The chromatic index of simple graphs

Thesis

THE CHROMATIC INDEX OF SIMPLE GRAPHS

by

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To

my wife,

JOAN

ABSTRACT

The object of this thesis is twofold:

(i) to study the structural properties of graphs which are critical with respect to edge-colourings;

(ii) to apply the results obtained to the classification problem arising from Vizing's Theorem.

Chapter 1 contains a historical, non-technical introduction, general graph-theoretic definitions and notation, a discussion of Vizing's Theorem as well as a survey of the main results obtained to date in Vizing's classification problem. Chapter 2 introduces the notion of criticality in the first section; the second section contains both well-known and new constructions of critical graphs which will be used in later chapters. The third and final section contains new results concerning elementary properties of critical graphs. Chapter 3 deals with uniquely-colourable graphs and their relationship to critical graphs. Chapter 4 contains results on the connectivity of critical graphs, whereas Chapter 5 deals with bounds on the number of edges of these graphs. In particular, bounds improving those given by Vizing are presented. These results are applied to problems concerning planar graphs. In Chapter 6, critical graphs of small order are discussed. All such graphs of order at most 8 are determined, while the 'critical graph conjecture' of Beineke & Wilson and Jakobsen is shown to be true for all graphs on at most 10 vertices. The seventh and final chapter deals with circuit length properties of critical graphs. In particular, the minimal order of certain critical graphs with given girth and maximum valency is determined. Results improving Vizing's estimate of the circumference of critical graphs are also given. The Appendix includes a computer programme which generates critical graphs from simpler ones using a constructive algorithm given in Chapter 2.

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PREFACE

This thesis presents an account of my research for the Ph.D. degree as a student at the Open University. I should like to express my gratitude to Dr. R. J. Wilson, my supervisor, for his constant encouragement and disinterested help throughout my course of study.

Most of the material in this thesis has been submitted for publication. Results on elementary properties of critical graphs and some results on the connectivity and bounds on the number of edges of these graphs appear in:

S. Fiorini & R. J. Wilson, On the chromatic index of a graph, I, Cahiers du Centre d'Études de Recherche Opérationnelle, 15 (1973), 253-262.

S. Fiorini & R. J. Wilson, On the chromatic index of a graph, II, in Combinatorics: Proceedings of the British Combinatorial Conference, Aberystwyth (T. P. McDonough & V. C. Mavron, eds.) (1973) (to appear).

The results of Chapter 3 on uniquely-colourable graphs are to appear in:

S. Fiorini, On the chromatic index of a graph, III: Uniquely-edge-colourable graphs, Quart. J. Math. (Oxford).

Parts of Chapters 5 and 7 on bounds on the number of edges and the circumference of critical graphs and results on planar graphs are to appear in:

S. Fiorini, On the chromatic index of outerplanar graphs, J. Combinatorial Theory.

S. Fiorini, Some remarks on a paper by Vizing on critical graphs, Proc. Cambridge Phil. Soc.

The results of Chapter 7 on the girth of critical graphs are to be published in:

S. Fiorini, On the girth of graphs critical with respect to edge-colourings, Bull. London Math. Soc.

Finally, the results on critical graphs of small order have been submitted for publication in Discrete Mathematics as a joint paper with L. W. Beineke.

During 1973, I attended conferences on Graph Theory at Manchester, at Aberystwyth, and in Rome, where I met a number of graph theorists with whom I discussed relevant topics. In my subsequent work I took these discussions into account. In particular, I should like to thank L. W. Beineke, R. Eggleton, G. H. J. Meredith, C. Thomassen, and W. T. Tutte.

Finally, I should also like to thank the Open University for their financial and practical support without which this research would not have been possible.

CHAPTER 1: INTRODUCTION

1) Historical

The beginnings of chromatic graph theory go back about 120 years when the Four-Colour Conjecture was first formulated by Guthrie and discussed in print by Cayley, Kempe, Tait, and Heawood. Since then, a great deal of research has developed around the problem of face-colouring of maps and the dual problem of vertex-colouring of planar graphs. Tait [38] showed in 1880 that the problem of face-colouring a cubic map with four colours is equivalent* to that of colouring its edges with three colours in such a way that edges meeting at a vertex are assigned different colours. In view of Tait's result, it is surprising that the edge-colouring approach to the Four-Colour Conjecture has been practically ignored until quite recently. In fact the only significant result that appeared in the literature between 1880 and 1964 was due to Shannon [37], who showed in 1949 that if ρ denotes the maximum valency of a graph, i.e. the maximum number of edges meeting at a vertex, then the least number of colours required to colour the edges of the graph does not exceed $[\frac{3}{2}\rho]$. For completeness' sake, we must also mention a result due to Johnson [22], who showed in 1963 that every cubic graph can be edge-coloured with four colours.

The real breakthrough came in 1964, when Vizing [41] showed that, for a simple graph, the least number of colours required to

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* To be historically accurate we must state that Tait only proved the sufficiency of this condition, since he accepted Kempe's erroneous proof of the Four-Colour Conjecture.

colour the edges must be equal either to ρ or to $\rho + 1$, where ρ is defined as above.

Vizing's result naturally partitions the set of all graphs into two disjoint classes. The first consists of those graphs with maximum valency ρ which can be edge-coloured with ρ colours, whereas the second consists of those which require $\rho + 1$ colours. It is not difficult to see that a complete solution of the problem of classifying graphs into one or other of these classes implies a solution to the Four-Colour Problem. This is just an indication of how difficult the problem is. However, since 1964, many attempts have been made to provide partial solutions to this question. We shall be surveying the main results in a later section of this chapter.

Just as in problems related to vertex-colourings of graphs, the notion of a graph which is critical in some sense plays an important rôle in most approaches to the problem. This is not surprising, since problems about graphs in general can often be reduced to ones about critical graphs which are less arbitrary, contain more structure and so more can be known about them.

The scope of this thesis is mainly to study the structure of critical graphs and apply this knowledge to give a partial solution to the classification problem.

2) Basic definitions and notation

In this section we shall give the basic definitions and notation to be used in the thesis. This seems the most natural way of proceeding since there is yet no standard terminology. However, the following list is not complete since more specialised terms will be defined at the beginning of the chapter in which they first appear or as they are needed.

A <u>graph</u> G is a pair $(V(G),E(G))$ or simply (V,E), where V is a finite set of <u>vertices</u> and E is a set of [illegible] unordered pairs of vertices; the elements of E are called <u>edges</u>. We shall be interested primarily in <u>simple</u> graphs, i.e. ones for which a pair of vertices (v,w) can define at most one edge. A <u>subgraph</u> H of G is a graph $(V(H),E(H))$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If W is a subset of $V(G)$, then the <u>subgraph induced</u> by W, denoted by $\langle W \rangle$, is the graph H such that $V(H) = W$ and $(v,w) \in E(H)$ if and only if $(v,w) \in E(G)$. If H is a subgraph of G and $V(H) = V(G)$, then H is called a <u>spanning</u> subgraph.

If X and Y are two disjoint subsets of $V(G)$ and $e = (x,y)$ is an edge such that $x \in X$ and $y \in Y$, then we say that e is of type $X \times Y$. Edges which are pairwise not adjacent (incident with the same vertex) are called <u>independent</u>. The <u>valency</u> of a vertex v is the number of edges incident with v and is denoted by $\rho(v)$. The maximum valency of G is denoted by $\rho(G)$ or simply by ρ. Similarly, the minimum valency of G is denoted by $\sigma(G)$ or simply by σ. $\tau(G)$, the <u>total deficiency</u> of G is the sum

$$\sum_{v \in V(G)} (\rho(G) - \rho(v)).$$

If each vertex has the same valency ρ, then G is called <u>regular</u> or <u>ρ-valent</u>. A 3-valent graph is called <u>cubic</u>. Vertices adjacent to a vertex v are called its <u>neighbours</u> and the set of neighbours of v is denoted by $N(v)$. The <u>order</u> of G is the number of vertices of G. It is denoted by $n(G)$ or simply by n. $m(G)$ or m denotes the number of edges of G.

We define a walk in a graph G to be a sequence: $v_0,e_1,v_1,e_2,v_2,\ldots,v_{n-1},e_n,v_n$, where the v_i's are vertices, the e_i's are edges and $e_i = (v_{i-1},v_i)$ for each $i = 1,2,\ldots,n$. The length of the walk is n, the number of edges. If all the vertices (except possibly v_0 and v_n) are distinct, then the walk is called a chain. If $v_0 = v_n$, the chain is said to be closed and it is then called a circuit; otherwise it is open. A chain of length k is called a k-chain (or k-circuit if closed) and is denoted by P_k or C_k according as to whether it is open or closed. If G contains a spanning circuit, then G is said to be Hamiltonian.

If v is a vertex of G, then $G - v$ is the graph G' such that $V(G') = V(G) \setminus \{v\}$ and $E(G') = E(G) \setminus \{(v,x): x \in N(v)\}$. If e is an edge of G, then $G - e$ is the graph G'' such that $V(G'') = V(G)$ and $E(G'') = E(G) \setminus \{e\}$. G' and G'' are said to be obtained from G by the deletion (or removal) of v and of e respectively. If G is a graph and v a vertex not in $V(G)$, then G' is said to be obtained from G by the insertion (or introduction) of v into an edge (x,y) in $E(G)$ if $V(G') = V(G) \cup \{v\}$ and $E(G') = E(G) \cup \{(v,x),(v,y)\} \setminus \{(x,y)\}$. If G and H are two graphs whose vertices are labelled, then the union of G and H, written $G \cup H$, is the graph K such that $V(K) = V(G) \cup V(H)$ and $E(K) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then K is called the sum of G and H and we write $K = G + H$.

We shall often encounter various types of graphs. Throughout we shall adopt the following notation: K_n denotes the complete graph on n vertices and $K_{m,n}$ the complete bipartite graph on m and n

vertices. If G is a graph, then a graph H is called the line-graph of G if there exists a one-one correspondence $\Phi: V(H) \longrightarrow E(G)$ such that $(v,w) \in E(H)$ if and only if $\Phi(v)$ and $\Phi(w)$ are adjacent in G. The line-graph of G is denoted by $\hat{G}$. $\bar{G}$ denotes the complement in the complete graph of the same order. A graph is planar if it can be embedded in the plane in such a way that no two edges intersect except at a vertex to which they are both incident. A planar graph that can be so embedded is called plane. An open region of $E_2 \setminus G$ defined by a plane graph G is called a face of G. If all the vertices of a plane graph lie on the same face, then the graph is called outerplanar.

Two graphs G and H are said to be isomorphic if there exists a one-one correspondence between $V(G)$ and $V(H)$ which preserves adjacency. We then write $G \simeq H$. G and H are said to be homeomorphic if they can both be obtained from the same graph K by the insertion of vertices into the edges of K.

We define a vertex-colouring of a graph G to be a mapping $\phi: V(G) \longrightarrow S$ such that S is a set whose elements are called colours, and if v and w are a pair of adjacent vertices, then $\phi(v) \neq \phi(w)$. If h is the least number of colours required to colour G, then h is called the chromatic number of G and is denoted by $\chi_v(G)$. G is then said to be h-chromatic. We define an edge-colouring or simply a colouring of G to be a mapping $f: E(G) \longrightarrow S$ such that S is a set whose elements are called colours and if e and e' are a pair of adjacent edges, then $f(e) \neq f(e')$. If k is the least number of colours required to colour G, then k is called the chromatic index of G and is denoted by $\chi_e(G)$. If t is any integer not less than $\chi_e(G)$, then G is said to be t-colourable. The set of edges coloured with any one colour is called a colour-class. If v is a vertex of a graph G which has been t-coloured, then Θ_v denotes the set of colours of edges incident with v, whereas $\overline{\Theta_v}$ denotes the complement of Θ_v in the set of t colours. If α and β are two distinct colours used in some colouring of G, then $C_{\alpha,\beta}$ denotes the subgraph of G consisting of edges coloured α or β and the vertices defining these edges.

Any colour-class is an independent setof edges since no pair of edges are adjacent. A set of independent edges is called a matching. The size of any maximal matching in a graph G is called the edge-independence number of G and is denoted by $\alpha_e(G)$. A k-valent spanning subgraph of a graph is called a k-factor. In particular, a matching covering all vertices is called a 1-factor.

Further Notation: If x is a real number, then [x] denotes the largest integer not greater than x and {x} denotes the smallest integer not less than x. If S is a set, |S| denotes the cardinality of S. := and =: mean 'is defined to be' and 'defines' respectively. // indicates the end or absence of a proof.

3) The Theorem of Vizing

We enunciate formally the theorem of Vizing [41] referred to earlier and give a proof which is somewhat different from other known proofs (see, for example [2, p. 211] or [32, p. 248]) but which, like them, depends crucially on a 'Kempe-chain' type of argument.

1.1 Theorem

If G is a simple graph with maximum valency ρ, then

$$\rho \leq \chi_e(G) \leq \rho + 1.$$

Proof

Among all graphs for which the statement is false, let G be one with a minimum number of edges. Thus, if e is any edge of G, G - e is $(\rho + 1)$-colourable. Let v be a vertex of maximum valency and assume that all edges of G except one edge e incident with v have been coloured with $\rho + 1$ colours. We shall show that e can also be coloured with one of the colours already used.

Without loss of generality, we can assume that $\rho(v) \geq 3$. Let N(v) be the set $\{v_1, v_2, \ldots, v_\rho\}$ and let the uncoloured edge be (v,v_1). Let $b \in \overline{\Theta_v}$ and $a_1 \in \overline{\Theta_{v_1}}$. Such a colour a_1 exists since $|\overline{\Theta_{v_1}}| = (\rho + 1) - (\rho(v_1) - 1) \geq 2$.

Also, $a_1 \in \Theta_v$, for otherwise we could colour (v,v_1) with colour a_1. Suppose a_1 colours (v,v_2). We can colour (v,v_1) with colour a_1 and un-colour (v,v_2). The vertices v, v_1, v_2 belong to the same component of $C_{a_1,b}$ for otherwise we could change the colours alternately on the open chain with initial vertex v_2 without affecting the colouring of (v,v_1). This would enable us to colour (v,v_2) with colour b. Let $a_2 \in \overline{\Theta_{v_2}}$. Again, $a_2 \in \Theta_v$ and moreover, $a_2 \neq a_1$. Suppose a_2 colours (v,v_3). Then we can colour (v,v_2) with a_2 and un-colour (v,v_3). Vertices v, v_2, v_3 belong to the same component of $C_{a_2,b}$ by the same argument as before.

We repeat this process until we arrive at a vertex v_k such that the edge (v,v_k) is un-coloured, $a_h \in \overline{\Theta_{v_k}}$ and h satisfies $1 \leq h \leq k - 2$. We note that this is possible since $k \geq 3$. Without loss of generality let $h = 1$. We have seen that v, v_1, v_2 belong to the same component of $C_{a_1,b}$. This component can only be an open chain Γ with initial vertex v, initial edge (v,v_1) coloured a_1 and terminal edge (x,v_2) coloured b, where $x \in N(v_2) \setminus \{v\}$. Consider a chain $\tilde{\Gamma}$ coloured a_1 and b, of maximum length and with initial edge (v_k,y) coloured b, where $y \in N(v_k) \setminus \{v\}$; clearly such a chain must exist. Since $a_1 \in \overline{\Theta_{v_2}}$, $\Gamma \cap \tilde{\Gamma} = \emptyset$. Thus, we can change the colours alternately on $\tilde{\Gamma}$ to allow us to colour the edge (v,v_k) with colour b. This completes the proof. //

The strength of this theorem is evident from the fact that it imposes very sharp, strict bounds on the chromatic index. It is also evident from the impossibility to strengthen it, at least in the following sense: Let $\hat{G}$ be the line-graph of a graph G and let $\omega(G)$ denote the <u>clique number</u> of G, i.e. the size of any largest maximal complete subgraph of G. Clearly, $\rho(G) = \omega(\hat{G})$ if $G \neq K_3$. Since $\chi_e(G) = \chi_v(\hat{G})$, Vizing's theorem implies that $\chi_v(\hat{G}) \leq \omega(\hat{G}) + 1$. This suggests the question: Can we construct an alternative proof of this theorem by first showing that $\chi_v(H) \leq \omega(H) + 1$ for all graphs H and obtaining Vizing's theorem as a corollary ? In fact, this statement is false, since House [15] has shown that for all integers ω, k such that $1 < \omega \leq k$ there exist k-chromatic graphs with clique number ω.

Moreover, we have been able to show the following:

<u>1.2 Theorem</u>

The Four-Colour Conjecture is true if and only if $\chi_v(G) \leq \omega(G) + 1$ for every planar graph G.

<u>Proof</u>

Assuming that every planar graph is 4-colourable and noting that for planar graphs $\omega(G) \leq 4$, we consider the following four cases:

(i) If $\omega(G) = 4$, then $\chi_v(G) \leq 5 = \omega(G) + 1$, by Heawood's 5-colour theorem (cf. [14]);

(ii) if $\omega(G) = 3$, then $\chi_v(G) \leq 4 = \omega(G) + 1$, by hypothesis;

(iii) if $\omega(G) = 2$, then $\chi_v(G) \leq 3 = \omega(G) + 1$, by Grötzsch's theorem (cf. [12]); and

(iv) if $\omega(G) = 1$, then $\chi_v(G) \leq 2 = \omega(G) + 1$, trivially.

Conversely, assume $\chi_v(G) \leq \omega(G) + 1$ for all planar graphs G. The result clearly holds if $\omega(G) \leq 3$. So let us assume that $\omega(G) = 4$ and apply induction on the order of G. If G is K_4 the result holds trivially. If G is not K_4, then G contains a triangle T, say, such that there are vertices of G both inside and outside T. Thus, by the inductive hypothesis, we can colour its interior Int(T), the subgraph of G induced by all vertices on and inside T, with four colours. If we define Ext(T) similarly, then we can also colour the exterior of T with four colours. Having chosen the notation in such a way that the vertices of T are coloured α, β, and γ in both cases, we can re-combine Int(T) and Ext(T) to give a 4-colouring of the graph G. //

For various other conditions equivalent to the Four-Colour Conjecture the reader is referred to [35].

4) A survey of classified graphs

In this section we survey the main results obtained in the classification problem arising from Vizing's theorem. One of the earliest results in this direction is due to König [23]:

1.3 Theorem

All bipartite graphs are in class 1. //

This was also proved by Vizing [42] and somewhat generalised by Welsh, who noted that if a graph which is not an odd circuit has all its circuits of the same parity, then it must be of class 1.

The complete graphs have also been investigated by various people. So, for example, Vizing [42], Wilson [45], Berge [4], and others have noted the following:

1.4 Theorem

K_n is of class 1 or of class 2 according as n is even or odd. //

To show that K_{2k} is of class 1, one can give an explicit colouring of the edges. That K_{2k+1} is of class 2 follows as a corollary from the following more general result proved explicitly by Beineke & Wilson [3] and found implicitly in the work of Vizing [42].

1.5 Theorem

If G is a graph of odd order and if its total deficiency is less than its maximum valency, then G is of class 2. //

We list some corollaries of this result to which we shall be referring later.

1.6 Corollary

If G is regular of odd order, then G is of class 2. //

1.7 Corollary

If G is regular and has a cut-vertex, then G is of class 2.//

1.8 Corollary

If G is obtained from a regular graph of even order by inserting a vertex into any one of its edges, then G is of class 2. //

Before proceeding with other established results in the classification problem, we include here one or two results of our own on regular graphs since this seems to be the most suitable place to insert them. We shall refer to the following theorem in Chapter 2.

1.9 Theorem

The graph obtained from a complete graph of even order by removing a 1-factor is of class 1.

Proof

We first exhibit a decomposition of K_{2k+2} into k Hamiltonian circuits and a 1-factor. We label the vertices a, b, 1, 2, 3,..., 2k and for i = 1, 2, 3,..., k we obtain the i'th Hamiltonian circuit H_i by taking the sequence of edges $\{(i,i+1), (i+1,i-1), (i-1,i+2), (i+2,i-2),\ldots,(i+k-1,i+k+1), (i+k+1,i+k)\}$ (where all arithmetic is worked modulo 2k) and joining a to i and to i + k. We then introduce b into the unique edge $(i+r,i-s)$, where $r + s \equiv k \pmod{2k}$.

Illustration

H_4 in K_{12}:

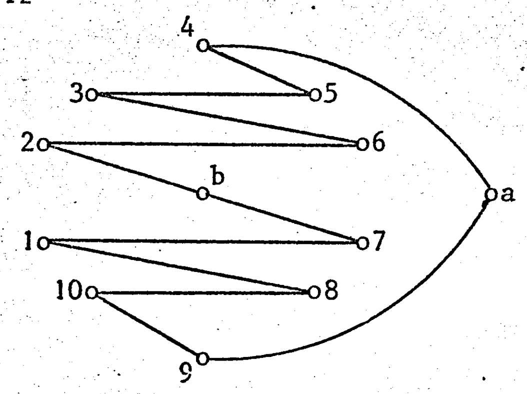


Figure 1.1

The 1-factor is then the set of edges $\{(a,b),(1,1+k),(2,2+k),\ldots(k,2k)\}$.

Now let an arbitrary 1-factor of K_{2k+2} be given. We label the vertices of the first edge 1 and 1 + k respectively, those of the second edge 2 and 2 + k respectively,...,those of the k'th edge k and 2k respectively, and those of the (k+1)'th edge a and b respectively. Then we can decompose the edges of K_{2k+2} which are not in the 1-factor into k edge-disjoint Hamiltonian circuits, each of which is

2-colourable. Thus, the graph obtained from K_{2k+2} by deleting an arbitrary 1-factor is of class 1, as required. //

The next result generalizes (1.7).

1.10 Theorem

If G is a ρ-valent graph which contains a set of t edges ($t < \rho$) whose removal yields a disconnected graph having an odd component, then G is of class 2. ?

Proof

Let the separating set of t edges be S. Let $G' := G - S$ and let C be the odd component of G'. If p is the order of C, then

$$p\rho = \sum_{v \in V(C)} \rho(v) \quad ,$$

since G is ρ-valent. Thus,

$$p\rho \leq p(p-1) + t < p(p-1) + \rho \ ,$$

i.e.

$$p > \rho .$$

Thus, C contains at least one vertex of valency ρ and its total deficiency is less than ρ. Hence, by (1.5), C is not ρ-colourable and so neither is G. //

Returning to the survey of established results in the classification problem we note that some rather special classes of graphs have been investigated. It is well known that the Petersen graph is of class 2. Watkins [13, p. 171] suggested the following generalization of the Petersen graph. We denote the generalized Petersen graph by P(n,k), where P(n,k) is defined for each pair of integers k and n satisfying $1 \leq k \leq n$ as follows:

$V(P(n,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and

$E(P(n,k)) = \{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+k}) \colon i \text{ an integer}\}$, and where all arithmetic is worked modulo n. Thus, for example, the Petersen graph is P(5,2). We call the subgraph induced by the vertices $\{u_0, u_1, \ldots, u_{n-1}\}$ the <u>outer rim</u>, that induced by the remaining vertices the <u>inner rim</u>, and the edges of the form (u_i, v_i) the <u>spokes</u> of P(n,k). Clearly, P(n,k) is isomorphic to P(n,n-k).

Watkins [13] conjectured that all graphs in this family are of class 1 with the exception of the Petersen graph itself. Watkins himself settled a few cases of this conjecture but the coup de grace was dealt recently by Castagna & Prins [6] , who showed:

<u>1.11 Theorem</u>

P(n,k) is of class 2 if and only if n = 5 and k = 2. //

A generalization of another type of graph, the circuit, has also been investigated. Parker [33] defines a <u>generalized circuit</u> C(n,k) as follows: C(n,k) is composed of k copies of the totally disconnected graph on n vertices. These copies are arranged in a k-cycle and two vertices are joined if and only if they lie in adjacent members of the k-cycle. In this set-up we have the following result:

<u>1.12 Theorem</u>

C(n,k) is in class 2 if and only if n and k are both odd. //

A similar result has also been established recently by Laskar & Hare [25].

<u>1.13 Theorem</u>

Let K(n,r) denote the complete r-partite graph each of whose parts has n vertices. Then K(n,r) is of class 2 if and only if n and r are both odd. //

Another type of graph has been considered by various authors: Balaban [1] , Biggs [5] , Meredith & Lloyd [31] . The family of <u>odd</u> graphs whose k'th graph is denoted by O_k is defined as follows: The vertices of O_k are indexed by the (k-1)-subsets of the (2k-1)-set $\{1,2,\ldots,2k-1\}$. Two vertices are adjacent if and only if their indexing sets are disjoint. Thus, O_k is k-valent and is of odd order if and only if k is a power of 2. In particular, O_2 is the 3-circuit and O_3 is the Petersen graph. It is conjectured that O_k is of class 1 for all $k \geq 5$ and $k \neq 2^n$. Meredith & Lloyd have proved the following result:

<u>1.14 Theorem</u>

O_5 and O_6 are the edge-disjoint unions of two Hamiltonian circuits and a 1-factor and three Hamiltonian circuits respectively and hence O_5 and O_6 are of class 1. O_7 is Hamiltonian. //

One other general result due to Vizing [43] is worth mentioning here although the problem related to it will be discussed in Chapter 5. This result is concerned with planar graphs.

<u>1.15 Theorem</u>

If G is a planar graph whose maximum valency is at least 8, then G is of class 1. //

As can be seen, not many general classes of graphs have been classified. However, some other results on particular graphs are known. Thus, for example, it is not difficult to see that all the Platonic graphs are of class 1. Also, a case-by-case analysis of all connected graphs of order at most 6, enabled Beineke & Wilson to show that out of a total of 143 such graphs only the eight in Figure 1.2 are of class 2; this seems to indicate that graphs of class 2 are rather fewer in number than those of class 1.

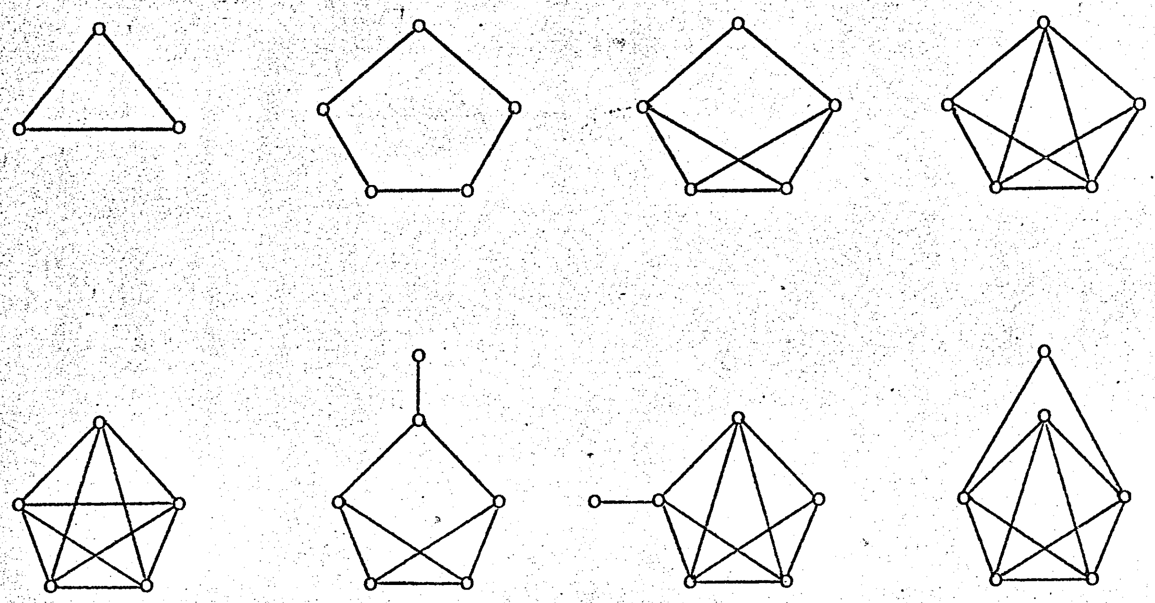


Figure 1.2

One rather important contribution to the solution of the problem has been the construction of class 2 graphs from simpler class 2 graphs. Thus, for example, Meredith [30] uses a 'vertex-replacement' method to construct n-valent, n-connected, non-Hamiltonian class 2 graphs based on the Petersen graph.

Two constructions of cubic class 2 graphs are due to Rufus Isaacs (private communication).

Construction 1

Let G and H be two cubic graphs of class 2, such that H contains the following induced subgraph on six vertices:

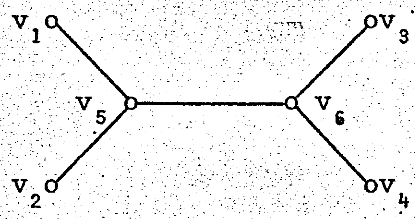


Figure 1.3

Now, let (w_1,w_2) and (w_3,w_4) be a pair of non-adjacent edges of G. Then the graph obtained from G and H by deleting the vertices v_5 and v_6 from H and the edges (w_1,w_2) and (w_3,w_4) from G and joining w_i with v_i $(i = 1,2,3,4)$ by an edge, is also cubic and of class 2.

Illustration

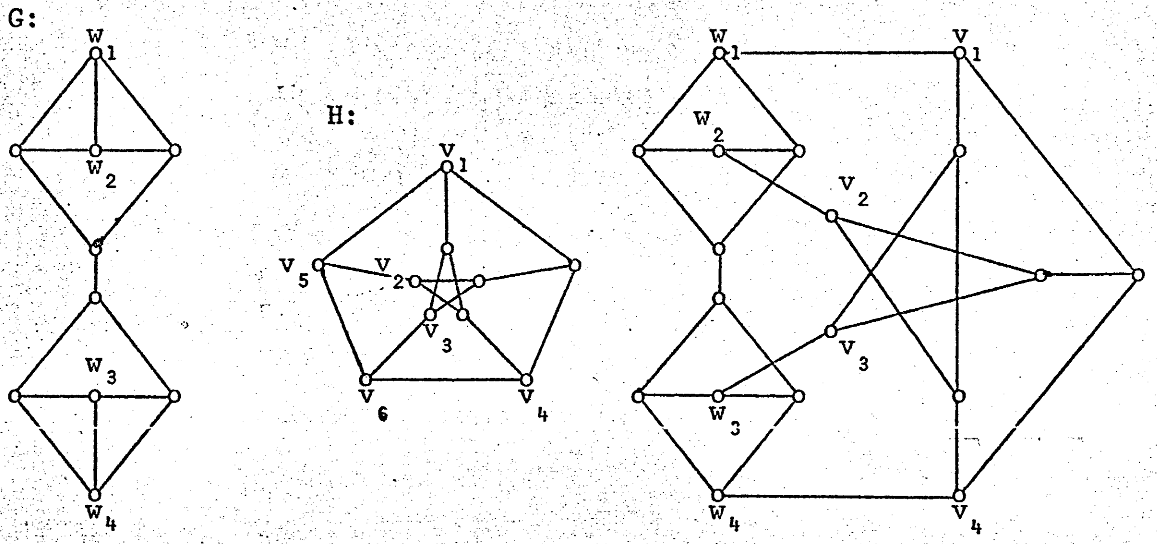


Figure 1.4

Construction 2

Let n be an odd positive integer. Take C_{2n} and label the vertices $1,2,\ldots,2n$. Take a set of n other vertices $\{a_1,a_2,\ldots,a_n\}$ and join a_i with vertices i and $i + n$ on C_{2n}, where all arithmetic is worked modulo 2n. Now take C_n and label its vertices $b_1, b_2, \ldots, b_n$. Join b_i with a_i cyclicly. The resulting graph is cubic and of class 2. Figure 1.5 illustrates this construction for the case n = 3.

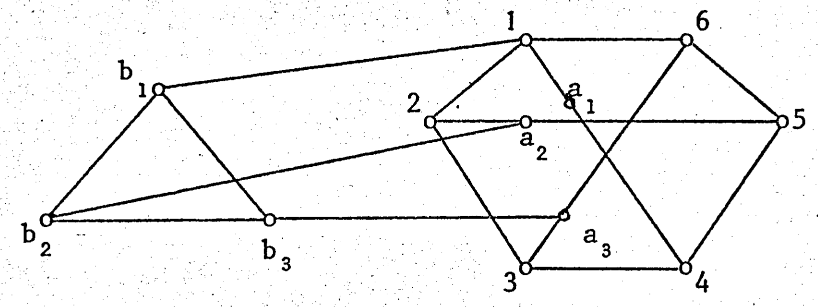


Figure 1.5

Making use of (1.6), we can construct an infinite family of iterated line-graphs all of which are of class 2. This can be achieved as follows: Let G be an arbitrary regular graph of odd order and valency $\rho \equiv 2 (\mathrm{mod}\ 4)$, (such as K_7), and let $(\hat{G})^k$ denote the k'th iterated line-graph of G. We shall show that $(\hat{G})^k$ is of class 2 for each k. It is clear that $(\hat{G})^k$ is regular. Denote by ρ_k the valency of each vertex of $(\hat{G})^k$. Thus, $\rho_0 = \rho(G)$ and $\rho_1 = \rho(\hat{G})$. Using the fact that, by hypothesis, $\rho_0 = 2(2t + 1)$ for some positive integer t, we can establish inductively that $\rho_k = 2(2^{k+1}t + 1)$, which is congruent to 2 (mod 4). Also,

$$n_k := n((\hat{G})^k) = \tfrac{1}{2}\rho_k . n_{k-1} = (2^k t + 1).n_{k-1},$$

which is odd since n_0 is odd. This completes the proof.

In the same vein, Jaeger [18] proved the following theorem:

1.16 Theorem

If G is a regular graph of class 1 with an even number of edges, then $\hat{G}$ is also of class 1. //

We conclude this section by giving a construction of class 2 graphs from others of smaller order. First we need the following definition:

A <u>Hajós union</u> of two graphs G and H is the graph obtained from G and H by the following construction:

(i) a vertex v in G is identified with a vertex w in H;

(ii) some edge (u,v) is deleted from G and some edge (w,x) is deleted from H;

(iii) the vertices u and x are joined by an edge.

