Aspects of equivalence relations in the school curriculum and the development of the concept in young children

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ASPECTS OF EQUIVALENCE RELATIONS IN THE SCHOOL CURRICULUM AND THE DEVELOPMENT OF THE CONCEPT IN YOUNG CHILDREN.

BY

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ABSTRACT

This thesis considers some aspects of equivalence relations, especially in areas outside mathematics and in the development of children's thinking.

The aim of Section 1 is to show that equivalence classes (and by implication equivalence relations) are an essential mode of thinking for adult English speakers in a variety of activities.

As children have their own patterns of thinking which are developing toward adult form, Section 2 is devoted to establishing a framework within which observations about the development of the concept of equivalence relation can be organised.

The relevant factors of Piaget's work are taken as the starting point. These are reviewed alongside more recent American studies. Some recent reformulations of Piaget's theory of groupings by German writers are also considered.

This review identifies difficulties arising from

(a) diversity of interpretation of (i) Piaget's work
(ii) terminology used,

(b) gaps between the psychological models and the behavioural counterparts which they were designed to represent,

(c) lack of agreed criterion for concept attainment.

Points arising from (a) and (c) have been considered in greater detail in the context of

- the identification and modification of points of weakness in the hypothesis that seriation implies transitivity,

- an attempt to specify the characteristics of a test of conservation of a quantitative relation.

The review also shows gaps in the research, notably, in the study of the growth of the understanding of symmetric relations; proposals for further tests to clarify the stages in the development of the concept of symmetry are put forward. The feasibility of these tests has been studied in the classroom.

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- to partition (i.e. to separate the large number of 'objects' in the collection into subsets such that each 'object' is in one and only one subset. →

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- Music
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- Chemistry
- Linguistics

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(1) explaining how mental operations basic to mathematical thought develop,

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(3) forming a theoretical basis for certain curricular decisions and experiments in the learning of mathematics.

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- the preliminary grouping of equalities
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Outline for Section 2 (contd)

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↓

Outline for Section 2 (contd)

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2. If the concept of transitivity is broken down into components, is it the case that in every physical context these components are acquired in an invariant order?

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← use of 'bigger therefore heavier' type arguments resulted?"

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Outline for Section 2 (contd)

(b)

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Outline for Section 2 (contd)

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Outline for Section 2 (contd)

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(b) the relationships between the use of properties of the relations including conservation. → See Section 2.8.

(c) the effects of the training used. → Summary of the effects of training.

Outline for Section 2 (contd)

2.8. Relationships between the use of properties of relations

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Steffe & Carey

- use of the reflexive and antireflexive properties is not a necessary or sufficient condition for use of transitivity of length relations.
- conservation of length relations is necessary for transitivity.

D.T. Owens

- No indication that conservation of matching relations precedes conservation of length relations.
- No evidence that the ability to conserve relations precedes the ability to use the transitive property.

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(ii) Various interpretations and uses of the word "conservation"

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→ (iii)

Outline for Section 2 (contd)

(iii) An attempt to specify the characteristics of a test of conservation of a quantitative relation based on Steffe and Carey's interpretation →

Consequences of this definition, with respect to

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(b) interpretation of "conservation of identity". →

The resulting confusion between a test of "conservation of identity" and a test of the "reflexive and anti-reflexive properties".

(iv) An analysis of points requiring consideration if the definition of conservation of identity is to be derived from the general case by putting x = y.

An attempt to resolve the above confusion by identifying the following sources of error.

(a) failure to recognize that the definitions of reflexivity and antireflexivity imply at least two levels of application.
(b) the application of quantity-preserving transformation(s) is not a design-feature for a test of reflexivity or antireflexivity, whereas the introduction of quantity-preserving transformation(s) is necessary to test conservation. →

(v) An attempt to clarify the main issue raised in Section 2.8 (i)

An attempt at identification of

↓

similarities in interpretation and use of the word "conservation" by D.T. Owens and Steffe and Carey.

Outcome:
Insufficient evidence on which to base Owen's interpretation.
Hence, confirmation or rejection of the hypothesis
"conservation of a set of quantitative relations is a necessary condition for restricted-transitivity of the same set of quantitative relations",
fails because of lack of clarity in interpretation and use of the terminology.

2.9. Partition - its role in the development of the concept of equivalence relation

The relevance of investigating the development of the ability to partition a given set and other associated classificatory skills of young children.

M.L. Johnson's summary of the main findings of a number of studies on the classificatory behaviour of young children.

The concern of M.L. Johnson over the lack of information on the relationship which may exist between the child's knowledge of the mathematical properties of an equivalence relation and his classification skills based on that relation.

The subsequent inclusion of the objective
"to determine if the subject's ability to use the transitive property of the equivalence relation 'same length as'

was related to his ability to classify on the basis of this relation"
in the study undertaken by M.L. Johnson.

The contribution of the study by

M.L. Johnson - The hypothesis of a relationship between the child's classification ability and his ability to use the transitive property of the equivalence relation 'same length as' was not confirmed.

Discussion of the possibility that the behavioural counterpart of the ability to recognize distinct pairs (x, y) where $x, y \in A$ such that xRy is ALL the child needs to successfully partition a set of concrete materials A, into equivalence classes.

2.10. Concluding remarks

SECTION 1

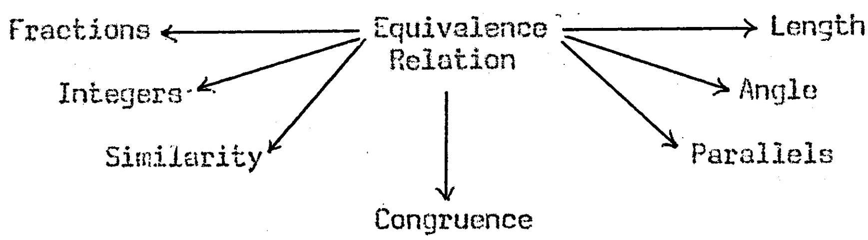
1.1. The role of the concept of equivalence relation in mathematics

"The concept of equivalence relation is basic to many ideas in mathematics. It enables us to find common ground for many topics and is easily illustrated by concrete examples."

(M. Bruckheimer and N. Gowar: "Equivalence relations and Compatibility", Mathematics Teaching, No. 34, Spring 1966, p. 60)

Part of the evidence which supports the above quotation is summarized by the following diagram:

A selection of topics from a "traditional" O-Level Mathematics Syllabus which are underpinned by an equivalence relation.



At least two of the topics in the diagram present difficulties for pupils, both conceptually and manipulationally, which perceptive teachers have appreciated for a long time. These topics are fractions and integers, where the difficulties centre particularly around addition and multiplication, respectively. Any methods which offer possibilities of improving the teaching of these topics merit further study. The construction of integers as equivalence classes of ordered pairs of natural numbers and the explicit recognition of the logical status of rational numbers as equivalence classes of ordered pairs of integers (or natural numbers if only the positive rationals are defined) goes back at least to Landau's classic exposition (1930) (1); but these structures were for a long time appreciated only by relatively advanced mathematicians who were interested in the foundations of the subject.

introducing them as part of the systematic line of development in school mathematics, or at least regarding them as essential background knowledge for teachers, only became part of current thinking in the early '60s. See, for example, Mansfield and Bruckheimer (1965) (2), and the School Mathematics Project (1965) (3).

An essential step in the development of mathematics along these lines is the careful definition of certain equivalence classes and the performance of operations on the classes as a whole, regarding them as new individual entities. These same ideas have applications elsewhere, later on in mathematics - for example, in the teaching of vectors and in more advanced topics such as operational calculus and topology. This means that this new material was proposed not only as an isolated innovation but, in part, to improve the teaching in long-standing areas of difficulty. It was an important function of the "new" to make the "old" more intelligible.

Readers wishing to follow up a discussion of how three major steps in mathematical education - the introduction of "fractions", the introduction of "directed numbers" and the introduction of free vectors - need not be regarded as three problems provided that they are seen from a suitably abstract algebraic (i.e. equivalence relation) point of view, are referred to two articles by B. Fletcher (1970) (4), and by T.J. Fletcher (1970) (5).

Further examples of how some of those devising new curricula considered that more explicit recognition of equivalence relations could assist the teaching of some other traditional topics are provided by Skemp's treatment of lengths of line segments (1965) (6) and Choquet's treatment of angle (1969) (7). Very many mathematical entities <u>can</u> be regarded as equivalence classes - although to advocate teaching them from this point of view at present might be neither natural nor expedient.

However, it is not the purpose of the present paper to justify our initial quotation. The point at issue is: can a similar statement be made about this concept with respect to subjects other than mathematics? In other words, are we justified in stating that

> "The concept of equivalence relation is basic not only in mathematics but also in other parts of the curriculum?"

1.2. The role of the concept of equivalence relation in subject areas other than mathematics

It appears that the answer to the question posed at the end of the previous sub-section is "yes". We begin our justification of this by considering words such as

congress, fleet, pride, library, herd, audience.

We see that each word in this list is, or can be, used to refer to a collection of objects, for example

library - a collection of books for study or reading.

When used in this context, these words are described as collective nouns. They are a fundamental part of the classificatory system on which the linguistic structure of English and other European languages is based.

But whenever large numbers of objects are being classified, simple classifications are not usually enough. Frequently, further sub-classifications are employed. For example, on returning to our library example, we note that the Dewey classification system has been developed to provide a sorting process whereby the books in the library are allocated to a particular set of shelves. This system divides the collection of books in the library into subsets according to the subject matter of the book and specifies their location to the potential borrower.

Similar systems of classification and sub-classification can be found in almost any area of human activity. But whichever sub-classification is being used in a given context, an attempt is being made to separate the large number of "objects" in the collection into subsets such that each "object" of the original collection is in one and only one subset. In other words, an attempt is being made to partition the original collection. Moreover, if no ambiguities arise in the sorting process under consideration, then, as we shall see later, the resulting partition defines an equivalence relation and every equivalence relation defines a partition.

This separation into equivalence classes by an equivalence relation is important in mathematics because the classes are used to build up further logical systems. Thus, at one end of the spectrum we have the precise classifications of mathematics in contrast with the less precise classifications of everyday speech. For in spite of the fact that a large number of classifications used in everyday speech appear to be precise, closer examination reveals weaknesses and exceptions which would lead to considerable difficulties should these classifications be handled by the methods of mathematical logic. Breakdown frequently occurs because the sets under discussion appear to be well-defined when closer examination shows that they are not.

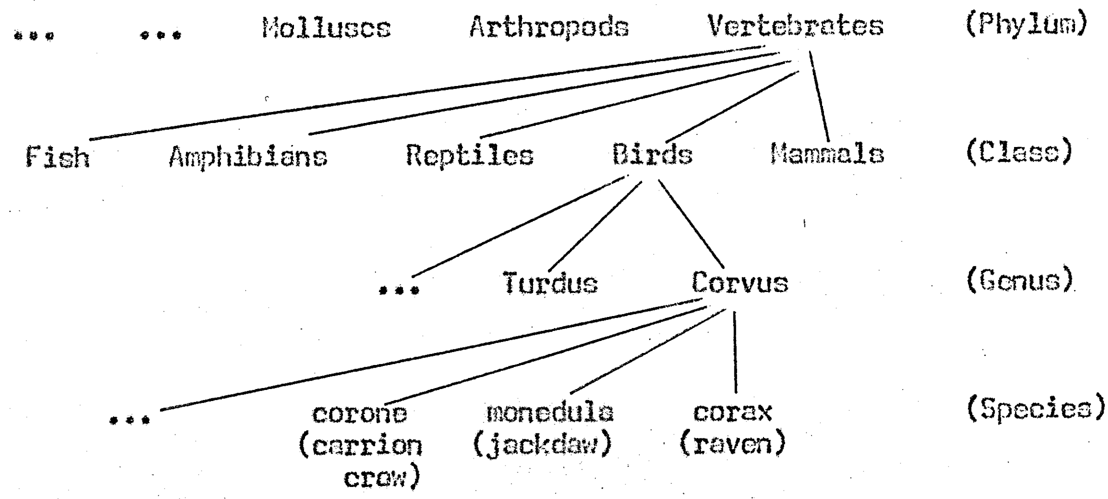
For example, at first sight it would appear that the human race is partitioned into two equivalence classes by sex; but until we have legislated for transvestites and pathological borderline cases, the sets are not well-defined. Even then, as the terms are normally used, over a period of time certain rare individuals transfer from one class to another.

Nationality appears to be a partitioning of living members of the human race into equivalence classes. Closer examination shows that these classes are not well-defined. The laws of nationality differ from country to country, some people are stateless and at any particular moment the status of some individuals may not be properly defined.

In everyday speech the attempt is often made to classify people by race, but again closer examination shows that these divisions can be very ill-defined indeed. However, the concept of race in the animal kingdom is more precisely defined in Biology. In fact, modern Biology depends upon systems of classification which are, ideally, precise and go back to Linnaeus and beyond. But prior to 1735, confusion had arisen amongst biologists because the same name had been used for different plants (and animals) and different names had been given to the same plant (or animal), and so Linnaeus introduced a system of naming animals and plants which uses two words:

- The first word in the name of every animal and plant is the Latin name of the genus to which it belongs. (This defines its closest relationship with other species, e.g. Felis - cats and mammals like them.)
- The second word in the name is the Latin name of the species. (A species is roughly a group of individuals able to breed among themselves if one disregards geographical separation, but not to breed with organisms of other groups.)

Many other biological systems of classification depend much on similar sequences of sub-divisions which lead to a diagram like an inverted tree. For example, consider the extension of our previous example:



Comparatively little use seems to be made of this type of classification in mathematics. But one of the most eminent developmental psychologists, Jean Piaget, pursued his early studies as a biologist. Consequently, the mathematical models of thinking which we find in his theories of cognitive development, particularly his theory of groupings (see Section 2.2.), draw heavily on this kind of biological thinking. Here, we see parallels with the point of view that members of an equivalence class in this kind of biological classification can often be regarded as "equal" in the sense that they serve equally well to exemplify the properties involved in the partition. But in the Biology lesson this latter point should not be pressed too far as the child may see the characteristics much more easily from some members of a species than from others.

In all of the examples just discussed the principles underlying the classification have been non-numerical. However, partitioning is often brought about by numerical relationships and we will now give some examples of these from various subject areas. The extent to which "calculations" are done with the equivalence classes varies and, generally speaking, the examples to follow show a progression - the "calculations" with equivalence classes being increasingly important in the later examples.

In the musical scale notes an octave apart (or an integral number of octaves apart) have a particularly simple relationship between their frequencies and this is the underlying reason for their being denoted by the same letter. Practically, the existence of this close relationship means that when a piece of music is arranged for different instruments, a melody may be transposed an octave if this is more convenient, without affecting the harmony. For, to a large extent, the notes occurring in chords are representatives of equivalence classes, another member of the class could replace them, although rather special rules apply to the base notes of chords.

Music at another level shows equivalence classes of <u>relations</u> - as distinct from equivalence classes of <u>elements</u>. Musicians think in terms of intervals and an interval is an equivalence class of pairs of notes just as a rational number is an equivalence class of ordered pairs of integers. For example, the major scale incorporates the intervals:

tone, tone, semitone, tone, tone, tone, semitone.

These are relationships between frequencies and equivalent relationships occur in every major key. (See Budden (8)).

The ideas considered above are part of musical theory and the academic musician has to work with proper regard for the grammar of these ideas. He will not usually think of this process as being one of calculation, although students who pass harmony examinations by using the rules, without mentally hearing the notes they are writing down, must surely be performing a process closely resembling calculation.

Many systems of classification are used in Geography, but we will mention one in which the partitioning of points into equivalence classes takes an unusual form which has interesting geometrical properties. Isopleths are lines drawn on a map through places having the same value of some measurement. Thus contour lines are isopleths because they are lines drawn on a map through places having the same height. Isotherms join places having the same temperature over a certain period. Isogonic lines join places having equal magnetic declination. Geographers also use isobars (pressure); isobaths (depth below sea-level); isohalines (salinity); isohels (duration of sunshine); isohyets (depth of rainfall) and isonephs (cloudiness).

In every case the equivalence classes correspond to lines on a map only some of which are drawn, and in a sense the geographer "calculates" with these in an intuitive way. For certain geometrical features indicate related aspects of the variable concerned. Thus, where the isopleths are closer together the quantity is changing more rapidly; closed loops surround local maxima and local minima; a saddle point has its own peculiarities, etc.

Chemistry has made progress by recognizing equivalences. Initially, the chemist appears to be confronted with an infinite variety of substances. As a result of the experience of centuries, these became classified as a certain number of elements and their compounds. Thus chemists decided that there was not an infinite variety of atoms, but only (in the first place) 92 different kinds. In addition, it was also recognized that any one atom of say, hydrogen could replace any other atom of hydrogen without the change being chemically noticeable. Thus the fundamental components of matter, as seen at the time, were put into 92 equivalence classes. Under this system some apparently different things are classified as equivalent. Thus certain physically different substances are all classified as sulphur; charcoal, graphite and diamond are all classified as carbon, and so on. This view, however, has had to be adapted to deal with isotopes and with the internal structure of the atom, and it continues to be adapted to accommodate new discoveries on particle physics. But these refinements do not alter the fundamental strategy of organizing the fundamental constituents of matter into some specific number of classes, the members of which are in some sense equivalent to each other.

The periodic table had the advantage of grouping together elements with similar physical and chemical properties. These properties are largely dependent upon the number of electrons in the outer shell of the atom, and proceeding down a group of the periodic table, there is an increase in the tendency:

(a) to form electrovalent compounds containing positive ions,

(b) to show metallic character,

(c) to be a reducing agent,

(d) to form basic oxides and hydroxides.

In addition, the periodic table produces a grouping of elements which to some extent can replace one another in compounds. For example, consider some of these chemical relationships between the elements in Group 1 and Group 7.

Group 1		Group 7			
3 Li	Lithium	Fluorine	F	9	Bromine is able to
11 Na	Sodium	Chlorine	Cl	17	displace iodine: e.g.
19 K	Potassium	Bromine	Br	35	$2KI + Br_2 = 2KBr + I_2$
37 Rb	Rubidium	Iodine	I	53	Chlorine is able to
55 Cs	Caesium	Astatine	At	85	displace bromine: e.g.
87 Fr	Francium				$2KBr + Cl_2 = 2KCl + Br_2$
					Fluorine is able to
					displace chlorine: e.g.
					$2KCl + F_2 = 2KF + Cl_2$.

Thus we see that the arrangement of elements in the periodic table was an attempt to produce equivalence classes and subsequently to order them. It must have looked initially as if this classification would account completely for valency; but unfortunately this was not to be. If it had been possible to explain valency by attributing to every element a unique (small) integer, this would have been a further triumph for equivalence classes.

The examples discussed are just a few of many that could have been chosen to illustrate the differences which occur between the classifications which are associated with equivalence relations and those which are associated with near-equivalence relations (i.e. relations like "is a synonym of" on the set of all English words which tend to be spoken of as if they were equivalence relations but which do not in practice entirely satisfy the mathematical criteria for an equivalence relation (see Section 1.3.)), and between those which have numerical and non-numerical principles underlying them. Further examples could have been given from the fields of

Art	(shape, colour, material employed, ... etc.)
Handicraft	(techniques employed, tolerances, ... etc.)
History	(political affiliation, dynasties, ... etc.)
....	

to highlight these differences.

But there is one further example which should be discussed in fuller detail. In the following paragraphs we will consider some linguistic ideas usually associated with Chomsky. These concern the grammatical structure of speech, and we will see that they make use of certain partitions of words and phrases into equivalence classes. Furthermore, a certain algebraic structure relates these classes to one another.

For the purpose of introducing the ideas to be discussed, we will begin by considering some of the points made by Ruth Strickland (1962) in "The Language of Elementary School Children: Its Relationship to the Language of Reading Textbooks and the Quality of Reading of Selected Children". Here Strickland, who is concerned with the development of language in young children, describes simple methods of constructing sentences. This starts with a "fixed slots" approach in which numbers are assigned to types of element as follows:

subject	verb	copula (e.g.is)	indirect object	direct object	complement
1	2	2b	3	4	5

so that

I saw the cat ⟶ (124)

John ran ⟶ (12)

Mary is pretty ⟶ (12b5)

He gave me a sweet ⟶ (1234)

This means that acceptable sentences are of certain prescribed patterns, e.g. (124), (12b5), and that particular sentences are obtained by replacing "variables" such as "4" by particular "values" such as "cat" or "sweet". The variables are therefore equivalence classes and the values they take are elements which are equivalent to one another in the sense that they are equally acceptable from a grammatical point of view. It is important to note that this type of analysis is concerned all the time with grammatical form and not with meaning.

Chomsky's phrase-structure grammar can be regarded as an extension of the above method. An example of one such system is

(i) Sentence ⟶ NP + VP

(ii) NP ⟶ T + N

(iii) VP ⟶ V + NP

(iv) T ⟶ the

(v) N ⟶ {man, ball, . . .}

(vi) V ⟶ {hit, took, . . .}

(See Lyons (9), p.59).

Note that each one of this set of rules is of the form X ⟶ Y, where X is a single symbol, Y is a <u>string</u> consisting of one or more symbols and ⟶ denotes "rewrite X as Y". (We regard NP and VP as single symbols). And so, on starting with the symbol "Sentence" and applying rule (i) we obtain

NP + VP,

by (ii) and (iii) we obtain

T + N + V + NP.

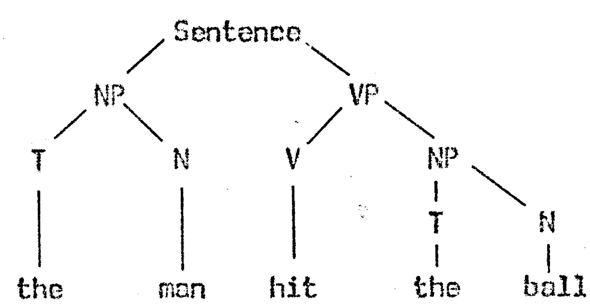
On applying (ii) again, we obtain

T + N + V + T + N.

Finally, on applying (iv) and (v) twice and (vi) once, we are able to obtain the terminal string

the + man + hit + the + ball.

This process, which has generated the sentence "The man hit the ball", can be summarized by a tree-diagram as follows:



This particular example given by Lyons of a phrase-structure grammar is rather trivial as it will generate only one type of sentence. Other similar systems are much richer and will, in fact, generate indefinitely long sentences of the type occurring in "The house that Jack built", for example.

We will now give some examples which illustrate this aspect in a context which shows that the type of thinking employed by Chomsky in linguistics is also employed in mathematics.

Example 1

Reversed Polish notation is employed in mathematical logic and computing. In fact, many pocket calculators use it rather than the conventional algebraic notation.

Working with some particular system employing Polish notation, all the acceptable expressions can be generated by the following rules, which are written in the notation employed by Lyons.

(i) exp ⟶ var

OR (ii) exp ⟶ exp + exp + binop

OR (iii) exp ⟶ exp + unop

(iv) var ⟶ {x, y, z, . . .}

(v) binop ⟶ {A, M, . . .}

(vi) unop ⟶ {N, R, . . .}

The interpretation of these symbols is as follows:

exp denotes an expression

var denotes a variable from the set x, y, z, . . .

binop denotes a binary operator, and A, M, . . . are the binary operators such as "add", "multiply", etc.

unop denotes a unary operator, and N, R, . . . are the unary operators such as "negate", "reciprocate", etc.

Hence, successive application of the above rules can generate

exp + exp + binop

from "exp" by rule (ii). On applying rule (iii) twice, we obtain

exp + unop + exp + unop + binop.

Application of rule (iii) to the above expression as a whole, gives

exp + unop + exp + unop + binop + unop.

And so by rule (i) twice, we now have

$$\text{var} + \text{unop} + \text{var} + \text{unop} + \text{binop} + \text{unop}.$$

On choosing x as replacement for the first "var", y as replacement for the second "var", R for "unop" and A for "binop", and on dropping the addition signs as we have now chosen the symbols for our terminal string, we obtain

$$x\ R\ y\ R\ A\ R.$$

In ordinary notation the expression obtained is $\dfrac{1}{\frac{1}{x}+\frac{1}{y}}$

<u>Example 2</u>

The official international definition of the much used programming language ALGOL 60 is given in this form. (See Naur (10).)

However, the limitations of the above approach in analysis of language are several. We have already noted that we are not concerned with meaning. Moreover, this approach is not really adequate for handling such aspects of language as inflexion, active and passive voice or changes of mood. To cope with these features Chomsky extended his ideas to transformational grammar. Transformational grammars are more complicated systems which consist of transformation rules that are applied to the phrase-structures derived from the phrase-structure grammar. The transformation rules are often sensitive to context and they modify the simple classifications into equivalence classes around which phrase-structure grammar is built. For example, if we try to proceed by phrase-structure we might in some system generate Pro + V, (i.e. pronoun followed by verb). For V we might seek to substitute "sang". This would be acceptable if for Pro we substitute any pronoun. Problems now arise if we try to transform Pro + V from the past to the present tense, for "sang" has to become "sing" if Pro is "I", "you", "we" or "they", but "sings" if Pro is "he" or "she". Thus we see that context is involved in a way which phrase-structure grammar in its basic form is insufficient to handle.

Recognition of these limitations, however, need not detract our interest from the fact that for a particular phrase-structure grammar, to each of the symbols such as NP, VP, etc., there corresponds a substitution set which is derived from the fundamental substitution set N, V, etc. Thus, as in traditional algebra, the symbols N, V, . . . can be regarded as place-holders for elements from the substitution sets and these substitution sets are equivalence classes with respect to grammatical acceptability. They are in no way equivalence classes with respect to meaning, but this analysis is not concerned with meaning.

Following this line of argument, the linguist might be said to "calculate" with equivalence classes in the sense that combinatorial analysis is undertaken or performed with classes as wholes.

Many of the above ideas have been applied to the teaching of foreign languages. As an elementary example we may give Longman's Audio Visual French, intended for lower secondary school children, which uses many examples of fixed slot patterns with rather limited substitution sets.

