Topological methods in geometry and discrete mathematics

by

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July, 2020

A thesis presented to the
Graduate School
of the
Institute of Science and Technology Austria, Klosterneuburg, Austria
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

I|S|T AUSTRIA
Institute of Science and Technology

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IST Austria Thesis, ISSN: 2663-337X

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Abstract

We present solutions to several problems originating from geometry and discrete mathematics: existence of equipartitions, maps without Tverberg multiple points, and inscribing quadrilaterals.

Equivariant obstruction theory is the natural topological approach to these type of questions. However, for the specific problems we consider it had yielded only partial or no results.

We get our results by complementing equivariant obstruction theory with other techniques from topology and geometry.

List of Publications

The thesis is based on the following papers:

1. Avvakumov, S. and Kudrya, S., 2019. Vanishing of all equivariant obstructions and the mapping degree. Preprint, arXiv:1910.12628.
2. Avvakumov, S., Mabillard, I., Skopenkov, A. and Wagner, U., 2015. Eliminating higher-multiplicity intersections, III. Codimension 2. Preprint, arXiv:1511.03501.
3. Avvakumov, S., Karasev, R. and Skopenkov, A., 2019. Stronger counterexamples to the topological Tverberg conjecture. Preprint, arXiv:1908.08731.
4. Avvakumov, S. and Karasev, R., 2019. Envy-free division using mapping degree. Preprint, arXiv:1907.11183.
5. Akopyan, A., Avvakumov, S. and Karasev, R., 2018. Convex fair partitions into an arbitrary number of pieces. Preprint, arXiv:1804.03057.
6. Akopyan, A. and Avvakumov, S., 2018. Any cyclic quadrilateral can be inscribed in any closed convex smooth curve. In Forum of Mathematics, Sigma (Vol. 6). Cambridge University Press.

Table of Contents

1 Introduction

In this text we present solutions to several questions originating from geometry and discrete mathematics. Most of the tools we use come from topology.

A common topological approach to such questions is the *equivariant obstruction method*. It has been used with great success to solve a variety of problems, including Kneser's conjecture [57], the Square Peg conjecture for smooth curves [109], the Splitting Necklace problem [3], and the Topological Tverberg conjecture for primes and, more generally, prime powers [14; 75; 107], etc. It had also been applied to the questions we deal with here, but only partial or no results had been achieved.

In this chapter we briefly introduce the equivariant obstruction method and explain when it can fail. Then we give an overview of the results presented in this text. Each of the subsequent chapters is on its own problem. They can be read independently, however, some of the results from earlier chapters are used later.

1.1 The Equivariant Obstruction method

Let us briefly describe the method. One starts by defining a suitable **configuration space** of potential solutions to the problem. Then a **test map** from the configuration space to the **test space** is defined. Informally, the test map measures how far the given potential solution is from the **target**, a certain subspace of the test space. A point in the configuration space is a valid solution to the problem if and only if its image under the test map intersects ("hits") the target. Typically, a certain symmetry group defined by the problem acts both on configuration and test spaces, and the test map is equivariant with respect to this action. So, one can now restate the problem in topological terms: Is it there an equivariant map from the configuration space to the test space missing the target? If the answer is “no”, then the original problem always has a solution. The reader can find more details and examples in [64].

As a very simple example to illustrate this, consider the well-known ham sandwich theorem, which asserts that any d sufficiently nice probability measures $\mu_1, \ldots, \mu_d$ in $\mathbb{R}^d$ can be simultaneously equipartitioned by an affine hyperplane. In our example, the configuration space is the sphere S^d, which naturally parametrizes (oriented, affine) hyperplanes in $\mathbb{R}^d$. The test map $F\colon S^d \to \mathbb{R}^d$ sends an oriented hyperplane parametrized by $u \in S^d$ to the point $F(u) \in \mathbb{R}^d$ whose i-th coordinate is the difference $\mu_i(H_u^+) - \mu_i(H_u^-)$ of the values of the i-th measure on the two corresponding halfspaces. This map is equivariant with respect to the natural $\mathbb{Z}_2$-action on both spaces, i.e., $F(-u) = -F(u)$, and the classical Borsuk–Ulam Theorem guarantees that any such map must have a zero, which yields the desired simultaneous equipartition.

So, the method allows one to reduce a suitable question from geometry or discrete mathematics to a topological problem of existence of a certain equivariant map missing the target. If no such map exists, the original problem always has a solution. However, if such a map does exist, the method fails. A priori, the existence of a map missing the target does not imply the existence of a counterexample to the original geometric problem. Thus, one needs to go beyond the equivariant obstruction method and get more insight into the geometric nature of the problem. Either to develop more refined obstructions that capture more geometric information, for example see Chapters 6 and 7. Or, on the other hand, to show that the original obstruction is, in fact, *complete*, i.e., that its vanishing does in fact imply the existence of counterexamples, see Chapters 3 and 4.

1.2 Overview of the results

We briefly describe the content of the remaining chapters. See the corresponding chapters for the citations which we don’t give in this section.

Chapter 2 is based on [7]. The standard method of deciding whether an equivariant map between the configuration and test spaces exists requires computing a series of so called “equivariant cohomology obstructions”, the length of the series depending on the specific problem, see [25, Chapter II]. Typically, computing even the second obstruction in the series can be challenging or sometimes impossible. In this chapter we present results which allow to completely avoid these difficulties for some "popular" combinations of test spaces and symmetry groups. The chapter also contains results originally proved in [5] and [6].

Chapter 3 is based on [8]. The Topological Tverberg conjecture was originally proved for prime power multiplicities using the equivariant obstruction method. The method failed for non-prime powers, but as mentioned above, that didn't imply that the conjecture was wrong. The first counterexamples were constructed using a modification of Whitney trick for high multiplicity intersections. As the original, this modification only worked in codimension ≥ 3. We show how to make it work in codimension ≥ 2, which gives new counterexamples to the conjecture.

Chapter 4 is based on [6]. Together with the Topological Tverberg conjecture we consider a more general question, when a map without r-tuple points exists from a simplicial complex K to $\mathbb{R}^d$? Compared to Chapter 3, we now allow the set of r-tuple points to have positive dimension. We combine the results on vanishing of equivariant obstructions from Chapter 2 with previous geometrical results, to get new answers.

Chapter 5 is based on [5]. We consider a problem with economic & game theoretical flavor. Given a finite resource and a several players with different preferences, can the players divide the resource among themselves so that everyone is satisfied, i.e., doesn't want anyone else's share? We apply the results of Chapter 2 and other topological techniques. Surprisingly, the answer depends on the number theoretical properties of the number of players.

Chapter 6 is based on [2]. We consider the following problem: given a convex body K in the plane can we cut K into m parts of equal area and perimeter? The problem had previously been solved for prime powers $m = p^k$ using the equivariant obstruction method. For all other m there exists an equivariant map from the relevant configuration space missing the target space, and so the method fails. We prove the existence of the desired equipartition of K for all m using more refined topological methods. We also consider generalizations to higher dimensions and functions different from "area" and "perimeter".

Chapter 7 is based on [1]. We consider the following geometrical question: Can a given cyclic (i.e., inscribed in a circle) quadrilateral can be inscribed into any Jordan curve in the plane? This question was first asked for squares more than a hundred years ago. Equivariant obstruction theory gives a positive solution for squares (and “nice” curves), but fails for other quadrilaterals. Approaching the problem geometrically, we solve it positively for convex curves.

2 Vanishing of all equivariant obstructions and the mapping degree

2.1 Main results

Applying the equivariant obstruction method to problems from geometry, one often finds that the natural symmetry group of the problem is the symmetric group $\mathfrak{S}_n$. At the same time, the corresponding test space often is $(\mathbb{R}^m)^n$ with the diagonal $\delta_n := \{(x, x \dots, x) \in \mathbb{R}^n | x \in \mathbb{R}^m\}$ as the target; the symmetric group $\mathfrak{S}_n$ acts on these space by permutations of coordinates. For example, this is the case in the Splitting Necklace problem and the Topological Tverberg conjecture mentioned above, and fair [51; 22] or envy-free [34; 86; 71] division problems. This is also the case for many problems considered in the later chapters of this thesis.

For this popular combination of the symmetry group and the test space, the main results of this chapter allow us to bypass the difficult calculations of the equivariant obstruction (see [25, Chapter II]) entirely:

Theorem 2.1. *Suppose that $n \neq p^k$ and $n \neq 2p^k$ for all k and all primes p. Then for any Hausdorff compactum[1] X with a free action of $\mathfrak{S}_n$ there exists an equivariant map $X \to \mathbb{R}^n \setminus \delta_n$.*

Remark 2.2. From the conclusion of the theorem it easily follows that there exists an equivariant map $X \to (\mathbb{R}^m)^n \setminus \delta_n$ for any $m > 1$ (note, that δ_n depends on m).

[1]Hausdorff compactum is a topological space which is both Hausdorff and compact.

Theorem 2.3. *Suppose that $n \neq p^k$ for all k and all primes p. Then for any Hausdorff compactum X with a free action of $\mathfrak{S}_n$ there exists an equivariant map $X \to (\mathbb{R}^2)^n \setminus \delta_n$.*

Remark 2.4. With an additional assumption $\dim X = d(n-1)$ Özaydin proved that there exists an equivariant map $X \to (\mathbb{R}^d)^n \setminus \delta_n$ for $d \geq 1$, see [75] and the survey [98, Theorem 3.5]. In light of Remark 2.2, Theorem 2.3 is a generalization of Özaydin's Theorem. The assumption $\dim X = d(n-1)$ is crucial in Özaydin's proof, since it reduces the problem to the vanishing of the *primary* equivariant cohomology obstruction, see [25, Chapter II].

Remark 2.5. Some weaker results are proved with a harder technique in [17].

Remark 2.6. Theorem 2.1 and 2.3 are similar to, but are not particular cases of [16, Theorem 3.6]. Indeed, [16, Theorem 3.6] takes a group G from a certain class and proves that there exists *some* representation W of G, for which there exists a G-equivariant map $X \to S(W)$ from any fixed point free G-space X. In Theorem 2.1, by contrast, we prove that for a specific group $G = \mathfrak{S}_n$ and a *specific* representation sphere of G, $S(W_n)$, where W_n is the orthogonal complement to $\delta_n \subset \mathbb{R}^n$, there exists a G-equivariant map $X \to S(W_n)$ from any free G-space X. The group $G = \mathfrak{S}_n$ does not satisfy the hypothesis of [16, Theorem 3.6] because it contains a subgroup (the alternating group) of prime index. The discussion in [16, the paragraph after Theorem 3.6] also hints that our specific representation W_n cannot be the one constructed in the proof of [16, Theorem 3.6], since W_n has the property $W_n^H = 0$ whenever a subgroup $H \subset \mathfrak{S}_n$ acts transitively on the indices $1, \dots, n$.

The space X in the statements is a substitute for the configuration space. The restrictions on X are not significant, in practice a configuration space can usually be equivariantly contracted to a compact polyhedron.

The proofs of the theorems above rely on the following lemma. The lemma (together with its proof) was communicated to us by Alexey Volovikov. It is a particular case of [16][Lemma 3.9], but we present a short proof of the case we need here for completeness.

Lemma 2.7. *Let G be a finite group and S be a sphere with an action of G. If there exists a G-equivariant map $f : S \to S$ of zero degree then any Hausdorff compactum X with a free action of G has a G-equivariant map $X \to S$.*

Proof. A zero degree map of spheres $S \to S$ is null-homotopic and can be continuously extended to a cone over the sphere S. Consider the join $G * S$ as a union of $|G|$ such cones glued together along their bases and extend the map from one cone to all other cones by equivariance with respect to the diagonal action of G on the join, obtaining an equivariant map $g : G * S \to S$. Then take joins of g with identity maps of G and compose them to extend the chain of equivariant maps

$$\cdots \to G * G * G * S \to G * G * S \to G * S \to S.$$

Since every component of the join embeds into the join, we may drop S in the domain and eventually have an equivariant map as a composition:

$$\underbrace{G * G * \cdots * G}_{N} \to \underbrace{G * G * \cdots * G}_{N} * S \to S$$

for any N.

The join in the domain of the last map is the $(N-2)$-connected $(N-1)$-dimensional approximation $E_N G$ to the classifying space EG of the group G. By standard properties of the classifying spaces it follows that, given a Hausdorff compactum X with a free action of G, there exists[2] an equivariant map $X \to E_N G$ for sufficiently large N, hence there exists an equivariant map $X \to S$ as a composition of $X \to E_N G \to S$. $\square$

Now, Theorem 2.1 follows as a combination of Lemma 2.7 and the following:

Theorem 2.8. *For $n > 1$, let S be the unit sphere of the orthogonal complement to $\delta_n \subset \mathbb{R}^n$. The symmetric group $\mathfrak{S}_n$ acts on S by permuting the coordinates of $\mathbb{R}^n$.*

Let d be the degree of a $\mathfrak{S}_n$-equivariant map $S \to S$. Then:

(a) if $n = p^k$ for some prime $p \neq 2$ then d can attain any value $d \equiv 1 \pmod p$ and only such values,

(b) if $n = 2p^k$ for some prime p then d can only attain values $d \equiv \pm 1 \pmod p$,

[2]Let us sketch a proof of this fact. Since X is Hausdorff, it follows that each point $x \in X$ has a G-equivariant neighborhood U disjoint with gU for all $g \neq 1$. Denote $U_x := GU$. From the compactness of X it follows, that X can be covered by a finite number of such orbits, $X = U_{x_1} \cup \ldots \cup U_{x_N}$. For each U_{x_i} pick a G-equivariant map $f_i : U_{x_i} \to G$. As a Hausdorff compactum, X is normal. So, there exists a subordinate to the cover G-equivariant partition of unity $\rho_1 + \cdots + \rho_N = 1$. Now, the map $x \mapsto \rho_1(x) f_1(x) \oplus \cdots \oplus \rho_N(x) f_N(x)$ maps X to $E_{N-1}G$ and is G-equivariant.

(c) if n is odd and $n \neq p^k$ for all primes p then d can attain any value,

(d) if n is even and $n \neq 2p^k$ for all primes p then d can attain 0.

Indeed, by parts (c) and (d) of Theorem 2.8 and by Lemma 2.7, there is a $\mathfrak{S}_n$-equivariant map $X \to S \subset \mathbb{R}^n \setminus \delta_n$, so Theorem 2.1 follows.

Note, that only parts (c) and (d) of Theorem 2.8 are required to prove Theorem 2.1. The "only" part of Theorem 2.8(a) was probably known before, and Theorem 2.8(c) was first proved in [5].

Likewise, Theorem 2.3 follows as a combination of Lemma 2.7 and the following:

Theorem 2.9. *Suppose that $n \neq p^k$ for all primes p. Let S be the unit sphere of the orthogonal complement to $\delta_n \subset (\mathbb{R}^2)^n$. The symmetric group $\mathfrak{S}_n$ acts on S by permuting the coordinates of $(\mathbb{R}^2)^n$.*

Then there exists a $\mathfrak{S}_n$-equivariant from $S \to S$ of degree 0.

Interestingly, the (almost) converse of Theorem 2.1 holds for $n = p^k$:

Theorem 2.10. *Suppose that $n = p^k$ for a prime p. Then for any $(n-2)$-connected topological space X with a free action of $\mathfrak{S}_n$ there is no $\mathfrak{S}_n$-equivariant map $X \to \mathbb{R}^n \setminus \delta_n$.*

For a proof of Theorem 2.10 see [107, §2, the Lemma] (to get the theorem from the lemma notice that $\mathfrak{S}_n$ contains $(\mathbb{Z}_p)^k$ as a subgroup acting on $\mathbb{R}^n \setminus \delta_n$ without fixed points), although the theorem was probably known earlier.

In the rest of this chapter we prove Theorems 2.8 and 2.9.

2.2 Lemmas

In this section we prove lemmas required for the proofs of Theorems 2.8 and 2.9.

Until the end of this section, let S be a unit sphere of a euclidean space W and let G be a finite subgroup of the orthogonal group of W. The group G naturally acts on S.

Lemma 2.11. *There exists a G-equivariant map $S \to S$ of degree d if and only if*

$$d = 1 - \sum_{i=1}^{k} d_i \frac{|G|}{|G_i|},$$

where for each $i = 1, \ldots, k$

(1) a subgroup $G_i \subseteq G$ is the stabilizer of some point $x_i \in S$,

(2) the G-orbits of all x_i are pairwise disjoint,

(3) there is a G_i-equivariant map $S \to S$ of degree d_i which is an identity in a neighborhood of x_i.

Proof of the "if" part of Lemma 2.11. Let $f_0 : S \to S$ be the identity map.

Let $g_1 : S \to S$ be a G_1-equivariant map which is the identity in a small neighborhood U of x_1.

Choose a smaller circular neighborhood $x_1 \in V \subset U$, i.e., V is the intersection of U with a round ball of a small radius centered at x_1. Clearly, $G_1(V) = V$ and g_1 is the identity in V.

Let $\varphi : V \to S \setminus V$ be a G_1-equivariant homeomorphism which is the identity on ∂V. One can construct V and φ as follows. Denote by C the point outside S and lying on the line connecting 0 with x_1 and such that any line connecting C to any point in ∂V is tangent to S. Define $\varphi : V \to S \setminus V$ to be the radial projection with center C.

Define a map $f_1' : S \to S$ as follows:

- f_1' equals to f_0 on $S \setminus V$,
- f_1' equals to $g_1 \circ \varphi$ on V.

Clearly, $\deg f_1' = \deg f_0 - \deg g_1 = 1 - d_1$.

Now, there is a unique G-equivariant way to redefine f_1' on $G(V) \setminus V$ to get a G-equivariant map f_1. The degree of f_1 is $1 - d_1 \frac{|G|}{|G_1|}$.

Repeating this process for $x_2, x_3, \dots, x_k$ we get a G-equivariant map of degree $1 - \sum_{i=1}^{k} d_i \frac{|G|}{|G_i|}$. □

To prove the "only if" part of Lemma 2.11 we need the following technical statement:

Lemma 2.12. *Assume G is a finite group acting on a polyhedron P and acting linearly on a finite vector space V. Assume that for any subgroup $H \subseteq G$ the inequality $\dim P^H \leq \dim V^H$ holds for the subspaces of H-fixed points. Then for any G-invariant triangulation of P its barycentric subdivision has the following property: The set of G-equivariant PL* *maps $f : P \to V$, linear on faces of the barycentric subdivision, has an open C^1-dense subset consisting of maps with finite fibers $f^{-1}(y)$ for any $y \in V$.*

Proof. Assume that P is triangulated G-invariantly and consider G-equivariant maps, linear on faces of the barycentric subdivision P'. We show that a dense open subset of such maps (that is a *generic map* of this kind) has the required property. Such a map $f : P \to V$ is defined whenever we define it equivariantly on vertices of the subdivision P', and we argue by induction on the poset of the vertices of P', which is the same as the poset of faces of P.

Assume we have a vertex $\varphi \in P'$ and consider possible values $f(\varphi)$. Let H be the stabilizer of φ, then $f(\varphi)$ must be chosen in V^H and $f(\varphi) \in V^H$ is the only constraint needed to extend f to the orbit $G\varphi$ equivariantly. For any face of P', given by a chain of vertices of P'

$$\varphi_1 < \varphi_2 < \cdots < \varphi_k < \varphi$$

of faces of P, we assume by induction that generically $f(\varphi_1), \dots, f(\varphi_k)$ are affinely independent and form a $(k-1)$-dimensional simplex in V. The dimension assumption of the lemma means that $k \leq \dim \varphi \leq \dim V^H$ (speaking of dimension, we consider φ as a face of P), hence for a generic choice of $f(\varphi) \in V^H$ the points $f(\varphi_1), \dots, f(\varphi_k), f(\varphi)$ are affinely independent. This applies to all chains that end in τ and completes the induction step and the proof is complete. □

Proof of the "only if" part of Lemma 2.11. Consider any G-equivariant map $S \to S$ and compose it with the inclusion $S \subset W$ to obtain a G-equivariant map

$$f_1 : S \to W.$$

Let $f_0 : S \to W$ be the standard inclusion. Connect f_0 and f_1 by a G-equivariant homotopy

$$h : S \times [0,1] \to W,$$

which can be chosen as $h(x,t) = (1-t)f_0(x) + tf_1(x)$.

Note that the difference in the degrees of f_0 and f_1 as maps of S to itself equals the degree of h over $0 \in W$. This follows from the fact that the degree of a map between closed connected oriented manifolds with boundary $h : M \to N$ satisfying $h(\partial M) \subset \partial N$ is well defined and equals the degree of the restriction $h|_{\partial M} : \partial M \to \partial N$ if ∂N is connected. Here $M = S \times [0,1]$ and $N \subset W$ is the unit ball.

Now we would like to make h generic by applying Lemma 2.12. We can think of S as a polyhedron, by identifying it with the boundary of the convex hull of several orbits in S. Lemma 2.12 applies because for any subgroup $H \subseteq G$ we have

$$(S \times [0,1])^H = (S)^H \times [0,1].$$

So, we after a small perturbation of h, we may assume that $h^{-1}(0)$ is finite and is still linear in t. The degree of h over $0 \in W$ can now be counted geometrically as the sum of local degrees at the points in $h^{-1}(0)$.

Split $h^{-1}(0)$ into disjoint orbits and let (x_i, t) be a point in the ith orbit. Let $-d_i$ be the degree of h at (x_i, t). The degree at any other point (gx_i, t) for $g \in G$ is also $-d_i$, because g acts on the orientation of the domain and the range of h in the same way (i.e., g either changes the orientation both in the domain and the range of h, or preserves the orientation both in the domain and the range of h). So, the total degree corresponding to the ith orbit is $-d_i\frac{|G|}{|G_i|}$ where $\frac{|G|}{|G_i|}$ is the size of the orbit. It remains to prove that d_i satisfies (3).

Let $U \subset S$ be a neighborhood of x_i such that $G_i(U) = U$. We take U sufficiently small so that $U \times [0,1]$ contains no points of $h^{-1}(0)$ except for (x_i, t); this is possible because h is linear in $t \in [0,1]$ and so $|h^{-1}(0) \cap (x_i \times [0,1])| = 1$. Clearly, d_i equals the degree of the map

$$\varphi : \partial(U \times [0,1])) \xrightarrow{h} W \setminus 0 \xrightarrow{\text{pr}} S,$$

where $\text{pr} : W \setminus 0 \to S$ is the standard radial projection. The map φ is G_i-equivariant as a composition of two G_i-equivariant maps. The restriction of φ to $U \times 0$ is the identity.

There exists a G_i-equivariant homeomorphism $\psi : \partial(U \times [0,1])) \to S$ which is the identity on $U \times 0$. For example, one can construct ψ as follows. Let $\psi' : \partial(U \times [0,1])) \to W$ be the map which is the identity on $U \times 0$, maps every $y \times 1 \in U \times 1$ to $y - 2x_i$ (here we consider y and x_i as vectors in W), and is linear in $t \in [0,1]$ on $\partial U \times [0,1]$. Let ψ be the composition of ψ' with the projection $\text{pr} : W \setminus 0 \to S$.

So, $\varphi \circ \psi^{-1} : S \to S$ is a G_i-equivariant map of degree d_i and the identity on $U \times 0 \ni x_i$, hence d_i satisfies (3).

$\square$

Lemma 2.13. *Let $x \in S$ be a point stabilized by a subgroup $H \subset G$. Let the sphere S_x be the boundary of a small H-invariant neighborhood of x. Let $f_1, \dots, f_\ell : S_x \to S_x$ be H-equivariant maps with degrees $d_1, \dots, d_\ell$, respectively.*

Then for any choice of the numbers $\varepsilon_i \in \{0, 1\}$ there exists a H-equivariant map $S \to S$ which is the identity in a neighborhood of x and whose degree is

$$1 + \varepsilon_1(d_1 - 1) + \varepsilon_2(d_2 - d_1) + \cdots + \varepsilon_\ell(d_\ell - d_{\ell-1}) - \varepsilon_{\ell+1} d_\ell$$

Corollary 2.14. *Using the notation from the statement of Lemma 2.13, suppose there exists a H-equivariant map $S_x \to S_x$ of degree -1. Then for any d there exists a H-equivariant map $S \to S$ which is the identity in a neighborhood of x and whose degree is d.*

Proof. The identity map $S_x \to S_x$ has degree 1 and is H-equivariant. So, we can use both $+1$ and -1 for d_i applying Lemma 2.13.

Suppose we were able to achieve some degree applying Lemma 2.13 using some values for $d_1, \dots, d_\ell$ and $\varepsilon_1, \dots, \varepsilon_\ell$. It's sufficient to prove that we can change the achieved degree by 1 and by -1 by incrementing ℓ and making a correct choice of $d_{\ell+1}$ and $\varepsilon_{\ell+2}$.