This construction is illustrated in Figure 1.6. Using it, Jakobsen [19] proved the following result:

<u>1.17 Theorem</u>

If G and H are two class 2 graphs with $\rho(G) = \rho(H) = \rho$, then any Hajós union K of G and H, in which the sum of the valencies of the identified vertices does not exceed $\rho + 2$, is also of class 2 and $\rho(K) = \rho$. //

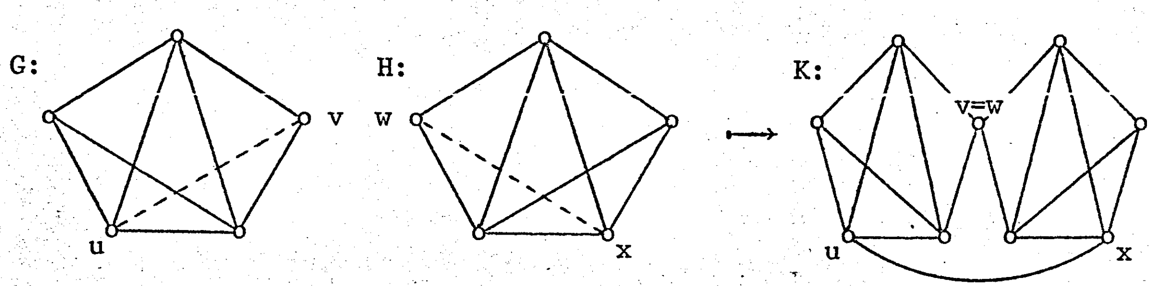


Figure 1.6

CHAPTER 2: CRITICAL GRAPHS

1) Definitions and examples

It has been noted in Chapter 1 that graphs which are critical in some sense have played an important part in the classification problem discussed there. Two different definitions of critical graphs have been put forward. The first one was used by Beineke & Wilson [3] , who defined a vertex-critical graph G to be a graph of class 2 such that the removal of any vertex yields a graph with smaller chromatic index. The second one was introduced by Vizing [42] and adopted by Jakobsen [19] and others: A graph G is defined to be edge-critical if it is connected and of class 2, but the removal of any edge reduces the chromatic index. It is clear that every edge-critical graph is necessarily vertex-critical, but that the converse is not true in general. Thus, for example, K_5 is vertex-critical but not edge-critical.

For graphs which are edge-critical, Vizing [43] proved the following fundamental result. Here we present our version of the proof, but we first require a definition:

Let x be a vertex of a graph whose edges have been coloured in such a way that adjacent edges are assigned distinct colours. A fan-sequence at x with initial edge (x,x_1) is a sequence $((x,x_j))$ of distinct edges incident with x such that, for each $j \geq 1$, the colour of (x,x_{j+1}) does not belong to Θ_{x_j}, where as before, Θ_v is the set of colours present at a vertex v.

2.1 Theorem

Let G be an edge-critical graph and let v and w be a pair of adjacent vertices such that $\rho(v) = k$. Then w is adjacent to at least $\rho - k + 1$ other vertices of maximum valency ρ.

Proof

Consider a ρ-colouring of $G' := G - (v,w)$. It is clear that w is incident to edges coloured with each of the $\rho - k + 1$ colours not present at v, since otherwise, we should get a ρ-colouring of G. Thus, $\rho(w) \geq \rho - k + 2$. We partition the set of ρ colours into three disjoint sets: $\Theta_w \setminus \Theta_v$, $\Theta_v \setminus \Theta_w$, and $\Theta_v \cap \Theta_w$. Note that the first two sets are necessarily non-empty, whereas the third may well be empty. We now make two assertions:

(i) In G', any pair of fan-sequences at w with distinct initial edges coloured with some colour from $\Theta_w \setminus \Theta_v$ must have empty intersection.

For suppose not. Let $((w,a_1),(w,a_2),\ldots,(w,a_p))$ and $((w,b_1),(w,b_2),\ldots,(w,b_q))$ be a pair of fan-sequences at w with $a_p = b_q$ and of minimal length, in the sense that each contains no proper subsequence having initial edge coloured with some colour from $\Theta_w \setminus \Theta_v$ and that all edges except a_p and b_q are distinct. At least one of p and q is at least 2; without loss of generality, suppose $p \geq 2$. Let the colours of the first sequence be $\alpha_1,\alpha_2,\ldots,\alpha_p$ and let the colours of the second sequence be $\beta_1,\beta_2,\ldots,\beta_{q-1},\alpha_p$, if $q \geq 2$. If $q = 1$, then $\beta_1 = \alpha_p$. Thus, α_p is missing both from a_{p-1} and from b_{q-1}. Since these fan-sequences are of minimal length, none of their colours except α_1 and β_1 are in $\Theta_w \setminus \Theta_v$. Let γ belong to $\Theta_v \setminus \Theta_w$ and consider a chain Γ of maximal length consisting of edges alternately coloured α_p and γ and having initial edge $(w,a_p) = (w,b_q)$. We consider two distinct cases:

(1) $q \geq 2$: Γ may terminate at either a_{p-1} or at b_{q-1} or at neither of these. In the last two instances, by interchanging the colours of Γ and re-colouring (w,a_i) with colour α_{i+1}, for

$i = 1,2,\ldots,p-1$, we can colour (v,w) with α_1. In the first instance, by interchanging the colours of Γ and re-colouring (w,b_j) with colour β_{j+1}, for $j = 1,2,\ldots,q-1$, we can colour (v,w) with colour β_1.

(2) $q = 1$: If Γ does not terminate at v, then by interchanging the colours of Γ, we can colour (v,w) with β_1. If Γ terminates at v, then by interchanging the colours of Γ and re-colouring (w,a_i) with colour α_{i+1}, for $i = 1,2,\ldots,p-1$, we can colour (v,w) with colour α_1.

This proves our first assertion. The second is the following:

(ii) No fan-sequence $((w,a_1),(w,a_2),\ldots,(w,a_p))$ with edges coloured $\alpha_1,\alpha_2,\ldots,\alpha_p$ respectively and such that α_1 lies in $\Theta_w \setminus \Theta_v$ can contain an edge (w,a_k) such that $\overline{\Theta_{a_k}}$ contains some α_i, where $i< k$.

Suppose on the contrary that this is the case and let γ belong to $\Theta_v \setminus \Theta_w$. If we assume that $i \geq 2$, then there must exist a chain with edges alternately coloured γ and α_i, having initial vertex a_{i-1} (incident with an edge coloured γ) and having terminal edge (a_i,w) (coloured α_i); otherwise, by colouring (w,a_{i-1}) with colour γ, interchanging the colours on the maximal (γ,α_i)-chain starting at a_{i-1} and re-colouring edge (w,a_j) with colour α_{j+1}, for $j = 1,2,\ldots,i-2$ (if this set is non-empty), we can re-colour (v,w) with colour α_1.

Similarly, there exists a (γ,α_i)-chain with initial vertex a_k (and initial edge coloured γ) and terminal vertex either w (and terminal edge (w,a_i) coloured α_i) or a_{i-1} (and terminal edge coloured γ); otherwise, we can re-colour (w,a_k) with colour γ. We can then interchange the colours on the maximal (γ,α_i)-chain starting at a_k and re-colour (w,a_j) with colour α_{j+1} for $j = 1,2,\ldots,k-1$. This allows us to colour (v,w) with colour α_1.

However, the simultaneous existence of these two (γ,α_i)-chains is clearly impossible. There remains to deal with the case when $i = 1$. In this case, by re-colouring (w,a_j) with colour α_{j+1} for $j = 1,2,\ldots,k-1$, colouring (w,a_k) with colour γ and interchanging the colours on the maximal (γ,α_i)-chain starting at a_k (and which terminates at a_1), we can colour (v,w) with colour α_1. This concludes the proof of our second assertion.

We now complete the proof of the theorem as follows: Consider a fan-sequence of maximal length at w and starting with an edge coloured with a colour from the set $\Theta_w \setminus \Theta_v$. By our second assertion, such a fan-sequence must end with an edge (w,z) such that $\overline{\Theta_z}$ is empty and thus $\rho(z) = \rho$. By our first assertion, there are at least $\rho - k + 1$ such fan-sequences ending in as many edges, which are pairwise distinct. Thus, w is adjacent to at least $\rho - k + 1$ vertices of maximum valency other than v, as required. //

This theorem has the following corollaries:

2.2 Corollary

If G is an edge-critical graph with maximum and minimum valencies ρ and σ respectively, then

(i) every vertex of G is adjacent to at least two vertices of maximum valency;

(ii) G has at least

$$\max\{3, \rho - \sigma + 2\}$$

vertices of maximum valency. //

2.3 Corollary

If G is an edge-critical graph, then for each edge e,

$$\rho(G - e) = \rho(G). \ //$$

2.4 Corollary

If G is an edge-critical graph, then for each edge e,

$$\chi_e(G - e) = \rho. \ //$$

In view of this discussion, we shall adopt the second definition of criticality, and define a graph G to be critical if G is connected and of class 2, but the deletion of any edge yields a graph of class 1. We also call such a graph ρ-critical when we wish to emphasize the fact that its maximum valency is ρ. Occasionally, we shall also use the first definition of criticality, but when we do so, we explicitly call such graphs vertex-critical.

It is not difficult to see that ρ-critical graphs exist for each ρ. One way of seeing this is by noting that ρ-valent graphs exist for each ρ. For each ρ we consider a ρ-valent graph which may be of odd order or of even order. In the former case, the graph is of class 2, by (1.6). In the latter case we can obtain a graph of class 2 by inserting a vertex into any one of the edges. Thus, for each ρ there exist class 2 graphs having maximum valency ρ.

Now consider an arbitrary graph G which is of class 2 and let e be any edge. If G - e is of class 1, then we call such an edge essential; otherwise, we call it non-essential. Thus, a critical graph is a class 2 graph all of whose edges are essential. Moreover, if from G we remove the (possibly empty) set of all non-essential edges, then we obtain a critical graph with the same maximum valency. This shows that ρ-critical graphs exist for each ρ.

Vizing [43] strengthened this last assertion as follows:

2.5 Theorem

If G is a class 2 graph with maximum valency ρ, then G contains a k-critical subgraph for each integer k satisfying $2 \leq k \leq \rho$. //

We illustrate these ideas by dealing with the following example in some detail. Consider the generalised Petersen graph P(8,3). This graph is illustrated in the following diagram and was shown to be of class 1 in (1.11).

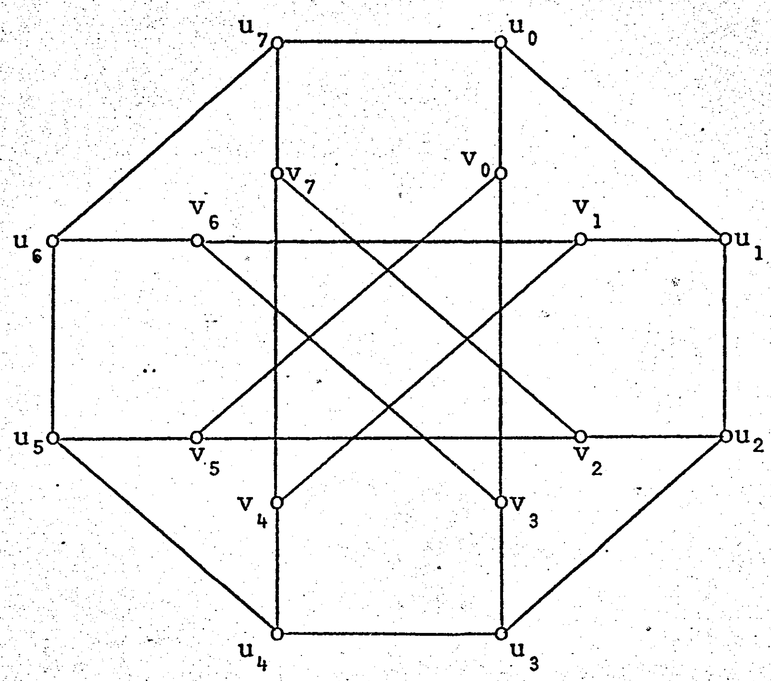


Figure 2.1

We obtain a class 2 graph from P(8,3) by introducing a vertex w into the edge (u_0,u_7). Clearly, each of the edges (w,u_0), (w,u_7) and (u_i,u_{i+1}) $(i = 0,1,2,\ldots,6)$ is essential, since the deletion of any such edge allows us to colour the remaining edges of this set and those of the inner rim alternately α and β, whereas the spokes can be coloured γ. Moreover, the following five 3-colourings show that all other edges are also essential. This shows that the graph obtained from P(8,3) by the insertion of a vertex into any edge of the outer rim is in fact 3-critical. Note that each of these colourings enables us to say only that the edges marked & are essential. The conclusion that the edges marked @ are also essential follows by symmetry.

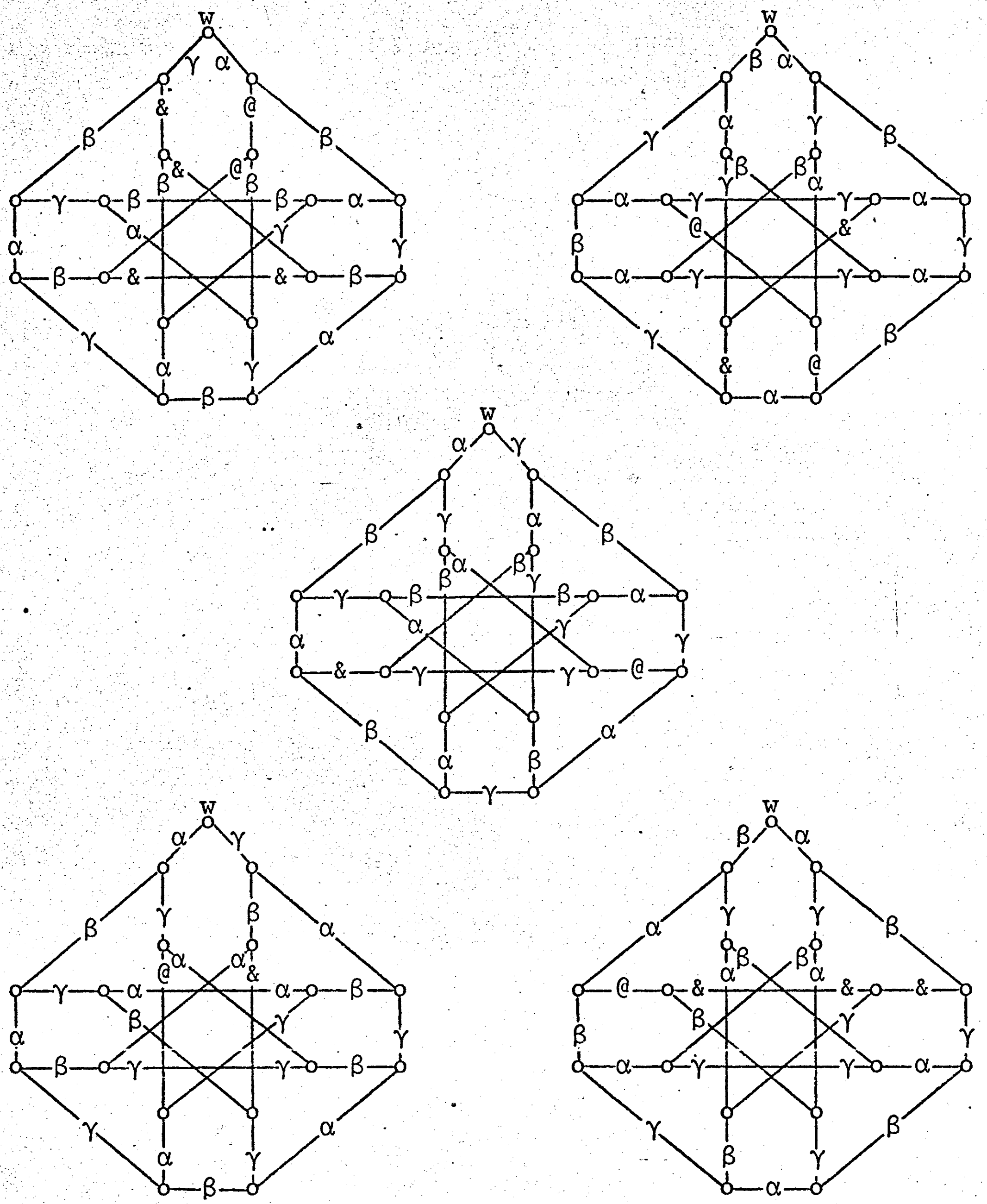


Figure 2.2

We conclude this section by giving a few examples of critical graphs to which we shall be referring in later chapters.

<u>2.6 Theorem</u>

The graph G obtained from the complete bipartite graph $K_{\rho,\rho}$ by inserting a vertex into an arbitrary edge is ρ-critical.

Proof

It follows from (1.8) that G is of class 2. We now show that the deletion of any of the edges of G yields a graph of class 1. It is readily seen that any two edges of $K_{\rho,\rho}$ lie on a Hamiltonian circuit, so that if v is the new vertex of G and e is any edge of G, there is a Hamiltonian circuit H containing both v and e. In G − e, the remainder of H can be 2-coloured, and the graph G − H is a bipartite graph which can be $(\rho - 2)$-coloured, by (1.3). It follows that $\chi_e(G - e) = \rho$. //

Before proving our next two theorems we need to establish the following lemma.

2.7 Lemma

There exists a decomposition of K_{2k+2} into k edge-disjoint Hamiltonian circuits and a 1-factor in such a way that

for any pair of distinct edges not in the 1-factor, one of the Hamiltonian circuits contains both of them.

Proof

We use the construction and notation of (1.9), and consider various cases according to the ways in which the prescribed edges e and f are adjacent to the edges of the 1-factor F.

Case 1: e and f have a common vertex v, say.

Let $e = (v,x)$ and let $f = (v,y)$.

(i) If $(x,y) \in F$, we assign the new label a to the old vertex v, the new label 1 to the old vertex x, and the new label 1 + k to the old vertex y. We then label b the vertex adjacent to v in F and label the vertices of the remaining edges of F $(2,2+k),(3,3+k),(4,4+k),\ldots,(k,2k)$ respectively. Thus, e and f belong to H_1 in the decomposition of K_{2k+2} of (1.9).

(ii) If $(x,y) \notin F$, then we assign the new label 1 to the old vertex v, the new label a to the old vertex x and the new label 2 to the old vertex y. We label vertices adjacent to a, 1, and 2 in F, b, 1 + k, and 2 + k respectively. We then label the vertices of the remaining edges of F (3,3+k), (4,4+k),..., (k,2k) respectively. As before, e and f belong to H_1 in the decomposition of K_{2k+2} of (1.9).

Case 2: e and f have no common vertex, but are adjacent to the same edge (v,w) in F. Let e = (v,x) and f = (w,y).

(i) If $(x,y) \in F$, then we assign the new labels 1, 1 + k, 2 + k, and 2 to the old vertices v, w, x, and y respectively. We label the vertices of the remaining edges of F (3,3+k), (4,4+k),..., (k,2k), and (a,b) respectively. Thus, e and f belong to $H_{\{\frac{1}{2}k\}+1}$ in the decomposition of K_{2k+2} of (1.9).

(ii) If $(x,y) \notin F$, then we assign the new labels 1, 1 + k, a, and 2 + k to the old vertices v, w, x, and y respectively. We label b and 2 the vertices adjacent in F to a and to 2 + k respectively. We then label the vertices of the remaining edges of F (3,3+k), (4,4+k),..., (k,2k) respectively. Thus, e and f belong to H_1 in the decomposition of K_{2k+2} of (1.9).

Case 3: e and f are not adjacent to the same edge in F.

Let e = (v,w), f = (x,y), and vertices adjacent in F to v, w, x, and y be v', w', x', and y' respectively. We assign the new labels 1, 1 + k, a, b, 2, 2 + k, k, and 2k to the old vertices v, v', w, w', y, y', x', and x respectively. We then label the vertices of the remaining edges of F (3,3+k), (4,4+k),..., (k-1,2k-1) respectively. Thus, e and f belong to H_1 in the decomposition of K_{2k+2} of (1.9). This completes the proof. //

2.8 Theorem

The graph G obtained from K_{2k} by inserting a vertex into an arbitrary edge is critical.

Proof

It follows from (1.8) that G is of class 2. We now show that the deletion of any of the edges of G yields a graph of class 1. Let the new vertex v be inserted into the edge e. It is trivial to see that if f is any edge incident with v, then G - f is of class 1. So let f be an edge not incident with v. By (2.7), there exists a Hamiltonian circuit H in G including v and f, such that G - H is (2k-3)-colourable. But H - f is 2-colourable. Thus, G - f is (2k-1)-colourable, as required. //

2.9 Theorem

The graph G obtained from K_{2k} by the deletion of an arbitrary 1-factor and the insertion of a vertex into any one of the remaining edges is critical.

Proof

By (1.8), the graph G is of class 2. To see that it is in fact critical, let the vertex v (of valency 2) be inserted into the edge e. It is trivial to see that if f is any edge incident with v in G, then G - f is of class 1. So let f be any edge not incident with v. By (2.7), there exists a Hamiltonian circuit H in K_{2k} including e and f, such that $K_{2k} - (H \cup F)$ is (2k-4)-colourable for any arbitrary 1-factor F. But H - f is 2-colourable. Thus, G - f is (2k-2)-colourable, which completes the proof. //

Finally we show that each graph in the infinite family of graphs, the k'th member of which we denote by L_k, is 3-critical. L_1 and L_2 are shown in Figure 2.3. For $k \geq 2$, we obtain L_k from L_{k-1} by inserting a vertex labelled 2k - 1 into the edge (a,2k-3) and a vertex

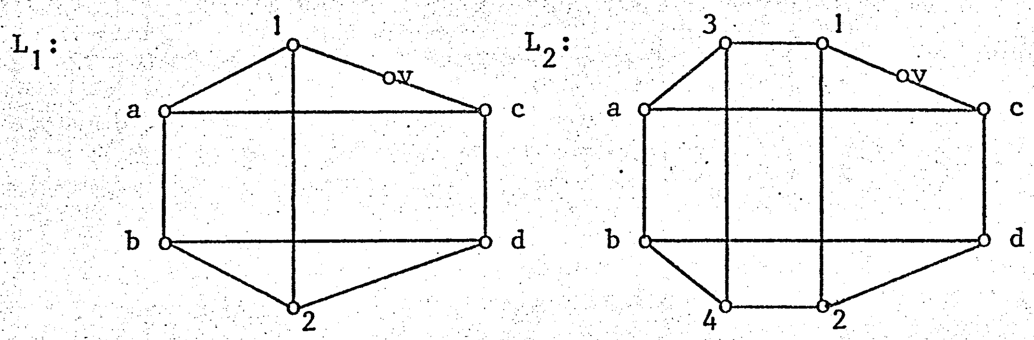


Figure 2.3

labelled 2k into the edge (b,2k−2); we then join the new vertices 2k and 2k − 1 by an edge. We note that L_k consists of a Hamiltonian circuit, which we denote by H_k, and a set of k + 2 independent edges, which we denote by S_k.

It follows from (1.8) that for each k, L_k is of class 2. We now show that the deletion of any edge of L_k results in a 3-colourable graph. This is clearly true if the edge e is deleted from H_k, for then we can colour $H_k - e$ with two colours and the edges of S_k with the third colour. So let e be an edge of S_k. To show that $L_k - e$ is 3-colourable, we use induction on k. It is easy to start the induction and check that the statement holds for L_1, L_2, and L_3. Now we assume that the statement holds for all L_k with $k < k_0$. We note that L_{k_0} $(k_0 \geq 4)$ contains at least four edges of the type (2r−1,2r). Moreover, $L_{k_0} - e$ has at least two "consecutive" edges of this type. Let L' be the graph obtained from $L_{k_0} - e$ by deleting two "consecutive" edges of the said type and contracting all edges incident with the resulting vertices of valency 2. Then L' has at least one edge of type (2r−1,2r) and is 3-colourable, by the inductive hypothesis. Moreover, any 3-colouring of L' induces a 3-colouring of $L_{k_0} - e$, as indicated in the following diagrams:

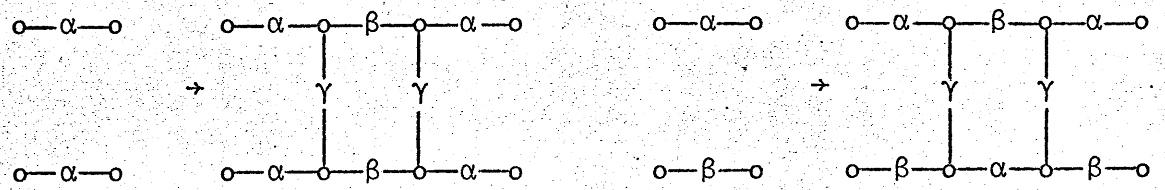


Figure 2.4

This completes the induction and the proof.

2) Constructions of critical graphs

In this section we consider two types of constructions of ρ-critical graphs:

Type (A) We construct ρ'-critical graphs from ρ-critical graphs of the same order, where $\rho' > \rho$.

Type (B) We construct ρ-critical graphs from other ρ-critical graphs of smaller order.

Type (A)

We first establish a lemma.

2.10 Lemma

Let G_i $(i = 1,2)$ be a pair of 3-critical graphs formed respectively by taking an odd circuit C_{2s+1} and a set S_i of s independent edges. If $S_1 \cap S_2 = \emptyset$, then $G := G_1 \cup G_2$ is a 4-critical graph.

(Remark. This construction is illustrated in Figure 2.5)

Proof

From (1.5) it follows that G is of class 2. We now show that every edge e of G is essential. We consider two cases:

(i) $e \in C_{2s+1}$: In this case we colour the edges of $C_{2s+1} - e$ with two colours and each of S_1 and S_2 with the third and fourth colours respectively.

(ii) $e \in S_i$: Here we colour $G_i - e$ with three colours and the edges of S_j $(j \neq i)$ with the fourth colour. //

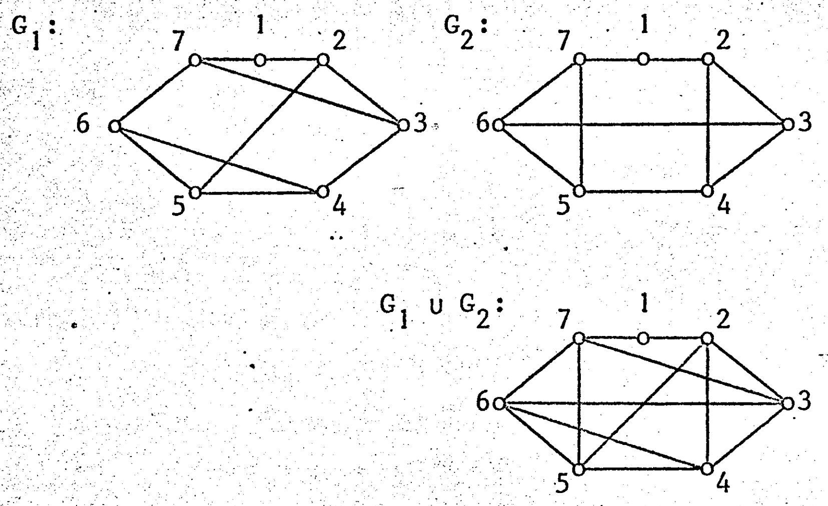


Figure 2.5

An inductive argument easily generalizes Lemma 2.10 to give the following theorem

2.11 Theorem

Let $G_1, G_2, \ldots, G_t$ be 3-critical graphs such that for each $i = 1,2,\ldots,t$, G_i is formed by taking an odd circuit C_{2s+1} and a set S_i of s independent edges. If $S_i \cap S_j = \emptyset$ for each i, j $(i \neq j)$, then $G := G_1 \cup G_2 \cup \ldots \cup G_t$ is a (t+2)-critical graph. //

To illustrate this theorem, we first show how to obtain an infinite family of 3-critical graphs. We take an odd circuit C_{2s+1} $(s \geq 2)$ and let $G^{(s)}$ be the graph obtained from it by adding a set of s independent edges as in Figure 2.6.

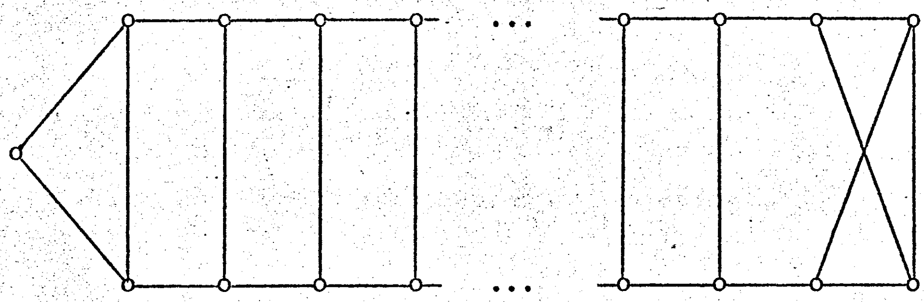


Figure 2.6

By induction on s, it is easy to check that $G^{(s)}$ is in fact 3-critical.

We now take the graph of order 2s + 1 and label its vertices 1,2,3,...,2s+1 in a counterclockwise manner along the (2s+1)-circuit. We call this graph G_i and we denote the set of edges of G_i not on the (2s+1)-circuit by S_i. We now let G_{i+1} be a copy of G_i except that the vertex formerly labelled j is now labelled j + 2 (mod 2s+1) for j = 1,2,3,...,2s+1. The graphs in the set $\{G_1,G_2,G_3,\ldots,G_{s+1}\}$ then satisfy the conditions of Theorem 2.11. We can therefore construct ρ-critical graphs of odd order 2n + 1 for each ρ satisfying $2 \leq \rho \leq \{\frac{1}{2}n\} + 2$. Figure 2.7 illustrates this for the case n = 3.

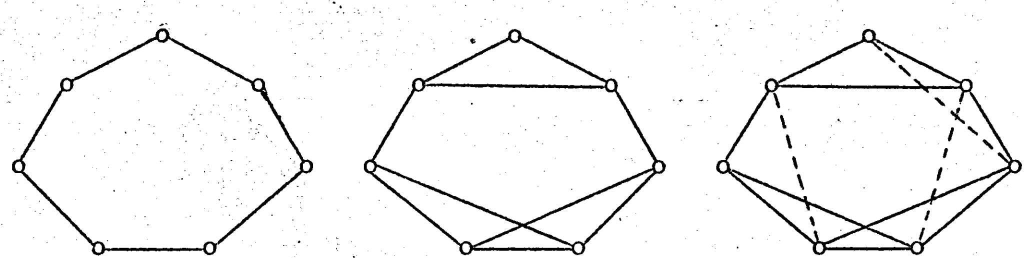


Figure 2.7

This can be considerably improved by the use of more complicated constructions. The statement and proof of these are deferred till Chapter 4 by which time the relevant preliminary concepts will have been introduced.

The reader is referred to a computer programme included in the Appendix. This programme generates ρ'-critical graphs from ρ-critical graphs ($\rho' > \rho$) of the same order, using the algorithm exhibited in (2.11). The programme was written jointly with C. Galea and A. Buttiġieġ.

Jakobsen [21] gives another construction of a type similar to that of (2.11). This can be formulated as follows:

2.12 Theorem

Let G be a ρ-critical graph of odd order $2s + 1$ and satisfying the following conditions:

(i) G does not contain three independent edges;

(ii) the number m of edges of G satisfies $\{\frac{m}{s}\} = \rho + 1$.

Let G' be any graph obtained from G by adding any new set of s independent edges; then G' is $(\rho+1)$-critical. //

For future reference we also include here a similar construction for vertex-critical graphs due to Beineke & Wilson [3]:

2.13 Theorem

If G is a graph obtained from an odd circuit C_{2s+1} by adding t mutually disjoint sets, each consisting of s independent edges, then G is vertex-critical. //

Type (B)

We now turn to constructing ρ-critical graphs from other ρ-critical graphs of smaller order. One such construction makes use of the Hajós union of two graphs and is again due to Jakobsen [19]:

2.14 Theorem

Let G and H be two ρ-critical graphs and let K be any Hajós union of G and H obtained by identifying two vertices the sum of whose valencies does not exceed $\rho + 2$; then K is also ρ-critical. //

If we restrict ourselves to odd ρ, then we can give another similar construction of our own.

2.15 Theorem

Let ρ be odd, let G be a ρ-critical graph and let H be either $K_{\rho,\rho}$ or $K_{\rho+1}$. Then the graph K obtained from G and H by the following construction is also ρ-critical:

Construction

(i) Choose a vertex v in G which has valency ρ and label its neighbours $v_1, v_2, v_3, \ldots, v_\rho$ respectively;

(ii) choose a vertex w in H and label its neighbours $w_1, w_2, \ldots, w_\rho$ respectively;

(iii) delete v and w from G and from H respectively and join the v_i's with the w_j's in a one-one manner.

(The proof of this construction will be given in Chapter 4).

Illustration

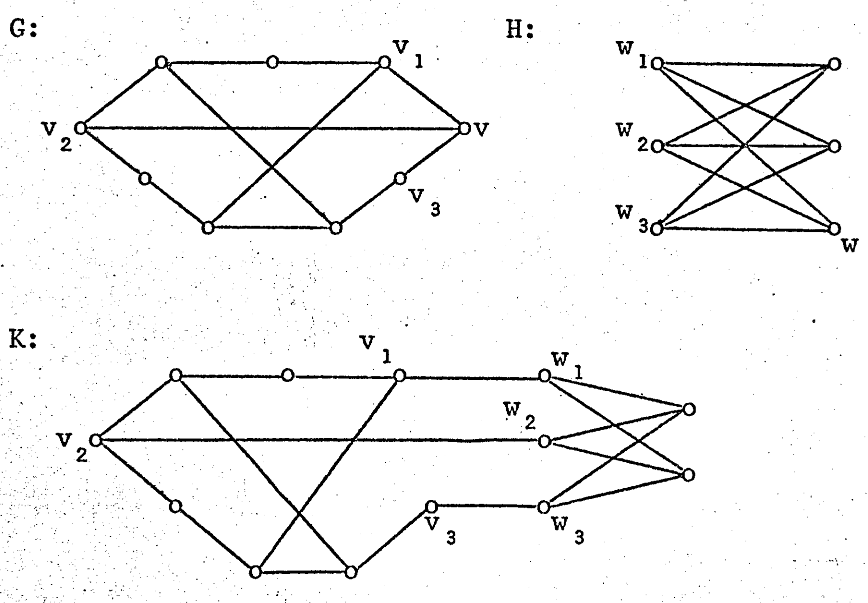


Figure 2.8

3) Basic properties of critical graphs

Before attempting a study of structural properties of critical graphs it is illuminating to analyze in some depth critical graphs of small order. This will give us some idea of what sort of properties to look for in the general case.