At this point we should note that equivalence classes with respect to a phrase-structure grammar in one language do not necessarily carry over into other languages. For the structures of languages are often very sensitive to context (e.g. German). If we try and convert the diagram on page 12 into German, the two 'T's have to be replaced by two different things, the first by "der" and the second by "den". These two German words are not equivalent. In order to produce grammatically acceptable sentences one, or the other, or even some other variant, has to be substituted for 'T'. Hence

The man hit the ball ⟶ Der Mann schlug den Ball

Thus we see that the set of rules with their associated equivalence classes, given on Page 11, which produces grammatically acceptable sentences in English does not produce grammatically acceptable sentences in German if the rules and the equivalence classes are translated as they stand. So phrase-structure grammar may be of very limited help in problems of translation. In addition, meaning is sensitive to context and as there is certainly not a one-to-one correspondence between words and phrases in different languages examples frequently occur where simple words such as "box" in English and "boîte" in French correspond in certain contexts but not in others.

Before concluding this review of linguistics, we must also point out that within a particular language Chomsky and his followers have argued that certain patterns are fundamental in grammatical speech, and that these patterns enable all fluent speakers of that language to produce and understand sentences which they have never heard before. In other words, the "creativity" within a language appears to imply the fundamental importance of equivalence classes.

We have given merely a small selection from an enormous range of possible examples of the uses of equivalences and near equivalences. It is to be hoped that this selection, small though it is, is sufficient to indicate that equivalence classes (and by implication partition with the associated equivalence relation) are an essential mode of thinking for adult English speakers and indeed for adult speakers of all the familiar languages of developed countries.

Children, however, have their own patterns of thinking which are developing towards adult form, but which at various stages of growth display more or less stable configurations with a logic of their own. Consequently, we must now turn our attention to cognitive development theory, for as pointed out by L.P. Steffe (11)

" . . . *Cognitive development theory can contribute to an understanding of how it is a child acquires knowledge of the mathematical systems through its descriptions of cognitive operations children acquire and the mechanism through which children acquire them. A mathematical educator cannot stop there, however, because the cognitive operations demanded by mathematical systems may be distinguishable from (but include) the cognitive operations described in cognitive-development psychology. Mathematics educators do not yet know how to utilize the cognitive operations studied in cognitive development psychology in the further acquisition of cognitive operations demanded by the mathematical systems mentioned. In fact, few attempts have been made toward the identification of relationships between the cognitive operations studied in developmental psychology and the cognitive operations demanded by the mathematical systems.* . . ." ((11), p. 3).

Thus, on taking up the challenge introduced by Steffe, we see that an essential preliminary to any discussion of the ways in which equivalence relations and the associated ideas of partition and equivalence classes are and could be used, is an investigation arising from the following question:

> If a child is or is not in possession of the cognitive operations associated with the properties of an equivalence relation, what does this say about his knowledge or acquisition of an equivalence relation?

In other words, we require an investigation of the probable growth of the concept from initial germination to explicit recognition and confident use. Section 2 of this paper is therefore devoted to discussion of the psychogenetic development of the concept of equivalence relation.

But so far the term "equivalence relation" has been undefined. Our immediate requirement is an agreed set of definitions and results associated with equivalence relation on which to base the discussion to be undertaken in Section 2. It is therefore proposed that the following definitions and results be taken as the agreed foundation. We shall use them throughout except in direct quotations.

1.3. Definitions and results

SET will be taken as an undefined term.

Intuitively a set is seen as any collection of objects, which may be concrete objects (e.g. dogs, chairs, Manchester United Football team (seen as a specific set of players)), or abstract objects such as other sets previously defined (e.g. Football teams in the First Division).
Sets can sometimes be defined by explicitly listing their elements. In general, we say that each object in the set is an ELEMENT of the set.
We also use the nomenclature that each element BELONGS to the set.
To avoid having to write in full that any element x either belongs or does not belong to a set, we use the following notation:

x	$\in$	A
x	belongs to (or is a member of)	the set A
x	$\notin$	A
x	does not belong to (or is not a member of)	the set A

At this point we should note that set theory, in elementary teaching, is usually introduced by what may be termed "unformalized description" as exemplified by the above paragraph. It can be objected that such unformalized description only conveys anything to the reader because he implicitly appreciates certain equivalences. Thus, this form of presentation runs the risk of a certain kind of circularity. The elements of a set are in some particular relation to one another (if only in that they have been ascribed to the same set) and it would seem that we cannot identify the set without at the same time recognizing the relation and we cannot describe the relation without at the same time recognizing the set.

However, even with fully formalized axiomatic set theory, in the most rigourous mathematical formulations so far achieved, somewhat similar objections apply. Because if a symbol 'A' is used, the reader has to regard various symbol 'A's on different parts of the page, each differing from the others in microscopic detail as well as position and so recognizably distinct, as denoting the same logical 'object' - that is to say as being in some way equivalent. It would be difficult to conceive of any formulation to which this does not apply.

But this objection clearly involves a confusion of the theory and the meta-theory. It involves a confusion between the well-defined system under scrutiny and the incompletely defined system, potentially capable of indefinite extension and modification, within which the system under study is embedded. We have to avoid circularity in the theory, we cannot guarantee to avoid it in the meta-theory.

When the various symbols 'A' on a page of set theory or logic are recognized as being "the same", this recognition (i.e. this use of an equivalence relation, or use of an equivalence class - whichever way it is regarded) is outside the theory. There are many equivalent 'A's, but there is only one A.

This being said we will now proceed with our description of set theory at an introductory level.

When every element of the set A is an element of a set B also, we say that A is a SUBSET of B and denote this as follows:

$$A \subset B$$

Thus A is said to be a subset of B if and only if, each element in A also belongs to B, i.e. A is a subset of B if $x \in A$ implies $x \in B$. Note also that this definition of a subset does not exclude the possibility that the two sets are equal; indeed, it leads to a convenient definition of equality. Two sets E and F say, are said to be equal if and only if each is a subset of the other. Hence, we write $E = F$ if and only if $E \subset F$ and $F \subset E$.

$A \cap B$ is read as "A intersection B" and is used to denote the set of elements which belong to BOTH A and B, i.e.

$$A \cap B = \{x: \ x \in A \ \text{ and } \ x \in B\}$$

which is read "A intersection B is equal to the set of all elements x such that x belongs to A and x belongs to B".

NB. This definition uses a style of set description which is open to mathematical objection but is usually found more readily intelligible than a more correct form. Here, technically $A \cap B = \{x \in A: \ x \in B\}$ is better.

$A \cup B$ is read as "A union B" and is used to denote the set of elements which belong to A or B or both, i.e.

$$A \cup B = \{x: \ x \in A \ \text{ or (inclusive) } \ x \in B\}.$$

The set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$ is called the CARTESIAN PRODUCT of A by B and it is denoted by $A \times B$.

$$A \times B = \{(x, y): x \in A, \ y \in B\}$$

Thus the ordered pair (x, y) is an ELEMENT of the Cartesian product if $x \in A$, $y \in B$.

Any subset R of such ordered pairs (i.e. a subset of the Cartesian product) defines a CORRESPONDENCE denoted by the ordered triple of sets (A, B, R) from A to B with

$$R \subset A \times B.$$

As a special case A and B may coincide, in which case we speak of a RELATION R IN A. The relation is the ordered pair of sets (A, R) where

$$R \subset A \times A.$$

For every set A there exists

$$D_A = \{(x, x): \; x \in A\}$$

which is called the DIAGONAL of A. Thus we see that D_A is a subset of A x A, i.e.

$$D_A \subset A \times A.$$

Further relations on a given set may be defined by introducing two operations, inversion and composition.

Every relation (A, R), where $R \subset A \times A$, has the inverse (A, R^{-1})

$$R^{-1} = \{(y, x): \; (x, y) \in R\}.$$

If (A, S) and (A, T) are two relations in A, their composition is also a relation (A, SoT) in A given by

$$S \circ T = \{(x, y): \; (x, z) \in S, \; (z, y) \in T\}.$$

A relation (A, R) in A is said to be REFLEXIVE if

$$D_A \subset R.$$

A relation (A, R) in A is said to be ANTIREFLEXIVE(1) if

$$D_A \cap R = \emptyset$$

where ∅ denotes the empty set.

---

(1) These definitions correspond to the terminology as used in "Travaux Pratiques de Mathématique - Serie II: Les Relations", by Duvert, Gauthier and Glaymann, O.C.D.L., 1968.

A relation (A, R) in A is said to be NON-REFLEXIVE(1) if

$$D_A \not\subset R.$$

A relation (A, R) in A is said to be SYMMETRIC if

$$R^{-1} = R.$$

A relation (A, R) in A is said to be ANTISYMMETRIC if

$$R \cap R^{-1} \subset D_A.$$

A relation (A, R) in A is said to be NON-SYMMETRIC(1) if

$$R^{-1} \neq R.$$

A relation (A, R) in A is said to be ASYMMETRIC if

$$R \cap R^{-1} = \emptyset.$$

A relation (A, R) in A is said to be TRANSITIVE if

$$R \circ R \subset R.$$

A relation (A, R) in A is said to be ANTITRANSITIVE if

$$(R \circ R) \cap R = \emptyset.$$

A relation (A, R) in A is said to be NON-TRANSITIVE if

$$R \circ R \not\subset R.$$

These definitions are not necessarily universally accepted.

The above type of formulation may be unfamiliar to some readers. What follows may seem less obscure because it is an attempt at a more direct model of ordinary speech and reasoning. Unfortunately it is also somewhat less precise. However, we are not so much concerned with definitions as with indicating correspondences between ordinary language and an idealized model.

A relation in A may be denoted by the letter P say, which replaces the verb or verbal clause in a statement. Here

$$aPb$$

has to be a meaningful statement for all $a, b \in A$ which is either true or false for any $a, b \in A$, but never both.

The relation P in A is REFLEXIVE if and only if for all x, $x \in A$, the statement xPx is true.

The relation P in A is ANTIREFLEXIVE if and only if for all x, $x \in A$, the statement xPx is false.

The relation P in A is NON-REFLEXIVE if and only if, for some but not all x, $x \in A$, the statement xPx is true.

The relation P in A is SYMMETRIC if and only if, whenever xPy is true then yPx is true $(x, y \in A)$.

The relation P in A is ANTISYMMETRIC if and only if whenever xPy and yPx are both true then $x = y$ is true $(x, y \in A)$. (By $x = y$ we mean that x and y are both the same element of A.)

The relation P in A is NON-SYMMETRIC if and only if for some but not all $x, y \in A$, xPy is true and yPx is false $(x, y \in A)$.

The relation P in A is ASYMMETRIC if and only if whenever xPy is true yPx is false $(x, y \in A)$.

The relation P in A is TRANSITIVE if when xPy and yPz are both true then xPz is true $(x, y, z \in A)$.

The relation P in A is ANTITRANSITIVE if when xPy and yPz are both true then xPz is false $(x, y, z \in A)$.

The relation P in A is NON-TRANSITIVE if for some but not all $x, y, z \in A$, xPy and yPz are both true but xPz is false $(x, y, z \in A)$.

We can now define equivalence relation:

A relation which is at one and the same time reflexive, symmetric and transitive is an EQUIVALENCE RELATION.

A PARTITION of a set A is a separation of the elements of A into subsets such that each element of A is in one and only one subset.

These last two definitions give rise to a very important result:

Any equivalence relation R in A partitions the set in that x and y belong to the same subset if and only if xRy, and conversely, given a partition of a set A, xR^1y if and only if x and y belong to the same subset of the given partition of A, defines an equivalence relation R^1 in A.

(See Appendix 1).

The subsets of a partition of A are called EQUIVALENCE CLASSES.

The terms defined above will assist discussion in the following section. It will be seen that we have defined ten possible properties of a relation yet our final definition of equivalence relation requires only three of them. This is done partly for clarity (because variations in the terminology do occur), partly because when discussing examples of a particular property one also needs to discuss the various types of counter-examples, and partly for completeness. Relations possessing other combinations of these properties (i.e. other than the specific three properties of equivalence relations) are by no means without importance and relevance as we shall see.

SECTION 2

2.0. Aim of the section

In this section an attempt will be made to establish a framework within which observations about the development of the concept of equivalence relation can be organized.

2.1. The contribution of Piaget

As no investigation into any aspect of concept development can ignore the tremendous contribution made to this field by Jean Piaget, a review of the relevant factors in his work will be taken as our starting point. Further justification for this line of approach, in view of the aim of this section, is provided by D.C. Johnson (12).

> " . . . *the research literature surrounding the work of the Geneva school provides a framework for*
> (1) *explaining how mental operations basic to mathematical thought develop,*
> (2) *identifying structural characteristics of thought as they undergo change with age, and*
> (3) *forming a theoretical basis for certain curricular decisions and experiments in the learning of mathematics.*"
> ((12) p. 123).

But on examining the details of this structure which relate to the concept of equivalence relation, we may find that the framework constructed to date is too coarse to provide sufficient help for the classroom teacher. It is possible that there are large gaps in our knowledge which need to be filled. This cautionary note is even more appropriate when we also take into consideration the fact that Piaget has done and said so much in fifty years of work on cognitive development that foci for contention and disagreement abound.

However, the widely recognized and substantially uncontested parts of Piaget's work are his obversations of children and his descriptive accounts of the stages of development their thinking goes through. Less well known, but of great importance to Piaget himself, are the theoretical models of cognition he has devised to describe the characteristics of thought which appear at different stages of development. The predominant part of this is the theory of groupings, which provides algebraic models of various aspects of thinking much as the more recent work by Thom (13) provides topological models of other aspects of thinking.

2.2. Piaget's theory of groupings

The three defining properties of an equivalence relation, reflexivity, symmetry and transitivity, are attributes which have been incorporated into Piaget's theory of groupings. There are nine distinct groupings which Piaget and his associates have derived in their attempts to find adequate models of cognition in the concrete-operational subperiod of child development. Of these one is regarded as a minor, preliminary grouping as it occurs as a special case in the remaining eight more complex structures. But all of Piaget's groupings are seen as possessing the attributes of a group* and a lattice**.

---

* A group (G, o) is a set G with a binary operation o defined on it with the following properties

(i) o is closed,

(ii) o is associative,

(iii) there is an identity element $e \in G$ such that for all $a \in G$,

$$a \circ e = a = e \circ a,$$

(iv) for any element $a \in G$, there is an inverse element $b \in G$ such that

$$a \circ b = e = b \circ a.$$

** A lattice is a partially ordered set in which a subset composed of any two elements has both a least upper bound and a greater lower bound.

For the eight major group/lattice structures (i.e. groupings) conceived by Piaget, the following quotation from "The Developmental Psychology of Jean Piaget" by J.H. Flavell summarizes their role.

"*These groupings are viewed as models for cognition in several different realms of intellectual endeavor. First, they describe the organization of logical operations proper, i.e. operations dealing with logical classes and relations. Four of the major groupings relate to class operations and the other four to relation operations. Second, these same groupings also fit the organization of what Piaget calls infralogical operations (i.e. cognitive actions bearing on position and distance relationships and part-whole relationships apropos of concrete spatiotemporal objects or configurations)*". ((14), p. 171).

In particular, groupings I - IV concern operations performed on sets (referred to as logical classes above). On the other hand, groupings V - VIII involve operations upon the relations which may exist between two or more elements or between two or more sets. But fundamental to each is the hybrid structure between a group and a lattice.

The formal properties for the composition of operations in a grouping, as given by Piaget in "La Psychologie de l'intelligence" (1947), produce an unsatisfactory mathematical formalization. In particular, we note that

(i) $x + y = z$ (composability)

(ii) $z - y = x$

or $z - x = y$ (reversibility)

(iii) $(x + y) + z = x + (y + z)$ (associativity)

(iv) $x - x = 0$ (identity)

where x, y, and z represent grouping elements, and "+" and "-" represent grouping operations, more or less describes the group-structure. By including

$$(v) \quad x + x = x \qquad \text{(tautology)}$$

however, Piaget's grouping reduces to a singleton-group only, as there can only be one idempotent element in a group, namely, the identity element.

Moreover, Piaget's account of his groupings is far from clear. (See (15), (16) and (17)). For example, his account of the preliminary grouping of equalities does not appear in these three references but it is described in (18). This is reported by Flavell (14) who says (p. 187).

> "*The Preliminary Grouping of Equalities.*
> *Brief mention may be made of this extremely simple but fundamental grouping which is said to occur in disguised form as a special case in all the preceding major groupings. ((18), p. 33-34). It closely resembles Grouping VI, inasmuch as it involves the addition of a particular type of symmetrical relation: equality or, as Piaget sometimes calls it, "pure equivalence". Its compositions are of the form (A = B) + (B = C) = (A = C); such compositions are clearly associative; the inverse of an operation (A = B) is, analogous to Grouping VI, (B = A); the general identity is (A = A); and each equality plays the role of special identity with itself and every other equality, e.g. (A = B) + (A = B) = (A = B) and (A = B) + (C = D) = (C = D)".*

But this account appears to contain a number of notational obscurities and confusions.

(a) Basic to these is the lack of definition of the set involved. Is $\{A, B, C, \ldots\}$ the set under consideration or have we to regard $(A = B)$ as a typical element of the set?

(b) The equals sign is used to denote both 'equality' or 'pure equivalence' between two elements of a set, and also to denote the deduction of one statement from another, or the deduction of a third statement from two given ones. To accommodate this, the statements are re-written below using $\sim$ to denote 'equality' and $\Longrightarrow$ for implication. It may also be observed that 'equality' or 'pure equivalence' is a symmetric relation whereas 'implication' is not.

(c) Composition of relations is confused with logical 'and' (i.e. conjunction). But composition and conjunction differ in nature in that composition is not in general commutative (i.e. $a \circ b \neq b \circ a$), whereas conjunction is, and also that when elements have inverses (as here) composition can be 'undone', meaning

$$a \circ b \circ b^{-1} = a,$$

whereas there is no corresponding process with conjunction. However, in the note on "The Preliminary Grouping of Equalities" there is the effort to combine together relational statements as if they were elements of a group - which they are not. They are not because they only obey a restricted law of composition, much like bound vectors in formulations of vector algebra (or near-vector algebra) in which $AB + BC = AC$ but where we may not say $AB + CD$ equals anything. However, it would be possible to legitimatize this by a construction analogous to the one employed to turn the algebra of line segments into the algebra of vectors.

It could be the intention to deal with equivalence classes of statements, but this is nowhere stated. Once again the set under discussion is not clear.

It would seem possible to re-write the above equations in an acceptable mathematical form as follows:

(d) $(A = B) + (B = C) = (A = C)$
becomes $(A \sim B) \wedge (B \sim C) \Longrightarrow (A \sim C)$

(e) the inverse of $(A = B)$ is $(B = A)$ becomes $(A \sim B) \Longrightarrow (B \sim A)$

(f) the general identity is $(A = A)$ becomes, for all A, $A \sim A$.

(g) The final two equations concerning the special identity properties of 'equality' may be reformulated as follows:

$(A = B) + (A = B) = (A = B)$
becomes $(A \sim B) \wedge (A \sim B) \Longrightarrow (A \sim B)$

and

$(A = B) + (C = D) = (C = D)$
becomes $(A \sim B) \wedge (C \sim D) \Longrightarrow (C \sim D)$.

These two equations suggest that the elements of the set under consideration are equalities of the form $(A = B)$, and they are an attempt to force a group structure on the set of elements of this form, which they do not necessarily have for the reasons considered in (c) above.

Moreover (d), (e) and (f) correspond to the properties of transitivity, symmetry and reflexivity - the basic properties of an equivalence relation. Thus, given that Flavell has accurately translated the relevent section on p. 33-34 of "Classes, relations et nombres: essai sur le 'groupement' de la logistique et la réversibilité de la pensée", we see that Piaget's preliminary grouping "simple, but fundamental", "which is said to occur in disguised forms as a special case in all the preceding major groupings" seems to be nothing but the idea of an equivalence relation in a malformed notation.

At this point we should note that Piaget was not attempting to give a fully formalized account of his ideas and it is hardly appropriate to criticize him for omitting mathematical detail where it is clear how it may be filled in, but there are some places where it is not clear how it is to be filled in. It is not clear how sets and relations are to be defined, which of the two is fundamental, and which (if either) is to be defined in terms of the other. This means that there is a serious risk of circularity in the fundamental concepts.

Unfortunately, circularity seems to be regarded as an unavoidable problem in this field of psychology, as the following argument by Lesh (19) shows:

> "*Mathematicians can formalize a mathematical structure (e.g. define a strict partial ordering relation*) by starting with*

---

*In this American paper a strict partial ordering relation has been defined as follows:

> "*$<$ is a strict partial ordering on a set S if $<$ is a set of ordered pairs of elements in S such that*
>
> 1. *For every element a in S, (a, a) is not in $<$, (nonreflexive property),*
> 2. *For every pair of elements a, b in the set S, if (a, b) is in $<$ then (b, a) is not in $<$, (asymmetric property),*
> 3. *For any three elements a, b and c in the set S, if (a, b) is in $<$, and if (b, c) is in $<$ then (a, c) is in $<$, (transitive property).*" ((19), p. 98).

This definition differs somewhat from the standard English definition of a strict partial ordering relation, but of greater significance to the present discussion is the fact that the term 'nonreflexive' has been used to name the property which we, following Duvert et al, have called the <u>antireflexive</u> property. (We repeat that except in quotations we shall use the terminology given on pages 21-24.)

certain axioms, undefined terms, or accepted rules of logic, and construct theorems and definitions on the basis of these. That is, axiomatics terminates endless regression by beginning with undefined terms and it avoids circularity by arbitrarily choosing a starting point which has not been demonstrated. Psychologically, however, one is not afforded the luxury of beginning with indefinables, axioms or accepted rules of logic.

For example, in the case of the ordering relation <, the nonreflexive, asymmetric and transitive properties cannot be used as self-evident concepts. Before the relation < has been coordinated with its inverse, each of these properties is repeatedly and often emphatically denied by children[1]*. Even such mathematically primitive concepts as Hilbert's order axiom (if B is between A and C then it is also between C and A) are not a priori intuitions for children until the betweeness relation has been subsumed within a system of relations*[2]*."* ((19), p. 99).

Is the author trying to say that the logical analysis of any psychological situation must necessarily be circular? This claim would obviously be far too strong. Moreover, greater sympathy with the author's point of view would have been achieved if the arguments used had been based on evidence of attempts to identify agreed primitive terms and axioms on which to base definitions and theorems for this branch of psychology. It is hard to believe that any mathematician

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(1) Inhelder, B. & Piaget, J. "The early growth of logic in the child: Classification and seriation", translated by E.A. Lunzer, Routledge and Paul, 1964.

(2) Piaget, J. & Inhelder, B. "The mental imagery of the child", (contd)

would wish to use non-reflexivity, asymmetry or transitivity as self-evident concepts! Many applications of mathematics have to find sequences of development which may not correspond to the axiomatic sequence by which the mathematical model might be developed on its own, and they succeed by avoiding circularity, not acquiescing to it. The psychologist's plea that he is dealing with some unsequenced totality may be merely an admission that he has not yet succeeded in recognizing a suitable sequence in terms of which to analyze the situation. But as we are concerned with psychogenetic development, it would seem to be the case that later stages are structurally richer than earlier stages and therefore that some things precede others. Hence, if one is seeking to construct a mathematical model, then the problem is to ensure that the mathematical counterparts of the psychogenetically prior concepts precede (in a logical sense) the psychogenetically subsequent concepts. Such a model would avoid circularity.

Fortunately, the incipient circularity in Piaget's presentation of the theory of groupings was avoided in the reformulation of grouping theory developed by E. Wittmann (20). This has been summarized by H.G. Steiner (21) and this summary is reproduced here, with slight notational changes to facilitate direct comparison with Wittmann's original formulation.

2.3. Wittmann's and Steiner's reformulations

" . . . *a grouping is a 5-tuple* $(M, M \times M, \triangle, o, \triangleleft)$ *with the following data and properties:*

(i) M *is a non-empty set, whose elements* $a, b, c,$ . .

(contd)

Translated by P.A. Chilton, New York: Basic Books Inc., 1971.

are called states,

(ii) M x M is the set of all ordered pairs (a, b) where a, b $\in$ M. The elements (a, b) of M x M are called operations,

(iii) Δ is a subset of M x M whose elements are called elementary operations,

(iv) o is the canonical partial composition on M x M defined by

$$(a, b) \circ (b, c) = (a, c),$$

(v) $\lhd$ is a relation in M, defined by means of Δ in the following way:

a $\lhd$ b if and only if there are (a_1, b_1), (a_2, b_2) . . . $(a_n, b_n) \in \Delta$ such that

$$(a, b) = (a_1, b_1) \circ (a_2, b_2) \circ \ldots \circ (a_n, b_n).$$

$\unlhd$ is the union of $\lhd$ and the identity relation in M.

(vi) The following postulates shall hold

(a) M x M is generated by $\Delta \cup \Delta^{-1}$ relative to o (where $\Delta^{-1} = \{(c, d): (d, c) \in \Delta\}$)

(b) (M, $\unlhd$) is a lattice." ((21), p. 242).

However, Steiner omits to point out that from this complex structure Wittmann does derive the five psychologically important properties which Piaget sought, as the following quotation shows:

"C.1. *Composability of operations:*

(i) within a natural restriction operations are arbitrarily composable,

(ii) as a rule, an operation can be represented as a product of operations in several different ways.

G.2. *Associativity:* *the partial composition o is trivially associative.*

G.3. *Reversibility:* $(a, b) \circ (b, c) \circ (c, b) = (a, b)$ *for all* $a, b, c \in M$.

G.4. *Identical operations:*

(*i*) $(a, b) \circ (b, a) = (a, a)$ *for all* $a, b \in M$,

(*ii*) $(a, b) \circ (b, b) = (a, b)$.

G.5. *Tautology:* *If* $a \sqcup b$ *denotes the least upper bound of* a, b *in the lattice (or semi-lattice) then, for all* $a, b \in M$ *such that* $a \trianglelefteq b$

(*i*) $a \sqcup a = a$ (*tautology*)

(*ii*) $a \sqcup b = b$ (*absorption*)."

((20) p. 127-128).

In addition, Wittmann acknowledges that his account is a redundant formulation, and in fact H.G. Steiner (21) has shown how Wittmann's axioms can be simplified to give the following definition:

"*A relational system* $(M, \triangle)$ *is called a grouping if and only if* $(M, RT(\triangle))$, *where* $RT(\triangle)$ *is the reflexive transitive hull of* $\triangle$, *is a lattice.*" (21), p. 243).