When we increase ℓ by 1 the degree changes by $w := \varepsilon_{\ell+1} d_{\ell+1} - \varepsilon_{\ell+2} d_{\ell+1} = (\varepsilon_{\ell+1} - \varepsilon_{\ell+2}) d_{\ell+1}$. For any value of $\varepsilon_{\ell+1} \in \{0, 1\}$, we can choose $\varepsilon_{\ell+2}$ so that $|\varepsilon_{\ell+1} - \varepsilon_{\ell+2}| = 1$. Then choosing $d_{\ell+1}$ to be either 1 or -1, we can get $w = 1$ and $w = -1$. □

Corollary 2.15. *Using the notation from the statement of Lemma 2.13, suppose there exists a H-equivariant map $S_x \to S_x$ of degree d. Then there exists a H-equivariant map $\Sigma_n \to \Sigma_n$ which is the identity in a neighborhood of x and whose degree is also d.*

Proof. In the statement of Lemma 2.13, put $\ell = 1$, $d_1 = d$, $\varepsilon_1 = 1$, $\varepsilon_2 = 0$. The corollary follows. □

Proof of Lemma 2.13. Draw the diameter containing x. Draw $\ell + 1$ different hyperplanes orthogonal to the diameter and intersecting its interior. The hyperplanes cut S into two spherical caps U_1 and U_2 which are H-equivariantly homeomorphic to a cone over S_x, where U_1 contains x and U_2 contains the point opposite to x; and ℓ cylinders C_i, each H-equivariantly homeomorphic to $S_x \times [0, 1]$. For each i, let the end $S_x \times 1$ of C_i be those end which is further away from x.

Let us construct a required map $f : S \to S$. Define the restriction of f to U_1 to be the identity. Define the restriction of f to the end $S_x \times 1$ of the cylinder C_i to be f_i.

Extend f to every cylinder C_i from its boundary by some map going to either U_1 or its complement $S \setminus U_1$ according to the value $\varepsilon_i = 1$ or $\varepsilon_i = 0$, respectively. The spherical caps U_1 and $S \setminus U_1$ are H-equivariantly contractible, hence such an extension is always possible and can be made H-equivariantly.

Likewise, extend f to U_2 from its boundary by some map going to either U_1 or its complement $S \setminus U_1$ according to the value $\varepsilon_{\ell+1} = 1$ or $\varepsilon_{\ell+1} = 0$, respectively.

Clearly, f is H-equivariant and is the identity on $U_1 \ni x$.

Let us compute the degree of f over x. The degree of $f|_{U_1}$ is 1. The degree of $f|_{C_i}$ is the difference $d_i - d_{i-1}$ (where $d_0 = 1$) of degrees with which the boundary components of the cylinder are mapped to ∂U_1 in case of $\varepsilon_i = 1$ and 0 in case of $\varepsilon_i = 0$. Likewise, the degree of $f|_{U_2}$ is $0 - d_\ell = -d_\ell$ in case of $\varepsilon_{\ell+1} = 1$ and 0 in case of $\varepsilon_{\ell+1} = 0$. So, the total degree of f over x is as required. $\square$

The last lemma we need is used only in the proof of part (d) of Theorem 2.8.

Lemma 2.16. *Let n be a positive integer which is not a prime power and not a twice prime power. Then there exist integers $d_1, d_2, \ldots, d_{n-1}$ such that*

- $1 - \sum_{k=1}^{n-1} d_k \binom{n}{k} = 0$,
- $d_{q^\alpha} = 0$ *or* $d_{q^\alpha} \equiv 1 \pmod q$ *for any prime q and $\alpha > 0$,*
- $d_1 \equiv 1 \pmod p$ *if $n = p^t + 1$ for some prime p.*

Proof. Consider all distinct representations of n as a sum of two powers of the same prime, $n = p_1^{s_1} + p_1^{t_1} = p_2^{s_2} + p_2^{t_2} = \ldots = p_\ell^{s_\ell} + p_\ell^{t_\ell}$, $0 \le s_i < t_i$ for each $i = 1, \ldots, \ell$. Note, that it is possible that $s_i = 0$ for some i. Clearly, $p_i \neq p_j$ for all $1 \le i < j \le \ell$.

Put

- $d_k = 0$ if $k = p_i^{t_i}$ for some i,
- $d_k = 1 + p_i b_k$ if $k = p_i^{s_i}$ for some i,
- $d_k = 1 + q b_k$ if $k \neq p_i^{s_i}$ and $k \neq p_i^{t_i}$ for all i and $k = q^\alpha$, $\alpha > 0$ for some prime q,

- $d_k = b_k$ otherwise,

where integers b_k will be chosen later. It is easy to see that the last two conditions on d_k in the statement of the lemma are satisfied by this assignment.

Define the number

$$N = 1 - \sum_{k=p_i^{s_i}} \binom{n}{k} - \sum_{k \neq p_i^{s_i},\, k \neq p_i^{t_i},\, k=q^\alpha,\, \alpha>0} \binom{n}{k}.$$

Here the summation is over k satisfying the second or the third case above. Define numbers c_k as follows:

- $c_k = 0$ if $k = p_i^{t_i}$ for some i,
- $c_k = p_i\binom{n}{k}$ if $k = p_i^{s_i}$ for some i,
- $c_k = q\binom{n}{k}$ if $k \neq p_i^{s_i}$ and $k \neq p_i^{t_i}$ for all i and $k = q^\alpha$, $\alpha > 0$ for some prime q,
- $c_k = \binom{n}{k}$ otherwise.

Plugging in these definitions we get

$$1 - \sum_{k=1}^{n-1} d_k \binom{n}{k} = N - \sum_{k=1}^{n-1} b_k c_k.$$

It remains to prove that we can choose b_k so that the right-hand expression becomes 0. This will follow if we prove that $\mathrm{GCD}(c_1, \ldots, c_\ell)$ divides N. To do that we first prove that N is divisible by $p_1 p_2 \ldots p_\ell$ and then prove that $\mathrm{GCD}(c_1, \ldots, c_\ell)$ divides $p_1 p_2 \ldots p_\ell$.

By Lucas's theorem, for every p_i and every $1 \leq k \leq n-1$ the binomial coefficient $\binom{n}{k}$ is divisible by p, unless $k = p_i^{s_i}$ or $k = p_i^{t_i}$, in which case $\binom{n}{k}$ is equal 1 modulo p_i. In the definition of N above, for each i there is a single summand $\binom{n}{p_i^{s_i}}$ and no summands $\binom{n}{p_i^{t_i}}$. Hence, N is divisible by p_i for every i.

Let us prove that $\mathrm{GCD}(c_1, \ldots, c_\ell)$ divides $p_1 p_2 \ldots p_l$. Fix i. If $k = p^{s_i}$, then $c_k = p_i\binom{n}{p^{s_i}}$ is not divisible by p_i^2 because $\binom{n}{p^{s_i}}$ is not divisible by p_i. Hence, $\mathrm{GCD}(c_1, \ldots, c_\ell)$ is not divisible by p_i^2 for every i.

It remains to prove that $\mathrm{GCD}(c_1, \ldots, c_\ell)$ is not divisible by any prime q which is not equal to any of p_i. To do so we find c_k which is not divisible by q.

Suppose that $q > n$. Then $\binom{n}{k}$ is not divisible by q for all k and hence all the non-zero c_k are also not divisible by q.

Suppose now that $q < n$. Write the base q expansion of n and decrease the leftmost digit by 1, denoting the obtained number by k. Since $n > q$, the expansion had more than 1 digit and so $n - k$ is divisible by q. On the other hand, $n - p_i^{t_i} = p_i^{s_i}$ is not divisible by q, meaning that $k \neq p_i^{t_i}$ for all i.

Also, k is not a positive power of q, though it's possible that $k = 1$. Indeed, assume the contrary. Then, by the definition of k, either $n = 2k$, which is impossible because n is not a twice prime power; or n is the sum of k and a larger positive power of q, which is impossible because q is different from all p_i.

So, $k \neq p_i^{t_i}$ for all i and k is not a positive power of q. Hence, either $c_k = \binom{n}{k}$ or $c_k = q'\binom{n}{k}$ for some prime $q' \neq q$. Both numbers are not divisible by q by Lucas's theorem. We have established that $\mathrm{GCD}(c_1, \dots, c_\ell)$ divides $p_1 p_2 \dots p_l$.

$\square$

2.3 Proof of Theorem 2.8

In this section W is the orthogonal complement to $\delta_n \subset \mathbb{R}^n$, and S is the unit sphere in W. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permuting the coordinates. This action induces an action on W and S. This way $\mathfrak{S}_n$ can be considered as a subgroup of the orthogonal group of W.

There is a $\mathfrak{S}_n$-equivariant homeomorphism between S and the boundary $\partial\Delta^{n-1}$ of the standard simplex, where $\mathfrak{S}_n$ acts on $\partial\Delta^{n-1}$ by permuting the barycentric coordinates. We use this homeomorphism to identify S with $\partial\Delta^{n-1}$. This way we can talk of barycentric coordinates of points in S and of vertices and simplices of S.

Proof of the "only" part of Theorem 2.8(a) and of Theorem 2.8(b). Suppose that $n = p^k$ for some prime p. Consider any point of S and split its barycentric coordinates into blocks of equal coordinates. Suppose the sizes of the blocks are $\alpha_1, \dots, \alpha_\ell$. Then the orbit of the point under $\mathfrak{S}_n$ has size

$$\frac{n!}{\alpha_1! \cdot \ldots \cdot \alpha_\ell!} = \binom{n}{\alpha_1, \dots, \alpha_\ell}.$$

The multinomial coefficient above is a product of binomial coefficients

$$\binom{n}{\alpha_1,\dots,\alpha_\ell} = \binom{n}{\alpha_1}\cdot\binom{n-\alpha_1}{\alpha_2}\cdot,\dots,\cdot\binom{n-\alpha_1-\cdots-\alpha_{\ell-1}}{\alpha_\ell}.$$

Hence, it is divisible by p, as the first factor is divisible by p by Lucas's theorem, [58]. So, the size of every orbit is divisible by p. Hence, by the "only if" part of Lemma 2.11, the degree of any $\mathfrak{S}_n$-equivariant map $S \to S$ is 1 modulo p. This finishes the proof of the "only" part of Theorem 2.8(a).

Suppose now that $n = 2p^k$ for some prime $p \neq 2$. Then there is only one $\mathfrak{S}_n$ orbit whose size is not divisible by p. More precisely, it is the orbit of the center x of the subsimplex of S on the first p^k vertices. Indeed, considering the product of binomial coefficients above and applying Lucas's theorem we see that the first factor is not divisible by p only if $\alpha_1 = p^k$ (note, that $\alpha_1 = 2p^k$ is impossible). Then the second factor is not divisible by p only if $\alpha_2 = \alpha_1 = p^k$. The stabilizer of x is $\mathfrak{S}_{p^k} \times \mathfrak{S}_{p^k} =: G$. The orbit of x has size $\frac{|\mathfrak{S}_n|}{|G|} = \binom{2p^k}{p^k}$ which by Lucas's theorem equals 2 modulo p.

So, by the "only if" part of Lemma 2.11, the degree of any $\mathfrak{S}_n$-equivariant map $S \to S$ is equal modulo p to $1 - \deg(f)\frac{|\mathfrak{S}_n|}{|G|} \equiv 1 - 2\cdot\deg(f) \pmod p$, where $f : S \to S$ is some G-equivariant map which is the identity in a neighborhood of x. It remains to prove that $\deg(f)$ is either 0 or 1 modulo p.

Let x' be the center of the subsimplex of S on the last p^k vertices. Points x and x' are opposite to each other and are the only points of S fixed by G. The size of the G-orbit of any other point of S is divisible by p. Indeed, consider any point of S different from x and x'. As was said above, the size of its $\mathfrak{S}_n$ orbit is divisible by p. Its G orbit is smaller by a factor which divides $\frac{|\mathfrak{S}_n|}{|G|}$. And $\frac{|\mathfrak{S}_n|}{|G|}$ is not divisible by p.

Consider the G-equivariant homotopy $h : S \times [0,1] \to W$ such that

- $h|_{S\times 0} = f$,
- $h(S \times 1) = f(x')$,
- h is linear in $t \in [0,1]$, i.e., $h(x,t) = (1-t)h(x,0) + th(x,1)$.

The degree of the constant map $h|_{S\times 1}$, considered as a map to S, is zero. So, the degree of f is equal to the degree of h over $0 \in W$. Since f is the identity in small neighborhood of U of x, then $(h|_{U\times[0,1]})^{-1}(0)$ is finite. By the same argument as in the proof of the "only if" part of Lemma 2.11, we may assume, after a small G-equivariant perturbation of h outside of $U \times [0,1]$, that $h^{-1}(0)$ is finite and the degree can be counted geometrically as the sum of local degrees at the points in $h^{-1}(0)$.

By the definition, $h(x',t) = f(x') \neq 0$ for every $t \in [0,1]$. So, the point (x',t) is not in $h^{-1}(0)$ for any t.

Since h is linear in t on $U \times [0,1]$ (recall, that we didn't perturb h on $U \times [0,1]$), there is at most one t such that $(x,t) \in h^{-1}(0)$. For such t, the local degree of h at (x,t) over $0 \in W_n$ is 1 since f is the identity on $U \ni x$.

For any other $y \in S$, $y \neq x, x'$ the size of the G-orbit of $(y,t), t \in [0,1]$ is divisible by p. So, the degree of h over $0 \in W_n$, and hence the degree of f, is either 1 or 0 modulo p, depending on whether (x,t) is in $h^{-1}(0)$ for some $t \in [0,1]$ or not. This finishes the proof of Theorem 2.8(b). $\square$

Proof of Theorem 2.8(a,c,d). For every $k = 1, \dots, n-1$ pick the center c_k of some $(k-1)$-dimensional face of S. The $\mathfrak{S}_n$-orbit of c_k contains $\binom{n}{k}$ points and the stabilizer of c_k in the permutation group is the subgroup $G_k := \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n$. A small G_k invariant neighborhood of c_k is bounded by the sphere $S_k := \Sigma_k * \Sigma_{n-k}$, where Σ_k is the boundary of the standard simplex with k vertices.

Parts (a) and (c). In (a) and (c) we have that n is odd. Since n is odd, one of the numbers k and $n-k$ is even. The join of the antipodal map of the even dimensional factor and the identity map of the odd dimensional factor gives a G_k-equivariant map $S_k \to S_k$ of degree -1. By Corollary 2.14, for any integer d_k there exists a G_k-equivariant map $S \to S$ which is the identity in a neighborhood of c_k and whose degree is d_k. By Lemma 2.11, there exists a $\mathfrak{S}_n$-equivariant map $S \to S$ of degree

$$d = 1 - \sum_{k=1}^{n} d_k \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_k \times \mathfrak{S}_{n-k}|} = 1 - \sum_{k=1}^{n} d_k \binom{n}{k}.$$

If n is not a prime power, by Lucas's theorem the GCD of the binomial coefficients in question is 1. So, after an appropriate choice of d_k, the resulting degree d can attain any integer value. This proves part (c) of the theorem.

Likewise, if n is a prime power with the base p, by Lucas's theorem the GCD of the binomial coefficients in question is p. So, after an appropriate choice of d_k, the resulting degree d can attain any integer value which is 1 modulo p. This finishes the proof of part (a) of the theorem, the "only" part of (a) was proved earlier.

Part (d). Let d_k be some numbers whose existence is guaranteed by Lemma 2.16. By Lemma 2.11, it is sufficient to prove that for each k such that $d_k \neq 0$ there is a G_k-equivariant map $f_k : S \to S$ of degree d_k which is the identity in a neighborhood of c_k. By Corollaries 2.14 and 2.15, this means that it is sufficient to find a G_k-equivariant map $S_k \to S_k$ of degree -1 or d_k.

Finally, it is sufficient to find a $\mathfrak{S}_k$-equivariant map $\Sigma_k \to \Sigma_k$ or a $\mathfrak{S}_{n-k}$-equivariant map $\Sigma_{n-k} \to \Sigma_{n-k}$ of degree -1 or d_k. Indeed, using the join operation with the identity map $\Sigma_{n-k} \to \Sigma_{n-k}$ or $\Sigma_k \to \Sigma_k$, respectively, we can get a required map $S_k \to S_k$ of degree -1 or d_k.

Consider now all the possibilities for k.

k is even: As noted above, then there is a $\mathfrak{S}_k$-equivariant map $\Sigma_k \to \Sigma_k$ of degree -1.

$k > 1$ is odd and not a prime power: Then there is a $\mathfrak{S}_k$-equivariant map $\Sigma_k \to \Sigma_k$ of any degree, including -1, by part (c) of the theorem.

$k > 1$ is odd and is a prime power with the base p: Then by the definition either $d_k = 0$ and there is nothing to prove; or $d_k \equiv 1 \pmod p$. In the latter case, by part (a) of the theorem, there is a $\mathfrak{S}_k$-equivariant map $\Sigma_k \to \Sigma_k$ of degree d_k.

$k = 1$ and $n = p^t + 1$ for some prime p: Then $d_k \equiv 1 \pmod p$ by the definition. So, by part (a) of the theorem, there is a $\mathfrak{S}_{n-k}$-equivariant map $\Sigma_{n-k} \to \Sigma_{n-k}$ of degree d_k.

$k = 1$ and $n \neq p^t + 1$ for any prime p: Then $n - k$ is odd and not a prime power. So, by part (c) of the theorem, there is a $\mathfrak{S}_{n-k}$-equivariant map $\Sigma_{n-k} \to \Sigma_{n-k}$ of any degree including -1.

□

2.4 Proof of Theorem 2.9

In this section W is the orthogonal complement to $\delta_n \subset (\mathbb{R}^2)^n$, and S is the unit sphere in W. The symmetric group $\mathfrak{S}_n$ acts on $(\mathbb{R}^2)^n$ by permuting the coordinates. This action induces an action on W and S. This way $\mathfrak{S}_n$ can be considered as a subgroup of the orthogonal group of W.

For any k, let Σ_k be the boundary of the standard simplex with k vertices. There is a $\mathfrak{S}_n$-equivariant homeomorphism between S and the join $\Sigma_n * \Sigma_n$; the group $\mathfrak{S}_n$ acts on Σ_n by permuting the barycentric coordinates and acts on $\Sigma_n * \Sigma_n$ diagonally. We use this homeomorphism to identify S with $\Sigma_n * \Sigma_n$. This way we can talk of barycentric coordinates of points in S and of vertices and simplices of S.

Proof of Theorem 2.9. For every $k = 1, \dots, n-1$ pick the center c'_k of some $(k-1)$-dimensional face of Σ_n. Denote $c_k := (\frac{c'_k}{2}, \frac{c'_k}{2}) \in \Sigma_n * \Sigma_n$. The $\mathfrak{S}_n$-orbit of c_k contains $\binom{n}{k}$ points and the stabilizer of c_k in the permutation group is the subgroup $G_k := \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n$. Let sphere S_k be the boundary of a small G_k invariant neighborhood of c_k.

The case of n odd of the theorem is already covered by Theorem 2.8(c). So, we may assume that n is even.

Let us prove that there is a G_k-equivariant map $S_k \to S_k$ of degree -1. Without the loss of generality we may assume that the last barycentric coordinate of c'_k is 0. Consider the map $\sigma : \Sigma_n * \Sigma_n \to \Sigma_n * \Sigma_n$ which swaps the last coordinate in the first factor with the last coordinate in the second factor. This map is G_k-equivariant. Since n is even, the map σ reverses the orientation of $\Sigma_n * \Sigma_n$. On the other hand, $\sigma(c_k) = c_k$. So, σ restricted to S_k is the requiredG_k-equivariant map of degree -1.

By Corollary 2.14, for any integer d_k there exists a G_k-equivariant map $S \to S$ which is the identity in a neighborhood of c_k and whose degree is d_k. By Lemma 2.11, there exists a $\mathfrak{S}_n$-equivariant map $S \to S$ of degree

$$d = 1 - \sum_{k=1}^{n} d_k \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_k \times \mathfrak{S}_{n-k}|} = 1 - \sum_{k=1}^{n} d_k \binom{n}{k}.$$

Since n is not a prime power, by Lucas's theorem the GCD of the binomial coefficients in question is 1. So, after an appropriate choice of d_k, the resulting degree d can attain any integer value, including zero. $\square$

3 Eliminating higher-multiplicity intersections

3.1 Main results

3.1.1 The topological Tverberg conjecture and almost r-embeddings

Throughout this chapter, let r and d be positive integers, and let K be a finite simplicial complex; later we omit 'finite simplicial'. A map $f\colon K \to \mathbb{R}^d$ is an **almost r-embedding** if $f\sigma_1 \cap \ldots \cap f\sigma_r = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint simplices of K. (We stress that this definition depends on the complex, i.e., a specified triangulation of the underlying polyhedron.)

The well-known *topological Tverberg conjecture*, raised by Bajmoczy and Bárány [9] and Tverberg [40, Problem 84] asserts that the $(d+1)(r-1)$-dimensional simplex does not admit an almost r-embedding in $\mathbb{R}^d$. This was proved in the case where r is a prime [9; 14] or a prime power [75; 107], but the case of arbitrary r remained open and was considered a central unsolved problem of topological combinatorics.

Recently and somewhat unexpectedly, it turned out that for r not a prime power and $d \geq 3r$ there are counterexamples to the topological Tverberg conjecture. The construction of these counterexamples follows an approach proposed in [59], which is based on

- a general algebraic criterion for the existence of almost r-embeddings in *codimension* at least 3 [59; 60] (the *deleted product criterion*, cf. Theorem 3.5 and Proposition 3.7 below), and

- a result of Özaydin [75] that guarantees that the hypothesis of this criterion is satisfied whenever r is not a prime power (see [98, §3.3 Proof of the Özaydin Theorem 3.5: localization modulo a prime] for a suitable reformulation and simplified exposition of Özaydin's theorem).

There seemed to be a serious obstacle to completing this approach: maps from the $(d+1)(r-1)$-dimensional simplex to $\mathbb{R}^d$ do not satisfy the codimension 3 restriction. (In a sense, the problem is rather a codimension zero problem.) Frick [32] was the first to realize that this obstacle can be overcome by a beautiful combinatorial trick (Constraint Lemma 3.4) discovered by Gromov [38] and independently by Blagojević–Frick–Ziegler [20], and that thus the results of [75], [38; 20] and [60] combined yield counterexamples to the topological Tverberg conjecture for $d \geq 3r+1$ whenever r is not a prime power, cf. [32; 21]. Using a more involved method (‘*prismatic maps*’) to overcome the obstacle, the dimension for the counterexamples was lowered to $d \geq 3r$ in [60]. The topological Tverberg conjecture is still open for low dimensions $d < 12$, in particular, for $d = 2$.

For more detailed accounts of the history of the counterexamples, see the surveys [13], [23, §1 and beginning of §5], [98], [114, §21.4.5], [15] and the references therein.

Here, we improve this and show that counterexamples exist for $d \geq 2r + 1$ (see also Remark 3.21.a):

Theorem 3.1. *There is an almost* 6*-embedding of the 70-dimensional simplex in* $\mathbb{R}^{13}$.

More generally, if r *is not a prime power and* $d \geq 2r + 1$*, then there is an almost* r*-embedding of the* $(d+1)(r-1)$*-dimensional simplex in* $\mathbb{R}^d$.

Any sufficiently small perturbation of an almost r-embedding is again an almost r-embedding. So the existence of a *continuous* almost r-embedding is equivalent to the existence of a *piecewise linear (PL)* almost r-embedding.

A result closely related to the topological Tverberg conjecture is the following theorem, which generalizes a classical theorem (the case $r = 2$) of Van Kampen and Flores [105], see also Lemma 3.19 below.

Theorem 3.2 (r-fold van Kampen–Flores Theorem; [82], [108, Corollary in §1])**.** *If* r *is a prime power and* $k \geq 1$*, then there is no almost* r*-embedding of the* $k(r-1)$*-skeleton of the* $(kr+2)(r-1)$*-dimensional simplex in* $\mathbb{R}^{kr}$.

The first ingredient for the proof of Theorem 3.1 is Part (a) of the following result, which shows that Theorem 3.2 fails in a strong sense whenever r is not a prime power.

Theorem 3.3. (a) *If $k \geq 2$ and r is not a prime power, then every $k(r-1)$-dimensional complex admits an almost r-embedding in $\mathbb{R}^{kr}$.*

(b) *For every fixed $k, r \geq 2$, $k + r \geq 5$, almost r-embeddability of $k(r-1)$-dimensional complexes in $\mathbb{R}^{kr}$ is decidable in polynomial time.*

For $k \geq 3$, Theorem 3.3 is a consequence of [75] and [60]; for $k = 2$, it is a result of this chapter. Theorem 3.3 is deduced from Theorem 3.5 below in §3.1.2.

The second ingredient for the proof of Theorem 3.1 is the following lemma, which was proved in [38, 2.9.c], [20, Lemma 4.1.iii and 4.2] (see also [32, proof of Theorem 4], [21, proof of Theorem 3.2] and the surveys [98, Constraint Lemma 3.2], [23, §4, §5]).

Lemma 3.4 (Constraint)**.** *If k, r are integers and there is an almost r-embedding of the $k(r-1)$-skeleton of the $(kr+2)(r-1)$-dimensional simplex in $\mathbb{R}^{kr}$, then there is an almost r-embedding of the $(kr+2)(r-1)$-dimensional simplex in $\mathbb{R}^{kr+1}$.*

Before we proceed, we first show how to derive counterexamples to the topological Tverberg conjecture from these results:

Proof of Theorem 3.1. It is well-known that the general case $d \geq 2r + 1$ follows from the 'boundary' case $d = 2r + 1$ [27, Proposition 2.5], [98, Lemma 3.1]. To prove the boundary case, suppose r is not a prime power and let $k = 2$. By Theorem 3.3 (a), there is an almost r-embedding of the $2(r-1)$-skeleton of the $(2r+2)(r-1)$-dimensional simplex in $\mathbb{R}^{kr}$. Thus, by Lemma 3.4, there exists an almost r-embedding of the $(2r+2)(r-1)$-simplex in $\mathbb{R}^{kr+1}$. $\square$

The proof of Theorem 3.3 is based on Theorem 3.5 below, which is an extension of a general algebraic criterion for the existence of almost r-embeddings in *codimension* ≥ 3 [59; 60] to *codimension* 2.