Jakobsen [20] was the first to have a look at such graphs, but he limited himself to ones with chromatic index 4. We summarize his results in the following theorem:

2.16 Theorem

If G is a 3-critical graph, then

(i) $\rho(v) = 2$ or 3 for each vertex v;

(ii) the distance between any two vertices of valency 2 is at least 3;

(iii) G cannot have exactly two vertices of valency 3;

(iv) $\frac{4}{3}\, n(G) \leq m(G) \leq \frac{1}{2}(3n(G) - 1)$. //

Using these results, Jakobsen produced a list of all 3-critical graphs on at most 10 vertices. We include the list here for future reference as Table 2.17.

In view of these results and others discussed so far, Beineke & Wilson [3] and, independently, Jakobsen [21], were led to make the following conjecture:

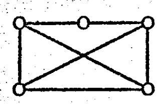
Critical Graph Conjecture

There do not exist any critical graphs of even order.

2.17 Table All 3-critical graphs of order at most 10

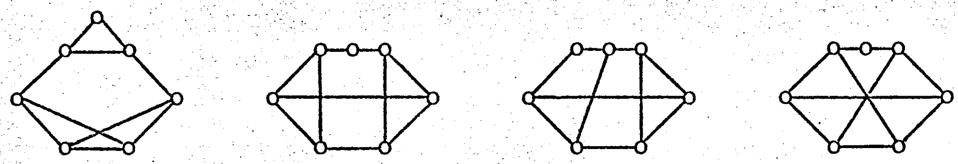
n=4 None

n=5



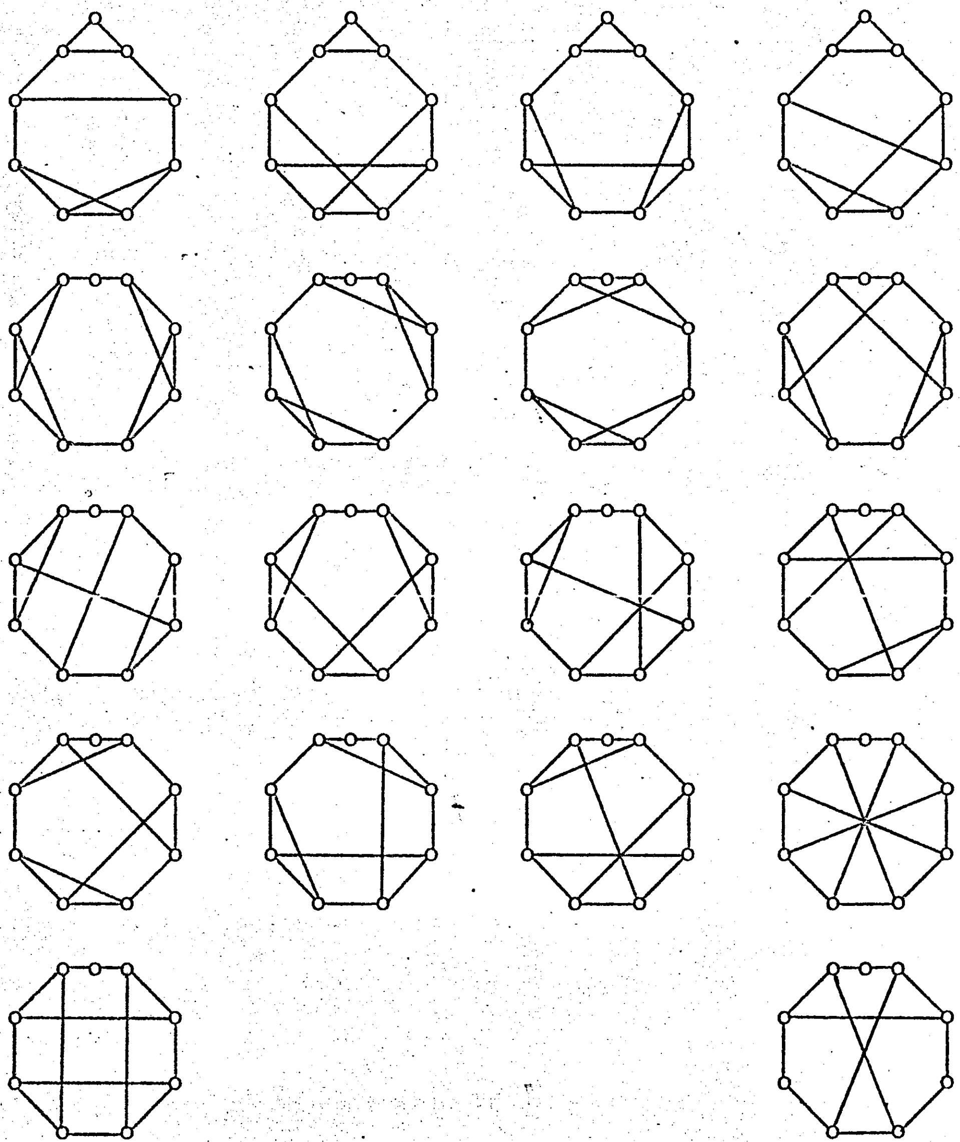
n=6 None

n=7



n=8 None

n=9



n=10 None.

We shall be discussing the Critical Graph Conjecture in Chapter 6. Here we limit ourselves to establishing some elementary properties of critical graphs. We have already proved one such property in (2.1). As a corollary to that theorem we then obtain the following:

2.18 Theorem

Let $\lambda(G)$ denote the Szekeres-Wilf number of a graph G, i.e. the integer max $\sigma(G')$, where the maximum is taken over all induced subgraphs G' of G and σ denotes the smallest vertex valency. Then G is of class 1 if $\rho(G) \geq 2\lambda(G)$.

Proof

Assume the contrary, i.e. that there exists a graph G satisfying $\rho(G) \geq 2\lambda(G)$ and $\chi_e(G) = \rho + 1$. Without loss of generality we can assume that G is critical. Let S be the set of vertices of G whose valency does not exceed λ. Suppose that there exists at least one vertex in $V(G) \setminus S$ of valency at most λ in the subgraph induced by $V(G) \setminus S$ and let v' be any such vertex of valency at most λ in this subgraph. From the definition of S it follows that v' is adjacent in G to at least one vertex of S which has valency in G at most λ. Hence, by (2.1) and since $\rho \geq 2\lambda$ and $V(G) \setminus S$ contains all vertices of valency ρ in G, it follows that v' is adjacent to at least $\rho - \lambda + 1 \geq \lambda + 1$ vertices in $V(G) \setminus S$, contrary to the above supposition. This proves the result. //

This in turn has the following corollary:

2.19 Corollary

Let $h_{max}(G)$ be the largest eigenvalue of the adjacency matrix of G. Then G is of class 1, if $\rho(G) \geq 2.h_{max}(G)$.

Proof

Let G' be an induced subgraph of G for which $\sigma(G') = \lambda(G)$.

Then

$$h_{max}(G) \geq h_{max}(G') \geq \sigma(G') = \lambda(G).$$

The result then follows by (2.18). //

We shall often make use of the following simple property of critical graphs:

2.20 Theorem

Let the edges of a graph G be labelled $e_1, e_2, \ldots, e_m$. Then G is a ρ-critical graph if and only if G is of class 2 and there exist ρ-colourings $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ such that in Γ_i, one colour-class consists precisely of e_i for each $i = 1,2,\ldots,m$.

Proof

If G is ρ-critical, then G is of class 2 and for each $i = 1,2,\ldots,m$, $G_i := G - e_i$ is ρ-colourable. We obtain Γ_i from any ρ-colouring of G_i by re-introducing e_i and colouring it with the $(\rho+1)$'th colour.

Conversely, if G is of class 2, then G_i is ρ-colourable for each $i = 1,2,\ldots,m$. Thus, $G - e$ is ρ-colourable for each edge e, i.e. G is ρ-critical. //

2.21 Corollary

Let G be a ρ-critical graph and let J be an arbitrary set of independent edges. Then there exists a $(\rho+1)$-colouring of G in which J is a colour-class.

Proof

Let e be an arbitrary edge of J. (2.20) allows us to colour G with $\rho + 1$ colours in such a way that $\{e\}$ is one colour-class. We now obtain a $(\rho+1)$-colouring of G with J as one colour-class, by changing the colours of $J \setminus \{e\}$ to the colour of e. //

2.22 Corollary

If G is a ρ-critical graph and J is an arbitrary set of independent edges, then $\chi_e(G - J) = \chi_e(G) - 1$.

Proof

By (2.21), $\chi_e(G - J) \leq \rho$.

Now, if we assume that $\chi_e(G - J) < \rho$, then G would be ρ-colourable. Thus, $\chi_e(G - J) = \rho = \chi_e(G) - 1$. //

2.23 Corollary

If G is a ρ-critical graph and F is a 1-factor of G, then $G - F$ is of class 2. //

If we denote by $\alpha_e(G)$ (or simply by α) the edge-independence number of G, then we have the following result:

2.24 Theorem

If G is a ρ-critical graph, then $\alpha\rho \geq m - 1$.

Proof

Let J be a maximal independent set of edges and let e be an edge not in J. $G' := G - e$ has maximum valency ρ and satisfies the equality $\alpha_e(G') = \alpha_e(G)$. Thus,

$$\rho = \chi_e(G') \geq \frac{m(G')}{\alpha_e(G')} = \frac{m(G) - 1}{\alpha_e(G)},$$

since G' is of class 1. Hence the result follows. //

The following remark shows that this result is in a sense best possible. We note that there exist ρ-critical graphs satisfying $\alpha\rho = m - 1 + k$ for each non-negative integer k. The first graph of Table 2.17 shows that this is true for $k = 0$. To see that the statement is also true for $k \geq 1$ we construct recursively the infinite family of 3-critical graphs $\{G_k\}$ as follows: G_0 is the last graph of Table 2.17. G_k is obtained from G_{k-1} by taking a Hajós union of G_0

and G_{k-1} where a vertex of valency 2 in G_{k-1} is identified with a vertex of valency 3 in G_0. Figure 2.9 illustrates this construction.

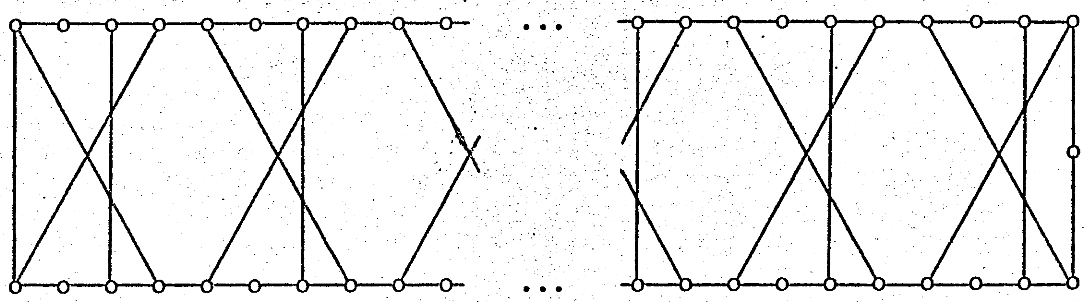


Figure 2.9

Note that $m(G_k) = (8k + 1) + 3k$ and $\alpha_e(G_k) = 4k$ since G_k is Hamiltonian and of odd order for each k.

Theorem 2.24 can be re-formulated as follows:
The total deficiency of a ρ-critical graph of order n is at least $\rho(\frac{1}{2}n - \alpha) - 1$. Such bounds on the deficiency of critical graphs turn out to be very useful in applications, as will be seen later. We now present one other such bound.

2.25 Theorem

Let G be a ρ-critical graph with minimum valency σ. Then $\tau(G)$, the total deficiency of G, is at least

$$\begin{array}{ll} \rho - 2 & \text{if } n \text{ is odd} \\ 2(\rho - \sigma + 1) & \text{if } n \text{ is even.} \end{array}$$

Proof

If n is odd, then the result follows from (1.5). So, let n be even and let v be a vertex of valency σ. Let v_1 be a vertex of valency ρ which is adjacent to v. The graph obtained by deleting the edge (v,v_1) is ρ-colourable, and in such a colouring, one colour is missing from v_1 and is used for some edge (v,v_2). Thus, the graph G' obtained from G by deleting v and adding an edge (v_1,v_2) (which may possibly be a double edge) is ρ-colourable. In the case that (v_1,v_2) is a double edge coloured α and β, say, we can transform the multigraph G' into a simple graph by effecting the following transformation:

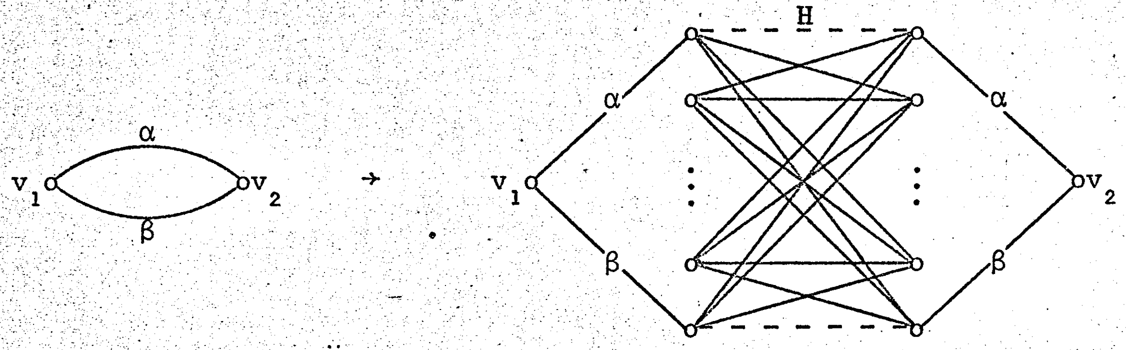


Figure 2.10

Here H is the graph $K_{\rho,\rho}$ with two independent edges removed. Since, for $\rho \geq 3$, $K_{\rho,\rho}$ is of even order and ρ-colourable with two given edges having different colours, in any case we get a new graph which is both of class 1 and of odd order. Consequently,

$$\tau(G') = \tau(G) - (\rho - \sigma) + (\sigma - 2)$$

and

$$\tau(G') \geq \rho, \text{ by } (1.5).$$

Therefore,

$$\tau(G) \geq 2(\rho - \sigma + 1), \text{ as required. //}$$

<u>2.26 Corollary</u>

If G is ρ-critical and $\rho > 2$, then G is not regular. //

For completeness' sake we state:

<u>2.27 Theorem</u>

G is 2-critical if and only if G is an odd circuit. //

We conclude this section by listing two results on critical graphs which are due to Vizing [43].

2.28 Theorem

Let G be a ρ-critical graph. Then

(i) G contains a circuit of length at least $\rho + 1$;

(ii) G has at least $\frac{1}{8}(3\rho^2 + 6\rho - 1)$ edges. //

It is to be noted that if $(n - \rho)$ is 'small', then the bounds given by this theorem are very good. However, there are critical graphs (see, for example, the graph of Figure 2.9) for which they are not particularly good. Hence, possible directions of investigation are the improvements of such bounds. This is what we propose to do in later chapters, where results complementing Vizing's conclusions are presented.

CHAPTER 3: UNIQUELY-COLOURABLE GRAPHS

Since the writing of the material in this chapter, it has come to our attention that some of the results obtained here have since been proved independently by D. Greenwell & H. Kronk [11]. Unlike the approach adopted by these authors, our motivation for looking at uniquely-colourable graphs stems from the close relationship these graphs have to critical graphs. We investigate this relationship and proceed to analyze the structure of uniquely-colourable graphs, which in turn sheds more light on our problem. In particular, we are interested in connectivity properties of these graphs in the same way as similar properties of critical graphs are investigated in Chapter 4; similarly we look for bounds on the number of edges and for circuit length properties of uniquely-colourable graphs with the corresponding results for critical graphs appearing in Chapters 5 and 7 respectively. However, like Greenwell & Kronk, we base our investigation of structural properties of uniquely-colourable graphs on their one fundamental property expressed in Theorem 3.1 below. For this reason, our proofs of Theorems 3.6, 3.16, 3.17 and 3.18 are rather similar. We note that our example of a non-planar uniquely-3-colourable graph of order 18 discussed on p. 46 is a counterexample to Greenwell & Kronk's Conjecture 1.

We define a graph G to be <u>uniquely-k-colourable</u> (or simply <u>uniquely-colourable</u>) if its chromatic index is k, and any colouring of G with k colours induces the same partition of E(G).

Uniquely-colourable graphs exist as is exhibited by the following examples: All even circuits and all open chains of length at least 2 are uniquely-2-colourable; the graphs K_3, K_4, $K_4 - a$, and $K_4 - \{a,b\}$ are all uniquely-3-colourable for any pair of adjacent edges a and b. All star-graphs $K_{1,t}$ are uniquely t-colourable.

One way of obtaining an infinite family of uniquely-3-colourable graphs is by taking any such graph, other than K_3, and replacing any one of its vertices of valency 3 by a triangle. So, for example, we can start by taking the first graph of Figure 3.1 and obtain a new graph (the second in the diagram) by replacing the vertex v by the triangle T. We then repeat the process on some vertex v of valency 3 in the second graph, and so on.

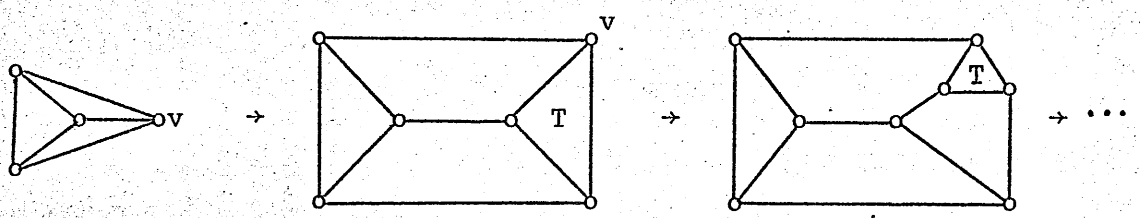


Figure 3.1

From the examples and construction just given one might infer that all uniquely-colourable graphs are planar. However, this is not the case. We propose to consider the following (counter-)example in some detail because it gives us some insight into the structure of such graphs. The graph in question is the generalised Petersen graph on 18 vertices, P(9,2), shown in Figure 3.2.

We note that P(9,2) has girth 5. If we assume that P(9,2) is planar and has f faces, then Euler's polyhedron formula gives:

$$f = 27 + 2 - 18 = 11.$$

But $5f \leq 54$, since the graph has girth 5. This contradicts the previous statement and proves that the graph is non-planar.

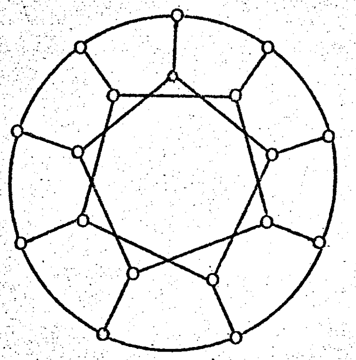


Figure 3.2

On the other hand, a case-by-case analysis shows that (not counting rotations) the following are the only 1-factors of P(9,2):

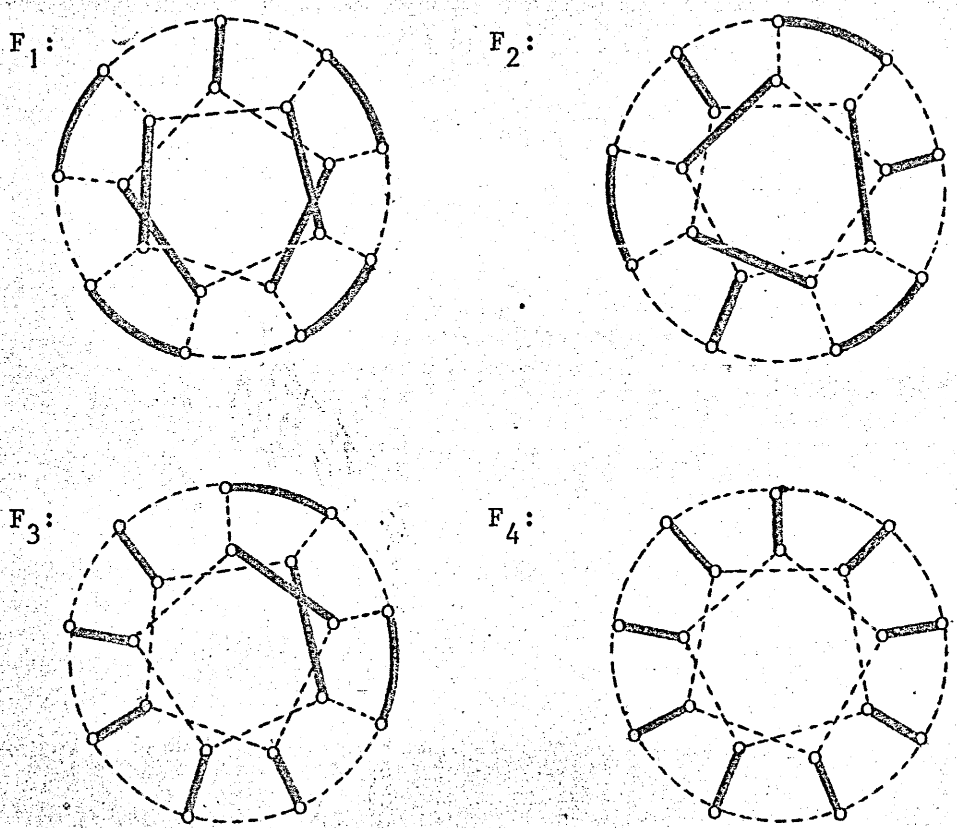


Figure 3.3

In this figure, solid lines indicate edges of the 1-factor. Now, F_1 partitions its complement in $E(P(9,2))$ into a 5-circuit and a 13-circuit, F_3 partitions its complement in $E(P(9,2))$ into a 7-circuit and an 11-circuit, whereas F_4 partitions its complement in $E(P(9,2))$ into two 9-circuits. Thus, none of these 1-factors can be a colour-class in some 3-colouring of $P(9,2)$.

However, the complement of F_2 is a Hamiltonian circuit (of even length) and so we can colour the edges of F_2 with the first colour and the edges of the Hamiltonian circuit alternately with the remaining two. Moreover, all rotations of F_2 and colourings induced by them yield the same partitioning of the edges, so that $P(9,2)$ is indeed uniquely-colourable.

We shall come to this example later on. For the time being we shall concentrate on the motivation for studying uniquely-colourable graphs. We start by giving a simple property of these graphs which comes in very useful in applications.

<u>3.1 Theorem</u>

Let G be a uniquely-k-colourable graph and let the colours be $\{c_1,c_2,\ldots,c_k\}$. For each i, any edge coloured c_i is adjacent to at least one edge of every other colour.

<u>Proof</u>

Assume not. i.e. there exists some edge coloured c_1, say, which is not adjacent to any edge coloured c_2, say. We can then re-colour the first edge with colour c_2 to obtain a different partition of $E(G)$. This contradicts the fact that G is uniquely-colourable. //

<u>3.2 Corollary</u>

If, for each i,j, $C_{i,j}$ denotes the subgraph of a uniquely-k-colourable graph G induced by the edges coloured c_i or c_j, then $C_{i,j}$

is an open chain or an even circuit.

Proof

We first show that $C_{i,j}$ is connected. Assume not, and let S and T be two distinct components of $C_{i,j}$. At least one of them, S say, contains edges of both colours. By interchanging the colours of these, thereby not affecting the colouring of the edges of T, we obtain a new partition of E(G). This contradicts the uniqueness of colourability of G. Now the maximum valency of $C_{i,j}$ cannot exceed 2, and so $C_{i,j}$ is a chain. If the chain is closed, then it has to be of even order, otherwise it is not 2-colourable. //

3.3 Corollary

Let C_i be the i'th colour-class in the colouring of a uniquely-colourable graph. Then we have:

$$0 \leq \big|\, |C_i| - |C_j| \,\big| \leq 1.$$

Proof

Method 1: Follows from (3.2).

Method 2: De Werra [8,9] and MacDiarmid [26] proved independently that if G is an arbitrary graph with chromatic index k, then for any integer t ($t \geq k$), there exists a t-colouring of G in which two distinct colour-classes have cardinality differing by at most unity. Now uniquely-colourable graphs have essentially one colouring. Thus, all colour-classes in a uniquely-colourable graph enjoy this property. //

3.4 Corollary

Let C_i be the i'th colour-class in the colouring of a uniquely-k-colourable graph G having m edges. Then

$$\frac{1}{k}(m - k + 1) \leq |C_i| \leq \frac{1}{k}(m + k - 1).$$

Proof

Let C_1 be a colour-class of minimum cardinality. Then.

$$k.|C_1| \leq \sum_{j=1}^{k} |C_j| = m \leq |C_1| + (k-1)(|C_1| + 1), \text{ by (3.3).}$$

Thus, $|C_1| \geq \frac{1}{k}(m - k + 1)$.

Similarly, if C_2 is a colour-class of maximum cardinality, then

$$m = \sum_{j=1}^{k} |C_j| \geq |C_2| + (k-1)(|C_2| - 1), \text{ by } (3.3).$$

Hence, $|C_2| \leq \frac{1}{k}(m + k - 1)$.

This proves the result. //

Given m and k as above, if we let $q = [\frac{m}{k}]$ and r to be the unique integer satisfying $m = qk + r$, $0 \leq r < k$, then we could have the following alternative formulation of (3.4):

3.4* Corollary

In a uniquely-k-colourable graph there are $k - r$ colour-classes of size q and r of size $q + 1$. //

The connection, or rather contrast, between critical graphs and uniquely-colourable graphs is evident from the following theorem:

3.5 Theorem

If G is a graph which is both critical and uniquely colourable, then G is K_3.

Proof

If G is critical, then there exists a colouring in which one colour-class consists precisely of any one pre-assigned edge, by (2.20). Hence, with each edge of G we can associate a colouring of G, thereby obtaining at least as many partitions of E(G) as there are edges, contradicting the fact that G is uniquely-colourable. This argument holds provided that not all of the colour-classes consist of exactly one edge, i.e. not all edges are pairwise adjacent, in which case each edge forms a colour-class of its own. This can happen in either of two ways: $G = K_3$, which we are admitting, or $G = K_{1,t}$, which is in class 1 and thus not critical. //

In fact this result can be considerably strengthened as follows:

3.6 Theorem

If G is a uniquely-colourable graph and G is not K_3, then G is of class 1.

Proof

Suppose G is of class 2. Let v_0 be a vertex of maximum valency ρ, and let $v_1, v_2, \ldots, v_\rho$ be its neighbours. Without loss of generality we can assume that the edges $(v_0,v_1),(v_0,v_2),\ldots,(v_0,v_\rho)$ are coloured $c_1,c_2,\ldots,c_\rho$ respectively. For each $i = 1,2,\ldots,\rho$, the subgraph $C_{i,\rho+1}$ induced by edges coloured either c_i or $c_{\rho+1}$, is an open chain having v_0 as an end-vertex. It follows from (3.1) that each vertex v_i $(i = 1,2,\ldots,\rho)$ is incident with some edge coloured $c_{\rho+1}$. This implies that $|V(C_{i,\rho+1})| \geq \rho + 1$.

We also claim that $|V(C_{i,\rho+1})| \leq \rho + 1$, for each i. This can be seen as follows: Assume that $|V(C_{1,\rho+1})| > \rho + 1$, and let the ordered set of vertices of the chain $C_{1,\rho+1}$ be

$\Lambda := \langle v_0,v_1,a_2,a_3,\ldots,a_{\lambda-1},a_\lambda \rangle$ $(\lambda \geq \rho + 1)$, as shown in Figure 3.4:

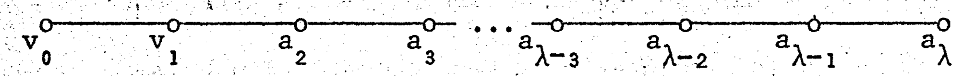


Figure 3.4

Since $|\{C_{j,\rho+1}\}_{j=2}^{\rho}| = \rho - 1 < \rho \leq |\{v_1,a_2,a_3,\ldots,a_{\lambda-1}\}|$, then there exists at least one vertex in the set $\Lambda \setminus \{v_0,a_\lambda\}$ which is not an end-vertex of any of the chains $\{C_{j,\rho+1}\}_{j=2}^{\rho}$. Thus,

$\mu := \bigcap_{j=1}^{\rho} V(C'_{j,\rho+1}) \neq \emptyset$, where $C'_{j,\rho+1}$ is obtained from $C_{j,\rho+1}$ by deleting the two end-vertices. If $x \in \mu$, then x has valency 2 in each of the chains $C_{j,\rho+1}$, $j = 1,2,3,\ldots,\rho$. This means that $\rho(x) = \rho + 1$, which contradicts the fact that $\rho(G) = \rho$. Hence $|V(C_{j,\rho+1})| = \rho + 1$.

Now, $A := \{v_i\}_{i=0}^{\rho} \subseteq V(C_{j,\rho+1})$ since each v_i is adjacent to some edge coloured $\rho + 1$. Also, $|A| = |V(C_{j,\rho+1})| = \rho + 1$. Hence, $A = V(C_{j,\rho+1})$ for each $j = 1,2,3,\ldots,\rho$. Moreover,

$$V(G) = \bigcup_{j=1}^{\rho} V(C_{j,\rho+1}) = \bigcup_{j=1}^{\rho} A = A.$$

Finally we show that G must be of the form K_{2k+1}. Note that since each of the vertices $\{v_i\}_{i=1}^{\rho}$ is incident with an edge coloured $c_{\rho+1}$, ρ must be even, say $\rho = 2k$, and there are k edges coloured $c_{\rho+1}$. Also, since $C_{i,\rho+1}$ is an open chain of length $\rho + 1$, there are k edges coloured c_i for each $i = 1,2,3,\ldots,\rho$. So in all, G has $k + k.2k$ edges, and if we write $2k + 1 = n = |V(G)|$, then G has $\frac{1}{2}n(n-1)$ edges. This means that G is K_n.

Now it is not difficult to see that K_{2k+1} is not uniquely-colourable except for the case $k = 1$, which we are excluding. This completes the proof. //

3.7 Theorem

If G is a ρ-valent $(\rho \geq 3)$ and uniquely-ρ-colourable graph, then the graph G' obtained from G by the insertion of a vertex into any one of the edges is ρ-critical.

(Remark: Compare this construction with that in (1.8))

Proof

By virtue of (3.6), we know that G is of even order, and so, G' is of class 2, by (1.8). We have to show that for any edge e, G' - e is of class 1. Let the vertices of G be labelled $w_1, w_2, \ldots, w_n$ and let the new vertex v_0 be adjacent to w_1 and to w_2 in G'. Consider the graph H obtained from G' by splitting v_0 into two (end-)vertices v_1, v_2 such that v_i is adjacent to w_i $(i = 1,2)$ (see Figure 3.5).

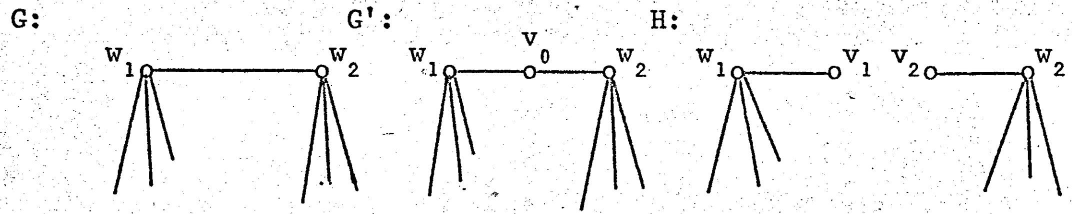


Figure 3.5

Any ρ-colouring of G induces a ρ-colouring of H in which (v_1,w_1) and (v_2,w_2) are both coloured α. Let e be any other edge coloured β, and define ϕ to be equal to β if $\beta \neq \alpha$, and let ϕ be any other colour not α, if $\beta = \alpha$. Then $C_{\alpha,\phi}$ is an open chain and so (v_1,w_1) and (v_2,w_2) belong to different components of $C_{\alpha,\phi}$ in H - e. This implies that (v_1,w_1) can be re-coloured ϕ whereas (v_2,w_2) can still remain coloured α. This in turn implies that G' - e is of class 1. The case when e is either (v_1,w_1) or (v_2,w_2) is trivial. //

The graph in Figure 3.6, in which the labelling is the same as that used in the above discussion, illustrates the construction of Theorem 3.7.

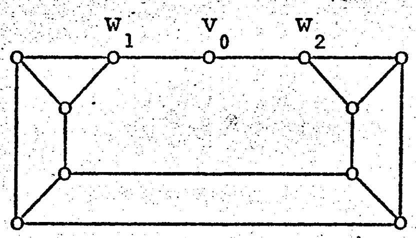


Figure 3.6

3.8 Theorem

If G is a ρ-critical graph, then G cannot contain a uniquely-ρ-colourable subgraph H such that $G - E(H)$ is a disconnected graph with at least two non-trivial components.

Proof

Assuming the contrary, let G be both ρ-critical and contain such a subgraph H. Let $G - E(H) = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$ and $E(H_i) \neq \emptyset$ $(i = 1,2)$. Then for $i = 1,2$, $H_i \cup E(H)$ is ρ-colourable and moreover, each ρ-colouring of $H_i \cup E(H)$ induces the same partition of $E(H)$. Thus, we can re-combine the colourings of $H_1 \cup E(H)$ and of $H_2 \cup E(H)$ to give a ρ-colouring of G. This establishes the required contradiction. //

Another motivation for studying uniquely-colourable graphs stems from the following observation. The proof of the theorem follows very closely that of (3.7) and will be left to the reader.