(See Appendix 2a for notes on the reflexive, transitive hull of $\triangle$)

From the above definition the following properties were also derived by Steiner to provide a comparison with Piaget's laws (i) - (v). (See page 27).

(I) $(a, b) \circ (b, c) = (a, c)$

(II) $((a, b) \circ (b, c)) \circ (c, b) = (a, b)$

(III) $((a, b) \circ (b, c)) \circ (c, d) = (a, b) \circ ((b, c) \circ (c, d))$

(IV) (a, b) o (b, a) = (a, a)

(V) a⊔a = a.

But as already indicated, Wittmann deliberately maintained the extended formulation which is also close to Piaget's original in deference to the psychological application for which groupings were intended. Nor must we lose sight of the fundamental reason for which they were conceived, that is, to answer the question: to what extent can 7-11-year-old children operate to grouping specification and hence justify the grouping as a model of their cognition?

In response to this question Piaget has devised a variety of experiments with children to see if it is possible to bring to the surface behavioural analogues or counterparts of one or other differentiating component of a given grouping. For example, Piaget has created tests to tap and probe for the presence or absence of

- the ability to effect transitive compositions of asymmetric relations (Grouping V - see Appendix 2b),
- the capacity to grasp the symmetry of symmetric relations (Grouping VI).

2.4. <u>The relevance of the experiments associated with Groupings V and VI</u>

As indicated above, grouping VI involves compositions of several distinct and different kinds of symmetric relations: some transitive, some non-transitive, some reflexive, some non-reflexive or anti-reflexive, whereas grouping V is specifically concerned with asymmetric relations whose compositions are transitive. Consequently, grouping VI has been taken as the model for the cognitive actions present when the child is using a symmetric relation. Similarly, grouping V has been taken as the model for the cognitive actions present in the act of seriating objects at stage three level*, for combinativity has been interpreted in terms of relation composition to produce the required transitive property, as the following quotation shows:

"*In Grouping V, Addition of Asymmetrical Relations, consider the seriation $0 < A < B < C < D$, etc. If $0 < A$, $0 < B$, $0 < C$, etc, are denoted by a, b, c, etc, and $A < B$, $B < C$, $C < D$, etc, are denoted by a', b', c', etc, respectively, then combinativity $(a + a' = b)$ is interpreted as transitivity of the relation when written as given. (Beth and Piaget, 1966, p. 177)*[1]." ((22), p. 48).

Moreover on the basis of this argument Beth and Piaget (1966)[1] hypothesize that transitivity is necessarily present when a child exhibits behaviour characterized as stage three (operational) seriation* behaviour.

Thus we see that it is appropriate to the present investigation to consider the experiments and results associated with groupings V and VI. But the question as to which set of experiments should be considered first, now arises.

By following the suggestion

"*. . . the teacher should first see how the child grasps each of these properties* (i.e. reflexivity, symmetry and transitivity), *independently of the others in situations where they can be clearly illustrated.*" ((23), p. 28),

---

*Operational seriation (stage three) is distinguished by

1. the discovery of a systematic way of forming a series,
2. the ability systematically to insert new elements in an existing series.

(1) Beth, E.W. & Piaget, J. "Mathematical epistemology and psychology", Dordrecht-Holland: D. Reidel, 1966.

our attention is immediately drawn to grouping V which, according to Piaget, via seriation, focuses on the transitive property. Moreover, this grouping is considered by Piaget to be one of a pair of groupings which are the first to emerge.

"In fact, the operational groupings which become established at the age of round about 7 or 8 (a little before sometimes) end up with the following structures. First, they lead to the logical operations of class inclusion (the question of brown beads A being less numerous than the wooden beads B is solved about 7) and of seriation of asymmetric relations. From this comes the discovery of the transitivity on which are based the deductions: A = B; B = C therefore A = C; or A < B; B < C therefore A < C. Further, as soon as these additive groupings have been acquired the multiplicative* groupings are at once understood as correspondences: knowing how to seriate objects according to the relations $A_1 < B_1 < C_1 < \ldots$ the subject has no further difficulty in seriating two or more collections such as $A_2 < B_2 < C_2 < \ldots$ which correspond term by term: to one sequence of dolls of increasing size which he has already ordered the 7-year-old will know how to match a sequence of sticks or bags, and even be able to find,*

---

*The test situations associated with Grouping I - Primary Addition of Classes, and Grouping V - Addition of Asymmetrical Relations, focus on the child's ability

- to think of a set and subsets of that set simultaneously,
- to build up elements into an asymmetrical, transitive series,

respectively. In contrast, the test situations associated with Grouping III - Bi-univocal Multiplication of Classes, and Grouping VII - Bi-univocal Multiplication of Relations, focus on the child's capacity

- to find the intersection (logical product) of two or more sets,

(contd)

after everything has been mixed up, the element of the one sequence which corresponds to some arbitrary element of the other (the multiplicative character of the grouping does not add any difficulty to the additive operations of seriation which have already been discovered.)"

(Translation of (16), p. 158).

Let us therefore consider experiments associated with grouping V first.

2.5. Grouping V

(i) Piaget's transitivity studies

The core operation of grouping V (i.e. the building up of elements into a transitive, asymmetric series) has been studied via

(i) the ability to seriate 10 sticks (A - J) of varying lengths and then insert 9 more sticks (a - i) in their proper places. (Piaget, J. "The child's conception of number", New York: Humanities, 1952, ch. 6).

(ii) the ability to seriate three objects by weight, two at a time only, where volume is not a reliable clue to weight. Piaget, J. & Inhelder, B. "Le developpment des quantites chez l'enfant", Neuchatel: Delachaux et Niestle, 1941, ch. 10).

(iii) the "Conservation of weight and transitivity of the relation ' . . . weighs more than . . .' experiment". (Nuffield Mathematics Project, "Checking Up II", Chambers/

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* (contd)

- to build a double-entry matrix with respect to two asymmetric, transitive relations,

respectively.

Piaget regards the setting up of a one-to-one correspondence between two sets of

(contd)

Murray/Wiley, 1972, Summary Check-up No. 3),

to name but three. (See also Appendix 2c for further instruments designed to test a child's ability to use the transitive property of matching and length relations.)

To illustrate the general tenor of these experiments, let us examine more closely one of these experiments for investigating transitivity of weight, namely (ii) above. As outlined, this experiment entails placing before the child three objects of different weight (but weight uncorrelated with volume). The child is then asked to seriate them by weight (e.g. lightest, middle, heaviest) but under the condition that he can compare the weight of only two objects at a time. It turns out that young children in the preoperational subperiod of development have considerable difficulty in solving this problem. Typical responses of such a child are as follows:

(a) he establishes only that A is lighter than B and A is lighter than C, and then concludes that

(i) A is lighter than B which is lighter than C

or (ii) A is lighter than C which is lighter than B,

(b) he is unable to "see" that A is lighter than C is a necessary conclusion from the knowledge that A is lighter than B and B is lighter than C.

These responses indicate that in the first case (a) the child is drawing an invalid conclusion from evidence which does not permit a conclusion to be drawn, whereas in the second case (b) the child does not draw any conclusion from evidence which permits a conclusion to be drawn.

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* (contd)

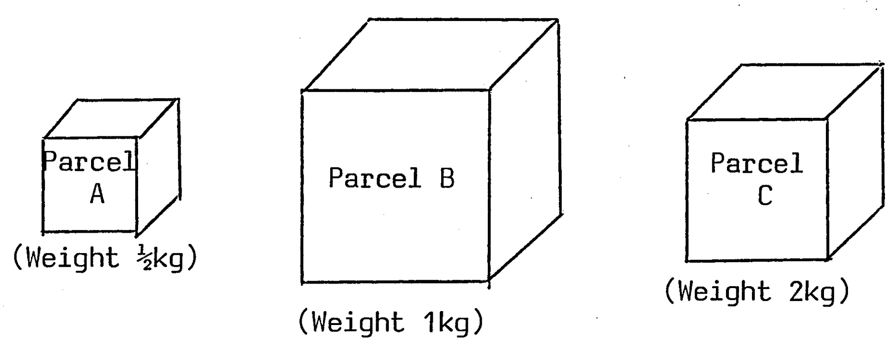
unseriated elements as the basis of all Grouping III operations.
seriated elements as the basis of all Grouping VII operations.

J.H. Flavell (14) summarizes Piaget's interpretation of this type of response as follows:

> "*The central difficulty underlying these diverse pre-operational failures, Piaget believes*((18), p. 301-302.), *is the inability to see that each element in an asymmetrical series must be simultaneously conceived in terms of both a direct* ($<$) *and an inverse* ($>$) *relational operation: the element B must be both larger than A and smaller than C to be inserted between them in the series. Piaget feels that the failure to grasp this reversibility inherent in systems of asymmetrical relations lies behind the younger child's occasional willingness to conclude* $B < C$ *from* $A < B$ *and* $A < C$, *his occasional reluctance to conclude* $A < C$ *from* $A < B$ *and* $B < C$, *and his general inability to create and manipulate asymmetrical series.*" ((14), p. 193)

However, one of the problems which arise when presenting an overview even of a limited set of experiments is that details that could be significant in a particular situation can be lost. For example, one reason why, for young children, weight and volume are apparently not seen as distinct and different properties which can vary independently, stems from the fact that weight and volume are often correlated in nature. This fact is often relied on by sighted adults as can be tested by asking any sighted person to judge the weight of two suitcases which are very different in volume (one large and bulky, the other small and compact), but which are approximately the same weight. After lifting both suitcases, the odds are in favour of his response being that he found the larger suitcase lighter, for on seeing the two suitcases, he sizes them up and anticipates that the bulky one will be heavier because of its volume and prepares himself accordingly, only to find it lighter than expected.

Thus, if no balance is used, it is possible that "bigger therefore heavier" type reasoning is operating for young children in the situation where the three objects used in experiment (ii) are three distinct cubical parcels as illustrated below:



Here, the child establishes that "A is lighter than B and A is lighter than C" and since this does not contradict the "bigger therefore heavier" type argument the child continues to use that argument and gives the response corresponding to $A < C < B$ without checking the relationship between B and C.

Thus we see that when devising a test situation which is intended to focus on a child's ability to effect transitive compositions of an asymmetric, transitive relation, such as " . . . is lighter than . . .", there are at least three points requiring careful consideration.
There is the need to check

(i) that the child has had sufficient experience in handling weight so that the likelihood of his recognizing the possibility of deducing something from $A < B$, $B < C$ is increased,

(ii) whether any of the key attributes are undifferentiated by the child in his everyday conversation (e.g. age and size as exemplified by the remark "he's bigger than me" made by a 4ft 2in 7-year-old boy of his 3ft 10in. 8-year-old friend.),

(iii) that there is no attribute of the materials selected other than the one on which the experiment is based, which could dominate the child's perception (e.g. length when transitivity of weight is under investigation.)

Concern over the use of distracting perceptual cues has also been expressed by T.P. Carpenter (24). He criticizes the studies by Piaget, Inhelder and Szemenska (1960) which relate to the logical inter-dependence of conservation and measurement, on this point:

> *". . . in all comparisons distracting cues were perceptual . . ."* ((24), p. 145)

Moreover, Carpenter maintains

> *"There is evidence that certain conclusions of Piaget et al (1960) resulted from this lack of experimental variability.*
>
> *They conclude that young children are dominated by the immediate perceptual qualities of the situation. However, the results of another investigation (Carpenter, 1971)*[1] *indicate that young children respond to numerical cues with about the same degree of frequency as perceptual cues."*
>
> ((24), p. 145)

Consequently

> *". . . the question as to whether conservation and measurement failures are primarily the result of a dependence on perceptual cues, the order of the cues or an interaction of*

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(1) Carpenter, T.P. "The role of equivalence and order relations in the development and coordination of the concepts of unit size and number of units in selected conservation type measurement problems", Technical Report No. 178, Wisconsin Research and Development Center for Cognitive Learning, Madison: The University of Wisconsin, 1971.

the two was investigated. That is, an attempt was made to determine whether young children respond differently to visual and numerical cues in conservation and measurement problems or whether they simply respond to the last cue available to them." ((24), p. 151),

was a main purpose in Carpenter's investigation "The Performance of First- and Second-Grade Children on Liquid Conservation and Measurement Problems Employing Equivalence and Order Relations", and the conclusions arrived at from this investigation which relate to this purpose are

"*It appears that it is not simply the perceptual properties of the stimuli that produce errors in conservation problems. There is no significant difference in difficulty between conservation problems and corresponding measurement problems in which the distracting cues are numerical. The position of Piaget (1952, 1960)*[1], [2], *Bruner, Olver and Greenfield*[3] *and others that young children are highly dependent on perceptual properties of events and that conservation problems occur because the immediate perceptual properties of the conservation problems override the logical properties that imply conservation, has been based on tasks in which distracting visual cues always appeared last. The results of the current investigation, however, demonstrate that*

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(1) Piaget, J. "The child's conception of number", Routledge and Kegan Paul, 1952.

(2) Piaget, J. "Equilibration and the development of logical structures", in J.M. Turner & B. Inhelder (eds.) "Discussions on child development", Vol. 4, Tavistock, 1960.

(3) Bruner, J.S., Olver, R.R. & Greenfield, P.M. et al. "Studies in cognitive growth", New York: John Wiley and Sons, 1966.

> *misleading numerical cues produce the same errors as misleading visual cues.*
>
> . . .
>
> *Thus it appears that the most significant factor in determining which cues young children attend to is the order in which the cues appear. . . . however, the order of the cues was not the only factor that was found to affect responses.*
>
> . . .
>
> *Thus, of the factors under consideration in this study it appears that centering on a single dominant dimension is the major reason for most conservation and measurement failures and the development of conservation and measurement concepts can be described in terms of increasing ability to decenter. In the earliest stage children respond on the basis of a single immediate dominant dimension. The dimension may be either visual or numerical, depending on the problem. . . .*
> ((24), p. 167-169).

However, in the present investigation, the conjecture that centering on a single dominant dimension is also a major reason for most transitivity failures, which underlies point (iii) above, arose from discussion of the situation involving the three cubical parcels. This situation highlighted the need for caution over the use of distracting perceptual cues when we saw the possibility that a child could be using "bigger therefore heavier" type arguments.

But the possible use of the "bigger therefore heavier" type argument in this situation also suggests that transitivity of volume could emerge before transitivity of weight, whereas with respect to conservation, conservation of weight occurs before conservation of volume. (See

(14), p. 299). But it is answers to questions such as

> Is there a natural order of concept formation which is substantially unaffected by teaching, or can the order be changed by appropriate experience/teaching?

that are needed if we are to achieve our original goal of building a framework within which observations about the development of the concept of equivalence relation can be organized. In fact, there are two issues which demand attention when applying the above question to the acquisition of the concept of transitivity.

1. If we consider transitivity with different physical quantities (e.g. length, volume, weight), is it the case that these are always acquired in some specific order, substantially independent of the experience/teaching given, or can the order be affected by the experience/teaching given?
2. If the concept of transitivity is broken down into components, is it the case that in every physical context these components are acquired in an invariant order?

At this point we should note that the objective of identifying the order of emergence of transitive relations, as outlined by Question 1 above, was not one of Piaget's major goals for the experiments he devised. When the children were working with a transitive relation Piaget was looking to see if they used the five properties of his grouping V, particularly reversibility, for as soon as reversibility appears in the solution of a particular problem

> ". . . *the child's thought (for this one problem at least) has passed beyond the level of preoperational representation into the subperiod of concrete operations.*" ((14), p. 165).

In other words, Piaget was focusing his investigations on part of the answer to Question 2.

(ii) The extensions of Piagetian-type transitivity experiments

(a) Recent research in the U.S.A.

Many experimenters in following up Piaget's investigations have, however, extended the scope of the tests used in an attempt to find answers for Question 1. For example, in the investigation by D.T. Owens (22), questions asked of disadvantaged five- and six-year-old children after formal instruction on

(i) establishing matching relations (i.e. "as many as", "more than" and "fewer than") and length relations (i.e. "longer than", "shorter than" and "as long as") only,

(ii) establishing matching and length relations (as above), conserving matching relations and transitivity of matching relations*

included

- To what extent does an experimentally induced capability to conserve and use transitivity of matching relations transfer across relational categories to conservation and transitivity of length relations?
- Is the ability to use transitivity of matching relations related to the ability to use transitivity of length relations?

(See Appendix 2c for notes on the transitivity tests used in this investigation.)

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*The chief method of the transitivity training was what has been termed fixed practice with empirical control (Smedslund, J. "The acquisition of transitivity of weight in five- to seven-year-old children", Journal of Genetic Psychology, 1963, 102, p. 245-255). The instructor gave explicit instructions for comparing sets A and B, then B and C. Sets A and C were compared after the child made a prediction of the relation between them.

In the discussion of these results which relate to the transitivity of matching and length relations, we find

"*The mean performance of the children in the full treatment group* (i.e. those given formal instruction on transitivity of matching relations) *was significantly greater than the mean performance of the children in the partial treatment group* (i.e. those not given formal instruction on the transitivity of matching relations) *on the Transitivity of Matching Relations Test. This was an indication that the treatment was effective in improving the ability of the children in using the transitive property of these relations. However, the results from the Transitivity Problem indicated no relationship between a student's membership in a treatment group and his level of performance on the Transitivity Problem. This apparent discrepancy may be interpreted by an examination of the tasks and the instructional activities. In the instructional setting the children were instructed to establish the relation between two sets, say A and B, and between B and a third set C. The sets were constructed in such a way that the same relation existed between B and C as between A and B. The children were then asked to predict the relation between A and C and were given an opportunity to verify their prediction. Each item of the structured transitivity test followed this same procedure except that on the test the child did not have the opportunity to verify his conclusion. Also in the testing situation the objects were screened at the time of the transitive inference, whereas this was not always the case in instruction. In the Transitivity Problem the child was required to compare sets A and B, and sets A and C where A contained two more* objects than B or C. He then was required to remove (either physically or mentally) two objects from the set A to form a new set which was equivalent to B and C before applying the transitive property of "as many as", and to conclude that B was equivalent to C. The reasonable conclusion then, is that the treatment improved the ability of the children to perform tasks very much like the treatment activities, but this improvement did not generalize to the Transitivity Problem, a higher order task.

These results are consistent with previous transitivity training studies. In a study with five- to seven-year-old children, Smedslund[1] found that none of the children acquired transitivity of weight due to practice. In another study, he (Smedslund[2]) found that about 30% of a group of eight-year-old children acquired transitivity of weight by practice, while only 12.5% of a control group acquired transitivity. Thus, behaviour indicative of transitivity has been obtained in some transitivity studies, but it appears to be difficult to induce transitivity by practice.

It appears from Piaget's theory that if a child's cognitive structure contains the grouping of addition of asymmetrical, transitive relations, he can use the transitive property of any such relations, regardless of concrete embodiment.

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(1) Smedslund, J. "The acquisition of transitivity of weight in five- to seven-year-old children", Journal of Genetic Psychology, 1963, 102, p. 245-255.

(2) Smedslund, J. "Patterns of experience and the acquisition of concrete transitivity of weight in eight-year-old children", Scandinavian Journal of Psychology, 1963, 4, p. 251-256.

Piaget[1] has indicated, on the contrary, that a formal structure of transitivity is not acquired all at once, but it must be reacquired every time a new embodiment is encountered. Sinclair[2] has further suggested that properties of the concrete embodiments (such as discrete or continuous) will affect the attainment of psychologically parallel concepts.

In the present study, experiences in length relations were given to introduce an embodiment of the transitive relations in addition to the matching relations, but no instruction was given in transitivity of the length relations. The results indicate that while the treatment improved the ability to use transitivity of matching relations, there was no corresponding improvement in the ability for the children to use transitivity of length relations. Thus, the conclusion was reached that the treatment was rather task specific and no generalized scheme of transitivity was induced.

This conclusion is consistent with Piaget's conjecture, and with the results of training studies in conservation. For example, Beilin's[1] subjects improved in conservation of number and length when experiences were given. However, the training was not sufficient to foster generalization to conservation of area." ((22), p. 69-70).

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(1) Piaget, J. "The child's conception of number", Routledge and Kegan Paul, 1952, p. 204.

(2) Sinclair, H. "Numbers and measurement", in M.F. Rosskopf, L.P. Steffe and S. Taback (Eds.) "Piagetian cognitive-development research and mathematics education", Washington, D.C.: National Council of Teachers of Mathematics, 1971.

(Beilin's (1) overleaf)

But the investigation by D.T. Owens outlined above does not provide conclusive evidence that transitivity of matching relations necessarily precedes transitivity of length relations, and this is duly acknowledged.

> *"These data gave no indication that, for the subjects in this study, the ability to use the transitive property in one relational category consistently preceded the ability to use the transitive property in the other relational category."* ((22), p. 69).

(b) Commentaries based on Piaget's work

The last of Owen's statements quoted above seems to contradict the general tenor of observations on the order of emergence of transitive relations that are to be found in a number of commentaries written in the late 1960s, and which are based on the work of the Geneva school. Typical of such commentaries are Chapters 1 - 8 of "Primary Mathematics Today" by E.M. Williams and H. Shuard (25) as the introduction indicates:

> *"The book begins with a child's first experiences of objects and events, and traces the growth of mathematical ideas in the light of the findings of research workers like Piaget who have studied the development of children's thinking."* ((25), p.2).

In fact, Chapter 2 is devoted to a summary of the stages of growth identified by Piaget and his associates, and all references to aspects of concept development discussed in these eight chapters lead to one of the following books:

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(1) Beilin, H. "Learning and operational convergence in logical thought development", Journal fo Experimental Child Psychology, 1965, 2, p. 317-339.

Piaget, J. & Inhelder, B. "The Child's Conception of Space"

Piaget, J. "Logic and Psychology"

Inhelder, B. & Piaget, J. "The Early Growth of Logic in the Child"

Piaget, J. "The Child's Conception of Number"

Piaget, J., Inhelder, B. & Szemenska, A. "The Child's Conception of Geometry".

But in order to gauge the extent of the contradiction between Owen's statement and the conclusions that can be drawn on the order of emergence of transitive relations from this source, we require an appropriately deduced sequence of such conclusions. Let us therefore consider the following set of quotations:

"*A relation which children recognize at a very early age is that of 'bigger than' or 'smaller than'.*" (p. 36).

"*. . . seriation depends on using the relation 'bigger' (or smaller) to connect each successive pair of things in a sequence. Such relations can also be added. If one tin is taller than another, and the second tin is taller than a third, then the child putting the two relations together, will be able to say that the first tin is taller than the third*'. (p. 18).

"*. . is bigger than . .*"

↓ at a later stage, refined to

"*. . is taller than . .*"
(Height)

or (if appropriate)

"*. . is longer than . .*"
(Length)

"*At the preoperational stage a child is unable to hold in mind more than one relation at a time, so that he is unable to compare, for instance, the*

later

capacities of two jugs which differ in width as well as in height, . . . At a later stage he is able to take into account at the same time both the greater height and the smaller base, and so to recognize that the volume is unaltered by the change in its shape. This grasp of the logical multiplication of relations is a characteristic of the concrete-operational stage of thinking." (p. 36)

"The ordering of weights is more difficult than forming a sequence of sets, lengths or capacities, since each pair must be balanced until the correct ordering is found." (p. 42).

". . holds more than . ."

(Capacity)

or

". . takes up more space than . ."

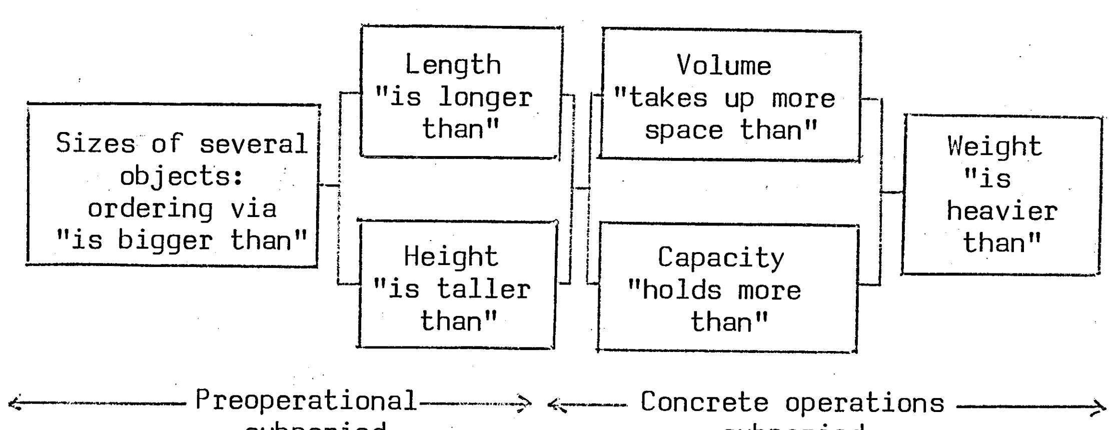
(Volume)

later still

". . is heavier than . ."

(Weight)

We obtain



where 'volume' is associated with the 'amount of material in the solid object' and 'capacity' is associated with the 'space inside a container'.

Clearly, order in the acquisition of transitivity with different physical quantities is implied by these quotations. Hence, Owen's statement does conflict with the overall trend implied by the quotations from "Primary Mathematics Today". But on taking into account the specific context of Owen's statement, namely, the relationship between transitivity of matching and length relations, we see that no contradiction has in fact occurred because of the coarseness of the framework that we were able to set up from the quotations used. Clearly, more information is required to close these gaps.

Fortunately, there has been a dramatic Piagetian renaissance in mathematics education in the United States during the past decade. This stems from the recognition that Piaget's theory and data were not generated by researchers primarily interested in the establishment of scientific pedagogy, so that it cannot be indiscriminately applied in the hope that, somehow, such application will improve the state of affairs in mathematics education. However, the Americans are assuming that applications of cognitive-development can be made to mathematics education in which learning-instructional models can be formulated and tested empirically, on the understanding that such a model may not attain the status of a theory, but that it can be used to describe and prescribe learning-instructional phenomena concerning mathematics until it proves unusable in terms of desired objectives and/or learning process. It is against this background that the studies undertaken by D.C. Johnson (12), Lesh (19), Owens (22) and Carpenter (24), which have already contributed to the discussion in this section, and those of Steffe and Carey (26) and M.L. Johnson (27), should be viewed.