Assume that $\dim K = k(r-1)$ for some $k \geq 1$, $r \geq 2$, and that $f\colon K \to \mathbb{R}^{kr}$ is a PL map in general position. Then preimages $y_1, \ldots, y_r \in K$ of any r-fold point $y \in \mathbb{R}^{kr}$ (i.e., of a point having r preimages) lie in the interiors of $k(r-1)$-dimensional simplices of K. Choose arbitrarily an orientation for each of the $k(r-1)$-simplices. By general position, f

is affine on a neighborhood U_j of y_j for each $j = 1, \dots, r$. Take a positive basis of k vectors in the oriented normal space to oriented fU_j. **The r-fold intersection sign** of y is the sign ± 1 of the basis in $\mathbb{R}^{kr}$ formed by r such k-bases.[1] The *algebraic r-fold intersection number* $f(\sigma_1) \cdot \dots \cdot f(\sigma_r) \in \mathbb{Z}$ is defined as the sum of the r-fold intersection signs of all r-fold points $y \in f\sigma_1 \cap \dots \cap f\sigma_r$. We call a PL map f in general position a **$\mathbb{Z}$-almost r-embedding** if $f\sigma_1 \cdot \dots \cdot f\sigma_r = 0$ whenever $\sigma_1, \dots, \sigma_r$ are pairwise disjoint simplices of K. The sign of the algebraic r-fold intersection number depends on an arbitrary choice of orientations for each σ_i, but the condition $f\sigma_1 \cdot \dots \cdot f\sigma_r = 0$ does not.

Theorem 3.5. *If $k \geq 2$, $k + r \geq 5$ and a $k(r-1)$-dimensional complex is $\mathbb{Z}$-almost r-embeddable in $\mathbb{R}^{kr}$, then it is almost r-embeddable in $\mathbb{R}^{kr}$.*

The case $r = 2$ is a classical result of van Kampen, Shapiro and Wu [97, Lemma 4.2]. For $k \geq 3$ Theorem 3.5 is the main result of [60]. In the present chapter, we generalize this to $k \geq 2$.

The proof of Theorem 3.5 for $k \geq 3$ in [60] is based on a higher multiplicity generalization [60, Theorem 17] of the classical Whitney trick [111] (see, e.g., [81, Whitney Lemma 5.12] for a proof of the Whitney trick in the piecewise-linear setting). Our proof of Theorem 3.5 for $k \geq 2$ is based on a further generalization of the higher-multiplicity Whitney trick that works for $k \geq 2$, namely, the Local and Global Disjunction Theorems 3.9 and 3.11 that we will formulate in the next subsection (§3.1.2). Some readers may consider the resulting proof for $k \geq 2$ simpler than the proof for $k \geq 3$ in [60]. See also Remarks 3.12 and 3.17 below for further discussion of the proof ideas and related work.

The analogue of Theorem 3.5 for $r = 2$ and $k = 1$ is a classical result of graph theory (the Hanani-Tutte Theorem [26; 104]). This analogue holds in a stronger form: a mod2-analogue of $\mathbb{Z}$-almost 2-embeddability in $\mathbb{R}^2$ implies planarity. For $r = 2$ and $k \geq 3$ see Remark 3.21.b.

The following Theorem 3.6 shows that the analogue of Theorem 3.5 fails for $k = r = 2$. Freedman, Krushkal, and Teichner [31] proved that there is a 2-dimensional complex that admits a $\mathbb{Z}$-almost 2-embedding in $\mathbb{R}^4$, but not an embedding in $\mathbb{R}^4$. (This implies that the Van Kampen obstruction to embeddability, whose definition we recall before Proposition 3.7 below, is incomplete for 2-dimensional complexes in $\mathbb{R}^4$.) Here, we strengthen their

[1] This is classical for $r = 2$ [24] and is analogous for $r \geq 3$, cf. [60, § 2.2].

result and show that their 2-dimensional complex does not even admit an almost 2-embedding in $\mathbb{R}^4$.

Theorem 3.6. *There exists a* 2*-dimensional complex that admits a* $\mathbb{Z}$*-almost* 2*-embedding in* $\mathbb{R}^4$ *but does not admit an almost* 2*-embedding in* $\mathbb{R}^4$*.*

Theorem 3.6 is deduced from the Singular Borromean Rings Lemma 3.18 below in §3.2.2. This deduction is essentially known [31], [97, §7].

To conclude this subsection, we state a reformulation of $\mathbb{Z}$-almost r-embeddability in $\mathbb{R}^{kr}$, which allows one to deduce Theorem 3.3 from Theorem 3.5.

Let *the simplicial* r*-fold deleted product* $K^{\times r}_{\Delta}$ of K be

$$K^{\times r}_{\Delta} := \bigcup\{\sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \text{ a simplex of } K, \sigma_i \cap \sigma_j = \emptyset \text{ for all } i \neq j\},$$

on which the symmetric group $\mathfrak{S}_r$ acts by permuting the factors.

Recall that d and r denote positive integers (see the first line of §1.1). The group $\mathfrak{S}_r$ acts on the set of real $d \times r$-matrices by permuting the columns. Denote by $S^{d(r-1)-1}_{\mathfrak{S}_r}$ the subset of such matrices for which every row sums up to zero and the sum of squares of the matrix elements is equal to 1. This set is homeomorphic to the sphere of dimension $d(r-1)-1$, and it is invariant under the action of $\mathfrak{S}_r$. In what follows, we will use this in the special case $d = kr$.

For any general position PL map $f\colon K \to \mathbb{R}^{kr}$, the *generalized van Kampen obstruction* is represented by the *intersection cocycle* that assigns to each $kr(r-1)$-cell $\sigma_1 \times \ldots \times \sigma_r$ of $K^{\times r}_{\Delta}$ the algebraic intersection number $f\sigma_1 \cdot \ldots \cdot f\sigma_r$. The obstruction is an element of the equivariant cohomology group $H^{kr(r-1)}_{\mathfrak{S}_r}(K^{\times r}_{\Delta}; \mathcal{Z})$, where $\mathcal{Z}$ denotes the integers with a suitable action of $\mathfrak{S}_r$ (equivalently, this cohomology group is isomorphic to the cohomology of the quotient space $K^{\times r}_{\Delta}/\mathfrak{S}_r$ with twisted integer coefficients); see [60, §4] for details. The obstruction is zero if and only if the intersection cocycle is null-cohomologous.

Proposition 3.7. *[60] Let* K *be a* $k(r-1)$*-dimensional complex. The following conditions are equivalent:*

(1) K *is* $\mathbb{Z}$*-almost* r*-embeddable in* $\mathbb{R}^{kr}$*.*

(2) *The generalized van Kampen obstruction to* $\mathbb{Z}$*-almost* r*-embeddability of* K *in* $\mathbb{R}^{kr}$ *is zero.*

(3) *There exists a* $\mathfrak{S}_r$*-equivariant map* $K^{\times r}_\Delta \to S^{kr(r-1)-1}_{\mathfrak{S}_r}$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. The implication (2) $\Rightarrow$ (1) is [60, Corollary 44]. The equivalence (2) $\Leftrightarrow$ (3) is proved using equivariant obstruction theory, see [60, Theorem 40], [98, Proposition 3.6]. $\square$

Proposition 3.8. *[75; 60] If* r *is not a prime power, then every* $k(r-1)$*-dimensional complex admits a* $\mathbb{Z}$*-almost* r*-embedding in* $\mathbb{R}^{kr}$.

Proof. Denote the complex by K. Since $\dim K = k(r-1)$ we have $\dim K^{\times r}_\Delta \leq kr(r-1)$. We recall the following theorem of Özaydin [75] [98, the Özaydin' Theorem 3.5] (also following from Theorem 2.3): *If* r *is not a prime power and* X *is a* $d(r-1)$*-dimensional complex with a free PL action of* $\mathfrak{S}_r$*, then there is a* $\mathfrak{S}_r$*-equivariant map* $X \to S^{d(r-1)-1}_{\mathfrak{S}_r}$. Now the proposition follows from Özaydin's theorem (applied for $d = kr$ and $X = K^{\times r}_\Delta$) and the implication (3) $\Rightarrow$ (1) of Proposition 3.7. $\square$

Proof of Theorem 3.3. Part (a) follows from Theorem 3.5 together with Proposition 3.8.

Part (b) follows because by Theorem 3.5 (together with its trivial converse) *for each* $k \geq 2$, $k + r \geq 5$, *almost* r*-embeddability of a* $k(r-1)$*-dimensional complex* K *in* $\mathbb{R}^{kr}$ *is equivalent to each property of Proposition 3.7.* Of these, Property (2) is decidable in polynomial time, see [60, p. 32, Proof of Corollary 9] (this is based on algorithms for solving system of linear equations over the integers [101]). $\square$

3.1.2 Ideas of the proof of Theorem 3.5: Disjunction Theorems

We first formulate the simpler Local Disjunction Theorem, which we consider interesting in itself and which illuminates in simple terms 'the core' of the proof of Theorem 3.5.

Let $B^d := [0,1]^d$ denote the standard PL ball and $S^{d-1} = \partial B^d$ the standard PL sphere. We need to speak about PL balls of different dimensions and we will use the word 'disk' for lower-dimensional objects and 'ball' for higher-dimensional ones in order to clarify the distinction (even though, formally, the disk D^d is the same as the ball B^d). We denote by ∂M, respectively $\mathring{M}$, the boundary, respectively the interior, of a manifold M. A map $f\colon M \to B^d$ from a manifold with boundary to a ball is called **proper**, if $f^{-1}S^{d-1} = \partial M$. In this chapter we work in the PL category, in particular, all disks, balls and maps are PL.

Denote by

$$D = D_1 \sqcup \ldots \sqcup D_r$$

the disjoint union of r disks of dimension $k(r-1)$.

Theorem 3.9 (Local Disjunction)**.** *If $k \geq 2$ and $f : D \to B^{kr}$ is a proper general position PL map such that $fD_1 \bullet \ldots \bullet fD_r = 0 \in \mathbb{Z}$, then there is a proper general position PL map $f' : D \to B^{kr}$ such that $f' = f$ on ∂D and $f'D_1 \cap \ldots \cap f'D_r = \emptyset$.*

The condition $fD_1 \bullet \ldots \bullet fD_r = 0$ can be called *algebraic triviality*, and the condition $fD_1 \cap \ldots \cap fD_r = \emptyset$ can be called *geometric triviality*.

The case $r = 2$ of Theorem 3.9 is known, see Remark 3.21.c. The case $k \geq 3$ is essentially proved in [60, Theorem 17] (in fact, the case $k \geq 3$ of Theorem 3.9 is the only part of quite technical [60, Theorem 17] required to prove Theorem 3.5 for $k \geq 3$). The case $r \geq 3$, $k = 2$ is a result of this chapter.

Theorem 3.9 for $r \geq 3$ follows from the Global Disjunction Theorem 3.11.(a)-(b) below.

The analogue of Theorem 3.9 for $k = 1$ clearly holds when $r = 2$ and fails for each $r \geq 3$:

Theorem 3.10. *For each $r \geq 3$ take $k = 1$ in the definition of D. Then there is a proper general position PL map $f : D \to B^r$ such that $fD_1 \bullet \ldots \bullet fD_r = 0$ but there is no proper general position PL map $f' : D \to B^r$ such that $f' = f$ on ∂D and $f'D_1 \cap \ldots \cap f'D_r = \emptyset$.*

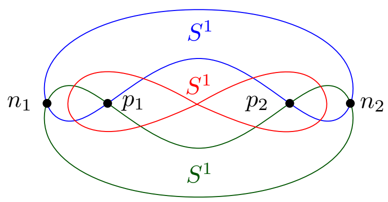

Figure 3.1: The boundary of an example corresponding to Theorem 3.10 for $r = 3$.

As an example corresponding to Theorem 3.10 one can take an extension of the map $f|_{\partial D}$ constructed in the proof of Lemma 3.20 below. For $r = 3$ see Figure 3.1; this construction might be known. For $r = 3$ Theorem 3.10 could be reproved using Figure 3.1 and [70].

The Local Disjunction Theorem 3.9 can be *globalized*, i.e. generalized to other connected orientable manifolds instead of disks and balls, including closed manifolds in $\mathbb{R}^d$

rather than proper manifolds in B^d. For $k \geq 3$ see [60, Theorem 17], for $k = r = 2$ see [84] and references therein. Let us state a *polyhedral* global version required to prove Theorem 3.5. (For *ornamental* global versions see §3.1.3.)

We call a point $y \in \mathbb{R}^d$ a **global r-fold point** of a map $f\colon K \to \mathbb{R}^d$ if y has r preimages lying in pairwise disjoint simplices of K, i.e., $y \in f\sigma_1 \cap \ldots \cap f\sigma_r$ and $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$.

(Thus, f is an almost r-embedding if and only if it has no global r-fold points.)

Assertion ($D_{k,r}$). *Let*

- K *be a* $k(r-1)$*-dimensional complex,*
- $f\colon K \to B^{kr}$ *a general position PL map,*
- $\sigma_1, \ldots, \sigma_r$ *pairwise disjoint simplices of* K*,*
- $x, y \in f\sigma_1 \cap \ldots \cap f\sigma_r \subset \mathring{B}^{kr}$ *two global* r*-fold points of opposite* r*-fold intersections signs.*

Then there is a general position PL map $f'\colon K \to B^{kr}$ *such that* $f = f'$ *on* $K - (\mathring{\sigma}_1 \sqcup \cdots \sqcup \mathring{\sigma}_r)$*, and the set of global* r*-fold points of* f' *(with signs) is equal to the set of global* r*-fold points of* f *(with signs) minus* $\{x, y\}$.

This can be informally described as 'cancelation of a pair of global r-fold points of opposite sign'. The Local Disjunction Theorem 3.9 is such a cancelation *in a restricted local situation.* So these are partial analogues of the Whitney trick, but we prefer a self-descriptive name.

Assertion $(D_{1,2})$ is a version of 'redrawing of a graph in the plane' [83, §4]. It would be interesting to know if it is true.

Theorem 3.5 ($\mathbb{Z}$-almost r-embeddability implies almost r-embeddability), as well as Theorem 3.13 below (classification of ornaments) follow from the following Global Disjunction Theorems 3.11.(a)-(b).

Theorem 3.11 (Global Disjunction). (a) *[60] Assertion* $(D_{k,r})$ *is true for each* $k \geq 3$ *and* $r \geq 2$.

(b) *Assertion* $(D_{2,r})$ *is true for each* $r \geq 3$.

(c) *Assertion* $(D_{2,2})$ *is false.*

(d) *Assertion* $(D_{1,r})$ *is false for each* $r \geq 3$.

Proof of Theorem 3.5 assuming the Global Disjunction Theorems 3.11.(a)-(b). Let $f\colon K \to \mathbb{R}^{kr}$ be a $\mathbb{Z}$-almost r-embedding. Take pairwise disjoint simplices $\sigma_1, \dots, \sigma_r$ of K with $f\sigma_1 \cap \dots \cap f\sigma_r \neq \emptyset$. Since f is a $\mathbb{Z}$-almost r-embedding, $f\sigma_1 \cap \dots \cap f\sigma_r$ consists of pairs of global r-fold points of opposite sign. By assertion $(D_{k,r})$, we eliminate these pairs one by one, without introducing any new global r-fold points in the process. By repeating this for every r-tuple of pairwise disjoint simplices, we obtain an almost r-embedding $K \to \mathbb{R}^{kr}$. □

The Global Disjunction Theorems 3.11.(a)-(b) are proved in §3.2.1. The Global Disjunction Theorem 3.11.c follows because assertion $(D_{2,2})$ implies the negation of Theorem 3.6 analogously to the above proof. The Global Disjunction Theorem 3.11.d follows because assertion $(D_{1,r})$ implies the negation of Theorem 3.10 analogously to the above proof.

Remark 3.12. It is well-known that the Whitney trick works in codimension ≥ 3 and fails in codimension 2 [52] without an assumption of simple connectivity [81, Whitney Lemma 5.12.2 and p. 72, the first condition (2)], which is not satisfied in our applications.

Usually it is non-trivial to make 'Whitney-trick-arguments' work for codimension 2; a famous example is Freedman's proof of the Poincaré conjecture in dimension 4 [53, Chapter 13]. The non-triviality of Theorem 3.5 for $k = 2$ is also seen from Theorem 3.6 (which shows that the analogous result for $r = 2$ is false) and from Theorems 3.9 and 3.13.a (which show that the analogous result for *ornaments* is true even for $r = 2$). In other words, the codimension 2 situation is sufficiently delicate to provide different results for different r and for different conditions on r-fold intersections.

A crucial insight for making a version of the Whitney trick work in our context is that, unlike in the classical case of embeddings, we can permit singularities (as long as they are of multiplicity less than r). This allows us to make modifications by *homotopy* as opposed to isotopy, which gives us more flexibility. Together with a restructuring of the arguments, this also leads to a simpler proof of the codimension 3 result, which is presented here and in [98, §3.5], and which some readers may wish to read before studying the proof for codimension $k \geq 2$ in §3.2.1. For further comments on the proof ideas and related work, see also Remark 3.17 below.

3.1.3 Classification of ornaments and doodles

In this subsection we describe another application of our methods in the topological context of higher multiplicity linking.

Throughout this subsection $S = S_1 \sqcup \ldots \sqcup S_r$ will denote a disjoint union of r copies of S^n and $D = D_1 \sqcup \ldots \sqcup D_r$ a disjoint union of r copies of D^{n+1}; the dimensions of S, D will be clear from the context.

An **r-component n-ornament in S^d** is a general position PL map $f : S \to S^d$ such that $fS_1 \cap \ldots \cap fS_r = \emptyset$.

Let $r \geq 2$ and f be an r-component $(k(r-1)-1)$-ornament in S^{kr-1}. Extend f to a general position PL map $g : D \to B^{kr}$ (the extension is constructed e.g., by 'coning' each $f|_{S_i}$ to interior point of B^{kr}, a distinct cone point for each component). Define **the r-linking number** of f by

$$\operatorname{lk} f := gD_1 \bullet \ \ldots \bullet gD_r \in \mathbb{Z}.$$

This definition is a natural generalization of the classical linking number (obtained for $r = 2$), and μ-invariant of [30] (obtained for $r = 3$ and $k = 1$).[2] Analogously to the case $r = 2$ one can check that $\operatorname{lk} f$ is well-defined, i.e., is independent of the choices of the extension g.[3]

Clearly, if an r-component $(k(r-1)-1)$-ornament in S^{kr-1} bounds a map $g : D \to B^{kr}$ such that $gD_1 \cap \ldots \cap gD_r = \emptyset$, then the ornament has zero r-linking number. The converse

[2]It is also similar in spirit, but different from, the *Massey-Milnor triple linking number* [77], [89, §4.5 'Massey-Milnor number modulo 2'], which distinguishes Borromean rings from the standard link. The 3-linking number of Borromean rings is not defined, because they do not form an r-component $(k(r-1)-1)$-dimensional ornament in S^{kr-1} for any k, r. For the relation see [30, Theorem 3].

[3]By induction, it suffices to prove this for two extensions g and g' that agree on all but one disk, say $g|_{D_i} = g'|_{D_i}$ for every $i < r$. Then $gD_r \cup (-g'D_r)$ carries an integer cycle in B^{kr}. This cycle is the boundary of some integral $(k(r-1)+1)$-dimensional chain C in B^{kr} with $C \cap S^{kr-1} = fS_r$. Since $fS_{r-1} = -\partial gD_{r-1} \subset S^{kr-1}$, $gD_i \cap S^{kr-1} = fS_i$, and f is an ornament, by [60, Definition 29 and Lemma 28] it follows that $gD_1 \bullet \ \ldots \bullet gD_r - g'D_1 \bullet \ \ldots \bullet g'D_r = gD_1 \bullet gD_{r-2} \bullet fS_{r-1} \bullet C = 0$.

is true for every $k \geq 2$, which is a generalization of the Local Disjunction Theorem 3.9 and a particular case of Theorem 3.13.a below. For $k = 1$, the converse clearly holds when $r = 2$ and fails for each $r \geq 3$ by Theorem 3.10.

Denote $I := [0, 1]$. An **ornament concordance** is a map $F : S \times I \to S^d \times I$ such that

$$F(\cdot, t) \subset S^d \times \{t\} \quad \text{for each} \quad t = 0, 1, \quad \text{and} \quad F(S_1 \times I) \cap F(S_2 \times I) \cap \ldots \cap F(S_r \times I) = \emptyset.$$

We remark that in the special case $r = 2$, ornaments and ornament concordance are commonly referred to as link maps and link concordance, respectively. Analogously to the case $r = 2$ [87, §77], $\mathrm{lk}\, f$ is invariant under ornament concordance.

An ornament is called a **doodle** if its restriction to each connected component is an embedding. Likewise, a **doodle concordance** is an ornament concordance such that its restriction to each connected component is an embedding.

An ornament [doodle] is called **trivial** if it is concordant to an ornament [doodle] whose components lie in pairwise disjoint balls. For $(r-1)d > rn+1$ ($\Leftrightarrow (r-1)(d+1) > r(n+1)$) any r-component n-ornament in S^d is trivial by general position.

Theorem 3.13. *The r-linking number defines a 1-1 correspondence between $\mathbb{Z}$ and the set of*

(a) *ornament concordance classes of r-component $(k(r-1)-1)$-ornaments (or doodles) in S^{kr-1} for each $r, k \geq 2$.*

(b) *doodle concordance classes of r-component $(k(r-1)-1)$-doodles in S^{kr-1} for each $r \geq 2$, $k \geq 3$.*

Theorem 3.13 for $r = 2$ is well-known. The case $k \geq 3 = r$ of Theorem 3.13.a is due to Melikhov [68, p. 7]. For $k \geq 3$ and each r, Theorem 3.13 can be derived from [60, Theorem 17], for Part (b) using Remark 3.17.b. Theorem 3.13.a for $r \geq 3$, $k = 2$ is a result of this chapter. Our proof (§3.2.3) works for any $r, k \geq 2$.

The analogue of Theorem 3.13.a for $k = 1$ and $r = 2$ is clearly true, for $k = 1$ and each $r \geq 3$ it is false by Theorem 3.10. See Remark 3.22 below for further comments on ornaments.

The Local Disjunction Theorem 3.9 is a particular case of the following 'ornamental' analogue of Theorem 3.5. The existence of an ornament is trivial, so we state a non-trivial *relative* version.

Theorem 3.14. *Assume that* $k, r \geq 2$,

- $K = K_1 \sqcup \ldots \sqcup K_r$ *is a* $(k-1)r$*-dimensional complex,*
- $f : K \to B^{kr}$ *is a general position map,*
- $L := f^{-1}S^{kr-1} \subset K$ *is a subcomplex and* $f|_L$ *is an* r*-component ornament in* S^{kr-1}*,*
- $f\sigma_1 \cdot \ \ldots \cdot f\sigma_r = 0 \in \mathbb{Z}$ *whenever* $\sigma_1 \subset K_1, \ldots, \sigma_r \subset K_r$ *are* $(k-1)r$*-simplices of* K*.*

Then there is an r*-component ornament* $f' : K \to B^{kr}$ *such that* $f' = f$ *on* L*.*

For $r = 2$ this is known [90]. For $r \geq 3$ this follows from the Global Disjunction Theorems 3.11.(a)-(b) analogously to the above proof of the injectivity in Theorem 3.13.a. A proof which works for any $k, r \geq 2$ could perhaps be given by stating and proving the ornamental version of the Global Disjunction Theorems 3.11.(a)-(b) (which works even for $k = r = 2$). It is interesting to compare the case $k = r = 2$ of Theorem 3.14 to the Global Disjunction Theorems 3.11.(c).

3.2 Proofs

3.2.1 Proof of the Global Disjunction Theorems 3.11.(a)-(b)

Informally speaking, the first step in the proof of the Global Disjunction Theorems 3.11.(a)-(b) is to make the $(r-1)$-fold intersection $f\sigma_1 \cap \ldots \cap f\sigma_{r-1}$ connected. See the following Lemmas 3.15 and 3.16.

Throughout this section, let us fix orientations on balls B^d and disks D^m.

Lemma 3.15 (Surgery of Intersection)**.** *Assume that* $d - 2 \geq p, q$ *and that* $f : D^p \to B^d$, $g : D^q \to B^d$ *are proper embeddings in general position such that* $fD^p \cap gD^q$ *is a proper submanifold (possibly disconnected) of* B^d *containing points* x, y*.*

(a) *If* $p + q > d$ *then there is a proper general position map* $f' : D^p \to B^d$ *with the following properties:*

- *$f' = f$ on ∂D^p and on a neighborhood of $\{f^{-1}x, f^{-1}y\}$;*
- *x, y lie in the interior of an embedded $(p+q-d)$-disk contained in $f'D^p \cap gD^q$.*

(b) *If $p+q = d \geq q+3$, $\{x, y\} = fD^p \cap gD^q$ and x, y have opposite double intersection signs, then there is a general position map $f' : D^p \to B^d$ such that $f' = f$ on ∂D^p and $f'D^p \cap gD^q = \emptyset$.*

This lemma is known for $d-3 \geq p, q$ (then Part (b) is the classical Whitney trick, and for Part (a) see Remark 3.17.a), and Part (b) is known also for $q = 2$ [100, Lemma 2.4]. Passage to $d-2 \geq p, q$ in (a), or to $d-2 = p$ in (b), requires losing the injectivity properties of f, g. Part (b) $d-2 = p \geq 3$ is proved by seeing that $f|_{\partial D^p}$ is null-homotopic in $B^d - gD^q$ (an analogue for $d-2 = p = 2$ is discussed in Remark 3.21.c).