3.9 Theorem

Let G be a uniquely-colourable graph which has two end-edges e_1 and e_2 belonging to the same colour-class, and let G' be a graph obtained from G by the deletion of an edge which is not an end-edge. Then G' can be coloured in such a way that e_1 and e_2 belong to different colour-classes. //

Closely related concepts have been investigated by Izbicki [16,17]. In his notation, a graph whose vertex-valencies satisfy $\rho(v) = \rho$ or 1 for each vertex v, is called pseudo-regular. Also, edges incident with a vertex of valency 1 (i.e. end-edges) in a pseudo-regular graph, are called external edges; otherwise they are called internal. With this terminology the following theorem is shown to hold.

3.10 Theorem

Let G be a pseudo-regular graph with chromatic index k. If f_i denotes the number of external edges coloured c_i $(i = 1,2,\ldots,k)$, then all the f_i's have the same parity. //

The above discussion yields the following construction for critical graphs: Let G be a graph and let (v_1,v_2) be an edge of G. Let H be any other graph having two end-edges (a_1,b_1) and (a_2,b_2). Define the operation Ψ on G and H as follows: Identify a_i with v_i $(i = 1,2)$ and replace the edge (v_1,v_2) in G by the graph H to give a new graph K of order $n(G) + n(H) - 2$. This construction is illustrated in Figure 3.7.

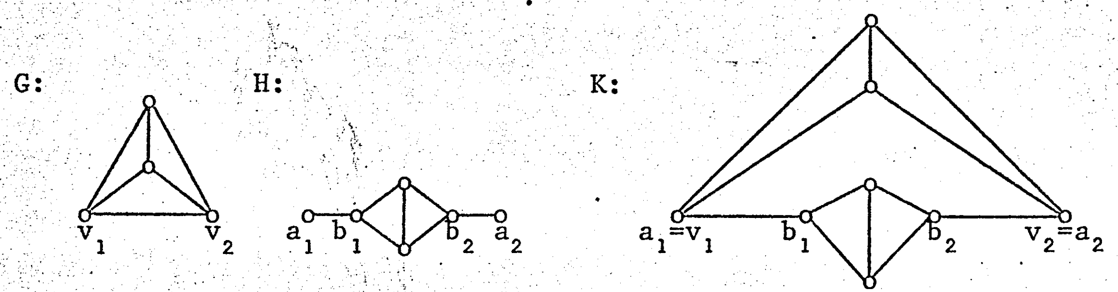


Figure 3.7

For brevity, in what follows we shall speak of 'replacing an edge of G by the graph H' when we are referring to this construction. This discussion has the following application:

3.11 Theorem

Let G be a ρ-critical graph and let H be a uniquely-ρ-colourable graph with two end-edges in the same colour-class. If K is any graph obtained by replacing an edge of G by H using the above construction, then K is also ρ-critical.

Proof

K clearly is of class 2, since any ρ-colouring of K forces (a_1,b_1) and (a_2,b_2) to be in the same colour-class. This induces a ρ-colouring of G. Also, since (a_1,b_1) and (a_2,b_2) lie in the same colour-class, then the deletion of any edge in the set $(E(G) - (v_1,v_2)) \cup \{(a_1,b_1),(a_2,b_2)\}$ yields a graph of class 1. Now let e be an arbitrary edge in $E(H) - \{(a_1,b_1),(a_2,b_2)\}$. Then, by (3.9), the removal of e enables us to re-colour (a_1,b_1) and (a_2,b_2) differently. Thus, the notation of the ρ-colourings of $G - (v_1,v_2)$ and of $H - e$ can be so chosen as to give a ρ-colouring of $K - e$. //

One important application of this construction is the following:

3.12 Theorem

There exist critical graphs which are not Hamiltonian.

Proof

Such graphs can be constructed in general by replacing three adjacent edges of a ρ-critical graph ($\rho \geq 3$) by three uniquely-ρ-colourable graphs, as exhibited in the construction of (3.11). The resulting graph H then contains an induced subgraph homeomorphic to the first graph of Figure 3.8, but not to the second. This implies that H is not Hamiltonian. //

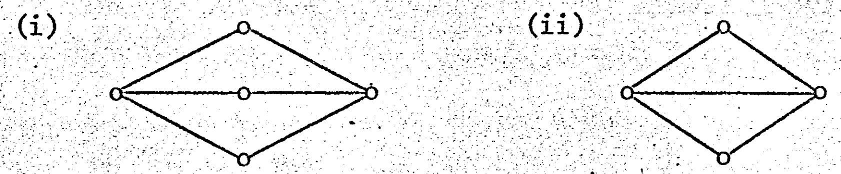


Figure 3.8

This is illustrated by the following 3-critical graph which is not Hamiltonian:

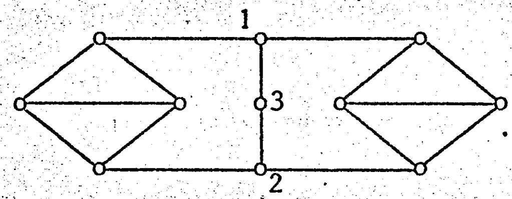


Figure 3.9

Table 2.17 shows that all 3-critical graphs on at most 10 vertices are Hamiltonian. Since this graph is of order 11, it is a smallest such graph. It is constructed by taking the multigraph of Figure 3.10 and replacing each of the two edges joining vertices 1 and 2 by the uniquely-3-colourable graph H of Figure 3.7.

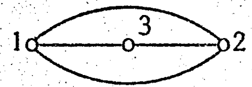


Figure 3.10

In fact this construction enables us to say more about circuit length in critical graphs. However, we defer this discussion to Chapter 7. At present, having seen the intimate relationship that exists between critical graphs and uniquely-colourable graphs, we propose to have a closer look at the structure of the latter. This will in fact yield further information about the former.

First we try to obtain bounds on the number of edges of uniquely-colourable graphs. The next theorem follows as a direct corollary of (3.1).

3.13 Theorem

For any pair of adjacent vertices v, w of a uniquely-k-colourable graph, $\rho(v) + \rho(w) \geq k + 1$. //

In fact the next theorem tells us exactly by how much $\rho(v) + \rho(w)$ exceeds $k + 1$. Let Θ_x denote the set of colours of edges incident with a vertex x in some colouring of a graph. Then we have the following result:

3.14 Theorem

If G is a uniquely-k-colourable graph, then

(i) $|\Theta_v \cup \Theta_w| = k$,

(ii) $|\Theta_v \cap \Theta_w| = \rho(v) + \rho(w) - k$,

(iii) $|\Theta_v \setminus \Theta_w| = k - \rho(w)$,

(iv) $|\Theta_w \setminus \Theta_v| = k - \rho(v)$,

for each pair of adjacent vertices v and w.

Proof

(i) follows from (3.1). It is also easy to see that $k = |\Theta_v \cup \Theta_w| = |\Theta_v \cap \Theta_w| + |\Theta_w \setminus \Theta_v| + |\Theta_v \setminus \Theta_w|$.

Also, $|\Theta_v| = |\Theta_v \setminus \Theta_w| + |\Theta_v \cap \Theta_w| = \rho(v)$ and $|\Theta_w| = \rho(w)$.

The result follows by straightforward manipulation of these equalities. //

Let Q_a be a colour-class of maximum cardinality in a uniquely-k-colourable graph G and let $Q_1, Q_2, \ldots, Q_p$ be all of the colour-classes of cardinality strictly less than that of Q_a. (Note that there need not be any such colour-classes). Corollary 3.3 implies that each of these colour-classes has cardinality $|Q_a| - 1$ and the rest all have cardinality $|Q_a|$. Thus,

$$m = p(|Q_a| - 1) + (k - p)|Q_a| = k|Q_a| - p.$$

Hence,

$$|Q_a| = \frac{1}{k}(m + p).$$

Now, all vertices covered by colour-classes of cardinality $|Q_a| - 1$ are also covered by Q_a, and hence there can be at most $k - p - 1$ vertices not covered by Q_a. Thus,

$$n - (k - p - 1) \leq 2|Q_a| \leq \frac{2}{k}(m + p).$$

This implies that

$$m \geq \tfrac{1}{2}k(n - k + 1) + \tfrac{1}{2}p(k - 2)$$

$$\geq \tfrac{1}{2}k(n - k + 1).$$

We also have trivially that $m \leq \frac{1}{2}nk$. Note that the lower bound is attained by the star-graphs $K_{1,k}$ whereas the upper bound is attained by all uniquely-k-colourable graphs which are k-valent. Thus, these bounds are best possible and we obtain the following result:

3.15 Theorem

If G is a uniquely-k-colourable graph, then

$$\tfrac{1}{2}nk - \binom{k}{2} \leq m(G) \leq \tfrac{1}{2}nk$$

and there exist uniquely-k-colourable graphs that attain these bounds. //

Regarding the connectivity of uniquely-k-colourable graphs we have the following result:

3.16 Theorem

Let G be a uniquely-k-colourable graph whose vertex-valencies are either k or $k - 1$. Then G is $(k - 1)$-edge connected.

Proof

Let G contain a set S of at most $k - 2$ edges whose removal disconnects the graph, i.e. $G - S = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$. Note that since we are requiring the vertex-valencies to be at least $k - 1$, then $E(H_i) \neq \emptyset$ $(i = 1,2)$. Let C_j denote the set of edges coloured c_j. Then, since $|S| \leq k - 2$, there are at least two colours, c_1 and c_2 say, such that $(C_1 \cup C_2) \cap S = \emptyset$. By (3.2), $C_{1,2}$ the subgraph induced by those edges coloured c_1 or c_2, is connected. Thus, $C_{1,2} \cap H_i = \emptyset$ for some $i \in \{1,2\}$. Say $C_{1,2} \cap H_1 = \emptyset$ and let e be any edge of H_1. (Such an edge exists as shown above). If e is coloured with c_3 say, then we have obtained an edge coloured c_3 which is not adjacent to any edge coloured c_1 or c_2. This contradicts (3.1). //

By studying the examples of uniquely-colourable graphs given at the beginning of this chapter, we are led to expect that quite a lot can be said about circuit length of such graphs. In fact we have the following theorem:

3.17 Theorem

If G is a uniquely-colourable and regular graph, then G is Hamiltonian.

Proof

Either G is K_3, in which case the result holds trivially, or G is not K_3, in which case G is of class 1, by (3.6). Thus the union of any two colour-classes is a regular spanning subgraph of valency 2. But this has to be connected, by (3.2), and hence G is Hamiltonian. //

For the special case of the cubic graph we have more to say.

3.18 Theorem

If G is a cubic and uniquely-colourable graph, then G has exactly three Hamiltonian circuits.

Proof

The number of Hamiltonian circuits is at least three, since the union of any two colour-classes constitutes such a circuit and there are three such unions. Call these circuits $C_{1,2}$, $C_{2,3}$, and $C_{3,1}$. Now let C be a fourth Hamiltonian circuit. $C \cap C_{i,j} \neq \emptyset$, $(i \neq j)$ $(i,j = 1,2,3)$. Thus, we obtain a new colouring of G by colouring the edges of C alternately α and β and the rest of the edges γ. This implies that G is not uniquely-colourable. //

We conclude this chapter by showing how the construction of (3.11) supports the Critical Graph Conjecture discussed in the previous chapter (see p. 37). In view of our construction, the conjecture would be disproved if, for example, one could construct a uniquely-3-colourable graph with an odd number of vertices and two external edges in the same colour-class. This would imply the existence of a uniquely-3-colourable pseudo-regular graph with an odd number of external edges, two of which are in the same colour-class. However, since two external edges are in the same colour-class, there are no other external edges, by (3.2). It would therefore be of interest to investigate whether there exist uniquely-k-colourable graphs for $k \geq 4$. The only ones known so far to exist are the trivial ones, the star-graphs $K_{1,k}$, which are not suitable for our construction. We are led to the following conjecture:

Conjecture 1

With the exception of the star-graphs $K_{1,k}$, there do not exist uniquely-k-colourable graphs with $k \geq 4$.

At any rate, the following theorem holds. Its proof also follows from (3.2) and is left to the reader.

3.19 Theorem

There do not exist any uniquely-k-colourable pseudo-regular graphs having at least two end-edges in the same colour-class and satisfying: $n_k \equiv n_1 \equiv k \equiv 1 \pmod 2$, where n_i is the number of vertices of valency i. //

Finally, we take another look at the beginning of this chapter and state the following conjecture:

Conjecture 2

If G is a planar, cubic, and uniquely-3-colourable graph, then G contains a triangle.

If this conjecture were true, then we should have characterized planar, cubic, and uniquely-3-colourable graphs as those graphs obtained from K_4 by a (possibly empty) sequence of transformations of the type: replace a vertex by a triangle.

CHAPTER 4: THE CONNECTIVITY OF CRITICAL GRAPHS

In this chapter we consider some connectivity properties of critical graphs. We have already dealt with one such property in the previous chapter, namely that ρ-critical graphs are not separable by uniquely-ρ-colourable subgraphs. Now we look for properties analogous to those enjoyed by vertex-critical k-chromatic graphs. In particular we note that these graphs cannot be separated by fewer than $k - 1$ edges (see, for example, [32, p. 165]). We also note that G is a ρ-critical graph if and only if its line-graph $\hat{G}$ is vertex-critical and $(\rho+1)$-chromatic. These last two remarks lead us to expect that critical graphs have quite 'high' connectivity properties. Surprisingly this is not the case, as we shall establish in this chapter. We shall also look at certain ρ-critical graphs separable by ρ independent edges and characterize those for which $\rho = 3$.

1) General connectivity properties

To begin with, we note that if H is a vertex-critical vertex-coloured graph, then H cannot be written as $H_1 \cup H_2$, where $H_1 \cap H_2 = K_p$ for some p. Now, if H is the line-graph $\hat{G}$ of some critical graph G, then since $\widehat{K_{1,p}} = K_p$, we obtain the following result:

4.1 Theorem

If G is a critical graph, then G is connected, has no cut-vertex and is therefore bridgeless. //

Thus we have that critical graphs are at least 2-connected. We shall now show that, in general, this result is best possible. We do this by proving the following result:

4.2 Theorem

For each odd integer n and for all possible ρ, there exists a Hamiltonian ρ-critical graph of order n with minimum valency 2.

Proof

We note first that ρ can be at most $n - 2$ and that the cases $\rho = n - 2$ and $\rho = n - 3$ have already been proved in Theorems 2.8 and 2.9 respectively. The case $\rho = 2$ is trivial.

We shall prove the remainder of the theorem by explicitly constructing $n - 6$ 3-critical graphs G_i $(i = 1,2,\ldots,n-6)$ each consisting of an odd circuit C_n and an independent set S_i of $[\frac{1}{2}n]$ edges, such that $S_i \cap S_j = \emptyset$ if $i \neq j$. We shall then apply Theorem 2.11 to the union of any t of these graphs $(1 \leq t \leq n-6)$ to give the result.

The construction:

Given an odd circuit C_{2k+1} with vertices labelled $\langle 1,2,3,\ldots,2k,2k+1,1\rangle$ we construct three families of 3-critical graphs as follows:

(A) The first family consists of $k - 4$ graphs $X_1, X_2, \ldots, X_{k-4}$. For each r, the independent set of edges of X_r is obtained by joining the vertices 1 and $r + 2$, $r + 2 - i$ and $r + 2 + i$ (for $i = 1,2,\ldots,r$), $2k$ and $k + r$, $2k - 1$ and $k + r + 1$, $k + r + 1 + j$ and $k + r - j$ (for $j = 1,2,\ldots,k-r-3$). The graph X_r is shown in Figure 4.1.

To see that X_r is critical for each r, we use an inductive argument. To start the induction we note that X_1 is a Hajós union of the graph L_{k-4} described in Chapter 2, pp. 30 et seq., and the first graph of Table 2.17, where the identified vertices have valency 2 in both cases. Thus, X_1 is 3-critical, by (2.14). Assuming that X_{r-1} is 3-critical, we see that X_r is also 3-critical since X_r is obtained from X_{r-1} and K_4 by the construction of Theorem 2.15. With the notation of that theorem, G is X_{r-1}, H is K_4, and the deleted vertex v of G is the vertex labelled r + 1 in X_{r-1}. This establishes that X_r is 3-critical for each r.

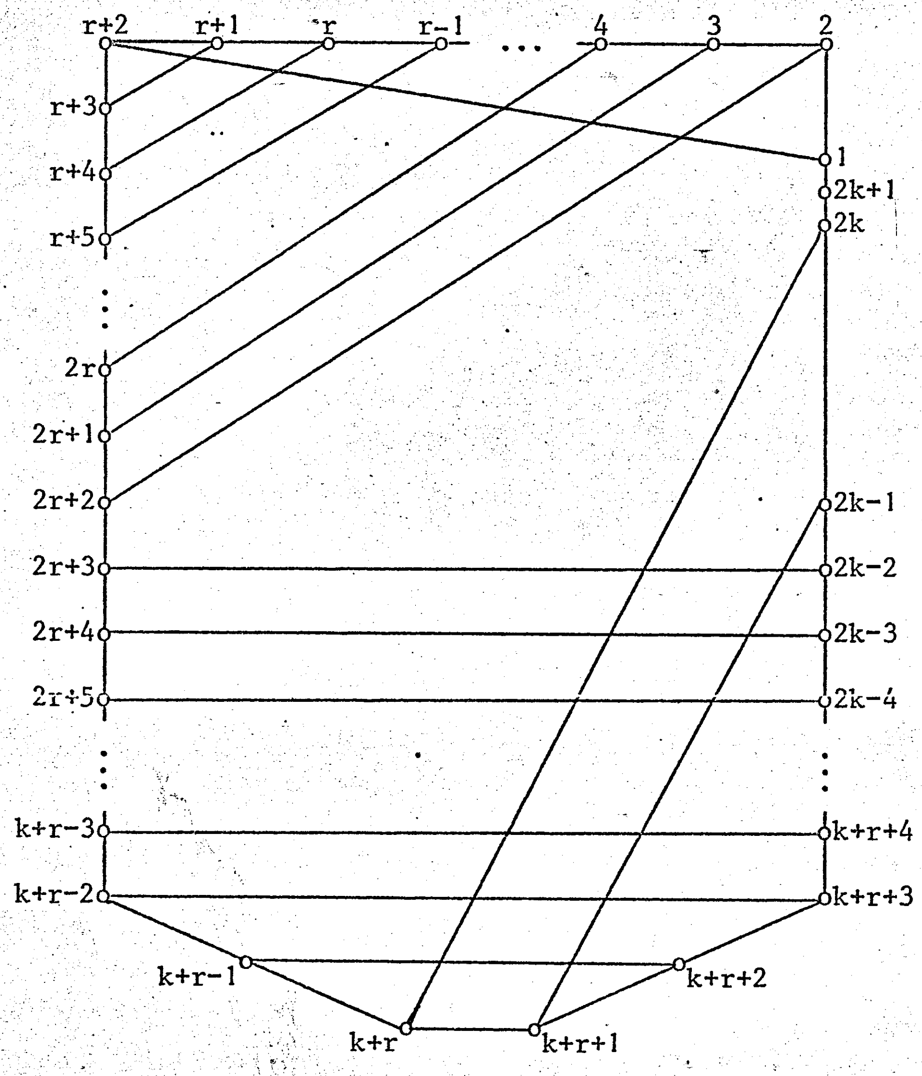


Figure 4.1

(B) The second family consists of another k - 4 graphs $Y_1, Y_2, \ldots, Y_{k-4}$. For each s, the independent set of edges of Y_s is obtained by joining the vertices 1 and $k + s + 1$, $k + s + 1 - i$ and $k + s + 1 + i$ (for $i = 1,2,\ldots,k-s-3$), $2k$ and $s + 2$, $2k - 1$ and $s + 3$, $s + 2 - j$ and $s + 3 + j$ (for $j = 1,2,\ldots,s$). The graph Y_s is shown in Figure 4.2.

To see that Y_s is 3-critical for each s, we use an inductive argument similar to the one given above. We note that Y_1 is obtained from the graph L_{k-3}, referred to above, and K_4 by the construction of Theorem 2.15. In this case, if G is Y_{s-1} and H is K_4, then the vertex v deleted from G is the vertex labelled 2 in Y_{s-1}, for all $s \geq 1$. This shows that Y_s is 3-critical for all s.

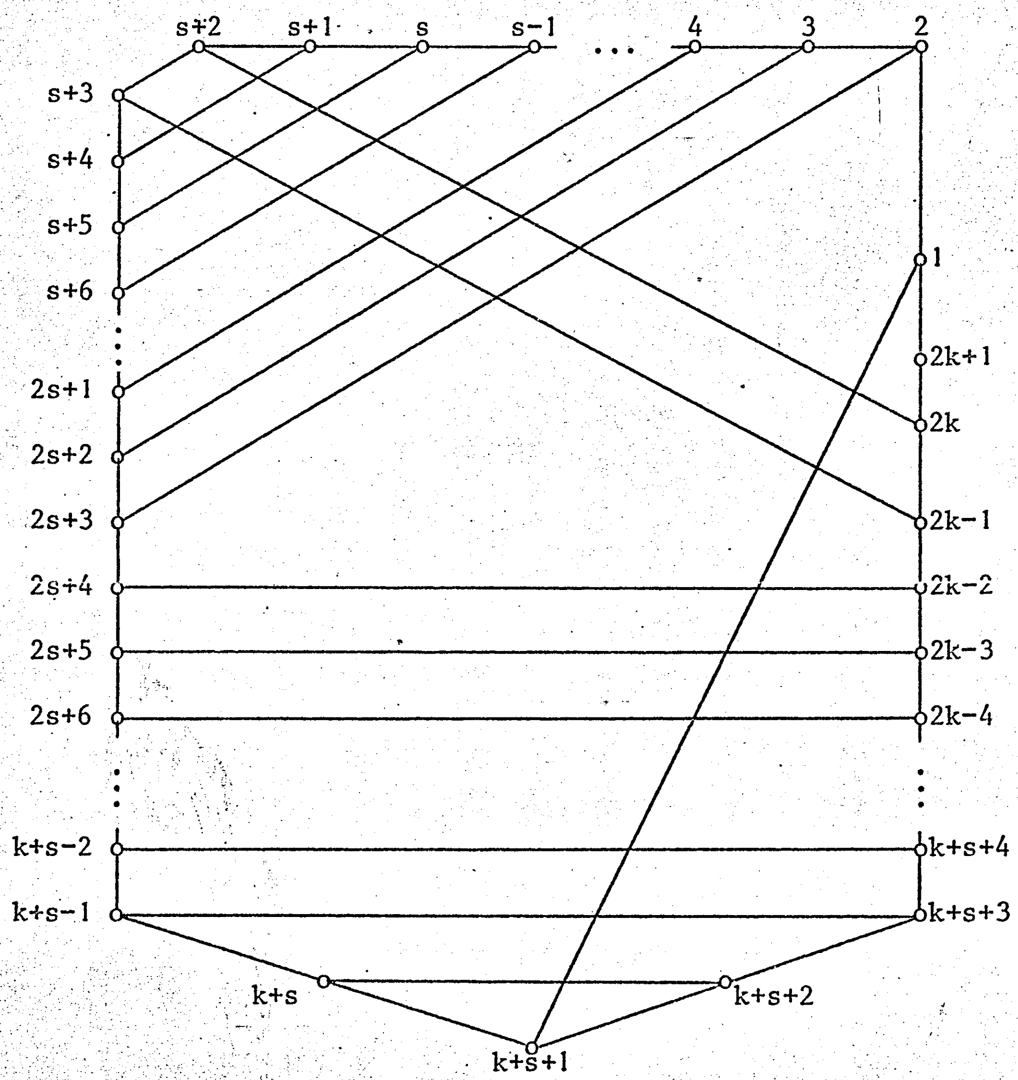


Figure 4.2

(C) The last family consists of the three graphs Z_1, Z_2, Z_3 shown in Figure 4.3.

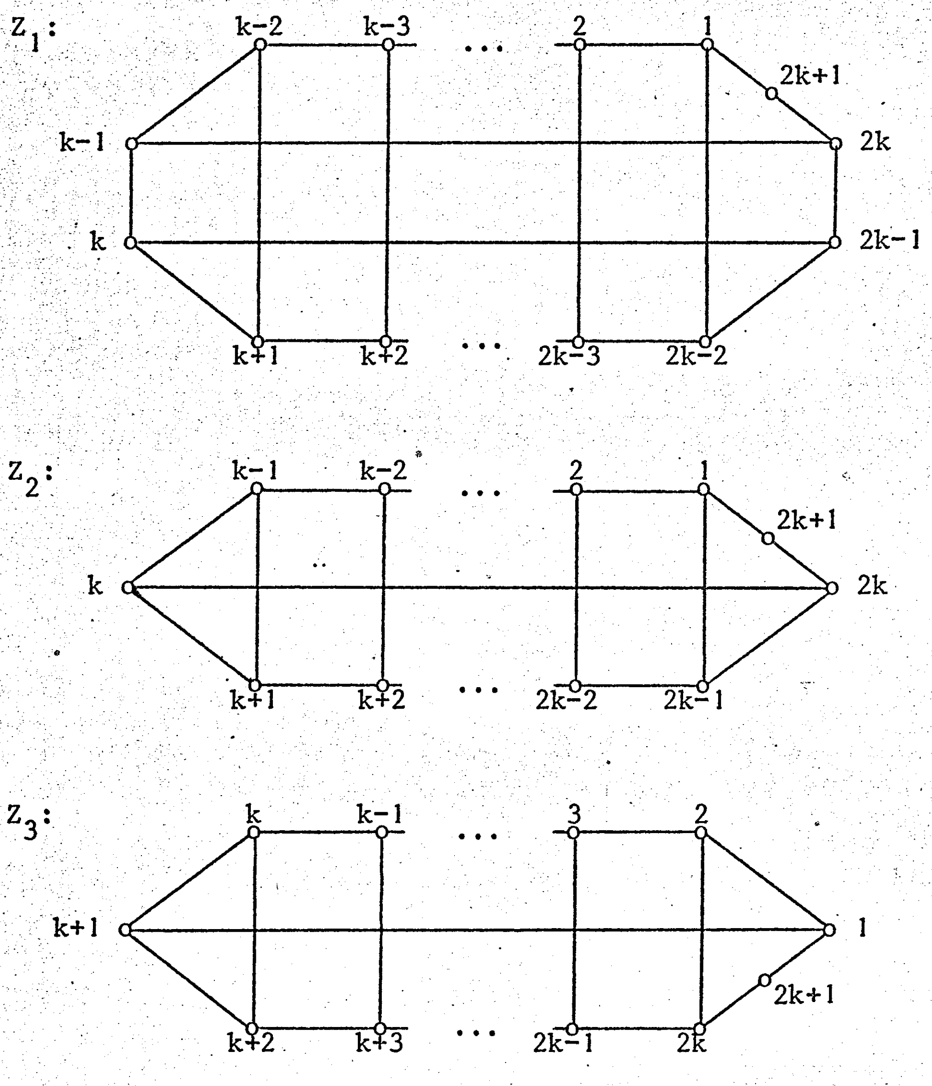


Figure 4.3

Z_1 is simply the graph L_k we have been considering and which we know is 3-critical. Z_2 and Z_3 are easily seen to be 3-critical by an inductive argument making use of Theorem 2.15. This completes the proof. //

2) Separability by independent edges

From Theorem 4.2 we deduce that in general not much can be said about the connectivity of critical graphs. However, we can look for connectivity properties of particular critical graphs. One class of ρ-critical graphs we propose to consider is that in which the graphs are separable by t independent edges. It turns out that for the values of t we consider, the graphs are constituted of smaller ρ-critical graphs. This implies that when we are looking for ρ-critical graphs of minimal order, as we shall often have the opportunity to do in later chapters, we can exclude those which are separable in this way.

One result in this direction is due to Jakobsen [19], who proves the following theorem:

4.3 Theorem

A ρ-critical multigraph G is separable by two independent edges if and only if G is a Hajós union of two other ρ-critical multigraphs G_1 and G_2, in which a vertex of valency 2 in G_1 is identified with some vertex in G_2. //

Now we consider ρ-critical graphs which are separable by ρ independent edges. Given any graph G with maximum valency ρ and which is separable by ρ independent edges, we can associate with it two pairs of graphs (H_1,H_2) and (J_1,J_2) obtainable from G as follows: Let the separating set S of ρ independent edges be $\{(a_j^1,a_j^2)\}_{j=1}^{\rho}$ and let $G - S = T_1 \cup T_2$, where $T_1 \cap T_2 = \emptyset$. Also, let $\{a_j^i\}_{j=1}^{\rho} \in T_i$ for $i = 1,2$.

We obtain J_1 from T_1 by appending ρ end-edges $\{(a_j^1,b_j^1)\}_{j=1}^{\rho}$.

H_1 is obtained from J_1 by identification of the b_j's to a single vertex v_1. H_2 and J_2 are similarly defined. We then write $G = H_1 \cup_\alpha H_2$.

Illustration

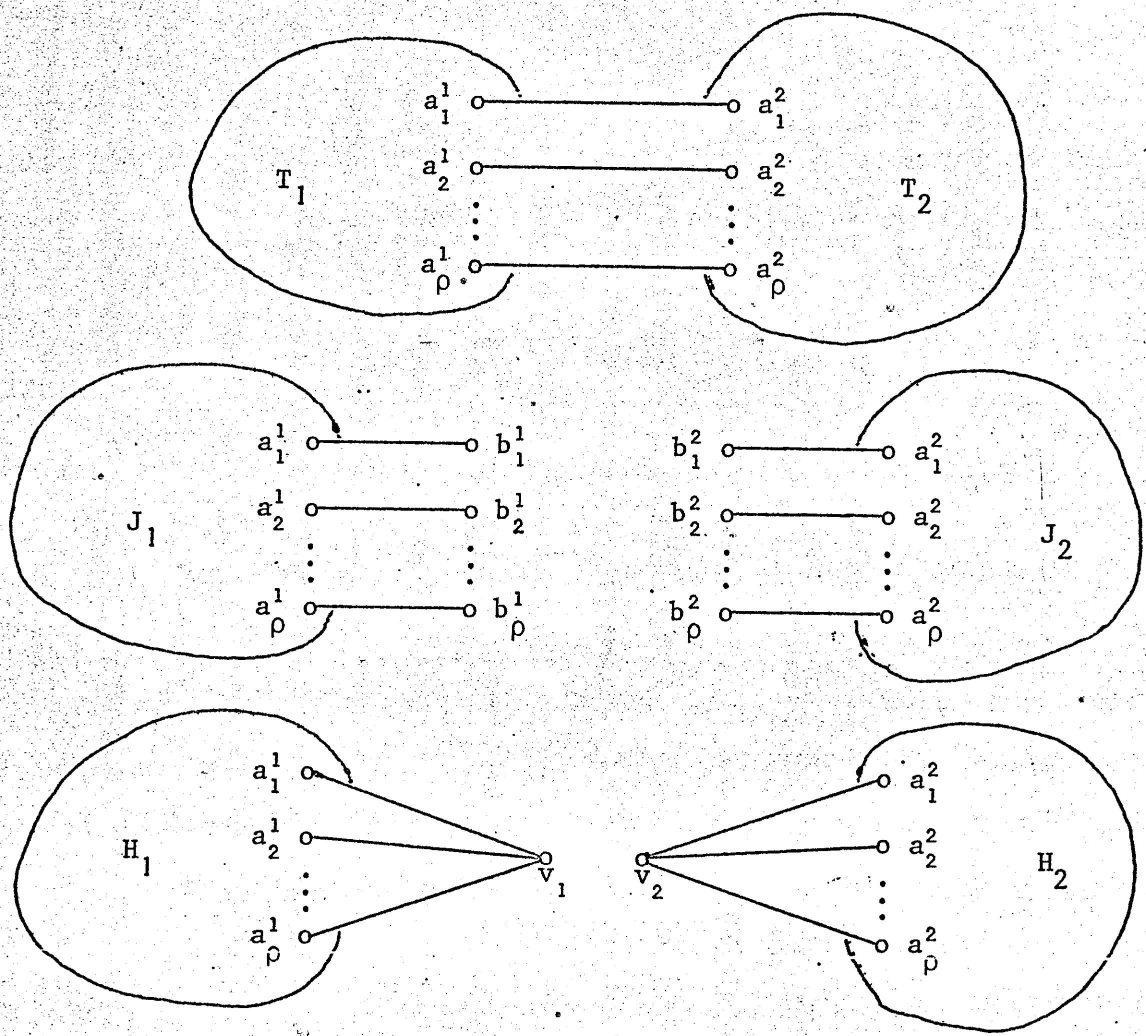


Figure 4.4

Conversely, given any two graphs having maximum valency ρ, we can obtain a graph G from them by reversing the above process. The graphs H_i, J_i $(i = 1,2)$ just defined, will be referred to throughout the rest of this chapter.

