(f_0,f_4,h^{\alpha},f_4) < (4+\rho)\lambda L(r)$.

12*. $t \equiv g_2fh_1 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g_1ph_1^{-1}f^{-1},h)^{II} \xrightarrow{f^{-1}S^{II}} (1,f^{-1}g_1ph_1^{-1},h)^{II}$$

$$\xrightarrow{S_0^{II}} (1, f^{-1}g_1ph_1^{-1}, h^{-1})^{I} \xrightarrow{S^{Ih_1}} (h_1, f^{-1}g_1p, h_2^{-1})^{I}$$

(Note that the Type of factorization changes).

But unless $L(g_2) \leqslant L(p)$, $L(f,g,h) > L(h_1, f^{-1}g_1p, h_2^{-1})$.

Similarly for h_1.

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (g_2^{-1}ph_1^{-1}, g, h)^{II} \xrightarrow{g_2S^{II}h_1} (p, g_2g_1, h_2h_1)^{II}$$

But unless $L(f) \leqslant L(p)$, $L(f,g,h) > L(p, g_2g_1, h_2h_1)$

As a consequence, unless {ii} occurs,

$L(t) = L(g_2, f, h_1) < 3\rho\lambda L(r)$.

13*. $t \equiv g_2fhh_1 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g_1ph_1^{-1}h^{-1}f^{-1}, h)^{II} \xrightarrow{f^{-1}S^{II}} (1, f^{-1}g_1ph_1^{-1}h^{-1}, h)^{II}$$

$$\xrightarrow{S_0^{II}} (1, f^{-1}g_1ph_1^{-1}h^{-1}, h^{-1})^{I} \xrightarrow{S^{Ih}} (h, f^{-1}g_1ph_1^{-1}, 1)^{I}$$

$$\xrightarrow{S_0^{I}} (1, h, f^{-1}g_1ph_1^{-1})^{II} \xrightarrow{S_0^{II}} (1, h^{-1}, f^{-1}g_1ph_1^{-1})^{I}$$

$$\xrightarrow{S^{Ih_1}} (h_1, h_2^{-1}, f^{-1}g_1p)^{I}$$

(Note that the Type of factorization changes). But unless $L(g_2) \leqslant L(p)$, $L(f,g,h) > L(h_1, h_2^{-1}, f^{-1}g_1p)$

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (g_2^{-1}ph_1^{-1}h^{-1}, g, h)^{II} \xrightarrow{g_2S^{II}hh} (p, g_2g_1, h_2h_1)^{II}$$

But unless $L(f) \leqslant L(p)$, $L(f,g,h) > L(p, g_2g_1, h_2h_1)$.

Also

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g, f^{-1}g_2^{-1}ph_1^{-1})^{II}$$

$$\xrightarrow{S^{IIf-1}} (1, g, g_2^{-1}ph_1^{-1}f^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1, g, fh_1p^{-1}g_2)^{I} \xrightarrow{S^{Ig_2-1}} (g_2^{-1}, g_1, fh_1p^{-1})^{I}$$

(Note that the factorization Type changes.) But unless $L(h_2) \leqslant L(p)$, $L(f,g,h) > L(g_2{}^{-1}, g_1, fh_1p^{-1})$

If h_1 is not a piece, then

$h_1p^{-1}g_2fh \equiv h_1h_2h_1p^{-1}g_2f$,

so that as fh and g_2f are non-trivial, $g_2fh^{-1}f^{-1}$ is not reduced.

As a consequence, unless (ii) occurs,

$L(t) = L(g_2, f, h_1, h_2, h_1\ ) < (2+3\rho)\lambda L(r)$.

14. $t \equiv g_2fh^{\alpha_1}h_1 =_G p$, $\alpha_1 > 1$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g_1ph_1{}^{-1}h^{-\alpha_1}f^{-1}, h)^{II}$$

$$\xrightarrow{{}^{f-1}S^{II}} (1, f^{-1}g_1ph_1{}^{-1}h^{-\alpha_1}, h)^{II}$$

$$\xrightarrow{S_0^{II}} (1, f^{-1}g_1ph_1{}^{-1}h^{-\alpha_1}, h^{-1})^{I}$$

$$\xrightarrow{S^{Ih}} (h, f^{-1}g_1ph_1{}^{-1}h^{1-\alpha_1}, 1)^{I}$$

$$\xrightarrow{S_0^{I}} (1, f^{-1}g_1ph_1{}^{-1}h^{1-\alpha_1}, h)^{II}.$$

these last three mappings are repeated α_1-1 times until:

$$(1, f^{-1}g_1ph_1{}^{-1}, h)^{II} \xrightarrow{S_0^{II}} (1, f^{-1}g_1ph_1{}^{-1}, h^{-1})^{I}$$

$$\xrightarrow{S^{Ih_1}} (h_1, f^{-1}g_1p, h_2{}^{-1})^{I}$$

(Note that the Type of factorization changes). But unless $L(g_2) \lessgtr L(p)$, $L(f,g,h) > L(h_1, f^{-1}g_1p, h_2{}^{-1})$

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (g_2{}^{-1}ph_1{}^{-1}h^{-\alpha_1}, g, h)^{II}$$

$$\xrightarrow{{}^{g_2}S^{II}h^{\alpha_1}h} (p, g_2g_1, h_2h_1)^{II}$$

But unless $L(f) \leqslant L(p)$, $L(f,g,h) > L(p,g_2g_1,h_2h_1)$

As in the previous case (13) since $(f,g,h)^{II}$ is reduced, $h^{\alpha_1 -1}h$ is a piece, and therefore h is also a piece.

Thus unless {ii} occurs,

$L(t) = L(g_2,f,h^{\alpha_1}h_1) < (2+2p)\lambda L(r)$.

15*. $\quad t \equiv g_2fhf_2^{-1} =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g_1pf_2h^{-1}f^{-1},h)^{II} \xrightarrow{f^{-1}S^{II}} (1,f^{-1}g_1pf_2h^{-1},h)^{II}$$

$$\xrightarrow{S_0^{II}} (1,f^{-1}g_1pf_2h^{-1},h^{-1})^{I}$$

$$\xrightarrow{S^{I}h} (h,f^{-1}g_1pf_2,1)^{I}$$

$$\xrightarrow{S_0^{I}} (1,f^{-1}g_1pf_2,h)^{II} \xrightarrow{f_2S^{II}h} (f_2,f_1^{-1}g_1p,h)^{II}$$

But unless $L(g\ ) \leqslant L(p)$, $L(f,g,h) > L(f_2,f_1^{-1}g_1p,h)$

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (g_2^{-1}pf_2h^{-1},g,h)^{II}$$

$$\xrightarrow{g_2S^{II}h} (pf_2,g_2g_1,h)^{II}$$

But unless $L(f_1) \leqslant L(p)$, $L(f,g,h) > L(pf_2,g_2g_1,h)$

Also

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f,g,f^{-1}g_2^{-1}pf_2)^{II}$$

$$\xrightarrow{S^{II}f^{-1}} (1,g,g_2pf_1^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1,g^{-1},g_2^{-1}pf_1^{-1})^{I}$$

$$\xrightarrow{g_2S^{I}} (g_2,g_1^{-1},pf_1^{-1})^{I}.$$

(Note that the Type of factorization changes.) But unless $L(h,f_2) \leqslant L(p)$, $L(f,g,h) > L(g_2,g_1^{-1},pf_1^{-1})$.

If f_2 is not a piece, then $h \equiv h^{-1}$, which is not possible.

As a consequence if {ii} does not occur

$L(t) = L(g_2, f_1, f_2, h, f_2) < (1+3\rho)\lambda L(r)$.

16. $t \equiv g_2 f h^{\alpha} f_2^{-1} =_G p$, $\alpha > 1$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g_1 p f_2 h^{-\alpha} f^{-1}, h)^{II}$$
$$\xrightarrow{f^{-1}S^{II}} (1, f^{-1} g_1 p f_2 h^{-\alpha}, h)^{II}$$
$$\xrightarrow{S_0^{II}} (1, f^{-1} g_1 p f_2 h^{-\alpha}, h^{-1})^{I}$$
$$\xrightarrow{S^{I}h} (h, f^{-1} g_1 p f_2 h^{-\alpha+1}, 1)^{I}$$
$$\xrightarrow{S_0^{I}} (1, f^{-1} g_1 p f_2 h^{-\alpha+1}, h)^{II}$$

These last three mappings are repeated $\alpha-1$ times, until:

$$(1, f^{-1} g_1 p f_2, h)^{II} \xrightarrow{f_2 S^{II}} (f_2, f_1^{-1} g_1 p, h)^{II}$$

But unless $L(g_2) \leqslant L(p)$, $L(f,g,h) > L(f_2, f_1^{-1} g_1 p, h)$

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (g_2^{-1} p f_2 h^{-\alpha}, g, h)^{II}$$
$$\xrightarrow{g_2 S^{II} h^{\alpha}} (p f_2 g_2 g_1, h)^{II}$$

But unless $L(f\ ) \leqslant L(p)$, $L(f,g,h) > L(pf_2, g_2 g_1, h)$.

If $h^{\alpha-1}$ is not a piece, then

$h^{\alpha} f_2^{-1} p^{-1} g_2 f \equiv h^{\alpha-1} f_2^{-1} p^{-1} g_2 f h$,

so that as h and f_2 are non-trivial, $f_2 h$ is not reduced,

and thus $(f,g,h)^{II}$ is not reduced.

As a consequence, unless {ii} occurs,

$L(t) = L(g_2, f_1, f_2, h^{\alpha}, f_2) < (2\rho+4)\lambda L(r)$.

17*. $t \equiv g_0 g_4 f h f^{-1} g_4^{-1} =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g_3 p g_4 f h^{-1} f^{-1}, h)^{II}$$
$$\xrightarrow{{}^{f^{-1}}S^{II}} (1, f^{-1} g_3 p g_4 f h^{-1}, h)^{II}$$
$$\xrightarrow{S_0^{II}} (1, f^{-1} g_3 p g_4 f h^{-1}, h^{-1})^{I}$$
$$\xrightarrow{S^{Ih}} (h, f^{-1} g_3 p g_4 f, 1)^{I}$$
$$\xrightarrow{{}^{f}S_0^{I}} (1, f^{-1} g_3 p g_4 f, h)^{II}$$
$$\xrightarrow{{}^{f}S^{II}} (f, g_3 p g_4, h)^{II}$$

But unless $L(g_0) \leqslant L(p)$, $L(f,g,h) > L(f, g_3 p g_4, h)$

In addition

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g, f^{-1} g_4^{-1} g_0^{-1} p g_4 f)^{II}$$
$$\xrightarrow{{}^{g_4}S^{II f^{-1} g_4^{-1}}} (1, g_0^{-1} g_3^{-1} g_4^{-1}, g_0^{-1} p)^{I}$$
$$\xrightarrow{{}^{g_0}S^{I}} (g_0, g_3^{-1} g_4^{-1}, p)^{I}$$

(Note that the Type of factorization changes). But unless $L(h,f) \leqslant L(p)$, $L(f,g,h) > L(g_0, g_3^{-1} g_4^{-1}, p)$

If $g_4 f$ is not a piece, then $h \equiv h^{-1}$, which is not possible.

As a consequence, unless {ii} occurs,

$L(t) = L(g_0, g_4 f, h, g_4 f) < (2+2\rho)\lambda L(r)$

18. $t \equiv g_0 g_4 f h^{\alpha} f^{-1} g_4^{-1} =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g_3 p g_4 f h^{-\alpha} f^{-1}, h)^{II}$$
$$\xrightarrow{{}^{f^{-1}}S^{II}} (1, f^{-1} g_3 p g_4 f h^{-\alpha}, h)^{II}$$
$$\xrightarrow{S_0^{II}} (1, f^{-1} g_3 p g_4 f h^{-\alpha}, h^{-1})^{I}$$
$$\xrightarrow{S^{Ih}} (h, f^{-1} g_3 p g_4 f h^{-\alpha+1}, 1)^{I}$$

$$\xrightarrow{S_0^I} (1, f^{-1}g_3 p g_4 f h^{-\alpha+1}, h)^{II}.$$

Repeat these last three mappings α-1 times until:

$$(1, f^{-1}g_3 p g_4 f, h)^{II} \xrightarrow{{}^{f}S^{II}} (f, g_3 p g_4, h)^{II}.$$

But unless $L(g_0) \leqslant L(p)$, $L(f,g,h) > L(f,g_3pg_4,h)$.

If $h^{\alpha-1}$ is not a piece, then

$h^{\alpha}f^{-1}g_4^{-1}p^{-1}g_0g_4f \equiv h^{\alpha-1}f^{-1}g_4^{-1}p^{-1}g_0g_4fh$,

so that as h and $g_4 f$ are non-trivial, g_4fh^{-1} is not reduced,

and therefore $(f,g,h)^{II}$ is not reduced.

As a consequence, unless {ii} occurs,

$L(t) = L(g_0; g_4f, h^{\alpha}, g_4f) < (4+\rho)\lambda L(r)$

19. $t \equiv g_4fhf^{-1}g_3 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g_3g_0pg_3^{-1}fh^{-1}f^{-1}, h)^{II}$$

$$\xrightarrow{{}^{f-1}S^{II}} (1, f^{-1}g_3g_0pg_3^{-1}fh^{-1}, h)^{II}$$

$$\xrightarrow{S_0^{II}} (1, f^{-1}g_3g_0pg_3^{-1}fh^{-1}, h^{-1}$$

$$\xrightarrow{S^{Ih}} (h, f^{-1}g_3g_0pg_3^{-1}f, 1)^{I}$$

$$\xrightarrow{S_0^{I}} (1, f^{-1}g_3g_0pg_3^{-1}f, h)^{II}$$

$$\xrightarrow{{}^{g_3-1f}S^{II}} (g_3^{-1}f, g_0p, h)^{II}$$

But unless $L(g_4) \leqslant L(p)$, $L(f,g,h) > L(g_3^{-1}f, g_0p, h)$

Similarly for g_3. In addition,

$$(f,g,h)^{II} \xrightarrow{S_G^{II}} (f, g, f^{-1}g_4^{-1}pg_3^{-1}f)^{II}$$

$$\xrightarrow{S^{IIf-1}} (1, g, g_4^{-1}pg_3^{-1})^{II}$$

$$\xrightarrow{S_0^{II}} (1, g^{-1}, g_4^{-1}pg_3^{-1})^{I}$$

$$\xrightarrow{g_4 S^I g_3} (g_4 g_3 g_0^{-1}, p)^I$$

But unless $L(h,f) \leqslant L(p)$, $L(f,g,h) > L(g_4 g_3, g_0, p)$

If f is not a piece, then $h \equiv h^{-1}$, which is not possible.