In what follows, we first use the Surgery of Intersection Lemma 3.15 to prove the following Lemma 3.16 and the Global Disjunction Theorem 3.11.(a)-(b). The proof of the Surgery of Intersection Lemma 3.15 is then given at the end of this subsection.

In the rest of this section, we abbreviate B^{kr} to B.

Lemma 3.16. *Assume that $k, r \geq 2$,*

- *K is a $k(r-1)$-dimensional complex,*
- *$f\colon K \to B$ a general position PL map,*
- *$\sigma_1, \ldots, \sigma_r$ are pairwise disjoint top-dimensional simplices of K,*
- *$x, y \in f\sigma_1 \cap \ldots \cap f\sigma_r \subset \mathring{B}$ are two global r-fold points of opposite r-fold intersections signs.*

Then for each $n = 1, \ldots, r-1$ there is a general position PL map $f'\colon K \to B$ such that

- *$f = f'$ on $K - (\mathring{\sigma}_1 \sqcup \cdots \sqcup \mathring{\sigma}_r)$,*
- *x, y lie in the interior of an embedded $k(r-n)$-disk contained in $f'\sigma_1 \cap \ldots \cap f'\sigma_n$, and*
- *f' has the same global r-fold points with the same signs as f.*

Proof. The proof is by induction on n. The base $n = 1$ follows by setting $f' = f$. The required disk is then a small regular neighborhood in $f\sigma_1$ of a path in $f\sigma_1$ joining x to y and avoiding the self-intersection set $\{x \in K \ : \ |f^{-1}fx| \geq 2\}$ of f.

In order to prove the inductive step assume that $n \geq 2$ and the points x, y lie in the interior of an embedded $k(r-n+1)$-disk $\sigma_- \subset f\sigma_1 \cap \ldots \cap f\sigma_{n-1}$. By general position

$$\dim(\sigma_- \cap f\sigma_n) \leq k(r-n+1) + k(r-1) - kr = k(r-n).$$

Since f-preimages of x lie in the interiors of $\sigma_1, \ldots, \sigma_r$, the intersections of $f\sigma_i$ and small regular neighborhoods of x, y in B equal to the intersections of affine spaces and the neighborhoods. Hence the regular neighborhoods of x, y in $\sigma_- \cap f\sigma_n$ are $k(r-n)$-balls.

Take points x', y' in such balls. Take general position paths $\lambda_+ \subset f\mathring{\sigma}_n$ and $\lambda_- \subset \sigma_-$ joining x' to y'. By general position dimension of the self-intersection set of f does not exceed $2k(r-1) - kr < k(r-1) - 1$. So the union $\lambda_+ \cup \lambda_-$ is an embedded circle in $\mathring{B}$. Since $k, r \geq 2$, we have $kr \geq 4$. Hence by general position this circle bounds an embedded 2-disk $\delta \subset \mathring{B}$. Since $k \geq 2$, we have $k(r-1) + 2 \leq kr$. Hence by general position

$$\delta \cap fK = \lambda_+ \cup \lambda_- \sqcup \{fp_1, \ldots, fp_s\}$$

for some points $p_1, \ldots, p_s \in K$ outside the self-intersection set of f and the $(k(r-1)-1)$-skeleton of K, and $s = 0$ for $k \geq 3$.

Let $O\delta$ be a small regular neighborhood of δ in $\mathring{B}$. Then $O\delta$ is a kr-ball and $f^{-1}O\delta$ is the union of

- a regular neighborhood $D_n \cong D^{k(r-1)}$ of the arc $f|_{\sigma_n}^{-1}\lambda_+$ in σ_n;
- regular neighborhoods $D_i \cong D^{k(r-1)}$ of the arcs $f|_{\sigma_i}^{-1}\lambda_-$ in σ_i for each $i = 1, \ldots, n-1$;
- pairwise disjoint $k(r-1)$-disks that are regular neighborhoods of p_j in the $k(r-1)$-simplices of K containing them, for each $j = 1, \ldots, s$; these disks are disjoint from the self-intersection set of f, and their f-images are disjoint from $fD_1 \cup \ldots \cup fD_n$.

Then $f|_{D_i} : D_i \to O\delta$ is proper for each $i = 1, \ldots, n$, and $\sigma_- \cap O\delta$ is a proper $k(r-n+1)$-ball in $O\delta$. Since the regular neighborhoods of x, y in $\sigma_- \cap f\sigma_n$ are $k(r-n)$-balls, the set $\sigma_- \cap O\delta \cap fD_n$ is a proper $k(r-n)$-submanifold of $O\delta$. Hence we can apply the Surgery of Intersection Lemma 3.15.a to fD_n and $\sigma_- \cap O\delta$ in $O\delta$. For the obtained map $f' : D_n \to O\delta$ the points $x, y \in f'\sigma_1 \cap \ldots \cap f'\sigma_r \subset \mathring{B}$ are two global r-fold points of opposite r-fold intersections signs, lying in the interior of an embedded $k(r-n)$-disk contained in $\sigma_- \cap f'D_n$. Extend f' by f outside D_n.

Clearly, the first two bullet points in the conclusion of Lemma 3.16 are fulfilled. All the global r-fold points of f lie outside $O\delta$, and $f = f'$ outside $O\delta$. Therefore all the global r-fold points of f are also global r-fold points of f', and they have the same sign. It remains to check that f' does not have new global r-fold points inside $O\delta$. In $O\delta$ the map f' can have global points of multiplicity at most n in $f'D_n \cap fD_1 \cap \ldots \cap fD_{n-1}$, or of multiplicity 2 in the intersection of $f'D_n$ with the f-image of a small neighborhood of some p_j. Since $r > n \geq 2$, none of these global points are r-fold.

Thus the map f' is as required. $\square$

Proof of the Global Disjunction Theorems 3.11.(a)-(b). By Lemma 3.16 for $n = r-1$ we may assume that the points x, y lie in the interior of an embedded k-disk $\sigma_- \subset f\sigma_1 \cap \ldots \cap f\sigma_{r-1}$. Choose orientations of $\sigma_1, \ldots, \sigma_{r-1}$. These orientations define an orientation on σ_- (this is analogous to the definition of the r-fold intersection sign given before Theorem 3.5, cf. [60, §2.2] for a longer formal exposition). Since $x, y \in f\sigma_1 \cap \ldots \cap f\sigma_r$ have opposite r-fold intersections signs, $x, y \in \sigma_- \cap f\sigma_r$ have opposite double intersections signs [60, Lemma 27.cd].

Analogously to the proof of Lemma 3.16 (except that we start from x, y not from x', y') we construct a kr-ball $O\delta \subset \mathring{B}$ and $k(r-1)$-disks $D_i \subset \mathring{\sigma}_i$ for $i = 1, \ldots, r$, such that $x, y \in O\delta$ are the only global r-fold points in $O\delta$ and $f|_{D_i} : D_i \to O\delta$ is proper.

Since either $r \geq 3$ or $k \geq 3$, we have $kr \geq \dim \sigma_- + 3$. So we can apply the Surgery of Intersection Lemma 3.15.b to fD_r and $\sigma_- \cap O\delta$ in $O\delta$. For the obtained map $f' : D_r \to O\delta$ we have $\sigma_- \cap f'D_r = \emptyset$. Extend f' by f outside D_r.

Clearly, $f = f'$ on $K - (\mathring{\sigma}_1 \sqcup \ldots \sqcup \mathring{\sigma}_r)$. Since $f = f'$ outside of D_r, all the global r-fold points of f except x, y are also global r-fold points of f', and they have the same sign. It remains to check that $f'D_r$ contains no global r-fold points of f'. Recall the description of $f^{-1}O\delta$ from the bullet points in the proof of Lemma 3.16.

If $r = 2$, then $k \geq 3$, so $s = 0$. Also $O\delta \cap \sigma_- = f(D_1) = f'(D_1)$. So $f'(K) \cap O\delta = f'(D_1) \sqcup f'(D_2) = (O\delta \cap \sigma_-) \sqcup f'(D_2)$, where the union is disjoint by the construction of f'. Therefore $f'D_2 = f'D_r$ contains no global 2-fold points of f'.

If $r > 2$, then $f'|_{K \setminus D_r}$ has no $(r-1)$-fold point in $O\delta$ except for $\sigma_- \cap O\delta$. By construction $\sigma_- \cap f'D_r = \emptyset$, so again $f'D_r$ contains no global r-fold points of f'. $\square$

Proof of the Surgery of Intersection Lemma 3.15.a. To simplify notation, let us write

$$Q := gD^q \quad \text{and} \quad M := fD^p \cap Q$$

throughout this proof. Furthermore, let $m := \dim M = p + q - d$. Note that the assumptions on the dimensions p, q, d imply that $m + 2 \leq p, q$ and $d \geq 5$.

The chosen orientations of B^d, D^p, and D^q define an orientation on M (this is analogous to the definition of the r-fold intersection sign given before Theorem 3.5, cf. [60, §2.2] for a longer formal exposition).

Let us first assume that x and y lie in different connected components of M. We proceed in two steps to reduce this case to the case where x and y lie in the same connected component of the intersection; it will then be easy to deal with the latter situation.

Step 1. Ambient 1-surgery. ("piping".) Pick two generic points $a, b \in M$ such that a lies in the same connected component of M as x, and b lies in the same connected component of M as y. Pick a general position path $\ell \subset Q$ connecting a and b.

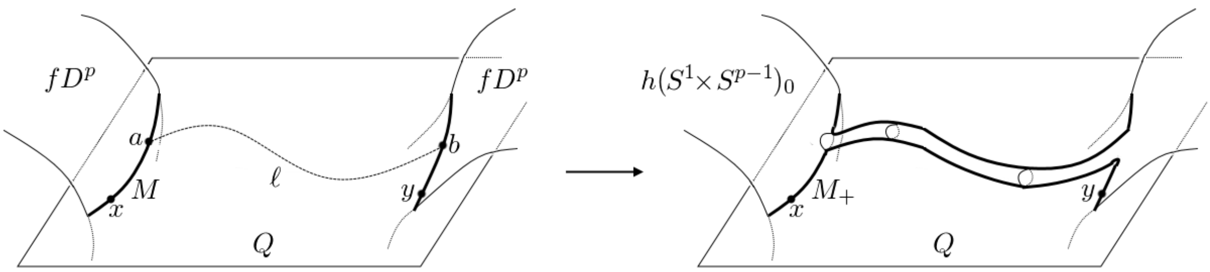

Figure 3.2: Piping

By general position, ℓ is disjoint (and hence at a positive distance from) the set of points at which Q is not locally flat in B^d (see [81, p. 50] for the definition of local flatness); this follows because *the set of non-locally flat points of the codimension ≥ 2 submanifold $Q \subset B^d$ has codimension ≥ 2 in Q.*[4]

[4]Let us prove the latter statement. Take a triangulation of B^d such that Q is a subcomplex of this triangulation. For each point $c \in Q$ the pair $(\operatorname{lk}_{B^d} c, \operatorname{lk}_Q c)$ is a codimension ≥ 2 pair of spheres. If c is outside the codimension 2 skeleton of this subcomplex, then $\dim \operatorname{lk}_Q c \in \{-1, 0\}$. Hence the pair $(\operatorname{lk}_{B^d} c, \operatorname{lk}_Q c)$ is unknotted. Thus, Q is locally flat in B^d at c.

We now perform ambient 1-surgery on M in Q as described in [81, pp. 67-68] (where this procedure is called "piping") to obtain connected manifold M_+; more precisely, take an embedding $L : I \times D^m \to Q$ that satisfies the following properties (where we use $m \le q-2$ for the second property):

- $L(I \times 0) = \ell$,
- $M \cap L(I \times D^m) = L(\{0,1\} \times D^m)$ is a regular neighborhood of $\{a,b\}$ in M,
- the orientation of M on this neighborhood is compatible with the 'boundary' orientation of $L(\{0,1\} \times D^p)$, and
- $L(I \times D^m)$ is disjoint from x, y and from any non-locally flat points of Q in B^d.

We define

$$M_+ := \big(M \setminus L(\{0,1\} \times \mathring{D}^m)\big) \cup L(I \times \partial D^m).$$

By construction, x and y lie in the same component of M_+, and M_+ is orientable. We give M_+ the orientation induced by that of M.

By general position fD^p and Q are transverse at $\{a,b\}$. Since ℓ does not contain non-locally flat points of Q in B^d, the submanifold Q is locally flat in B^d in a neighborhood of ℓ. Hence, we can extend L to an embedding $L : I \times D^p \to B^d$ such that

- $Q \cap L(I \times D^p) = L(I \times D^m)$,
- $fD^p \cap L(I \times D^p) = L(\{0,1\} \times D^p)$ is a regular neighborhood of $\{a,b\}$ in fD^p, and
- the orientation of fD^p on this neighborhood is compatible with the 'boundary' orientation of $f|_{D^p}^{-1}(L(\{0,1\} \times D^p))$.

Denote by $(S^1 \times S^{p-1})_0$ the manifold $S^1 \times S^{p-1}$ with an open p-disk removed. Let

$$h \colon (S^1 \times S^{p-1})_0 \to B$$

be the proper embedding obtained by adding the embedded 1-handle $L(I \times \partial D^p)$ to fD^p; thus,

$$\operatorname{im} h = \big(fD^p \setminus L(\{0,1\} \times \mathring{D}^p)\big) \cup L(I \times \partial D^p).$$

By construction, the intersection $\operatorname{im} h \cap Q = M_+$ is connected. (Note that h is an embedding of $(S^1 \times S^{p-1})_0$, not of D^p; this will be repaired in the next step.)

Step 2. Ambient 2-surgery. (*"unpiping"*). We now perform ambient 2-surgery on $\operatorname{im} h$ in Q to obtain a proper embedding $f'\colon D^p \to B$ such that $\operatorname{im} h \cap Q = M_+ \subseteq f'D^p \cap Q$ (this is analogous to [60, Lemma 38], where the corresponding operation is called "unpiping").

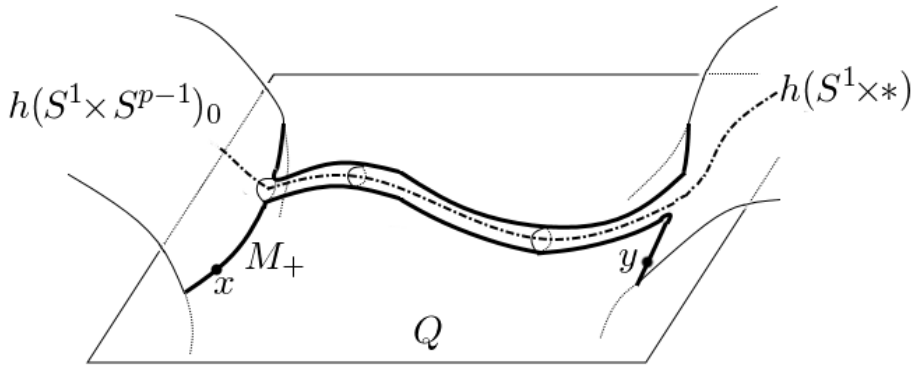

Figure 3.3: Unpiping

Pick a point $* \in S^{p-1}$ in general position with respect to h, and a general position embedded 2-disk $\delta \subset B^d$ such that $\partial\delta = h(S^1 \times *)$. By general position, $\partial\delta$ is disjoint from Q, and δ intersects Q in a finite set (empty if $q \le d-3$) of points disjoint from $\operatorname{im} h$.

Denote by $O\delta$ a small regular neighborhood of δ in B. Take a small regular neighborhood $U \cong D^{p-1}$ of $*$ in S^{p-1}. We may assume that $h(S^1 \times U) \subset O\delta$. Since $O\delta \cong B^d$, the restriction $S^1 \times \partial U \to O\delta$ of h extends to a map $j\colon D^2 \times \partial U \to O\delta$.

Let

$$\Delta := \big((S^1 \times S^{p-1})_0 \setminus (S^1 \times \mathring{U})\big) \cup (D^2 \times \partial U) \cong D^p.$$

Define

$$f'\colon \Delta \to B^d \quad \text{by} \quad f'(x) := \begin{cases} h(x) & \text{if} \quad x \in N \setminus (S^1 \times \mathring{U}), \\ j(x) & \text{if} \quad x \in D^2 \times \partial U. \end{cases}.$$

By construction of f', $f = f'$ on ∂D^p and in a neighborhood of x, y (identifying Δ with D^p). Moreover, $f'\Delta \cap Q$ consists of the manifold $\operatorname{im} h \cap Q$ plus possibly some additional further components. In particular, x and y lie in the same connected component $f'\Delta \cap Q$, and this component is a manifold of dimension $m = p + q - d$.

To complete the proof, take a general position path $\ell \subset f'\Delta \cap Q$ connecting x and y. Then a regular neighborhood of ℓ in $f'\Delta \cap Q$ is then an m-disk that contains x and y. $\square$

Proof of the Surgery of Intersection Lemma 3.15.b. Denote $X := B^d - gD^q$. Consider the composition

$$\pi_{p-1}(X) \xrightarrow{h} H_{p-1}(X) \xrightarrow{\cong} H_0(D^q) \cong \mathbb{Z}$$

of the Hurewicz homomorphism and the (homological) Alexander duality isomorphism. This composition carries $[f|_{S^{p-1}}]$ to $fD^p \cdot gD^q$. [5] The assumptions of part (b) imply that $fD^p \cdot gD^q = 0$. By general position X is $(p-2)$-connected. Since $p \geq 3$, we have $p-2 \geq 1$, so by the Hurewicz theorem h is an isomorphism. Hence the restriction $f : S^{p-1} \to X$ is null-homotopic. Thus there is an extension $f' : D^p \to X$ of the restriction. This is the required map. $\square$

Remark 3.17. (a) Lemmas and Lemma 3.16 are generalizations, to $(r-1)$-multiplicity and to codimension 2, of the 'high-connectivity' version of the Whitney trick [45], [42, Lemma 4.2], [43, Theorem 4.5 and appendix A], [80, Theorem 4.7 and appendix].The lemmas are proved by ambient surgery, i.e. by first adding to $f\sigma_{r-1}$ 'an embedded 1-handle' along a path joining x to y in $f\sigma_1 \cap \ldots \cap f\sigma_{r-1}$ (which is assumed by induction to be already connected), and then cancelling 'an embedded 2-handle' along the 'Whitney disk', which for codimension ≥ 3 was done in [44, §3] $(r=2)$, [68, proof of Theorem 1.1 in p. 7] $(r=3)$.

For a generalization to the 'metastable' version see [61; 94].

(b) Applying the Disjunction Theorems in the form presented here may introduce new r-fold points (albeit no global ones). On the other hand, for $k \geq 3$, the higher-multiplicity Whitney trick in [60, Theorem 17] does not create any new r-fold points at all. This difference is immaterial for the study of almost r-embeddings or *ornaments* (see §3.1.3), but it is important in for the study of *doodles* (see §3.1.3).

For $k \geq 3$, our proof can perhaps be modified to show that in the Local Disjunction Theorem 3.9, under the additional assumption that f embeds each disk, we may obtain additionally that the resulting map f' embeds each disk, as in [60, Theorem 17].

[5] This is one of the equivalent definitions of Alexander duality isomorphism, see Alexander Duality Lemmas of [91; 92]. Another equivalent definition is as follows. Take a small oriented disk $D^p_g \subset B^d$ whose intersection with gD^q consists of exactly one point of sign $+1$ and such that $\partial D^p_g \subset X$. Then the Alexander duality carries the generator ∂D^p_f of $H_{p-1}(X)$ to the generator of $H_0(D^q)$ defined by the orientation of D^q.

Such an improvement might be obtained by an application of the corresponding (known) 'injective' version of the Surgery of Intersection Lemma 3.15.

If $k = r = 2$, we cannot obtain this (as e.g. disks extending the Whitehead link show). It would be interesting to know if we can obtain this for $k = 2$, $r \geq 3$.

3.2.2 The Singular Borromean Rings and proof of Theorem 3.6

We consider the following lemma (required for Theorem 3.6) interesting in itself.

Lemma 3.18 (Singular Borromean Rings). *For each $n = 2l$ let $T := S^l \times S^l$ be the $2l$-dimensional torus with meridian $m := S^l \times \cdot$ and parallel $p := \cdot \times S^l$, and let S^n_p and S^n_m be copies of S^n. Then there is no PL map $f\colon T \sqcup S^n_p \sqcup S^n_m \to \mathbb{R}^{n+l+1}$ satisfying the following three properties:*

(i) *the f-images of the components are pairwise disjoint;*

(ii) *fS^n_p is linked modulo 2 with fp and is not linked modulo 2 with fm,* [6] *and*

(iii) *fS^n_m is linked modulo 2 with fm and is not linked modulo 2 with fp.*

Proof. The proof uses a 'triple intersection' homology argument analogous to the classical proof showing that Borromean rings are linked [77], [89, §4.5 'Massey-Milnor number modulo 2']. The reader might want to read this proof first for $n = 2$ and $l = 1$.

Assume to the contrary that the map f exists. Without loss of generality, we may assume that f is in general position.

Throughout the proof all the chains and cycles are assumed to have $\mathbb{Z}_2$ coefficients, and all the equalities are congruences modulo 2. Since all the chains below are represented by general position polyhedra, chains could be identified with their supporting bodies. We denote by ∂ the boundary of a chain.

We can view $f(T)$, $f(S^n_p)$, and $f(S^n_m)$ as $2l$-, n- and n-dimensional PL cycles in general position in $\mathbb{R}^{n+l+1}$. Denote by C_T, C_p, and C_m singular cones in general position over

[6]See the well-known definition of 'linked modulo 2' e.g. in [87, §77] or in [89, §4.2 'Linking modulo 2 of curves in space'].

$f(T)$, $f(S^n_p)$, and $f(S^n_m)$, respectively. We view these cones as $(2l+1)$-, $(n+1)$- and $(n+1)$-dimensional PL chains. The contradiction is

$$0 \underset{(1)}{=} |\partial(C_T \cap C_p \cap C_m)| \underset{(2)}{=} |\underbrace{\partial C_T}_{=f(T)} \cap C_p \cap C_m| + |C_T \cap \underbrace{\partial C_p}_{=f(S^n_p)} \cap C_m| + |C_T \cap C_p \cap \underbrace{\partial C_m}_{=f(S^n_m)}| \underset{(3)}{=} 1+0+0 = 1.$$

Here (1) follows because $C_T \cap C_p \cap C_m$ is a 1-dimensional PL chain, so its boundary is 0. Equation (2) is Leibniz formula. So it remains to prove (3).

Proof of (3). For $X \in \{T, S^n_m, S^n_p\}$ denote $f_X := f|_X$.

For the *second term* we have

$$|C_T \cap f(S^n_p) \cap C_m| \overset{(*)}{=} |(f^{-1}_{S^n_p} C_T) \cap (f^{-1}_{S^n_p} C_m)| \overset{(**)}{=} 0, \quad \text{where}$$

(*) holds because $(n+1)+(2l+1)+2n < 3(n+l+1)$, so by general position $C_T \cap C_m$ avoids self-intersection points of $f(S^n_p)$,

(**) holds by the well-known higher-dimensional analogue of [93, Parity Lemma 3.2.c] (which is proved analogously) because the intersecting objects are general position cycles in S^n_p; they are cycles because $\partial(C_T \cap f(S^n_p)) = 0 = \partial(C_m \cap f(S^n_p))$ and $n \le 2l \Leftrightarrow (n+1)+2n < 2(n+l+1)$, so by general position both C_T and C_m avoid self-intersection points of $f(S^n_p)$.

Analogously $|C_T \cap C_p \cap f(S^n_m)| = 0$.

For the *first term* we have

$$|f(T) \cap C_p \cap C_m| \overset{(***)}{=} |(f^{-1}_T C_p) \cap (f^{-1}_T C_m)| \overset{(****)}{=} m \cap p = 1, \quad \text{where}$$

(***) holds because $n \ge l \Leftrightarrow 2(n+1)+4l < 3(n+l+1)$, so by general position $C_p \cap C_m$ avoids self-intersection points of $f(T)$,

(****) is proved as follows:

The l-chain $f^{-1}_T C_p$ is a cycle in T because $\partial(C_p \cap f(T)) = 0$ and $n \ge 2l \Leftrightarrow n+1+4l < 2(n+l+1)$, so by general position C_p avoids self-intersection points of $f(T)$. By conditions (b) and (c) of Lemma 3.18 we have

$$|p \cap f^{-1}_T C_p| = |f(p) \cap C_p| = 1 \quad \text{and} \quad |m \cap f^{-1}_T C_p| = |f(m) \cap C_p| = 0.$$

I.e. the cycle $f_T^{-1}C_p$ intersects the parallel p and the meridian m at 1 and 0 points modulo 2, respectively. Therefore $f_T^{-1}C_p$ is homologous to the meridian m. Likewise, $f_T^{-1}C_m$ is homologous to the parallel p. This implies (****).

$\square$

Construction of the 2*-complex in Theorem 3.6.* We begin by recalling the construction of the 2-complex K from [31]. Let P be the 2-skeleton of the 6-simplex whose vertices are $\{p_1, \dots, p_7\}$. Let $p := \partial[p_1, p_2, p_3]$ denote the boundary of the 2-simplex $[p_1, p_2, p_3]$. Denote by P_- the complement in P to (the interior of) the 2-simplex $[p_1, p_2, p_3]$. The remaining four vertices p_4, p_5, p_6, p_7 span a 'complementary' 2-sphere $S^2_p := \partial[p_4, p_5, p_6, p_7] \subset P$ that is the boundary of the 3-simplex $[p_4, p_5, p_6, p_7]$ (this 3-simplex itself is not contained in P).