Before proceeding with the discussion for arbitrary ρ, we first restrict ourselves to the case $\rho = 3$ and impose the further restriction that H_2 should be 3-valent. We motivate our investigation by the construction of (2.15) and by the following typical examples:

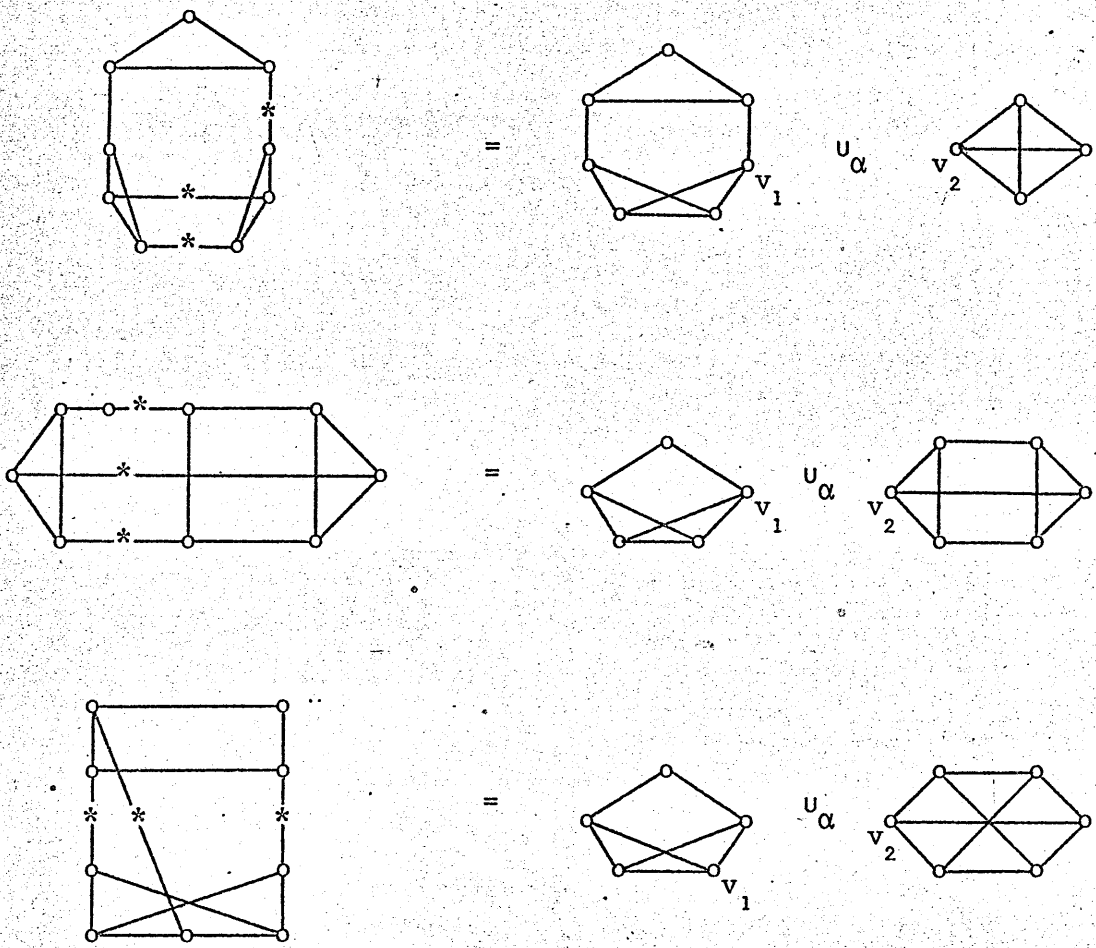


Figure 4.5

Let H_1 be a 3-critical graph and let v_1 be a vertex of valency 3 incident with edges $\{a,b,c\}$. Consider the graph J_1 obtained from H_1 by splitting v_1 into three end-vertices v_a, v_b, and v_c incident with edges a, b, and c respectively. J_1 is 3-colourable and we can choose our notation in such a way that among all 3-colourings of J_1 there exists a colouring with a and b in the same colour-class, there exists one with b and c in the same colour-class, but there need not exist one with a and c in the same colour-class. This is illustrated in the following example:

Figure 4.6

In this example, a and c always lie in distinct colour-classes. Thus a 3-critical graph always has a distinguished edge - we may call a root edge - defined to be that edge b incident with v_b in J_1 (as defined above), which may be coloured with the same colour as a or c in some 3-colourings of J_1. Note that it may well happen that all three edges a, b, and c have this property, as in the following example:

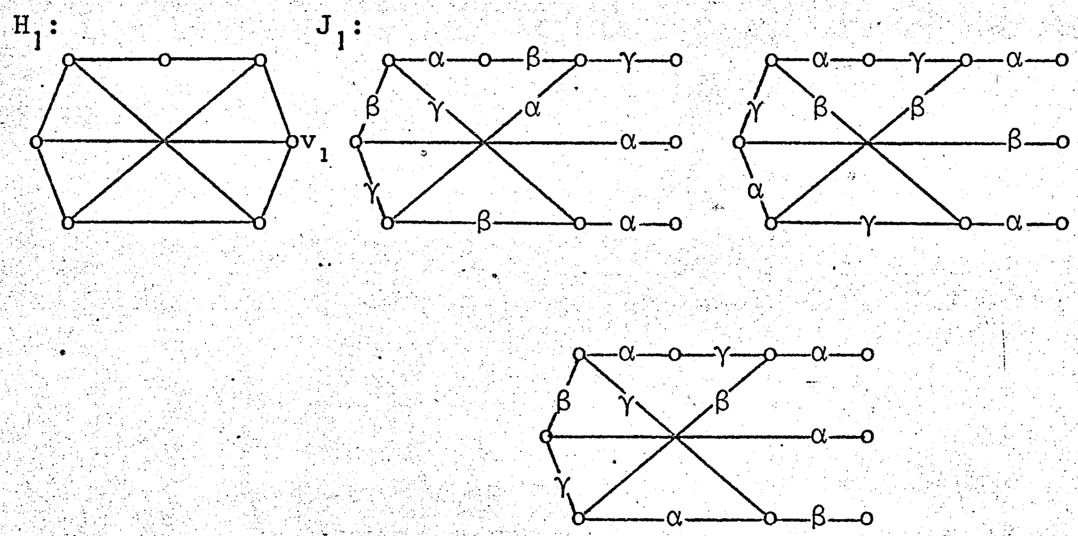


Figure 4.7

Note also, that if two of the edges have this property, then so does the third.

Let J_2 be a class 1 pseudo-regular graph which has maximum valency 3 and which has exactly three external edges. By (3.10), these external edges lie in distinct colour-classes. We say that J_2 has the weak-pairing property if for any internal edge e, there exists some 3-colouring of J_2 - e with some pair of the three external edges in the same colour-class. We say that J_2 has the strong-pairing property **if there exists an external edge f such that for any internal edge e there exists some** 3-colouring of J_2 - e with f and some other external edge in the same colour-class. f is then called a root. Thus, for example, if H_2 is uniquely-3-colourable, then J_2 has the strong-pairing property.

A 3-critical graph H_1 and a class 1, 3-valent graph H_2 are said to be compatible if either (i) J_1 has more than one root and J_2 has the weak-pairing property, or (ii) J_1 has a unique root and J_2 has the strong-pairing property.

If H_1 and H_2 are compatible, we define the α-union of H_1 and H_2 to be the graph $G = H_1 \cup_\alpha H_2$, where in the case that H_1 has a unique root, then this is identified with some root of H_2. In this setting we have the following result:

4.4 Theorem

G is a 3-critical graph having a set S of three independent edges satisfying $G - S = T_1 \cup T_2$, $T_1 \cap T_2 = \emptyset$ and $\rho_G(v) = 3$ for each $v \in V(T_2)$ if and only if G is the α-union of H_1 and H_2, as defined.

Proof

(A) Necessity: We assume that G is 3-critical and separable by a set S of three independent edges as described. We have to show the following:

(i) H_2 is of class 1: If $e \in E(T_1)$, then $G - e$, and hence J_2, is of class 1. But if J_2 is of class 1, then by (3.10), all external edges lie in distinct colour-classes. This implies that H_2 is of class 1.

(ii) H_1 is 3-critical: Not all external edges of J_1 lie in distinct colour-classes, for otherwise we should obtain a 3-colouring of G. Thus, H_1 is of class 2. Now if $e \in E(H_1)$, $G - e$, and hence J_2, is of class 1. As before, this implies that all the end-edges of J_2, and hence also those of $J_1 - e$, lie in distinct colour-classes. Thus, $H_1 - e$ is of class 1.

(iii) H_1 and H_2 are compatible: Suppose H_1 has a unique root, then without loss of generality, in every 3-colouring of J_1, edge a is coloured α, edge b is coloured β, whereas edge c is coloured α or β. Let e be any internal edge of J_2. $G - e$, J_1, and $J_2 - e$ are all 3-colourable. Thus, in each 3-colouring of $J_2 - e$, a' is coloured α, b' is coloured β, whereas c' is coloured α or β, where the external edges a', b', and c' of J_2 are identified respectively with the edges a, b, and c of J_1. But this means that J_2 has the strong-pairing property with c' as root which, moreover, is identified with the unique root of J_1.

If H_1 has more than one root, then it has three, as remarked above. In a typical 3-colouring of J_1, the end-edges are partitioned as follows: a is coloured α, b is coloured β, and c is coloured α or β. Arguing in exactly the same manner as before, we conclude that J_2 has at least the weak-pairing property, that H_1 and H_2 are compatible, and that G is the α-union thereof.

<u>(B) Sufficiency:</u> Conversely, we assume that H_1 is 3-critical, H_2 is cubic and of class 1, and H_1 and H_2 are compatible. Thus, the α-union of H_1 and H_2 is clearly of class 2, since in every 3-colouring of J_1 two of the end-edges lie in the same colour-class whereas all those of J_2 lie in distinct colour-classes.

There remains to show that the deletion of any edge e of G results in a 3-colourable graph. If $e \in E(H_1)$, then $H_1 - e$ is 3-colourable and hence all end-edges of $J_1 - e$ lie in distinct colour-classes. This induces a 3-colouring of $G - e$. On the other hand, if $e \in E(H_2)$ and if H_1 has a unique root, c say, then, since H_1 and H_2 are compatible, some root of H_2 is identified with c. This yields a 3-colouring of $G - e$. The same holds if H_1 has more than one root, which concludes the proof. //

We now consider the case of ρ odd but otherwise arbitrary.

Let J_1 and J_2 be a pair of graphs which are of class 1, have maximum valency ρ, and have exactly ρ end-edges. Moreover, suppose that J_2 is pseudo-regular. If Φ is a one-one correspondence between the end-edges of J_1 and those of J_2, we define J_1 and J_2 to be compatible with respect to Φ if there exist ρ-colourings of J_1 and of $J_2 - e$ which partition the end-edges of J_1 and those of $J_2 - e$ in the same way, for each internal edge e of J_2. Using the same notation as above, we can then prove the following result:

4.5 Theorem

Let ρ be odd. Then a graph G is ρ-critical and has a set S of ρ independent edges satisfying $G - S = T_1 \cup T_2$, $T_1 \cap T_2 = \emptyset$, $\rho_G(v) = \rho$ for each $v \in V(T_2)$ if and only if $G = H_1 \cup_\alpha H_2$, where H_1 is ρ-critical, H_2 is of class 1, and J_1 and J_2 are compatible with respect to the one-one correspondence Φ induced by this union.

Proof

(A) Necessity: We assume that G is ρ-critical and separable by a set S of ρ independent edges as described. We have to show the following:

(i) H_2 is of class 1: Let e be an edge of T_1. Then $G - e$, and hence J_2, is ρ-colourable. Thus, by (3.10), all external edges of J_2 lie in distinct colour-classes and hence H_2 is of class 1.

(ii) H_1 is ρ-critical: Not all external edges of J_1 lie in distinct colour-classes, for otherwise we should get a ρ-colouring of G. Thus, H_1 is of class 2. There remains to show that the deletion of any edge e from H_1 yields a ρ-colourable graph. Now, $G - e$, J_2, and $J_1 - e$ are all ρ-colourable for each $e \in E(H_1)$. Thus, the end-edges of J_2, and hence also those of $J_1 - e$, lie in distinct colour-classes, by (3.10). Hence, $H_1 - e$ is of class 1, as required.

(iii) J_1 and J_2 are compatible: Let e be any internal edge of J_2. Then $G - e$ is ρ-colourable and hence so are J_1 and $J_2 - e$. Moreover, any ρ-colouring of $G - e$ induces the same partition of the end-edges of J_1 and of $J_2 - e$. Thus, J_1 and J_2 are compatible with respect to Φ.

(B) Sufficiency: Conversely, assume that H_1 is ρ-critical and that H_2 is ρ-valent and of class 1. Moreover, let the associated graphs J_1 and J_2 be compatible with respect to some one-one correspondence Φ between their end-edges. We claim that the graph G obtained from J_1 and J_2 by the identification of their end-edges according to Φ is ρ-critical. To this end we show the following:

(i) G is of class 2: If G is of class 1, then there exist ρ-colourings of J_1 and of J_2 which partition their end-edges in the same way. But there is a unique way of partitioning the external edges of J_2, namely to partition them into distinct colour-classes. But this implies that H_1 is of class 1. Thus, G is of class 2.

(ii) $G - e$ is ρ-colourable for each edge e of G: Let e be any edge of H_1. $H_1 - e$ is of class 1 and hence there exists a ρ-colouring of $J_1 - e$ in which all of the end-edges lie in distinct colour-classes. This and any ρ-colouring of J_2 induce a ρ-colouring of $G - e$. Finally, let e be any edge of J_2. The compatibility of J_1 and of J_2 ensures the same partitioning of the end-edges of J_1 and of $J_2 - e$. This in turn induces a ρ-colouring of $G - e$, thereby completing the proof. //

Remarks

(i) The usefulness of this theorem is twofold: In one direction it enables us to construct ρ-critical graphs from 'simpler' ρ-critical graphs; in the other direction it allows us, under certain circumstances, to focus on that part of a critical graph without which the graph would not be critical.

(ii) The compatibility conditions of these theorems are not very restrictive. There are a number of ρ-valent graphs of class 1 which satisfy more stringent conditions and which are in fact compatible with any ρ-critical graph. One such sufficient condition is the following:

For any prescribed pair of external edges e_1 and e_2, and any internal edge e of a pseudo-regular graph J_2 having exactly ρ external edges, there exists a ρ-colouring of $J_2 - e$ in which e_1 and e_2 belong to the same colour-class whereas all other external edges lie in distinct colour-classes different from that of e_1 and of e_2.

If H_2 is any of the following regular, class 1 graphs, then any associated pseudo-regular graph J_2 satisfies the above condition:
(a) $H_2 = K_{2k}$, (b) $H_2 = K_{t,t}$.
A proof of this statement implies a proof of Theorem 2.15 which, with the notation we have been using in this chapter, states the following:

For each odd ρ, if H_1 is a ρ-critical graph and H_2 is either $K_{\rho+1}$ or $K_{\rho,\rho}$, then $H_1 \cup_\alpha H_2$ is ρ-critical.

We now proceed to prove this theorem by showing that $K_{\rho+1}$ and $K_{\rho,\rho}$ satisfy the compatibility condition stated above.

Proof of Theorem 2.15

(a) Given K_ρ, any two vertices v and w, and any edge $e \neq (v,w)$, then there exists a spanning open chain P, which includes e and has initial and terminal vertices v and w respectively. The graph obtained from K_ρ by deleting P is of class 1, since it contains exactly two vertices of maximum valency $\rho - 2$ (This follows from (2.2) (ii) and (2.5)). If we append end-edges to each of the vertices except v and w, we obtain a pseudo-regular graph which is of class 1, has maximum valency $\rho - 2$, and has exactly $\rho - 2$ end-edges which are all in distinct colour-classes. Let P' be the graph obtained from P by deleting e and adding an end-edge e_v at v and an end-edge e_w at w. We can now re-introduce P'. Since P' is the disjoint union of open chains, we can two-colour it with e_v and e_w in the same colour-class, different from any of those of the remaining external edges.

To deal with the exceptional case $e = (v,w)$, we append end-edges to all the vertices and let those edges incident to v and w be e_v and e_w respectively. Let e_x be the end-edge in the same colour-class as e and let the colour of e_x be α. If e is deleted, then we can re-colour e_v and e_w with colour α. Moreover, if e_v was previously coloured β, then we can re-colour e_x with colour β.

(b) Let ρ be odd and let e_1, e_2, e_3 be any three edges of $K_{\rho,\rho}$ exactly two of which (e_1 and e_2 say) are adjacent, at the vertex v say. Then there exists a Hamiltonian circuit H including all three edges. The graph G obtained from $K_{\rho,\rho}$ by the deletion of H is a $(\rho - 2)$-valent graph. Thus, the pseudo-regular graph obtained from G by splitting v into $\rho - 2$ end-vertices has $\rho - 2$ end-edges in different colour-classes. Let H' be obtained from the Hamiltonian circuit by splitting it at v and deleting the edge e_3. In G we can re-introduce H' which is the disjoint union of two open chains and hence can be two-coloured with the end-edges e_1 and e_2 in the same colour-class and different from that of any of the other external edges. //

CHAPTER 5: SOME BOUNDS ON THE NUMBER OF EDGES OF CRITICAL GRAPHS AND APPLICATIONS TO PLANAR GRAPHS

In Chapter 2 we obtained bounds on the total deficiency of critical graphs which are equivalent to upper bounds on the number of edges of these graphs. We also discussed a lower bound due to Vizing and noted that there is room for improvement on this bound. In the first part of this chapter we propose to determine bounds on the number of edges which complement Vizing's result. In the second part we apply these results to planar graphs and discuss a problem raised by Vizing.

1) Some bounds on the number of edges

We note first that, if G is a ρ-critical graph and $\hat{G}$ is its line-graph, then $\hat{G}$ is vertex-critical. Thus, by a well-known result (see, for example, [32, p. 164]), $\sigma(\hat{G}) \geq \rho$. Moreover, if $(v,w) = e \in E(G)$ and if Φ is the one-one correspondence between $E(G)$ and $V(\hat{G})$, then $\sigma(\hat{G}) \leq \rho(\Phi^{-1}(e)) = \rho(v) + \rho(w) - 2$. This implies the following result:

5.1 Theorem

If G is a ρ-critical graph, then for each pair of adjacent vertices v and w,

$$\rho(v) + \rho(w) \geq \rho + 2. \quad //$$

In fact, Berge [4, p. 254]) tells us exactly by how much $\rho(v) + \rho(w)$ exceeds $\rho + 2$. He calls this result the 'Uncoloured Edge Lemma'. An equivalent formulation is the following:

5.2 Lemma

We assume that G is a ρ-critical graph, that $e = (v,w)$ is an edge of G and that $G - e$ has been ρ-coloured. If Θ_v and Θ_w denote respectively the set of colours of edges incident with v and w in this colouring, then

(i) $|\Theta_v \cup \Theta_w| = \rho$,

(ii) $|\Theta_v \cap \Theta_w| = \rho(v) + \rho(w) - \rho - 2$,

(iii) $|\Theta_v \setminus \Theta_w| = \rho + 1 - \rho(w)$,

(iv) $|\Theta_w \setminus \Theta_v| = \rho + 1 - \rho(v)$. //

Note the similarity of these last two results with Theorems 3.13 and 3.14 on uniquely-colourable graphs. This is not surprising in view of the construction of Theorem 3.11.

Using these results, we can deduce the following bounds on the number of edges of a ρ-critical graph:

5.3 Theorem

Let G be a ρ-critical graph and let $S = \sum_{v \in V(G)} (\rho(v))^2$ and $T = \frac{\rho}{\rho+1}$. Then

$$(T^2 + ST)^{\frac{1}{2}} - T \leq m(G) \leq \frac{S}{\rho+2} - \tfrac{1}{2}(\rho - 2).$$

Proof

Let n_ρ be the number of vertices of valency ρ in G, and let $\sigma(G)$ be the minimum vertex-valency of G. Then

$$\sigma(\hat{G}) \geq \max\{\rho, 2\sigma(G) - 2\},$$

since $\hat{G}$ is a line-graph which is vertex-critical. Also, by (2.2) (ii),

$$n_\rho \geq \rho - \sigma(G) + 2.$$

Thus,

$$\begin{aligned} 2m(\hat{G}) &\geq n_\rho(2\rho - 2) + (m(G) - n_\rho).\sigma(\hat{G}) \\ &\geq (\rho - \sigma(G) + 2)(2\rho - 2 - \sigma(\hat{G})) + m(G).\sigma(\hat{G}) \\ &\geq m(G).\rho + \tfrac{1}{2}(\rho^2 - 4). \end{aligned}$$

Now, a straightforward computation shows that

$$m(\hat{G}) + m(G) = \tfrac{1}{2}S.$$

These last two results yield the upper bound of the theorem.

The following inequality is due to Turán [39]:

$$m(\hat{G}) \leq \frac{(m(G))^2(\rho - 1)}{2\rho}.$$

This and the former result yield the lower bound of the theorem and complete the proof. //

Note that this theorem again implies the fact (shown in (2.26)) that, apart from the odd circuits, critical graphs cannot be regular. It is also worth noting that for certain critical graphs these bounds are best possible. Thus, for example for the graph of order 5 in Table 2.17 both bounds are exact. This result is also quite good for testing particular graphs, but more generally applicable bounds which involve just n and ρ are desirable. Bounds of this type can be obtained for small values of ρ. Thus, for example, Jakobsen [20] showed that if $\rho = 3$, then

$$\tfrac{4}{3}n \leq m \leq \tfrac{1}{2}(3n - 1).$$

A similar analysis establishes analogous bounds for $\rho = 4$. Here and in what follows we use an extension of an argument due to Vizing [43].

5.4 Theorem

If G is a 4-critical graph, then

$$\tfrac{5}{3}n \leq m \leq 2n - 1.$$

Proof

If v is a vertex of G of valency 2, and if u and w are the vertices adjacent to v, then (2.1) implies that each of u and w has valency 4, and that neither of them is adjacent to any vertex of valency 3 or 2.

Now denote by n_j the number of vertices of valency j, and let $n_4(p,q)$ denote the number of vertices of valency 4 adjacent to exactly p vertices of valency 2 and q vertices of valency 3. With this notation we have:

$$2n_2 = n_4(1,0)$$

and

$$2n_3 \le n_4(0,1) + 2n_4(0,2).$$

Thus,

$$2n_2 + n_3 \le n_4(1,0) + \tfrac{1}{2}n_4(0,1) + n_4(0,2) \le n_4.$$

But

$$4(n_2 + n_3 + n_4) = 4n$$

and

$$2n_2 + 3n_3 + 4n_4 = 2m,$$

Hence,

$$2n_2 + n_3 = 4n - 2m.$$

Thus

$$n_4 \ge 4n - 2m.$$

But

$$2m \ge 4n_4 + (n - n_4).2 = 2n_4 + 2n.$$

These last two statements show that

$$2m \ge 8n - 4m + 2n,$$

which proves the lower bound of the theorem. The upper bound follows from (2.25). //

A similar argument can be given for $\rho = 5$. In this case we have

$$2n_2 = n_5(1,0,0)$$

$$2n_3 \le n_5(0,1,0) + n_5(0,1,1) + 2n_5(0,2,0)$$

$$2n_4 \le n_5(0,1,1) + n_5(0,0,1) + 2n_5(0,1,2) + 2n_5(0,0,2) + 3n_5(0,0,3),$$

where $n_5(p,q,r)$ denotes the number of vertices of valency 5 adjacent to exactly p vertices of valency 2, to q of valency 3 and to r of valency 4. Thus,

$$n - n_5 = n_2 + n_3 + n_4 \le \tfrac{3}{2} \sum n_5(p,q,r) \le \tfrac{3}{2} n_5.$$

This implies that $5n_5 \ge 2n$.

Since we also have that $n_2 \le \frac{1}{2}n_5$, then provided that $n - \frac{3}{2} n_5 \ge 0$,

$$2m \ge 5n_5 + 2.\tfrac{1}{2}n_5 + 3.(n - \tfrac{3}{2}n_5)$$

$$= 3n + \tfrac{3}{2}n_5$$

$$\ge \tfrac{18}{5} n.$$

On the other hand, if $n - \frac{3}{2} n_5 < 0$, then

$$2m \geq 5.n_5 + 2.(n - n_5)$$
$$\geq 4n.$$

So, in all cases $m \geq \frac{9}{5} n$.

This technique can be generalized to give quite a good lower bound for m involving only n and ρ. We first need the following lemma.

5.5 Lemma

Let G be a ρ-critical graph. Then for all k satisfying $2 \leq k \leq \rho - 1$, we have

$$\sum_{j=2}^{k} \left\{ \frac{n_j}{j-1} \right\} \leq \tfrac{1}{2} n_\rho .$$

Proof

With each vertex v of valency ρ of G we associate a (k-1)-tuple $(i_2, i_3, \ldots, i_k)$, where for $2 \leq t \leq k$, i_t denotes the number (possibly zero) of vertices of valency t joined to v. For any such (k-1)-tuple we have the following: Let q denote the smallest index of all non-zero elements of the (k-1)-tuple. Then v is joined to i_q (> 0) vertices of valency q, and since by (2.1) v is joined to $\rho - q + 1$ vertices of valency ρ, then the vertex v is joined to at most $\rho - (\rho - q + 1)$ vertices of valency less than ρ. Consequently,

$$i_2 + i_3 + \ldots + i_k \leq q - 1.$$

Now, let $n_\rho(i_2, i_3, \ldots, i_k)$ denote the number of vertices of valency ρ associated with the (k-1)-tuple $(i_2, i_3, \ldots, i_k)$. Then

$$2n_j \leq \sum_{\langle k \rangle} i_j . n_\rho(i_2, i_3, \ldots, i_j, \ldots, i_k)$$

where the summation extends over all (k-1)-tuples associated with any vertex of valency ρ of G. In this summation every vertex of valency j is counted at least twice, since by (2.2) (i), each vertex is joined to at least two vertices of valency ρ. Hence, a lower bound for this sum must be $2n_j$. From this it follows that

$$\sum_{j=2}^{k}\left\{\frac{2n_j}{j-1}\right\} \le \sum_{j=2}^{k}\sum_{<k>}\left\{\frac{i_j}{j-1}\right\}.n_\rho(i_2,\ldots,i_j,\ldots,i_k)$$

$$= \sum_{<k>} n_\rho(i_2,\ldots,i_k)\sum_{j=2}^{k}\left\{\frac{i_j}{j-1}\right\}$$

$$\le \sum_{<k>} n_\rho(i_2,\ldots,i_k)\sum_{j=2}^{k}\left\{\frac{i_j}{q-1}\right\}$$

$$\le \sum_{<k>} n_\rho(i_2,\ldots,i_k)$$

$$\le n_\rho,$$

where we have used the fact that $i_2 + i_3 + \ldots + i_k \le q - 1$. This completes the proof of the lemma. //

5.6 Corollary

If G is a ρ-critical graph, then for each integer k satisfying $2 \le k \le \rho - 1$, we have

$$\sum_{j=2}^{k} n_j \le \tfrac{1}{2}(k-1)n_\rho.$$

Proof

The result follows from (5.5) by noting that $\frac{n_j}{k-1} \le \frac{n_j}{j-1}$ for each j satisfying $2 \le j \le k$. //

The next corollary follows directly from the fact that each vertex of a ρ-critical graph is adjacent to at least two vertices of maximum valency ρ. We include it here for future reference and since it also follows from (5.6).

5.7 Corollary

If G is a ρ-critical graph of order n, then $\rho n_\rho \ge 2n$. //

5.8 Corollary

A ρ-critical graph of order n has at least $\frac{2}{\rho}n(\rho - 1)$ edges.

Proof

$2m \ge \rho n_\rho + 2.(n - n_\rho)$. The result then follows from (5.7).//

We can now give a lower bound for the number of edges m in terms of n and ρ as follows. Let $w = [\frac{1}{2}(\rho + 1)]$. Arguing in a way similar to that given in Lemma 5.5, for each h satisfying $0 \le h \le w-2$, with each vertex v of valency $\rho - h$ of G, we associate a $(w - 1)$-tuple

$$(i_2^{(\rho-h)}, i_3^{(\rho-h)}, \ldots, i_w^{(\rho-h)}),$$

where for $2 \le t \le w$, $i_t^{(\rho-h)}$ denotes the number (possibly zero) of vertices of valency t joined to v. Let

$$n_{\rho-h}(i_2^{(\rho-h)}, i_3^{(\rho-h)}, \ldots, i_w^{(\rho-h)})$$

denote the number of vertices of valency $\rho - h$ associated with the $(w - 1)$-tuple

$$(i_2^{(\rho-h)}, i_3^{(\rho-h)}, \ldots, i_w^{(\rho-h)}).$$

Then, by virtue of (5.1), we have the following *EQUALITY* for j satisfying $2 \le j \le w$:

$$j.n_j = \sum_{h=0}^{w-2} \sum_{<w_{\rho-h}>} i_j^{(\rho-h)}.n_{\rho-h}(i_2^{(\rho-h)}, \ldots, i_j^{(\rho-h)}, \ldots, i_w^{(\rho-h)})$$

where the first summation extends over all $(w - 1)$-tuples associated with any vertex of valency $\rho - h$ in G. This implies the following:

$$\sum_{j=2}^{w} n_j = \sum_{j=2}^{w} \sum_{h=0}^{w-2} \sum_{<w_{\rho-h}>} n_{\rho-h}(i_2^{(\rho-h)}, \ldots, i_j^{(\rho-h)}, \ldots, i_w^{(\rho-h)}).\left\{ \frac{i_j^{(\rho-h)}}{j} \right\}$$

$$= \sum_{h=0}^{w-2} \sum_{<w_{\rho-h}>} n_{\rho-h}(i_2^{(\rho-h)}, \ldots, i_w^{(\rho-h)}) \sum_{j=2}^{w} \left\{ \frac{i_j^{(\rho-h)}}{j} \right\}.$$

Now, for each h satisfying $0 \le h \le w - 2$, let q_h denote the smallest index of all non-zero elements of the $(w - 1)$-tuple

$$(i_2^{(\rho-h)}, \ldots, i_w^{(\rho-h)}).$$

Then, by (2.1), we have

$$\sum_{j=2}^{w} i_j^{(\rho-h)} \leq (\rho - h) - (\rho - q_h + 1) = q_h - h - 1.$$

Thus,

$$\sum_{j=2}^{w}\left\{\frac{i_j^{(\rho-h)}}{j}\right\} \leq \frac{1}{q_h}\sum_{j=2}^{w} i_j^{(\rho-h)} \leq \frac{q_h - h - 1}{q_h} \leq \frac{w - h - 1}{w}.$$

Hence,

$$\sum_{j=2}^{w} n_j \leq \sum_{h=0}^{w-2} \sum_{<w_{\rho-h}>} n_{\rho-h}(i_2^{(\rho-h)},\ldots,i_w^{(\rho-h)}).\frac{w-h-1}{w}$$

$$= \sum_{h=0}^{w-2} \frac{w-h-1}{w} \sum_{<w_{\rho-h}>} n_{\rho-h}(i_2^{(\rho-h)},\ldots,i_w^{(\rho-h)})$$

$$\leq \sum_{h=0}^{w-2} n_{\rho-h}.\frac{w-h-1}{w}.$$

But

$$\sum_{j=2}^{w} n_j = n - \sum_{j=w+1}^{\rho} n_j,$$

which implies that

$$n \leq \sum_{h=1}^{w-1} \frac{2w-h}{w}.n_{\rho-h+1} + \zeta$$

where

$$\zeta = \begin{cases} 0 & \text{if } \rho \text{ is odd,} \\ n_{w+1} & \text{if } \rho \text{ is even.} \end{cases}$$

This implies that:

$$\frac{nw\rho}{2w-1} \leq \sum_{h=1}^{w-1} \frac{2w-h}{2w-1}.\rho.n_{\rho-h+1} + \frac{w\zeta\rho}{2w-1}.$$

Now,

$$\frac{(2w-h)\rho}{2w-1} \leq \rho - h + 1, \text{ by definition of } w.$$

Thus,

$$\frac{nw\rho}{2w-1} \leq \sum_{h=1}^{w-1} (\rho-h+1).n_{\rho-h+1} + \frac{w\zeta\rho}{2w-1} \leq \sum_{j=w+1}^{\rho} j.n_j \leq 2m.$$

This implies that m is at least $\frac{nw\rho}{2(2w-1)}$ and proves the following result:

5.9 Theorem

If G is a ρ-critical graph of order n, then the number of edges of G is at least $\frac{1}{4}n(\rho + 1)$. //

In fact, this result can be slightly sharpened as follows:

5.9* Theorem

If G is a ρ-critical graph of order n, then the number of edges of G is at least $\frac{1}{4}n(\rho + i)$, where $i = 1$ or 2 according as ρ is odd or even. //

One corollary of this result is the following:

5.10 Corollary

If G is a ρ-critical graph, then its edge-independence number α satisfies the inequality

$$4\alpha\rho \geq n(\rho + 1) - 4.$$

Proof

Theorem 2.24 implies that $\alpha\rho \geq m - 1$. This and (5.9) together imply that $\alpha\rho \geq \frac{1}{4}n(\rho + 1) - 1$, whence the result follows. //

If α is known, then we can obtain a lower bound for m by applying (5.1). In particular, we get the following corollary:

5.11 Corollary

If G is a Hamiltonian ρ-critical graph, then

$$m(G) \geq \tfrac{1}{4}(n - 1)(\rho + 2). \text{ //}$$

This is slightly better than the bound given by (5.9).

These bounds are by no means the last word on the subject. Vizing [44] has made the following conjecture:

Conjecture

Every ρ-critical graph has at least $\frac{1}{2}(n(\rho - 1) + 3)$ edges.

One interesting conclusion that could be drawn should this bound be proved correct, would be that all planar graphs with maximum valency at least 7 are of class 1. This leads naturally to the second part of this chapter which deals with the problem of classifying *PLANAR* graphs into either class 1 or class 2. Before discussing this problem we make a resumé of the bounds on the number of edges treated in this first section. We present the following table which compares the various bounds for the first few values of ρ.