Therefore if {ii} does not occur,

$L(f) = L(g_4, f, h, f, g_3) < (1+3\rho)\lambda L(r)$

20. $t = g_4 f h^{\alpha} f^{-1} g_3 =_G p$. Then

$$(f,g,h)^{II} \xrightarrow{S^{II}_G} (f, g_3 g_0 p g_3^{-1} f h^{-\alpha} f^{-1}, h)^{II}$$

$$\xrightarrow{f^{-1} S^{II}} (1, f^{-1} g_3 g_0 p g_3^{-1} f h^{-\alpha}, h)^{II}$$

$$\xrightarrow{S^{II}_0} (1, f^{-1} g_3 g_0 p g_3^{-1} f h^{-\alpha}, h^{-1})^{I}$$

$$\xrightarrow{S^I h} (h, f^{-1} g_3 g_0 p g_3^{-1} f h^{-\alpha+1}, 1)^{I}$$

$$\xrightarrow{S^I_0} (1, f^{-1} g_3 g_0 p g_3^{-1} f h^{-\alpha+1}, h)^{II}$$

These last three mappings are repeated $\alpha-1$ times, until:

$$(1, f^{-1} g_3 g_0 p g_3^{-1} f, h)^{II} \xrightarrow{g_3^{-1} f S^{II}} (g_3^{-1} f, g_0 p, h)^{II}$$

But unless $L(g_4) \leqslant L(p)$, $L(f,g,h) > L(g_3^{-1} f, g_0 p, h)$

Similarly for g_3

If f is not a piece, then $h \equiv h^{-1}$, which is not possible.

If $h^{\alpha-1}$ is not a piece, then

$h^{\alpha} f^{-1} g_3 p^{-1} g_4 f \equiv h^{\alpha-1} f^{-1} g_3 p^{-1} g_4 f h$

and as $g_4 f$ and h are non-trivial, $g_4 f h^{-1}$ is not reduced, so that $(f,g,h)^{II}$ is not reduced.

As a consequence, unless{ii} occurs,

$L(t) = L(g_4, f, h^{\alpha}, f, g_3) < (4+2\rho)\lambda L(r)$

A GUIDE TO THE PROOF OF EACH CASE REQUIRED FOR THEOREM 2.2

Because of the exceptionally large number of cases it is not practical to provide a detailed proof in each instance. However a simple coding system will indicate, in each case, the strategy required so that the reader may fill in the details, following the methods described in Section 2.2. For each case, we divide t into subwords $t_1, t_2, \ldots, t_n$, where $t \equiv t_1 t_2 \ldots t_n$. Then it can be shown that if W^* is $\hat{S}$-minimal (so that {i} cannot occur,) either $(f,g,h)^J$ is weakly related to a smaller factorization, or the lengths of the subwords $t_1, t_2, \ldots, t_n$ are bounded in size. The precise limit depends on the method used. We indicate which method should be used by writing A,B or C above each subword. These coincide with the detailed description of the methods in Section 2.2. For convenience we do not make use of method D. This is because method C can always be applied in those cases that could (under certain conditions) be covered by method D.

In particular, the length of a subword of type A is either bounded by $\rho\lambda L(r)$, or $(f,g,h)^J$ is related to a smaller factorization.

A subword of type B is always a piece.

If a subword t_i is of type C, $(1 \leqslant i \leqslant n)$, then t_i is a piece unless (u,v) is not root-closed (rel R) and either

(a) W^* is not S-minimal and {i} occurs, or

(b) $(f,g,h)^J$ is weakly related to a smaller factorization, so that {ii} occurs.

We assume W^* is S-minimal (rel R)

By assuming that $(f,g,h)^J$ is not weakly related to a smaller factorization or if (u,v) is root-closed (rel R), by assuming $(f,g,h)^J$ is not related to a smaller factorization, we can find, for each case defined below, a total bound T on the length of t. This is obtained by adding together the individual limits on the lengths of the subwords $t_1,\ldots,t_n$.

We will use a similar notation to that used for Theorem 2.1. That is, if Y is any word in $W(x)$ then we write

$Y \equiv Y_1Y_2 \equiv Y_3Y_0Y_4$, where $Y_2, Y_3, Y_4 \not\equiv 1$.

We use $\alpha, \alpha_1, \alpha_2$ to denote non-negative integers, and $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3$ to denote +1 or -1.

The cases are divided into two parts according to the Type J of factorization of (f,g,h).

A. *The factorization* J = I

We do not list below those cases which are similar. For it is possible to construct from each of the expressions listed below, 11 further unlisted expressions by means of the transformation S_0^I. These expressions can be found by replacing the elements of the triple (f,g,h) by the elements of a permutation of $(f^\varepsilon, g^\varepsilon, h^\varepsilon)$.

We consider in the first 20 cases the possible values for t, where t is a subword of W^*, and where g is not an F-subword of tvt^{-1}. The cases obtained by assuming h or f is not an F-subword of tvt^{-1} are similar.

Suppose first that f, g and h are not F-subwords of $t\upsilon t^{-1}$, then t or t^{-1} is a subword of $f^{-1}g, gf^{-1}$, $f^{-1}h, hf^{-1}$, gh^{-1} or $h^{-1}g$, and this gives cases A1 and A2 of Theorem 2.1.

Secondly let f^{-1} be F-subword of t, but h and g not F-subwords of $t\upsilon t^{-1}$. Then $t \subset hf^{-1}h$, $hf^{-1}g$ or similar F^*-subwords of W^*. This gives cases A3, 4 and 5 of Theorem 2.1.

In the third place, let h and f^{-1} be subwords of t, but g is not an F-subword of $t\upsilon t^{-1}$. This gives cases A6 - 18 of Theorem 2.1, and two additional cases, 19 and 20. These extra cases arise when g is a subconjugate of t, but not an F-subword of $t\upsilon t^{-1}$. The value for T in these cases shows that the conclusion for Theorem 2.1 is in a sense maximal, and in fact these are limiting cases for Theorem 2.2.

C A C AC A C
19. $t \equiv g_0 g_4 f^{-1} h f^{-1} \; g_3 g_0$, $T = (3\rho+4)\lambda L(r)$

C A C C A C
20. $t \equiv g_0 g_4 \overline{(f^{-1} \; h)^{\alpha} f^{-1}} \overline{hf^{-1}} \; g_3 g_0$, $T = (2\rho+4)\lambda L(r)$.

We consider in cases 21-24 the possible expressions for t where f^{-1}, g, h each occur as F-subwords of tvt^{-1} only once. In the first instance suppose $t \subset f^{-1} h f^{-1} g f^{-1}$

C A C A C A C
21. $t \equiv f_3^{-1} h f_4^{-1} \; f_0^{-1} \; f_3^{-1} \; g f_4^{-1}$, $T = (3\rho+4)\lambda L(\; r)$

C A C C A C
22. $t \equiv \overline{f_0^{-1} f_3^{-1}} h f_4^{-1} \overline{f_0^{-1} f_3^{-1}} g \overline{f_4^{-1} f_0^{-1}}$, $T = (2\rho+4)\lambda L(r)$

Secondly suppose $t \subset g^{-1} h f^{-1} g f^{-1}$

B A C A B A C
23. $t \equiv g_1^{-1} h \; f_2^{-1} \; f_1^{-1} \; g_1 g_2 f_2^{-1}$, $T = (3\rho+4)\lambda L(r)$

Lastly suppose $t \subset g^{-1} h f^{-1} g h^{-1}$

B A B A B A B
24. $t \equiv g_1^{-1} h_1 h_2 f^{-1} \; g_1 g_2 h_2^{-1}$, $T = (3\rho+4)\lambda L(r)$

We consider in cases 25-36 the possible expressions for t where g and h each occur as F-subwords of tvt^{-1} only once, and f occurs at least twice as an F-subword of tvt^{-1}.

In the first instance suppose f^{-1} is an F-subword of t only twice. If $t \subset f^{-1} h f^{-1} g f^{-1} h$, then

C A C A C C
25. $t \equiv \overline{f_1^{-1}} h_1 h_2 f^{-1} g f^{-1} h_1$, $T = (2\rho+4)\lambda L(r)$.

If $t \subset f^{-1} h f^{-1} g f^{-1} g$, then

C A C A C
26. $t \equiv f_1^{-1} \; h\overline{f^{-1}} \; g_1 g_2 \overline{f^{-1} \; g_1}$, $T = (2\rho+4)\lambda L(r)$

If $t \subset g^{-1} h f^{-1} g f^{-1} h$, then

27. $t \equiv \overset{B}{g_1^{-1}}\overset{C}{h_1}\overset{A}{h_2}\overset{C}{f^{-1}}\overset{B}{g_1}\overset{A}{g_2}\overset{C}{f^{-1}}\overset{C}{h_1}$, $T = (2\rho+6)\lambda L(r)$.

If $t \subset g^{-1}hf^{-1}gf^{-1}g$, then

28. $t \equiv \overset{B}{g_3^{-1}}\overset{A}{h}\overset{C}{\overline{f^{-1}g_3g_0}}\overset{A}{g_4}\overset{C}{\overline{f^{-1}g_3g_0}}$, $T = (2\rho+3)\lambda L(r)$

29. $t \equiv \overset{B}{g_0^{-1}}\overset{B}{g_3^{-1}}\ \overset{A}{h}\overset{C}{\overline{f^{-1}g_3}}\overset{B}{g_0}\overset{A}{g_4}\overset{C}{\overline{f^{-1}g_3}}$, $T = (2\rho+5)\lambda L(r)$.

Secondly suppose f^{-1} is an F-subword of t at least 3 times then either $t \subset gf^{-1}hf^{-1}gf^{-1}g$, and

30. $t \equiv \overset{C}{\overline{g_4f^{-1}}}\overset{A}{h}\overset{C}{f^{-1}}\overset{A}{g_3g_0}\overset{C}{\overline{g_4f^{-1}}}\overset{C}{g_3}$, $T = (2\rho+4)\lambda L(r)$

31. $t \equiv \overset{C}{\overline{g_0g_4f^{-1}}}\overset{A}{h}\overset{C}{\overline{f^{-1}g_3}}\overset{C}{\overline{g_0g_4f^{-1}}}\overset{C}{\overline{g_3g_0}}$, $T = (\rho+4)\lambda L(r)$

or $t \subset hf^{-1}hf^{-1}gf^{-1}g$ and

32. $t \equiv \overset{C}{\overline{h_2f^{-1}}}\overset{A}{h_1}\overset{C}{\overline{h_2f^{-1}}}\overset{C}{g_1}\overset{A}{g_2}\overset{C}{\overline{f^{-1}g_1}}$, $T = (2\rho+4)\ L(r)$

Lastly suppose f and f^{-1} are F- subwords of t then either $t \subset g^{-1}fg^{-1}hf^{-1}g$, and

33. $t \equiv \overset{B}{\overline{g_4f^{-1}}}\overset{A}{g_3}\overset{B}{\overline{g_0g_4}}\overset{B}{\overline{fg_4^{-1}}}\overset{B}{g_0^{-1}}$, $T = (\rho+4)\lambda L(r)$

34. $t \equiv \overset{C}{g_0}\overset{B}{\overline{g_4f^{-1}}}\overset{A}{g_3}\overset{C}{\overline{g_0g_4}}\overset{B}{\overline{fg_4^{-1}}}$, $T = (\rho+4)\lambda L(r)$

or $t \subset gf^{-1}hg^{-1}fh$, and

35. $t \equiv \overset{B}{g_2}\overset{B}{f^{-1}}\overset{B}{h_1}\overset{A}{h_2}\overset{B}{g_2^{-1}}\overset{A}{g_1^{-1}}\overset{B}{f}\overset{B}{h_1^{-1}}$, $T = (2\rho+6)\lambda L(r)$

or $t \subset hf^{-1}hg^{-1}fg^{-1}$, and

36. $t \equiv \overset{C}{h_2}\overset{B}{f^{-1}}\overset{A}{h_1}\overset{C}{h_2}\overset{C}{g_2^{-1}}\ \overset{A}{g_1^{-1}}\ \overset{B}{f}\overset{C}{g_2^{-1}}$, $T = (2\rho+6)\lambda L(r)$

We consider in the rest of the cases the possible expressions for t, where g occurs as an F-subword of $t\upsilon t^{-1}$ only once, but f and h occur as distinct F-subwords of $t\upsilon t^{-1}$ at least twice each.

First suppose g occurs in $t\upsilon t^{-1}$ as an F-subword only once, and t does not begin or end in the F-subwords g or g^{-1}. Then $t \subset (hf^{-1})^{\alpha_1}\, g(f^{-1}h)^{\alpha_2\varepsilon}$

We get a number of cases here, according to the position of t in this word, and the value of ε.

37. $t \equiv \overset{C}{\overline{(hf^{-1}\ldots hf^{-1})_4\, hf^{-1}}}\,\overset{C}{\overline{h\, f^{-1}}}\,\overset{A}{g}\,\overset{C}{\overline{(f^{-1}h)_3}}$, $T = (\rho+3)\lambda L(r)$

38. $t \equiv \overset{C}{\overline{(\ldots hf^{-1})_4}}\ \overset{C}{h}\overset{C}{f^{-1}}\ \overset{A}{g}\ \overset{C}{f^{-1}}\overset{C}{h}\ \overset{C}{\overline{(f^{-1}h\ldots)_3}}$, $T = (\rho+6)\lambda L(r)$

39. $t \equiv \overset{C}{\overline{(\ldots hf^{-1})_4\, hf^{-1}}}\,\overset{C}{\overline{hf^{-1}}}\,\overset{A}{g}\,\overset{B}{h_2^{-1}}$, $T = (\rho+3)\lambda L(r)$

40. $t \equiv \overset{C}{\overline{(\ldots f^{-1}h)_4\ f^{-1}}}\ \overset{C}{h}\overset{C}{f^{-1}}\ \overset{A}{g}\ \overset{B}{\overline{h^{-1}f_3}}$, $T = (\rho+4)\lambda L(r)$

41. $t \equiv \overset{C}{\overline{(\ldots hf^{-1})_4}}\,\overset{B}{h}\overset{B}{f^{-1}}\overset{A}{g}\overset{B}{h^{-1}}\overset{B}{f}\ \overset{C}{\overline{(h^{-1}f\ \ldots)_3}}$, $T = (\rho+6)\lambda L(r)$

Secondly suppose g occurs in $t\upsilon t^{-1}$ as an F-subword only once and t begins but does not end in the F-subwords g or g^{-1}.