Let M_- denote a copy of P_- on a disjoint set of vertices $\{m_1, m_2 \dots, m_7\}$, and let $m := \partial[m_1, m_2, m_3]$ and $S^2_m := \partial[m_4, m_5, m_6, m_7]$.

The 2-complex K then is defined by the formula

$$K := (P_- \underset{p_1=m_1}{\cup} M_-) \underset{p=S^1\times\cdot,\ m=\cdot\times S^1}{\cup} T,$$

where T is the torus $S^1 \times S^1$ with any triangulation for which $S^1 \times \cdot$, $m = \cdot \times S^1$ are subcomplexes.

Lemma 3.19. *[105, Satz 5] Let $g\colon P \to \mathbb{R}^4$ be a PL map in general position of the 2-skeleton of the 7-simplex. Then the number $v(g)$ of intersection points of f-images of disjoint triangles (i.e., the total number of global 2-fold points of g) is odd.* [7]

Proof of Theorem 3.6. Analogously to [31, §3.3], K admits a $\mathbb{Z}$-almost 2-embedding in $\mathbb{R}^4$. Suppose to the contrary that there is a PL almost 2-embedding $f\colon K \to \mathbb{R}^4$. We may assume it is in general position. Let us show that $f|_{S^2_p \sqcup S^2_m \sqcup T}$ satisfies the conditions (a), (b) and (c) of the Singular Borromean Rings Lemma 3.18 (this is essentially proved in [31, Lemma 6]). This would give a contradiction by Lemma 3.18.

[7] The lemma implies that the Van Kampen obstruction of P is nonzero even modulo 2, or equivalently, P does not admit a '$\mathbb{Z}_2$-almost 2-embedding' in $\mathbb{R}^4$. For an elementary exposition and an alternative proof see [93].

Condition (a) is satisfied because f is an almost 2-embedding and because any simplex in the triangulation of T is vertex-disjoint from any simplex in S^2_p and from any simplex in S^2_m.

The complex K contains the cone $p_4 * p$, which is a disk disjoint from S^2_m. Since f is an almost 2-embedding, $f(p_4 * p) \cap f(S^2_m) = \emptyset$. Then $f(p)$ and $f(S^2_m)$ are unlinked modulo 2. Analogously, $f(m)$ and $f(S^2_p)$ are unlinked modulo 2.

Extend $f|_{P_-}$ to a general position PL map $g : P \to \mathbb{R}^4$. Then the sphere $f(S^2_p) = g(S^2_p)$ and the circle $f(p) = g(p)$ are linked modulo 2 because

$$|g(S^2_p) \cap g[p_1, p_2, p_3]| = \sum_{\{i,j,k\} \subset \{4,5,6,7\}} |g[p_i, p_j, p_k] \cap g[p_1, p_2, p_3]| \overset{(1)}{\equiv} v(g) \overset{(2)}{=} 1 \in \mathbb{Z}_2, \quad \text{where}$$

(1) holds because $f|_{P_-}$ is an almost 2-embedding, so $f(\sigma) \cap f(\tau) = \emptyset$ for all 'other' pairs σ, τ;

(2) holds by Lemma 3.19.

Analogously the sphere $f(S^2_p) = g(S^2_p)$ and the circle $f(p) = g(p)$ are linked modulo 2. $\square$

3.2.3 Proof of Theorems 3.10 and 3.13.a

Proof of Theorem 3.13.a. The case $r = k = 2$ is known, cf. Remark 3.21.c. We present the proof for $r = 3$, the generalization to arbitrary $r \geq 3$ or to $r = 2 \leq k - 1$ is obvious (because by the Global Disjunction Theorem 3.11.(a)-(b) assertion $(D_{k,r})$ is true for arbitrary $k \geq 2$ and $k + r \geq 5$).

We first prove surjectivity, i.e., that for any integer l there is an ornament (actually a doodle) f such that $\operatorname{lk} f = l$.

The case $l = 0$ is trivial, we can take any doodle such that the images of its connected components lie in 3 pairwise disjoint balls.

Consider now the case $l = \pm 1$. Identify B^{3k} with $B^k \times B^k \times B^k$. Define the *Borromean ornament (doodle)* $f : \bigsqcup_{i=1}^3 S_i^{2k-1} \to S^{3k-1} = \partial B^{3k}$ by

$$fS_1^{2k-1} = \partial(B^k \times B^k \times \cdot), \quad fS_2^{2k-1} = \partial(B^k \times \cdot \times B^k), \quad \text{and} \quad fS_3^{2k-1} = \partial(\cdot \times B^k \times B^k).$$

Clearly, $|\operatorname{lk} f| = 1$. By composing f with the reflection of one of the spheres S_i^{2k-1} we get a new ornament f' such that $\operatorname{lk} f' = -\operatorname{lk} f$. So $\{\operatorname{lk} f, \operatorname{lk} f'\} = \{-1, 1\}$.

Let f_0, f_1 be two ornaments, their images lying in disjoint balls. Connect each of the connected components of f_0 with the respective connected component of f_1 by a thin tube and denote the obtained doodle by f_2. Clearly, $\mathrm{lk}\, f_2 = \mathrm{lk}\, f_0 + \mathrm{lk}\, f_1$. So the case of general l follows from the cases $l = 0$ and $l = \pm 1$.

We now prove injectivity. We have to prove that if $f_0, f_1 : \bigsqcup_{i=1}^3 S_i^{2k-1} \to S^{3k-1}$ are two ornaments such that $\mathrm{lk}\, f_0 = \mathrm{lk}\, f_1$, then f_0 and f_1 are ornament concordant.

Take a general position PL map $F : (\sqcup_{i=1}^3 S_i^{2k-1}) \times I \to S^{3k-1} \times I$ such that $F(\cdot, 0) = f_0(\cdot) \times 0$ and $F(\cdot, 1) = f_1(\cdot) \times 1$. Since $\mathrm{lk}\, f_0 = \mathrm{lk}\, f_1$, the set $F(S_1^{2k-1} \times I) \cap F(S_2^{2k-1} \times I) \cap F(S_3^{2k-1} \times I)$ consists of pairs of 3-fold points of opposite signs. Each such pair can be eliminated by the Global Disjunction Theorem 3.11.(a)-(b) applied to $K = \bigsqcup_{i=1}^3 S_i^{2k-1} \times I$. $\square$

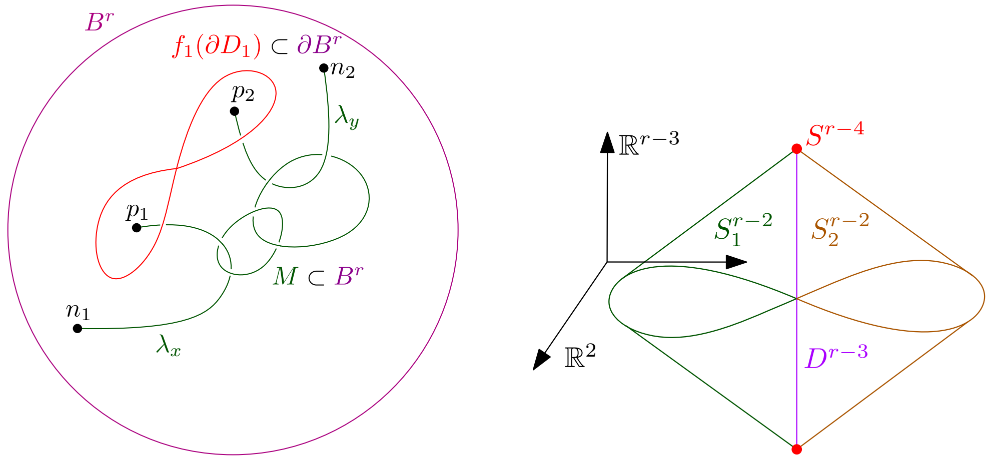

Figure 3.4: (a) To Lemma 3.20. (b) To the proof of Lemma 3.20.

Lemma 3.20. *For each $r \geq 3$ there is a proper general position PL map $f : \partial D_1 \sqcup D_2 \sqcup \ldots \sqcup D_r \to B^r$, where D_j is a copy of $(r-1)$-disk, such that*

1. *$M := fD_2 \cap \ldots \cap fD_r$ is a proper oriented submanifold of B^r and $\partial M = \{p_1, p_2, n_1, n_2\} \subset \partial B^r$, where the points p_1, p_2 have positive sign and the points n_1, n_2 have negative sign (the signs are defined as the signs of intersection points of $r-1$ oriented $(r-2)$-dimensional spheres in S^{r-1}),*

2. *for any generic oriented path λ in B^r from p_j to n_i and any proper extension $g : D_1 \sqcup D_2 \sqcup \ldots \sqcup D_r \to B^r$ of f we have $gD_1 \cdot \lambda = (-1)^j$.*

Proof. It is easy to define the map f on $D_2 \sqcup \ldots \sqcup D_r$ so that the property (1) is satisfied. Let us now define f on ∂D_1.

Identify $S^{r-1} = \partial B^r$ with $S^2 * S^{r-4}$, and $S^{r-2} = \partial D_1$ with $S^1 * S^{r-4}$. (This works for $r = 3$, when $S^{r-4} = \emptyset$.) Without loss of generality we may assume that $\{p_1, p_2, n_1, n_2\} \subset S^2 * \emptyset \subset S^{r-1}$. Let $8 : S^1 \to S^2$ be a map whose image is figure "8" winding 1 time around p_1, -1 time around p_2, and 0 times around n_1 and n_2, i.e. $\mathrm{lk}(8, n_i) = 0$ and $\mathrm{lk}(8, p_j) = (-1)^{j+1}$. Now define $f|_{\partial D_1} : \partial D_1 \to S^{r-1}$ by $f := 8 * \mathrm{id}\, S^{r-4}$ (see Figure 3.4.b).

Let us prove that f satisfies (2). Let π be a generic oriented path in $S^2 = S^2 * \emptyset \subset S^{r-1}$ from p_j to n_i. Then

$$gD_1 \cdot \lambda = gD_1 \cdot (\lambda \cup -\pi) + gD_1 \cdot \pi = 0 + f\partial D_1 \cdot \pi = 8 \cdot \pi = \mathrm{lk}(8, n_i - p_j) = 0 - (-1)^{j+1} = (-1)^j.$$

□

Proof of Theorem 3.10. Let us prove that for each map f given by Lemma 3.20 the ornament $f|_{\partial D}$ is as required. Extend f to D_1 properly and generically in an arbitrary way (e.g., by coning over a generic point).

Proof that $fD_1 \cdot \ldots \cdot fD_r = 0$. By the property (1) of Lemma 3.20 (and possibly by exchanging n_1, n_2) we may assume without the loss of generality that M consists of generic oriented paths λ_j from p_j to n_j, $j = 1, 2$, and a union ω of disjoint embedded circles (see Figure 3.4.a). Then by the property (2) of Lemma 3.20 we have $fD_1 \cdot \ldots \cdot fD_r = fD_1 \cdot (\lambda_1 \sqcup \lambda_2 \sqcup \omega) = -1 + 1 + 0 = 0$.

Proof that $gD_1 \cap \ldots \cap gD_r \neq \emptyset$ for any other proper generic map $g\colon D \to B^r$ such that $f = g$ on ∂D. Since $f = g$ is generic on the boundary, we have that $M' := gD_2 \cap \ldots \cap gD_r$ is a relative 1-dimensional integer homology cycle in B^r and $\partial M' = \{p_1, p_2, n_1, n_2\}$. Without the loss of generality (and possibly by exchanging n_1, n_2), we may assume that M' contains an oriented path λ_1 from p_1 to n_1. By the property (2) of Lemma 3.20, p_1 and n_1 are in the different connected components of $B^r \setminus gD_1$. So

$$\emptyset \neq gD_1 \cap \lambda_1 \subset gD_1 \cap M' = gD_1 \cap \ldots \cap gD_r.$$

□

3.3 Discussion and open problems

Remark 3.21. (a) Analogously to [60, §5], it can perhaps be shown that the analogue of Theorem 3.1 holds for $d \geq 2r$, by using the results in the present chapter (the Global Disjunction Theorem 3.11 below) to rewrite the proofs of [60, Thm. 11] with $k \geq 2$ instead of $k \geq 3$. Since the necessary facts about prismatic maps are not gathered in one easily citable statement in the current version of [60, §5] but dispersed throughout the text, for simplicity of presentation we focus here on the shorter argument for $d \geq 2r + 1$.

(b) Theorem 3.5 for $r = 2$ was a step in the proof of a classical algebraic criterion of van Kampen, Shapiro and Wu for embeddability of n-complexes into $\mathbb{R}^{2n}$ [105; 88; 112], see survey [97, Theorem 4.1]. Both this criterion and Theorem 3.5 for $r = 2$ were generalized by Haefliger and Weber who showed that an n-complex K embeds into $\mathbb{R}^d$ iff there is a $\mathbb{Z}_2$-equivariant map from the deleted product $K^{\times 2}_{\Delta}$ to S^{d-1}, provided $d \geq 3(n+1)/2$ [45; 110], see survey [97, Theorem 8.1 and Proposition 8.4]. One might conjecture that the dimension restriction $d \geq 3(n+1)/2$ can be weakened if one uses the configuration space of r-*tuples* of distinct points, and that methods of this chapter would allow to prove such a conjecture. Surprisingly, this is not so, see [97, end of §5]. An explanation is that the notion of an embedding is more subtle than the notion of almost r-embedding.

On the other hand, for a generalization of Theorem 3.5 to n-complexes in $\mathbb{R}^d$ keeping r arbitrary see [61, Theorem 2], [94, Theorems 1.1-1.3].

(c) The case $r = 2$, $k \geq 3$ of the Local Disjunction Theorem 3.9 is a version of the Whitney trick; the subcase $k = 2$ is an exercise on elementary link theory. Here is a well-known proof for $r = k = 2$ for the general position case when $f|_{\partial D}$ is an embedding. Given a 2-component 1-dimensional link in S^3, one can unknot one component in the complement of the second by crossing changes (or by finger moves, guided along arcs) [78, Theorem 3.8]. By the assumption the linking number is zero. The linking number is preserved under crossing changes. So after crossing changes we obtain a link formed by the unknot and the component which shrinks in the complement of the unknot. For such link the assertion is trivial.

(d) In [85] it is shown that *for each* (n, d) *such that* $n+2 \le d \le \frac{3n}{2}+1$ *there exists a finite* n*-complex* K *that admits an almost* 2*-embedding in* $\mathbb{R}^d$ *but that does not embed into* $\mathbb{R}^d$*.* (This example was used to show, for such n, d, the incompleteness of deleted product obstruction, which is defined before Proposition 3.7.) For $d = 2n = 4$ this improves [31] in a different direction than Theorem 3.6: *there exists a finite* 2*-complex* K *that admits an almost* 2*-embedding in* $\mathbb{R}^4$ *but that does not embed into* $\mathbb{R}^4$*.*

Remark 3.22. (a) Assume that $(d, n, r) = (2, 1, 3)$ (hence $2d = 3n + 1$). In this case, a triviality criterion for ornaments is given in [70]; it would be interesting to know if it is algorithmic and if it extends to a classification. The r-linking number is not a complete invariant for doodles, e.g., there is a non-trivial $(2, 1, 3)$-doodle with zero 3-linking number.[8] Thus the analogue of Theorem 3.13.b for $k = 1$ and $r = 3$ is false. (We conjecture that such an analogue is also false for $k = 1$ and each $r \ge 4$, cf. Theorem 3.10.) See [11] for a study of ornaments which are PL immersions, up to regular ornament homotopy (they were called doodles, which is different from terminology of this chapter).

(b) There is a concordance version of Theorem 3.14 for p-ornaments in S^{kr-1}, where $p := k(r-1) - 1$. It involves a complete invariant in $H^p(K_1; \mathbb{Z}) \times \ldots \times H^p(K_r; \mathbb{Z})$.

(c) An *s-component r-multiplicity ornament in* S^d is a PL general position map $f : K_1 \sqcup \ldots \sqcup K_s \to S^d$ of disjoint union of s complexes such that the intersection of any r objects among $fK_1, \ldots, fK_s$ is empty. Although here we only consider the case $s = r$, our results have straightforward generalizations to $s > r$. 2-multiplicity ornaments were widely studied under the name of *link maps*, mostly for the case when each K_j is a sphere, see [90] and references therein.

Remark 3.23. (a) Our proof of the Singular Borromean Rings Lemma 3.18 for $n = 2$ and $l = 1$ gives a shorter, elementary proof of the result from [31] mentioned before Theorem 3.6.

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[8] This is written in [30, bottom of p. 39 and fig.4] with a reference to a later chapter. It would be interesting to have a published proof. It might be easier to obtain the proof using the 'intersection' language, see §3.2.2, rather than 'commutators' language [30].

(b) In [96], the Singular Borromean Rings Lemma 3.18 is used to study algorithmic aspects of almost 2-embeddability of complexes in $\mathbb{R}^d$.

(c) The analogue of the Singular Borromean Rings Lemma 3.18 for $n = l+1 = 1$ is true, although our proof does not work for this case. The analogue of Lemma 3.18 for $n = l$ is false, but would conjecturally become true if we add an additional condition that $f(S^n_p)$ and $f(S^n_m)$ are unlinked modulo 2. For the corresponding construction of Borromean rings in $\mathbb{R}^3$ see [89, §4.4 'Borromean rings and commutators'].

(d) Lemma 3.18 for $n = 2$, $l = 1$ and *embedded* torus $f(T)$ was proved in [56, Theorem 1 and the middle paragraph on page 53] (in a much more general form). We are grateful to S. Krushkal and P. Teichner for explanation of how the proof of [56] works for the case of *non-embedded* torus, as well as for sketching a short direct proof of Lemma 3.18 for $n = 2$, $l = 1$ involving the Milnor group of the complement. It would be interesting to know if these arguments can be generalized to higher dimensions.

Remark 3.24 (Open problems). (a) Does the analogue of Theorem 3.5 holds for $k = 1$ and large enough r? Cf. Theorems 3.10 and 3.11.d.

(b) Is there an example for Theorem 3.10 for which $f|_{\partial D}$ is an embedding?

(c) Does the analogue of Theorem 3.13.b hold for $k = 2$?

By a result of Melikhov [69, Theorem 1.3], in Theorem 3.13, ornament concordance can be replaced by ornament homotopy; it would be interesting to know whether analogously, doodle concordance can be replaced by doodle homotopy.[9] Here, an ornament [doodle] concordance F is an ornament [doodle] homotopy if it is 'level preserving', i.e., if $F(\cdot, t) \subset S^m \times \{t\}$ for each $t \in I$.

(d) **Gromov's problem** [38, 2.9.c]. *Is it correct that if r is not a prime power, then for each compact subset K of $\mathbb{R}^m$ for some m, having Lebesgue dimension* $\dim K = (r-1)k$*, there is a continuous map $X \to \mathbb{R}^{kr}$ each of whose point preimages contains less than r points?*

[9] Our terminology for ornaments and doodles follows the one of Fenn–Taylor–Vassiliev [30; 70; 68]. By contrast [69] uses a different terminology and does *not* require each component of a doodle to be embedded; in other words, the word "doodle" in [69] corresponds to "ornament" in our terminology.

The analogue of this problem for polyhedra K and almost r-embeddings instead of maps without r-fold points is true by Theorem 3.3.a.

(e) Let X be a compact subset of $\mathbb{R}^m$ for some m. Is it correct that $\dim(X\times X\times X) < 6n$ if and only if any continuous map $X \to \mathbb{R}^{3n}$ can be arbitrary close approximated by a continuous map without triple points? This is interesting for 'fractal' $2n$-dimensional compacta X, for which $\dim(X \times X \times X) < 3\dim X$.

4 Eliminating higher-multiplicity intersections of positive dimension

Denote by Δ_N the N-dimensional simplex. We omit 'continuous' for maps. A map $f\colon K \to \mathbb{R}^d$ of a union K of closed faces of Δ_N is an **almost r-embedding** if $f\sigma_1 \cap \ldots \cap f\sigma_r = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces of K.

Theorem 4.1. *If r is not a prime power and $N := (d+1)r - r\left\lceil \dfrac{d+2}{r+1} \right\rceil - 2$, then there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$.*

Remark 4.2 (tightness). In a recent preprint [33] it was shown that the converse of Theorem 4.1 holds for $N \geq (d+1)r - 1$. In that sense the result of Theorem 4.1 is rather tight.

Remark 4.3 (motivation). (a) A counterexample to the topological Tverberg conjecture asserts that *if r is not a prime power and $d \geq 2r+1$, then there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$.* See the surveys [23; 98] and the references therein. There naturally appears more general problem: *For which a, d there is an almost r-embedding $\Delta_a \to \mathbb{R}^d$?*

This problem was considered in [21, §5]. Lemma 5.2 of [21] provides a simple procedure of constructing higher-dimensional counterexamples by 'taking k-fold join power' of lower-dimensional ones. According to a private communication by F. Frick the bound of [21, Theorem 5.4] together with the counterexample in [8, Theorem 1.1] gives an almost r-embedding $\Delta_F \to \mathbb{R}^d$ for r not a prime power, d sufficiently large, and F some integer close to $(d+1)r - \dfrac{r+\frac{1}{2}}{r+1}(d+1)$. Presumably $F - (d+1)(r-1)$ can be arbitrarily large.

Theorem 4.1 provides even stronger counterexamples to the topological Tverberg conjecture: for d large compared to r we have $N > (d+1)(r-1)$ and even $N > F$. Theorem

4.1 is a step towards the part of [21, Conjecture 5.5] saying that *For $r < d$ not a prime power there is an almost r-embedding $\Delta_{(d+1)r-2} \to \mathbb{R}^d$.* Observe that for $r < d$ we have $N \le dr - 2$.

(b) We think counterexamples of Theorem 4.1 are mostly interesting because their proof requires non-trivial ideas, see below. Thus we do not spell out even stronger counterexamples which presumably could be obtained by combining the procedure of [21, §5] with Theorem 4.1.

(c) Let us illustrate Theorem 4.1 by numerical examples. Earlier results gave almost 6-embeddings $\Delta_{280} \to \mathbb{R}^{55}$ and $\Delta_{275} \to \mathbb{R}^{54}$, as well as almost r-embeddings $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ for $d \ge 2r+1$, $\Delta_{d(r-1)} \to \mathbb{R}^{d-1}$ for $d \ge 2r+2$ and $\Delta_{(d+1-s)(r-1)} \to \mathbb{R}^{d-s}$ for $d \ge 2r+s+1$. Corollary 4.4 below gives an almost 6-embedding $\Delta_{280} \to \mathbb{R}^{54}$ and almost r-embeddings $\Delta_{(d+1)(r-1)} \to \mathbb{R}^{d-s}$ for certain r, d, s.

Corollary 4.4. *Assume that r is not a prime power.*

(a) For $q \ge r+2$ and $d = (r+1)q - 1$ there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^{d-1}$.

(b) If $d \ge (s+2)r^2$ for some integer s, then there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^{d-s}$.

Proof. Part (a) follows by Theorem 4.1 because $q \ge r+2$, so $((r+1)q-1)r - rq - 2 \ge (r+1)q(r-1)$. Part (b) follows by Theorem 4.1 because $d \ge (s+2)r^2 \ge (s+1)r^2 + r - 1$, hence

$$(d+1)(r-1) \le (d-s+1)r - r\frac{d-s+2+r}{r+1} - 2 \le (d-s+1)r - r\left\lceil \frac{d-s+2}{r+1} \right\rceil - 2.$$

□

Remark 4.5 (motivation). A **complex** is a collection of closed faces (=simplices) of some simplex. The *body* (or geometric realization) $|K|$ of a complex K is the union of simplices of K. Thus continuous or piecewise-linear (PL) maps $|K| \to \mathbb{R}^d$ and continuous maps $|K| \to S^m$ are defined. Below we abbreviate $|K|$ to K; no confusion should arise.

By general position, any k-complex admits an almost r-embedding in $\mathbb{R}^{k+\left\lceil \frac{k+1}{r-1} \right\rceil}$. A counterexample to the r-fold van Kampen–Flores conjecture asserts that *if r is not a prime power and k is divisible by $r-1$, then any k-complex admits an almost r-embedding* *in* $\mathbb{R}^{k+\frac{k}{r-1}}$. This is a combination of results of Özaydin [75] and Mabillard–Wagner [60], see [59, §1, Motivation & Future Work, 2nd paragraph] and the survey [98]. The following result produces stronger counterexamples to the conjecture.

Theorem 4.6. *If r is not a prime power, then any k-complex admits an almost r-embedding in $\mathbb{R}^{k+\lceil\frac{k+3}{r}\rceil}$.*

Theorem 4.6 follows from Theorems 4.7 and 4.8 below.

The main new ingredient in the proof of Theorem 4.1 is the following Theorem 4.7.

Denote by $\mathfrak{S}_r$ the permutation group of r elements. Let $\mathbb{R}^{d\times r} := (\mathbb{R}^d)^r$ be the set of real $d \times r$-matrices. The group $\mathfrak{S}_r$ acts on $\mathbb{R}^{d\times r}$ by permuting the columns. Denote

$$\delta_r = \delta_{r,d} := \{(x, x, \dots, x) \in \mathbb{R}^{d\times r} \mid x \in \mathbb{R}^d\}.$$

The following theorem follows immediately from Theorem 2.3:

Theorem 4.7. *If r is not a prime power and X is a complex with a free action of $\mathfrak{S}_r$, then there is a $\mathfrak{S}_r$-equivariant map $X \to \mathbb{R}^{2\times r} - \delta_r$.*

For a complex K let

$$K^{\times r}_\Delta := \bigcup\{\sigma_1 \times \cdots \times \sigma_r\ : \sigma_i \text{ a simplex of } K,\ \sigma_i \cap \sigma_j = \emptyset \text{ for every } i \neq j\}.$$

The group $\mathfrak{S}_r$ has a natural action on the set $K^{\times r}_\Delta$, permuting the points in an r-tuple $(p_1, \dots, p_r)$. This action is evidently free.