On referring to Appendix 2c, we see that four of these six studies have involved transitivity. The results of the investigation by D.T. Owens which relate to this property have already been considered. It remains to consider the contribution of the studies by D.C. Johnson (12), Steffe and Carey (26) and M.L. Johnson (27), to our knowledge of transitivity.

(iii) The contribution of the studies by D.C. Johnson, Steffe and Carey, and M.L. Johnson

The study by D.C. Johnson (12) was designed to include the following purpose:

to investigate that if specific instructional conditions improve abilities to

(a) form classes

(b) establish selected equivalence or order relations

whether transfer occurs to the transitive property of the selected equivalence and order relations.

Hence, activities were designed to define operationally the relations "more than", "fewer than" and "as many as". The equivalence relations "same shape as" and "same colour as" were also included in the investigation.

The results showed that the instructional activities produced a positive transfer to the transitive property of the equivalence and order relations used in the study. But this was attributed to clarity of language rather than to usage of the transitive property as the items based on the relations of shape and colour contributed greatly to the rather high mean scores of the Transitivity Test (TR). (See Appendix 2c). Mean scores for control and experimental groups on matching relations were 30% and 55% respectively, whereas the analogous means for the shape and colour relations were 86% and 97% respectively.

Although it was noted that

". . . relations such as "same shape as" and "same colour as" and the transitive property of these relations were very easy even for kindergarteners. Very little, if any, instruction is required in kindergarten for such relations."

((12), p. 143),

no attempt was made to relate this to possible differences in nature between the concepts underlying the equivalence relations "same shape as", "same colour as" and "as many as". For Lesh (19) has in fact identified at least two subcategories within the class of concepts:

"An example of the first of these types is the concept of "red". This type of concept may be referred to as a <u>concrete concept</u> since all of the information that is necessary in order to distinguish instances from non-instances is directly given in the preceptual field. Another type of concept may be referred to as an <u>operational concept</u> in that it involves abstractions, not just from directly perceived properties of objects, but also from relations between objects, or from operations (or transformations) that are performed on objects (Piaget, 1971, p. 26)[1]." ((19), p. 95).

These definitions reveal a fundamental difference in the methods required to teach concrete concepts and operational concepts. In order to teach a concept such as "red" or "triangle", the child can simply be shown examples and counterexamples of red or triangular objects, whereas in order to give a child an intuitive understanding of the relation "as many as" or "same length as", the situation is not so simple.

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(1) Piaget, J. "Science of education and the psychology of the child", translated by D. Coltman, New York: Viking, 1971.

The complexity of developing an intuitive understanding of the length relations "same length as", "longer than" and "shorter than" is discussed in the introduction to the study undertaken by Steffe and Carey (26). Here they establish a case to justify the following point of view:

> "*Before presenting length relations to children below six years of age, it seems necessary then, to define the relations on a basis that does not assume number. Such a definition follows. Let A, B and C be segments. A is the same length as B, if and only if, when segments (or their transforms) lie on a line in such a way that two endpoints coincide (left or right), the two remaining endpoints coincide. A is longer than B if and only if the remaining endpoint of B coincides with a point between the endpoints of A. Also in this case, B is shorter than A.*"
> ((26), p. 20),

and the operational counterpart of this definition was used as a basis for the instructional sequence designed to develop the ability of children to establish a length relation between two curves.

Concerning the main investigation we find that one of the questions asked of four- and five-year-old children after formal instruction on

(i) establishing length relations only,

(ii) establishing length relations, conserving length relations and using properties and consequences of length relations,

was

> Are children able to use the transitive property of length relations?

In the discussion of results which relate to this question, we find

"Few five-year-old children were able to use the transitive property after only instructional experience in establishing length relations. At this point in time, only 16% of the five-year-olds used the transitive property. At the same point in time the distribution of total scores for the four-year-olds did not statistically depart from a binomial distribution based on random responses, so no four-year-old was considered able to use the transitive property of length relations. Some children performed poorly because of their inability to establish the two initial comparisons, an inability Smedslund (1963)(1) considers as a reason for failure of some young children to use the transitive property.

Instructional Sequences II and III (designed to develop the ability of children to use the reflexive and nonreflexive* properties; to conserve length relations, use the asymmetric property and logical consequences respectively), *did increase the ability of five-year-olds to use the transitive property, since the percent of five-year-olds able to use the transitive property increased to 31. These same experiences did not increase the ability of four-year-old children to use the transitive property because again the distribution of total scores for the four-year-olds did not statistically depart from a binomial distribution based on guessing. The number of five-year-olds that used transitivity of length relations*

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* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

(1) Smedslund, J. "Development of concrete transitivity of length in children", Child Development, 1963, 34, p. 389-405.

is below that found by Braine[1] but above that found by Smedslund (1964)[2]. It appears that these experiences were not logical-mathematical experiences that readily increase children's ability to use the transitive property. All the children may not have had a mental structure sufficient to allow assimilation of the information. The mean Verbal Maturity and I.Q. of five-year-old children who were able to use the transitive property appeared to be slightly higher than for those who do not use this property. However, the correlations between these two variables and transitivity scores earned by the total sample was not statistically different from zero. Also, there appears to be little, if any, relationship between the variables Age and Social Class and the ability of four- and five-year-old children to use the transitive property." ((26), p. 41-42).

These results can be used to argue that there is no case at all for attempting any instruction using similar populations with a view to improving the use of the transitive property of length relations before five years of age. Moreover, this line of argument is consistent with the views of Beth and Piaget (1966)[3], who point out that although seriation behaviour can be found in children from the sensory-motor

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(1) Braine, M.D.S. "The ontogeny of certain logical operations: Piaget's formulation examined by nonverbal methods", Psychological Monographs: General and Applied, 1959, 73, (5, Whole No. 475).

(2) Smedslund, J. "Concrete reasoning: A study of intellectual development", Monographs of the Society for Research in Child Development, 1964, 9 (Serial No. 93).

(3) Beth, E.W. and Piaget, J. "Mathematical epistemology and psychology", Dordrecht-Holland: D. Reidel, 1966.

stage onwards, it is only when seriation becomes 'operational'* at about eight years of age that transitivity emerges. (See also the translation of (16), p. 158 on page 38 of this section.)

However, concern over

(i) the lack of information on the relationship which may exist between seriation ability and properties of order relations, and

(ii) the small amount of research reported in which training procedures were used in an attempt to facilitate seriation ability,

led M.L. Johnson (27) to investigate

1. the influence of training on the ability of first and second grade children to classify and seriate objects on the basis of length,
2. the influence of such training on the child's ability to conserve and use the transitive properties of the relations "same length as", "longer than" and "shorter than".

Additional objectives included an investigation of the relationship between the child's ability to use the transitive property of the relations "longer than" and "shorter than" and his ability to seriate on the basis of these relations; and to determine if the ability to seriate linear objects was material specific or relational specific.

In the discussion of the results an important question emerges:

"The extent of the subjects' seriation ability, in terms of being operational in a Piagetian sense, must be questioned when one considers the overall performance on

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*See footnote on page 37.

the transitivity test. *In particular, the treatment* (i.e. training in classification and seriation on the basis of length) *appears to have had no effect on the children's ability to use the transitive property of the order relations involved in this study. In fact, no significant relationship could be detected between transitivity of "longer than" and "shorter than" and the ability to seriate using these relations. This finding is not consistent with the hypothesis presented by Beth and Piaget (1966)*[1] *and confirmed by Elkind (1964)*[2] *that transitivity is necessarily present when a child exhibits behaviour characterized as stage three seriation behaviour. The question is raised concerning what is 'operational' seriation behaviour. In this study, children were able to seriate strings and sticks, as well as insert additional sticks into a series already formed without any trouble but could not use the transitive property of "longer than". Such responses would indicate that the seriation training was successful in training the children to use an algorithm which was not part of an operational scheme. If this was the case, it would be expected that the relationship*

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(1) Beth, E.W. & Piaget, J. "Mathematical epistemology and psychology", Dordrecht-Holland: D. Reidel, 1966.

(2) Elkind, D. "Discrimination, seriation and numeration of size and dimensional differences in young children: Piaget replication study VI", Journal of Genetic Psychology, 1964, 104, p. 275-296.

between seriation and transitivity would be negligible. If, however, the children were now 'operational' then these findings suggest that contrary to Piaget's hypothesis, seriation behavior does not necessarily imply transitivity. In any case, it is clear that we need additional guidelines as to what constitutes operational behavior and more effective ways of measuring such behavior." ((27), p. 90-91)

To suggest that a more precise study of the relationship between seriation and transitivity will resolve the possible contradiction between Beth and Piaget's hypothesis and the results of this study, is an easy option to take. It is in effect no more than an indication that someone else should tackle the problem as the proposer of this suggestion has been unable to find possible reasons for the apparent contradiction.

Let us therefore attempt the more difficult option: to identify possible flaws in the components of the arguments and to suggest appropriate modifications.

(iv) <u>The identification and modification of points of weakness in the hypothesis: seriation implies transitivity</u>

As it stands the hypothesis presented by Beth and Piaget (1966), namely,

> transitivity is necessarily present when a child exhibits behaviour characterized as stage three seriation behaviour

is clearly false when transitivity is defined as follows:

> The relation R on a set S is TRANSITIVE if, whenever xRy and yRz, then xRz, for all x, y, z $\in$ S.

For this definition covers not only the cases where the set S contains three distinct elements x, y, z, only, but also the cases where

(i) x, y, z may be just three distinct elements of the set S which contains more than three distinct elements

(ii) x, y, z need not be distinct elements of the set S,

and we note that none of the investigations discussed in this section cover any of the special cases which occur under (ii). Nor have we found any evidence which suggests that Beth and Piaget took account of these special cases before their hypothesis was presented. Thus, we can argue that the core operation for grouping V as interpreted by Beth and Piaget and investigated by these studies, is restricted-transitivity which we now define as follows:

A relation R in a set S has restricted-transitivity if whenever xRy and yRz, with x, y, z, all distinct, then xRz ($x, y, z \in S$).

At this point we also note that this definition of restricted-trancitivity contains at least two levels of applications:

Level I: when the set S contains three distinct elements x, y, z only,

Level II: when the set S contains more than three distinct elements,

for all of the studies discussed in detail in Sections 2.5 (i)-(iii) except Carpenter's, were concerned with Level I. The extent to which a child was able to apply restricted-transitivity to situations involving four (or more) cubical parcels, four (or more) sticks, or four (or more) collections, was not included in these studies.

It is therefore suggested that the first modification of Beth and Piaget's hypothesis should be

Restricted-transitivity (Level I) is necessarily present when a child exhibits behaviour characterized as stage three seriation behaviour.

But the argument which led to the above amendment did not take into account the fact that on establishing

"heavier than"	"lighter than"	"same weight as"
"longer than"	"shorter than"	"same length as"
"more than"	"fewer than"	"as many as"
		"same colour as"
		"same shape as"

on the appropriate triples, ALL these relations give rise to instances of restricted-transitivity (Level I). If, therefore, we attempt to apply the strict criterion for concept attainment suggested by Lesh (19), namely

> "*A concept has been attained when the child can, within a given universe of experience, distinguish instances from noninstances of the concept.*" ((19), p. 95),

within the context of the weight, length or matching relations, we have, for example

> Restricted-transitivity (Level I) has been attained when the child can, with respect to length relations, distinguish instances from noninstances of restricted-transitivity (Level I),

which is impossible. Only by extending the universe of experience to noninstances, i.e. to relations such as "lives next door to", on an appropriate triple of persons, can we ensure concept attainment of restricted-transitivity (Level I).

Unfortunately, none of the relevant investigations discussed in this section presented any evidence that such counterexamples had been taken into consideration. It seems to be the case that all of these studies involved situations in which it was impossible for the child to attain the concept of restricted-transitivity (Level I) as specified above. If so, then not even the first modification of Beth and Piaget's hypotheses was being tested by M.L. Johnson (27), and so "Restricted-transitivity (Level I)" must now be deleted from the modification.

Thus we see that the second modification of Beth and Piaget's hypothesis should take the following form:

> ________ is necessarily present when a child exhibits behaviour characterized as stage three seriation behaviour.

But now the question arises as to what should fill the gap left by the deletion of "Restricted-transitivity (Level I)".

Remembering that identification of behavioural counterparts of one or other differentiating component of a given grouping was a major factor in the design of Piaget's experiments, is it possible that "the behavioural counterpart of restricted-transitivity (Level I)" is the required gap-filler?

If this is the case, then there is a plausible argument which accounts for the discrepancies such as

> "*. . . children were able to seriate strings and sticks as well as insert additional sticks into a series already formed without any trouble but could not use the transitive property of "longer than".*" ((27), p. 91),

where we interpret "use the transitive property of 'longer than'" to mean "use restricted-transitivity (Level I) of longer than". The argument is that the acquisition and use of the behavioural counterpart of restricted-transitivity (Level I) is analogous to the acquisition of an unconscious habit or to perfectly correct use of grammar by a young child, in that the acquisition and use of the behavioural counterpart of restricted-transitivity (Level I) occurs before the child is explicitly aware of and can verbalize his analysis of the operations and relations that are implicit in his seriation activities. In addition, we also note that restricted-transitivity (Level I) is formulated as an implication and the proper use of an implication is more than the enunciation of ideas in sequence.

Thus, on the basis of the above discussion, it is proposed that the hypothesis underlying experiments to follow the question raised by M.L. Johnson, should be

> The behavioural counterpart of restricted-transitivity (Level I) is necessarily present when a child exhibits behaviour characterized as stage three seriation behaviour,

for the decision to take stage three seriation behaviour as the actual behavioural counterpart of restricted-transitivity (Level I) must be left to the educational psychologists. But until a decision is made on this point, any attempt to rebuild a framework which takes into account the levels of application of transitivity considered above, will be incomplete.

2.6. Grouping VI

(i) Piaget's symmetry studies

On turning our attention to grouping VI, we find that

> *"There is very little direct experimental evidence on this grouping. What there is concerns almost exclusively the acquisition of the symmetry property of symmetrical relations . . ."* ((14), p. 194).

This comment is still applicable, but during his pre-1930 studies, Piaget did show that children in the preoperational subperiod of development tend not to see the symmetry which may or may not exist in relations such as " . . is the brother of . . ", ". . is the enemy of . .", and so on.

For example, in "Judgement and Reasoning in the Child", first published in 1928 by Harcourt Brace, New York, we find an experiment based on finding the absurdity in each of five absurd sentences drawn from the Binet-Simon intelligence test (1917), which included the sentence

I have three brothers: Paul, Ernest and myself (Tester: Male)
sisters: Pauline, Jeanne & myself (Tester: Female)

Unfortunately, this sentence highlights the conflict that exists between a possible logical interpretation of the word "brother" as "male and has the same parents as" which produces a reflexive relation (any male has the same parents as himself), on a set of men or boys, and colloquial usage in which a male cannot be his own brother. Thus, colloquial usage gives rise to an antireflexive relation on a set of men or boys, and hence to an absurdity in the above sentence, whereas the above logical interpretation of "brother" does not. However, the purport of the above sentence was interpreted by Piaget as follows:

> *"The three brothers test requires that the child should find a contradiction between the existence of three brothers in one family (Paul, Ernest and myself) and the proposed judgement "I have three brothers, (Paul, Ernest and myself)."*
> ((28), p. 74).

It is therefore against colloquial usage that the following analysis of results should be judged.

Piaget found that of the 44 boys aged 9 to 12 years and 3 aged 14, only 13 succeeded in finding the absurdity. For the 72% who did not succeed, some failed because they did not view "myself" (i.e. the male tester) as a brother to Paul and Ernest, although they readily asserted that Paul and Ernest are the brothers of "myself". Thus, for these boys, the total number of "brothers" in the family is two: Paul and Ernest. And from Piaget's point of view, this meant that these boys had found the "wrong absurdity". Other boys assimilated the relational "I have" into a classificatory "there are" in the sentence and so found nothing absurd about it. There was also a third group of boys for which differentiation and coordination between relational and classificatory "brother" was made but not sustained throughout their reasoning.

From these observations Piaget argues that the various types of incorrect answers given by the boys, indicates

(i) their inability to differentiate between two points of view

(a) that of "brother" as a SET with set members ("we are three brothers", "I am a brother", etc)

(b) that of "brother" as a RELATION between individuals ("I have three brothers", "he is my brother", etc)

and more generally

(ii) their difficulty in handling relations as opposed to sets.

But this preliminary study of the brother concept was in fact followed up by a second, larger-scale investigation in which about 240 children aged 4 to 12 years were asked the following set of questions:

"1. *How many brothers have you? And how many sisters?*

(Let us suppose that the child has a brother A and a sister B.)

And how many brothers has A? And how many sisters?

And how many brothers has B? And how many sisters?

2. *How many brothers are there in the family?*
How many sisters?
How many brothers and sisters altogether?

3. *There are three brothers in a family: Auguste, Alfred and Raymond. How many brothers has Auguste? And Alfred? And Raymond?*

4. *Are you a brother? What is a brother?*
sister? sister?

5. *Ernest has three brothers, Paul, Henry and Charles. How many brothers has Paul? And Henry? And Charles?*

6. *How many brothers are there in this family?"* ([28], p. 98)

In this second investigation the principal findings were as indicated below:

<u>Table showing the age when at least 75% of the children in that age group answered the question correctly</u>

Age	No. of the question(s) answered correctly					
4	-					
5	-					
6	2					
7	2					
8	2,	3				
9	2,	3,	4			
10	1,	2,	3,	4,	5,	6
11	1,	2,	3,	4,	5,	6
12	1,	2,	3,	4,	5,	6

Question 1: Children had difficulty in seeing themselves as brothers or sisters of their own siblings. The extent of their difficulty is indicated by the following table which shows the percentage of right answers given by the

different age groups

Years:	4 - 5	6 - 7	8 - 9	10 - 11	12
Percentage:	19%	24%	55%	87%	100%

Question 2: Some children also had trouble including themselves in the total number of brothers and sisters in their family. However, the success rate of 75% achieved at 6 years of age when compared with the results obtained for Question 1, suggest that

"*the child has far oftener had occasion to take up the point of view of his family as a whole than that of each one of his brothers and sisters.*" ((28), p. 101)

Questions 3, 5 and 6:

The difficulties highlighted by Questions 1 and 2 were augmented when parallel questions were asked about a hypothetical family. Question 3, however, was easier for most of the children than either Question 1 or Question 5. Piaget suggests that the explanation for this is

"*In the case of test 1 the child has more difficulty in entering into the point of view of his brothers than into that of the three brothers of test 3, because in the case of his own family it is not enough for him to enter into the point of view of others, he must also look at himself from the point of view of others, which is twice as difficult. Now in test 5 the child is placed straight away at the privileged point of view, that of Ernest. The difficulty is therefore analogous in a sense to that of test 1. These considerations explain why test 3, which does not involve these* *peculiar difficulties, is found to be easier than test 1."* ((28), p. 103).

Second half of Question 4: (i.e. the question calling for definition of the word 'brother' or 'sister'.)	An interesting sequence of responses emerged:

<u>Stage 1</u>: The most primitive definition simply states that a boy is a brother.

e.g. *"Lo (age 5): "A sister is a girl you know."*

- "Are all the girls you know sisters?"

- "Yes, and all the boys are brothers."

((28), p. 104).

<u>Stage 2</u>: The child realizes that there must be two or more children in the family in order to call one of them a brother, but the concept is not yet genuinely relational for the child does not assign the title to all the appropriate children.

e.g. *"Hal (age 9): "When there is a boy and another boy, when there are two of them."*

- "Has your father got a brother?"

- "Yes".

- "Why?"

- "Because he was born second."

- "Then what is a brother?"

- "It is the second brother that comes".

- "Then the first is not a brother?"

- "Oh no. The second brother that comes is called brother".

Piaget comments: *"It would be impossible to show more clearly the absence of relativity from the word* *'brother'."*

((28), p. 105).

Stage 3: The child was able to give a definition which implies the idea that there must be at least two in the same family for there to be a brother or sister, and which includes a fair to good grasp of the relational meaning of the word.

e.g. *"Mi (age 7½): "A brother is . ."*

- *"a boy"*
- *"Are all boys brothers?"*
- *"Yes"*
- *"Is a boy who is the only one in the family a brother?"*
- *"No"*
- *"Why are you a brother?"*
- *"Because I have sisters"*
- *"Am I a brother?"*
- *"No"*
- *"How do you know?"*
- *"Because you are a man."*
- *"Has your father got brothers?"*
- *"Yes"*
- *"Is he a brother?"*
- *"Yes"*
- *"Why?"*
- *"Because he had a brother when he was little."*
- *"Tell me what a brother is."*
- *"When there are several children in*

a family."

N.B. According to Piaget's data about 60% of 7-year-olds and 75% of 9-year-olds were able to give "correct" (i.e. Stage 3 type) definitions.

Thus we see that the above analysis of responses to Question 4 supports the conjecture:

The semantic features that are more general, more central to the meaning of the word are acquired first,

with respect to the word 'brother' (or 'sister'). This evidence also implies that the features associated with the colloquial use of the word 'brother' are acquired as follows:

male and not adult ⟶ Stage 1.

↓ + (later)

implies more than one in the same generation but the title is not assigned to all the children where appropriate ⟶ Stage 2.

↓ + (later still)

if a boy has a brother, then he is a brother also (i.e. the symmetric aspect of the definition) ⟶ Stage 3.

<u>(ii) Confirmation of these results by Danziger</u>

A parallel set of results was also obtained by K. Danziger (29) in an investigation conducted with 41 Australian children (20 boys, 21 girls) between the ages of 5 and 8.

"In order to study the development of the understanding of relational terms a group of Australian children between the ages of five and eight were asked to give definitions of a number of kinship terms like brother, sister, daughter, uncle, cousin. They were also asked a series of questions relating to the use of each term. The findings indicate a clear separation between two conceptual levels which show a correlation with age. At the first level, the terms are used attributively and do not imply a relationship, while at the second level they are used relationally. Further, two distinct levels in the relational use of these terms appeared. At the lower level the term expresses a relationship that is both concrete and isolated, while at the higher level the relationship is linked up with others to form a system and its definition derives from its position in this system. The kinship term is now defined in a general way."
((29), p. 231).

(iii) Critique of the above studies

Unfortunately, these investigations by Piaget and Danziger not only ignore the possibility of children interpreting the word 'brother' in its logical form (see page 67) but they do not take into account the possible application of a previously encountered relational property which could be influencing the type of response made by some of the children to the questions used. This second relational property is restricted-transitivity.

In section 2.5 (iv), the term restricted-transitivity was deliberately chosen because the children who were able to make the transitive inference in the studies discussed, had been working with concrete embodiments in which a, b and c were distinct. These children had yet to encounter the impact on their understanding resulting from the application of the definition of transitivity (see page 23) to situations summarized by the following:

Consider a relation R on a set, and a pair of elements {a, b} such that aRb and bRa are true.

If R is transitive, this entails the truth of aRa also, for as aRb and bRa then we must have aRa.

Similarly, bRb is true also, since bRa and aRb implies bRb.

Hence, within a transitive relation, any two elements of a pair such as {a, b} for which aRb and bRa are true, must be elements for which aRa and bRb are also true.

For these children "transitivity" meant "if aRb and bRc, then aRc where a, b and c are distinct".

Let us therefore consider some of the possibilities associated with the emergence of restricted-transitivity.

Suppose the experimenter follows up a correct response to Question 3 with supplementary questions such as

Is Raymond the brother of Alfred? Is Alfred the brother of Auguste?

Is Raymond the brother of Auguste?

What else can you tell me about another family when I tell you that Robert is the brother of David and David is the brother of Paul?,

evidence could be obtained about this additional aspect.

However, evaluation of the responses given to these questions will require care. For example, let us suppose that as 75% of the children aged 8 and upwards gave correct answers to Question 3, that Hal (aged 9) was one of them. The two responses

"Because he was born second"

and

"It is the second that comes",

that Hal gave in response to Question 4 (see page 71), suggest that the additional relation ". . is younger than . ." is associated with his interpretation of the word 'brother'. If this is so, the order of the names in Question 3 (i.e. Auguste, Alfred, Raymond) could be taken by Hal as information on the order of birth so that

"Is Raymond the brother of Alfred?"

is interpreted as

"Is Raymond younger than Alfred?"

etc. With this mental set in operation, Hal's response

"Robert is the brother of Paul",

really means

"Robert is younger than Paul".

But even when contamination by an additional relation such as ". . is younger than . ." has been eliminated, the response "Robert is the brother of Paul" does not provide conclusive evidence on the child's ability to use restricted-transitivity in similar situations. This response is concerned with a situation in which the child is outside the family under consideration. Where appropriate, the child's ability to use restricted-transitivity with respect to his own family should also be tested. The evidence collected when the child is part of the family under consideration could show what effect, if any

(i) the size and composition of the family

(ii) his position in the family

has on this ability. Furthermore, if as Piaget suggests, it is twice as difficult for the child to see his family from the point of view of his own siblings and to look at himself from their point of view also, there is the possibility that a child will be able to analyse correctly the restricted-transitivity in a hypothetical family, but that he will be unable to transfer and extend this analysis to his own family. Hence, the following stages in a child's ability to handle restricted-transitivity seem possible:

Restricted-transitivity	Hypothetical family	Own family
Stage A	x	x
Stage B	/	x
Stage C	/	/

Note that it is also possible that the child's response to the question

> What else can you tell me about another family when I tell you that Robert is the brother of David and David is the brother of Paul?,

could also provide additional evidence on the child's capacity to grasp the symmetry in the above situation. This in turn could lead to the confirmation (or rejection) of a parallel set of stages in the child's capacity to grasp symmetry which is implied by Piaget's suggestion.

But whether or not it is confirmed that the stages A - C outlined above occur for both restricted-transitivity and symmetry, questions concerning the extent of the interactions between these two properties remain. For example, is it true that both properties emerge together, resulting in confusion in arguments used by the child, as first one and then the other dominates his thinking at a particular moment? Answers to this question and others which highlight the nature of the inter-action between restricted-transitivity and symmetry are required before we can complete the evidence on the order of acquisition of the features associated with the colloquial use of the word 'brother'. In other words, the possibility that

the child is able to see his own family from the point of view of his own siblings and to look at himself from their point of view	=====>	the child is able to use restricted-transitivity and symmetry as appropriate with confidence,

requires further investigation.