Then either $t \subset gf^{-1}hf^{-1}\ldots hf^{-1}g(f^{-1}h\ldots f^{-1}h)^{\varepsilon}$ (cases 42-47)

or $t \subset g^{-1}hf^{-1}\ldots hf^{-1}g(f^{-1}h\ldots)^{\varepsilon}$ (cases 47-51)

42. $t \equiv \overset{C}{\overline{g_2f^{-1}}}\ \overset{A}{g_1}\overset{C}{\overline{g_2f^{-1}}}\ \overset{C}{hf^{-1}}\ \overset{C}{\overline{h(f^{-1}h\ldots)_1}}$, $T = (\rho+4)\lambda L(r)$

43. $t \equiv \overset{C}{\overline{g_2f^{-1}h}}\overset{C}{f^{-1}}\overset{A}{g_1}\overset{C}{\overline{g_2f^{-1}h}}\overset{C}{\overline{(f^{-1}h\ldots)_1}}$, $T = (\rho+4)\lambda L(r)$

44. $t \equiv \overset{C}{g_2}\overset{C}{\overline{(f^{-1}h)^{\alpha}f^{-1}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{A}{g_1}\overset{C}{\overline{g_2(f^{-1}h}}\overset{C}{\overline{\ldots)_1}}$, $T = (\rho+5)\lambda L(r)$

45. $t \equiv \overset{C}{g_2}\overset{B}{f^{-1}}\,\overset{A}{g_1}\overset{C}{g_2}\overset{C}{h^{-1}}\,\overset{B}{fh^{-1}}\,\overset{C}{\overline{(fh^{-1}\ \ldots)}}_1,\quad T = (\rho+6)\lambda L(r)$

46. $t \equiv \overset{C}{g_2}\overset{C}{\overline{(f^{-1}h)^{\alpha}f^{-1}}}\overset{B}{h}\overset{C}{f^{-1}}\overset{A}{g_1}\overset{C}{g_2}\overset{B}{h^{-1}}\overset{B}{(}\overset{C}{\overline{fh^{-1}\ldots)}}_1 \quad T = (\rho+8)\lambda L(r)$

47. $t \equiv \overset{C}{g_2}\overset{C}{\overline{(f^{-1}h)^{\alpha}f^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{A}{g_1}\overset{C}{g_2}\overset{B}{\overline{(h^{-1}}}\ \overset{C}{\overline{f\ldots)}}_1,\quad T = (\rho+6)\lambda L(r)$

48. $t \equiv \overset{B}{g_1^{-1}}\overset{C}{h}\overset{C}{f^{-1}}\overset{B}{g_1}\overset{A}{g_2}\overset{C}{f^{-1}}\overset{C}{h}\overset{C}{\overline{(f^{-1}h\ldots)}}_1,\quad T = (\rho+7)\lambda L(r)$

49. $t \equiv \overset{B}{g_1^{-1}}\overset{C}{\overline{(hf^{-1})^{\alpha}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{B}{g_1}\overset{A}{g_2}\overset{C}{\overline{(f^{-1}}}\overset{C}{\overline{h\ldots)}}_1,\quad T = (\rho+6)\lambda L(r)$

50. $t \equiv \overset{B}{g_1^{-1}}\overset{B}{h}\overset{B}{f^{-1}}\overset{B}{g_1}\overset{A}{g_2}\overset{B}{h^{-1}}\overset{B}{f}\overset{C}{\overline{(h^{-1}f\ldots)}}_1,\quad T = (\rho+7)\lambda L(r)$

51. $t \equiv \overset{B}{g_1^{-1}}\overset{C}{\overline{(hf^{-1})^{\alpha}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{B}{g_1}\overset{A}{g_2}\overset{B}{\overline{(h^{-1}}}\overset{C}{\overline{f\ldots)}}_1,\quad T = (\rho+6)\lambda L(r).$

Lastly suppose that g occurs in tvt^{-1} as an F-subword only once, but t begins in the F-subword g or g^{-1} and ends in the F-subword g or g^{-1}. Then

$t \subset g(f^{-1}h)^{\alpha_1}f^{-1}g(f^{-1}h)^{\alpha_2}f^{-1}g;\ \alpha_1+\alpha_2>1;\ \alpha_1\geqslant\alpha_2,$

(Cases 52-55), or

$t \subset g(f^{-1}h)^{\alpha_1}f^{-1}g(h^{-1}f)^{\alpha_2}h^{-1}g;\ \alpha_1+\alpha_2\geqslant 1;\ \alpha_1\geqslant\alpha_2,$

(Cases 56-59), or

$t \subset g(f^{-1}h)^{\alpha_1}f^{-1}g(f^{-1}h)^{\alpha_2}g^{-1};\ \alpha_1+\alpha_2>1;\ \alpha_2>0,$

(Cases 60-63), or

$t \subset g(f^{-1}h)^{\alpha_1}f^{-1}g(h^{-1}f)^{\alpha_2}g^{-1};\ \alpha_1+\alpha_2>1;\ \alpha_2>0,$

(Cases 64-67), or

$t \subset g^{-1}(hf^{-1})^{\alpha_1}g(f^{-1}h)^{\alpha_2}g^{-1};\ \alpha_1,\alpha_2>0;\ \alpha_1\geqslant\alpha_2,$

(Cases 68-71), or

$t \subset g^{-1}(hf^{-1})^{\alpha_1}g(h^{-1}f)^{\alpha_2}g^{-1};\ \alpha_1,\alpha_2>0;\ \alpha_1\geqslant\alpha_2,$

(Cases 72-75).

52. $t \equiv \overset{C}{g_4}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-2}f^{-1}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{C}{g_3}\overset{A}{g_0}\overset{C}{g_4}\overset{C}{\overline{f^{-1}}}g_3$, (where $\alpha_2 = 0$),

$T = (\rho+6)\lambda L(r)$

53. $t \equiv \overset{C}{g_4}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}}}\overset{C}{\overline{f^{-1}hf^{-1}}}\overset{A}{g_3}\overset{C}{g_0}g_4\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{C}{\overline{f^{-1}hf^{-1}}}g_3$,

(where $\alpha_2>0$), $T = (\rho+6)\lambda L(r)$.

54. $t \equiv \overset{C}{\overline{g_0g_4}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-2}f^{-1}hf^{-1}}}\overset{C}{h}\overset{C}{\overline{f^{-1}g_3}}\overset{C}{\overline{g_0g_4}}\overset{C}{\overline{f^{-1}g_3g_0}}$, (where $\alpha_2 = 0$),

$T = 6\lambda L(r)$.

55. $t \equiv \overset{C}{\overline{g_0g_4}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-2}f^{-1}}}\overset{C}{\overline{hf^{-1}g_3g_0}}\overset{C}{\overline{g_4(f^{-1}h)^{\alpha_2-1}}}\overset{C}{\overline{f^{-1}hf^{-1}g_3g_0}}$,

(where $\alpha_2>0$), $T = 5\lambda L(r)$.

56. $t \equiv \overset{C}{g_4}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{BC}{hf^{-1}}\overset{C}{g_3}\overset{A}{g_0}\overset{C}{g_4}\overset{B}{h^{-1}}\overset{C}{g_3}$, (where $\alpha_2 = 0$),

$T = (\rho+8)\lambda L(r)$

57. $t \equiv \overset{C}{g_4}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{B}{\overline{hf^{-1}}}\overset{C}{\overline{g_3g_0}}\overset{C}{g_4}\overset{C}{\overline{h^{-1}}}\overset{C}{\overline{fh^{-1}(fh^{-1})^{\alpha_2-1}}}\overset{C}{\overline{g_3g_0}}$,

(where $\alpha_2>0$), $T = 8\lambda L(r)$.

58. $t \equiv \overset{C}{\overline{g_0g_4}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{B}{h}\overset{C}{f^{-1}}\overset{C}{g_3}\overset{C}{\overline{g_0g_4}}\overset{B}{h^{-1}}\overset{C}{\overline{g_3g_0}}$,

(where $\alpha_2 = 0$), $T = 8\lambda L(r)$.

59. $t \equiv \overset{C}{\overline{g_0g_4}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{B}{\overline{hf^{-1}}}\overset{C}{g_3}\overset{C}{\overline{g_0g_4}}\overset{B}{\overline{h^{-1}}}\overset{C}{\overline{fh^{-1}(fh^{-1})^{\alpha_2-1}}}\overset{C}{\overline{g_3g_0}}$,

(where $\alpha_2>0$), $T = 8\lambda L(r)$.

60. $t \equiv \overset{C}{\overline{g_0g_4f^{-1}}}\overset{A}{g_3}\overset{C}{\overline{g_0g_4}}\overset{C}{\overline{f^{-1}hf^{-1}}}\overset{C}{\overline{h(f^{-1}h)^{\alpha_2-2}}}\overset{B}{g_4}^{-1}$,

(where $\alpha_1 = 0$), $T = (\rho+5)\lambda L(r)$.

61. $t \equiv \overset{C}{\overline{g_4f^{-1}}}\overset{A}{g_3}\overset{B}{\overline{g_0g_4}}\overset{C}{\overline{f^{-1}hf^{-1}}}\overset{C}{\overline{h(f^{-1}h)^{\alpha_2-2}}}\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$,

(where $\alpha_1 = 0$), $T = (\rho+5)\lambda L(r)$.

62. $t \equiv \overset{C}{\overline{g_0g_4f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{A}{g_3}\overset{C}{\overline{g_0g_4f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{B}{g_4}^{-1}$

(where $\alpha_1>0$), $T = (\rho+5)\lambda L(r)$.

63. $t \equiv \overset{C}{\overline{g_4f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{A}{g_3}\overset{B}{\overline{g_0g_4}}\overset{C}{\overline{f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$,

(where $\alpha_1>0$), $T = (\rho+6)\lambda L(r)$.

64. $t \equiv \overset{C}{\overline{g_0 g_4}}\overset{B}{f^{-1}}\overset{A}{g_3}\overset{C}{\overline{g_0 g_4}}\overset{C}{\overline{h^{-1}f}}\overset{C}{\overline{h^{-1}f(h^{-1}f)^{\alpha_2-2}}}\overset{B}{g_4^{-1}}$,

(where $\alpha_1 = 0$), $T = (\rho+6)\lambda L(r)$

65. $t \equiv \overset{C}{g_4}\overset{B}{f^{-1}}\overset{A}{g_3}\overset{B}{\overline{g_0 g_4}}\overset{C}{\overline{h^{-1}f}}\overset{C}{\overline{h^{-1}f(h^{-1}f)^{\alpha_2-2}}}\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$

(where $\alpha_1 = 0$), $T = (\rho+6)\lambda L(r)$

66. $t \equiv \overset{C}{\overline{g_0 g_4 f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{A}{g_3}\overset{C}{\overline{g_0 g_4 f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{B}{g_4^{-1}}$

(where $\alpha_1>0$), $T = (\rho+5)\lambda L(r)$

67. $t \equiv \overset{C}{\overline{g_4 f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_1-1}f^{-1}}}\overset{A}{g_3}\overset{B}{g_0}\overset{C}{\overline{g_4 f^{-1}h}}\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$,

(where $\alpha_1>0$), $T = (\rho+6)\lambda L(r)$

68. $t \equiv \overset{B}{g_3^{-1}}\overset{C}{h}\overset{C}{f^{-1}}\overset{B}{g_3}\overset{A}{g_0}\overset{B}{g_4}\overset{C}{f^{-1}}\overset{C}{h}\overset{B}{g_4^{-1}}$, $T = (\rho+8)\lambda L(r)$

69. $t \equiv \overset{B}{g_3^{-1}}\overset{C}{\overline{(hf^{-1})^{\alpha_1-2}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{B}{g_3}\overset{A}{g_0}\overset{B}{g_4}\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{C}{\overline{f^{-1}h}}\overset{B}{g_4^{-1}}$

where $\alpha_1>1$, $T = (\rho+8)\lambda L(r)$

70. $t \equiv \overset{B}{\overline{g_0^{-1}g_3^{-1}}}\overset{C}{h}\overset{C}{f^{-1}}\overset{B}{\overline{g_3 g_0}}\overset{B}{g_4}\overset{C}{f^{-1}}h\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$

$T = 8\lambda L(r)$

71. $t \equiv \overset{B}{\overline{g_0^{-1}g_3^{-1}}}\overset{C}{\overline{(hf^{-1})^{\alpha_1-2}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{B}{\overline{g_3 g_0}}\overset{B}{g_4}\overset{C}{\overline{(f^{-1}h)^{\alpha_2-1}}}\overset{C}{\overline{f^{-1}h}}\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$,

(where $\alpha_1>1$), $T = 8\lambda L(r)$

72. $t \equiv \overset{B}{g_3^{-1}}\overset{B}{h}\overset{B}{f^{-1}}\overset{B}{g_3}\overset{A}{g_0}\overset{B}{g_4}\overset{B}{h^{-1}}\overset{B}{f}\overset{B}{g_4^{-1}}$, $T = (\rho+8)\lambda L(r)$

73. $t \equiv \overset{B}{g_3^{-1}}\overset{C}{\overline{(hf^{-1})^{\alpha_1-2}hf^{-1}}}\overset{C}{\overline{hf^{-1}}}\overset{B}{g_3}\overset{A}{g_0}\overset{B}{g_4}\overset{C}{\overline{(h^{-1}f)^{\alpha_2-1}}}\overset{C}{\overline{h^{-1}f}}\overset{B}{g_4^{-1}}$,

(where $\alpha_1>1$) , $T = (\rho+8)\lambda L(r)$

74. $t \equiv \overset{B}{\overline{g_0^{-1}g_3^{-1}}}\overset{B}{h}\overset{B}{f^{-1}}\overset{B}{\overline{g_3 g_0}}\overset{B}{g_4}\overset{B}{h^{-1}}\overset{B}{f}\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$, $T = 8\lambda L(r)$

75. $t \equiv \overset{B}{\overline{g_0^{-1}g_3^{-1}}}\,\overset{C}{\overline{(hf^{-1})^{\alpha_1-2}hf^{-1}}}\,\overset{C}{\overline{hf^{-1}}}\,\overset{B}{g_3}\,\overset{B}{\overline{g_0g_4}}\,\overset{C}{\overline{(h^{-1}f)^{\alpha_2-1}}}\,\overset{B}{\overline{h^{-1}f}}\,\overset{B}{\overline{g_4^{-1}g_0^{-1}}}$,
(where $\alpha_1>1$), $T = 8\lambda L(r)$

We have listed all the cases where g,f or h do not occur twice as distinct F-subwords of $t\upsilon t^{-1}$. If g occurs twice as an F-subword of $t\upsilon t^{-1}$, then t or t^{-1} has the subword

$g(f^{-1}h)^{\alpha}f^{-1}g$ where $\alpha\geqslant 1$

or $g(h^{-1}f)^{\alpha}h^{-1}g$ where $\alpha\geqslant 1$

or $g(f^{-1}h)^{\alpha}g^{-1}$ where $\alpha>1$

and for each of these it can be seen by using method B or C of Section 2.2 that g is a piece. Similarly if f(respectively h) occurs twice as distinct F-subwords of t, then f (respectively h) is a piece. This completes the analysis of the cases, when the factorization is of Type I.