Theorem 4.8 ([61; 94; 95])**.** *Assume that K is a k-complex and $rd \geq (r+1)k+3$. There exists an almost r-embedding $f : K \to \mathbb{R}^d$ if and only if there exists a $\mathfrak{S}_r$-equivariant map $K^{\times r}_\Delta \to \mathbb{R}^{d\times r} - \delta_r$.*

Theorem 4.8 is a generalization of the Mabillard–Wagner theorem (see [60], [8] and the survey [98, Theorem 3.3]).

Lemma 4.9 (Constraint Lemma)**.** *For every integers $r, d, k > 0$ and $N = (k+2)r-2$ if there is an almost r-embedding of the union of k-faces of Δ_N in $\mathbb{R}^{d-1}$, then there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$.*

Lemma 4.9 is a straightforward generalization of the Gromov–Blagojević–Frick–Ziegler Constraint Lemma (see [38, 2.9.c], [20, Lemma 4.1.iii and 4.2], [32, proof of Theorem 4] and the survey [98, Lemma 3.2]).

Proof of Theorem 4.1. We may assume that $d \geq 3$. Denote $k := d-1-\left\lceil \dfrac{d+2}{r+1} \right\rceil$. Since r is not a prime power, by Theorem 4.7 there is a $\mathfrak{S}_r$-equivariant map $(\Delta_N^{(k)})^{\times r}_\Delta \to \mathbb{R}^{2\times r} - \delta_r$. The composition of this map with the r-th power of the inclusion $\mathbb{R}^2 \to \mathbb{R}^{d-1}$ gives a $\mathfrak{S}_r$-equivariant map $(\Delta_N^{(k)})^{\times r}_\Delta \to \mathbb{R}^{(d-1)\times r} - \delta_r$. We have $r(d-1) \geq (r+1)k + 3$. Hence by Theorem 4.8 there is an almost r-embedding $\Delta_N^{(k)} \to \mathbb{R}^{d-1}$. Since $N = (k+2)r - 2$, by the Constraint Lemma 4.9 there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$. $\square$

5 Envy-free division using mapping degree

5.1 Introduction

Consider a situation when n players want to divide a resource X, which is "continuous" in certain sense, among themselves. We assume that, for each partition of X into n pieces (some possibly empty), each player would be satisfied to take one of the partition pieces, where the choice for each player need not be unique. When no player prefers an empty piece of the resource, the existence of an equilibrium, where every player receives one piece of the partition and is satisfied, is guaranteed by Gale's theorem (see Theorem 5.3 below for the precise statement). For such situations, when every player receives what she/he prefers from a given partition, the term *envy-free partition* is usually used.

Making one step from the classical situations, we may try to make a generalization, following [86; 71]. The resource might come with some cost, so it might naturally happen that for certain partitions the cost of all the non-empty pieces is too high for a player. Then some of the players might prefer to take an empty piece. As in Gale's theorem and other classical results, we make a natural assumption on player's preferences, mathematically speaking, a player prefers a part if in another, but arbitrarily close to given partition configuration she/he also prefers this part.

We will mostly have in mind the segment partition problem, for a unit interval $[0, 1]$, we consider its partitions into n closed (possibly empty) segments with pairwise disjoint interiors, see the details in Section 5.4. As a simple example, every player may rate the parts with her/his own integrable "value" function f_i on $[0, 1]$, and prefers any of those segments which maximize the value of the integral of f_i over them.

Following the classical works, we consider a more general setting than the "value" function; we allow any player to rate the pieces of a given partition with more complicated logic. The very term "envy-free partition" is motivated by the fact that a player's preference of a certain piece may depend on how the rest of the resource is partitioned, and in the solution for the problem no player has envy to take a different piece than she/he is given.

In the special case of the segment partitioning problem, in [86] it was proved that envy-free segment partitions exist for $n = 3$ (the case $n = 2$ is an easy exercise). In [71] the result was extended to $n = 4$, or any prime n. In this work we give a complete solution to the problem in the setting of [86; 71]: We prove that if n is a prime power then an envy-free segment partitioning always exists (Theorem 5.7). Conversely, for every n which is not a prime power, there exists an instance of the segment envy-free partition problem with no solution (Theorem 5.9).

Remark 5.1. The assumption that X is the unit segment is in fact not very restrictive, once we speak about positive solutions. For example, if our resource to partition is a compact set in a Euclidean space, then we may just project it to a line segment and then partition. Therefore, our result implies that for a prime power number of players n any compact set in a Euclidean space can be envy-free partitioned into strips with parallel hyperplanes.

Section 5.2 of the chapter contains an outline of the classical results and techniques, then we prove the existence of solutions or existence of counterexamples. We start from the mapping version of the Knaster–Kuratowski–Mazurkiewicz theorem, Theorem 5.2 and then proceed to Gale's theorem, Theorem 5.3, to some easy results in Section 5.3 that we provide for reader's convenience, and then to substantially new results in subsequent sections.

For classical results in Section 5.2 and for new results in Section 5.4 we emphasize that the natural way to handle the envy-free partition problem is to analyze necessary and sufficient conditions that a continuous map of a simplex to itself hits its center; which amounts to determining possible mapping degrees of maps between spheres under some additional assumptions, analogous to *equivariance* with respect to a group action.

5.2 Classical KKM-type results and partition problems

Let us recall some classical results around the Knaster–Kuratowski–Mazurkiewicz theorem [55] with modifications from [34; 10]. Let us introduce some notation, let Δ^{n-1} be the $(n-1)$-dimensional simplex, which we usually parametrize as

$$\Delta^{n-1} = \{(t_1, \dots, t_n) \mid t_1, \dots, t_n \geq 0,\ t_1 + \dots + t_n = 1\}.$$

We also denote Δ_i^{n-1} the facet of Δ^{n-1} given by the additional constraint $t_i = 0$. Sometimes, when we know the dimension n, we will denote these objects simply as the simplex Δ and its facets Δ_i.

In the above notation the KKM theorem reads: If $A_1, \dots, A_n$ are closed subsets of Δ^{n-1}, covering the simplex, such that for every $i = 1, \dots, n$ the intersection $\Delta_i^{n-1} \cap A_i$ is empty then the intersection $A_1 \cap A_2 \cap \dots \cap A_n$ is not empty. We will also use the KKM theorem in the mapping form:

Theorem 5.2 (The mapping KKM theorem)**.** *Assume $f : \Delta^{n-1} \to \Delta^{n-1}$ is a continuous map such that for all i we have $f(\Delta_i^{n-1}) \subset \Delta_i^{n-1}$. Then f is surjective.*

Proof. Let us approximate f with a PL map having the same property that any facet (and hence any face of arbitrary dimension) is mapped to itself. Considering Δ as a PL manifold with boundary we notice that f takes the boundary to the boundary. Therefore the mapping degree of f is well defined and is equal to the mapping degree of its restriction $f|_{\partial\Delta}$.

Then we prove by induction on the dimension that the mapping degree of f equals 1. The case of dimension $n = 1$ is clear. For the induction step we note $f|_{\Delta_i}$ satisfies the same assumptions and hence we assume its degree equals 1. But this is the same as the degree of $f|_{\partial\Delta}$, which in turn equals the degree of f. □

Reduction of the classical KKM to its mapping version. Replace A_i by a continuous function $g_i : \Delta \to \mathbb{R}$, such that $g_i(A_i) = 1$ and $g_i(x) = 0$ for x outside an ε-neighborhood of A_i. When $\varepsilon > 0$ is sufficiently small, we will have $g_i(\Delta_i) = 0$ from the assumption $\Delta_i \cap A_i = \emptyset$.

Since the A_i cover the simplex, we conclude that $g_1(x) + \cdots + g_n(x) > 0$ for every $x \in \Delta$. Dividing every g_i by this sum, we obtain non-negative continuous functions $f_1, \ldots, f_n$ with unit sum everywhere in the simplex. Such f_i are coordinates of a map

$$f : \Delta \to \Delta,$$

and the property $f_i(\Delta_i) = 0$ means that any facet goes to itself. Hence by the mapping KKM theorem f is surjective and therefore there exists $x \in \Delta$ such that $f_i(x) = 1/n$ for any i. Such a point x is in the ε-neighborhood of each A_i. Passing to the limit $\varepsilon \to 0$ and using compactness of Δ and closedness of the A_i yields the result. □

Now we proceed to a generalization of the KKM theorem, useful in proving existence of equilibria in economic questions.

Theorem 5.3 (Gale's theorem). *Let A_{ij} be closed subsets of Δ^{n-1}, indexed by $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Assume that for every fixed j the sets $\{A_{ij}\}_{i=1}^n$ cover the simplex, and $A_{ij} \cap \Delta_i^{n-1}$ is empty for every i and j.*

Then there exists a permutation σ of size n such that the intersection $\bigcap_i A_{i\sigma(i)}$ is not empty.

Proof. We essentially reproduce the (sketch of the) proof in [34], giving more details. Replace each set A_{ij} by a function g_{ij}. Using the covering assumption, we may normalize g_{ij} to obtain f_{ij} such that

$$f_{1j} + \cdots + f_{nj} = 1$$

at any point of the simplex and any j, and also $f_{ij}(\Delta_i) = 0$. Now introduce non-negative functions

$$h_i = \frac{f_{i1} + \cdots + f_{in}}{n},$$

which still satisfy $h_1 + \cdots + h_n = 1$ everywhere in the simplex, and $h_i(\Delta_i) = 0$. Hence there appears a continuous map $h : \Delta \to \Delta$ sending each facet to itself and by the mapping KKM theorem we conclude that there exists $x \in \Delta$ such that $h_i(x) = 1/n$ for every i.

Evaluating our original matrix of functions f_{ij} at the point x, we conclude that

$$\sum_i f_{ij}(x) = 1, \quad \sum_j f_{ij}(x) = 1.$$

This matrix is doubly stochastic and the Birkhoff–von Neumann theorem [19] asserts that this matrix is a convex combination of permutation matrices. In particular, there exists a permutation σ such that $f_{i\sigma(i)}(x) > 0$ for every i; alternatively, this can also be deduced with a little effort from Hall's marriage theorem [46]. Going to the limits and using the compactness, we again obtain $\bigcap_i A_{i\sigma(i)} \neq \emptyset$. $\square$

For far-reaching generalizations of these theorems, see [72]. Theorem 3.1 there provides a Gale-type theorem corresponding to homotopy classes of maps from topological spaces to spheres, of which the degree of a map between spheres of equal dimensions is a particular case.

The economic meaning of Gale's theorem is as follows. The simplex Δ^{n-1} (sometimes) parametrizes partitions of a certain resource into n parts, the set A_{ij} corresponds to the partitions where the player j would be satisfied to take the ith part of the resource and leave the rest to the other players. The other assumptions of the theorem mean that in every partition every player would be satisfied with some part, and nobody will be satisfied to take the empty part with $t_i = 0$. The conclusion of the theorem then means that there exists a partition and an assignment σ of the parts to the players such that every player will be satisfied.

5.3 When some players may choose nothing

5.3.1 Assume that some parts may be dropped

What happens when $A_{ij} \cap \Delta_i$ is non-empty in Gale's theorem, or, in economic terms, if some players sometimes prefer to take nothing from the resource partition? This question was left as an exercise to the reader in [71, middle of page 3], let us perform this exercise here.

We may obtain a result about this by adjusting the situation to the assumption of Gale's theorem. Let us remove from A_{ij} the part where $t_i < \varepsilon$. This will satisfy the assumption $A_{ij} \cap \Delta_i = \emptyset$ of Gale's theorem, but will break the assumption that $\{A_{ij}\}_{i=1}^n$ cover the simplex for every j.

In order to fix the covering assumption, given j, let us add $t \in \Delta$, which did not belong to any A_{ij}, to $A_{i_{max}j}$ where $t_{i_{max}}$ is a maximal coordinate of the point t, there may be several maximal coordinates. Such a modification of A_{ij} keeps the assumption that the coordinate t_i is no smaller than ε on A_{ij}.

Now apply Gale's theorem to the modified sets to obtain a permutation σ and a point $x_\varepsilon \in \bigcap A_{i\sigma(i)}$. If all the coordinates of x_ε are greater than ε then we are in the range where we did not modify anything and the problem is solved.

Otherwise there exist coordinates of x_ε that are at most ε. In this case we are going to the limit $\varepsilon \to +0$, from the compactness we may assume that $x_\varepsilon \to x$ and the permutation is all the time the same. In the coordinates $x_1, \dots, x_n$ of the limit configuration some coordinates x_i will then be zero, otherwise we are in the first case.

In this limit configuration, speaking in economic terms, some player $j = \sigma(i)$ may be dissatisfied with the assignment of the part i to her/him. But this may only happen in the situation when this player preferred parts with some $t_{i'} < \varepsilon$ in the neighborhood of x, we may assume i' fixed here. By the closedness of the preference set $A_{i'j}$ we obtain that $x_{i'} = 0$ for the limit point x and that the player j does prefer the emptyset in the partition x.

Now we conclude:

Corollary 5.4. *Under the assumptions of Gale's theorem, modified so that some players may sometimes prefer nothing, it is possible to find a partition, assign some parts to the players, drop some unwanted parts, and assign nothing to some of the players, so that all players will be satisfied.*

5.3.2 General observations when no part may be dropped

In our argument it is crucial that whenever the player is satisfied with the part i such that $t_i = 0$, he/she will also be satisfied with any other part i' such that $t_{i'} = 0$. In other words, there is only one sort of "nothing".

Now we return to the setting when it is not allowed to drop parts in a partition. Let us explain why any economic problem of KKM–Gale type is roughly equivalent to the study of continuous maps $f : \Delta^{n-1} \to \Delta^{n-1}$. We will always use the covering assumption, in economic terms, in every partition any player is satisfied with some of the parts.

In one direction, we start from the preference sets A_{ij} and pass to functions f_{ij}, as in the proof of Theorem 5.3 above. If certain assumptions on A_{ij} imply certain other assumptions on f_{ij} that, in turn, allow us to conclude that the map hits the center of the simplex, then we are done by essentially the same argument.

In the other direction, having a continuous map $f : \Delta^{n-1} \to \Delta^{n-1}$, we put

$$A_{ij} = \left\{t \in \Delta^{n-1} \mid \forall i' \; f_i(t) \geq f_{i'}(t)\right\}.$$

This definition does not depend on j, that is the players have precisely the same preference, hence we put $A_i = A_{ij}$. The family of closed sets $A_1, \dots, A_n$ covers the simplex. Note that in the case, when all the players have the same preference, the setting of Gale's theorem degenerates to the setting of the KKM theorem. Now we observe that the A_i have a common point if and only if

$$f_1(t) = \cdots = f_n(t) = \frac{1}{n}$$

for some t.

Since it is easy to build a continuous map $f : \Delta^{n-1} \to \Delta^{n-1}$ missing the center of the simplex, it is now clear that in order to have a Gale-type theorem, we need some assumption like "no player is satisfied with an empty part". Here we give a very explicit example:

Example 5.5. One may ask if it is sufficient to have the assumption "if somebody prefers nothing then he/she does not care on which position this nothing occurs" and prove a KKM–Gale-type theorem, without using any equivariance assumptions or other similar assumptions. This is not the case already for the KKM theorem. Take the triangle Δ^2 and put

$$A_1 = \Delta^2, \quad A_2 = \{t_1 = t_2 = 0\}, \quad A_3 = \{t_1 = t_3 = 0\}.$$

In economic terms, in all cases the player prefers part 1. When parts 1 and 2 are empty, the player also prefers part 2. When parts 1 and 3 are empty, the player also prefers part 3. But there is no configuration where the player prefers all three parts; or in case of Gale's theorem, where the preferences of three identical players are met.

5.3.3 Using permutation equivariance

One possible way is to introduce an assumption of "equivariance on the boundary" with respect to the action of the permutation group $\mathfrak{S}_n$ on the simplex Δ^{n-1} by permuting the coordinates. For example, in Gale's theorem we may require

$$\sigma A_{ij} \cap \partial\Delta^{n-1} = A_{\sigma(i)j} \cap \partial\Delta^{n-1}, \quad \forall i, j.$$

In economic terms this means that when a partition has empty parts (the boundary of the simplex) and the parts of a partition are permuted, then the players trace the parts they prefer and continue preferring them. When a partition has n non-empty parts, then the players may take the order into account. Perhaps, the economic formulation here is not very natural, but it may serve to us as a mathematically natural example, which we can handle. Here we give a positive result for this setting:

Theorem 5.6. *The KKM theorem and Gale's theorem are valid when it is allowed to choose empty parts if we impose the "equivariance on the boundary" assumption and also assume that n is a prime power.*

Proof. Recall, that in the reduction of the classical KKM to its mapping version, we used the mapping version to prove that the map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ hits the center of the simplex.

Assumptions of this theorem mean that f is $\mathfrak{S}_n$-equivariant on the boundary $\partial\Delta^{n-1}$ and n is a prime power.

Assume that the center of the simplex is not in the image of f. Then the composition of f with the central projection gives us a map $g : \Delta^{n-1} \to \partial\Delta^{n-1}$ which is also $\mathfrak{S}_n$-equivariant on the boundary $\partial\Delta^{n-1}$. The restriction of g to the boundary has degree zero, since it can be extended to the whole simplex. But this is impossible by Theorem 2.8(a)(b). $\square$

5.4 A segment partition problem with choosing nothing

One particular setting, which we borrow from [86; 71], is when a point $(t_1, \dots, t_n) \in \Delta^{n-1}$ is interpreted as a partition of a unit segment, in this case different points of the simplex in fact give the same partition. More precisely, in the vector $(t_1, \dots, t_n)$ we may move zero coordinates of this vector to any position, only keeping the order of positive coordinates, the actual partition of the segment will be the same. Hence the preferences of the players have to follow these permutations, which gives us a modification of the equivariance assumptions.

5.4.1 Pseudo-equivariance assumptions

Now it is natural to introduce *the segment partition problem with the possibility of choosing nothing* so that preferences are in accordance with the above described identifications. Those identifications can be described by identifying the proper faces of Δ^{n-1} by linear maps. Those maps $\sigma_{FGZ} : F \to G$ may be viewed as permutations of the coordinates $\sigma_{FGZ} : \Delta^{n-1} \to \Delta^{n-1}$ of the simplex, that move the nonzero coordinates of a face F to the nonzero coordinates of another face G preserving their order, and move the zero coordinates of a face F to zero coordinates of a face G with an arbitrary bijection, which we denote by Z. In particular, for given F and G of dimension k there are $(n-k-1)!$ bijections Z. The possibility to permute the zero coordinates arises because those permutations do not change the actual partition of the segment.

We also assume that a player is not allowed to take nothing in the presence of n non-empty parts, otherwise we would have to drop a part, as we did in the previous section. This keeps the covering property

$$\Delta^{n-1} = \bigcup_i A_{ij}, \quad \forall j$$

and allows, as in the proof of Theorem 5.3, to pass to the continuous map $f : \Delta^{n-1} \to \Delta^{n-1}$ setting. In terms of the continuous map, we then have the restrictions

$$f \circ \sigma_{FGZ} = \sigma_{FGZ} \circ f \quad \text{valid on the face} \quad F. \tag{5.1}$$

Let us clarify these relation. For given F, G, Z this relation is only applied to points $x \in F \subset \Delta^{n-1}$. The image $\sigma_{FGZ}(x)$ on the left hand side then belongs to G, and then f applies to it. On the right hand side we first apply f to x to obtain a point in the simplex that need not belong to any specific facet; after that we apply σ_{FGZ} defined as a permutation, taking its Z part into account.

Note that this setting resembles a certain equivariance assumption on the map f, at least on the boundary of Δ^{n-1}. But this is not quite that, because the permutations

σ_{FGZ} do not constitute a group and the commutation restrictions (5.1) are only applied for points lying on the facet F. For briefness, let us call a continuous $f : \Delta^{n-1} \to \Delta^{n-1}$ satisfying the commutation restrictions (5.1) *pseudo-equivariant.*

Of course, we need to explain, how to pass from sets to continuous functions in the pseudo-equivariant case. Relations (5.1) in terms of closed sets A_{ij} read

$$\sigma_{FGZ}\left(A_{ij} \cap F\right) = A_{\sigma_{FGZ}(i)j} \cap G, \tag{5.2}$$

which assumes the form (5.1), when we pass from the closed sets A_{ij} to their upper semicontinuous indicator functions $\chi_{ij} = \chi_{A_{ij}}$. If we approximate the indicator functions by continuous functions without due caution, the assumptions (5.1) may fail at a point x in a face F, because during the approximation of the χ_{ij} by continuous functions f_{ij} the values $f_{ij}(\sigma_{FGZ}(x))$ may be influenced by nearby points not belonging to F and not subject to the relation (5.1).

In order to pass to continuous functions correctly, we put our Δ into a slightly enlarged concentric simplex $\widetilde{\Delta}$, and first extend the upper semicontinuous indicator functions χ_{ij} to $\widetilde{\Delta}$ by composing them with the metric projection $\pi : \widetilde{\Delta} \to \Delta$, $\chi_{\widetilde{A}_{ij}} = \chi_{ij} \circ \pi$. This does not affect the existence of solutions for the partition problem, but allows us to conclude that (5.1) will now hold not only on a face $\widetilde{F} \subset \widetilde{\Delta}$, but also in some ε-neighborhood of $\widetilde{F}$, for some $\varepsilon > 0$, because the new $\widetilde{F}$ projects to the corresponding original F along with its neighborhood. After that we choose a single $\varepsilon > 0$ for all faces, take continuous functions

$$g_{ij}(x) = \max\left\{1 - \frac{\operatorname{dist}(x, \widetilde{A}_{ij})}{\varepsilon}, 0\right\},$$

and then normalize

$$f_{ij}(x) = \frac{g_{ij}(x)}{\sum_{i'} g_{i'j}(x)}.$$

The relations (5.1) will hold for such functions on respective faces of $\widetilde{\Delta}$, since they only depend on the behavior of $\widetilde{A}_{ij}$ in the ε-neighborhood of x.

5.4.2 A positive solution when n is a prime power

The arguments in the previous section reduce the segment partition problem with the possibility of choosing nothing to proving that a pseudo-equivariant map $f : \Delta^{n-1} \to \Delta^{n-1}$ sends some point to the center of the simplex.

Theorem 5.7. *When $n = p^\alpha$, for a prime p, any pseudo-equivariant map $f : \Delta^{n-1} \to \Delta^{n-1}$ in the sense of* (5.1) *hits the center $c \in \Delta^{n-1}$.*

Proof. We fix $n = p^\alpha$ and omit it from the notation where appropriate. In order to prove what we need, it is sufficient to show that $f(\partial\Delta)$ either has nonzero linking number with the center of Δ, or touches the center. If it touches the center then the problem is solved; hence assume that the center is not touched by $f(\partial\Delta)$ and study the linking number.

In order to have information about the linking number we start with the identity map $f_0 : \Delta \to \Delta$, which is pseudo-equivariant and has the linking number of $f(\partial\Delta)$ with the center equal to 1. It then remains to show that once we deform this f_0 to arbitrary f_1 pseudo-equivariantly, the linking number may only change by a multiple of p, thus remaining always nonzero.

The linking number changes when a point in the boundary $x \in \partial\Delta$ passes through the center c under a pseudo-equivariant homotopy h_t with parameter t. If x lies in the relative interior of a k-dimensional face F of Δ then we may apply the relations (5.1) to x with different G and Z. Those relations show that in total $\binom{n}{k+1}$ images $h_t(\sigma_{FG}(x))$ pass through c together with x. Let us call the points $\sigma_{FGZ}(x)$ for different G of dimension k (they do not depend on Z) the *pseudo-orbit* of x.

The change in the linking number corresponds to the sum of mapping degrees of the homotopy

$$h : \partial\Delta \times [0, 1] \to \Delta$$

at the points of $h^{-1}(0)$. To make the argument correct, we may assume h piece-wise linear and perturb it generically, keeping the pseudo-equivariance conditions. For any point x in the relative interior of a face F, the relations (5.1) restrict the image $h(x, t)$ to the linear span of F ("linear" in the sense that we put the origin to the center of Δ), which has dimension no less than $F \times [0, 1]$. Hence, exactly as in the proof of Lemma 2.12, a generic pseudo-equivariant PL map h has the property that the preimage of the center under h is a discrete point set, consisting of several pseudo-orbits; and the local mapping degrees are correctly defined.

If we had an equivariance for h under a group action making this pseudo-orbit a real orbit, and permuting their neighborhoods in $\partial\Delta$ accordingly, then we would have that the change in the linking number equals $\binom{n}{k+1}$ times an integer, which would do the job since such a binomial coefficient is divisible by p when $n = p^\alpha$. But we only have pseudo-equivariance in (5.1), whose equations with σ_{FGZ} are only applied on the respective face F.

In order to use the pseudo-equivariance correctly, we notice that any point of the considered pseudo-orbit belongs to $n-k-1$ facets of Δ and its disk neighborhood in $\partial\Delta$ splits into $n-k-1$ parts. Some of those parts of neighborhoods of the points in the pseudo-orbits are identified by the maps $\sigma_{\Delta_i\Delta_j}$, corresponding to pairs of facets (the bijection Z in this case is always unique). Since we have n facets in total, we in fact split the parts of neighborhoods of the pseudo-orbit to identified n-tuples.