Thus we see that on reviewing an example of

> ". . . *one of Piaget's favourites for this grouping: the symmetrical relations found within a genealogical hierarchy.*" ((14), p. 182),

we find that symmetry alone is not necessarily the only relational property that could be in use with relations such as

. . is the brother (sister) of . .

. . is the cousin of . .

. . has the same grandfather as . .

on the appropriate sets. Moreover, when we also take into consideration

(i) the differences between the logical and colloquial interpretations of the word 'brother' (or 'sister'),

(ii) the difficulties surrounding the comprehension of secondary kinship terms such as cousin, nephew, etc, experienced by most 10-year-olds, because these words involve more semantic features and are thus more complex,

the suitability of these relations for studying the child's capacity to grasp 'symmetry' is suspect. It is therefore somewhat surprising to find the following in "Checking Up III", *"We now focus on the symmetric property. Relations established between the members of a family provide interesting situations for check-ups. . . . The relation 'p is the sister of q' in an all-girl family is symmetric, as would be 'x is the brother of y' in a family of boys. If we consider the children of several brothers and sisters, the relation 'm is the cousin of n' is symmetric when considering both boys and girls."* ((23), p. 29),

when on the facing page we find the quotation already given on page 37, namely,

"It is suggested that the teacher should first see how the child grasps each of these properties independently of the others, in situations where they can be clearly illustrated."

In order to eliminate the objections raised above, we require a relation in a set which is

(i) symmetric

(ii) either antireflexive (i.e. aRa never occurs) or nonreflexive (i.e. aRa does not occur for all a)

(iii) non-transitive,

and which is within the child's everyday experience. Let us therefore consider experimental situations involving a set of three (or four) Action-Man type dolls and the relation ". . is wearing a different coloured shirt from . . ".

<u>(iv) Proposals for further tests - to clarify the stages in the development of symmetry</u>

<u>Materials required:</u> Four Action-Man type dolls

Six shirts (3 red, 2 blue, 1 yellow)

A small suitcase or box, which represents the dolls' house.

At the start of each sequence of tests the child is given the following information:

"Four dolls called John, Paul, David and Robert want to play a new game with us. Here are their rules:

1. Sometimes only three dolls will play in a round of the game; sometimes all four dolls will play in a round.
2. The dolls are not allowed to tell us all of their names. Instead, we are allowed to look at their name-labels. This is to help us to remember which dolls are playing in a particular round of the game.
3. The dolls are allowed to tell us something about the shirts they are wearing. They want to see if we can tell which doll has which name from what they tell us about their shirts.
4. We are allowed to pin the name-label on a doll when we are sure we know the doll's name.
5. After each round of the game all of the dolls are allowed to go into their house so that some of them can change their shirts for the next round of the game."

Before commencing the test sequence, the experimenter must check that the child sees the same colour similarities and differences as the experimenter, so it is suggested that the introductory dialogue continues with

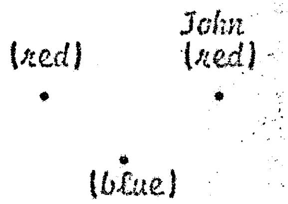
"Here are the four dolls who will be playing the game and all of the shirts they will be wearing."

(Experimenter displays four dressed dolls plus two extra shirts).

"Show me a red shirt? What are the colours of the other shirts here?"

Test A: (Three dolls - John, Paul and David)

Two of the dolls are wearing red shirts and one is wearing a blue shirt.



The experimenter picks up a doll wearing a red shirt and asks the following sequence of questions:

Question 1(a): This is John. (Experimenter attaches John label and spreads out the other two labels.) John is wearing a different coloured shirt from Paul. Which doll is called Paul?

Question 2: Is there another doll wearing a different coloured shirt from Paul?

If the child says "Yes", ask

Question 3: What is his name?

Question 4: Are John and David wearing shirts which are the same colour or are they different?

N.B.
As a preliminary to Test B when given on a separate occasion, Test A can be repeated with Question 1(a) replaced by

Question 1(b): This is John. (Experimenter attaches John label and spreads out the other two labels.) Paul is wearing a different coloured shirt from John. Which doll is called Paul?

Differences in the length of the hesitation (if any) before answering Questions 1(a) and 1(b) will give some indication as to which of the two possible ways of using this relation the child finds easiest to handle.

Test B: (Four dolls - John, Paul, David and Robert)

Three of the dolls are wearing red shirts and one is wearing a blue shirt.

John (red) • (red) •

• (blue) • (red)

The experimenter picks up a doll wearing a red shirt and asks the following sequence of questions:

Question 1: This is John. (Experimenter attaches John label and spreads out the other three labels.) He is wearing a different coloured shirt from David. Which doll is called David?

Question 2: Is David wearing a different coloured shirt from Paul?

If the child says "Yes" ask:

Question 3: Tell me the names of two dolls that are wearing different coloured shirts. Can you tell me the names of another two dolls who are wearing different coloured shirts?

Test C: (Four dolls - John, Paul, David and Robert)

Three of the dolls are wearing red shirts and one is wearing a blue shirt.

David (red) • (red) •

• (red) • (blue) Paul

The experimenter picks up a doll wearing a red shirt and says:

This is David (experimenter attaches David label) and this is Paul. (Experimenter attaches Paul label and spreads out the other two labels.) I am going to say the names of two dolls and I want you to tell me whether their shirts are the same colour or whether they are different:

1. David and Paul
2. John and Paul
3. David and John

4. Paul and Robert
5. Robert and David
6. John and Robert

<u>Test D</u>: (Four dolls - John, Paul, David and Robert)

Two of the dolls are wearing red shirts, one is wearing a blue shirt and one is wearing a yellow shirt.

John *(red)*	*(blue)*
(yellow)	*(red)*

The experimenter picks up a doll wearing a red shirt and asks the following sequence of questions:

<u>Question 1</u>: This is John. (Experimenter attaches John label and spreads out the other three labels. John is then replaced in the group of four dolls.) John, Paul and David are all wearing different coloured shirts. Will you please put John, Paul and David sitting together in a group in front of you. Which doll is called Robert?

<u>Question 2</u>: Are Robert and John wearing shirts which are the same colour or are they different?

<u>Question 3</u>: Are Robert and Paul wearing shirts which are the same colour or are they different?

<u>Question 4</u>: Are David and Paul wearing shirts which are the same colour or are they different?

<u>Test E</u>: (Four dolls - John, Paul, David and Robert)

Three of the dolls are wearing red shirts and one is wearing a blue shirt.

(red)	*(red)*
(blue) *Paul*	*(red)*

The experimenter picks up the doll wearing a blue shirt and says:

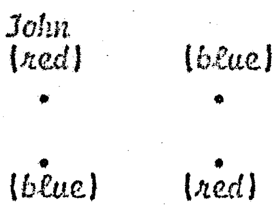
This is Paul. (Experimenter attaches Paul label and spreads out the other three labels.)

I am going to say the names of two dolls and I want you to tell me whether their shirts are the same colour or whether they are different.

1. Paul and David
2. John and Robert
3. John and Paul
4. David and Robert

<u>Test F</u>: (Four dolls - John, Paul, David and Robert)

Two of the dolls are wearing red shirts and two of the dolls are wearing blue shirts.



The experimenter picks up a doll wearing a red shirt and asks the following sequence of questions:

<u>Question 1</u>: This is John. (Experimenter attaches John label and spreads out the other three labels.) John is wearing a different coloured shirt from David. Can we tell which doll is called David?

If the child says "No", ask:

<u>Question 2</u>: Which doll might be called David?

<u>Question 3</u>: John is wearing a different coloured shirt from Paul. Are Paul and David wearing shirts which are the same colour or are they different?

<u>Question 4</u>: Which doll is called Robert?

<u>Question 5</u>: Are the shirts of Paul and Robert the same colour or different?

<u>Test G</u>: (Three dolls - John, Paul and David)

Two of the dolls are wearing red shirts and one is wearing a blue shirt.

(red) • (red) • The experimenter asks the following questions:

• (blue)

Question 1: John is wearing a different coloured shirt from Paul and Paul is wearing a different coloured shirt from David. Which doll is called Paul?

Question 2: Are John and David wearing shirts which are the same colour or are they different?

Test H: (Four dolls - John, Paul, David and Robert)

Three of the dolls are wearing red shirts and one is wearing a blue shirt.

(red) • (red) •

• (red) • (blue)

(a) The experimenter forms two distinct pairs with the four dolls and asks the following question:

Question 1: John and Paul are sitting together and Robert and David are sitting together. Robert and David are wearing different coloured shirts. Where are the dolls called Robert and David?

(b) The experimenter allows the four dolls to go to their 'house' and a red shirt is changed for a blue shirt. The dolls are once again placed in front of the child so that they now form two blue/red pairs, and the questioning continues as follows:

Question 2: John and Paul are sitting together and Robert and David are sitting together. Robert and David are still wearing different coloured shirts. Can we tell where

Robert and David are sitting?

The experimenter then picks up a doll wearing a blue shirt and asks:

<u>Question 3</u>: This is Robert. (Experimenter attaches Robert label and re-establishes the pair Robert - David). Just now one of the dolls changed his shirt from a red one to a blue one. Can you tell me which of the dolls changed his shirt?

Underlying the design of the above tests are two basic factors:

1. The relation ". . is wearing a different coloured shirt from . ." in the sets of dolls, really does have symmetry without being embedded in an equivalence relation.
2. It is assumed that the significant stage in the development of symmetry is the ability to pick out a pair (or pairs) of dolls without being bothered that you do not know which doll of a solution pair corresponds to x and which to y in xRy or yRx.

N.B.

The assumption stated in (2) above is based on the hypothesis that the subject's ability to disassociate himself from the need to know which doll corresponds to x and which doll corresponds to y in xRy or yRx is an indication that the subject has recognised and can use the symmetry in the situation as appropriate.

Consequently, a small pilot study was undertaken to see whether or not this assumption was ill-founded. In fact, the questions used in this small pilot study were designed to extend the above assumption in the following way:

To see if the subjects' responses indicated the following three stages of development.

<u>Stage 1</u>: The child recognizes a pair (x, y) such that "x is wearing a different coloured shirt from y", when certain

about both individuals.

<u>Stage 2</u>: The child recognizes a pair (or pairs) (x, y) such that "x is wearing a different coloured shirt from y", when certain about one individual only.

<u>Stage 3</u>: The child recognizes a pair (or pairs) (x, y) such that "x is wearing a different coloured shirt from y", when not certain about either individual.

(See Appendix 2d for the list of questions used and the results of this pilot study.)

The results of the pilot study appeared to support the conjecture that the subjects' responses indicate at least three stages in the development of symmetry which correspond to the stages specified above. A review of the tests used was therefore undertaken. This highlighted a number of points of weakness in the overall design of the items included, should these items only be used in a larger scale follow-up study, the purpose of which would be the confirmation or rejection of the existence of these three stages in the development of symmetry. Consequently, additional items which were similar in structure to the items used in the pilot study were included in the proposals for further tests given at the beginning of this sub-section.

There are two further observations to make about the sequence of tests as proposed. Not all of the questions focus specifically on identifying Stage 1-3 responses. The intention is to incorporate the decisive questions in the context of a more general conversation about the dolls in a particular situation.

It may be noted also that other logical notions may be involved in the child's deductions; notably there may be arguments by elimination.

For example, if we have three dolls and we know they are John, Paul and David, then there may be arguments of the form

"<u>This</u> is John, <u>that</u> is Paul, so <u>this</u> must be David."

This may be said quite independently of the shirt colours. We have assumed that all children chosen as subjects for an investigation into these stages in the development of symmetry are capable of this form of argument, so our classification of responses does not involve it. This could be a design fault, but at this stage no conclusion can be drawn on this particular point. Similarly, at this stage, no conclusions can be drawn on the other points raised in this sub-section. Clearly, further development of the test items is required.

But even if the three stages outlined above are confirmed there are still important aspects of the development of the concept of symmetry which have yet to be taken into account. For example, we need to distinguish between two levels of recognition by the child before we say that the concept of a symmetric relation is fully developed. These two levels of recognition are

<u>Level A</u>: Given a set of objects x, y, . . . and a relation R (which is symmetric, e.g. ". . is wearing a different coloured shirt from . ."), the child recognizes instances of xRy and x$\not R$y (where x$\not R$y denotes a non-exemplar of the basic relation, e.g. "x is NOT wearing a different coloured shirt from y") with respect to Stages 1-3.

<u>Level B</u>: Given a set of relations R, S, T, . . . (e.g. ". . is wearing a different coloured shirt from . .", ". . is taller than . .", ". . same colour as . .", ". . is older than . .", etc) on appropriate sets, the child recognizes instances and non-instances of relations

which are symmetric.

When at the first level, Level A, the child is in the position of recognizing whether pairs of objects of a given set have or have not a given property; but only on attaining the second level, Level B, does he see the given property as a thing in itself, that may be compared with other things of a similar kind and classified according to some higher level concepts.

The doll experiment discussed in this sub-section is an experiment designed to identify the degree of confidence that the subject has when working at Level A only. And so before we can ensure attainment of the concept of symmetry, the child's universe of experience must be extended to include situations based on relations which are not symmetric.

2.7 Reflexivity

(i) Piagetian type reflexivity check-ups

To date, the only information to be found concerning Piaget's views on the growth of the remaining defining property of an equivalence relation, namely, reflexivity, seems to be in "Checking Up III" of the Nuffield Mathematics Project. Here, all of the check-ups have been prepared by a team from the Institut des Sciences de l'Education in Geneva under the general supervision of Piaget, and on page 28 we find

> *"For many children, the reflexive property of a relation, although it may look self-evident at times, is the most difficult to understand. The following is a check-up for this idea.*
>
> *The teacher should collect together a group of children who are wearing pullovers with either skirts or trousers.*

There should be at least two girls who each have a skirt and pullover of the same colour. A large sheet of paper and lots of coloured pencils will be needed. All the children are asked to stand around the paper and to print their names on it at the nearest place to them. The boys are asked to write under their names the word 'pullover' followed by the colour of the pullover they are wearing; the girls do the same but for both pullover and skirt, e.g. 'pullover red, skirt green'.

One of the children is asked to point to the name of each child in turn to see if he can find any girls who have a skirt which is the same colour as that particular child's pullover. He then draws one or more arrows pointing from the name of that child to that of the girl, or girls, concerned. Several children may be asked to do this, each drawing only a few of the arrows.

The teacher watches the children playing the game to see what they do about the two little girls each wearing a jumper the same colour as her skirt. One of the children might say: "She is wearing a blue skirt and a blue pullover so she can point to herself". If none of the children mention this spontaneously the teacher may ask: "Can we draw another arrow from this girl?" ... "Where would it be pointing?" ... "What colour of pullover and skirt does this girl have?" The teacher could draw an arrow like this:

Jane
Pullover green
Skirt green

and could ask: "Why is this girl pointing to herself?" ...

"What does this arrow mean?"

The 'first-name/surname' game given by Papy in his book Mathématique Moderne is also excellent for checking up on the child's grasp of the reflexive property. . . .".

(ii) Critique of the above check-ups

The above extract raises a number of points of concern:

(a) Lines 1-3: No evidence is given to support the statement given. Whilst no evidence is given in support, it would appear that the statement is true since reflexivity seems to be psychogenetically subsequent to the other properties of equivalence relations. Strong arguments in support of this conjecture can be found in the history of mathematics itself, where on a number of occasions properties of relations are investigated - the relations not being reflexive initially but being redefined subsequently so that they become so, once the convenience of reflexivity is realized. As an example, consider the set of lines in the Euclidean plane and the following Euclidean definition:

Two lines are said to be parallel if they have no points in common.

This definition 'partitions' the set of lines into 'classes' - not equivalence classes however, but we have produced a set of subsets of lines with the following properties:

- the intersection of any pair of subsets is empty,
- the union of all these subsets is the set of lines in the Euclidean plane.

By definition, therefore, we can drop the quotes surrounding the word partition above, and we see that we can partition with something less than an equivalence relation. Having done this, however, there is an induced equivalence relation on the set of lines, namely, ". . is in the same class as . .". Thus we see that for centuries parallelism was based on antireflexivity. It became an equivalence relation retrospectively when we adopted the convention that lines are parallel to themselves, i.e. when we adopted the convention of reflexivity and made appropriate change in the definition of parallelism. And so it could be argued that reflexivity is often a useful mathematical convention applied to relations rather than an intrinsic property which some relations possess. More often than not we have a choice of convention to make, rather than an externally imposed constraint to accommodate.

(b) No attempt is made to give explicit formalization of the relation under consideration. Lines 17-19 suggest that the relation is in fact

> ". . is wearing a skirt which is the same colour as . . . .'s pullover."

But this relation is quite complex, and might be more difficult for a child to grasp than an adult might think.

(c) The set on which the relation is to be used is not made explicit. The authors are implying that attention should be focused on the subset of girls who have skirts of the same colour as their pullovers. Over this subset, the relation is reflexive. Over the whole set of girls in the group of children selected, however, the relation will not necessarily be reflexive, and over any set involving boys the relation cannot be reflexive. These observations highlight the necessity to specify the set under consideration to avoid misinterpretation.

(d) Similar criticism can also be made about the presentation of the second example in this section of "Checking Up III". The variation of the 'first name/surname' game (see (30), p. 88-91) in which the first names have the same initial as the accompanying surname (e.g. Kevan Keegan) does in fact give a reflexive relation on the appropriate set. But again, the relation is difficult to grasp and difficulties may arise for the children through complicating factors which are not themselves the objects of study. Moreover, this type of situation will not have immediate relevance for the children - is this why an assignment card has been suggested as the appropriate place for this exercise?

Thus, on taking into account the points raised under the four headings (a)-(d), namely

(a) the psychogenetically subsequent nature of reflexivity in equivalence relations,

(b) the need to give an explicit formalization of the relation under consideration,

(c) the need to specify the set on which the relation is to be used,

(d) the desirability of relevance for the children,

the following conclusion has been reached. With a class of children, a more appropriate context to begin the study of the reflexive property is given by the challenge to point to someone in your class who satisfies the relation " . . lives in the same house as . . ". Subsequent activities would also include non-instances of the concept of reflexivity to satisfy the strict criterion for concept attainment that was suggested by Lesh (19).

<u>(iii) Recent American studies</u>

On searching for further guidelines from recent American studies, we find that additional information concerning children's use of the reflexive property is limited. The study by Steffe and Carey (26) does, however, include the question

Are children able to use the reflexive and antireflexive properties?

as one of the questions asked of four- and five-year-old children after formal instruction on

(i) establishing length relations only,

(ii) establishing length relations, conserving length relations and using properties and consequences of length relations.

In addition, the following question was also considered:

Does formal instruction on conserving length relations; on the reflexive, antireflexive, and asymmetric properties; and on consequences of length relations, improve the ability to use the reflexive and antireflexive properties of length relations?

To measure the pupils' capabilities a Reflexive and Antireflexive Test was designed. This consisted of six items: three of the items involved the reflexive property of "same length as" and three of the items involved the antireflexive property of "longer than" or "shorter than". In administrating the test, the items were assigned at random to each child so that each had a different sequence of the same six items.

In the discussion of the results which relate to the above questions, we find

"Very few four- and five-year-old children were able to use the reflexive and nonreflexive property. . . . Instructional*

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* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

experiences on length comparison appear to be sufficient for such children to exhibit the reflexive property, 14% of the sample were able to use the reflexive property on the first test administration (i.e. after exposure to Instructional Sequence I - 7 sessions of 20-30 minutes designed to develop the ability of children to establish a length relation between two curves) *as compared to 4% who were able to use both properties. Instructional Sequences II and III* (designed to develop the ability of children to use the reflexive and nonreflexive* properties; to conserve length relations and use the asymmetric property and logical consequences, respectively) *significantly increased the ability of four- and five-year-old children to use both properties. On the second test administration 41% of the sample were able to use only the reflexive property and 30% of the sample were able to use both. Only 29% of the sample did not display an ability to use the reflexive or nonreflexive* properties. These conclusions substantiate Piaget's theory that experience is a necessary but not sufficient condition for the development of logical thought processes because all the children received the same selected experiences. Certainly, the data substantiate that the ability to use the reflexive property is different from and precedes the ability to use the nonreflexive* property.*

There appears to be little, if any, relation between the student variables Verbal Maturity, I.Q., Age and Social

---

* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

Class and scores earned by four- and five-year-old children on the Reflexive and Nonreflexive Test. Only correlations involving Social Class were significantly different from zero, but these correlations were low."* ((26), p. 41)

The observations contained in the last paragraph of the above quotation together with the statement

"Instructional Sequences II and III significantly increased the ability of four- and five-year-old children to use both properties.",

could be used to argue a case that the appropriate instructional activities may profitably be undertaken with similar populations of four- and five-year-olds. But such an argument is not taking into account at least two qualifying factors.

1. The information given by the results of this investigation do not enable us to specify, in advance, which children will benefit from such instruction and which will not. All we are told is that some will benefit.

2. By applying the strict criterion for concept attainment in this context, namely,

   'reflexivity' has been attained when the child can, with respect to length relations, distinguish instances from non-instances of reflexivity,

   it can be argued that

   (a) as only 30% were able to use both properties, no more than 30% of the sample should be regarded as having attained reflexivity with respect to length relations.

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* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

(b) the use of the reflexive property only, by 41% of the sample in this study, suggests a 'learned response' to the relation "same length as" had occurred.

This last possibility was in fact acknowledged by Steffe and Carey, (see (26), p. 42), and it could only have been resolved by their undertaking an appropriate study of the temporal development of the concept. But without this additional information the question surrounding the psychogenetically subsequent nature of reflexivity remains.

Before concluding this review of studies concerning the development of the properties of transitivity, symmetry and reflexivity, undertaken in Sections 2.5 - 2.7, we note that one of our original aims was

> to see how the child grasps each of these properties independently of the others, in situations where they can be clearly illustrated.

This aim was, however, not strictly adhered to because not all of the studies considered chose to highlight just one of the three defining properties of an equivalence relation. In fact, the aims of the American studies covered a much broader base than that indicated in the discussion so far. Let us therefore redress this imbalance by summarizing the basic themes of the purposes and questions of these American studies.

Author(s)	Context	Basic themes of purposes/questions of the investigation
T.P. Carpenter (24)	Conservation and measurement of liquid.	1. Assessment of degree of development of ideas of measurement and conservation. 2. Identification of factors involved in this development.
D.C. Johnson (12)	Classification by and establishment of matching relations.	1. The effects of training. 2. The use of specific properties of the relation (e.g. transitivity). 3. The possibility of transfer of learning.

Author(s)	Context	Basic themes of purposes/questions of the investigation
M.L. Johnson (27)	Classification and seriation by length relations.	1. The effects of training. 2. The use of specific properties of the relation.
R.A. Lesh (19)	Interdependent development of classification seriation and number concepts	1. The effects of training on transfer of learning.
D.T. Owens (22)	Establishing matching and length relations.	1. The effects of training. 2. The effects of age. 3. The possibility of transfer of use of properties across relational categories. 4. The relationships between use of properties of the relations including conservation.
L.P. Steffe and R.L. Carey (26)	Establishing length relations.	1. The effects of training. 2. The use of specific properties of the relation. 3. The relationships between the use of properties of the relations including conservation.

This summary shows that the theme 'the use of specific properties of the relation' is one of three which are common to two or more investigations. However, the relevent details of this theme have already been discussed in Sections 2.5 (ii) and (iii), and 2.7 (iii). It remains to consider further details of the findings concerning the relationships between the use of properties of the relations including conservation. This will be undertaken in Section 2.8. But first, let us consider further details of the effects of the training used. The question of the effects of training was first raised in Section 2.5 (i) with respect to the transitive property only. In five of the American studies it is posed in a much broader context as the following summary by K. Lovell (31) shows.

Author(s)	Group	Nature of Training	Effects of Training
D.C. Johnson (12)	Kindergarten and first grade children with measured I.Q. 80-120. No precise details of social background.	To form classes, intersection and union of classes, complement of classes, relations between classes and between class elements.	Improved performance on all five direct achievement tests and on three of the transfer tests, although not on the test of Class Inclusion. Some doubt remains as to whether there is any improvement in regard to operativity.
M.L. Johnson (27)	First and second grade children: Negroes and middle class Caucasian pupils. No I.Qs given	To classify on basis of equivalence relation "same length as" and seriate on basis of order relations "longer than", "shorter than".	Improved performance on Seriation Test. No improvement on Classification Test, Conservation of Length Relations Test or Transitivity Test.
R.A. Lesh (19)	Aged 5:3 to 6:2. Drawn from small Indiana community. A spread of ability.	To classify and seriate.	Improved performance on number tests but not on tasks involving spatial transformations.
D.T. Owens (22)	Five- and six-year-olds. Disadvantaged Negro children.	To establish length relations to conserve matching relations, and to use the transitive property of matching relations.	Improved performance on transitivity of matching relations - a task similar to activities in treatment. No transfer to other tasks.
L.P. Steffe and R.L. Carey (26)	Four- and five-year-olds. Normal spread of I.Q. and social background.	To establish length relations between two curves, to use reflexive and nonreflexive* properties and to conserve length relations.	Improved ability to compare the lengths of two curves, in conservation of length relations, in use of reflexive and nonreflexive* properties. Limited improvement in use of transitive properties by 5-year-olds.

((31), p. 181).

By taking into account some of the serious reservations we have already noted concerning interpretation of results, we need not be misled by the positive weighting of the statements in the last column of the above table.

2.8. Relationships between the use of properties of relations

(i) The contribution of the studies by Steffe and Carey and D.T. Owens

On turning our attention to the theme 'the relationships between the use of properties of relations including conservation', the specific questions asked by Steffe and Carey (26) of four- and five-year-old children after formal instruction on

(i) establishing length relations only,

(ii) establishing length relations, conserving length relations and using properties and consequences of length relations,

were designed to produce answers to:

- Is the ability to use the reflexive and antireflexive properties necessary (or sufficient) for children to
  - (a) conserve relations,
  - (b) use the transitive property of length relations?
- Is the ability to conserve length relations necessary (or sufficient) for children to use the transitive property of length relations?

In the discussion of results which relate to these questions, we find

> ". . . *the ability to use the reflexive and nonreflexive* properties as measured here is not a necessary or a sufficient condition for the ability to use transitivity*

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* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

of length relations.