Part B The factorization (f,g,h) is of Type II

As in Part A, we do not list those cases which are similar. These cases can be found by replacing the letters (f,g,h) in the expressions by $f,g^{\varepsilon_1},h^{\varepsilon_2}$ or $f^{-1},h^{\varepsilon_2},g^{\varepsilon_1}$ where $\varepsilon_1,\varepsilon_2 = \pm 1$. By this, we can assume that the number of times that g occurs as a distinct F-subword of $t\upsilon t^{-1}$ is less than the number of times that h occurs.

We consider in the first 22 cases the possible values for t, where t is a subword of W^*, and where g or f is not an F-subword of $t\upsilon t^{-1}$.

Suppose first that none of the words f,g,h are F-subwords of $t\upsilon t^{-1}$, then t or t^{-1} is a subword of fh^{ε}, $f^{-1}g^{\varepsilon}$, hh or

gg, and this gives cases B1, B2 and B3 of Theorem 2.1.

Secondly let h be an F-subword of t, but f(and therefore g) not F-subwords of $t\upsilon t^{-1}$. Then $t \subset fh^{\alpha}f^{-1}$, or similar F-subwords of W^*. This gives cases B5-11 of Theorem 2.1.

In the third place let f be an F-subword of t, but g and h not F-subwords of $t\upsilon t^{-1}$. Then $t \subset gfh$, and this gives case B12 of Theorem 2.1.

Lastly let f and h be F-subwords of t but g not an F-subword of $t\upsilon t^{-1}$. This gives cases B13-20 of Theorem 2.1, and two additional cases, 21 and 22. These extra cases arise when g is a subconjugate of t but not an F-subword of $t\upsilon t^{-1}$. As in Part A, the value for T in these cases shows that the conclusion for Theorem 2.1 is in a sense maximal, and in fact these are limiting cases for Theorem 2.2.

21. $t \equiv \overset{C}{g_0}\overset{A}{g_4}\overset{B}{f}\ \overset{A}{h}\overset{B}{f^{-1}}\overset{A}{g_3}\overset{C}{g_0}, \quad T = (3\rho+4)\lambda L(r)$

22. $t \equiv \overset{C}{g_0}\overset{A}{g_4}\overset{B}{f}\overset{C}{h}\ldots\overset{C}{h}\overset{B}{f^{-1}}\overset{A}{g_3}\overset{C}{g_0}, \quad T = (2\rho+6)\lambda L(r)$

We consider in 23 and 24 those cases where f, g and h are F-subwords of t, once each. Then $t \subset f^{-1}gfhf^{-1}$, and

23. $t \equiv \overset{B}{f_3^{-1}}\ \overset{A}{g}\ \overset{B}{f_3}\overset{A}{f_0}\overset{B}{f_4}\overset{A}{h}\ \overset{B}{f_4^{-1}}, T = (3\rho+4)\lambda L(r)$

24. $t \equiv \overset{B}{\overline{f_0^{-1}\ f_3^{-1}}}\ \overset{A}{g}\ \overset{B}{\overline{f_3 f_0 f_4}}\ \overset{B\ A}{h}\ \overset{B}{\overline{f_4^{-1} f_0^{-1}}}, \quad T = (2\rho+4)\lambda L(r)$

In 25 and 26, we consider those cases, where g and f are F-subwords of t, once each, but h occurs as a distinct F-subword of t more than once. Then $t \subset f^{-1}gfh\ldots hf^{-1}$, and we have

$$25.\quad t \equiv \overset{B}{f_3^{-1}}\,\overset{A}{g}\,\overset{B}{f_3}\,\overset{A}{f_0}\,\overset{B}{f_4}\,\overset{C}{h}\overline{\ldots}\overset{C}{h}\,\overset{B}{f_4^{-1}}, \quad T = (2\rho+6)\lambda L(r)$$

$$26.\quad t \equiv \overset{B}{\overline{f_0^{-1}f_3^{-1}}}\,\overset{A}{g}\,\overset{B}{\overline{f_3}}\,\overset{B}{f_0}\,\overset{C}{f_4 h}\overline{\ldots}\overset{C}{\overline{h}}\,\overset{B}{\overline{f_4^{-1}f_0^{-1}}}, \quad T = (\rho+6)\lambda L(r)$$

In 27-31, we consider those cases where g and f are F-subwords of t, once each, but h occurs as a distinct F-subword of tut^{-1} more than once. If f occurs only twice then $t \subset f^{-1}gfhf^{-1}g^{\varepsilon}$ and we have

$$27.\quad t \equiv \overset{B}{f_1^{-1}}\,\overset{C}{g_1}\,\overset{ABA}{g_2fh}\,\overset{B}{f^{-1}}\,\overset{C}{g_1}, \quad T = (2\rho+5)\lambda L(r)$$

$$28.\quad t \equiv \overset{B}{f_1^{-1}}\,\overset{A}{g_1}\,\overset{B}{\overline{g_2f}}\,\overset{A}{h}\,\overset{B}{\overline{f^{-1}g_2^{-1}}}, \quad T = (2\rho+3)\lambda L(r)$$

If f occurs more than twice, then $t \subset h^{\varepsilon_1}f^{-1}gfhf^{-1}g^{\varepsilon_2}$ and we have

$$29.\quad t \equiv \overset{C}{\overline{h_2f^{-1}g_1}}\,\overset{ABA}{g_2fh_1}\,\overset{C}{\overline{h_2f^{-1}g_1}}, \quad T = (2\rho+3)\lambda L(r)$$

$$30.\quad t \equiv \overset{C}{\overline{h_2f^{-1}}}\,\overset{A}{g_1}\,\overset{B}{g_2}\,\overset{BA}{fh_1}\,\overset{C}{\overline{h_2f^{-1}}}\,\overset{B}{g_2^{-1}}, \quad T = (2\rho+5)\lambda L(r)$$

$$31.\quad t \equiv \overset{B}{\overline{h_1^{-1}f^{-1}}}\,\overset{A}{g_1}\,\overset{B}{g_2}\,\overset{B}{\overline{f}}\,\overset{A}{h_1h_2}\,\overset{B}{\overline{f^{-1}g_2^{-1}}}, \quad T = (2\rho+4)\lambda L(r)$$

In 32 and 33, we consider those cases where h and g are distinct F-subwords of t at least twice, but f is an F-subword of tut^{-1} only once.

Then $t \subset f^{-1}g\ldots gfh\ \ldots hf^{-1}$, and we get

32. $t \equiv \overset{B}{f_3^{-1}}\overset{C}{\overline{g\ldots}}\overset{C}{g}\overset{B}{f_3}\overset{A}{f_0}\overset{B}{f_2}\ \overset{C}{\overline{h\ldots}}\overset{C}{h}\ \overset{B}{f_2^{-1}}$, $T = (\rho+8)\lambda L(r)$

33. $t \equiv \overset{B}{f_0^{-1}}\overset{C}{g}\overline{\ldots}\overset{C}{g}\overset{B}{f_3}\overset{A}{f_0}\overset{B}{f_2}\overset{C}{h}\overline{\ldots}\overset{C}{h}\overset{B}{f_2^{-1}}$, $T = (\rho+8)\lambda L(r)$

We consider in 34-59 those cases where g is an F-subword of tut^{-1} only once, but f and h are distinct F-subwords of tut^{-1} at least twice.

If t begins in the F-subword h, and ends in the F-subword h^{ε}, then $t \subset h\ldots hf^{-1}gfh^{\varepsilon}\ldots h^{\varepsilon}$ and this is examined in cases 34-37.

If t begins in the F-subword $f^{\pm 1}$, and ends in h^{ε}, then $t \subset fhf^{-1}gfh^{\varepsilon}\ldots h^{\varepsilon}$ (examined in cases 38,40) or $t \subset fh\ldots hf^{-1}gfh^{\varepsilon}\ldots h^{\varepsilon}$(examined in cases 39,41).

If t begins in the F-subword $f^{\pm 1}$ and ends in g^{ε}, then $t \subset f^{-1}gfh\ldots hf^{-1}g^{\varepsilon}$, and this is examined in cases 42,43.

If t begins in the F-subword h^{ε} and ends in the F-subword g^{ε}, then $t \subset h^{\varepsilon}\ldots h^{\varepsilon}\ f^{-1}gfhf^{-1}g^{\varepsilon_2}$(examined in cases 44-47) or $t \subset h^{\varepsilon}\ldots h^{\varepsilon}f^{-1}gfh\ldots hf^{-1}g^{\varepsilon_2}$(examined in cases (48-51).

Finally if t begins in the F-subword g^{ε_1} and ends within the F-subword g^{ε_2}then $t \subset g^{\varepsilon_1}fh^{\alpha_1}\ f^{-1}gfh^{\alpha_2\varepsilon}f^{-1}g^{\varepsilon_2}$ where we can assume $\alpha_1 \geqslant \alpha_2 > 0$ (examined in cases 52-59).

34. $t \equiv \overset{C}{\overline{(\ldots h)_4h}}\overset{C}{h}\overset{B}{f^{-1}}\ \overset{A}{g}\ \overset{B}{f}\ \overset{C}{h_3}$, $T = (\rho+5)\lambda L(r)$

35. $t \quad \overset{C}{\overline{(h\ldots h)_4}}\overset{C}{h}\overset{B}{f^{-1}}\overset{A}{g}\overset{B}{f}\overset{C}{h}\overset{C}{\overline{(h\ldots h)_3}}$, $T = (\rho+6)\lambda L(r)$

C CB AB B

36. $t \equiv (\ldots h)_4 hhf^{-1}gfh_2{}^{-1}$, $T = (\rho+5)\lambda L(r)$

C BB A B C

37. $t \equiv (\ldots h)_4 hf^{-1}gfh^{-1}(\ldots h^{-1})_2$, $T = (\rho+5)\lambda L(r)$

B CB ABC C

38. $t \equiv f_2 hf^{-1}gfh(h\ldots)_1$, $T = (\rho+6)\lambda L(r)$

B C CB AB C C

39. $t \equiv f_2 h\ldots hf^{-1}gf(h\ldots h)_1$, $T = (\rho+7)\lambda L(r)$

B BB ABB C

40. $t \equiv f_2 hf^{-1}gfh^{-1}(h^{-1}\ldots)_1$, $T = (\rho+6)\lambda L(r)$

B C CB AB B C

41. $t \equiv f_2 h\ldots hf^{-1}gf(h^{-1}\ldots h^{-1})_1$, $T = (\rho+7)\lambda L(r)$

B A BC C B B

42. $t \equiv f_1{}^{-1}g_1g_2fh\ldots hf^{-1}g_1$, $T = (\rho+6)\lambda L(r)$

B A B C C B

43. $t \equiv f_1{}^{-1}g_1g_2fh\ldots hf^{-1}g_2{}^{-1}$, $T = (\rho+5)\ L(r)$

C C A B C

44. $t \equiv (h\ldots h)_3 hf^{-1}g_1g_2f\ hf^{-1}g_1$, $T = (\rho+4)\lambda L(r)$

C B C A B C

45. $t \equiv (h^{-1}\ldots h^{-1})_3 h^{-1}f^{-1}g_1g_2fhf^{-1}g_1$, $T = (\rho+5)\lambda L(r)$

C CB A B C B

46. $t \equiv (h\ldots h)_3 hf^{-1}g_1g_2fhf^{-1}g_2{}^{-1}$, $T = (\rho+6)\lambda L(r)$

C B A B B B

47. $t \equiv (h^{-1}\ldots h^{-1})_3 h^{-1}f^{-1}g_1g_2fhf^{-1}g_2{}^{-1}$, $T = (\rho+5)\lambda L(r)$

C C A BC C C

48. $t \equiv (h\ldots h)_3 f^{-1}g_1g_2fh\ldots hf^{-1}g_1$, $T = (\rho+6)\lambda L(r)$

C B C A BC C C

49. $t \equiv (h^{-1}\ldots h^{-1})_3 f^{-1}g_1g_2fh\ldots hf^{-1}g_1$, $T = (\rho+7)\lambda L(r)$

C C A BC C B

50. $t \equiv (h\ldots h)_3 f^{-1}g_1g_2fh\ldots hf^{-1}g_2{}^{-1}$, $T = (\rho+6)\lambda L(r)$

C B A B C C B B

51. $t \equiv (h^{-1}\ldots h^{-1})_3 f^{-1}g_1g_2fh\ldots hf^{-1}g_2{}^{-1}$, $T = (\rho+7)\lambda L(r)$

Note that in the remaining cases if $\varepsilon = +1$, then h^{α_2} is a piece using Method B, and if $\varepsilon = -1$, then h^{α_2} is a piece using Method C. We refer to this argument as Z, and use it in all the remaining cases to show that h^{α_2} and h are pieces.