$m \geq$ / $\rho =$	small values (p. 80 et seq.)	$\frac{2}{\rho} n(\rho - 1)$ (Cor. 5.8)	$\frac{1}{4}n(\rho + i)$ (Theorem 5.9*)	$\frac{1}{2}(n(\rho-1) + 3)$ Conjecture
2	n	n	n	$\frac{1}{2}(n+3)$
3	$\frac{4}{3} n$	$\frac{4}{3} n$	n	$n+2$
4	$\frac{5}{3} n$	$\frac{3}{2} n$	$\frac{3}{2} n$	$\frac{3}{2}(n+1)$
5	$\frac{9}{5} n$	$\frac{8}{5} n$	$\frac{3}{2} n$	$2n+2$
6	$2n$	$\frac{5}{3} n$	$2n$	$\frac{1}{2}(5n+3)$

2) Applications to planar graphs

In this section we discuss the classification problem for planar graphs mainly with the help of the techniques and results established in the previous section. To begin with, we note that it is easy to construct planar graphs of class 2 with maximum valency ρ for $2 \leq \rho \leq 5$. Examples of these are the odd circuits and the graphs obtained from the following three Platonic graphs by inserting a vertex into one of the edges: the tetrahedron, the octahedron, and the icosahedron. We also note that it follows as a direct corollary of (2.18) that all planar graphs with maximum valency at least 10 are of class 1. This was noted by Vizing [42]. In fact, in a later paper Vizing [43] proved the following stronger result:

5.12 Theorem

If G is a planar graph with maximum valency at least 8, then G is of class 1. //

In this connexion, we must also mention Mel'nikov work [28] which generalizes (5.12). In particular, he proved the following two theorems:

5.13 Theorem

If G is a graph that can be embedded in the projective plane and if its maximum valency is at least 8, then G is of class 1. //

5.14 Theorem

If G is a graph that can be embedded in a surface with non-positive Euler characteristic ξ and if

$$\rho(G) \;>\; \max\{[\tfrac{1}{2}(11 + (25 - 24\xi)^{\frac{1}{2}})],[\tfrac{1}{3}(8 + 2(52 - 18\xi)^{\frac{1}{2}})]\},$$

then G is of class 1. //

However, the problem of determining what happens when $\rho = 6$ or 7 remains open. We state the following conjecture due to Vizing [43]:

Planar Graph Conjecture

If G is a planar graph with maximum valency at least 6, then G is of class 1.

On the assumption that it is not easy to settle this conjecture, we look for various restrictions on the graph which enable us to solve the problem at least partially if not in its entirety.

(a) Restrictions on the maximum valency

5.15 Theorem

(i) If G is a planar, class 2 graph with maximum valency 7, then $n_7 \geq 6$.

(ii) If G is a planar, 7-critical graph, then

$$n_7 \geq \max\{6, [\tfrac{1}{6}(n + 21)]\}.$$

Proof

Euler's Theorem for connected, planar graphs implies the following inequality:

$$n_7 + 12 \leq 4n_2 + 3n_3 + 2n_4 + n_5 \quad \ldots\ldots\ldots (A)$$

(i) Without loss of generality, we can assume that G is critical. Thus, by Lemma 5.5, we have the following:

$$2n_2 + n_3 + \tfrac{2}{3}n_4 + \tfrac{1}{2}n_5 + \tfrac{2}{5}n_6 \leq n_7 \quad \ldots\ldots\ldots (B)$$

Conditions (A) and (B) together imply that

$$n_7 + 12 \leq 6n_2 + 3n_3 + 2n_4 + \tfrac{3}{2}n_5 + \tfrac{6}{5}n_6 \leq 3n_7.$$

Hence,

$$n_7 \geq 6.$$

(ii) Since G is critical, it follows from (5.9) that

$$4n \leq 7n_7 + 6n_6 + 5n_5 \quad \ldots\ldots\ldots (C)$$

Conditions (A) and (C) together imply that

$$4n + 84 \leq 28n_2 + 21n_3 + 14n_4 + 12n_5 + 6n_6.$$

Condition (B) implies that

$$48n_2 + 24n_3 + 16n_4 + 12n_5 + 9n_6 \leq 24n_7.$$

These last two inequalities together imply the conclusion of the theorem. //

Similar results hold for graphs of maximum valency 6. We state the corresponding theorem without proof since this follows closely the one just given.

5.16 Theorem

(i) If G is a planar, class 2 graph with maximum valency 6, then $n_6 \geq 4$.

(ii) If G is a planar, 6-critical graph, then

$$n_6 \geq \max\{4, [\tfrac{2}{9}(n+6)]\}. \quad //$$

(b) Restrictions on the minimum valency

The following is a direct corollary of the work of (5.9):

5.17 Theorem

If G is a planar graph with maximum valency 7 and minimum valency 4, and if $n_4 > \frac{1}{2}n - 3$, then G is not critical.

Proof

Assume on the contrary that G is critical. Then it follows from (5.9) that

$$4n \leq 7n_7 + 6n_6 + 5n_5 = 2m - 4n_4.$$

Thus,

$$2m > 4n + 4(\tfrac{1}{2}n - 3) = 6n - 12,$$

which contradicts Euler's Theorem for connected, planar graphs. //

(c) Restrictions on the girth

We can impose other restrictions on a planar graph to obtain similar partial results. One obvious such restriction is on the girth of the graph. We can ask the question: If the girth g of a planar graph is at least g_0, what is the smallest maximum valency ρ_0 such that all planar graphs with girth $g \geq g_0$ and maximum valency $\rho \geq \rho_0$ are of class 1?

We can follow the same line of proof as in the previous theorems and assume that we are given a planar graph with girth g, maximum valency ρ, and which is of class 2. We then consider a ρ-critical subgraph G whose girth is therefore at least g. If we denote by n, m, and f the number of vertices, edges, and faces of G, it follows from Euler's Theorem for connected, planar graphs that:

$$2gf = 2gm - 2gn + 4.$$

We also have:

$$2gm = \sum_{j=2}^{\rho} gj.n_j \quad ;$$

$$2gn = \sum_{j=2}^{\rho} 2g.n_j \quad ;$$

$$2gf \leq \sum_{j=2}^{\rho} 2j.n_j \quad .$$

It follows that:

$$4 + (g(\rho - 2) - 2\rho).n_\rho \leq \sum_{j=2}^{\rho-1} (2j - gj + 2g).n_j \quad \ldots\ldots(*)$$

We now consider a few cases:

I) <u>ρ = 3</u>

Condition (*) implies that

$$4 + (g - 6).n_3 \leq 4n_2 .$$

This contradicts (5.5) if $g \geq 8$.

II) <u>ρ = 4</u>

Condition (*) implies that

$$4 + 2(g - 4).n_4 \leq 4n_2 + (6 - g).n_3 .$$

This contradicts (5.5) if $g \geq 5$.

III) <u>ρ = 5</u>

Condition (*) implies that

$$4 + (3g - 10).n_5 \leq 4n_2 + (6 - g).n_3 + (8 - 2g).n_4 .$$

This contradicts (5.5) if $g \geq 4$.

If $\rho = 6$ or 7 we obtain a similar contradiction when $g \geq 4$ and if $\rho \geq 8$, we have Vizing's Theorem. We have thus established the following result:

5.18 Theorem

Let G be a planar graph whose girth is at least g_0 and whose maximum valency is at least ρ_0; then G is of class 1 if one of the following holds:

(i) $\rho_0 = 3$ and $g_0 = 8$; (ii) $\rho_0 = 8$ and $g_0 = 3$;

(iii) $\rho_0 = 4$ and $g_0 = 5$; (iv) $\rho_0 = 5$ and $g_0 = 4$. //

Cases (iii) and (iv) of this theorem have also been stated by Kronk, Radlowski & Franen [24].

Arguing in exactly the same way as in Theorem 5.15, we obtain the following result, the proof of which is left to the reader.

5.19 Theorem

(i) Let G be a class 2 graph which is planar, has maximum valency 4 and girth 4; then G has at least six vertices of maximum valency;

(ii) Let G be a class 2 graph which is planar, has maximum valency 3 and girth 7; then G has at least twenty-eight vertices of maximum valency. //

Similar reasoning about class 2 graphs which are planar, have maximum valency 5 and girth 3 leads us to conclude that they have at least three vertices of maximum valency. However, this holds for class 2 graphs in general.

(d) Restriction to outerplanar graphs

We conclude this chapter by imposing one further restriction on the graph. We assume that G is an outerplanar graph, i.e. that all the vertices of G lie on the same face. Without loss of generality this can be taken to be the infinite face. A typical graph is shown in Figure 5.1.

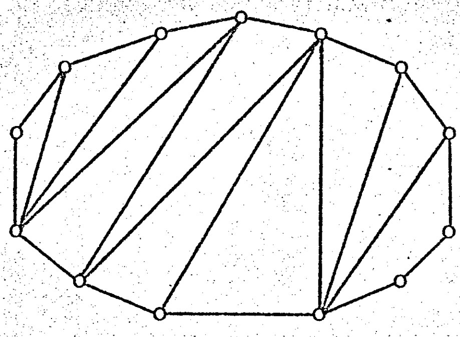


Figure 5.1.

In this case, the problem is solved completely. We first need to establish the following lemma.

5.20 Lemma

If G is a 2-connected outerplanar graph with maximum valency 3, then G has a vertex of valency 2 which is either (i) adjacent to another vertex of valency 2, or (ii) adjacent to two adjacent vertices both of valency 3.

Proof

Since G is 2-connected and outerplanar, it is clear that G is Hamiltonian. For any pair of vertices v, w define d(v,w) to be the shortest distance between v and w along the Hamiltonian circuit. Also define a chord of G to be any edge not belonging to the Hamiltonian circuit. Let (x_0,y_0) be a chord of G such that among all chords (x,y) of G, $d(x_0,y_0)$ is minimal. Note that since G has maximum valency 3, then G has at least one chord. Note also that $d(x_0,y_0) \geq 2$ and $d(x_0,y_0) = 2$ corresponds to case (ii) of our lemma. If $d(x_0,y_0)$ is at least 3, then, because G is outerplanar, we can write $G = H_1 \cup H_2$, where $H_1 \cap H_2 = (x_0,y_0)$. Without loss of generality, let $n(H_1) \leq n(H_2)$.

Now, either H_1 has a chord or not. If it has, we have a contradiction to the minimality of $d(x_0,y_0)$. If it has not, then this corresponds to case (i) of our lemma. //

We can now prove our result:

5.21 Theorem

An outerplanar graph is of class 1 if and only if it is not an odd circuit.

Proof

The necessity of the condition is obvious. So, assume that G is a connected outerplanar graph which is not an odd circuit. If $\rho = 1$ or 2, then the statement is trivially true. If ρ is at least 4, then the result follows by (2.18), since an outerplanar graph contains a vertex of valency 1 or 2.

The case $\rho = 3$ is the only one to consider. In this case, if G has a cut-vertex, then G contains some bridge e and if G - e is 3-colourable, then so is G. Thus, we need only consider 2-connected graphs. These are necessarily Hamiltonian.

We now proceed by induction on the number of edges t of the Hamiltonian circuit. The statement is clearly true for the case t = 4. So, assume it is true for all outerplanar graphs with $t \leq t_0$, and consider having $t = t_0 + 1$. By Lemma 5.20, G has either two vertices v and w say, which are adjacent and both have valency 2, or three mutually adjacent vertices, x, y, and z say, such that x has valency 2 and each of y and z has valency 3. In the first case, by contracting (v,w) to a single vertex, and in the second case, by contracting the 3-circuit (x,y,z) to a single vertex, we obtain an outerplanar graph having at most t_0 edges in the Hamiltonian circuit. This is therefore 3-colourable by the inductive hypothesis. It is not difficult to see that any 3-colouring of this graph can be extended to a 3-colouring of G. Since we have dealt with all cases, the result is proved. //

CHAPTER 6: SMALL CRITICAL GRAPHS AND THE CRITICAL GRAPH CONJECTURE

As stated in Chapter 2, it has been conjectured by Beineke & Wilson [3] and independently by Jakobsen [21], that critical graphs of even order do not exist. Some evidence for the truth of this conjecture has already been presented in Chapter 3. The object of this chapter is to provide some further evidence by considering critical graphs of small order. As a by-product, we shall obtain other interesting information about critical graphs of odd order. We divide the material in this chapter into two sections. In the first section we obtain results about critical graphs of small order; we then apply these results, in the second section, to the Critical Graph Conjecture. Here we show that there do not exist any critical graphs of even order not exceeding 10 and no 3-critical graphs of order 12. This extends Jakobsen's [20] conclusion that there are no 3-critical graphs of even order not exceeding 10.

1) Small critical graphs

As in the previous chapter, for a given graph G, we let n_j denote the number of vertices of valency j and if $a_1, a_2, \ldots, a_k$ are the valencies of G in ascending order, we call the (valency-)list of G the expression

$$a_1^{f_1} a_2^{f_2} \ldots a_k^{f_k},$$

where $f_j = n_{a_j}$.

We need the following results on matchings in critical graphs.

6.1 Theorem

If G is a 3-critical graph of even order $n \leq 26$, then G contains a 1-factor.

Proof

Let G be a 3-critical graph containing no 1-factor. Then, by Tutte's criterion [40], G contains a cut-set S of p vertices, such that $h(S)$, the number of components of odd order of $G - S$, is greater than p. A parity argument shows that there must be at least $p + 2$ such components. Let α and β be the number of isolated vertices of $G - S$ which have valency 2 and 3 respectively in G. Note that $\alpha \leq [\frac{1}{2}p]$. Let A be the set of isolated vertices of $G - S$ which have valency 2 in G. Thus $|A| = \alpha$. Let the total number of edges of the form $V(G - S) \times S$ be t. If all components of $G - S$ except elements of A are joined by at least three edges to S, then

$$\begin{aligned} 3p \geq t &\geq 2\alpha + 3(h(S) - \alpha) \\ &\geq 3(p + 2) - \alpha. \end{aligned}$$

Thus, $\alpha \geq 6$, which implies that $p \geq 12$, which in turn implies that $n \geq 26$. If $n = 26$, then each component of $G - S$ is trivial and G is therefore bipartite and thus not critical. Hence, either $n \geq 28$, or some component other than an element of A is joined by less than three edges to S. In the latter case, we have some non-trivial component joined by two edges to S and these edges must be independent, by the connectivity properties of G.

For each value of p, let k_p be the smallest order of a component of $G - S$ joined by two independent edges to S. Let $k = \max k_p$. We claim that $k \leq 5$.

To prove this, suppose that $k \geq 7$, and let G be a graph for which this occurs. Then

$$3p \geq 2\alpha + 3\sum_{i=1}^{k-1} \beta_i + 2\sum_{i \geq k} \beta_i ,$$

where β_j is the number of components of order j; $\beta_1 = \beta$.
Thus, if we denote $\sum_{i \geq k} \beta_i$ by B, we have

$$3p \geq 2h + (h - \alpha - B)$$
$$\geq 3p + 6 - \alpha - B,$$

and hence

$$\alpha + B \geq 6.$$

Then, since k is at least 7, we get following implications:

$B = 1 \Longrightarrow \alpha \geq 5 \Longrightarrow p \geq 10 \Longrightarrow n \geq 10 + 11 + 7 = 28;$

$B = 2 \Longrightarrow \alpha \geq 4 \Longrightarrow p \geq 8 \Longrightarrow n \geq 8 + 14 + 8 = 30;$

$B = 3 \Longrightarrow \alpha \geq 3 \Longrightarrow p \geq 6 \Longrightarrow n \geq 6 + 21 + 5 = 32;$

$B = 4 \Longrightarrow n \geq 2 + 28 = 30.$

Thus, $k \leq 5$.

Now, by (4.3), if G is separable by two independent edges, then G is a Hajós union of two 3-critical multigraphs, one of which has a vertex of valency 2 which is identified with a vertex in the other. Moreover, one of the constituent graphs has order 3 or 5. Call this graph G_1 and the other G_2. Since G is simple and since there are no 3-critical multigraphs of even order $n \leq 10$, the identified vertex of valency 2 belongs to G_2 which is necessarily simple, whereas G_1 can have at most one multiple edge. Thus, as shown in [20], G_1 can only be one of the following multigraphs:

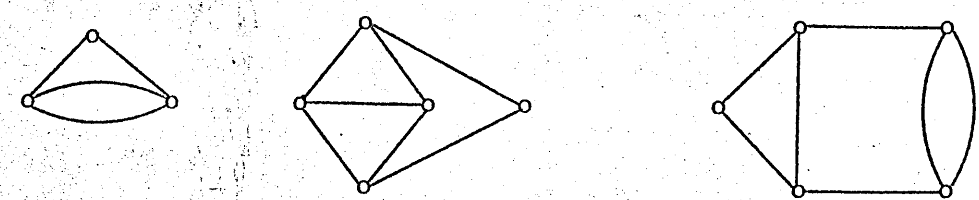


Figure 6.1

In all cases it is clear that a 1-factor of G_2, which exists by the minimality of G, can be extended to a 1-factor of G. //

The proof of the next theorem is omitted, since this follows closely the one just given.

6.2 Theorem

Let G be a 3-critical graph of odd order $n \leq 19$ and let v be any vertex of valency 2. Then G has a matching of $[\frac{1}{2}n]$ edges which covers every vertex except v. //

If we make no restriction on ρ, then we obtain the following result:

6.3 Theorem

If G is a ρ-critical graph of even order $n \leq 10$, then G contains a 1-factor.

Proof

In view of Theorem 6.1, we can assume that G is a ρ-critical ($\rho \geq 4$) graph of even order $n \leq 10$ which has no 1-factor. By Tutte's Theorem [40], there is a set S of p vertices for which G − S has at least p + 2 components of odd order. We consider the various possible values of p.

p = 2: By Theorem 2.1, no vertex can be adjacent to more than one of valency 2 so that at most one component of G − S has order 1. Since there must be at least four components, this is impossible.

p = 3: Similarly, a vertex of G can be adjacent to at most two of valency 3 (or less), so that G − S has at most two trivial components, which is impossible.

p = 4: Since G − S has at least six components, in this case all must be trivial. If $\rho = 4$, there can be at most 16 edges joining S to G − S, so that at least three vertices in G − S have valency less than 4 in G and this is impossible as before. Therefore $\rho \geq 5$, so that all vertices of maximum valency are in S. If σ denotes the minimum valency in G, this implies that $\sigma \geq \rho - 2$. But since each vertex in G – S is adjacent only to vertices in S and every vertex in S is adjacent to at least two others in S, $4\rho \geq 6\sigma + 8$. As this implies $\rho \leq 2$, we again have a contradiction. //

It is easy to see that there are no critical graphs of order 2 or 4. The only one of order 3 is the 3-circuit, and there are only three of order 5 (shown in Figure 6.2). We note that each of these last three is ρ-critical for a different value of ρ. These graphs are the only ones with the corresponding valency-lists, so we shall often refer to them by their lists: 2^5, 23^4, $3^2 4^3$.

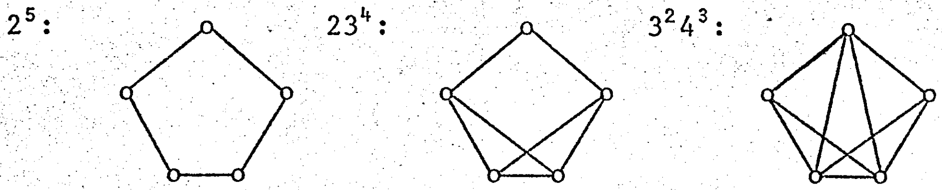


Figure 6.2

In our next theorem, we shall determine the valency-lists of all critical graphs of order 7. The proof involves a large number of cases by looking at possibilities for maximum and minimum valencies. In many cases, the problem is reduced to showing that the graph is of class 2 because of its total deficiency and then proving that it is ρ-critical because it could contain no other ρ-critical subgraph. For simplicity, we shall refer to this as the 'critical list argument'.

6.4 Theorem

Let G be a connected graph of order 7. Then G is ρ-critical if and only if it has exactly 3ρ + 1 edges.

Proof

Let G be ρ-critical with minimum valency σ and total deficiency τ. We note that G has 3ρ + 1 edges if and only if τ = ρ − 2. We shall consider all possible pairs (ρ,σ) with 2 ≤ σ < ρ ≤ 6 together with the first trivial case (2,2).

Case (2,2): The 7-circuit is the only 2-critical graph and the only connected graph with ρ = 2 and τ = 0.

Case (3,2): By (5.5), $n_3 \geq 2n_2$, so that the only possible valency-list is 23^6. (Here and in what follows we freely use the fact that the number of vertices of odd valency must be even). By the critical list argument, any such graph must be critical.

Case (4,2): Since in this case $n_4 \geq n_3 + 2n_2$, there are just three possible lists: 2^24^5, 23^24^4, and 24^6. Any graph with list 2^24^5 can be obtained by taking the only graph of order 6 which is 4-valent and splitting one vertex into two of valency 2. Since the original graph is 4-colourable, then so is the result. Thus, 2^24^5 cannot correspond to a critical graph. Because of the required adjacencies of all vertices to vertices of valency 4 (Theorem 2.1), there is only one possible graph with list 23^24^4 and this is 4-colourable (see Figure 6.3). It follows that every graph with list 24^6 is critical.

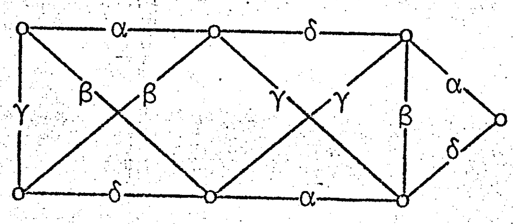


Figure 6.3

Case (4,3): Since we must have $n_3 \le n_4$, there is only one possible list, $3^2 4^5$, and any corresponding graph must be critical by the critical list argument.

Case (5,2): Since $n_5 \ge 5$, by Corollary 2.2, there are two lists to consider: 235^5 and 25^6. In a critical graph with the first list, the vertices of valency 2 and 3 cannot be adjacent. Thus, since K_6 is 5-colourable, so is this graph. Therefore the critical list argument implies that any graph with list 25^6 is critical.

Case (5,3): In this case $n_5 \ge 4$, so that $3^2 45^4$ and 345^5 are the only possible lists. If the first belongs to G, then each vertex of valency 5 is adjacent to all three others, and since the vertices of valency 3 cannot be adjacent, the vertex of valency 4 must be adjacent to one of them. There is only one graph meeting these conditions and it is 5-colourable (see Figure 6.4). As in earlier cases, we deduce that all graphs with list 345^5 are 5-critical.

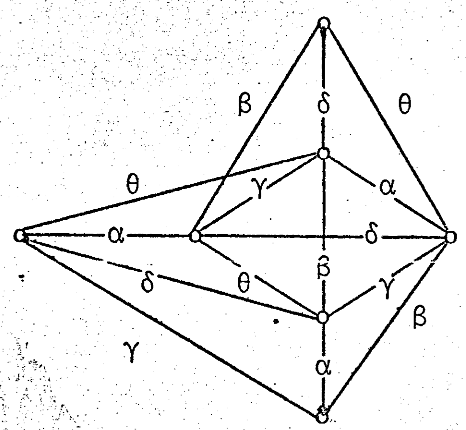


Figure 6.4

Case (5,4): Since $n_5 \ge 3$ and since the deficiency must be at least 3, there is only one list, $4^3 5^4$, and any corresponding graph is critical.

Cases (6,2) and (6,3): Since $n_6 \ge 6$ in the first case and $n_6 \ge 5$ in the second, there is no critical graph possible.

Case (6,4): Since $n_6 \ge 4$ and the deficiency is at least 4, there are three possible lists: $4^3 6^4$, $4^2 6^5$, and $45^2 6^4$. The second belongs to no graph and the first only to the complement of K_3 in K_7, which is readily seen to be 6-colourable. Therefore the critical graphs are precisely those with list $45^2 6^4$.

Case (6,5): In this case $n_6 \geq 3$ and, by Theorem 2.25, $n_5 \geq 4$. Therefore the only possible list is $5^4 6^3$ and all corresponding graphs are critical.

It is an easy matter to check that in each case, the ρ-critical graphs belong to lists with valency sum $6\rho + 2$. //

6.5 Corollary

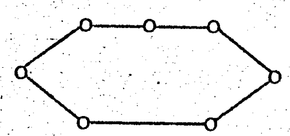
A connected graph of order 7 is critical if and only if its valency-list is in the set $\{2^7, 23^6, 24^6, 3^2 4^5, 25^6, 345^5, 4^3 5^4, 45^2 6^4, 5^4 6^3\}$.

We observe that analogous results also hold for n = 3 and n = 5. However, there is no comparable result for n = 9, as can be verified from Table 2.17.

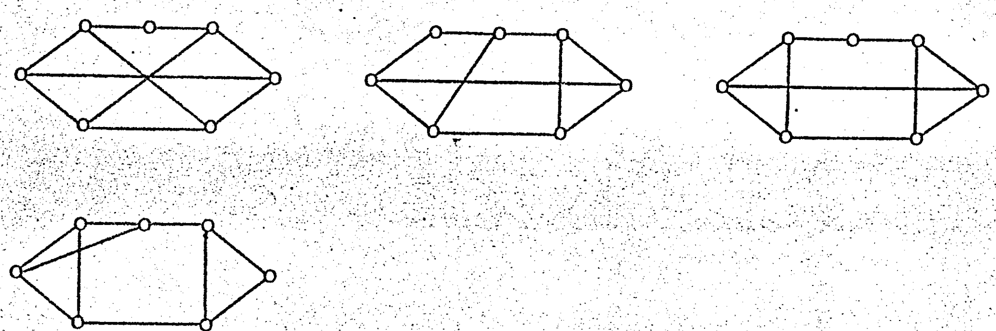
It is not difficult to find all critical graphs of order 7 using Corollary 6.5. We present them by list in Table 6.6.

6.6 Table   The critical graphs of order 7

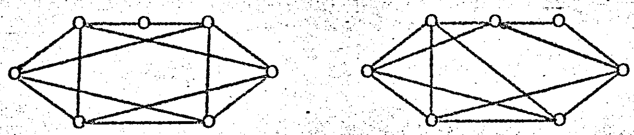
2^7:



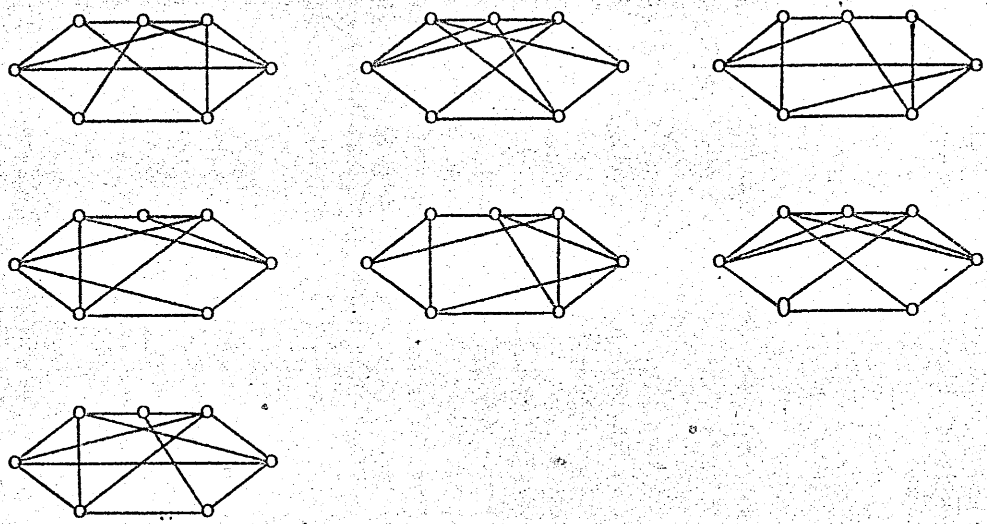
23^6:



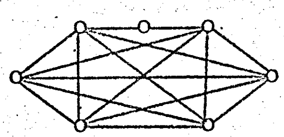
24^6:



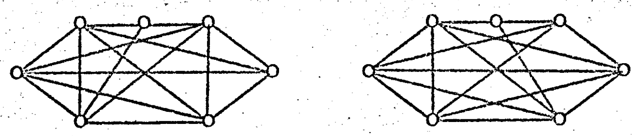
$3^2 4^5$:



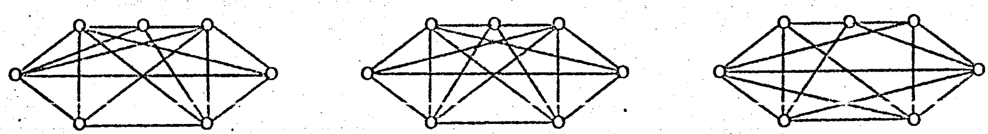
25^6:



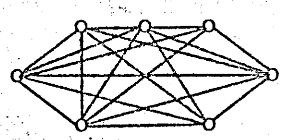
345^5:



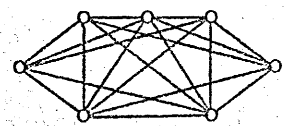
$4^3 5^4$:



$45^2 6^4$:



$5^4 6^3$:



We now apply this discussion to regular graphs of small order.

6.7 Theorem

All connected, regular graphs of even order less than 10 are of class 1.

Proof

Let G be a connected, regular graph of even order n and valency ρ. If $\rho = 2$, then G is an even circuit which is clearly of class 1. If $\rho \geq 3$, then $n \geq 4$. If $n = 4$, then G is K_4 which is of class 1.

If $n = 6$ and if G is of class 2, then G contains a ρ-critical subgraph H, which is either the graph 23^4 or the graph $3^2 4^3$ (see Figure 6.2). Hence $\rho(G) = 3$ or 4. If v is the vertex of G not in H, then $\rho(v) = 1$ or 2 according as $\rho(G) = 3$ or 4. In either case, we have a contradiction to the regularity of G.

So, let $n = 8$ and assume that G is of class 2. G contains a ρ-critical subgraph H whose order is 5 or 7. If H has order 7, then H has deficiency $\rho - 2$, by Theorem 6.4. This implies that the vertex v of G which is not in H has valency at most $\rho - 2$, contradicting the regularity of G. Thus, H has order 5, in which case H must again be one of the graphs 23^4 or $3^2 4^3$. In either case, there are three vertices of G not in H. In the first case, $\rho = 3$ and of the vertices of $G - H$, exactly one can be adjacent to a vertex in H. Thus, the remaining two have valency 1 or 2. This again contradicts the regularity of G. In the second case, $\rho = 4$ and of the vertices of $G - H$, exactly one can have valency 4. This final contradiction proves the result. //

6.8 Theorem

Apart from the Petersen graph, all cubic, bridgeless graphs of order 10, are of class 1.

Proof

Let G be a cubic, bridgeless graph of order 10 which is of class 2. G contains a 3-critical subgraph H whose order is 5, 7, or 9. With exactly one exception (see Table 2.17), H has total deficiency 1. The exceptional case gives rise to the Petersen graph. All other cases are either not realizable as subgraphs of a cubic graph of order 10, or give rise to the graph of Figure 6.5 which contains a bridge. //

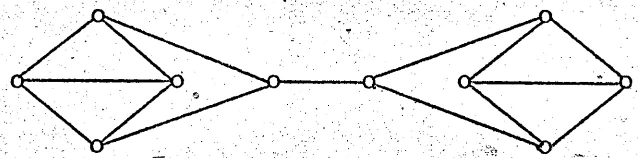


Figure 6.5

2) The Critical Graph Conjecture

In this section we shall prove that the smallest ρ-critical graph of even order must have at least 12 vertices and if $\rho = 3$ at least 14.

6.9 Theorem

There are no critical graphs of order 6.

Proof

Assume that G is a ρ-critical graph of order 6. We consider the three possible values of ρ separately.

Case $\rho = 3$: In this case $n_2 \geq 4$, by Theorem 2.25, and $n_3 \geq 3$, by Corollary 2.2 (ii), which is impossible.

Case $\rho = 4$: From Theorems 2.1 and 5.5 it follows that the only possible lists are $2^2 4^4$, 24^5, $23^2 4^3$, and $3^2 4^4$. The first three contradict Theorem 2.25, so G must have list $3^2 4^4$. By

Theorem 6.3, G has a 1-factor F, and the removal of F leaves a graph G', which is of class 2, by (2.23). Therefore G' must contain the 3-critical graph 23^4 of Figure 6.2, which is impossible for a graph with list $2^2 3^4$.

Case $\rho = 5$: In this case G is a subgraph of K_6 which is 5-colourable. Therefore G cannot be critical. //

6.10 Theorem

There are no critical graphs of order 8.

Proof

Assume that G is a ρ-critical graph which has order 8 and minimum valency σ. Since K_8 is 7-colourable and G cannot be regular, we assume $2 \leq \sigma < \rho \leq 6$. By Theorem 6.3, G has a 1-factor F, and $G - F$ contains a $(\rho-1)$-critical subgraph G'. We now consider all possible cases (ρ,σ), relying heavily on Theorems 2.25 and 5.5 which give inequalities for the number n_j of vertices of valency j.

Case (3,2): There are no such graphs since we must have $n_3 \geq 2n_2$ and $n_2 \geq 4$, which are irreconcilable.

Case (4,2): Similarly, we must have $n_4 \geq 2n_2 + n_3$ and $2n_2 + n_3 \geq 6$, again an impossible situation.

Case (4,3): A corresponding argument implies that G must have list $3^4 4^4$. Then $G - F$ must have list $2^4 3^4$ and hence G' cannot have 7 vertices. Therefore G' must have list 23^4 and any extension to G results in three vertices of valency 3 being mutually adjacent, which is impossible.

Case (5,2): Here $n_5 \geq 2n_2 + n_3$ and $3n_2 + 2n_3 + n_4 \geq 8$, so that the only possible list is $2^2 3 5^5$. By Theorem 2.1, such a list cannot belong to a critical graph.

Case (5,3): From the facts that $n_5 \geq 4$ and $2n_3 + n_4 \geq 6$, we see that $3^3 5^5$, $3^4 5^4$, and $3^2 4^2 5^4$ are the only possible lists.

Since G - F then has list $2^3 4^5$, $2^4 4^4$, or $2^2 3^2 4^4$, we see that the order of G' cannot be 7 and hence must be 5. However, this means that G' has list $3^2 4^3$. Any extension to G requires two vertices of valency 2 to be adjacent, which is impossible.

Case (5,4): Since $n_4 \geq 4$, G must have list $4^4 5^4$. Furthermore, G' must have list $3^2 4^3$ and thus cannot be a subgraph of G - F (which has list $3^4 4^4$).

Case (6,2): The deficiency of G must be at least 10 and yet n_6 must be at least 6. Thus, this case cannot occur.