. . . because some children could use the reflexive property but not the transitive property, there may be factors which enable children to use the reflexive property before they are able to use transitivity (e.g. spatial imagery or the definition of "the same length as"). In fact, the results indicate that the reflexive property may be necessary for transitivity. This observation may be due to the possibility that use of the reflexive property in this study was more of a 'learned response' than a logical-mathematical process. It also appears that use of the reflexive and nonreflexive* properties is not a necessary or sufficient condition for being able to conserve relations. . . . However, the data do not contradict the fact that being able to use only the reflexive property may precede an ability to conserve length relations. . . . The data in this study support the contention that conservation of identity is not unitary in nature. Certainly, if a child judges that a stick is the same length as itself, he must also judge that it is not longer or shorter than itself or a contradiction would be present. On a logical basis and on a psychological basis, when one considers "conservation" problems, it is necessary to consider the properties of the relations which may be involved.

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* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

(1) Smedslund, J. "Development of concrete transitivity of length in children", Child Development, 1963, 34, p. 389-405).

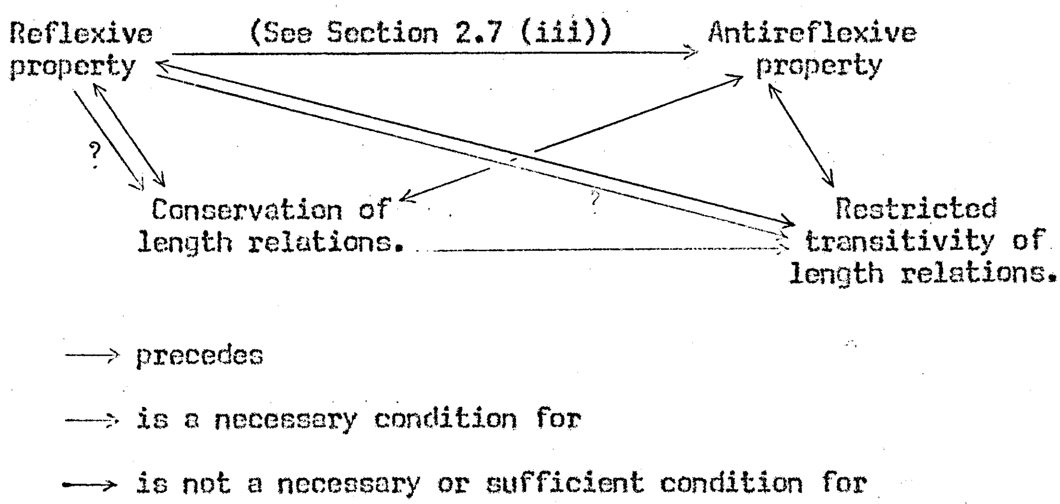
(1) See page overleaf ...

. . . it seems that conservation of length relations is necessary for transitivity.

. . . The above data are consistent with Smedslund's (1963)[1] observation that what he calls conservation of length is a necessary condition for what he calls transitivity."

((26), p. 42-43).

On remembering that we are interpreting 'transitivity' in the above context to be restricted-transitivity, it appears that the main points of the above discussion can be summarized as follows:



But before commenting on the above discussion of results, let us consider further evidence on the relationship between conservation and restricted-transitivity.

In the investigation by D.T. Owens (22), answers to the following questions were sought

- Is the ability to conserve matching relations related to the ability to use the transitive property of matching relations?
- Is the ability to conserve length relations related to the ability to use the transitive property of length relations?
- Is the ability to conserve matching relations related to the

ability to conserve length relations?

- Is the ability to solve a problem involving transitivity of a matching relation related to performance on a test of conservation or transitivity of matching relations which utilizes a standardized interview technique?

N.B.

The question concerning the relationship between transitivity of matching and length relations was discussed in Section 2.5 (iii).

In the discussion of the results which relate to the above questions, we find

> ". . . *no evidence is provided by these data that, for the children in this study, the ability to conserve relations preceded the ability to use the transitive property. The case is different, however, in the case of the Transitivity Problem.*
>
> . . .
>
> *The data gave no indication that conservation of matching relations precedes conservation of length relations for the children in this study.* . . . *This evidence is in opposition to the suggestion that the ability to conserve matching relations precedes the ability to conserve length relations.*" ((22), p. 68-69).

> "*The result that about one-half of the children who used the transitive property in each relational category failed to use the conservation of that respective category is at variance with results of previous studies. Smedslund*

(1964)[1] found only 4 out of 160 subjects who passed the test on transitivity and failed on conservation of discontinuous quantities and only 1 subject was in the corresponding cell for length. Owens and Steffe (1972)[2] observed only 4 of 126 instances (among 42 subjects) in which transitivity of a matching relation preceded conservation of that relation. Divers (1970)[3] found that in 87% of the cases where transitivity of a length relation was attained, the relation was also conserved. In the studies cited, the results consistently indicated that attainment of conservation preceded attainment of the transitive property. None of the studies involved instruction or practice, and the present results may be interpreted in terms of treatment effect. The treatment was effective in improving performance on the test of the transitive property while the treatment had no effect on conservation performance for matching relations. Thus some children in the treatment group met the criterion on the transitivity test who might otherwise not have attained transitivity. Only two children who used transitivity on the Transitivity Problem failed to exhibit conservation. This explanation applies, however, only to

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(1) Smedslund, J. "Concrete reasoning: A study of intellectual development", Monographs of the Society for Research in Child Development, 1964, 29 (Serial No. 93).

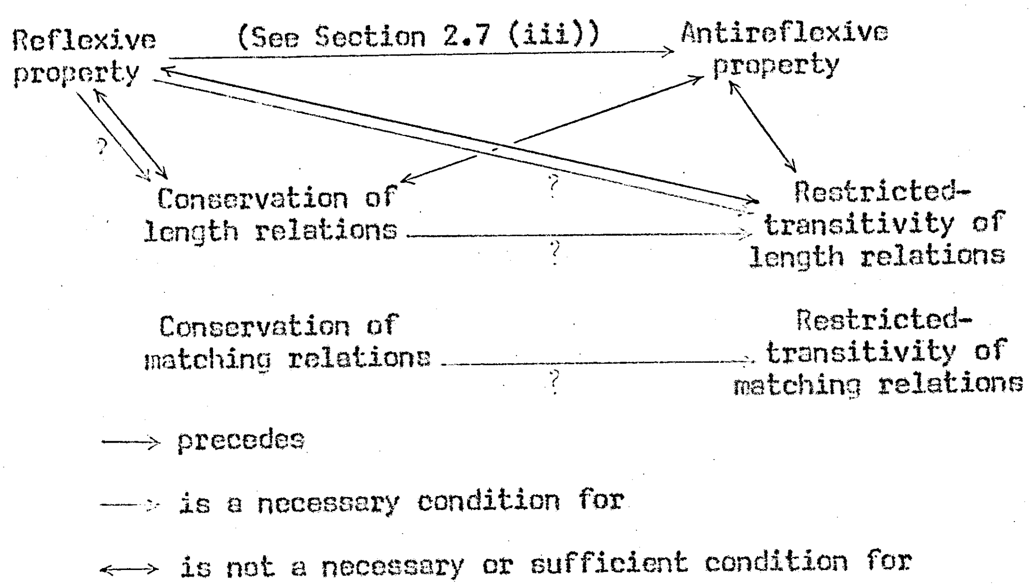
(2) Owens, D.T. and Steffe, L.P. "Performance of kindergarten children on transitivity of three matching relations", Journal for Research in Mathematics Education, 1972, 3, p. 141-154.

(3) Divers, B.P. Jr. "The ability of kindergarten and first grade children to use the transitive property of three length relations in three perceptual situations", Unpublished doctoral dissertation, University of Georgia, 1970.

the matching relational category because the treatment was not effective in improving the performance on transitivity of length relations.

Perhaps an interpretation can be made in terms of the characteristics of the children in the sample. Skypeck (1966)[1] conducted a study which involved both middle and lower socio-economic status children. It was found that among the lower status children the development pattern of cardinal number conservation was erratic. While the present study included no middle class group for comparison, it appears that the patterns of attainment of conservation and relational properties was irregular for these low economic subjects." ((22), p. 71-72).

On incorporating the main points of the above discussion into our summary diagram, we obtain



This second diagram differs from the first in two respects - there is the queried hypothesis about matching relations and a query is inserted on a hypothesis about length relations.

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(1) Skypeck, D.H. "The relationship of socio-economic status to the development of conservation of number", unpublished doctoral dissertation, University of Wisconsin, 1966.

Except for the reference made in the introduction to Piaget's earlier writing and to a paper by Northman and Gruen, which suggest that transitivity is being used in establishing equivalence (i.e. cardinal number) conservation, the results of this investigation by Owens (22) challenge the general conclusion of the previous studies he cited, namely,

> Conservation of a set of quantitative relations such as matching or length relations, is a necessary condition for restricted-transitivity of the same set of quantitative relations.

One reason not considered by Owens for this apparent contradiction may be the way in which the word "conservation" is interpreted and used by the investigators concerned. For example, variations in use of the terminology can be found in the following quotation from Owen's introduction to his investigation.

> ". . . *in a task given by Smedslund (1963)*[1] *a child was asked to establish that one stick was longer than a second stick and to maintain that the one stick was longer after a conflicting cue was introduced. While Smedslund called the task "conservation of length", a similar task in the present study is called "conservation of the relation 'longer than'"*." ((22), p. 51).
>
> (This passage is repeated, in its context, overleaf).

Let us therefore consider some of the ways in which the word "conservation" has been interpreted and used.

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(1) Smedslund, J. "Development of concrete transitivity of length in children", Child Development, 1963, 34, p. 389-405.

(ii) Various interpretations and uses of the word "conservation"

Limited information on the variations in points of view of Piaget, Van Engen, Smedslund, Northman and Gruen, and Owens concerning the concept of conservation, are to be found in Owen's introduction to his investigation.

> *"In Piaget's (1952)[1] classical conservation of number tasks, a child is asked to establish that there are as many objects in a set A as in set B. Then one of the collections, say A, is taken through a physical transformation. Then the child is asked "Are there as many a's as b's or does one have more?" Van Engen (1971, p. 43)[2] has argued that this task may be measuring whether or not the child conserves the one-to-one correspondence rather than conservation of number. In this study a task similar to the above example is considered to be a measure of conservation of the relation "as many as". It is not necessary that conservation be limited to cases of equivalence. For example, in a task given by Smedslund (1963)[3], a child was asked to establish that one stick was longer than a second stick and to maintain that the one stick was longer after a conflicting cue was introduced. While Smedslund called the task "conservation of length", a similar task in the present study is called "conservation of the relation 'longer than'". Thus, order*

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(1) Piaget, J. "The child's conception of number", Routledge and Kegan Paul, 1952.

(2) Van Engen, H. "Epistemology, research and instruction", in M.F. Rosskopf, L.P. Steffe and S. Taback (eds) "Piagetian cognitive-development research and mathematical education", Washington, D.C.: National Council of Teachers of Mathematics, 1971.

(3) Smedslund, J. "Development of concrete transitivity of length in children", Child Development, 1963, 34, p. 389-405.

relation conservation is also included.

Conservation is studied from the relational point of view and transitivity is necessarily a relational property. Thus the relationship between the development of conservation and attainment of transitivity is approached from the standpoint of relations. In his earlier writing, Piaget (1952, p. 205)[1] reported that as soon as children can establish a lasting equivalence (that is, conserve the equivalence), they can at once use the transitive property.

> ""The explanation is simple: the composition of two equivalences (transitivity) is already implied in the construction of a single lasting equivalence between two sets, since the different successive forms of the two sets seem to the child to be different sets."" (Piaget, 1952, p. 208)[1].

Similarly, Northman and Gruen (1970)[2] argue that transitivity is involved in equivalence conservation. Suppose the subject establishes A equivalent to B (A = B). When an equivalence-preserving transformation T is performed, the subject establishes (covertly) A = T(A). Then, transitivity is used in order to deduct T(A) = B or to conserve the equivalence of A and B.

Smedslund (1964)[3] has argued that from a logical point of

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(1) Piaget, J. "The child's conception of number", Routledge and Kegan Paul, 1952.

(2) Northmen, J.E. and Gruen, G.E. "Relationships between identity and equivalence conservation", Developmental Psychology, 1970, 2, 311.

(3) Smedslund, J. "Concrete reasoning: A study of intellectual development", Monographs of the Society for Research in Child Development, 1964, 29, (Serial No. 93).

view, conservation precedes transitivity in the child's development. Consider three quantities which are related by a transitive relation ρ. Assume that a child establishes A ρ B. B (or A) must undergo some transformation T before B is compared with C; otherwise A and C can be compared perceptually. Hence, B = T(B) (or A = T(A)) must hold from one comparison to the other."

(It might, incidentally, be asked whether Smedslund's argument focuses on the right point. It is not a matter of whether A and C can be compared directly but whether they are compared directly. But perhaps his comments concern test design and not the child's use of logic.)

"In a later discussion of training research Piaget (Beth and Piaget, 1966, p. 192)[1] *also alluded to an ordering in the attainment of conservation and transitivity. He reported that Smedslund easily induced conservation of weight by repeatedly changing the shape of a small clay ball and checking the weight on a scale. Smedslund was not successful in obtaining immediate learning of the transitive property."* ((22), p. 51-52).

To the above we may now add the following quotation from the introduction to the study by Steffe and Carey (26).

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(1) Beth, E.W. and Piaget, J. "Mathematical epistemology and psychology", Dordrecht-Holland: D. Reidel, 1966.

"Regardless of the content of these problems, they (i.e. conservation tests devised by Piaget) *involve presenting the subject with a variable (V) and a standard (S) stimulus that are initially equivalent in both the perceptual and quantitative sense. The subject is then asked to make a judgement regarding their quantitative equivalence. Once the judgement is made, the variable stimulus is subjected to a transformation V --> V', which alters the perceptual but not the quantitative equivalence between the variable and the standard. After completion of the transformation, the subject is asked to judge the quantitative equivalence between the standard and the transformed variable (p. 16)[1]."* ((26), p. 19).

When formulated in this way, Steffe and Carey point out that

". . . a judgement of conservation may be relative

(i) to the conservation of a quantitative relation, or

(ii) to the identity of V and V'." ((26), p. 19).

which was the basis for Elkind's categorization of Piaget's conservation tests. But Steffe and Carey also draw our attention to aspects of conservation which are not completely clarified by Elkind's categorization. In particular, they argue a case for the following statement

"a comprehension of relational terms is a prerequisite to problems in conservation of the relation." ((26), p. 20),

by considering conservation problems involving the relations "as many as" and "is longer than". These highlight the need to be assured that

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(1) Elkind, D. "Piaget's conservation problems", Child Development, 1967, 38, p. 841-848.

the child

(a) associates a one-to-one correspondence with the phrase "as many as",

(b) is taking into account both sets of endpoints when establishing a length relation.

In other words, we have an echo of Van Engen's point of view

> ". . . *one must be assured that a conservation problem is not a test of terminology.*" ((26), p. 19).

Note that Steffe and Carey have also extended their discussion to "conservation" problems in situations where the initial relation under consideration is an order relation, not just an equivalence relation. Their interpretation of "conservation" in this extended context has to be inferred from the following:

> ". . . *to conserve the* (length) *relation the child must realize that the relation obtains regardless of any length-preserving transformations on one or both curves. In other terms, the child must realize that, after such a transformation, if the curves are moved back side by side as in the original state, the ends will be still in the same relative manner.*" ((26), p. 21).

Even so, of the variations considered so far of what Piaget regards as a key concept, the strongest guidelines on how the word "conservation" should be interpreted and used are given by Steffe and Carey. Hence, their account will be taken as a starting point in an attempt to clarify the main issue raised in Section 2.8 (i). But first we require a more precise mathematical formulation of the characteristics of a conservation problem.

(iii) An attempt to specify the characteristics of a test of conservation of a quantitative relation based on Steffe and Carey's interpretation

The characteristics of a test of conservation of a quantitative relation as interpreted by Steffe and Carey appear to be as follows:

Let M be a domain of quantities modelled by a finite set of objects which represent quantities of M (e.g. a set of Cuisenaire rods - for the domain of lengths). The subject is presented with a pair (x, y) where $x, y \in M$, and is asked to identify the quantitative relation R with respect to a given physical context (e.g. matching, length, volume, weight) such that xRy.

The child having established xRy, the elements x and y are subjected to transformations S and T which alter perceptual aspects only, i.e. $x \dashrightarrow S(x)$, $y \dashrightarrow T(y)$ where S and T are transformations which preserve the quantitative relation under consideration. (N.B. S or T can be the identity transformation of the set of quantitative-preserving transformations $\mathcal{S}$ under consideration).

After completion of the transformations, the subject is asked to make a judgement concerning the truth of the statement S(x)R T(y).

To conserve the quantitative relation R, for the pair (x, y), the child must realize that there exist inverse transformations S^{-1} and T^{-1} such that

$$S^{-1}(S(x))R\ T^{-1}(T(y)) = xRy.$$

If the above interpretation is what Steffe and Carey had in mind, then

1. it would appear that conservation of length relations, for example, should be interpreted as meaning the child can conserve each of the length relations "same length as", "longer than" and "shorter than", i.e. the child realizes

that there exists a set of length-preserving transformations such that

$$S^{-1}(S(x))R_i\ T^{-1}(T(y)) = xR_iy \quad (i = 1, 2, 3)$$

where $S, T, S^{-1}, T^{-1} \in \mathcal{S}$, R_1 denotes the relation "same length as", R_2 denotes the relation "longer than", and R_3 denotes the relation "shorter than".

2. by taking $x = y$, "conservation of identity" can be interpreted as follows:

   To conserve identity, the child must realise that there exist $S, S^{-1} \in \mathcal{S}$ such that

   $$S^{-1}(S(x))R\ S^{-1}(S(x)) = xRx.$$

   Note that since y is now the same as x, and since we are concerned with physical transformations, T is necessarily the same as S. There is only one object, and two transformations cannot be performed on it simultaneously.

On comparing the statement

"... *what sometimes passes for a test of conservation of identity is no more than a test of the reflexive and nonreflexive* properties.*" ((26), p. 22)

with the equation

$$S^{-1}(S(x))R\ S^{-1}(S(x)) = xRx,$$

we see that it is possible that consideration of $x = y$ in the general case has given rise to this statement. Unfortunately, such a line of argument fails to take into account a number of important points. Let us therefore consider in greater detail the steps to be taken on putting $x = y$ in the general case, by comparing and commenting as appropriate.

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* For 'nonreflexive' read 'antireflexive' - see footnote on page 31.

(iv) An analysis of points requiring consideration if the definition of conservation of identity is to be derived from the general case by putting $x = y$

1. (a) "The subject is presented with a pair (x, y) where $x, y \in M$."

(a') "The subject is presented with a pair (x, x) where $x \in M$."

Comment:

In (a') the subject is being asked to consider a special case in which an element is to be paired with itself

2. (b) "The subject is asked to identify the quantitative relation R with respect to a given physical context such that xRy.

(b') "The subject is asked to identify the quantitative relation R with respect to a given physical context such that xRx.

Comment:

Affirmative responses to "Is x longer (shorter) than x?", for example, at this stage, indicate that these relational terms are not understood. In other words, the restriction $x = y$ implies that

(i) an exemplar of the antireflexive property for each of the order relations of the particular relational category under consideration is being tested indirectly,

(ii) an exemplar of the reflexive property of the equivalence relation of this particular relational category is also being tested,

for as noted by Steffe and Carey

> *"Certainly, if a child judges that a stick is the same length as itself he must also judge that it is not longer or shorter than itself, or a contradiction would be present."*
> ((26), p. 43).

But to ensure that the antireflexive and reflexive properties of the relations in a particular category are being tested, we must extend the set M under consideration to more than one element. This will entail asking the child to consider elements x_i ($i = 1, 2, \ldots n$, $n \geqslant 2$) in such a way that each is paired with itself and then to identify the quantitative relation R with respect to the given physical context such that $x_i R x_i$ for all $x_i \in M$, i.e. the child is being tested on his recognition of

- the antireflexive property for each of the order relations of the particular relational category,
- the reflexive property of the equivalence relation of the particular relational category.

3. (c) "The child having established xRy, the elements x and y are subjected to transformations S and T which alter perceptual aspects only. . . . After completion of the transformations, the subject is asked to make a judgement concerning the truth of the statement S(x)R T(y).

(c') "The child having established xRx, the element x is subjected to transformation S which alters perceptual aspects only. . . . After completion of the transformation, the subject is asked to make a judgement concerning the truth of the S(x)R S(x).

Comment:

If extension of M to include more than one element has not taken place, then the child is being tested on his recognition of the compatibility of a quantity preserving transformation S with an exemplar of the reflexive property of R. If extension of M has taken place, the two possibilities need to be considered.

(a) a single transformation S can be applied to each x_i and the child asked to consider the truth of the statement $S(x_i)R\ S(x_i)$ for all $x_i \in M$, i.e. the child is being tested on his recognition of the compatibility of the quantity-preserving transformation S with respect to the reflexive property of R.

(b) different transformations S_i where $S_i \in \mathcal{S}$ for each i, can be applied to each x_i and the child asked to consider the truth of the statement $S_i(x_i)R\ S_i(x_i)$ for each i, i.e. the child is being tested on his recognition of the compatibility of a set of quantity-preserving transformations with respect to the reflexive property of R.

4. (d) "To conserve the quantitative relation R for the pair (x, y), the child must realize that there exist inverse transformations S^{-1} and T^{-1} such that $S^{-1}(S(x))R\ T^{-1}(T(y)) = xRy$".

(d') "To conserve identity, the child must realize that there exists S^{-1} such that $S^{-1}(S(x))R\ S^{-1}(S(x)) = xRx$".

<u>Comment:</u>

Once again, three interpretations concerning recognition of inverse quantity-preserving transformations are possible.

If the extension of M to include more than one element has not taken place, the recognition that there exists S^{-1} such that

$$S^{-1}(S(x))R\ S^{-1}(S(x)) = xRx$$

can be interpreted as recognition of an exemplar of conservation of compatibility of a quantity-preserving transformation S with respect to an exemplar of the reflexive property of R.

If extension of M has taken place and a single transformation S has been applied to each x_i, the recognition that there exists S^{-1} such that

$$S^{-1}(S(x_i))R\ S^{-1}(S(x_i)) = x_i R x_i$$

for each $x_i \in M$, can be interpreted as recognition of an exemplar of conservation of compatibility of a quantity-preserving transformation S with respect to the reflexive property of R.

If extension of M has taken place and different transformations S_i have been applied to the x_i's, then recognition that there exists S_i^{-1} for each S_i such that

$$S_i^{-1}(S_i(x_i))R\ S_i^{-1}(S_i(x_i)) = x_i R x_i$$

for each $x_i \in M$, can be interpreted as recognition of conservation of compatibility of quantity-preserving transformations S_i with respect to the reflexive property of R.

From the above analysis it appears that confusion between "conservation of identity" and the "reflexive and antireflexive properties" could have arisen from two sources:

1. failure to recognize that the definitions of reflexivity and antireflexivity, as with transitivity, imply at least two levels of application

   Level 1: when x is the only element of the set M

   Level 2: when x is just one element of the set M, which contains more than one element,

   and that application at Level 2 must be attained before we can ensure understanding of the reflexive and antireflexive properties, hence the deliberate introduction of the word "exemplar" at stage (b') of the above analysis to highlight this point.

2. the application of quantity-preserving transformation(s) is not a necessary design-feature for a test of reflexivity or anti-reflexivity, whereas the introduction of quantity-preserving transformation(s) is necessary to test conservation.

Thus it seems that lack of clarity on interpretation and use of the terminology is the underlying source of confusion expressed by the statement quoted on page 113.

<u>(v) An attempt to clarify the main issue raised in Section 2.8 (i)</u>

At this point we note certain similarities in the interpretation and use of the word "conservation" by D.T. Owens to those of Steffe and Carey. Consider, for example, the following quotation

> ". . . *order relation conservation is also included. Conservation is studied from the relational point of view and transitivity is necessarily a relational property. Thus, the relationship between the development of conservation and attainment of transitivity is approached from the standpoint of relations.*" ((22), p. 52),

and the similarity in the design of the Tests of Conservation of Length Relations by these investigators. (See Appendix 2e.)

Unfortunately, we now have all the available evidence on which to base Owen's interpretation and use of the word "conservation". Hence, further attempts to seek points of similarity and difference would result in yet more conjectures with respect to the conjecture already applied in the attempt to clarify Steffe and Carey's interpretation and use of the word "conservation". Thus we see that without further details any attempt to confirm or reject the apparent contradiction surrounding the statement

> conservation of a set of quantitative relations is a necessary condition for restricted-transitivity of the same set of quantitative relations,

would be open to deserved criticism. Once again, lack of clarity in interpretation and use of the terminology is our stumbling block.

2.9 Partition - its role in the development of the concept of equivalence relation

In the list of definitions given at the end of Section 1.3 we included the following result

> Any equivalence relation R on a set A partitions the set, in that x and y belong to the same subset if, and only if, xRy and conversely, given a partition of a set A, xR^1y if and only if x and y belong to the same subset of the given partition of A, defines an equivalence relation R^1 in A.

Thus we see that the concepts of equivalence relation and partition are closely related. Clearly, an investigation into the development of the ability to partition a given set and other associated classificatory skills of young children would provide additional evidence on the development of the concept of equivalence relation.

Fortunately, classificatory behaviour of young children has been the subject of a number of recent research studies and a very useful overview of the main findings of these studies is provided by M.L. Johnson in his introduction to "Learning of Classification and Seriation by Young Children".

"Inhelder and Piaget (1964)[1] were among the first to systematically study the behavior of children as they attempted to form classes. These authors report behavior related to classificatory acts ranging from "graphic collection" (Stage I) in which the child forms spatial wholes, to true classification (Stage III). True classification appears when children are able to coordinate both the intension and extension of a class as shown by an ability to solve class inclusion problems - somewhere around 8-9 years of age. Lovell, Mitchell and Everett (1962)[2] found behavior similar to that found by Inhelder and Piaget with only Stage III children being able to group objects according to more than one criterion; such as color, shape or form. The fact that the bases of classification children use is age related was revealed by Olver and Hornsby[3]. Their research showed that collections made by very young children are based on perceptible properties of objects (color, shape, etc.) with an increase of functional based equivalence as children grow older. Other researchers (Maccoby and

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(1) Inhelder, B. and Piaget, J. "The early growth of logic in the child: Classification and seriation", Translated by E.A. Lunzer, Routledge and Paul, 1964.