If $\varepsilon_1 = \varepsilon_2 = 1$, then we have

52. $t \equiv \overset{C}{\overline{g_4 f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{C}{\overline{f^{-1}g_3}}\,\overset{A}{g_0}\,\overset{C}{\overline{g_4 f}}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{C}{\overline{f^{-1}g_3}}$, $T = (\rho+7)\lambda L(r)$

53. $t \equiv \overset{C}{\overline{g_0 g_4 f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{C}{\overline{f^{-1}g_3 g_0}}\,\overset{C}{\overline{g_4 f}}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{C}{\overline{f^{-1}g_3 g_0}}$, $T = 7\lambda L(r)$

If $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, we have

54. $t \equiv \overset{C}{\overline{g_4 f}}\,\overset{Z}{h}\,\overset{C}{\overline{h^{\alpha_1-1}f^{-1}}}\,\overset{B}{g_3}\,\overset{A}{g_0}\,\overset{B}{\overline{g_4 f}}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{B}{\overline{f^{-1}g_4^{-1}g_0^{-1}}}$, $T = (\rho+7)\lambda L(r)$

55. $t \equiv \overset{C}{\overline{g_0 g_4 f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{B}{f^{-1}g_3}\,\overset{A}{g_0}\,\overset{C}{\overline{g_4 f}}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{B}{\overline{f^{-1}g_4^{-1}}}$, $T = (\rho+7)\lambda L(r)$

If $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, we have

56. $t \equiv \overset{B}{\overline{g_0^{-1}g_3^{-1}f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{B}{\overline{f^{-1}g_3 g_0}}\,\overset{AB}{g_4 f}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{C}{\overline{f^{-1}g_3}}$, $T = (\rho+7)\lambda L(r)$

57. $t \equiv \overset{B}{\overline{g_3^{-1}f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{C}{\overline{f^{-1}g_3}}\,\overset{A}{g_0}\,\overset{B}{g_4 f}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{C}{\overline{f^{-1}g_3 g_0}}$, $T = (\rho+7)\lambda L(r)$

If $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, we have

58. $t \equiv \overset{B}{\overline{g_3^{-1}f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{B}{\overline{f^{-1}g_3}}\,\overset{A}{g_0}\,\overset{B}{\overline{g_4 f}}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{B}{\overline{f^{-1}g_4^{-1}}}$, $T = (\rho+7)\lambda L(r)$

59. $t \equiv \overset{B}{\overline{g_0^{-1}g_3^{-1}f}}\,\overset{Z}{h}\,\overset{C}{h^{\alpha_1-1}}\,\overset{B}{\overline{f^{-1}g_3 g_0}}\,\overset{B}{\overline{g_4 f}}\,\overset{Z}{h^{\alpha_2\varepsilon}}\,\overset{B}{\overline{f^{-1}g_4^{-1}g_0^{-1}}}$, $T = 7\lambda L(r)$

This completes all the cases where g, f or h do not occur twice as F-subwords of tvt^{-1}. If f occurs twice as an F-subword tvt^{-1}, then t or t^{-1} has the subword $f^{-1}g^{\alpha}f$, or $fh^{\alpha}f^{-1}$ $(\alpha \neq 0)$, so that f is a piece by method B. If g occurs twice as an F-subword of tvt^{-1}, then t or t^{-1} has the subword gg or $gfh^{\alpha}f^{-1}g^{\varepsilon}$ $(\alpha \neq 0)$ as a subword, in which case, using method C if the subword is gg or $\varepsilon = 1$ and method B if $\varepsilon = -1$, it can be seen that g is a piece. Similarly it can be shown that h is a piece if it occurs more than once as an F-subword of t. Therefore if f, g, and h all occur twice as F-subwords of $t \cup t^{-1}$ then f, g, and h are pieces.

CHAPTER III

COMMUTATORS AND GENERATORS

SECTION 1 SUMMARY

PRELIMINARY REMARKS:

A problem that occurs in group theory is that of determining whether or not a set of elements in a group, generate that group. A related problem is the isomorphism problem.(Given a set J of group presentations, is there an algorithm to decide for any pair of elements of J, whether or not they define isomorphic groups ?). As any isomorphism $G \longrightarrow H$, sends generating sets of G to generating sets of H, a study of generating sets is closely allied to a study of isomorphisms. We shall examine a connection between commutators and generating sets (in particular, for groups of rank 2).

INTRODUCTION:

Let X be a fixed generating n-tuple of a group G, and let Y be an arbitrary n-tuple of elements of G. Consider those sets Y which satisfy one or more of the following conditions:

a) Y belongs to the same NE class (or T-system) as X.

b) Y generates G

c) A higher commutator on Y is conjugate in G to a higher commutator on X or the inverse of such a commutator.

For certain types of groups, we will examine the relationship between those sets Y which satisfy a),b) or c). For if we can show that Y generates G iff Y belongs to the same NE class or T-system as X in G, or iff a higher commutator on Y is conjugate to a higher commutator on X (or its inverse), then the problem of determining the generators of G is changed to that of finding those sets Y which satisfy a) or c).

Our main interest will be in the case $n=2$. In this case we will write $X=(a\ b)$, $Y=(u,v)$. Then a),b),c) become:

a) (u,v) belongs to the same NE class (or T-system) as (a,b)

b) (u,v) generates G

c) $[u,v] \sim [a,b]^{\pm 1}$.

Until further notice we will assume $n=2$.

Consider the following properties (these properties are quantified over all pairs (u,v)

A : $[u,v] \sim [a,b]^{\pm 1}$ iff (u,v) generates G

B_N: $[u,v] \sim [a,b]^{\pm 1}$ iff (u,v) and (a,b) belong to the same NE class.

B_T: $[u,v] \sim [a,b]^{\pm 1}$ iff (u,v) and (a,b) belong to the same T-system.

C_N: (u,v) generates G iff (u,v) and (a,b) belong to the same NE class.

C_T: (u,v) generates G iff (u,v) and (a,b) belong to the same T-system.

These last two properties C_N, C_T can be shortened to saying G has 1 NE class or 1 T-system of generating pairs respectively. In general if G satisfies any two of the properties (A, B_N, C_N), then G satisfies all three. Similarly if G satisfies any two of the properties (A, B_T, C_T), then G satisfies all three. From the point of view of our preliminary remarks, the most interesting of these properties is A.

A well known theorem of Nielsen [34] (see also Malcev[31]) asserts that if G is free on a,b then G has A. In fact in this case G has all five properties. We remark in passing that Dicks [6] has verified a property analogous to A for the free algebra of rank 2 over a field. He has shown that if k is a field, then u and v generate $k\langle x,y\rangle$ (as a k-algebra) if the commutator $[u,v] = uv-vu$ is a non-zero scalar multiple of the commutator $xy-yx$.

We will consider the following two questions:

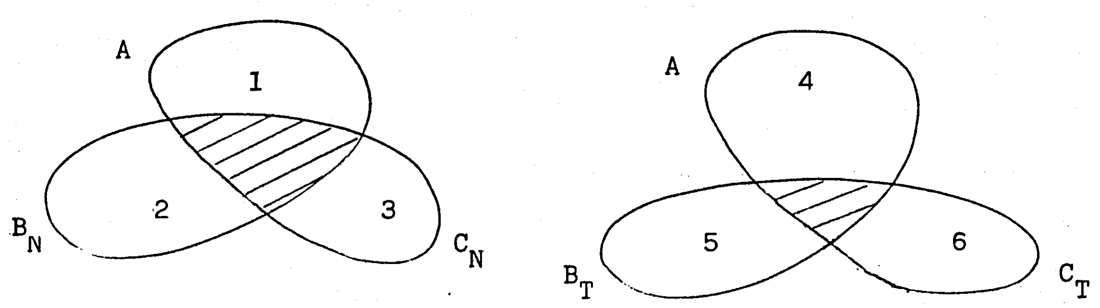
Question 1: Are there groups which satisfy one of the properties A, B_N or C_N, but not all three ? Similarly, are theregroups which satisfy one of the properties A, B_T or C_T, but not all three ?

Question 2: Can we show that certain types of groups posess properties A, B_N, C_N, B_T or C_T ?

In dealing with Question 2 we will consider almost exclusively infinite groups. More specifically, we will consider groups which from the point of view of combinatorial group theory are known to behave "like" the free group of rank 2.

QUESTION 1:

We can illustrate the question by means of diagrams



Then the question is, are the outer areas 1-6 non-empty ?
The following example shows that 3 and 6 are non-empty.

EXAMPLE 3.1 $G = \langle a,b;[a,b]\rangle$

Then G has one NE class and therefore one T-system.
But (a^2,b) does not generate G, and $[a^2,b] = [a,b] = 1$.

#

The next example shows that 2 and 5 are non-empty.

EXAMPLE 3.2 $G = \langle a,b;a^{16},b^{17},(ab)^8\rangle$

Then in Lemma 3.1 we will prove that G has Properties B_N and B_T, but more than one T-system.
(If one just requires an example showing 2 is non-empty then $G = \langle a,b;a^5\rangle$ is easier).

#

To show that region 1 is non-empty we have to find a group with more than one NE class which has Property A. In fact given $p \geqslant 2$, we find a group with exactly 2^{p-2} NE-classes which has Property A.

EXAMPLE 3.3 $G = \langle a,b;b^{2^p},[a,b]\rangle, p \geqslant 2.$

Then we will show, in Lemma 3.2 that G has 2^{p-2} NE classes and posesses Property A. Therefore for $p>2$, G lies in region 1 of the diagrams.

#

(It has been pointed out to us by J. Wiegold that the dihedral group of order 8, which as 2 NE classes, also lies in region 1 of the diagrams).

Unfortunately the group in Example 3 has one T-system. For we will see in Lemma 3.2 that

$$\{a, b^{j}; (j,2)=1,\ 0<j<2^{p-1}\}$$

is a set of representatives of the NE classes of G.

Let β_j be the map $a \mapsto a, b \mapsto b^{j}$, where $(j,2) = 1$, and $0<j<2^{p-1}$.

Then $\quad [a,b^2]\beta_j = [a,b^{2j}] =_G 1$

and $\quad b^{2^p}\beta_j = b^{2^p j} =_G 1$.

Thus β_j maps the relators in the given presentation to 1 in G. Also, as $(j,2) = 1$, there exists k such that $jk \equiv 1 \bmod 2$. It is clear that β_j and β_k define mutually inverse automorphisms of G. It now follows that G has one T-system.

QUESTION 2

(i) *Free products of cyclic groups:* $G=\langle a,b; a^s, b^t\rangle; s,t \geq 1$.

Then it is easily shown that G has MN NE classes.

where $\quad M = 1$ if $s = 1$ or 2, $\frac{1}{2}\phi(s)$ if $\infty>s>2$, 1 if $s = \infty$,

$\quad N = 1$ if $t = 1$ or 2, $\frac{1}{2}\phi(t)$ if $\infty>t>2$, 1 if $t = \infty$.

For by the Grushko-Newman Theorem, [27], any generating pair of G is in the same NE class as a pair of form (a^{i}, b^{j}) with $(i,s) = (j,t) = 1$. If we have two pairs (a^{i_1}, b^{j_1}), (a^{i_2}, b^{j_2}) of this form, then they are in the same NE class iff $\quad [a^{i_1}, b^{j_1}] \sim [a^{i_2}, b^{j_2}]^{\pm 1}$.

By the theory of normal forms for free products, this implies $i_1 \equiv \pm i_2 \pmod s$, $j_1 \equiv \pm j_2 \pmod t$.

Therefore G has one NE class if and only if

$s,t \in \{1,2,3,4,6,\infty\}$

However G has only one T-system. In order to see this, let $\beta_{i,j}$ be the mapping in G, $a \mapsto a^{i}, b \mapsto b^{j}$, where $(i,s) = (j,t) = 1$. Then this maps the relators a^s, b^t to a^{is}, b^{jt} respectively, and $a^{is}, b^{jt} = 1$ in G. Additionally as $(i,s) = (j,t) = 1$, there exists integers k, ℓ, where $ki \equiv 1 \bmod s$, $\ell j \equiv 1 \bmod t$ so that $\beta_{i,j}\, \beta_{k,\ell}$ is the identity map. Therefore $\beta_{i,j}$ is an automorphism as required.

As regards Property B_N, we have the following result.

THEOREM 3.1 *If* $G = \langle a,b; a^s, b^t\rangle; s,t>1$, *then* G *has Property* B_N.

This result is presumably well known. A proof can be found in Section 3.

#

Thus G has Properties A, B_N, B_T, C_N and C_T if and only if $s,t \in \{2,3,4,6,\infty\}$

One can ask about Properties A, B_N, B_T, C_N and C_T for two-generator amalgamated products of cyclic groups.

LEMMA 3.3 *Let* $G = \langle a,b; a^k = b^\ell\rangle$, k,ℓ *positive integers, then* G *does not have Properties* A, B_N *or* B_T.

In order to see this, let d be an integer greater than 1, such that $(k,d) = (\ell,d) = 1$ (e.g. $d=k\ell+1$). Then for certain integers i,j, $ik \equiv j\ell \equiv 1 \pmod d$.

Now since a^k, b^ℓ are central in G,

$$[a^{1-ik}, b^{1-j\ell}] = [a,b]$$

However McCool and Pietrowski have proved [32], that if r,s are non-zero integers, (x^r, y^s) is a generating pair of G if and only if $(r,s) = (r,k) = (s,\ell) = 1$.

Therefore as

$$(1-ik),(1-j\ell) \equiv 0 \pmod{d} \text{ and } d\neq 1,$$

$$(a^{1-ik}, b^{1-j\ell}) \text{ does not generate } G.$$

#

It has been shown by Zieschang [64] that when $k + \ell > 4$, G has an infinite number of NE classes, and when $k = \ell = 2$, G has only one NE class.

(ii) Small cancellation groups : $G = \langle a,b;R\rangle$

Using techniques similar to those in Chapter II we will prove the following:

THEOREM 3.2. *Let $G = \langle a,b;R\rangle$ be a non-cyclic group, where R satisfies $C_L'(1/16)$ or $C_L'(1/12)$ and $T(4)$ and where a,b are both pieces. Then G has Property B_N.*

A proof of this result can be found in Section 4.