We may calculate the sum of mapping degrees of h over the pseudo-orbit (or over all points mapped to the center of Δ) by choosing a radially symmetric differential form $\nu \in \Omega^{n-1}(\Delta)$ supported near the center of Δ with unit integral and integrating its pull-back over the neighborhoods of our pseudo-orbit points. The integration is possible, since we consider a piece-wise linear h. We essentially use the mapping degree formula (see [41, page 188], for example)

$$\int_{\partial\Delta\times[0,1]} h^*\nu = (\deg h)\int_\Delta \nu = \deg h,$$

taking in account that the image of the boundary of $\partial\Delta \times [0,1]$ does not hit the support of ν, the neighborhood of the center of Δ. From the assumption that the piece-wise linear map h is in general position, the integral on the left hand side is in fact the integral over neighborhoods of points in the preimage of the center of Δ, if we choose the support of ν sufficiently small. Hence we assume that we are now studying one pseudo-orbit of such points and integrate over a union of their neighborhoods, split into parts, in order to estimate the corresponding part of the mapping degree of h.

Once we split the neighborhoods into parts according to the facets of $\partial\Delta$, we may integrate $h^*\nu$ over every part P of a neighborhood of a point in the pseudo-orbit to obtain a *partial mapping degree* of P,

$$\deg_P h = \int_P h^*\nu.$$

Here we assume that the parts of neighborhoods P are oriented according to the orientation of $\partial\Delta$. Then the sum over all parts of neighborhoods will be the degree of h in the neighborhood of the pseudo-orbit in question. Note that a partial mapping degree is a real number, not necessarily an integer. The identifications $\sigma_{\Delta_i\Delta_j}$ show that among the numbers $\deg_P h$ obtained by such integration some are equal, the whole collection of these partial mapping degrees in fact split into n-tuples of equal real numbers. Those equalities appear with no sign, since ν is radially symmetric and only changes its sign according to the sign of a permutation of coordinates, which occurs simultaneously in the domain, where the orientation of $\partial\Delta$ also changes according to the sign of the permutation, and in the image of h.

Another relation for the partial mapping degrees $\deg_P h$ is that the sum of partial mapping degrees over the parts of the neighborhood of every point in the pseudo-orbit is an integer, possibly depending on the point, the ordinary local mapping degree.

We want to use the two types of equalities described above and show that the sum of all partial mapping degrees for the pseudo-orbit in question is an integer divisible by p. After the summation over all pseudo-orbits going to the center of Δ under h, this will show that the full mapping degree of h is divisible by p and therefore the degree of $f|_{\partial\Delta}$ as a map from $\partial\Delta$ to $\Delta \setminus \{c\} \sim \partial\Delta$ is always 1 modulo p, as it is for the identity map f_0. From this we can conclude that f, as a map $\Delta \to \Delta$, always touches the center of the simplex.

Let us introduce some notation in order to work with partial mapping degrees and their sum. Consider a point x in the pseudo-orbit, describe its *kind* by the sequence $[y_1, \dots, y_{k+2}]$, where y_i is the number of zero coordinates between the $(i-1)$th and ith nonzero coordinates of x. More precisely, if $x_{i_1}, \dots, x_{i_{k+1}}$ are the nonzero coordinates of x then the kind of x is $[i_1 - 1, i_2 - i_1 - 1, \dots, i_{k+1} - i_k - 1, n - i_{k+1}]$. For example, the point $(0, x_2, 0, 0, x_5)$ will have the kind $[1, 2, 0]$. For any sequence $y_1, \dots, y_{k+2}$ of non-negative integers summing up to $n - k - 1$ there corresponds a unique point of kind $[y_1, \dots, y_{k+2}]$ in the pseudo-orbit of a given point x from a relative interior of a k-dimensional face of the simplex. Hence we may use the kinds to enumerate points in a pseudo-orbit.

Let P be a part of the neighborhood of a point of the kind $[y_1, \dots, y_{k+2}]$ in the facet given by $t_i = 0$. The ith coordinate of the point is 0 and there is some y_j to which it corresponds. Hence P is uniquely described by $[y_1, \dots, y_{k+2}]$ with sum $n - k - 1$ and the choice of the index j of the position of zero. We may view the points of P as $k+1$ big coordinates, $n - k - 2$ small coordinates (which were zero for original pseudo-orbit points in k-faces), and one zero. The sequence $[y_1, \dots, y_{j-1}, y_j - 1, y_{j+1}, \dots, y_{k+2}]$ then describes the positions of small coordinates among big coordinates and ignores zero. The identifications of n such parts of neighborhoods in a pseudo-orbit corresponds to inserting zero into arbitrary position of a given sequence of big and small coordinates; therefore it is natural to call $[y_1, \dots, y_{j-1}, y_j - 1, y_{j+1}, \dots, y_{k+2}]$ the *kind* of a pseudo-orbit of parts of neighborhoods. Then to each sequence $y_1, \dots, y_{k+2}$ of non-negative integers summing up to $n - k - 2$ there corresponds a unique part of neighborhood kind.

Moreover, we denote by $\deg[y_1, \dots, y_{j-1}, y_j - 1, y_{j+1}, \dots, y_{k+2}]$ the partial mapping degree of any part of a neighborhood of the given kind, this degree indeed only depends on the kind. In order to prove the theorem, we need to show that the sum of all such degrees, multiplied by n, is an integer divisible by p. We split this sum into several parts, for any integer $0 \le r \le n - k - 2$, put

$$S_r = \sum_{r+y_2+\cdots+y_{k+2}=n-k-2} \deg[r, y_2, \dots, y_{k+2}],$$

and put $S_{-1} = 0$ for consistency. What we need to prove then translates to

$$n \sum_{r=0}^{n-k-2} S_r \equiv 0 \mod p. \tag{5.3}$$

Summing up the partial mapping degrees in the neighborhood of the point of the kind $[y_1, \dots, y_{k+2}]$ we get

$$\sum_i y_i \deg[y_1, \dots, y_i - 1, \dots, y_{k+2}] \in \mathbb{Z}. \tag{5.4}$$

Summing up formulas of (5.4) for different kinds with $y_1 = r$ we get

$$rS_{r-1} + (n - r - 1)S_r \in \mathbb{Z}. \tag{5.5}$$

Indeed, each $\deg[r - 1, y_2, \dots, y_{k+2}]$ contributes with coefficient r in (5.4) for the neighborhood of the point of the kind $[r, y_2, \dots, y_{k+2}]$. And each $\deg[r, y_2, \dots, y_{k+2}]$ contributes with coefficient $y_2 + 1$ in (5.4) for the neighborhood of the point of the kind $[r, y_2 + 1, \dots, y_{k+2}]$, with the coefficient $y_3 + 1$ in (5.4) for the neighborhood of the point of the kind $[r, y_2, y_3 + 1, \dots, y_{k+2}]$, and so on. Its total contribution then is

$$(y_2 + 1) + \cdots + (y_{k+2} + 1),$$

which is equal to $n - k - 2 - r + (k + 1) = n - r - 1$.

Let us prove by induction that

$$(r + 1)\binom{n-1}{r+1} S_r \in \mathbb{Z}. \tag{5.6}$$

The base $r = 0$ of induction follows from (5.5) with $r = 0$. Suppose we have proved (5.6) for some r. Writing (5.5) for $r + 1$, we get

$$(r + 1)S_r + (n - r - 2)S_{r+1} \in \mathbb{Z}.$$

Multiply by $\binom{n-1}{r+1}$ to get

$$(r + 1)\binom{n-1}{r+1} S_r + (n - r - 2)\binom{n-1}{r+1} S_{r+1} \in \mathbb{Z}.$$

By the induction assumption, we have

$$(n - r - 2)\binom{n-1}{r+1} S_{r+1} \in \mathbb{Z}.$$

Substituting $\binom{n-1}{r+1} = \frac{r+2}{n-r-2}\binom{n-1}{r+2}$, we get the desired result

$$(r + 2)\binom{n-1}{r+2} S_{r+1} \in \mathbb{Z}.$$

Since $n = p^\alpha$ is a prime power, then all digits of $n - 1$ in p-adic notation are $p - 1$. Hence, by the Lucas theorem [58] we get that $\binom{n-1}{r+1}$ is not divisible by p. This means that $(r + 1)\binom{n-1}{r+1}$ is not divisible by p^α for all $0 \le r \le n - k - 2$, since r is not divisible by p^α. Therefore, the least common multiple m of the numbers $(r + 1)\binom{n-1}{r+1}$ for all $0 \le r \le n - k - 2$ is also not divisible by p^α.

From (5.6) we conclude that

$$m \sum_r S_r \in \mathbb{Z}.$$

For each kind of a neighborhood there are exactly n partial neighborhoods of this kind, so we also know that

$$n \sum_r S_r = p^\alpha \sum_r S_r \in \mathbb{Z}.$$

Hence, $n \sum_r S_r$ is divisible by $\frac{n}{\gcd(n,m)}$, which in turn is divisible by p, because m is not divisible by $n = p^\alpha$. This establishes (5.3) and completes the proof. $\square$

5.4.3 Counterexamples when n is not a prime power

As it was shown above, in order to build a counterexample, where the segment partition problem with possibility to choose nothing and no part can be dropped has no solution, it is sufficient to build a pseudo-equivariant map $f : \Delta^{n-1} \to \Delta^{n-1}$ missing the center $c \in \Delta^{n-1}$ and put

$$A_{ij} = \left\{t \in \Delta^{n-1} \mid \forall i' \ f_i(t) \geq f_{i'}(t)\right\}$$

independent on the player index j.

The first observation is that it is sufficient to have a pseudo-equivariant map f such that the image of the boundary $f(\partial\Delta^{n-1})$ is not linked with the center $c \in \Delta^{n-1}$. Since the homotopy group $\pi_{n-2}(\Delta^{n-1} \setminus \{c\})$ is $\mathbb{Z}$, the possibility to (re)extend f continuously to the interior of the simplex Δ^{n-1} is fully governed by the linking number and any such continuous extension does not violate the pseudo-equivariance relations (5.1), because the relations are only applicable on the boundary of the simplex.

The second observation is that it is sufficient to find a continuous map $f : \partial\Delta^{n-1} \to \Delta^{n-1}$ having zero linking number of the image with the center of the simplex and equivariant with respect to the action of the full permutation group $\mathfrak{S}_n$. The full equivariance on the boundary implies the pseudo-equivariance we need, and a continuous extension of f to the interior of the simplex is possible provided the linking number is zero.

In what follows we will switch between the two points of view: To find $f : \partial\Delta^{n-1} \to \Delta^{n-1}$ with zero linking number with the center is the same as to find $f : \partial\Delta^{n-1} \to \partial\Delta^{n-1}$ with zero mapping degree. In order to see these are the same just compose f with a central projection from the center of the simplex to have its image contained in the boundary of the simplex; and note that such a projection preserves equivariance and pseudo-equivariance.

The following is a reformulation of Theorem 2.8(c). Here we give an alternative proof, since we use the same strategy later in the proof of Theorem 5.9.

Theorem 5.8. *If n is odd and not a prime power then there exists an $\mathfrak{S}_n$-equivariant continuous $f : \partial\Delta^{n-1} \to \Delta^{n-1}$ of zero linking number with the center of Δ^{n-1}.*

Proof. We fix n and omit n from the notation where appropriate. We will start with the identity $f_0 : \partial\Delta \to \partial\Delta$, considered also as the inclusion $\partial\Delta \to \Delta$. It definitely has degree 1 and we are going to modify it equivariantly so that its mapping degree will become 0.

A modification will consist in taking a dimension k, all the centers of the k-dimensional simplices $c_1, \dots, c_N$, $N = \binom{n}{k+1}$, and pulling the images $f(c_i)$ to the center of Δ (along with pulling their neighborhoods continuously and equivariantly). When the images $f(c_i)$ cross the origin, the linking number of $f(\partial\Delta)$ will change by either $+1$ or -1 at every point, and by $\pm\binom{n}{k+1}$ in total.

Of course, in such a modification the sign $+$ or $-$, at first glance, is fixed. But we may not only pull a point c_1 towards the origin, but also flip the mapping derivative image of the tangent space $T_{c_1}F$ to the k-face F containing c_1 on the way. Such a flip commutes with the stabilizer of c_i in the permutation group and can therefore be extended equivariantly to the neighborhood of the orbit $\{c_i\}$. Moreover, when k is odd, this flip will change the sign of the crossing and therefore we will be able to choose the sign of the modification by applying or not applying the flip before the crossing. See the details of this pulling and flipping moves, for $n = 3$, in Figures 5.1 and 5.2.

When k is even, the flip does not change the sign of the crossing, hence we are only able to make one crossing, and when we pull the point c_1 (and equivariantly its orbit) back through the center of Δ, we just make the opposite crossing and return to where we started from in terms of the linking number. When k is odd, we have much more freedom. We may pull the images $f(c_i)$ and their neighborhoods to the center $c \in \Delta$ once again and once again choose the sign of the crossing using or not using the equivariant flip before the crossing. In total, for odd k, this allows us to change the linking number by any multiple of $\binom{n}{k+1}$, positive or negative. Figure 5.3 shows how to make two successive changes of the linking number in the same direction.

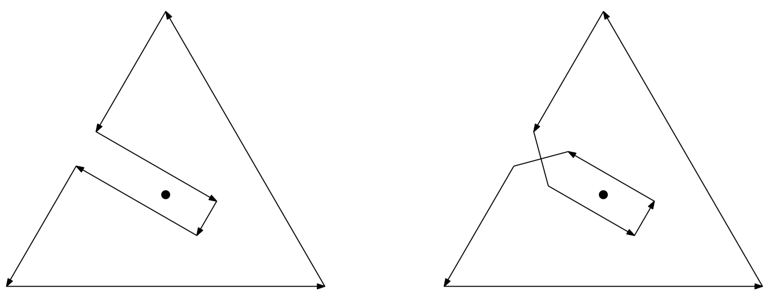

Figure 5.1: Pulling one point towards the center with/without a flip of signs.

Recall Ram's theorem [79] (or the Lucas theorem [58] that we have already used) that asserts that there exist integers $x_1, \dots, x_{n-1}$ such that

$$x_1\binom{n}{1} + x_2\binom{n}{2} + \cdots + x_{n-1}\binom{n}{n-1} = -1,$$

provided n is not a prime power. Note that in our case n is not a prime power.

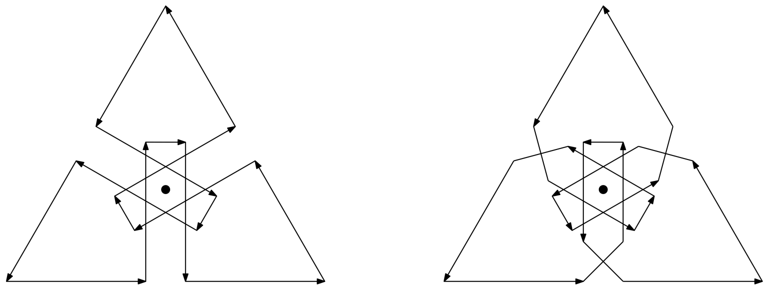

Figure 5.2: Pulling an orbit of points towards the center.

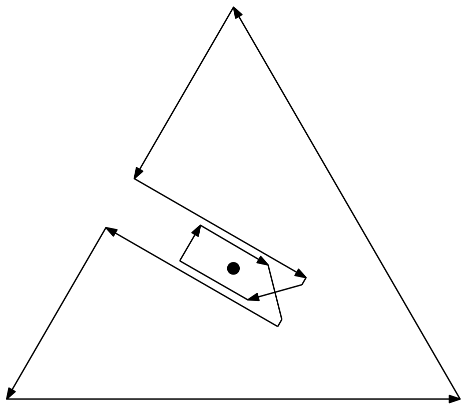

Figure 5.3: Pulling a point towards the center and then pulling it back with a flip. The other points in the orbit are not shown.

Moreover, n is odd and therefore, in view of the symmetry $\binom{n}{k+1} = \binom{n}{n-k-1}$, the set of the binomial coefficients is the same as the set of binomial coefficients with even $k+1$. Hence, if we repeatedly use our moves for odd k with possible flips then by Ram's theorem we will be able to modify the linking number of $f(\partial\Delta)$ with c from 1 to zero. $\square$

It remains to handle the case of even n, but this is less easy - Theorem 5.9 fails for n twice a prime power, as shown in Theorem 2.8(c). In the above argument we cannot change the crossing sign for even k and $n-k-2$, in particular, we can add or subtract $\binom{n}{k+1}$ from the linking number, but cannot repeat this operation, since when we move the orbit back to the center of Δ, we just change the linking number back. A flip was really needed in order to have a chance to repeat the change by $\pm\binom{n}{k+1}$ several times in the same direction.

What we are able to do now, is to do this in the setting of pseudo-equivariance instead of full equivariance. The following result shows that the segment partition problem with the possibility of choosing nothing has no solution if n is not a prime power.

Theorem 5.9. *If n is not a prime power then there exists a pseudo-equivariant, in terms of relations* (5.1)*, continuous $f : \partial\Delta^{n-1} \to \Delta^{n-1}$ of zero linking number with the center of Δ^{n-1}.*

Proof. We do the same modifications as in the previous proof, but we need to handle the case of even k. In view of the relations $\binom{n}{k+1} = \binom{n}{n-k-1}$ we may also assume that $k \geq n/2 - 1 \geq 2$.

Note that, for a k-face F, any composition of the pseudo-equivariance symmetries $\sigma_{F'G'Z}$ with $F' \supseteq F$ cannot take the face F to itself and induce a non-identity map on it, because all such symmetries preserve the order of the nonzero coordinates. Hence we can choose a direction $v_1 \in T_{c_1}F$ (because we only consider faces of positive dimension) in any point c_1 in the relative interior of F and we will have the well-defined defined pseudo-orbit $\{c_i\}$ of this point and this direction $v_i \in T_{c_i}F_i$, so that the pseudo-equivariance symmetries permute those points and those directions whenever they are defined on them.

Now we modify the original identity map f_0, we pull the images of the pseudo-orbit $f(c_i)$ towards the center c of Δ and on the way to the center we flip the tangent space $f_*(T_{c_1}F_1)$ along the chosen direction f_*v_1, if we need to switch the sign of the crossing. The corresponding flips around every point of the pseudo-orbit $\{f(c_i)\}$ will be made in the pseudo-equivariant fashion, in total allowing us to modify the linking number by $\pm\binom{n}{k+1}$ with a sign we choose.

It is possible to iterate such steps, moreover, in the absence of the true equivariance we are allowed to choose $c_1 \in F$ different from the center of F, making every step independent of the other steps. Having the possibility to choose the sign and iterate, in view of Ram's theorem for non-prime power n, we can obtain zero linking number. $\square$

6 Convex fair partitions into an arbitrary number of pieces

6.1 Introduction

In [73] a very natural problem was posed: Given a positive integer m and a convex body K in the plane, cut it into m convex pieces of equal areas and perimeters. Here we do not discuss any algorithm to provide such a cut, we only concentrate on the existence result.

The case $m = 2$ of the problem is proved using a simple continuity argument. The case $m = 2^k$ could be proved similarly using the Borsuk–Ulam-type lemma by Gromov [37] (see also [54]), which was used to prove another result, the waist theorem for the Gaussian measure (and the sphere). In [12] the case $m = 3$ was proved.

Further cases, $m = p^k$ for a prime p, were established in [51] and [22] independently (and a similar but weaker fact was established in [18; 99]). In both papers higher-dimensional analogues of the problem were stated and proved. Here we establish a new series of results:

Theorem 6.1. *Any convex body $K \subset \mathbb{R}^2$ can be partitioned into m parts of equal area and perimeter, for any integer $m \geq 2$.*

As in the previous work [51] and [22], a "perimeter" here may mean any continuous function of a convex body in the plane. More precisely, this real-valued function must be defined on convex bodies (convex compacta with non-empty interior) continuously in the Hausdorff metric; in particular, we never apply this function to degenerate convex compacta with empty interior.

An "area" may be measured with any finite absolutely continuous Borel measure with non-negative density in K; for a positive density the proof goes through literally and the non-negative density is obtained by a standard compactness argument.

In the rest of the chapter, we present the proof of the theorem. In Appendices 6.5 and 6.6 we present a higher-dimensional result that does not fully generalize the two-dimensional case, and an explanation of the difficulties of applying our tools to the true higher-dimensional generalization of the two-dimensional problem when m is not a prime power.

Compared to the previous work on this and similar problems, here we have found a way to go beyond the usual equivariant (co)homological argument that restricts the possible result to the prime power case. Our proof builds a solution recursively. To prove its validity we argue by induction and use a certain separation lemma that allows us to use standard homological arguments modulo different primes at different stages of the induction.

6.2 How the proof for $m = p^k$ works

We will essentially use the mechanism of the proof for the $m = p^k$ case; thus we recall the corresponding construction. Let $F_m(\mathbb{R}^2)$ be the configuration space of m-tuples $(x_1, \dots, x_m)$ of pairwise distinct points in the plane. To every such m-tuple we uniquely associate (following [4]) the weighted Voronoi partition of the plane,

$$\mathbb{R}^2 = V_1 \cup \dots \cup V_m,$$

with centers at $x_1, \dots, x_m$ such that the areas of the intersections $V_i \cap K$ are all equal. This can be done continuously in the configuration $F_m(\mathbb{R}^2)$. Then we produce the map (f is for perimeter here)

$$\sigma : F_m(\mathbb{R}^2) \to \mathbb{R}^m, \quad \sigma(x_1, \dots, x_m) \mapsto \left(f(V_1 \cap K), \dots, f(V_m \cap K)\right),$$

and then compose it with the quotient by the diagonal

$$\Delta = \{(t, t, \dots, t) : t \in \mathbb{R}\} \subset \mathbb{R}^m$$

to obtain

$$\tau : F_m(\mathbb{R}^2) \to \mathbb{R}^m/\Delta =: W_m.$$

The space W_m here can be interpreted as the $(m-1)$-dimensional irreducible representation of the permutation group $\mathfrak{S}_m$; with the natural action of $\mathfrak{S}_m$ of $F_m(\mathbb{R}^2)$ this map τ becomes $\mathfrak{S}_m$-equivariant.

The proof of the case $m = p^k$ is done by applying a Borsuk–Ulam-type theorem (essentially established in [106]) showing that any equivariant map $F_m(\mathbb{R}^2) \to W_m$ must hit the origin, this corresponds to equal perimeters in our partition. This is already good, but to move further we need more details.

For the case $m = 2^k$ (essentially in [37]) a subspace $T^{m-1} \subset F_m()$ was exhibited, this is a product of $m-1$ circles in one-to-one correspondence with all vertices of a binary tree with $m/2$ leaves. By choosing a sufficiently small $\varepsilon > 0$, for any leaf of this tree we consider the uniquely defined chain of its vertices (considered here as the corresponding unit vectors in the plane) $v_1, \dots, v_k$ from the root to this leaf, and add the two vectors

$$v_1 + \varepsilon v_2 + \dots + \varepsilon^{k-1} v_k, \quad v_1 + \varepsilon v_2 + \dots - \varepsilon^{k-1} v_k \tag{6.1}$$

to the list of points that totally makes m distinct points in the plane. This provides an embedding $T^{m-1} \subset F_m(\mathbb{R}^2)$, which is equivariant with respect to the action of $\mathfrak{S}_m^{(2)}$, the 2-Sylow subgroup of the permutation group, whose action on T^{m-1} is generated by sending $v_i \mapsto -v_i$ at one of the vertices and interchanging the two subtrees of this vertex. The map τ then restricts to a $\mathfrak{S}_m^{(2)}$-equivariant map

$$\tau_1 : T^{m-1} \to W_m.$$

It is easy to produce a particular case τ_0 of such an equivariant map by taking the first coordinates of the vectors in (6.1) and see that τ_0 is transverse to zero and $\tau_0^{-1}(0)$ consists of a single $\mathfrak{S}_m^{(2)}$ orbit (all vectors v_i pointing either either up or down). Hence τ_0^{-1} is nonzero in the 0-dimensional $\mathfrak{S}_m^{(2)}$-equivariant homology modulo 2 of T^{m-1}, and therefore for another equivariant map τ_1 transverse to zero the set $\tau_1^{-1}(0)$ is nonempty since it is homologous to τ_0^{-1} through $\chi^{-1}(0)$ for an appropriately perturbed equivariant homotopy

$$\chi : T^{m-1} \times [0,1] \to W_m, \quad \chi(t,s) = (1-s)\tau_0 + s\tau_1.$$

A more detailed explanation of this can be found in [54, Theorem 2.1].

For the odd prime power case, in [22] it was shown that there exists a polyhedron $P_m \subset F_m(\mathbb{R}^2)$ of dimension $m-1$, which is $\mathfrak{S}_m$-equivariantly homotopy equivalent to the whole $F_m(\mathbb{R}^2)$. This is useful because when we restrict the map to P_m, the solution set $\tau^{-1}(0)$ becomes generically (for a slightly equivariantly permuted τ) a finite set of points, so the proof can be interpreted as a statement about how many points are in $\tau^{-1}(0)$ for generic τ.

In [22, Section 4.1] it was shown that the top-dimensional and 1-codimensional cells of P_m can be oriented so that the action of $\mathfrak{S}_m$ changes the orientation according to the sign of the permutation, and with this orientation the top-dimensional cells produce a nontrivial cohomology class in the $\mathfrak{S}_m$-equivariant cohomology with coefficients in the sheaf $\pm\mathbb{Z}$, on which $\mathfrak{S}_m$ acts by its permutation sign.