(2) Lovell, K., Mitchell, B. and Everett, I.R. "An experimental study of the growth of some structures", British Journal of Psychology, 1962, 53, p. 175-188.

(3) Olver, R.R. and Hornsby, J.R. "On equivalence" in J.S. Bruner, R.R. Olver and P.M. Greenfield et al., "Studies in cognitive growth", New York: John Wiley and Sons, 1966.

Modiano, 1966[1]) reported that the choice of criteria for classification is a function of the child's culture. While this may be the case, Olmsted, Parks and Rickel (1970)[2] reported that the classification skills of culturally deprived children, including an increase in the variety of criteria used for classification, could be improved by involving the children in a systematic training procedure. Edwards (1969)[3] also reported an increase in classification performance of children due to training. Other investigators (Clarke, Cooper, and Loudon, 1969[4]; Darnell and Bourne, 1970[5]) reported that conditions of training such as making the child aware of natural relationships or orderings among a set of objects, may facilitate the learning of equivalence relations." ((27), p. 74).

Johnson's assessment of the current literature, however, led to the following conclusion:

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(1) Maccoby, M. and Modiano, N. "On culture and equivalence I" in J.S. Bruner, R.R. Olver and P.M. Greenfield et al., "Studies in cognitive growth", New York: John Wiley and Sons, 1966.

(2) Olmsted, P., Parks, C.V. and Rickel, A. "The development of classification skills in the pre-school child", International Review of Education, 1970, 16, p. 67-80.

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(5) Darnell, C.D. and Bourne, L. Jr. "Effects of age, verbal ability, and pretraining with component concepts on the performance of children in a bidimensional classification task", Journal of Educational Psychology, 1970, 61, p. 66-71.

> *". . . classification has been approached only as a general categorizing process not including the major action in classifying - the formation of equivalence classes. Hence, any relationship which may exist between the child's knowledge of the mathematical properties of an equivalence relation and his classification skills based on that relation has not been explicated."* ((27), p. 75).

Consequently, although the main purpose of Johnson's investigation was to determine the influence of training on the ability of first and second grade children to classify and seriate objects on the basis of length, an additional objective was

> *". . . to determine if the subject's ability to use the transitive property of the equivalence relation "same length as" was related to his ability to classify on the basis of the relation; . . ."* ((27), p. 75).

Associated with this additional objective were the following measuring instruments:

1. the two items designed by Johnson to test the child's ability to use the transitive property of the equivalence relation "same length as" that were included in the Transitivity of Length Relations Test (TLRT). (See Appendix 2c).
2. a three-item Classification Test:

   Item 1 required the child to find and sort into three distinct piles, sticks congruent to three given sticks.

   Item 2 required the child to discover the criteria for a given classification.

   Item 3 presented the child with the problem of forming a set containing one element.

(See Appendix 2f for further details).

For the subjects of this study, these measuring instruments produced the following results:

> ". . . *performance on items 1 and 2 was slightly related to transitivity ability of "same length as". No relationship could be detected between transitivity ability and classification performance on item 3. Perhaps transitivity was not needed to correctly perform the items on the classification test."* ((27), p. 87).

In the subsequent discussion, we also find

> *"The results of the classification test indicate that it was somewhat easier for children to classify sticks on the basis of self-selected criteria than to discover the criteria used for sticks already classified. While little difference was found in performance (as noted by frequencies of response) on items one and three, due to school and treatment, it was clear that second grade children did better on both of the items. On item three, the difference in response frequencies indicated that second grade children were able to form a class with only one element more consistently than the first graders. This finding was consistent with Piaget's observation that the concept of a singular class appears in a child around eight or nine years of age.*
>
> *The hypothesis of a relationship between the child's classification ability and his ability to use the transitive property of the equivalence relation of "same length as" was not confirmed. The lack of a relationship may be explained, at least partially, in two ways:*
>
> *1. A two-item test may not give a true assessment of transitivity ability. Past research reveals that much*

controversy exists over methodological issues and at the age at which children acquire the transitive property. Braine (1959)[1] using a non-verbal technique, reported that children can use the transitive property of length relations as early as four and one-half years of age. On the other hand Smedslund (1963)[2] reports that operational transitivity occurs around seven years of age and that Braine failed to assess transitivity.

2. *Transitivity was not needed to do the classification tasks. In the case of item one this could possibly have been the case since over one half of the subjects receiving a score of zero on the transitivity test (indicating failure to correctly answer both transitivity items), performed at the highest level on this item. On item 2 over 50% of the subjects performed at the lowest level of performance across transitivity scores. Over half of the subjects receiving zero on transitivity also performed at the lowest levels of performance on item 3. Such results suggest that transitivity was not necessary for the classification items in this test.*"((27), p. 91-92).

Apart from the now obvious comment that restricted-transitivity not the transitive property of "same length as" was under investigation, yet a

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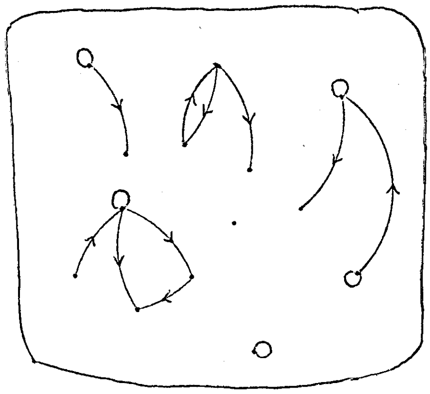
(1) Braine, M.D.S. "The ontogeny of certain logical operations: Piaget's formulation examined by nonverbal methods", Psychological Monographs: General and Applied, 1959, 73, (5 Whole No. 475).

(2) Smedslund, J. "Development of concrete transitivity of length in children", Child Development, 1963, 34, p. 389-405.

further possibility could account for the lack of any discernable relationship between the child's classification ability and his ability to use the "transitive" property of the equivalence relation "same length as". It is this. Given any equivalence relation R defined on a set A, successful partition of A into equivalence classes can be achieved by direct reference to the statement which defines R. In other words, the ability to recognize distinct pairs (x, y) where x, y ∈ A such that xRy is ALL the child needs to successfully partition into equivalence classes.

The source of this suggestion can be found in Section 2.7 (ii). Here, we noted that we can partition with something less than an equivalence relation, but on using this near-equivalence relation we induced on the set under consideration the equivalence relation ". . is in the same subset as . .". This idea can be generalized even further, for in fact, given any relation S defined on a set A, we can construct the partition defined by S as follows:

<u>Step 1</u>: Draw the arrow-diagram for the relation S defined on the set A.

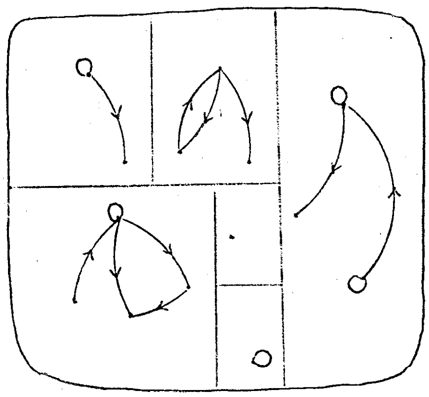


The relation defined on the set A.

<u>Step 2</u>: Put two distinct elements of A into the same subset of the partition if and only if they are connected by arrows of the arrow-diagram, i.e. if and only if we can go from either element to the other by following the arrows of the diagram

but without paying attention to the "sense" of the arrows. Each element, if any, of set A not associated with an arrow should be put in a class of its own.

This defines the partition P of the set A.



The relation S defined on the set A.

The partition P produced by S.

S ⟶ P

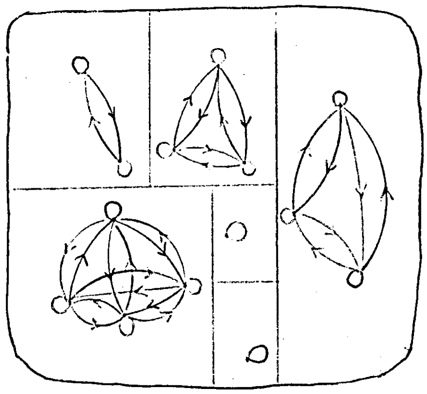
By the above process we now have

$$S \xrightarrow{\text{produces}} P$$

In addition, the partition P produces the equivalence relation Q ". . is in the same subset as . .". The arrow-diagram for this equivalence relation Q can be obtained by using all the arrows representing the ordered pairs of S and adding the minimum number of others to them so that the ordered pairs represented by these additional arrows together with the ordered pairs of S, satisfy the reflexive, symmetric and transitive properties*.

---

* The procedure by which P produces Q could also be described as forming the reflexive, symmetric, transitive hull of S, as an obvious extension of Appendix 2a. Q could also be defined as $\cap \mathcal{R}$ where $\mathcal{R}$ is the family of all equivalence relations on A each of which contains S.



The relation S defined on the set A.

The partition P produced by S.

The equivalence relation Q produced by P.

And so we now have

$$S \xrightarrow{\text{produces}} P \xrightarrow{\text{produces}} Q \xrightarrow{\text{produces}} P \xrightarrow{\text{produces}} Q \;.\;.\; \text{ad inf.}$$

i.e.

$$S \longrightarrow P \rightleftarrows Q$$

Note that if S is an equivalence relation, then we do not add to the arrow-diagram any additional arrows representing ordered pairs after Step 1 has been completed. But of greater significance to the present argument is the fact that if S is an equivalence relation, then by Step 2 above we can put pairs of distinct elements of A into the same subset of the partition by direct reference to the statement defining S and the job is done - the reflexive, symmetric and transitive properties are automatically satisfied.

Thus we see that it is possible for young children to use the behavioural counterpart of Step 2 with concrete materials and successfully partition by the equivalence relation under consideration without being aware of the reflexive, symmetric and transitive properties. Moreover, use of this process could account for one of the major difficulties experienced by some children that was reported by Johnson. This was dealing with a singleton subset, for we note that the process underlying Step 2 is dependent on pairs so that when faced with a singleton the child has to modify his previous strategy in some way.

It is also possible that this association of pairs is a factor in adult use of near-equivalence relations as true equivalence relations.

However, the hypothesis that the behavioural counterpart of the process underlying Step 2 is the one used by young children in partitioning a set A with respect to some equivalence relation S requires further investigation. For other procedures are possible. Consider the behavioural counterpart of Step 2 when it is specified as follows:

> Choose any element as first. Put it in a class. Choose any other element as second. If it is joined to the first by an arrow put it in the same class as the first. If not, put it in a new class. Then iterate with the following procedure until all the elements are classified:
>
> > If any element remains unconsidered, choose any one. If it is joined to any of the previously considered elements by an arrow of the arrow-diagram, put it in the same class as that element. If not put it in a new class.
> >
> > (For non-finite sets this algorithm will require modification.)

Note that this algorithm does not rely explicitly on recognition of pairs, nor does it require modification for elements not associated with arrows as did the original Step 2 procedure. But we now have two different approaches to the idea of partition. Still others may be possible, hence the request for further investigation to identify the behavioural counterpart(s) of the partitioning process(es) used by young children.

Whatever the outcomes of further research might be, it appears highly likely that explicit experiences which draw attention to the reflexive, symmetric and transitive properties of equivalence relations may be necessary before the child is able to 'see' that these three properties can be used to produce a partition of a set.

2.10 Concluding remarks

From the vast output of published work of Jean Piaget we have selected appropriate sections for the foundation of a framework within which observations about the development of the concept of equivalence relation can be organized. In doing so we have encountered widely differing interpretations of Piaget's work by individuals who have concentrated their efforts on different sections of it. The fundamental reasons for this diversity seem to be traceable to at least two sources. First, the complexity and occasional internal inconsistency that are to be found in Piaget's published work. Consider for example,

I. Piaget's psychological model known as the grouping,

II. the application of well-defined mathematical terms in restricted contexts.

The former (I) does not have a rigourous mathematical formulation, and although reformulations exist which appear to be logically satisfactory, recent American research in this area has been based only on the imprecise formulation by Piaget and not on this recent work by Wittmann and Steiner. The latter (II) appears to have led to confusions of ideas (e.g. conservation of identity and reflexivity) and to contradictions in the results of recent investigations. This suggests that the pay-off from the considerable amount of experimental work done might have had greater import had greater care been taken with respect to the terminology used. Second, the tie up between the psychological models that have been devised and the behavioural counterparts which they are supposed to represent is very slack. But, given that it is difficult to describe the behavioural counterparts of some of the simplest mathematical notions such as restricted-transitivity, reflexivity and conservation, this cannot be used to excuse the imprecision we have already noted.

Just as there are gaps between the psychological models and the behavioural counterparts which they are required to describe, so there are gaps between the mathematical notions and their pedagogical application in other parts of the school curriculum. Two inter-connected considerations appear to be involved. The first consideration is the lack of precision of ordinary language. This imprecision varies with the area of application and is related to the second consideration, which is the extent to which people "calculate", in some meaningful sense, in the classificatory systems in different subject fields. These matters are important because all subjects in the school curriculum should be contributing to the development of the logical use of language by the child, and we are here considering some of the difficulties of doing so. Teachers need to appreciate the traps, and to be aware of the need to make decisions on whether or not to discuss the traps explicitly with the children.

The lack of precision in everyday language is not necessarily a fault for which the user is to be criticized, as it may be brought about by unavoidable features of the matter under discussion. Many classificatory systems in everyday use can only be associated with near-equivalence relations. For example, there is among teachers a reluctance to discuss the frequent absence of the reflexive property. This absence is exemplified in the traditional view of parallelism and the colloquial usage of the word 'brother'. The teacher needs to consider the advantages and disadvantages of adapting usage in such a way that the relation becomes reflexive.

As examples of the difficulties - one cannot apply transitivity arguments to equivalences which are only approximately transitive without some modification; old style definitions in geometry (i.e. squares are not rhombuses) led to a situation in which general arguments fail to cover a number of inconvenient special cases which require special treatment. (If mathematicians themselves ran into such difficulties what may be expected of others?)

As noted earlier, the extent to which people "calculate" with classificatory systems varies very much in different subject areas. At one extreme there are systems in which no "calculation" is attempted at all; at the other extreme, as exemplified in the field of linguistics, we have something we can fairly call calculation, since linguistic theorists employ systems of ideas of precisely the same type as those employed in some parts of pure mathematics. The more people wish to "calculate" with classes, in the sense of manipulating them as if they were entities in themselves, the more necessary it is that the ideas are formulated in a precise quasi-mathematical way.

We also note that the diversity in interpretation of Piaget's work has produced implications requiring further consideration when, following Piaget's lead, we design experimental procedures which facilitate diagnosis and so maintain contact with the development we wish to study. In particular, when designing experimental procedures to investigate

1. how the child grasps each of the properties of reflexivity, symmetry and transitivity independently of the others,
2. the relationships which may exist between the child's use of properties of the relations under consideration,

we see that there is now the need to ensure

(a) a match between the terminology used and the content of the experiment,

(b) that strict criteria for concept attainment are applied.

Concerning this latter point of adopting as criterion the ability to distinguish instances from noninstances, we suggest that had the investigators used this criterion

- there would have been no need to identify levels of application for the concepts of reflexivity and transitivity (see Sections 2.5 (iv) and 2.8 (iv)),
- agreement on a proper definition of each concept could have resulted, thereby avoiding the blurring of meaning which we have encountered.

Additional design features which should also be incorporated are that

(i) the child should have had sufficient experience in working in the given physical context so that
  - with respect to 1 above, the likelihood of his recognizing the property concerned is increased,
  - with respect to 2 above, the likelihood of his use of the relationship between the properties is increased,

(ii) the key attributes under consideration should be differentiated by the child in his everyday conversation,

(iii) there are no attributes of the materials selected other than the ones on which the experiment is based which could be the source of failure for the child (e.g. use of distracting perceptual cues in tests of transitivity).

For example, on finding that a symmetric relation (outside of the context of an equivalence relation) had not been adequately studied, the suggested design for such an experiment, included in this paper (which has received a modest pilot in school), deliberately incorporated the following features:

- a set of Action-Man type dolls wearing shirts, trousers and boots,
- colour,
- dolls identical in every respect except for the colours of the shirts worn,

as exemplars of (i), (ii) and (iii) respectively. Use was also made of non-exemplars of the basic relation ". . is wearing a different coloured shirt from . ." for all three stages under investigation, to see if the child was in a position of recognizing whether pairs of dolls from the given set satisfied or did not satisfy this basic relation.

The aim underlying the above suggestions is the improvement of the effectiveness of both diagnostic/heuristic and clinical/experimental methods of enquiry when they are applied to this area of study; for severe criticism has been applied on this point:

> *"It is a sad commentary on the effectiveness of our methods of enquiry that after some eighty years of psychological investigation and a discontinuous history of forty years of laboratory experiments, our fund of accepted knowledge of the subject of conceptualization comprises so little of consequence that it is hardly worth compiling."*
>
> ((32), p. 198).

Although more than a decade old, this criticism still seems very relevant. The goal of further enquiry must be to bring about a state of affairs in which this criticism is no longer applicable, and we hope that this thesis will make a small contribution in this direction.

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APPENDIX 1

Theorem:

Any equivalence relation R in A partitions the set, in that x and y belong to the same subset if and only if xRy, and conversely given a partition of a set A, xR^1y if and only if x and y belong to the same subset of the given partition of A, defines an equivalence relation R^1 in A.

Proof:

Given an equivalence relation R in A, we can now define subsets of A by

x and y belong to the same subset of A if xRy.

As R is reflexive, xRx ($x \in A$), so each element belongs to at least one subset of A.

We now show that x cannot belong to two subsets of the partition.

Suppose $x \in B$ and $x \in C$ where B and C are subsets of the partition and $B \neq C$, then if b is any element of B and c is any element of C, we have

$$xRb \qquad \text{and} \qquad xRc$$

But R is symmetric and so bRx. Also R is transitive, hence bRx and xRc implies bRc, which in turn implies $b, c \in B$ and $b, c \in C$.

Thus we see that any element c of C belongs to B, and any element b of B belongs to C, i.e. B = C, which contradicts the hypothesis that $B \neq C$. Thus any equivalence relation R in A partitions the set A.

Conversely, given a partition of a set A we define R^1 so that xR^1y if and only if x and y belong to the same subset of the partition, then

(i) for all x, $x \in A$, xR^1x as x belongs to the same subset as itself, i.e. R^1 is reflexive,

(ii) yR^1x whenever xR^1y, i.e. R^1 is symmetric,

(iii) if xR^1y and yR^1z then xR^1z, because the subsets of the partition do not overlap by definition. Hence, R^1 is transitive.

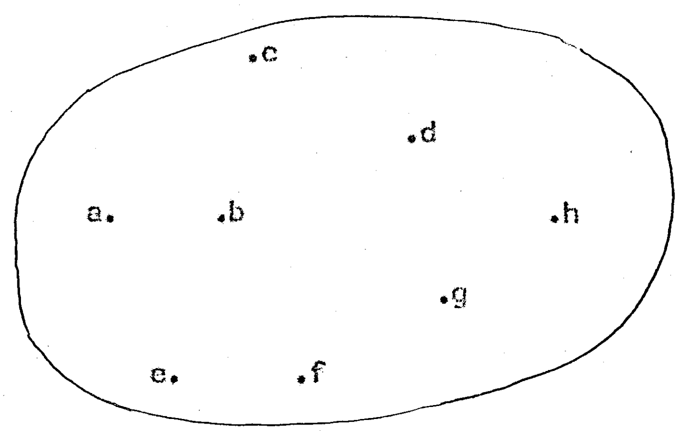
Thus we see that R^1 is an equivalence relation.

APPENDIX 2a

The reflexive, transitive hull of △

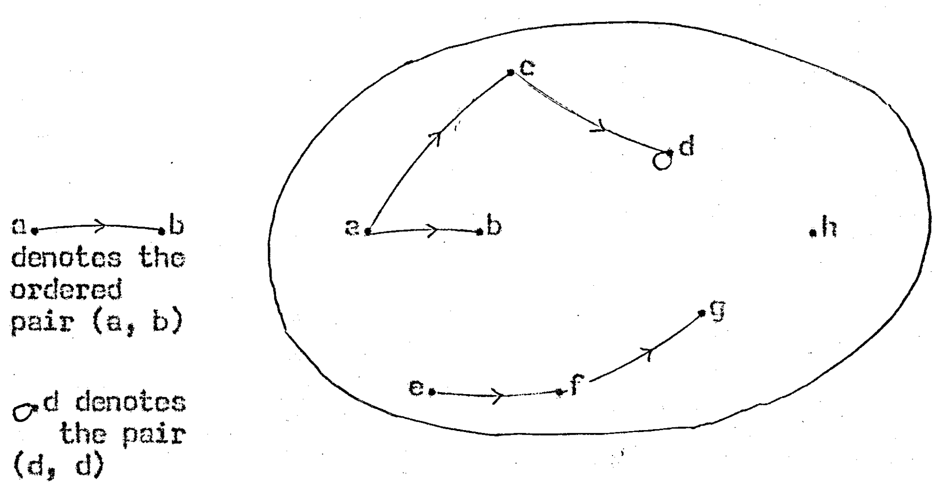
The reflexive, transitive hull of △ can be obtained as outlined below:

Suppose M is the set as illustrated



and that △ is defined by

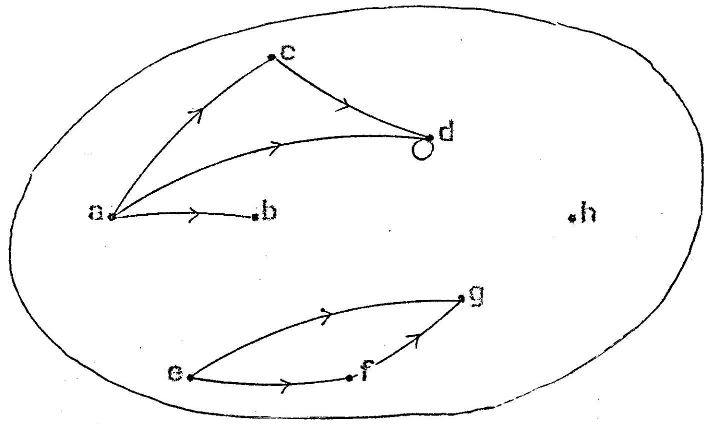
$$\triangle = \{(a, b), (a, c), (c, d), (d, d), (e, f), (f, g)\}$$



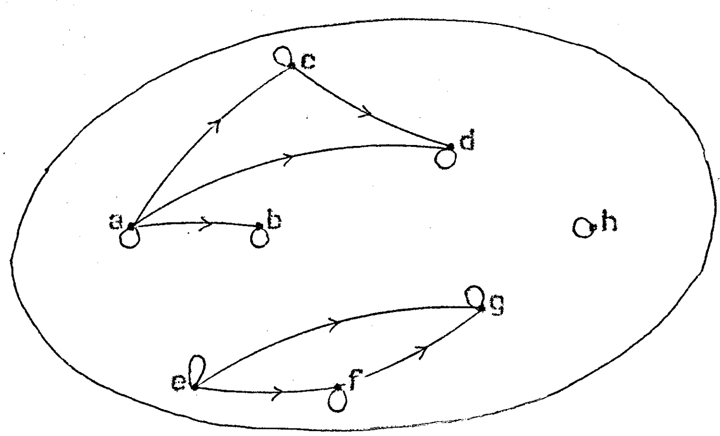
We now make all the compositions that are possible with the elements of △ , within the restriction imposed by (iv). For example

$$(a, c) \circ (c, d) = (a, d)$$

hence (a, d) becomes a member of the transitive hull we are constructing. This gives $\{(a, b), (a, c), (a, d), (c, d), (d, d), (e, f), (e, g), (f, g)\}$ as the transitive hull of △.



To obtain the reflexive, transitive hull of $\triangle$, we include all the ordered pairs of the form (x, x) where $x \in M$, as elements of the set i.e. $RT(\triangle) = \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (c, d), (d, d), (e, e), (e, f), (e, g), (f, f), (f, g), (g, g), (h, h)\}$.



Thus $RT(\triangle)$ is the smallest subset of $M \times M$ which is reflexive, transitive, and contains $\triangle$.

APPENDIX 2b

A notion used by German didacticians (though seemingly little discussed in England) is that of a domain of quantities (Grossenbereich). A domain of quantities is in fact the appropriate abstract model for the activities of weighing and measuring which are such a strong feature of the didactics of primary mathematics in England.

A domain of quantities is defined by Griesel (33) as

A set M, with a binary operation + and a relation $<$ which satisfies for all a, b $\in$ M

1. Commutativity: $a + b = b + a$
2. Associativity: $a + (b + c) = (a + b) + c$
3. Either $a < b$ or $b < a$ or $a = b$
4. $a < b$ if and only if there exists $c \in M$ such that

$$a + c = b.$$

(Note that M is assumed closed with respect to +; in axiom 3 the 'or' is exclusive. No reference is made to a zero element, but by implication such an element is excluded).

It is easy to show that axiom 3 ensures that $<$ is asymmetric, and that the associativity and closure of + lead to the transitivity of $<$.

A domain of magnitudes can be seen as a particular kind of grouping, and it corresponds to Piaget's grouping V.

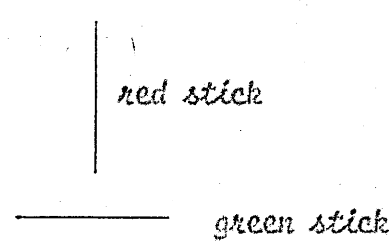
APPENDIX 2c

Instruments designed to test a child's ability to use the transitive property of matching and length relations

1. The Transitivity Test in Steffe, L.P. and Carey, R.L. "Learning of Equivalence and Order Relations by Four- and Five-Year-Old Children".

Sample item:

"*Materials:* *A red stick and a green stick of the same length attached to a cardboard as follows:*



A white stick the same length as the red and green sticks for the child's use.

Question: (a) *"Is the red stick the same length as your stick?"*

(b) *"Is the green stick the same length as your stick?"*

(c) *"Is the green stick shorter than the red stick?"*

((26), p. 46).