Cases (6,3), (6,4), and (6,5): It is not difficult to see that the only possible lists are: $3^2 4 6^5$; $4^4 6^4$, $4^3 6^5$, $4^2 5^2 6^4$; $5^4 6^4$. In no case can the list for G - F admit a 5-critical subgraph, since such a subgraph would have to be of order 7. //

6.11 Theorem

There are no critical graphs of order 10.

Proof

Assume that this is not the case, and let G be a ρ-critical graph of order 10 with ρ minimal. By Theorem 6.3, G has a 1-factor F, and the graph G' := G - F has a (ρ-1)-critical subgraph H which we take to have maximum possible order. Lemmas 6.12 and 6.13 below show that this order must be 9.

Let u be the vertex of G not in H, let k be the valency of u, and let σ be the minimum valency in G. Then the total deficiency of G - u is

$$\begin{aligned}\tau(G - u) &= \tau(G) - \rho + 2k \\ &\geq 2(\rho - \sigma + 1) - \rho + 2k, \text{ by Theorem 2.25} \\ &\geq \rho + 2.\end{aligned}$$

Therefore, $\tau(G' - u) \geq \rho + 1$. There must be at least one vertex w of maximum valency adjacent to u in G'. It follows that if t_H is the next-largest valency to $\rho - 1$ in H, then the valency of w is between t_H and $\rho - 2$, so that

$$\tau(H) \geq \tau(G' - u) + 2(\rho - 2 - t_H) \geq 3\rho_H - 2t_H,$$

where ρ_H (= $\rho - 1$) is the maximum valency in H. It follows from Lemma 6.14 below that H must have deficiency less than this. Therefore, no such graph H can exist. //

We now prove the three lemmas used in the proof of Theorem 6.11. To this end, we assume that G is a ρ-critical graph of order 10, with ρ minimum. If $\rho = 3$, then $n_2 \geq 4$ and $n_3 \geq 2n_2$, so $\rho > 3$. By Theorem 6.3, G has a 1-factor F, the deletion of which leaves a graph G' which must contain a $(\rho - 1)$-critical subgraph. Assume that H is one of maximum order. Since ρ is minimal, H must have odd order.

6.12 Lemma

The order of H is not 5.

Proof

Assume that the order of H is 5. Then the maximum valency ρ_H is either 3 or 4.

Case 1: $\rho_H = 3$: Then H has valency-list 23^4. Let J be the subgraph of G' induced by the other five vertices. Since each vertex in G' must be adjacent to at least one vertex of valency 3, J must contain at least two vertices of valency 3. Moreover, no vertex of valency 3 can be adjacent to more than one vertex of valency 1. So there are at most two vertices of valency 1. By Theorem 2.1, we also have that if G' has a vertex of valency 1, then G' has at least seven vertices of valency 3. All this implies that the only possible lists for G' are: $1^2 3^8$, $12^2 3^7$, $2^2 3^8$, and $2^4 3^6$. The first three cases clearly violate the deficiency condition of Theorem 2.25. The last case gives rise to the following disconnected graph and to no other:

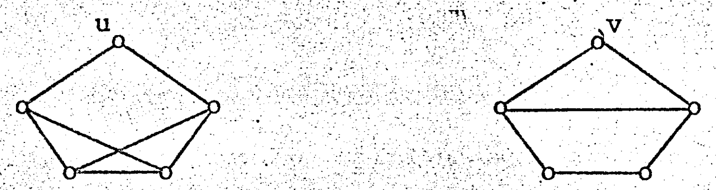


Figure 6.6

Moreover, since every vertex in G is to be adjacent to at least two vertices of valency 4, (u,v) must be an edge in F. But then one can clearly obtain a new 1-factor F' from G such that G − F' is not isomorphic to this configuration, which it must be since H is of maximum order.

Case 2: $\rho_H = 4$: Then H is the graph with list $3^2 4^3$. We consider subcases according to the number q of edges between H and J in G'.

q = 0: J has at least two vertices u_1, u_2 of valency 4 and these must be adjacent. Moreover, by Theorem 2.1, all other vertices have to be of valency 3 at least. The only possibility for G' is to consist of two disjoint copies of H. But then one can obtain a new 1-factor from G whose deletion does not yield the graph $3^2 4^3$.

q = 1: Again, J must contain two vertices of valency 4. Now either some vertex of valency 4 is adjacent to some vertex in H or not. In either case, each vertex of J has valency at least 3 in G' and they respectively give rise to the following two graphs and to no other:

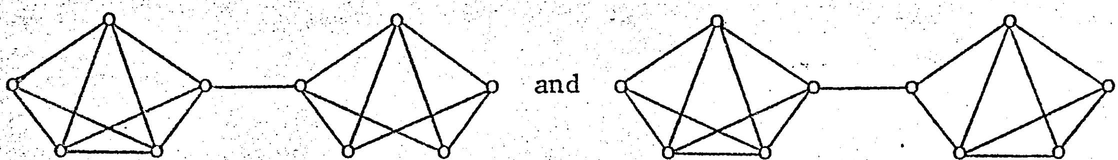


Figure 6.7

However, as before, we can obtain in each case a new 1-factor from G which does not contain H.

q = 2: If J has only one vertex of valency 4 in G', then Theorem 2.1 forces G' to have list $3^4 4^6$. Under these conditions, G' can only be the following graph:

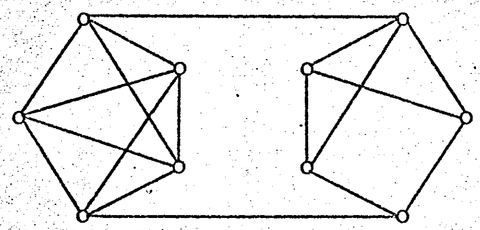


Figure 6.8

Again, we can obtain a new 1-factor from G whose deletion does not yield the graph $3^2 4^3$.

If J has exactly two vertices of valency 4 in G', then these must be adjacent to the vertices in H, since otherwise the deficiency condition is violated. Again by Theorem 2.1, $\sigma(G')$ is at least 2 and the only possible list for G' is $2^3 4^7$. Since a vertex of valency 5 in G which is adjacent to a vertex of valency 3 has to be adjacent also to at least three other vertices of valency 5, G' can only be the following graph:

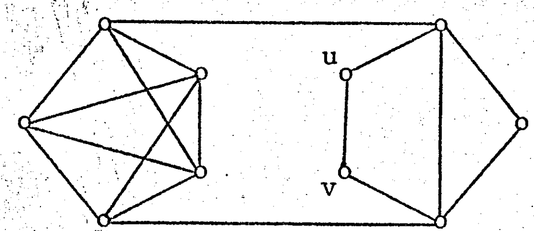


Figure 6.9

However, since this graph violates Theorem 5.1 since in G,

$$\rho(u) + \rho(v) = 6 < \rho(G) + 2.$$

Since, by Theorem 2.25, J cannot have three or more vertices of valency 4, the proof of the lemma is complete. //

6.13 Lemma

The order of H is not 7.

Proof

We consider individual cases of ρ, making repeated use of the fact that each of the three vertices u_1, u_2, u_3 in $G' - H$ must be adjacent to at least one vertex of valency $\rho - 1$ in G' which does not have valency $\rho - 1$ in H.

$\underline{\rho = 4}$: H must have list 23^6 and then $n_3(G')$ is at least 8, which contradicts the deficiency condition.

$\underline{\rho = 5}$: H must have one of the following lists: 24^6, $3^2 4^5$. Both of these cases contradict the deficiency condition.

$\underline{\rho = 6}$: H must have one of the following lists: 25^6, 345^5, $4^3 5^4$.

Hence, the list of G must be $x_1 x_2 x_3 6^7$ and each of u_1, u_2, u_3 has valency at most 4. Theorem 2.1 implies that no two of these can be adjacent and so G must have list $2^3 6^7$. Now, since no two of the u_i's can be adjacent to a common vertex, we can identify u_1, u_2, and u_3 to obtain a graph G" which is of order 8 and 6-valent. It follows that G", and hence G, are of class 1, by Theorem 6.7.

$\underline{\rho = 7}$: H must have one of the following lists: $45^2 6^4$, $5^4 6^3$. Also, in G , n_7 must be at least 6 and $n_6 + n_7$ must be at least 7 by the above condition. By Theorem 5.1, no two of u_1, u_2, u_3 can be adjacent. Thus, the only possible lists for G are $2^3 67^6$ and $2^2 37^7$. The second case violates the deficiency condition. The first yields a graph G", obtained by identification of u_1, u_2, and u_3, which is 7-colourable since it is a subgraph of K_8. This completes the proof. //

6.14 Lemma

If G is a ρ-critical graph $(3 \le \rho \le 7)$ of order 9 and total deficiency τ, then $\tau < 3\rho - 2t$, where t is the largest valency of G less than ρ.

Proof

We assume on the contrary that $\tau \ge 3\rho - 2t$ and proceed to get contradictions for all cases (ρ,σ), $2 \le \sigma < \rho \le 7$, where σ is as usual the minimum valency of G. We note that τ and ρ have the same parity.

Cases (5,4), (6,5), and (7,6): Here $\tau \ge \rho + 2$ and also, since $n_\rho \ge 3$, $\tau \le 6$, which is a contradiction.

Case (3,2): We have $n_2 \le 3$, $n_2 = \tau$ and $\tau \ge 5$, which is impossible.

Case (4,2): Here $n_4 \ge 2n_2 + n_3 = \tau \ge 12 - 2t$. If $t = 3$, this implies $n_4 \ge 6$ and in turn $\tau \le 4$, whereas if $t = 2$, $n_4 \ge 8$ and $\tau \le 2$, both of which are impossible.

Case (4,3): A contradiction follows from the inequalities $n_3 \le n_4$ (so $n_3 \le 4$) and $n_3 = \tau \ge 6$.

Case (5,2): On the one hand, $\tau \ge 15 - 2t$, and on the other hand, $\tau \le 14 - t$ (since $n_5 \ge 5$). It follows that if $t = 2$, $\tau = 11$; if $t = 3$, $\tau = 9$ or 11; and if $t = 4$, $\tau = 7$ or 9. The only lists which meet these conditions are 23^35^5, 234^25^5, and 2^245^6. It is not difficult to show that none can meet the adjacency condition of Theorem 2.1 for a 5-critical graph.

Case (5,3): Here $t = 4$, so $\tau \ge 7$. Also, $n_5 \ge 4$, so that $\tau \le 9$. The only possible lists are therefore $3^24^35^4$, 3^445^4, and 3^345^5, none of which is constructible as a 5-critical graph.

Case (6,2): Since $n_6 \ge 6$, we have $\tau \le 14 - t$ and $\tau \ge 18 - 2t$, so t is 4 or 5. If $t = 4$, $\tau = 10$ and the only list is 2^246^6, which cannot belong to a 6-critical graph. If $t = 5$, $\tau = 8$ and the only list is 2356^6. Such a graph has a Hamiltonian circuit by Pósa's

Theorem (see, for example, [4, p. 211]) so there is a set of four independent edges not meeting the vertex of valency 2. Then $G - F$ has list $2^2 45^6$ and must contain a 5-critical subgraph T, by (2.22). The order of T cannot be 9, so T must have 25^6, 345^5, or $4^3 5^4$ as its list. However, each of these requires the two vertices of valency 2 to be adjacent in $G - F$, which is impossible.

Case (6,3): Arguments similar to those given above show that the only possible lists are $3^4 6^5$, $3^3 56^5$, $3^2 4^2 6^5$, $3^2 5^2 6^5$, and $34^2 56^5$. In each of the first four cases, there must be at least twelve edges from vertices of valency 6 to other vertices, and yet each vertex of valency 6 must be adjacent to four others. This is clearly impossible. The graphs with list $34^2 56^5$ are handled using four independent edges as in the preceding case.

Case (6,4): In this case, $n_6 \geq 4$, $\tau = 2n_4 + n_5$ and $\tau \geq 18 - 2t$. The only possibilities are $4^5 6^4$ and $4^3 5^2 6^4$, but since any vertex adjacent to one of valency 4 must also be adjacent to at least three of valency 6, it is clear that such a critical graph cannot exist.

Case (7,2): Since $n_7 \geq 7$, $\tau \leq 12 - t$. But we must also have $\tau \geq 21 - 2t$, which is impossible.

Case (7,3): Here, $n_7 \geq 6$, so $\tau \leq 15 - t$. Again, $\tau \geq 21 - 2t$, so that $t = 6$ and $3^2 67^6$ is the only possible list. It is readily seen that the vertices of valency 3 are not adjacent and that their identification results in a simple subgraph of K_8, which is of class 1.

Cases (7,4) and (7,5): Because of bounds on n_7, we have $\tau \leq 16 - t$ in the first case and $\tau \leq 15 - t$ in the second. These inequalities, together with $\tau \geq 21 - 2t$, restrict τ and t so that the only possible lists are $4^3 57^5$, $4^2 567^5$, and $5^4 67^4$. Any such graph can be shown to have four independent edges whose deletion leaves a graph which cannot have a 6-critical subgraph. (cf. Case (6,2)).

This completes the proof of the lemma. //

Finally, we prove a result which extends the earlier work of Jakobsen [20] on 3-critical graphs.

6.15 Theorem

There are no 3-critical graphs of order 12.

Proof

Assume on the contrary, that G is a 3-critical graph of order 12. It follows from Theorems 2.25 and 5.5 that G has exactly four vertices of valency 2 and eight of valency 3. Furthermore, the vertices of valency 2 must be at distance at least 3 apart, by Theorem 2.1. By Theorem 6.1, G contains a 1-factor F, whose deletion leaves a class 2 graph G' with list $1^4 2^8$. It follows that, since G' must contain an odd circuit, it is the graph of Figure 6.10, in which the pairs joined by dotted lines cannot be adjacent, since otherwise G would contain a vertex of valency 2 contained in a 3-circuit. This implies that G is separable by two independent edges and hence is a Hajós union of two 3-critical graphs one of which is of even order at most 10, by Theorem 4.3. But this is contradictory.

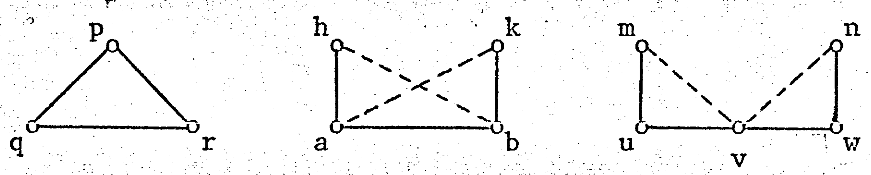


Figure 6.10

Since every vertex of valency 3 must be adjacent to one of valency 2, in G, a, b, u, v, w must generate a 5-circuit. Consequently, p, q, or r is adjacent to m or n, say p to m. without loss of generality. But then the edges (p,m), (q,r), (h,a), (k,b), (n,w), and (u,v) form a 1-factor whose deletion leaves a graph with no odd circuits. //

CHAPTER 7: CIRCUIT LENGTH IN CRITICAL GRAPHS

In this chapter we consider two problems related to the length of circuits in critical graphs. The first problem, discussed in the first section, deals with the girth (the length of any shortest circuit) of critical graphs. We showed in Chapter 5, how restrictions on the girth of graphs yield a partial solution to the planar graph problem. Here we show that ρ-critical graphs with arbitrary girth exist for each ρ. We also discuss the problem of determining $f(\rho,g)$, the minimum order of a ρ-critical graph having girth g. In the second section, we consider problems related to the circumference (the length of any longest circuit). We improve the lower bound for the circumference given by Vizing [43], and construct ρ-critical graphs with circumference not exceeding $t(n,\rho)$, a number depending on the order n and the maximum valency ρ of the graph.

1) The girth of critical graphs

One natural question to ask about the girth of critical graphs is the analogue of that asked by Faber & Mycielski [10] and by Meredith [30] about class 2 graphs: Do there exist ρ-critical graphs of arbitrary girth g for each ρ? If this is answered in the affirmative, one can go on to ask: Within what bounds can one expect to find ρ-critical graphs of given girth and of minimal order?

To answer the first question, one need look only at regular graphs and exploit the work that has already been done in this field.

In particular, we recall that Sachs [36] showed:

7.1 Theorem

For each $\rho \geq 3$, $g \geq 2$ and $h \geq 1$, there exists a Hamiltonian graph G which is ρ-valent, has girth g, and in which all g-circuits are mutually disjoint and constitute a 2-factor of G. Moreover, the number of g-circuits is divisible by h. //

Once we have a ρ-valent graph G of the required girth, it is easy to obtain a class 2 graph having maximum valency ρ and of the same girth, for either G is of odd order and so is itself of class 2, or G is of even order. In the latter case, the graph G' obtained from G by introducing a vertex into any one of the edges is of class 2. This follows from (1.8). Having obtained a class 2 graph G' having maximum valency ρ and of the required girth, we can consider a ρ-critical subgraph G" of G', which always exists, by (2.5). Now, the remark that if H is a subgraph of K, then $g(H) \geq g(K)$, enables us to make the following conclusion:

7.2 Theorem

For any integers $g \geq 3$ and $\rho \geq 3$, there exists a ρ-critical graph of girth at least g. //

However, we should like to sharpen this last statement and obtain ρ-critical graphs with girth exactly equal to g. This can be achieved using the construction of regular graphs which Sachs used to establish Theorem 7.1. The ρ-valent graph $G(\rho,g)$ of girth g in this construction has the property that the Hamiltonian circuit H includes $g - 1$ edges of each of the g-circuits, $G(\rho,g) - H$ has $\rho - 2$ 1-factors, and, except for the case $\rho = 2$ and g odd, $n(G(\rho,g))$ is even.

Now consider the graph G' obtained from $G(\rho,g)$ by inserting a vertex into an edge of the Hamiltonian circuit which is incident to some vertex of a g-circuit but does not belong to the g-circuit itself. By (1.8), G' is of class 2 and, by (2.13), G' is vertex-critical. Thus, if G'' is any ρ-critical subgraph of G', $n(G'') = n(G')$. Moreover, each of the edges of H is essential, for the removal of any of them allows us to colour the remaining edges of H with two colours and each of the $\rho - 2$ 1-factors with a distinct colour. Also, the edge of the g-circuit which is not on H is also essential, since a vertex of valency 2 in a ρ-critical graph can be adjacent only to vertices of maximum valency. Thus, G'' is a ρ-critical graph of girth g and we obtain the following result:

7.3 Theorem

For any $g \geq 3$ and $\rho \geq 3$, there exists a ρ-critical graph of girth g. //

The answer to the second question proposed above seems to be difficult to give in full generality. However, we can attempt a partial solution. Let $f(\rho,g)$ be the minimal order of a ρ-critical graph of girth g. Then we have the following result:

7.4 Theorem

$$f(\rho,3) = \begin{cases} \rho + 1 & \text{if } \rho \text{ is even} \\ \rho + 2 & \text{if } \rho \text{ is odd.} \end{cases}$$

Proof

Since a complete graph of even order is of class 1, it follows at once that $f(\rho,3) \geq \rho + 1$ if ρ is even and $f(\rho,3) \geq \rho + 2$ if ρ is odd. Now consider a complete graph of odd order, K_{2t+1}. It must contain critical graphs of order $2t + 1$ and maximum valencies $2t$ and $2t - 1$. We now show that they must contain a triangle. If we write $n = 2t + 1$, then for the former graph

$$m \geq \tfrac{1}{8}(3(n-1)^2 + 6(n-1) - 1), \text{ by Theorem 2.28 (ii),}$$
$$> [\tfrac{1}{4} n^2], \text{ for all } n \geq 2.$$

Thus, by Turán's Extremal Theorem [39], the graph contains a triangle. For the latter graph,

$$m \geq \frac{1}{8}(3(n-2)^2 + 6(n-2) - 1).$$

This exceeds $[\frac{1}{4}n^2]$ for all $n \geq 8$. For $n \leq 7$, we can verify the truth of the statement by considering the graph of order 5 in Table 2.17 and any one of the graphs with maximum valency 5 in Table 6.6. //

7.5 Theorem

$$f(\rho,4) = 2\rho + 1.$$

Proof

By Theorem 2.6, $f(\rho,4) \leq 2\rho + 1$. Let G be a ρ-critical graph with no triangles. By Theorem 2.1, there exist adjacent vertices u and v of valency ρ. As these have no common neighbours, G has at least 2ρ vertices and is not bipartite. If G has no other vertices, then there is an edge joining neighbours of u or v, which is impossible. Hence, $f(\rho,4) \geq 2\rho + 1$. //

We now consider $f(3,g)$ for small values of g. We define a (ρ,g)-cage to be a ρ-valent graph of girth g and of minimum order. From Table 2.17, it is clear that for g = 3, 4, and 5, $f(3,g) = 2g - 1$, the unique corresponding graphs being the following:

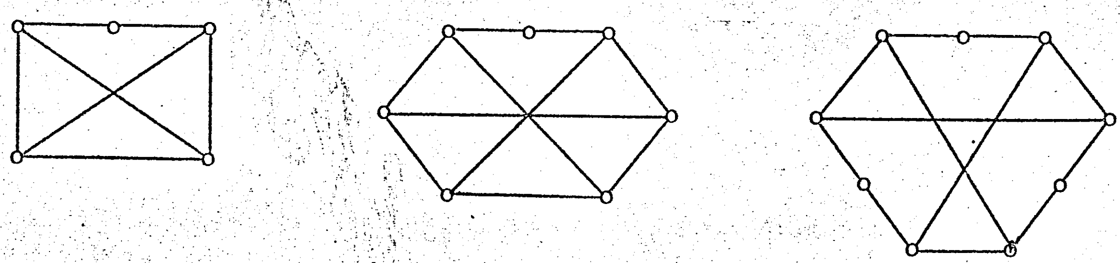


Figure 7.1

Note that, whereas the last graph is obtained from the Petersen graph by deleting a vertex, the first two are obtained from the (3,3)-cage and the (3,4)-cage respectively by inserting a vertex into one of the edges. In what follows, we shall use this technique to obtain upper bounds for $f(\rho,g)$ by exploiting the work that has already been done in determining the various (ρ,g)-cages.

We now determine $f(3,6)$. We are indebted to L.W. Beineke for the proof of the following theorem, since it considerably shortens our earlier proof of the same result.

7.6 Theorem

$$f(3,6) = 15.$$

Proof

The graph H in Figure 7.2 has $\rho = 3$ and $\chi_e = 4$. It has girth 6 since it is obtained from the (3,6)-cage, discovered by Heawood [14], by inserting a new vertex into one edge.

H:

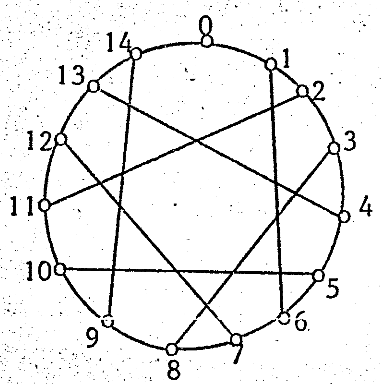


Figure 7.2

The circuits <0,1,2,3,4,...,12,13,14,0> and <0,1,6,7,12,13,4,5,10,11,2,3,8,9,14,0> are Hamiltonian and each edge of H lies in at least one of them. It follows that, for any edge e, the graph G − e is 3-colourable: We can now colour the remaining edges of a corresponding Hamiltonian circuit with two colours and the edges not on that circuit with a third. Thus, the graph H is critical and $f(3,6) \leq 15$.

Suppose now that $f(3,6) < 15$. Let H be a graph of order k attaining the minimum. From Table 2.17, it follows that $k \geq 11$. First assume k is even. As shown in Chapter 6, H must have exactly four vertices of valency 2, and must contain a 1-factor F. Furthermore, the distance between two vertices of valency 2 is at least three, and H − F must contain an odd circuit of length greater than 6. Thus, H − F must have at least three components, one of which is this odd circuit of order at least 7 and the other two of which are chains of order at least 4. This contradicts the fact that $k < 15$ and leaves only the cases $k = 11$ or 13.

From (5.5) it follows that H has either one or three vertices of valency 2. Furthermore, H has an odd circuit; let Z be such a circuit of shortest length. Clearly Z can have no chords nor can any vertex not on Z be adjacent to two vertices on Z, since Z is of minimal odd order. Hence, if Z has length at least 9, there are at least six other vertices, which is impossible. Therefore Z has length 7.

Since vertices of valency 2 cannot have a common neighbour, Z has at least five vertices of valency 3, and H must have order 13. Let A denote the set of vertices on Z, B those vertices not on Z but adjacent to Z, and C the remaining vertices, if any. Then A has order 7, with either one or two vertices of valency 2, and B has order 5 or 6 and at least four vertices of valency 3. But each of these vertices can be adjacent to at most one in A and one in B. Thus, there must be at least four edges joining a vertex in C to one in either A or B. Since C can have at most one vertex, this is a contradiction, i.e. $f(3,6) \geq 15$. //

We now prove the following lower bound for $f(3,7)$:

<u>7.7 Lemma</u>

$f(3,7) \;\geq\; 21.$

Proof

Let G be a 3-critical graph of girth 7 and of order $n \leq 20$. Let v be a vertex of valency 3 and let $A_i(v)$ be the set of vertices of G at distance i from v. Since the girth of G is 7, no element of $A_3(v)$ can be adjacent to two distinct elements of $A_2(v)$. Thus, $|A_1(v)| = 3$ and $|A_2(v)| \geq 5$, since the vertex v is adjacent to at least two vertices of valency 3, by Theorem 2.1. A similar argument applied to vertices of $A_1(v)$ shows that $|A_3(v)| \geq 8$. Thus $f(3,7) \geq 17$.

If $f(3,7) = 17$, then since 17 is not a multiple of 3, there is some vertex w of G which is adjacent to three vertices of valency 3. Hence, $|A_1(w)| = 3$, $|A_2(w)| = 6$, and $|A_3(w)| \geq 9$, which implies that $f(3,7) \geq 19$.

The same argument holds if $f(3,7) = 18$ and there is some vertex of valency 3 adjacent to three vertices of valency 3. So assume this is not the case, i.e. G contains exactly six vertices of valency 2 and twelve of valency 3. By Theorem 6.1, G contains a 1-factor F whose removal must result in an odd circuit C_t with $t \geq 7$ and three open chains each of order at least 4. But then G must have order at least 19.

Finally, we show that the order of G cannot be 19 or 20.

Case (i): Suppose $n(G) = 20$. By Theorem 6.1, G contains a 1-factor F and since G must have at least four vertices of valency 2, the graph $G' := G - F$ must consist of an odd circuit C_t $(t \geq 7)$ and at least two open chains each of order at least 4. If we denote an open chain of order j by P_j, and if G has six vertices of valency 2, then the graph $G'' := G' - C_t$ must be $P_4 + P_4 + P_5$ and t must be 7. In this case, there are three edges of F not incident with C_7. This implies that there are at least two adjacent vertices of G'' incident in F with C_7, which is inconsistent with the fact that the girth of G is 7. Hence, G has four vertices of valency 2 and so G'' has two components, with $t = 7, 9$, or 11.

If $t = 11$, then since no two vertices of C_{11} can be joined by a chord, we must have the order of G at least 22.

If $t = 9$, then there is exactly one edge of F not incident with C_9. This implies that G" contains a P_3 as subgraph all of whose vertices are joined in F to C_9. However, this reduces the girth of G.

If $t = 7$, then G" is one of the following: $P_4 + P_9$, $P_5 + P_8$, or $P_6 + P_7$ and there are exactly three edges of F not incident with C_7. In the first two cases, we always have two adjacent vertices of G" incident in F with C_7, which reduces the girth of G. In the third case, this can be avoided in a unique way, i.e. if the three edges cover the labelled vertices, as shown in Figure 7.3.

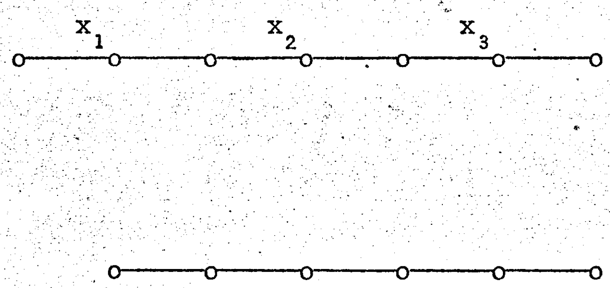


Figure 7.3

However, all remaining possibilities either reduce the girth of G or are inconsistent with properties of critical graphs.

Case (ii): Suppose $n(G) = 19$. If G has exactly one vertex x of valency 2, then by considering some vertex y of valency 3 at distance at least 3 from x, we obtain $|A_1(y)| = 3$, $|A_2(y)| = 6$, and $|A_3(y)| \geq 11$, which implies that $n(G) \geq 21$. Thus, the number of vertices of valency 2 in G is 3 or 5.

By Theorem 6.2, G has a matching M covering all vertices of G except one of valency 2. Then the graph $G' := G - M$ is of class 2, and consists of an odd circuit C_t $(t \geq 7)$ and one or two other components one of which is an open chain of order at least 4. Thus, t satisfies the inequalities: $7 \leq t \leq 15$.

It is to be noted, that for $t = 11 + 2k$ $(0 \leq k \leq 2)$, C_t can have at most 2k chords consistently with the girth of G being 7. Thus, for $0 \leq k \leq 2$,

$$n(G) \geq (11 + 2k) + ((10+2k) - 4k) = 21.$$

Hence, t = 7 or 9.

Suppose t = 9. Then note that if in any of the following configurations, all unlabelled vertices are joined in M to C_9, then the girth of G is reduced:

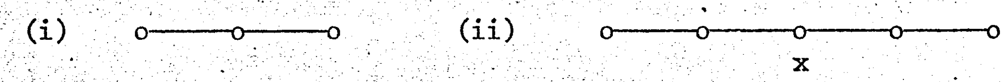


Figure 7.4

Now, $G'' := G' - C_9$ has ten vertices all of which except at most three are incident in M to C_9. Thus, G'' is one of the following: P_{10}, $P_6 + P_4$, or $P_5 + P_5$. Since some one of the configurations of Figure 7.4 must occur in each of these cases, t cannot be 9.

So finally we assume that t = 7. In this case, there cannot be two adjacent vertices of G'', which are both incident in M to C_7. (*)

Then $G'' := G' - C_7$ must be one of the following:

$$C_7 + P_5,\ C_8 + P_4,\ P_{12},\ P_8 + P_4,\ P_7 + P_5,\ P_6 + P_6.$$

The rest of the proof is dedicated to showing that each of these cases yields a contradiction.

There are six or seven edges from C_7 to G''. In all cases except $G'' = P_7 + P_5$, we get a contradiction to condition (*), if there are seven edges of this type. In the exceptional case, these adjacencies are uniquely determined consistently with the fact that the girth of G is 7. But then any other adjacency reduces the girth.

Thus, we can assume that there are six edges of the form $C_7 \times G''$, i.e. there are three edges of M with both end-vertices in G". In all cases except P_{12}, Theorem 2.1 forces each of these three edges of M not incident with C_7, to join one vertex in one component to a vertex in another component. In each of the cases $C_7 + P_5$, $C_8 + P_4$, and $P_8 + P_4$ we are left with at least two adjacent vertices in G" which are adjacent to vertices in C_7. This contradicts condition (*).

If G" is P_{12} or $P_6 + P_6$, then there is a unique way of fitting in the three edges of M so as to avoid a contradiction to condition (*) and to be consistent with properties of critical graphs. However, this reduces the girth of G.

Finally, if G" is $P_7 + P_5$, then there is a unique way of having the three edges of M incident with vertices in P_7 and not contradicting condition (*). However, all three adjacencies of the form $P_7 \times P_5$ which are consistent with properties of critical graphs reduce the girth of G.

This completes the proof. //

Now let us consider an upper bound for f(3,7). Consider the (3,7)-cage described by McGee [27]. This has order 24 and can be seen to be Hamiltonian in the representation of Figure 7.5. Note that <13,20,21,22,23,34,12,13> is a 7-circuit. Consider the graph G obtained from the McGee graph by inserting a vertex into the edge (13,14). By Theorem 2.13, G is vertex-critical. Thus, any 3-critical subgraph G' has the same order, 25.

Now, all edges of the Hamiltonian circuit of G are clearly essential. Moreover, by Theorem 2.1, the vertices 12 and 13 have valency 3 in G'. Thus, the removal of non-essential edges (if any)

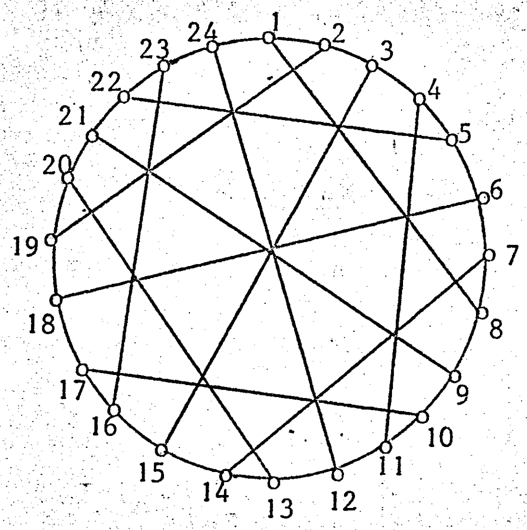


Figure 7.5

from G to yield G', leaves the 7-circuit <13,20,21,22,23,24,12,13> intact. We have therefore produced a 3-critical graph having girth 7 and order 25. This and the previous lemma together establish the following result:

7.8 Theorem

$$21 \le f(3,7) \le 25. //$$

An upper bound for f(4,5) can be similarly given. Consider the (4,5)-cage described by Robertson [34]. This is the following graph of order 19, which can be seen to be Hamiltonian in the representation of Figure 7.6.