#

REMARKS 1. If a is not a piece, then $G = \langle a,b;(ab^{\lambda})^{\ell},b^{m}\rangle$ for certain $\lambda,l,m\in\mathbb{Z}$ and this is a free product of cycles. So using Theorem 3.1 we could remove the condition that a,b both be pieces from the hypothesis of Theorem 3.2. However in our proof of Theorem 3.2 we will require that a and b are both pieces.

2. A new proof of Theorem 3.2 was obtained recently in joint work with Pride. This proof involves considering diagrams on a torus and is similar to work of

Schupp [59]. This geometric technique enables one to prove a result similar to Theorem 3.2 with R satisfying the non-metric condition C(15), or C(12), T(4).

In Chapter II we have shown that if R is finitely presented and satisfies conditions $C_L'(1/14)$ or $C_L'(1/10)$ and T(4), then G has a finite number of NE classes. We conjecture that for the "right" small cancellation conditions, if all elements of R are proper powers, and no element of R is a power of a primitive, then G has one NE class.

(iii) One relator groups with torsion : $G=\langle a,b;r^m\rangle, m>1$
Without loss of generality we can assume that r is not a proper power. We can also assume that r is not empty and is not primitive, otherwise G is a free product of cycles and so was dealt with above.
It is shown by Pride in [44] that G has one NE class. Pride has also proved the following result.(See [15]).
If G is as above and $m>3$ then G has Property B_N.
It follows from these two results of Pride that if $m>3$ then G has all the properties A, B_N, B_T, C_N, C_T.

When $m=2$ G need not have Property B_N:-
EXAMPLE 3.4 Let $G_t = \langle a,b;\{(ab^{-1}ab)^t(a^{-1}b^{-1}a^{-1}b)^t\}^2\rangle$.
Let $u = a$, $v = b(a^{-1}b^{-1}a^{-1}b)^{t-1}a^{-1}(b^{-1}aba)^{t-1}b^{-1}ab$.
Then $[u,v] \sim_{G_t} [a,b]$. For

$$[u,v] \equiv a^{-1}b^{-1}\ a^{-1}\ b(a^{-1}b^{-1}a^{-1}b)^{t-1}a(b^{-1}aba)^{t-1}b^{-1}$$
$$ab(a^{-1}b^{-1}a^{-1}b)^{t-1}a^{-1}(b^{-1}aba)^{t-1}b^{-1}ab$$

$$= (a^{-1}b^{-1}a^{-1}b)^t (ab^{-1}ab)^t (a^{-1}b^{-1}a^{-1}b)^t\, b^{-1}aba^{-1}(ab^{-1}ab)^t$$
$$=_{G_t} (a^{-1}b^{-1}a^{-1}b)^{-t}\, b^{-1}aba^{-1}\, (ab^{-1}ab)^t$$
$$\sim\ a^{-1}b^{-1}ab \equiv [a,b].$$

We can show that (u,v) does not generate G_t by using the following Lemma of Pride [44].

Let $B = \langle a,b;\ Q^m\rangle$, where $m>1$ and Q is a non-empty cyclically reduced word which is not a proper power. If (a,y) generates B, then y is expressible in the form $a^{\alpha_1}b^{\pm 1}a^{\alpha_2}$ for certain integers α_1,α_2, unless some cyclic permutation of $Q^{\pm 1}$ has the form ba^l.

Now if $by^{-1}a^{\alpha_1}b^{\pm 1}a^{\alpha_2} =_{G_t} b$ for certain integers α_1,α_2, then by the strengthened Freiheitssatz,[58], the freely reduced form W of the LHS would have to contain a cyclic permutation of

$$(ab^{-1}ab)^t\,(a^{-1}b^{-1}a^{-1}b)^t.$$

In particular W would have to contain at least $4t$ occurrances of b or b^{-1}, whereas it has at most $4t-1$ occurrances.

#

Note that $G_1 = \langle a,b;\ [a,b^{-1}ab]^2\rangle$ and G_1 is an HNN extension of

$$H = \langle a_0,a_1;\ [a_0,a_1]^2\rangle$$

with stable letter b, and associated subgroups sgp$\{a_0\}$, sgp$\{a_1\}$. It has been shown by Rosenberger [53] that H has Property A, so H is a group with properties A, B_N, C_N and C_T but with a 2-generator HNN extension with only Property C_N and C_T.

It remains an open question whether $G = \langle a,b;\ r^m \rangle$ has Property B_N when $m = 3$.

GENERALISATIONS

There are two natural ways one might generalise A. In the first case suppose one chooses a fixed but arbitrary pair (a,b) from a group G(so that (a,b) need not generate G). Then consider

Property A': $[u,v] \sim [a,b]^{\pm 1}$ iff sgp$\{u,v\}$ is conjugate to sgp$\{a,b\}$.

Then even if G is a free group A' need not hold. For consider the pairs (x^2,y) and (x^2,xy), then their commutators are equal, and Burns, Edmunds and Farouqi [2] have shown that they do not generate conjugate subgroups. However, the pairs are related by a transformation of type

$(u,v) \longrightarrow (u,sv)$ where $[u,s] = 1$.

Hmelevskiĭ [16] asked if all pairs whose commutators were conjugate were related by a sequence of Nielsen transformations together with transformations of type $(u,v) \longrightarrow (u,sv)$ where $[u,s] = 1$. Lyndon and Wicks [29] have constructed an example to show that this is not the case.

The second generalisation would be to consider more than two generators. Thus let G be an n-generator group with fixed generating n-tuple $(a_1,\ldots,a_n)$. Then is there a word $W(x_1,\ldots,x_n)$ on n-variables such that the solutions of

(1) $W(x_1,\ldots,x_n) \sim W(a_1,\ldots,a_n)^{\pm 1}$

are precisely the generating n-tuples of G?
When $G = F_n = \langle a_1,\ldots,a_n \rangle$, and W is a higher commutator, then Rips [51] has shown that all solutions to (1) are generating n-tuples of G. However when W is a higher commutator, $n > 2$, not all generating n-tuples of F_n are solutions to (1). For example if $F_3 = \langle a_1,a_2,a_3 \rangle$ and $W(x_1,x_2,x_3) = [[x_1,x_2]x_3]$, then (a_1,a_2a_1,a_3) is not a solution of (1).

We have seen that for $n = 2$, the free group $F = \langle a,b \rangle$ possesses Property B_N, so that the set of all conjugates of $[a,b]$ and $[a,b]^{-1}$ is fixed under automorphisms of F. However, it is well known that if $n \geqslant 3$, then there is no element w in F_n such that $w\alpha \sim w^{\pm 1}$ for all $\alpha \in \mathrm{Aut}(F_n)$, [30,p164-165]. In fact if we write $\bar{w}$ as the set of elements in F_n conjugate to w, then for $n \geqslant 2$, the orbit of $\bar{w}$ under some element $\alpha \in \mathrm{Aut}(F_n)$ is infinite unless $w \sim [x,y]^k$, and (x,y) generates F_n, $n = 2$,[27 p.44].

To show how bad things can be when $n \geqslant 3$, we prove the following.

THEOREM 3.3. *Let F be a free group of rank $n \geqslant 3$, and let N be the normal closure of an element w in F. Then*

$$\bigcap_{\alpha \in Aut(F)} N\alpha = 1$$

(For the proof of this see Section 5).

It follows immediately from Theorem 3.3, that if $G = \langle a_1, \ldots, a_n; w\rangle$ then F is residually G. Since any free group of rank greater than n is residually F, [36], we have:

THEOREM 3.4 *Let G be an n-generator, one-relator group with $n \geqslant 3$. Then any free group of rank greater than or equal to n is residually G.*

#

Pride [39] has shown that if $G = \langle a,b;\ r^k\rangle$, $k > 1$, then F_2 (and thus F_n for $n \geqslant 2$) is residually G, unless r is conjugate to $[a,b]^l$ for some $l \neq 0$. (If r is conjugate to $[a,b]^l$, then Pride (unpublished) has shown that F_3 ia residually G). The following question remains open: if $G = \langle a,b;\ r\rangle$ where r is not a proper power, and if G does not satisfy a non-trivial law, then is F_2 residually G?

SECTION 2 SOME GROUPS WITH MORE THAN 1 NE CLASS BUT PROPERTY A, B_N OR B_T

LEMMA 3.1. *Let $G = \langle a,b;\ a^{16}, b^{17}, (ab)^8\rangle$, then G G has more than one* T-*system, but satisfies Properties* B_N *and* B_T.

(Note that G is a small cancellation group satisfying C'(1/15) and T(4).)

(i) G has B_N by Theorem 3.2.

(ii) The pairs (a,b), (a^3,b) generate G and lie in different NE classes.

In order to see this, suppose (a,b) and (a^3,b) lie in the same NE class. Then there exists a solution to the equation

$$(1) \qquad \alpha^{-1}[a,b]\alpha[a^3,b]^{\varepsilon} = 1, \ \varepsilon = \pm 1.$$

From all possible solutions of (1) choose a word z so that the cyclically reduced form y of $z^{-1}[a,b]z[a^3,b]^{\varepsilon}$ is as small as possible. Then as there is no free solution to (1),

$Y \not\equiv 1$. Let $z = z_1 w z_2$ where $z_1^{-1}[a,b]z_1, z_2[a^3,b]^{\varepsilon}z_2^{-1}$ – freely reduce to give cyclic permutations A,B of $[a,b]$, $[a^3,b]^{\varepsilon}$ respectively, and $w^{-1}Aw$, wBw^{-1} are freely reduced. Then we can assume if $w \not\equiv 1$ that $Y \equiv w^{-1}AwB$. If $w \equiv 1$ then one may have to perform further cyclic permutations of A,B so that $Y \equiv A'B'$ where A',B' are subwords of A,B respectively.

If w or w^{-1} has a subword which is more than a half of

$a^{16}, b^{17}, (ab)^8$ or $(ba)^8$, then w is not minimal. Therefore the largest possible subwords of $a^{16}, b^{17}, (ab)^8$ or $(ba)^8$ in Y or Y^{-1} is $a^{12}, b^{12}, (ab)^6$ or $(ba)^6$ respectively. So Y cannot contain a 3-remnant, and by small cancellation theory $Y \equiv 1$, a contradiction.

(iii) The T-systems and NE-classes coincide in G. (See Rosenberger [53]).

LEMMA 3.2 *Let* $G = \langle a,b;\ b^{2^p},\ [a,b^2]\rangle$, $p \geqslant 2$, *then*

(i) G *has* 2^{p-2} NE *classes of generating pairs, for which a set of representatives is* $\{(b^\alpha, a) : 0 < \alpha < 2^{p-1}, \alpha\ \text{odd}\}$

(ii) G *has Property* A.

We note that G is a free product with amalgamation:

$$G = \langle a,c;\ c^{2^{p-1}},\ [a,c]\rangle \underset{c=b^2}{*} \langle b;\ b^{2^p}\rangle$$

From the theory of amalgamated products, each element of G can be represented uniquely in normal form.

$$b^{2\mu} d_1 d_2 \ldots d_m$$

where μ is a non-zero integer; d_i is either a^{γ_i} or b, where γ_i is an integer, $(i = 1,\ldots,m)$; d_i, d_{i+1} not both in the same factor, $(i = 1,\ldots,m-1)$. The number m is called the length of the element.

We first compute the NE classes of generating pairs. For this we use the following theorem of McCool and Pietrowski [32].

Let G_1 and G_2 be finitely generated groups with normal subgroups L_1 and L_2 respectively, such that L_1 and L_2 are isomorphic. Let P be the free product of G_1 and G_2 amalgamating the subgroups L_1 and L_2. Then any generating n-tuple of P is Nielsen equivalent to an n-tuple, each element of which has length $\leqslant 1$.

From this it follows that any generating pair of G belongs to the same NE class as one of the following types:

$$(b^{2\mu},\ b^{2\eta})$$
$$(b^{2\mu}b,\ b^{2\eta})$$
$$(b^{2\mu}a^{\gamma},\ b^{2\eta})$$
$$(b^{2\mu}b,\ b^{2\eta}b)$$
$$(b^{2\mu}a^{\gamma}, b^{2\eta}a^{\delta})$$
$$(b^{2}\ b,\ b^{2\eta}a^{\delta})$$

where $\gamma, \delta, \mu, \eta$ are integers. But the subgroup generated by one of the first five forms is abelian, and thus is not all of G. Consequently any generating pair of G belongs to the same NE class as the pair of the form.

$$(b^{2\mu}b, b^{2\eta}a^{\delta}).$$

Now such a pair is Nielsen equivalent in G to $(b^{\alpha}, b^{2\eta}a^{\delta})$ $0 < \alpha < 2^{p-1}$, α odd, and $2\mu+1 \equiv \pm\alpha \bmod 2^{p}$. But $(\alpha, 2^{p}) = 1$, so that for certain integers $i, j, i\alpha+j2^{p} = 2\eta$.

Therefore $b^{\alpha}, b^{2\eta}a^{\delta})$ is in the same NE class as

$$(b^{\alpha}, b^{-i\alpha-j2^{p}}b^{2\eta}a^{\delta}) =_{G} (b^{\alpha}, a^{\delta})$$

We require (b^{α}, a^{δ}) to generate G. Therefore as the sum of the indices of a in the relators of G is zero, $\delta = \pm 1$.

As a consequence, any pair of generators of G is in the same NE class as a pair

$$(b^{\alpha}, a),\ (0 < \alpha < 2^{p-1};\ \alpha \text{ odd}).$$

It remains to show that two pairs (b^{α}, a) and (b^{β}, a) $(0 < \alpha < \beta < 2^{p-1};\ \alpha, \beta \text{ odd})$ are in distinct NE classes.

Suppose (b^{α}, a) and (b^{β}, a) belong to the same NE class, then there is a primitive $U(x_1, x_2)$ of $F_2 = \langle x_1, x_2 \rangle$ such that $U(b^{\alpha}, a) = b^{\beta}$ in G. Since all relations of G have exponent sum zero in a, $\sigma_{x_2}(U) = 0$, so that $\sigma_{x_1}(U) = \pm 1$ and $\sigma_b(U(b^{\alpha}, a)) = \pm\alpha$. But any relator of G has the exponent sum of b divisible by 2^p, so we conclude that $\beta \equiv \pm\alpha \bmod 2^p$, a contradiction.

Thus the set $\{(b^{\alpha}, a) : 0 < \alpha < 2^{p-1}, \alpha \text{ odd}\}$ is a set of representatives of the NE classes of generating pairs of G.