Further details:

"*The Transitivity Test, consisted of six items where "Yes" was the correct response for three items. For these items each of the relations "longer than", "shorter than", and "the same length as" was included. "No" was the correct response for the remaining three items. Each of the latter three items involved transitivity of "the same length as". It was not possible for the child to use a non-transitive hypothesis to arrive at a correct response because all of the perceptual* *cues were biased against a correct response and the child was not allowed to directly compare the two curves under consideration.* ((26), p. 26)

Testing Procedure:

"*The children were tested on a one-to-one basis. The items were assigned at random to each child so that each had a different sequence of the same items. All tests were administered by specially trained evaluators.*

In the case of the Transitivity Test, unless a child established two correct comparisons no measure was obtained on his ability to use the transitive property of that relation." ((26), p. 26-27).

2. The Transitivity Tests in Owens, D.T. "Learning of Equivalence and Order Relations by Disadvantaged Five- and Six-Year-Old Children".

The Transitivity of Matching Relations (TMR) Test

"*The purpose of the Transitivity of Matching Relations (TMR) Test was to measure a child's ability to use the transitive property of matching relations. On a TMR item a child was presented three collections A, B, C of physical materials arranged in clusters. Suppose, for example, that there were fewer a's than b's and fewer b's than c's. The child was instructed to pair the a's and b's and was then asked*

"*Are there fewer a's than b's?*"

The examiner then put the a's into a cup which sat nearby and said

"*Pair the b's and c's.*"

After the pairing the examiner asked

"*Are there fewer b's than c's?*"

The examiner then placed the c's in another cup and asked

"Are there fewer a's than c's?"

and

"Are there more a's than c's?" (or "Are there as many a's as c's?")

Note that the sets A and C were not "paired" and that the objects were screened at the time of the transitive inference. ((22), p. 54-55).

The Transitivity of Length Relations (TLR) Test

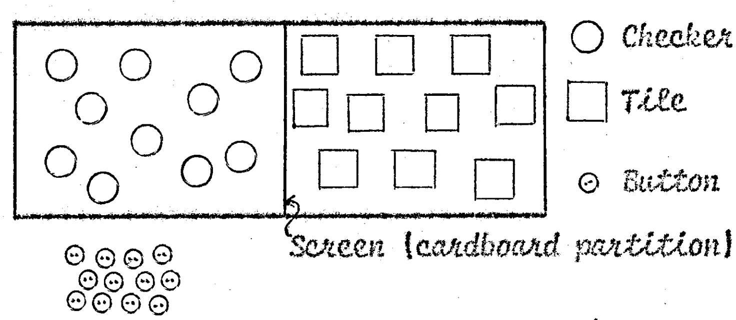
"The Transitivity of Length Relations (TLR) Test was designed to measure the ability of a child to use the transitive property of the length relations. On each item, as in the TMR test, a child was asked to establish the relation between two sticks A and B. Stick A was placed in a box and stick B was compared with another stick C such that the same relation held between B and C as between A and B. Then stick C was placed in a box and two questions, relative to A and C, were asked. ((22), p. 55).

The Transitivity Problem (TP)

"The Transitivity Problem (TP) was designed to measure the ability of a child to solve a problem which involved transitivity of a matching relation with minimum guidance from the examiner. The situation involved a cardboard box from which the front and top were removed. The box was divided into halves by a partition as shown in Figure 2. Ten checkers were attached to the bottom inside one half of the box and ten tiles were attached in the other side. Twelve buttons lay on the table in front of the box. After the objects were identified, the examiner said, "Find out if there are as many checkers as tiles. You may use the buttons to help you find out."

In general the examiner gave as little guidance as was possible, but if the child failed to respond at some point, the examiner directed the next step toward solution. When a response was given, the examiner asked for an explanation.

Figure 2



((22), p. 56).

Scoring Tests

"An item was scored "pass" provided that the child answered correctly all the questions contained in the item and "fail" otherwise. The number of items scored "pass" by a child on each test was considered to be his score on the test. For the purpose of comparing these data with other studies it was desirable to distinguish children for which evidence existed that they could use a property from those for which no such evidence existed. This was accomplished by setting a criterion score based on a random model. It was assumed that a child could use a relational property if and only if he met the criterion on a particular test. Four of the six items was the criterion set on each of the THR and TLR Tests. The probability of reaching this criterion by guessing was at most 0.038.

For the Transitivity Problem the following four levels of *ability to apply the transitive property were identified:*

1. *the child neither consistently established relations nor used the transitive property:*
2. *the child established relations but did not use the transitive property:*
3. *the child both established relations and used the transitive property without adequate justification:*
4. *the child established relations, used transitivity and gave adequate justification for his conclusion.*

The consensus of two of three judges' ratings, based on transcripts of audio tapes was taken as the child's rating on the Transitivity Problem. ((22), p. 57).

3. <u>Transitivity Test in Johnson, M.L. "Learning of Classification and Seriation by Young Children"</u>

<u>Transitivity of Length Relations Test (TLRT)</u>

"*This test consisted of six items; two each for the relations "same length as", "longer than" and "shorter than". Two perceptual stimuli were present: screened and conflictive. All materials in this test consisted of red, blue and green sticks all $\frac{3''}{8}$ in diameter and differing in length by $\frac{1''}{8}$. In each item the child had first to determine the relation that existed between the red and blue sticks, then the blue and green sticks. To make an inference about the relation that existed between the red and green sticks the child was asked three questions in random order,* (i.e. Is the red stick longer than the green stick? Is the red stick the same length as the green stick? Is the red stick shorter than the green stick?). *On the items with screened stimuli the final inference about the length of the red and green sticks had to be made with the sticks* *in boxes and not visible to the subjects.* *This test was used both as a pretest and a posttest with scoring.*

((27), p. 78).

4. <u>The Transitivity Test in Johnson, D.C. "Learning of Selected Parts of a Boolean Algebra by Young Children"</u>

<u>Transitivity Test (TR)</u>

"*This 10 item test designed to measure the ability of children to use the transitive property of the relations tested for in the Relation Achievement Test,* (i.e. "more than", "fewer than", "as many as", "same shape as" and "same colour as"). *Two items were designed to test for the transitive property of each of the five relations. A "left to right" and a "right to left" matching were used in the testing for the transitivity property of the relations "as many as", "more than" and "fewer than". The triplets of numbers of objects used for testing for the above three relations were (7, 7, 7) and (8, 8, 8); (8, 7, 6) and (9, 8, 7); and (6, 7, 8) and (7, 8, 9) respectively. The test was used as a transfer measure to determine if an ability to use transitivity is improved by instruction on the relations of concern.*

An example of a transitivity item for matching relations is where there were seven red discs and seven green discs mounted in rows on posterboard. The child was directed to match a pile of seven blue discs with the red discs and judge the relations between the two sets. The red discs were then covered. The child was then directed to match the blue discs with the green discs and judge the relations between the two sets. The green discs were then covered. Three questions *were then asked: "Are there as many red discs as green discs?" "Are there more red discs than green discs?" and "Are there fewer red discs than green discs?". An analogous procedure was used for transitivity of the equivalence relations involving color and shape, except only two questions were asked, one for the appropriate equivalence relation and one for its accompanying difference relation.* ((12, p. 130).

<u>Administration of TR</u>

"Items were arranged in a row on a low table. Administration of the six items for matching relations was conducted followed by the four items for the color and shape relations. Within this constraint the items were ransomized independently for each subject. A transitivity item was scored as correct only if all questions were correctly answered. ((12), p. 132).

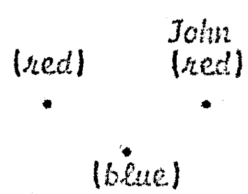
APPENDIX 2d

Further details of the pilot study undertaken to clarify the stages in the development of symmetry

List of questions used:

The questions below were included as part of a natural conversation with the child about the dolls. One of the purposes of the conversation was the establishment of a suitable rapport with the child. If a question had to be repeated, however, it was repeated exactly, as many times as required, without providing additional clues by comment or gesture. These questions differ from those listed on pages 81-86 in small respects only. However, it may be useful to have them listed completely here.

(See Test A)



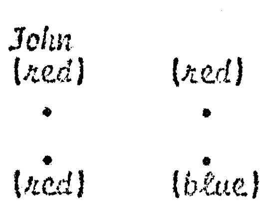
Question 1 (a) This is John. John is wearing a different coloured shirt from Paul. Which doll is called Paul?

OR Question 1 (b): This is John. Paul is wearing a different coloured shirt from John. Which doll is called Paul?

Question 2: Is there another doll wearing a different coloured shirt from Paul?

Question 3: What is his name?

(See Test B)

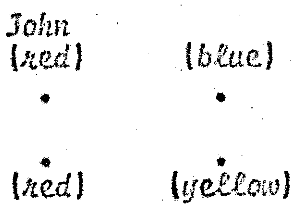


Question 1: This is John. He is wearing a different coloured shirt from David. Which doll is called David?

Question 2: Is David wearing a different coloured shirt from Paul?

Question 3: Tell me the names of two dolls that are wearing different coloured shirts. Can you tell me the names of another two dolls who are wearing different coloured shirts?

(See Test D)

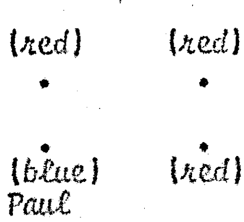


Question 1: This is John. John, Paul and David are all wearing different coloured shirts. Will you please put John, Paul and David sitting together in a group in front of you. Which doll is called Robert?

Question 2: Are Robert and John wearing shirts which are the same colour or are they different?

Question 3: Are Robert and Paul wearing shirts which are the same colour or are they different?

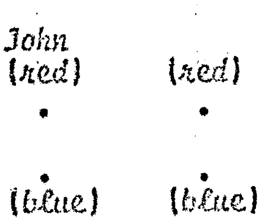
(See Test E)



This is Paul. I am going to say the names of two dolls and I want you to tell me whether their shirts are the same colour or whether they are different.

1. Paul and David
2. John and Robert
3. John and Paul
4. David and Robert.

(See Test F)



Question 1: This is John. John is wearing a different coloured shirt from David. Can we tell which doll is called David?

Question 2: Which doll might be called David?

Question 3: John is wearing a different coloured shirt from Paul. Are Paul and David wearing shirts which are the same colour or are they different?

Question 4: Which doll is called Robert?

Question 5: Are the shirts of Paul and Robert the same colour or different?

(See Test G)

(red) (red)

(blue)

Question 1: John is wearing a different coloured shirt from Paul and Paul is wearing a different coloured shirt from David. Which doll is called Paul?

Question 2: Are John and David wearing shirts which are the same colour or are they different?

(See Test H)

(a) (red) (red)

(red) (blue)

Question 1: John and Paul are sitting together and Robert and David are sitting together. Robert and David are wearing different coloured shirts. Where are the dolls called Robert and David?

(b) (red) (red)

(blue) (blue)

Question 2: John and Paul are sitting together and Robert and David are sitting together. Robert and David are wearing different coloured shirts. Can we tell where Robert and David are sitting?

Question 3: This is Robert (blue). Just now one of the dolls changed his shirt from a red one to a blue one. Can you tell me which of the dolls changed his shirt?

<u>Stages of development inferred from correct responses to the above questions</u>

It is convenient to classify the responses according to three stages of development which they may be taken to indicate. Still finer sub-divisions may be possible, but the three stages described below are a suitable initial classification as the responses of the children confirm.

At Stage 1, in a given problem situation, the child recognizes at least one instance of a pair (x, y) such that "x is wearing a different coloured shirt from y" when certain about both individuals. Furthermore, he recognizes non-exemplars of the basic relation, i.e. he recognizes at least one instance of a pair (x, y) such that "x is NOT wearing a different coloured shirt from y", when certain about both individuals.

At Stage 2, in a given problem situation, the child recognizes at least one instance of a pair (x, y) such that "x is wearing a different coloured shirt from y" when certain about one individual only. Furthermore he recognizes non-exemplars of the basic relation, i.e. he recognizes at least one instance of a pair (x, y) such that "x is NOT wearing a different coloured shirt from y", when certain about one individual only.

N.B.

A further distinction was made by referring to Stage 2^*, at which it was plain that the child could recognize more than one instance of the basic relation in a situation in which more than one instance was to be seen.

At Stage 3, in a given situation, the child recognizes at least one instance of a pair (x, y) such that "x is wearing a different coloured shirt from y" when not certain about either individual. Furthermore he recognizes non-exemplars of the basic relation, i.e. he recognizes at least one instance of a pair (x, y) such that "x is NOT wearing a different coloured shirt from y", when not certain about either individual.

N.B.

A further distinction was made by referring to Stage 3* at which it was plain that the child could recognize more than one instance of the basic relation in a situation in which more than one instance was to be seen.

Thus the answers to the various questions may be taken as indicating the stages of concept formation in the following way:

A correct response to

Question 1 (a), Test A Question 1 (b), Test A Questions 2 and 3, Test A Question 1, Test B	implies	the child recognizes a pair (x, y) such that "x is wearing a different coloured shirt from y", when certain about both individuals. (i.e. Stage 1).

A correct response to

Question 2, Test D	implies	the child recognizes a pair (x, y) such that "x is NOT wearing a different coloured shirt from y", when certain about both individuals. (i.e. Stage 1 on non-exemplar

of the basic relation).

A correct response to

Question 2, Test B	implies	the child recognizes a pair (or pairs) (x, y) such that "x is wearing a different coloured shirt from y", when certain about one individual only. (i.e. Stage 2)
Question 3, Test D		
Question 1, Test E		
Question 3, Test E		
Question 5, Test F		
Question 1, Test G		

A correct response to

Question 1, Test H	implies	the child recognizes a pair (or pairs) (x, y) such that "x is wearing a different coloured shirt from y", when not certain about either individual. (i.e. Stage 3)
Question 3, Test H		

A correct response to

Question 2, Test E	implies	the child recognizes a pair (or pairs) (x, y) such that "x is NOT wearing a different coloured shirt from y", when not certain about either individual. (i.e. Stage 3 on non-exemplar of the basic relation).
Question 4, Test E		
Question 3, Test F		
Question 2, Test G		

The response "John and David" only to Question 3, Test B, implies the child recognizes a pair (x, y) such that "x is wearing a different coloured shirt from y", when certain about both individuals.

A response which includes "John and David" and either "David and Paul" or "David and Robert" to Question 3, Test B, implies the child recognizes a pair (x, y) such that "x is wearing a different coloured shirt from y", when certain about one individual only, (i.e. Stage 2).

The response "David and Paul, and David and Robert" to Question 3, Test B, implies the child recognizes pairs (x, y) (i.e. more than one pair) such that "x is wearing a different coloured shirt from y", when certain about one individual only, (i.e. Stage 2*).

Similarly, a correct response to Questions 1 and 2 of Test F implies the child recognizes pairs (x, y) such that "x is wearing a different coloured shirt from y", when certain about one individual only, (i.e. Stage 2*).

Correspondingly, a correct response to Question 2, Test H, implies the child recognizes pairs (x, y) such that "x is wearing a different coloured shirt from y", when not certain about either individual, (i.e. Stage 3*).

Results

Stage 1:

		Alison (age 4)	Sarah (age 5)	Dean (age 5)	Jason (age 5)	Paul (age 5)	Daren (age 6)	Tracy (age 6)	Simon (age 6)	Brian (age 6)	Hayley (age 7)
Test A:	Question 1 (a)			/							/
	Questions 2 & 3			x							x
OR Test A:	Question 1 (b)	/	/		/	/	/	/	/	/	
	Questions 2 & 3	x	/		x	/	x	/	/	/	
Test B:	Question 1	/	/	/	x	/	x	/	/	/	/
Stage 1 on non-exemplar of the basic relation											
Test D:	Question 2	/	/	x	/	x	x	/	/	/	/
		1	0	2	2	1	3	0	0	0	1

Stage 2:

		Alison (age 4)	Sarah (age 5)	Dean (age 5)	Jason (age 5)	Paul (age 5)	Daren (age 6)	Tracy (age 6)	Simon (age 6)	Brian (age 6)	Hayley (age 7)
Test B:	Question 2	/	/	x	/	/	/	/	/	/	/
Test D:	Question 3	x	/	x	/	x	/	/	/	/	/
Test E:	Question 1	/	/	x	x	/	/	/	/	/	/
	Question 3	/	/	/	x	/	x	/	/	/	/
Test F:	Question 5	x	x	/	/	/	x	x	x	/	/
Test G:	Question 1	/	/	x	x	/	/	/	/	x	/
		2	1	4	3	1	2	1	1	1	0
Stage 2*:											
Test B:	Question 3	'	"	x	x	"	x	'	"	"	x
Test F:	Questions 1 and 2	x	"	"	"	"	"	"	"	"	"

Key: ' Stage 1 response

" Stage 2 response - Before giving a Stage 2 response, eye and hand movements indicated that each child considered both possibilities but only one of the two was selected.

Stage 3:

		Alison (age 4)	Sarah (age 5)	Dean (age 5)	Jason (age 5)	Paul (age 5)	Daren (age 6)	Tracy (age 6)	Simon (age 6)	Brian (age 6)	Hayley (age 7)
Test H:	Question 1	/	/	x	/	/	/	x	/	/	/
	Question 3	x	x	x	x	o	x	x	x	x	o
Stage 3 on non-exemplar of the basic relation											
Test E:	Question 2	/	/	x	x	/	/	/	/	/	/
	Question 4	/	/	/	x	/	x	/	/	/	/
Test F:	Question 3	x	x	x	x	/	x	/	/	x	x
Test G:	Question 2	/	/	x	/	/	x	/	/	x	/
		2	2	5	4	1	4	2	1	3	2
Stage 3*											
Test H:	Question 2	x	x	x	"'	"'	"'	x	"'	"'	"'

Key: o indicates that before selecting one individual, the child's eye and hand movements showed that both were considered.

"' Stage 3 response - Before giving a Stage 3 response, eye and hand movements indicated that each child considered both possibilities but only one of the two was selected.

Using an error count on the scoring items for each stage, we obtain

	No. of errors at		
	Stage 1	Stage 2	Stage 3
Simon (age 6)	0	1	1
Paul (age 5)	1	1	1
Hayley (age 7)	1	0	2
Tracy (age 6)	0	1	2
Sarah (age 5)	0	1	2
Alison (age 4)	1	2	2
Brian (age 6)	0	1	3
Daren (age 6)	3	2	4
Jason (age 6)	2	3	4
Dean (age 5)	2	4	5

These results appear to support the conjecture that the subjects' responses indicate at least three stages in the development of symmetry, which correspond to the stages already specified. For, apart from two exceptions (Hayley and Daren with more errors at Stage 1 than Stage 2), the subjects showed a steady state or increase in the number of errors from Stage 1 through to Stage 3.

Further points for consideration

Although the aim of this pilot study was merely to test the feasibility of an investigation along these lines and the suitability of the particular questions, it was noted that for each of the two sets of six items used to identify Stage 2 and Stage 3, we have 2^6 different sextuples with 'correct' or 'incorrect' as elements. In a very similar test situation with sextuples Steffe and Carey ((26), p. 28) pointed out that on the hypothesis that 'correct' and 'incorrect' answers were equiprobable, the probability that any one of the 2^6 possible sextuples occurred was 2^{-6}. This in turn would indicate that the probability that a child obtained at least 5 or 6 correct responses by guessing was approximately 0.11. This calculation is clearly open to criticism because of the assumption of the equiprobabilities. Whilst recognizing this weakness, we also note that in the present investigation there were usually more than 2 possible answers to the questions, so the actual probabilities of achieving a score of 5 or 6 by guessing should be substantially less than the 0.11 calculated above. And so, following Steffe and Carey, a total score of 5 or 6 was taken as the criterion score for Stage 2 and Stage 3. This gives

	Stage 2	Stage 3
Simon (age 6)	/	/
Paul (age 5)	/	/
Hayley (age 7)	/	x
Tracy (age 6)	/	x
Sarah (age 5)	/	x
Brian (age 6)	/	x
Alison (age 4)	x	x
Daren (age 6)	x	x
Jason (age 5)	x	x
Dean (age 5)	x	x

But before proposing that an appropriate follow-up study be undertaken to confirm or reject the existence of the three stages in the development of symmetry, a number of deficiencies in the design of the pilot study highlighted by the attempt to identify a criterion score for Stages 2 and 3, need to be rectified to satisfy this new purpose. These are

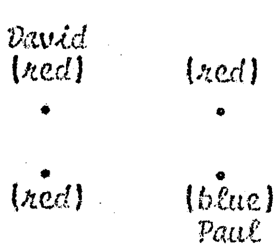
1. Insufficient scoring items (3 only) to test Stage 1 on the basic relation.
2. Insufficient scoring items (2 only) to test Stage 3 on the basic relation.
3. The need to check that the subject can distinguish between pairs which satisfy the basic relation and pairs which do not satisfy the basic relation, at all stages of development. Hence, the need to increase the number of items to test Stage 1 on non-exemplars of the basic relations and to include some items to test Stage 2 on non-exemplars of the basic relation.

It is therefore proposed that at least the following questions be included in the sequence of tests:

"Are John and David wearing shirts which are the same colour or are they different?"	Question 4, Test A (Stage 1 on non-exemplar of the basic relation).
"Are Paul and David wearing shirts which are the same colour or are they different?"	Question 4, Test D (Stage 3).

and that the following test be included as Test C of the sequence:

<u>Test C</u>: (Four dolls - John, Paul, David and Robert)
Three of the dolls are wearing red shirts and one is wearing a blue shirt.



The experimenter picks up a doll wearing a red shirt and says:
This is David (experimenter attaches David label) and this is Paul. (Experimenter attaches

Paul label and spreads out the other two labels).
I am going to say the names of two dolls and I want you to tell me whether their shirts are the same colour or whether they are different.

1. David and Paul (Stage 1)
2. John and Paul (Stage 2)
3. David and John (Stage 2 on non-exemplar of basic relation)
4. Paul and Robert (Stage 2)
5. Robert and David (Stage 2 on non-exemplar of basic relation)
6. John and Robert (Stage 3 on non-exemplar of basic relation).

All of these additional items are similar in structure to items which were included in the pilot study.

APPENDIX 2e

Comparison of instruments designed by Steffe and Carey and D.T. Owens to test conservation of length relations

Steffe and Carey

Conservation of Length Relations Test

Sample Items:

"*Level I* - *Longer than*

Materials: One green straw; 3 red straws, one being longer than, one shorter than and one the same length as the green straw.

Statement: Using these red straws, find a straw longer than this green straw.

Transformation:

green

red

(move the red straw)

Question: "Is this red straw still longer than this green straw?" .

((26), p. 44).

D.T. Owens

Conservation of Length Relations Test

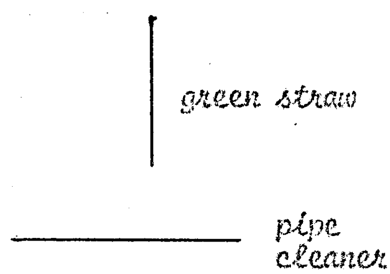
"The Conservation of Length Relations (CLR) Test was designed to measure the ability of a child to conserve length relations. In each item the child was asked to establish a length relation between two sticks (or straws) by answering two questions. Then the sticks were rearranged to produce a perceptual bias against the correct conclusion, and the questions were repeated. ((22), p. 54).

"<u>Level II</u> - <u>Longer than</u>

<u>Materials</u>: One green straw, 3 white pipe cleaners, one being longer than, one shorter than, and one the same length as the green straw.

<u>Statement</u>: Using these pipe cleaners, find a pipe cleaner longer than this green straw.

<u>Transformation</u>:



(move the green straw)

<u>Question</u>: "Now is the green straw longer than the pipe cleaner?"

((26), p. 45).

APPENDIX 2f

Further details of the Classification Test used by M.L. Johnson

Classification Test

"This test consisted of three items: two requiring the child to group sticks on the basis of length and one in which the child had to determine the criteria used for sticks already grouped.

The material for item 1 consisted of 12 green sticks, each $\frac{3}{8}$" diameter, with four of length 5", four of length $5\frac{1}{4}$" and four of length 5½". One stick of each length was mounted on a piece of paper board. The three mounted sticks were pointed out to the child who was then instructed to

"find all of the sticks that would go with this stick (5"), this stick ($5\frac{1}{4}$") and this stick (5½").

The nine sticks to be classified were in disorder before the child. A record of all sticks correctly and incorrectly placed was kept by the experimenter.

The materials of item 3 consisted of ten red sticks all $\frac{3}{8}$" diameter, three of length 4", three of length $4\frac{1}{4}$", three of length 4½", and one of length $4\frac{3}{4}$". The ten sticks were given to the child and he was instructed to

"put all of the sticks together that belong together".

A record of the child's actions was kept by the experimenter.

Item 2 required that the child determine the criteria used for grouping. The materials for this item consisted of fifteen sticks; five each at length 6", $6\frac{1}{4}$", and 6½". The sticks were placed into three distinct piles about 15" apart on a table. Within a pile, sticks differed in colour and diameter; with length being constant. The child was instructed to

"Tell me why I have all of these sticks together in this pile (6"), in this pile ($6\frac{1}{4}$") and in this pile (6½")."

If a correct answer was given, the child was asked to justify his answer. Upon justification, he was then asked

"Why do I have these sticks in different piles?"

Again a justification for a correct answer was asked for. A record of all answers was kept by the experimenter. ((27), p. 79).

<u>Scoring Test</u>

"From the children's responses to item 1, four performance categories were identified. They were:

(a) the child did not attempt to classify sticks;

(b) the child made some partial classes but did not exhaust the set of sticks to be classified;

(c) the child exhausted the set but made some incorrect choices; and

(d) the child correctly classified all sticks.

Item 2 ..... . Five distinct categories were identified. They were:

(a) the child did not discover the criteria;

(b) the child gave a correct reason for the piles being together but without justification;

(c) a correct reason was given with justification;

(d) in addition to justifying the reason for sticks belonging in distinct groups, the subject correctly gave a reason for sticks being in different groups but without justification for his reason.

(e) all of (d) with justification.

In item 3 ..... . Four categories of performance were identified;

(a) no attempt was made to group the sticks;

(b) the child made at least two piles with the sticks being placed incorrectly;

(c) the child put all sticks in correct piles according to length except the longest stick;

(d) the child correctly classified all sticks, including the longest stick.

((27), p. 85-86).