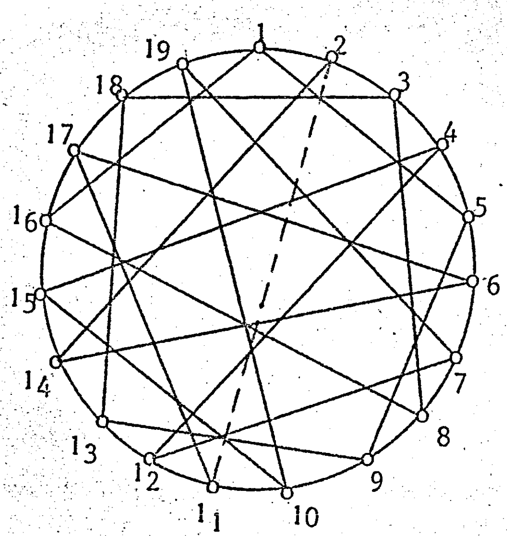


Figure 7.6

The graph G obtained from the Robertson graph by the deletion of the edge (2,11) is vertex-critical. This follows from Theorem 2.13 and the fact that G is obtained by taking a 19-circuit and the two disjoint sets, each of nine independent edges,

{(1,5),(2,14),(3,18),(4,12),(6,17),(7,19),(8,16),(9,13),(10,15)} and

{(1,16),(3,8),(4,15),(5,9),(6,14),(7,12),(10,19),(11,17),(13,18)}.

Now, let G' be any 4-critical subgraph of G. Then G' has order 19. Clearly, all edges of the Hamiltonian circuit are essential and hence belong to G'. Moreover, vertex 2 has valency at most 3 in G'. Hence, by Theorem 2.1, vertex 1 has valency at least 3 in G'. Thus, not both edges (1,5) and (1,16) are non-essential. This implies that either the circuit <1,2,3,4,5,1> or the circuit <1,16,17,18,19,1> is a 5-circuit in G'.

We can now prove the following result:

<u>7.9 Theorem</u>

$$15 \leq f(4,5) \leq 19.$$

<u>Proof</u>

The proof of the upper bound follows from the previous discussion. To prove the lower bound, let v be a vertex of valency 4, whose neighbours are w, x, y, and z. It follows from Theorem 2.1 that w and x, say, have valency 4, whereas y and z either have valencies 4 and 2 respectively, or both have valency at least 3. Moreover, w, x, y, and z can have no common neighbour except v, since the girth is 5. Thus, $f(4,5) \geq 15$. //

2) The circumference of critical graphs

As we mentioned in Chapter 2, Vizing [43] proved that if G is a ρ-critical graph, then G contains a circuit of length not less than $\rho + 1$. This means that $\rho + 1$ is a lower bound for the circumference of ρ-critical graphs. As with the lower bound for the number of edges obtained in the same paper, this estimate for the circumference seems to suffer from the same daficiency, i.e. no account is taken of the order of the graph. However, the analogy between the two estimates is not complete, as will be discussed in the next few pages.

In tackling the problem from a different angle, we notice that critical graphs need not be Hamiltonian. This was pointed out in Chapter 3, where we showed that the graph of Figure 3.9 is a non-Hamiltonian 3-critical graph of minimal order. The same argument can be used further to give an infinite family of ρ-critical graphs whose circumference does not exceed a certain number $t(n,\rho)$, which is a function of the order n and the maximum valency ρ of the graph.

To avoid repetition, we consider only the case when ρ is odd; the case when ρ is even is strictly analogous. Let G be the graph obtained from $K_{\rho+1}$ by inserting a vertex into any edge. (If ρ is even, then we use the graph obtained from $K_{\rho+2}$ by deleting a 1-factor). G has been shown to be ρ-critical in (2.8). Let H be the graph obtained from $K_{\rho+1}$ by deleting one edge and appending an end-edge at each of the two resulting vertices of valency $\rho - 1$. The graph K obtained from G and H by deleting an edge of G and replacing it by H is ρ-critical. In fact, it is a Hajós union of two copies of G, where the vertex of valency 2 in one copy is identified with a vertex of maximum valency in the other. We illustrate this for the case $\rho = 3$, in Figure 7.7. In what follows, we shall talk of ' replacing an edge of the given graph, G say, by another graph, H say ', when we refer to this construction.

We construct our infinite family of graphs $\{G_k\}$ recursively

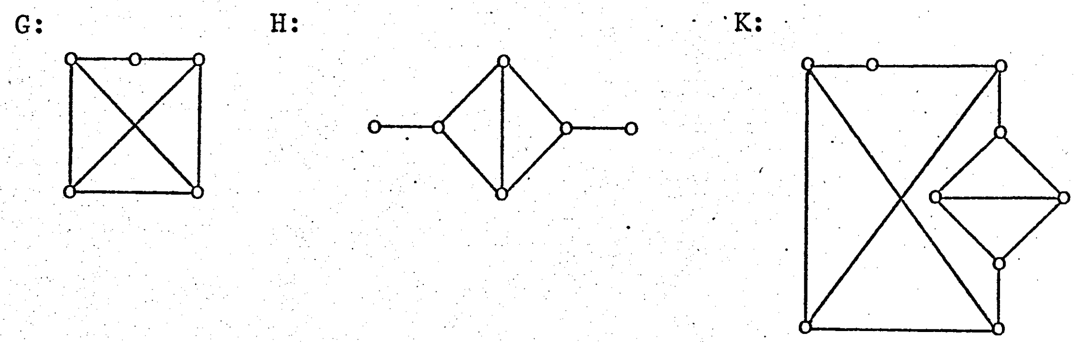


Figure 7.7

as follows: G_0 is the graph G just described. Let the vertex of valency 2 in G_0 be labelled v and let its neighbours (which have valency ρ) be labelled u and w. Note that u is not adjacent to w. Let T_u be the set of edges incident with u but not incident with v and let T_w be similarly defined. Thus there are $\rho - 1$ edges in each of T_u and T_w. We obtain G_1 by replacing each of the T_u and T_w edges by a copy of the graph H. Thus, if we denote the order and the circumference of G_k by n_k and c_k respectively, then

$$n_0 = (\rho + 2) = c_0$$

$$n_1 = n_0 + 2(\rho - 1)(\rho + 1)$$

$$c_1 \leq c_0 + 4(\rho + 1).$$

Now in G_1 there are $2(\rho - 1)$ copies of the graph H, each of which contains $2(\rho - 1)$ edges of type T_u. Thus, G_1 contains $2^2(\rho - 1)^2$ edges of this type. We obtain G_2 by replacing each of these $2^2(\rho - 1)^2$ edges by the graph H. This yields

$$n_2 = n_1 + 2^2(\rho - 1)^2(\rho + 1), \text{ and}$$

$$c_2 \leq c_1 + 2\ (\rho + 1).$$

Repeating this process, we obtain

$$n_{k-1} = 1 + (\rho + 1)(1 + t + t^2 + \ldots + t^{k-1}), \text{ where } t = 2(\rho - 1)$$

$$= 1 + \frac{(\rho + 1)(2^k(\rho - 1)^k - 1)}{(2\rho - 3)}$$

$$\geq 2^{k-1}(\rho - 1)^k.$$

Similarly,

$$\begin{aligned} c_{k-1} &\leq 1 + (\rho + 1)(1 + 2^2 + 2^3 + \ldots + 2^k) \\ &= 1 + (\rho + 1)(2^{k+1} - 3) \\ &\leq (\rho + 1)2^{k+1}. \end{aligned}$$

This implies that

$$\log (2n_k) \geq (k + 1) \log (2\rho - 2)$$

i.e.

$$k \leq q(n_k,\rho), \text{ where,}$$

$$q(n_k,\rho) = \frac{\log n_k - \log (\rho - 1)}{\log (2\rho - 2)}.$$

This establishes the following theorem:

7.10 Theorem

There exists an infinite family of ρ-critical graphs whose circumference c satisfies

$$c \leq 4(\rho + 1)2^{q(n,\rho)},$$

where $q(n,\rho)$ is as defined above. //

We now give an estimate for the circumference of ρ-critical graphs which does not depend solely on ρ. It was shown in (4.1) that critical graphs are 2-connected. Thus, given any pair of vertices v, w of a critical graph, there are two vertex-disjoint chains from v to w, by Menger's Theorem [29]. Hence, if the distance from v to w is $d(v,w)$, then G has a circuit of length at least $2d(v,w)$, implying the following result, which is true for any 2-connected graph:

7.11 Theorem

The circumference c of a critical graph with diameter d satisfies $c \geq 2d$. //

In view of this result, we look for lower bounds for the diameter of critical graphs, preferably in terms of n and ρ. One such bound can be given by the following standard method. Let v be a vertex of a ρ-critical graph G such that v is of minimum valency σ. The set V(G) can be partitioned into disjoint subsets $A_0, A_1, \ldots, A_t$, where A_i is the set of vertices of G at distance i from v. Thus, $A_0 = \{v\}$, $|A_1| = \sigma$, and $|A_j| \le \sigma(\rho - 1)^{j-1}$ for $2 \le j \le t$.

Thus,

$$n = \sum_{i=1}^{t} |A_i| \le 1 + \sigma(1 + s + s^2 + \ldots + s^{t-1})$$

$$= 1 + \frac{\sigma(s^t - 1)}{(s - 1)}, \text{ where } s = \rho - 1.$$

Hence,

$$\frac{1}{\sigma}(n - 1)(s - 1) + 1 \le s^t$$

i.e.

$$t \ge \left\{\frac{\log\{(n - 1)(\rho - 2) + \sigma\} - \log \sigma}{\log\{\rho - 1\}}\right\}.$$

Since the diameter $d \ge t$, we obtain the following corollary to (7.11):

<u>7.12 Corollary</u>

The circumference of a ρ-critical graph whose minimum valency is σ is at least

$$2\left\{\frac{\log\{(n - 1)(\rho - 2) + \sigma\} - \log \sigma}{\log\{\rho - 1\}}\right\}. \quad //$$

Already, this gives a better estimate than that of Theorem 2.28. So, for example, if we take ρ to be 3, Theorem 2.28 gives 4 as a lower bound, which is smaller than the estimate of Corollary 7.12 for all 3-critical graphs of order at least 9. In general, this estimate is better than Vizing's if $n \sim \rho^\rho$. However, there is still room for improvement. We think that a bound of the order $\rho.2^x$, where $x = \frac{\log (2n)}{\log (2\rho)}$, is nearer the truth than the one given.

APPENDIX

The following programme, written jointly with Galea and Buttiġieġ, produces t-critical graphs ($t \geq 4$) of some order n from 3-critical graphs of the same order by the algorithm of Theorem 2.11.

To illustrate the use of the programme, we have limited ourselves to graphs of order 7, thereby giving the opportunity to compare the graphs thus obtained with those obtained in Table 6.6 by other methods.

We do not claim to have produced all t-critical graphs with $t \geq 4$ which can possibly be generated by the algorithm of Theorem 2.11. To do this, we would have had to input all labelled isomorphic images of the four 3-critical graphs of Figure A.1.

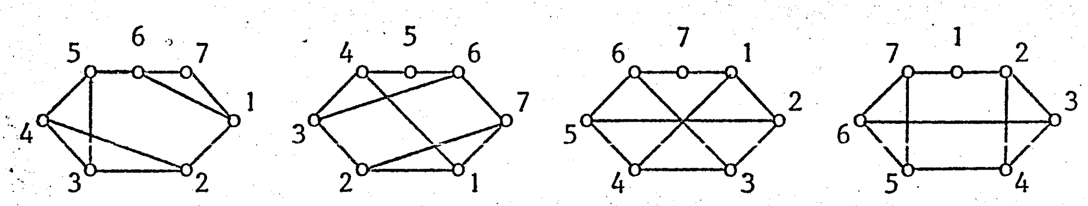


Figure A.1

We have limited ourselves to inputting the 28 graphs obtained by rotating each of the four graphs of Figure A.1 as labelled there. Thus, for example, the graph obtained by inserting a vertex into an edge of K_6 is glaringly absent. This was shown to be 5-critical in Theorem 2.8. It can be produced by the algorithm of Theorem 2.11, for example, by taking the union of the two graphs of Figure A.2. However,

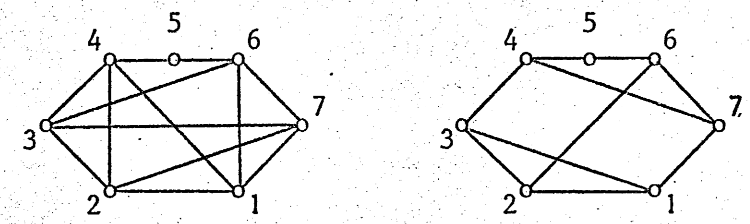


Figure A.2

the second graph in this figure is a reflection of the second graph of Figure A.1 but <u>not</u> a rotation of it.

Similarly, we can obtain all critical graphs of order 7 by the algorithm of Theorem 2.11. This can be verified from Table 6.6.

When dealing with the production of graphs by computer or otherwise, one has to face the difficult problem of weeding out isomorphic copies of the same graph. To do this, we make use of the algorithm of Corneil & Gotlieb [7], which we split into four subroutines: ISO1, ISO2, REFINE, and LEXORD.

Another feature of the programme is that space has been economized by working with a (1×14)-matrix instead of the usual (7×7)-adjacency matrix to represent the graph. Thus, for example, the second graph of Figure A.1 is represented in vector form by $\langle 0,1,0,0,0,0,0,1,0,1,0,0,0,0 \rangle$, where adjacencies $(7,1)$ and $(i,i+1)$ $(i = 1,2,\ldots,6)$ have been ignored since the 7-circuit is common to all these graphs. The remaining adjacencies are then labelled as follows:

$$\langle (1,3),(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(2,7),(3,5),(3,6),(3,7),(4,6),(4,7),(5,7) \rangle \quad \rightarrow \quad \langle a_1,a_2,a_3,\ldots,a_{14} \rangle,$$

exploiting the fact that the adjacency matrix of a graph is symmetric.

The matrix representation of a graph is only resorted to when the isomorphism subroutine is called.

The programme, which now follows, is written in FORTRAN.

```
      PROGRAMME (MFOR)
      DIMENSION NG(10,7,7),G(80,14),N(4),B(14),IB1(7,7)
      DIMENSION IB2(7,7),IQ(7,7),IND(7),IG(7,7)
      INTEGER G,Y,B
      DO 222 I = 1,22,7
      DO 222 J = 1,14
      G(I,J) = 0
222   CONTINUE
      G(1,2) = 1
      G(1,8) = 1
      G(1,10) = 1
      G(8,5) = 1
      G(8,10) = 1
      G(8,14) = 1
      G(15,2) = 1
      G(15,6) = 1
      G(15,10) = 1
      G(22,4) = 1
      G(22,5) = 1
      G(22,9) = 1
C     THE ORIGINAL SET CONSISTS OF THE ABOVE FOUR GRAPHS
C     THE 28 ROTATIONS OF THIS SET WILL BE OBTAINED BY DO
C     LOOP 170
      DO 170 K = 1,22,7
```

```
      L1 = K + 1
      L2 = K + 6
      DO 170 I = L1,L2
      J = I - 1
      Y = G(J,14)
      G(I,14) = G(J,12)
      G(I,12) = G(J,9)
      G(I,9) = G(J,5)
      G(I,5) = G(J,1)
      G(I,1) = G(J,8)
      G(I,8) = G(J,4)
      G(I,4) = Y
      Y = G(J,2)
      G(I,2) = G(J,11)
      G(I,11) = G(J,7)
      G(I,7) = G(J,3)
      G(I,3) = G(J,13)
      G(I,13) = G(J,10)
      G(I,10) = G(J,6)
      G(I,6) = Y
170   CONTINUE
1000  FORMAT(1H1,20X,17H,GRAPHS.OF.VALENCY,I 3 ///)
1001  FORMAT(8X,4I4 /)
1002  FORMAT(12X,4I4 /)
1003  FORMAT(16X,3I4 /)
1004  FORMAT(20X,2I4 /)
1005  FORMAT(24X,I4 ////)
C     IB1 IS THE GRAPH UNDER CONSIDERATION IN MATRIX FORM
      DO 65 J = 1,7
65    IB1(J,J) = 0
```

```
      DO 66 J = 1,6
   66 IB1(J,J+1) = 1
      DO 67 J = 2,7
   67 IB1(J,J-1) = 1
C   THE ABOVE ARE INVARIANT ELEMENTS OF IB1
      N(1) = 28
      M = 0
      I = 1
  300 I = I + 1
C   GRAPHS OF VALENCY I WILL BE GENERATED
      IF (I.EQ.5) GO TO 305
      IF (I - 3) 880,881,882
  880 KS = 1
      KT = 1
      I1 = 0
      I2 = 0
      GO TO 883
  881 KS = 1
      KT = 2
      I1 = 28
      I2 = 0
      GO TO 883
  882 KS = 2
      KT = 2
      I1 = 28
      I2 = 28
  883 WRITE (2,1000) I
C   N(I) IS THE CURRENT NUMBER OF GRAPHS OF VALENCY I
      N(I) = 0
      M = M + N(I - 1)
```

```
      C   M IS THE NUMBER OF GRAPHS OBTAINED SO FAR.  THESE
      C   INCLUDE THE ORIGINAL 28 ROTATIONS BUT EXCLUDE
      C   THE N(I)
          JJJ = N(KT)
          DO 250 IZ = 1,JJJ
      C   DO LOOPS 250 AND 302 TAKE A GRAPH OF VALENCY KT AND
      C   SUPERIMPOSE IT ON ONE OF VALENCY KS
      C   LL IS THE LABEL OF THIS GRAPH IN THE SET G
          LL = I1 + IZ
          JKJ = N(KS)
          DO 302 LS = 1,JKJ
          IF (I - 3) 886,884,887
  886     MAX = N(1)
          GO TO 888
  887     MAX = N(1) + N(2)
  888     MM = LL + LS
          IF(MM.GT.MAX) GO TO 250
          GO TO 885
  884     MM = I2 + LS
  885     DO 199 K = 1,14
  C   DO LOOP 199 LOOKS FOR COMMON EDGES
          B(K) = G(LL,K) + G(MM,K)
          IF (B(K).EQ.2) GO TO 302
  199     CONTINUE
          DO 68 J = 1,4
          IB1(1,J + 2) = B(J)
  68      IB1(J + 2,1) = B(J)
          DO 69 J = 5,8
          IB1(2,J - 1) = B(J)
  69      IB1(J - 1,2) = B(J)
```

```
      DO 70 J = 9,11
      IB1(3,J - 4) = B(J)
   70 IB1(J - 4,3) = B(J)
      IB1(4,6) = B(12)
      IB1(6,4) = B(12)
      IB1(4,7) = B(13)
      IB1(7,4) = B(13)
      IB1(5,7) = B(14)
      IB1(7,5) = B(14)
      IB1(1,7) = 1
      IB1(7,1) = 1
      KK = 7
      KKK = 7
  333 FORMAT (//,15H AN IB1 FOLLOWS,//,(7I2,10X,7I2))
C   DO LOOP 770 LOOKS FOR ISOMORPHIC GRAPHS
      CALL ISO1(IB1,IB2,IND,KK,KKK)
      IF(N(I).EQ.0) GO TO 304
      KJK = N(I)
      DO 770 L3 = 1,KJK
      DO 339 L1 = 1,7
      DO 339 L2 = 1,KKK
      IF(NG(L3,L1,L2) - IB2(L1,L2)) 770,339,770
  339 CONTINUE
      GO TO 302
  770 CONTINUE
  304 N(I) = N(I) + 1
      M1 = M + N(I)
      DO 370 K = 1,14
  370 G(M1,K) = B(K)
      WRITE (2,1001) (G(M1,K),K = 1,4)
```

```
           WRITE (2,1002) (G(M1,K), K = 5,8)
           WRITE (2,1003) (G(M1,K), K = 9,11)
           WRITE (2,1004)  G(M1,12),G(M1,13)
           WRITE (2,1005)  G(M1,14)
           DO 371 I6 = 1,7
           DO 371 I7 = 1,7
           IJK = N(I)
      371  NG(IJK,I6,I7) = IB2(I6,I7)
      C   DO LOOP 371 STORES THE IQ OF THE GRAPH OBTAINED
      302  CONTINUE
      250  CONTINUE
           IF(N(I).EQ.0) GO TO 305
           GO TO 300
      305  STOP
           END
```

END OF MAIN PROGRAMME

SUBROUTINE ISO1

```
           SUBROUTINE ISO1(IG,IQ,IND,N,JS)
           DIMENSION IND(N),IG(N,N),IQ(N,N),IND1(100)
      C   THE SUBROUTINE PRODUCES THE MATRIX IQ, GIVEN THE MATRIX
      C   IG.  ON EXIT THERE WILL BE A CORRESPONDENCE J TO IND(J)
      C   REPRESENTING THE CORRESPONDENCE BETWEEN THE VERTICES OF
      C   IG AND THOSE OF IQ.  IND1(J) SHOWS THE SIZE OF THE J'TH
      C   CELL
           K = 1
           DO 51 J = 1,N
           ISUM = 0
```

```
C   DO LOOP 52 FORMS ROW SUMS OF IG
      DO 52 I = 1,N
52    ISUM = ISUM + IG(J,I)
51    IQ(J,I) = ISUM
      MAXVAL = 0
      N1 = N + 1
C   DO LOOP 53 FINDS THE MAXIMUM VALUE OF THE ROW SUMS, AND
C   SETS IT EQUAL TO M
      DO 53 I = 1,N
      IT = IQ(I,1)
      IF(IT.LE.MAXVAL) GO TO 53
      MAXVAL = IT
53    CONTINUE
      M = MAXVAL
      ICOUNT = 0
      JJ = 0
C   SET IND, IND1 AND FIND JMAX FOR THE FIRST TIME
      DO 54 J = 1,N
      DO 55 I = 1,N
      IF(IQ(I,1).NE.M) GO TO 55
      IND(K) = I
      K = K + 1
      ICOUNT = ICOUNT + 1
55    CONTINUE
      M = M - 1
      IF(ICOUNT.EQ.0) GO TO 250
      JJ = JJ + 1
      IND1(JJ) = ICOUNT
250   IF(K.EQ.N1) GO TO 200
54    ICOUNT = 0
```

```
200  JMAX = JJ
301  CALL ISO2(IG,IQ,IND,IND1,N,JMAX,JS)
C    IF THIS CALL OF ISO2 HAS FAILED TO PRODUCE A REFINEMENT,
C    THEN ONE IS ARTIFICIALLY PRODUCED IN THE REST OF THIS
C    SUBROUTINE, BY SPLITTING THE FIRST CELL WHICH HAS SIZE
C    LARGER THAN ONE, AND THEN RE-LABELLING APPROPRIATELY
     IF(JMAX.EQ.N) GO TO 300
     DO 150 I = 1,JMAX
     IF(IND1(I).EQ.1) GO TO 150
     K = I
     GO TO 151
150  CONTINUE
151  ITEM = IND(K)
     IF(K.EQ.1) GO TO 400
     KK = K - 1
     DO 152 I = 1,KK
     KL = K - I + 1
152  IND(KL) = IND(KL - 1)
     IND(1) = ITEM
400  DO 153 I = K,JMAX
     KL = JMAX - I + K + 1
153  IND1(KL) = IND1(KL - 1)
     IND1(K + 1) = IND1(K) - 1
     IND1(K) = 1
     JMAX = JMAX + 1
     GO TO 301
300  RETURN
     END
```

END OF SUBROUTINE ISO1

SUBROUTINE ISO2

```
      SUBROUTINE ISO2(IG,IQ,IND,IND1,N,JMAX,JS)
      DIMENSION IG(N,N),IQ(N,N),IND(N),IND1(N)
      DIMENSION IND2(100),IND3(100),ISUP(100),ISIND2(100)
      DIMENSION IND4(100)
C     THIS SUBROUTINE REFINES AND REORDERS LEXICOGRAPHICALLY
C     UNTIL THE FIRST FAILURE OF FURTHER REFINEMENT IS
C     DETECTED
  201 CALL REFINE(IG,IQ,IND,IND1,N,JMAX)
      DO 500 I = 1,N
  500 IND4(I) = IND(I)
      CALL LEXORD(IQ,IND3,IND1,IND2,ISUP,ISIND2,N,JMAX,JS,1)
      DO 501 I = 1,N
      II = IND3(I)
  501 IND(I) = IND4(II)
      IF(JS.EQ.JMAX) GO TO 300
      JMAX = JS
      GO TO 201
  300 RETURN
      END
```

END OF SUBROUTINE ISO2

SUBROUTINE REFINE

```
      SUBROUTINE REFINE(IG,IQ,IND,IND1,N,JMAX)
      DIMENSION IG(N,N),IQ(N,N),IND(N),IND1(N)
      DO 70 I = 1,JMAX
      DO 70 J = 1,N
```

```
   70 IQ(J,I) = 0
      DO 67 J = 1,N
      K = IND(J)
      DO 60 I = 1,N
      IF(IG(K,I).EQ.0) GO TO 60
C     LOOP 61 FINDS THE VALUE OF IL SUCH THAT IND(IL) = I, AND
C     SETS ITEM = TO THAT VALUE, IFND IS THE NUMBER OF THE CELL
C     WHICH CONTAINS THE NUMBER IL
      DO 61 IL = 1,N
      IF(IND(IL).NE.I) GO TO 61
      ITEM = IL
      GO TO 62
   61 CONTINUE
   62 M = 0
      DO 65 L = 1,JMAX
      M = M + IND1(L)
      ITEST = M - ITEM
      IF(ITEST.LT.0) GO TO 65
      IFND = L
      GO TO 66
   65 CONTINUE
   66 IQ(J,IFND) = IQ(J,IFND) + 1
   60 CONTINUE
   67 CONTINUE
      RETURN
      END
```

END OF SUBROUTINE REFINE

SUBROUTINE LEXORD

```
      SUBROUTINE LEXORD(IA,INDEX,IB,IAA,IB1,INDEX1,N,IP,
                                              NOUT,INDIC)
      DIMENSION IA(N,IP),IB(N),INDEX(N),IAA(N),IB1(N),
                                              INDEX1(N)
C     THIS IS A GENERAL SUBROUTINE.  GIVEN A MATRIX IA
C     CONSISTING OF N ROW-VECTORS OF LENGTH IP, THEN ON EXIT
C     INDEX(G) WILL CONTAIN THE NUMBER OF THE ROW OF IA WHICH
C     CONTAINS THE J'TH LARGEST VECTOR.  IB IS AN INPUT VECTOR
C     SUCH THAT IB(J) CONTAINS THE SIZE OF THE J'TH CELL (IF
C     IA IS IMPLICITLY SPLIT INTO CELLS).  IF INDIC = 1, THEN
C     IT IS UNDERSTOOD THAT IA IS ALREADY SPLIT INTO CELLS,
C     AND THAT LEXICOGRAPHICAL ORDERING IS TO TAKE PLACE ONLY
C     WITHIN EACH CELL.  IT IS ASSUMED THAT EACH ENTRY OF IA
C     IS NON-NEGATIVE
      IF(INDIC.EQ.1) GO TO 200
      IB(1) = N
      L = 1
      GO TO 201
  200 L = IP
  201 DO 6 I = 1,N
      INDEX1(I) = I
      INDEX(I) = I
C     LOOP 10 LOOKS AT EACH OF THE IP CELLS
      DO 10 J = 1,IP
      L1 = 1
      K = 1
      DO 30 M = 1, L
      IF(M.EQ.1) GO TO 100
```

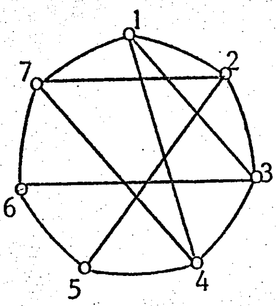
```
      LL = LL + IB(M - 1)
      LR = LR + IB(M)
      GO TO 101
100   LL = 1
      LR = IB(1)
101   IF(LR - LL) 102,31,102
31    IB1(L1) = 1
      L1 = L1 + 1
      K = K + 1
      GO TO 30
102   DO 9 I = LL,LR
      II = INDEX(I)
9     IAA(I) = IA(II,J)
50    ICOUNT = 0
      ISUM = - 1
      DO 11 I = LL,LR
      IT = IAA(I)
      IF(IT.GT.ISUM) ISUM = IT
11    CONTINUE
      IF(ISUM.EQ.-1) GO TO 30
      DO 12 I = LL,LR
      IF(IAA(I).NE.ISUM) GO TO 12
      INDEX1(K) = INDEX(I)
      K = K + 1
      IAA(I) = -1
      ICOUNT = ICOUNT + 1
12    CONTINUE
      IB1(L1) = ICOUNT
      L1 = L1 + 1
      GO TO 50
```

```
30    CONTINUE
      L = L1 - 1
      DO 17 I = 1,L
17    IB(I) = IB1(I)
      DO 18 I = 1,N
18    INDEX(I) = INDEX1(I)
      IF(L.EQ.N) GO TO 56
10    CONTINUE
56    NOUT = L
      RETURN
      END
```

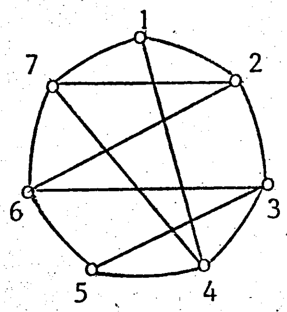
END OF SUBROUTINE LEXORD

GRAPHS OF MAXIMUM VALENCY 4

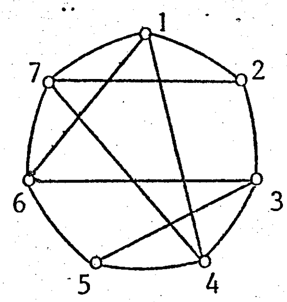
$$\begin{array}{ccccc} 1&1&0&0&0\\ &0&1&0&1\\ &&0&1&0\\ &&&0&1\\ &&&&0 \end{array}$$



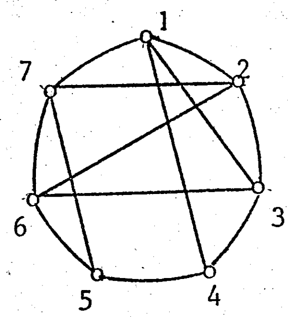
$$\begin{array}{ccccc} 0&1&0&0&0\\ &0&0&1&1\\ &&1&1&0\\ &&&0&1\\ &&&&0 \end{array}$$



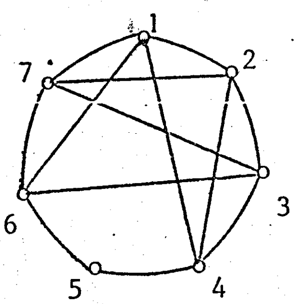
$$\begin{array}{ccccc} 0&1&0&1&0\\ &0&0&0&1\\ &&1&1&0\\ &&&0&1\\ &&&&0 \end{array}$$



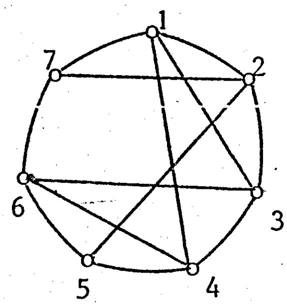
$$\begin{array}{ccccc} 1&1&0&0&0\\ &0&0&1&1\\ &&0&1&0\\ &&&0&0\\ &&&&1 \end{array}$$



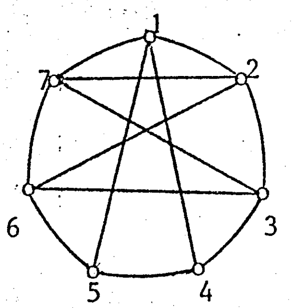
$$\begin{array}{ccccc} 0&1&0&1&0\\ &1&0&0&1\\ &&0&1&1\\ &&&0&0\\ &&&&0 \end{array}$$



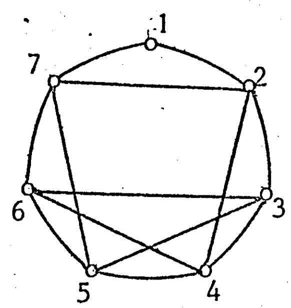
$$\begin{array}{ccccc} 1&1&0&0&0\\ &0&1&0&1\\ &&0&1&0\\ &&&1&0\\ &&&&0 \end{array}$$



$$\begin{array}{ccccc} 0&1&1&0&0\\ &0&0&1&1\\ &&0&1&1\\ &&&0&0\\ &&&&0 \end{array}$$

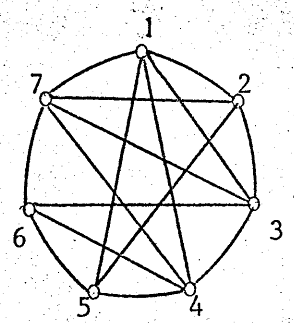


$$\begin{array}{ccccc} 0&0&0&0&0\\ &1&0&0&1\\ &&1&1&0\\ &&&1&0\\ &&&&0 \end{array}$$

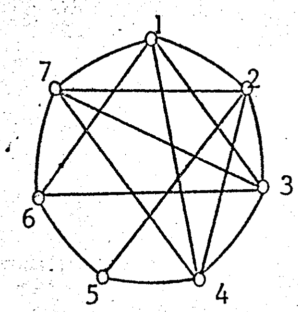


GRAPHS OF MAXIMUM VALENCY 5

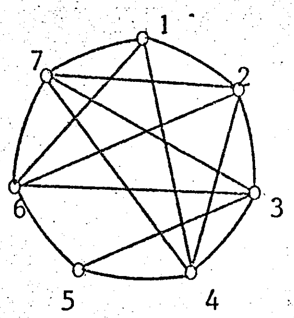
$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 1 \\ & & 0 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 0 \end{array}$$



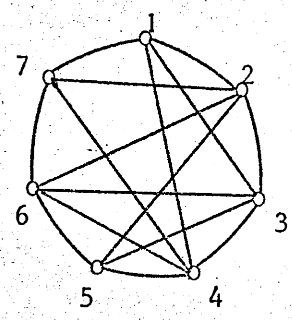
$$\begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 1 \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{array}$$



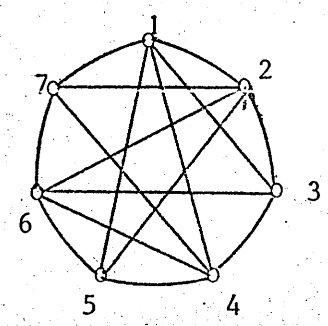
$$\begin{array}{ccccc} 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{array}$$



$$\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 1 \\ & & 1 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 0 \end{array}$$



$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 \\ & & 0 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 0 \end{array}$$



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