In order to show that G has Property A, let

(2) $$[x, y] \sim [a, b]^{\pm 1}$$

and let C denote the normal subgroup sgp$\{b^2\}$ of G. Then G/C is the free product of the infinite cyclic group sgp$\{aC\}$, and the cyclic group sgp$\{bC\}$ of order 2. Since

$$[xC, yC] \sim [aC, bC]^{\pm 1}$$

we deduce from Theorem 3.1 that $U(x_1, x_2)$, $V(x_1, x_2)$ is a generating pair of $F_2 = \langle x_1, x_2 \rangle$, such that $U(xC, yC) = aC$, $V(xC, yC) = bC$. Thus $U(x, y) = b^{2\mu}a$, $V(x, y) = b^{2\eta}b$ for certain integers μ, η. Therefore (x, y) is in the same NE class as the pair $(b^{2\mu}a, b^{2\eta}b)$, which generates G. As a consequence all solutions of (2) are generating pairs.

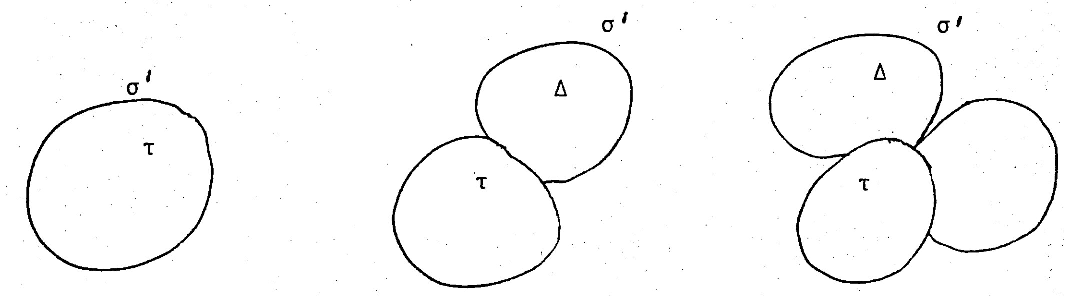
Now any generating pair of G is in the same NE class as a pair $(b^{2\gamma}b, a)$, and any pair of this form is a solution to (2). Thus any generating pair of G is a solution of (2).

#

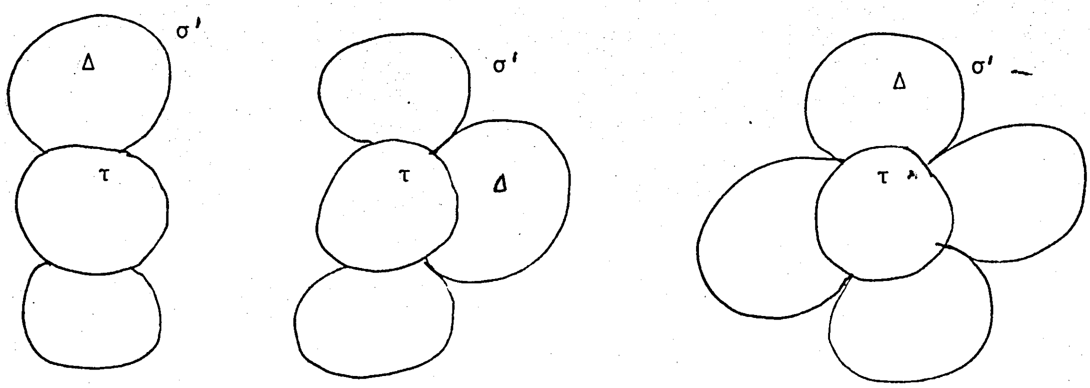
SECTION 3 FREE PRODUCTS OF CYCLES AND PROPERTY B_N

THEOREM 3.1 *If $G = \langle a,b;\ a^m b^n \rangle$, then G has Property* B_N.

Suppose $[u,v] \sim_G [a,b]$ where u,v are words in the generator of G. Then by [27,p.254] there exists a reduced annular R-diagram M, where the label of the outer boundary σ is a cyclically reduced cyclic permutation of $[u,v]^{\pm 1}$, and the label of the inner boundary is a cycle of $[a,b]^{\pm 1}$ But in this case we can have no internal edges. In addition $[a,b]$ has only 4 letters, so that there can be at most 4 vertics on τ. Therefore if we draw the subdiagram M' of M consisting of τ together with those regions Δ, where $\partial\Delta \cap \tau$ contains an edge, we have one of the following:



a) b) c)

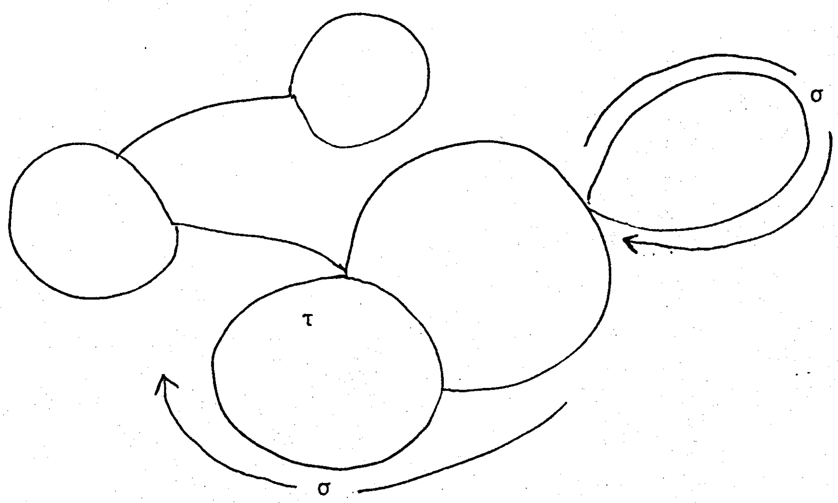


d) e) f)

In order to prove the Theorem, we use the notation and definitions of Chapter II. Let (f,g,h) be a factorization of Type I of (u,v). (i.e. $u \equiv f^{-1}g$, $v \equiv f^{-1}h$). We assume that (u,v) is chosen so that $L(f,g,h)$ is minimal for all factorizations of Type I of pairs of elements which are Nielsen equivalent to conjugates of (u,v). Then g and h are non-trivial and $g^{-1}fh^{-1}gf^{-1}h$ is cyclically reduced. Therefore a cycle of $(g^{-1}fh^{-1}gf^{-1}h)^{\pm 1}$ is the label on σ.

We show that for this (u,v), M' is all of M.

Suppose M' is not all of M. Then some subpath σ_1 of σ bounds a region of M. (The situation is illustrated when M' is as in diagram b.)



Therefore the label of σ contains a subword $a^{\pm m}$ or $b^{\pm n}$. In order to show that this is impossible we use the results in the detailed cases of Chapter II. If $a^{\pm m}$ or $b^{\pm n}$ is a subword of a cycle of $g^{-1}fh^{-1}gf^{-1}h$, then $a^{\pm m}$ or $b^{\pm n}$ is a subword of $g^{-1}fh^{-1}, fh^{-1}g, h^{-1}gf^{-1}, gf^{-1}h, f^{-1}hg^{-1}$ or $hg^{-1}f$, and this is shown to be not possible in cases A.1, A.2 and A.5 of Section II.3.

Thus M is one of the diagrams a),b),c),d),e),f). If Δ is a region of M, then the label of $\partial\Delta \cap \tau$ is a^{ε} or b^{ε}, $\varepsilon = \pm 1$. Thus in b),c), or e) either the exponent sum of a in the label of σ is congruent to 1 mod m, or the exponent sum of b in the label of σ is congruent to 1 mod n. In either case the label on σ could not be freely conjugate to $[u,v]$, so b), c) or e) are impossible. From diagrams a),d),f) it can be seen that $[u,v]$ is freely conjugate to $[a^{\mu},b^{\eta}]^{\pm 1}$ where $\mu = m-1$ or 1, $\eta = n-1$ or 1.

We will show that if $[u,v]$ is freely conjugate to $[a^{\mu},b^{\eta}]^{\pm 1}$, $(\mu,\eta > 0)$ then (u,v) is Nielsen equivalent (in the free group on a,b) to a conjugate of $(b^{\eta}a^{\mu_1},a^{\mu})$ or $(a^{\mu}b^{\eta_1},b^{\eta})$.

Now as $\mu = m-1$ or 1, $\eta = n-1$ or 1, μ is relatively prime to m, the order of a in G, and η is relatively prime to n, the order of b in G. Therefore (u,v) is in the same NE class in G as (a,b) as required.

In order to show that (u,v) is (freely) Nielsen equivalent to a conjugate of $(b^{\eta}a^{\mu_1},a^{\mu})$ or $(a^{\mu}b^{\eta_1},b^{\eta})$ we use the symmetry between a,b, so that we can assume
$g^{-1}fh^{-1}gf^{-1}h \equiv a^{-\mu'}b^{-\eta}a^{\mu}b^{\eta}a^{-\mu''}$, $(\mu'+\mu'' = \mu; \mu' > 0; \mu'' \geqslant 0)$
Therefore

(1) $\quad g^{-1}fh^{-1} \equiv a^{-\mu'}b^{-\eta}a^{\mu''}$

(2) $\quad gf^{-1}h \equiv a^{\mu'}b^{\eta}a^{-\mu''}$

If $\mu'' = 0$, then $g^{-1}fh^{-1} \equiv a^{-\mu}b^{-\eta}$,

$$gf^{-1}h \equiv a^{\mu}b^{\eta}.$$

Thus if $f \not\equiv 1$, either f begins with the letter a^{-1}, or f begins with the letter b^{-1}. We can assume, (using the symmetry between a,b) that f does not begin with b^{-1}. Then $f \equiv a^{-\mu_1}$, $g \equiv a^{\mu_2}$ and $h \equiv b^{\eta}$, $(\mu_1+\mu_2 = \mu)$.

If $\mu'' > 0$ then by (1) and (2), g begins and ends with the letter a. Therefore g is a subword of $a^{\mu'}$. Similarly h is a subword of $a^{-\mu''}$. Therefore by (1)

$$f \equiv a^{-\gamma_1}b^{-\eta}a^{\gamma_2}, \ (\gamma_1,\gamma_2 \geqslant 0).$$

However as g and h are non-trivial, and $gf^{-1}h$ is reduced, $\gamma_1,\gamma_2 = 0$. Thus

$f \equiv b^{-\eta}$, $g \equiv a^{\mu_1}$, $h \equiv a^{-\mu_2} (\mu_1+\mu_2 = \mu)$.

In both cases $(u,v) = (f^{-1}g, f^{-1}h)$ is Nielsen equivalent to a conjugate of $(b^{\eta}a^{\mu_1}, a^{\mu})$, as required.

#

SECTION 4 SMALL CANCELLATION GROUPS AND PROPERTY B_N

THEOREM 3.2 *Let* $g = \langle a,b;\ R \rangle$ *be a non-cyclic group, where* R *is symmetrized and satisfies* $C'_L(1/16)$ *or* $C'_L(1/12)$, $T(4)$, *and where* a,b *are both pieces. Then* G *has Property* B_N.

In order to prove the theorem we shall use the notation and terminology defined in Chapter II. Let (x,y) be a pair of generators of G. Let f,g,h be words in (a,b). Assume the length $L(f,g,h)$ is minimal, subject to the condition that $(u,v) = (f^{-1}g, f^{-1}h)$ is Nielsen equivalent in G to (x,y). As in Chapter II, we call (f,g,h) a factorization (of Type I) of (u,v). By Lemma 2.3, as $[u,v] \neq 1$ $g^{-1}fh^{-1}gf^{-1}h$ is cyclically reduced.

Let $(\dot f,\dot g,\dot h)$ be a permutation of (f,g,h) or (f^{-1},g^{-1},h^{-1}). Then $(\dot f,\dot g,\dot h)$ is a Type I factorization of a pair $(\dot u,\dot v)$ which is (freely) Nielsen equivalent to a conjugate of (u,v). Note that as $g^{-1}fh^{-1}gf^{-1}h$ is cyclically reduced, $\dot g^{-1}\dot f\dot h^{-1}\dot g\dot f^{-1}\dot h$ is also cyclically reduced. As a consequence it can be seen that there is a symmetry between the elements of the factorization, and the number of cases to be considered is thereby reduced.

By definition, (u,v) is Nielsen equivalent in G to (x,y), so that

$[x,y] \underset{G}{\sim} [u,v]^{\pm 1}$, and therefore $[u,v] \underset{G}{\sim} [a,b]^{\pm 1}$.

But it will be shown in Lemmas 3.4 and 3.5 that if $[u,v] \underset{G}{\sim} [a,b]^{\pm 1}$ then $[u,v]$ is freely conjugate to $[a,b]^{\pm 1}$, so that (u,v) and (a,b) lie in the same NE class.

LEMMA 3.4 *Let G,u,v be as defined above. Then no cyclically reduced conjugate of $[u,v]$*

(a) has a subword which is a 3-remnant

(b) is the product of two 4-remnants of equal length

(c) is an 8-remnant.

LEMMA 3.5 *Let G be as defined above. Let W be a word in G such that no cyclically reduced conjugate*

(a) has a subword which is a 3-remnant

(b) is the product of two 4-remnants of equal length

(c) is an 8-remnant.

Then if $W \underset{G}{\sim} [a,b]^{\pm 1}$, W is freely conjugate to $[a,b]^{\pm 1}$

PROOF OF LEMMA 3.4

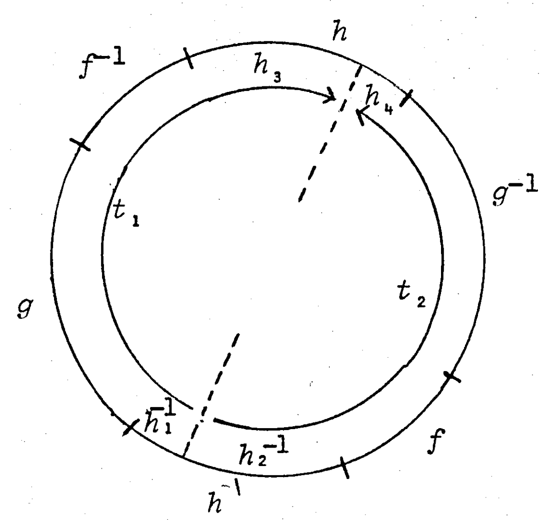
By definition $u \equiv f^{-1}g$, $v \equiv f^{-1}h$ and

$$[u,v] \equiv g^{-1}fh^{-1}gf^{-1}h$$

where the RHS is cyclically reduced.

(a) From the symmetry between the elements (f,g,h) we only need to consider 7 cases. That is, when the 3-remnant is a subword of m, but not $m-1$ $(1 \leqslant m \leqslant 7)$ elements of (f,g,h). But Cases A.1, A.2, A.5, A15, A.24, A35 and A.72 of Section II.3 deal with each of these possibilities, (although the notation is not always the same). By examining each of these cases, it can be seen that as there is no change in the Type of factorization (when considering the factorization transformation that might map the factorization (f,g,h) to a shorter factorization), (a) is true.

(b) In order to prove (b) we draw $g^{-1}fh^{-1}gf^{-1}h$ around a circle. At least two of the elements of (f,g,h) are non-trivial, so that since (f,g,h), (g,h,f), (h,f,g) are equivalent factorizations, we can assume g and h are non-trivial and the 4-remnants t_1, t_2 meet in the subwords h and h^{-1}.



We can assume $L(h_3) \geqslant L(h_1)$, for if $L(h_3) < L(h_1)$, consider the equivalent factorization $(\dot{f},\dot{g},\dot{h}) = (f^{-1},g^{-1},h^{-1})$. Then $L(\dot{h}_3) > L(\dot{h}_1)$.

Until further notice, we will assume $L(h_3) \geqslant L(h_1)$, and

$$h \equiv h_1h_2 \equiv h_3h_4 \equiv h_1h_0h_4$$

Let $t_1p_1^{-1} \equiv r_1$, $t_2p_2^{-1} \equiv r_2$; $r_1, r_2 \in R$.

Then

$$(1) \qquad r_1 \equiv h_1^{-1}gf^{-1}h_1h_0p_1^{-1}, \; r_2 \equiv h_0h_4f^{-1}gh_4^{-1}p_2^{-1}$$

But r_1, $r_2 \in R$ and satisfy the small cancellation condition $C_L^{\prime}(1/16)$ so that as (f,g,h) is reduced, and g,h are non-trivial f is a piece. In addition as h_1, h_1^{-1} are both subwords of r_1, h_1 is a piece, and similarly as h_4, h_4^{-1} are both subwords of r_2, h_4 is a piece.

We have shown in Case A.15 (of Chapter II) that the factorizations

$$(2) \qquad (f,g,h)^I, \; (h_4f^{-1}h_1,p_1^{-1},h_0^{-1})^I, \; (g,f,h_1p_1h_4)^I$$

are related. But by Lemma 2.2, they are factorizations of pairs of elements which lie in the same NE class. Since we assumed $L(f,g,h)$ was minimal,

$L(g) \leqslant L(p_1) < 4\lambda L(r_1)$, $L(h_0) \leqslant L(p_1) < 4\lambda L(r_1)$.

Thus, summarizing so far, we have

$L(f)$, $L(h_1) < L(r_1)$; $L(g)$, $L(h_0) < 4\lambda L(r_1)$

If, in fact g or h_0 is a piece, then g or h_0 $<\lambda L(r_1)$ and

$$L(r_1) = 2L(h_1) + L(g) + L(f) + L(h_0) + L(p_1)$$
$$< 2L(p_1) + 4\lambda L(r_1)$$
$$< 12\lambda L(r_1),$$

a contradiction of the small cancellation condition $C'_L(1/12)$

If h_0 is not a piece, then by (1), either $h_4 f^{-1}$ is an initial subword of p_1^{-1}, or p_1^{-1} is an initial subword of $h_4 f^{-1}$. But by (a), p_1 is more than $3\lambda L(r_1)$, so that as $L(h_4 f^{-1}) < 2\lambda L(r_1)$,

(3) $\qquad p_1^{-1} \equiv h_4 f^{-1} p_1'^{-1}$

Thus using the first two factorizations in (2), and (3) the factorizations

$(f,g,h)^I$, $(h_4 f^{-1} h_1, p_1^{-1}, h_0)^I$ and $(h_1, p_1'^{-1}, f h_4^{-1} h_0)^I$

are related.

Similarly using the first and last factorizations in (2) and (3), the factorizations

$(f,g,h)^I$, $(g,f,h_1 p_1 h_4)^I$ and $(gf^{-1}, 1, h_1 p_1')^I$

are related. Thus as we chose $(f,g,h)^I$ to be minimal, $L(h_0)$ and $L(g) \leqslant L(p_1') = L(p_1) - L(f) - L(h_4)$.

But we assumed $L(t_1) = L(t_2)$, so that $L(h_1) = L(h_4)$, and therefore

$$L(r_1) = 2L(h_1) + L(g) + L(f) + L(h_0) + L(p_1)$$
$$\leqslant 3L(p_1)$$
$$< 12\lambda L(r_1),$$

which contradicts the small cancellation condition $C'_L(1/12)$.

(c) Let $t \equiv h_2 f^{-1} g h f g^{-1} h_1$,

where $tp^{-1} \equiv r, r \in R$, and $h \equiv h_1 h_2$. Then by small cancellation theory f, g, h_1 and h_2 are all pieces, so that

$$L(t) < 8\lambda L(r)$$

#

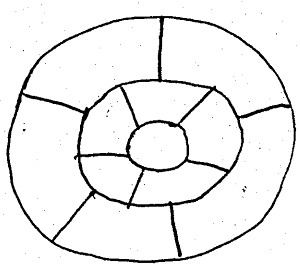
PROOF OF LEMMA 3.5

As $W \sim_G [a,b]$ by [27, p.254] there exists a reduced annular R-diagram M, where the label of the outer boundary σ is a cyclically reduced cyclic permutation of W, and the label of the inner boundary τ is a cycle of $[a,b]^{\varepsilon}$. Then by hypothesis (a), M satisfies:

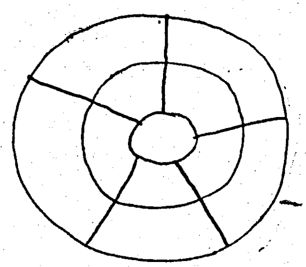
for all regions Δ of M with $\sigma_1 = \sigma \cap \partial\Delta$ connected, $\phi(\sigma_1)$ is not a $(p/q+1)$-remnant; and similarly with σ replaced by τ.

So by Theorem 1.3 (see also [27]) M has one of the following structures:

(1)



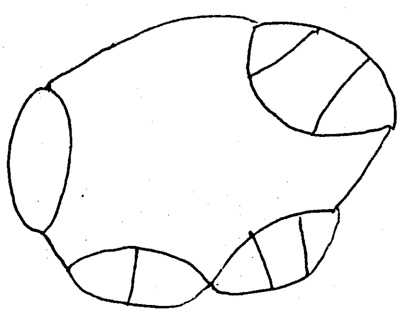
(If $(p,q) = (6,3)$)



(If $(p,q) = (4,4)$)

(The number of regions per layer, is variable).

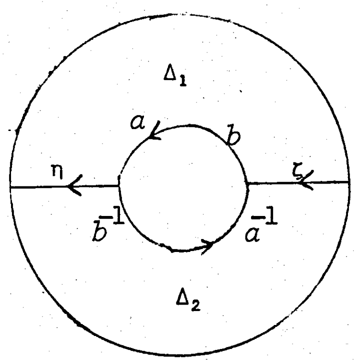
(2)



(The number of regions in each 'island', and the number of 'bridges' is variable. Every region has an edge on both inner and outer boundary, and has at most two internal edges.)

Now M cannot be as in (1). For then the label on a region Δ with an edge on the inner boundary τ would be the product of at most 8 pieces, (since the label on each internal edge is a piece, and the label on $\partial\Delta\cap\tau$ is at most 4 pieces).

Suppose M has the structure as in (2). If Δ is a region of M then the label on $\partial\Delta\cap\tau$ cannot be a piece, (and in particular cannot be a single letter), for otherwise the label on $\partial\Delta\cap\sigma$ would be a 3-remnant, contradicting hypothesis (a). Hence M can have at most 2 regions. If M has two regions, it must look like

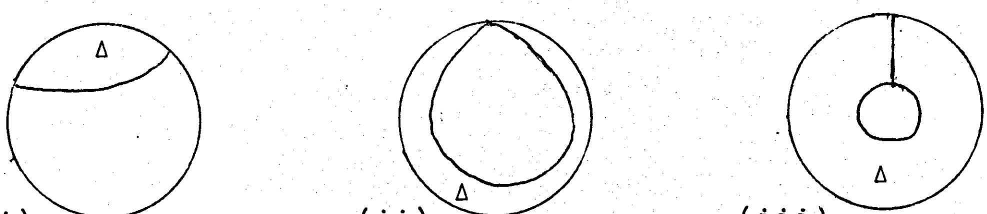


(wlog we assume $\partial\Delta_1\cap\tau$ has label ba, $\partial\Delta_2\cap\tau$ has label $b^{-1}a^{-1}$)
One or both of η ζ could be degenerate, i.e. single points.

The labels on $\partial\Delta_1\cap\tau$, $\partial\Delta_2\cap\tau$ have length 2, (and are each the product of two pieces). Thus we contradict hypotheses (a) if η or ζ is degenerate.

If η and ζ are not degenerate, let ck, ld be the labels on η, ζ respectively, where c, d are a, a^{-1}, b or b^{-1}. If c or $d \equiv a^{-1}$ or b^{-1}, then the diagram is not reduced. If $c \equiv a$, $d \equiv b$, then b^2a^2 and $(ba)^2$ are subwords of R, so that ba is a piece and the label on $\partial\Delta_1 \cap \sigma$ is a 3-remnant. Similarly if $c \equiv b$, $d \equiv a$, then ab is a piece and the label on $\partial\Delta_2 \cap \sigma$ is a 3-remnant. Both cases contradict hypothesis (a). If $c \equiv d \equiv a$ or if $c \equiv d \equiv b$, then ab is a subword of the labels of Δ_1 and Δ_2. However, if ab is a piece hypothesis (a) is contradicted, and if ab is not a piece, then the label on Δ_1 is a cyclic permutation of the label on Δ_2 or its inverse. Thus in this case the labels on $\partial\Delta_1 \cap \sigma$, $\partial\Delta_2 \cap \sigma$ have equal length so that hypothesis (b) is contradicted.

We are left with the case when M has just one region. The three possibilities are:



(i) (ii) (iii)

Now (i) is impossible since the label on $\partial\Delta \cap \tau$ is at most 3 pieces, so the label on $\partial\Delta \cap \sigma$ is a 3-remnant, contradicting (a). Also (ii) is impossible since the label on $\partial\Delta \cap \tau$ is at most 4 pieces, so the label on $\partial\Delta \cap \sigma$ is an 8-remnant; (using the fact that R satisfies $C'_L(1/12)$) contradicting hypothesis (c).

Suppose (iii) holds. Let z be the label on γ, then $a^{-1}b^{-1}abzWz^{-1}$ is a relator. Now R does not satisfy T(4). For $r \equiv a^{-1}b^{-1}abzWz^{-1}$ is cyclically reduced, so that z must begin with a or b.

But if $z \equiv a\hat{z}$ then

$\hat{z}^{-1}a^{-2}b^{-1}aba\hat{z}W$ is a cyclic permutation of r. Let

$r_1 \equiv aba\hat{z}W\hat{z}^{-1}a^{-2}b^{-1}$, $r_2 \equiv ba\hat{z}W\hat{z}^{-1}a^{-2}b^{-1}a$

$r_3 \equiv a^{-1}b^{-1}aba\hat{z}W\hat{z}^{-1}a-1$

then r_1, r_2, r_3 are all cyclic permutations of r, $r_i \not\equiv r_{i+1}^{-1}$, and $r_i' r_{i+1}$ is not reduced for all $i \in (1,2,3)$ where $i+1$ is reduced mod 3. Similarly if $z \equiv b\hat{z}$.

Now using the fact that the label on $\partial\Delta \cap \tau$ is the product of at most 4 pieces, and the fact that the label on γ is a piece (and remembering R satisfies $C'_L(1/16)$, we find that the label on $\partial\Delta \cap \sigma$ is an 8-remnant, contradicting hypothesis (c).

SECTION 5 A RESIDUAL PROPERTY OF FREE GROUPS

THEOREM 3.5 *Let F be a free group of rank $n \geqslant 3$, and let N be the normal closure of w in F, $w \neq 1$. Then*

$$\bigcap_{\alpha \in Aut(F)} N\alpha = 1$$

Let $(x_1, \ldots, x_n)$ generate F.

Since N contains all conjugates of W, we can assume W is cyclically reduced, and non-empty. By reordering the generators we assume W involves $x_1,\ldots,x_l$ only ($l \leqslant n$). Consider the set

$\{m\colon\ (x_2^{-m}x_1)^{\pm 1}$ or $(x_1x_2^{-m})^{\pm 1}$ is a subword of a cyclic permutation of $W; m > 0\}$.

Then let M be the maximal of this set, and let $\alpha_k \in \mathrm{Aut}(F)$, $k > 0$, be the map:

$$\begin{aligned} x_1 &\longmapsto x_2^{M+k}x_1x_2^{M+k}x_{l+1}\cdots x_n \\ x_2 &\longmapsto x_2 \\ &\vdots \\ x_n &\longmapsto x_n \end{aligned}$$

Let W_k be a cyclically reduced conjugate of $W\alpha_k$ so that $N\alpha_k = \langle W_k\rangle^F$. Then W_k involves all of $x_1,\ldots,x_n$, and any subword of a cyclic permutation of W_k which involves x_1 and x_i ($i \neq 1,2$) has either x_2^{k} or x_2^{-k} as a subword. Now if U is a non-empty freely reduced word, and $U \in N\alpha_k$, by the strengthened Freiheitssatz [58] U has a subword which involves all of $x_1,\ldots,x_n$, and is a subword of a cyclic permutation of $W_k^{\pm 1}$. Therefore U has length greater than k. Letting $k \longrightarrow \infty$ we deduce $U \notin \bigcap_{k=1}^{\infty} N\alpha_k$

and hence $\bigcap_{k=1}^{\infty} N\alpha_k = 1$.

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