In order to avoid orientation issues and twisted coefficients in a sheaf, we pass to the subgroup $G \subset \mathfrak{S}_m$ of even permutations. The above mentioned facts mean that the quotient P_m/G is a modulo p pseudomanifold, that is its two top-dimensional cells can be oriented so that the boundary of the corresponding chain is equal to zero modulo p. It is also shown in [22, Section 4.1] that a specially chosen test map $\tau_0 : P_m \to W_m$ has $\tau_0^{-1}(0)$ consisting of a single $\mathfrak{S}_m$-orbit, that corresponds to a pair of points in P_m/G with equal signs. The G-invariant orientations of P_m and W_m allow one to consider the solution points with signs.

Since P_m/G is a pseudomanifold modulo p, the quotient of the zero set $\tau^{-1}(0)/G$ of an equivariant map $\tau : T^{m-1} \to W_m$, considered as a 0-dimensional modulo p chain in P_m/G, changes homologously to itself (modulo p, since P_m/G is a modulo p pseudomanifold) when we change the map τ in a generic (transverse to zero) homotopy of equivariant maps. Since the test zero set $\tau_0^{-1}(0)$ has two points of P_m/G of the same sign, it is not homologous to a zero cycle, hence $\tau^{-1}(0)$ is generically a 0-cycle modulo p not homologous to zero and hence it is never empty.

It will be important for us to extract the following observation from the above exposition of the proof. If the problem (e.g. the set K) depends on a parameter $t \in [a, b]$ then we obtain a family of zero sets $\tau_t^{-1}(0)$. This can be naturally viewed as a preimage of zero under an $\mathfrak{S}_m$-equivariant map

$$\chi : P_m \times [a, b] \to W_m.$$

For a generic (transverse to zero) homotopy χ this preimage of zero $Z \subset P_m/G \times [a, b]$ will be a one-dimensional polyhedron. Since any continuous map can be approximated with a piece-wise linear map, the reader may assume all maps we consider piece-wise linear; in this case the transversality to zero is simply defined by the transversality to zero of the restriction of the map to any face of the triangulation of the polyhedron, for which the map is linear on faces of the triangulation.

From the orientation of W_m and the pseudomanifold modulo p structure of $P_m/G\times[a,b]$ it follows that Z is naturally a one-dimensional pseudomanifold modulo p with boundary. This means Z is an oriented graph with some vertices on $P_m/G \times \{a\}$ or $P_m/G \times \{b\}$, whose vertices lying in $P_m/G \times (a,b)$ have the number of incoming edges equal to the number of outgoing edges modulo p, similar to the structure of the pseudomanifold, where every condimension one face has zero modulo p attached top-dimensional faces counted with orientation. In particular, Z represents a modulo p cycle in $P_m/G \times [a,b]$ relative to $P_m/G \times \{a,b\}$, whose intersection with a generic subset $P_m/G \times \{t\}$ is a nontrivial 0-dimensional cycle modulo p. This is what we need to move further.

6.3 Proof for $m = 2p^k$

Let us start by considering the simplest particular case of our result, still exhibiting the main technique that we utilize. We consider an odd prime p and $m = 2p^k$. The full proof will be somewhat technical and is postponed to the next section.

Now we essentially use the last observation of the previous exposition of the case $m = p^k$. Take a parameter $t \in [0,\pi]$ and cut K by a straight line directed along $(\cos t, \sin t)$ into equal area halves, it is uniquely done given the direction t. K will be cut into L_t and M_t. Note that (see Figure 6.1)

$$L_\pi = M_0, \quad M_\pi = L_0. \tag{6.2}$$

Consider the problem of partitioning L_t and M_t into equal parts. They produce two families of problems with two solution sets $Z_L, Z_M \subset P_{p^k}/G \times [0,\pi]$. Perturbing the test map τ generically we make it transverse to zero and assume Z_L and Z_M are 1-cycles modulo p, as described above. Now for any solution, say, $(z,t) \in Z_L$ assign a point $(f(z),t) \in \mathbb{R} \times [0,\pi]$, where $f(z)$ is the common perimeter in the corresponding partition of L_t into m equal parts. This is a continuous map, hence we may view its image G_L as a 1-dimensional cycle modulo p in the strip $S = \mathbb{R} \times [0,\pi]$ relative to its boundary.

This means that generically G_L is an oriented graph drawing all of whose vertices in the interior of the strip have the number of incoming edges equal to the number of outgoing edges modulo p; and a generic vertical line $S_t = \mathbb{R} \times \{t\}$ intersects G_L a nonzero number of times modulo p, when counted with signs and multiplicities.

In a similar fashion we produce the 1-dimensional cycle G_M, coming from perimeters of the partitions of M_t. From (6.2) it follows that $G_L \cap S_0$ equals $G_M \cap S_\pi$ up to a horizontal shift and $G_M \cap \ell_0$ equals $G_L \cap \ell_\pi$ up to a horizontal shift, as zero-dimensional cycles. The crucial observation is:

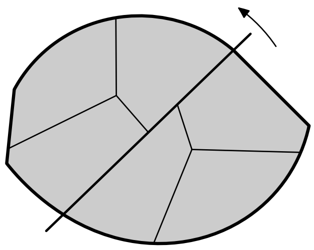

Figure 6.1

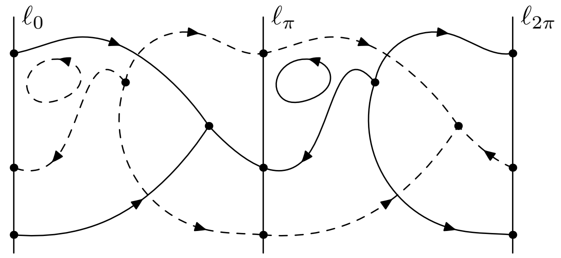

Figure 6.2

Lemma 6.2. *The assumptions described above on the cycles G_L and G_M guarantee that their supports intersect.*

The theorem follows from this lemma since a common point of the supports of G_L and G_M corresponds to a pair of partitions of L_t and M_t into m parts each such that the all areas in both partitions are equal to $\frac{\operatorname{area} K}{2m}$, and all the perimeters on both partitions coincide because the corresponding points of G_L and G_M are the same.

Proof of the Lemma. Double the strip S to have $C = \mathbb{R} \times [0, 2\pi]$, and consider it a cylinder by identifying $t = 0$ and $t = 2\pi$. Let R be the translation by π to the right modulo 2π, the half-rotation of the cylinder. The description of the boundary shows that the chains

$$G'_L = G_L + R(G_M), \quad G'_M = G_M + R(G_L)$$

are cycles modulo p with disjoint supports, intersecting a generic vertical line nonzero modulo p number of times, and such that $R(G'_L) = G'_M$ (see Figure 6.2 for an example of two such cycles, they are drawn with solid and dashed lines respectively). Those cycles are just full versions of the original cycles defined for arbitrary rotation angle t.

Since the modulo p cycle G'_L intersects a generic vertical line a nonzero modulo p number of times, it must also intersect any curve going from the infinite bottom to the infinite top of the cylinder by the homological invariance of the intersection (this curve is considered as a modulo p chain in the homology of the cylinder relative to its top and bottom). Hence G'_L splits the cylinder C into connected parts, one of which is infinite at the top and bounded at the bottom, call it A. The half-rotated $G'_M = R(G'_L)$ has the corresponding component of the complement $R(A)$. The strict inclusion $A \subset R(A)$ is impossible since it would follow that $R(A) \subset R(R(A)) = A$; the opposite strict inclusion $R(A) \subset A$ is also impossible. Since A and $R(A)$ evidently intersect somewhere at the top, this means that the boundaries of A and $R(A)$ must intersect, hence G'_L intersects G'_M.

□

6.4 Proof for arbitrary m

In our proof of the general case of Theorem 6.1, we are going to use induction, which also resembles the proof of a particular case of the Knaster problem by induction in [113].

First, for an odd prime p let P_p be the polyhedron used in the proof of [22]; it is a subset of the configuration space of distinct p-tuples of points in $\mathbb{R}^2$ and it therefore has a natural action of $\mathfrak{S}_p$, which we restrict to the group of even permutations $G_p \subset \mathfrak{S}_p$. The group G_p acts freely on P_p and preserves a certain orientation of its top-dimensional faces of dimension $p-1$, splitting those faces into two G_p-orbits. For $p = 2$, let P_2 be a circle with the antipodal action of $G_2 := \mathbb{Z}/2$. What we need is that, as in the proof of the prime power case, P_p will parameterize certain partitions of a planar convex body into p parts of equal area.

It turns out helpful to use the language of multivalued functions on the space of convex bodies $\mathcal{K}$. In fact, in our argument we will only use finite-dimensional subspaces of $\mathcal{K}$ build of the above-mentioned polyhedra P_p, hence we may always assume that $\mathcal{K}$ is a polyhendron, thus avoiding topological difficulties.

Definition 6.3. A *nice* multivalued function $\mathcal{K} \to (-1, 1)$ is determined by its closed graph in $\mathcal{K} \times [-1, 1]$, that is given by the equation

$$\varphi(C, y) = 0,$$

where $\varphi : \mathcal{K} \times [-1, 1] \to \mathbb{R}$ is a continuous single-valued function satisfying

$$\varphi(C, -1) < 0, \quad \varphi(C, 1) > 0$$

for all $C \in \mathcal{K}$.

Evidently, a nice multivalued function attains at least one value on every $C \in \mathcal{K}$, that is for every $C \in \mathcal{K}$ there exists y such that the pair (C, y) is on the graph of the multivalued function. This follows from the intermediate value theorem for the continuous function φ.

Here we restrict the values of a multivalued function to $(-1, 1)$, which in practice may be assumed after an appropriate scaling of its values, if the values were in a larger interval $(-L, L)$. Any continuous single-valued function $f : \mathcal{K} \to (-1, 1)$ may be considered as a nice multivalued function by putting

$$\varphi(C, y) = f(C) - y.$$

We will identify a nice multivalued function on $\mathcal{K}$ with the equation of its graph $\varphi(C, y)$, when we need to refer to this function by a name. Theorem 6.1 will follow from iterations of the following claim:

Lemma 6.4. *Assume φ is a nice multivalued function of $\mathcal{K}$ and p is a prime. Then there exists another nice multivalued function ψ of $\mathcal{K}$ such that whenever $C \in \mathcal{K}$ satisfies*

$$\psi(C, y) = 0$$

then there exists a partition $C = C_1 \cup \dots \cup C_p$ into convex bodies of equal area, such that

$$\varphi(C_1, y) = \dots = \varphi(C_p, y) = 0. \tag{6.3}$$

Proof of Theorem 6.1 assuming the lemma. Let φ_1 be the perimeter single-valued function. Apply the lemma to φ_1 and p_1 to obtain φ_2. Then apply the lemma to φ_2 and p_2 and so on, where $m = p_1 \cdots p_n$ is the decomposition of our given m into primes. The final function φ_{n+1} will be a nice mutlivalued function of a convex body.

From the intermediate value theorem, there exists $y \in (-1, 1)$ such that

$$\varphi_{n+1}(C, y) = 0$$

for the convex body C we are interested in. In means that C may be partitioned into p_n convex bodies of equal area and the same value y of the multivalued function φ_n. Each of these bodies may in turn be partitioned into p_{n-1} parts of equal area and the same value y of the multivalued function φ_{n-1}, and so on. Eventually, we obtain a partition of C into $m = p_1 \cdots p_n$ parts of equal area and the same value y of the multivalued function φ_1, which is in fact the single-valued perimeter function. $\square$

Proof of Lemma 6.4. Parametrize some of the partitions of $C \in \mathcal{K}$ into p convex parts of equal area with the space P_p, as in [22]. Equations (6.3) then define a closed subset $S \subset \mathcal{K} \times P_p \times [-1, 1]$.

The set S is G_p-invariant, where G_p acts on P_p as in [22] (P_p is a subset of the configuration space of p-tuples of pairwise distinct points in $\mathbb{R}^2$ and G_p permutes those points) and trivially acts on $\mathcal{K}$ and $[-1, 1]$. To be more precise, the set S is the preimage of zero under the G_p-equivariant continuous map

$$\Phi : \mathcal{K} \times P_p \times [-1, 1] \to \mathbb{R}^p, \quad \Phi(C, x, y) = (\varphi(C_1(x), y), \varphi(C_2(x), y), \dots, \varphi(C_p(x), y)),$$

where $C_i(x)$ denotes the ith part of the convex partition of C corresponding to the configuration $x \in P_p$. This map is G_p-equivariant if $\mathbb{R}^p$ is acted on by G_p by permutation of coordinates.

Fix a body C and study the structure of the fiber set

$$S_C = S \cap (\{C\} \times P_p \times [-1, 1]).$$

When the G_p-equivariant map

$$\Phi_C = \Phi|_{\{C\} \times P_p \times [-1,1]}$$

is transverse to zero, the solution set S_C is a finite number of points from the dimension considerations.

If we make a homotopy of Φ_C as a G_p-equivariant map with the boundary conditions on its components $\varphi(C_i, y)$ then the solution set S_C changes, but it changes in a definite way. If the homotopy $H : P_p \times [-1, 1] \times [0, 1] \to \mathbb{R}$ is transverse to zero (this can be achieved by a small perturbation) then $H^{-1}(0)$ represents a G_p-equivariant 1-dimensional cycle modulo p relative to $P_p \times [-1, 1] \times \{0, 1\}$. Indeed, under the transversality assumption

$H^{-1}(0)$ consists of smooth oriented segments in the top-dimensional faces of the domain $P_p \times [-1,1] \times [0,1]$ and isolated points of intersection with the 1-codimensional skeleton of the domain. Since the domain is a pseudomanifold, the segments are attached to every point 0 modulo p times, unless we are at the boundary pieces $P_p \times [-1,1] \times \{0\}$ and $P_p \times [-1,1] \times \{1\}$ of the domain, where the chain $H^{-1}(0)$ has the boundary modulo p coinciding with the zero sets of the initial $\Phi_C(\cdot,\cdot) = H(\cdot,\cdot,0)$ and the final $\Phi_C(\cdot,\cdot) = H(\cdot,\cdot,1)$. Hence the zero set of a transverse to zero Φ_C changes equivariantly homologously to itself under G_p-equivariant homotopies of the map Φ_C.

Let us present an instance of a transverse to zero map $\Phi_0 : P_p \times [-1,1] \to \mathbb{R}^p$ (a test map), which is G_p-equivariant and satisfies the boundary conditions that we impose on Φ_C, and for which the set $\Phi_0^{-1}(0)$ is homologically nontrivial. By the above homotopy consideration (connecting Φ_0 to Φ_C by convexly combining their coordinates), the existence of such a test map implies the homological nontriviality of S_C for any transverse to zero map Φ_C. In order to produce the needed test map, we may take the $\mathfrak{S}_p$-equivariant test map

$$\Psi_0 : P_p \to W_p$$

considered in [22]. Here it is convenient to consider $W_p \subset \mathbb{R}^p$ as the linear subspace of p-tuples with zero sum. The transverse preimage of zero $\Psi_0^{-1}(0)$ consists of the unique $\mathfrak{S}_p$-orbit of a point in the relative interior of a top-dimensional face of P_p. This solution set $\Psi_0^{-1}(0)$ is either a single G_p-orbit (for $p = 2$) or splits into two G_p-orbits (for odd p), but both with the same sign. This verifies the homological nontriviality of $\Psi_0^{-1}(0)$ as a 0-dimensional G_p-equivariant cycle.

We augment Ψ_0 to the map (assuming the coordinates of Ψ_0 are in the interval $(-1,1)$)

$$\Phi_0(x,y) = \Psi_0(x) + (y,\dots,y)\,.$$

Then $\Phi_0^{-1}(0) = \Psi_0^{-1}(0) \times \{0\}$ and this preimage is still a nontrivial G_p-equivariant 0-cycle modulo p. Hence, we obtain:

Claim 6.5. *For transverse to zero Φ_C, the set S_C is a nontrivial G_p-equivariant 0-cycle modulo p. Its projection to the segment $[-1,1]$ is a nontrivial 0-cycle modulo p.*

Note that the set S_C is always non-empty, since were it empty, the map Φ_C would be transverse to zero by definition and S_C would have to be non-empty by the claim. Assume now we change the convex body C in a continuous one-parameteric family $\{C(s) \mid s \in [a,b]\}$ and obtain a G_p-equivariant map with one more parameter

$$\widetilde{\Phi} : P_p \times [a,b] \times [-1,1] \to \mathbb{R}^p, \quad \widetilde{\Phi}(x,s,y) = (\varphi(C_1(x,s),y), \varphi(C_2(x,s),y), \dots, \varphi(C_1(x,s),y)),$$

where $C_i(x,s)$ is the ith part of the partition of $C(s)$ corresponding to P_p.

The solution set $\widetilde{\Phi}^{-1}(0)$ now generically (when $\widetilde{\Phi}$ is transverse to zero) represents a G_p-equivariant 1-dimensional cycle modulo p relative to $P_p \times \{a,b\} \times [-1,1]$. As in the above argument, under the transversality assumption $\widetilde{\Phi}^{-1}(0)$ consists of smooth oriented segments in the top-dimensional faces of the domain $P_p \times [a,b] \times [-1,1]$ and isolated points of intersection with the 1-codimensional skeleton of the domain; since the domain is a pseudomanifold, the segments are attached to every point 0 modulo p times, unless $s = a$ or $s = b$.

Projecting $\widetilde{\Phi}^{-1}(0)$ to the rectangle $[a,b]\times[-1,1]$ (every G_p-orbit goes to a single point), we get a 1-dimensional cycle modulo p relative to $\{a,b\} \times [-1,1]$, intersecting a generic line $s = c$ nontrivially modulo p by the previous claim, since this is the solution set of a generic problem without a parameter. It is crucial that any curve connecting the bottom $[a,b] \times \{-1\}$ to the top $[a,b] \times \{1\}$ of the rectangle is homologous to such a line, and it must intersect the cycle by the homological invariance of the intersection number. Hence we obtain:

Claim 6.6. *For a family of convex bodies $C(s)$, the set*

$$S_{C(s)} = S \cap (\{C(s) \mid s \in [a,b]\} \times P_p \times [-1,1])$$

separates top from the bottom when projected to $[a,b] \times [-1,1]$.

We have proved this for a transverse to zero $\widetilde{\Phi}$, but the transversality assumption is not necessary. Once we have a curve from $[a,b] \times \{-1\}$ to $[a,b] \times \{1\}$ not touching the projection of the solution set for an arbitrary G_p-equivariant $\widetilde{\Phi}$, satisfying the boundary conditions, this curve will not touch the projection of the solution set for a small generic (and therefore transverse to zero) perturbation of $\widetilde{\Phi}$; but the latter is already shown to be impossible.

Now we consider the "big cylinder" $\mathcal{K} \times [-1,1]$, where the graphs of multivalued functions live. Assume we have a continuous curve

$$\gamma : [a,b] \to \mathcal{K} \times [-1,1]$$

passing from the bottom $\mathcal{K} \times \{-1\}$ to the top $\mathcal{K} \times \{1\}$ in the cylinder and parametrized by a segment $[a, b]$. Its first coordinate may be considered as a one-parametric family of convex bodies $C(s)$. Hence applying the previous claim, we obtain:

Claim 6.7. *The projection Z of S to $\mathcal{K} \times [-1, 1]$ separates the top $\mathcal{K} \times \{1\}$ from the bottom $\mathcal{K} \times \{-1\}$.*

This is the crucial separation property of $Z \subset \mathcal{K} \times [-1, 1]$, considered as a graph of a multivalued function. We show that the separation property implies that this multivalued function is nice.

Take the distance to the set Z function, $\operatorname{dist}(\cdot, Z)$, under some metrization of $\mathcal{K} \times [-1, 1]$, it is continuous and positive on the complement of Z. Since the top and the bottom of $\mathcal{K} \times [-1, 1]$ belong to different connected components of the complement, we can flip the sign of this function on the bottom component to make it satisfy the signed boundary conditions of nice multivalued functions. In effect, we obtain a function

$$\psi : \mathcal{K} \times [-1, 1] \to \mathbb{R},$$

satisfying the boundary condition sufficient to call its corresponding multivalued function with the graph $\{\psi(C, y) = 0\}$ nice. Our construction ensures that whenever $\psi(C, y) = 0$, the pair (C, y) is in Z and corresponds to $(C, x, y) \in S$. The latter triple, in turn, provides a partition of C into p convex bodies $C_1, \dots, C_p$ satisfying

$$\varphi(C_1, y) = \dots = \varphi(C_p, y).$$

□

6.5 Appendix: A weaker higher-dimensional result

Now we are going to consider the case when we work in $\mathbb{R}^d$, have $d-1$ measures $\mu_1, \dots, \mu_{d-1}$ in a convex body K and want to partition K into m convex parts of equal μ_j measure (for every j) and equal surface area. As with the perimeter, the "surface area" may be any continuous function of a convex body in $\mathbb{R}^d$.

In Appendix 6.6 we explain why our approach is not suitable when we want to equalize two arbitrary functions and $d - 2$ measures of parts, which is why we only dare to handle one arbitrary function here. Let us state the result:

Theorem 6.8. *Assume $d-1$ finite non-zero Borel measures $\mu_1,\dots,\mu_{d-1}$ with non-negative density are given in a convex body $K \subset \mathbb{R}^d$ and f is a continuous function of a convex body. If $m \geq 2$ is an integer then it is possible to partition K into m convex parts $V_1,\dots,V_m$ so that for every i*

$$\mu_i(V_1) = \mu_i(V_2) = \cdots = \mu_i(V_m)$$

and

$$f(V_1) = f(V_2) = \cdots = f(V_m).$$

In [99] a similar result was proved, in the case of d measures and no arbitrary function. In terms of the previous section this is explained as follows: In the induction step we equalize d measures in p_1-tuples of parts of the bottom level of the hierarchical partition, but we do not need to work with "multivalued functions" because the measures are additive and once we equalize the measures we know the common value.

As for the proof, our d-dimensional theorem follows from the following analogue of Lemma 6.4. Let $\mathcal{K}^d$ be the space of d-dimensional convex bodies:

Lemma 6.9. *Assume φ is a nice multivalued function of $\mathcal{K}^d$ and p is a prime. Then there exists another nice multivalued function ψ of $\mathcal{K}^d$ such that whenever $C \in \mathcal{K}^d$ satisfies*

$$\psi(C, y) = 0$$

then there exists a partition $C = C_1 \cup \cdots \cup C_p$ into convex bodies of equal measure μ_i, for every $i = 1,\dots,d-1$, and such that

$$\varphi(C_1, y) = \cdots = \varphi(C_p, y) = 0. \tag{6.4}$$

The proof follows by considering the more general $(d-1)(p-1)$-dimensional pseudomanifolds modulo a prime p, $P_{p;d}$, introduced in [22], with the group of symmetry G_p as in the previous section. The map $\Phi_C : P_{p;d} \times [-1,1] \to \mathbb{R}^{(d-1)p}$ is then build from a configuration $x \in P_{p;d}$ of p points in $\mathbb{R}^d$, considered as Voronoi centers. The measure μ_{d-1} is equalized by finding appropriate Voronoi weights and establishing a partition $C = C_1 \cup \cdots \cup C_p$, the functions $\mu_i(C_j)$ $(i = 1,\dots,d-2)$ and $\varphi(C_i, y)$ then constitute the coordinates of Φ_C. Whenever such Φ_C is transverse to zero, the preimage of zero is a nontrivial 0-cycle modulo p; this is a version of Claim 6.5 in this more general situation.

The rest of the proof of Lemma 6.9 is essentially the same as the proof of Lemma 6.4. Theorem 6.8 follows as in the two-dimensional case, the measures are equalized since on every prime number stage the partition is a partition into parts of equal measures, the function φ is equalized as guaranteed by the lemma.

Remark 6.10. Of course, we were trying to find a generalization of this argument in order, for example, to equalize two arbitrary functions of the convex parts in $\mathbb{R}^3$ together with their volumes. A crucial obstacle, in our opinion, is that when we make an induction step and consider a "subfunction" of a multivalued function with a separation argument, then the procedure of restoring the subpartition (of a part in the hierarchy) corresponding to the chosen common value of this equalized function of the subpartition is not continuous. In particular the other function we want to equalize may not depend continuously (or be a nice multivalued function) on the first one after this choice.

6.6 Appendix: Difficulty of equalizing two arbitrary functions

In this section we point out some essential difficulties in the attempt to generalize our technique to the case when we need to equalize at least two arbitrary continuous functions of convex parts. We thank Sergey Melikhov for sharing with us his ideas that developed into the argument of this section.

Assume we have a convex body $K \subset \mathbb{R}^3$ and want to partition it into $m = 2p^s$ (p is an odd prime) convex parts with equal volumes, and equal values of two other continuous in Hausdorff metric functions F_1, F_2 of the parts. We would naturally start by partitioning K into two parts of equal volume; such partitions are parametrized by the normal of the oriented partitioning plane, that is by the sphere S^2. Then in the part of K the normal points to, we would apply the Blagojević–Ziegler result for $m = p^s$ to have a nonzero modulo p number of solutions for this half of the problem. Looking at the possible pairs of common values of F_1, F_2 we would obtain, as in the proof of the main result of this chapter, a multivalued function $S^2 \to \mathbb{R}^2$, whose graph in $S^2 \times \mathbb{R}^2$, under certain genericity assumptions, could be viewed as a 2-dimensional cycle modulo p, which we denote by Z, homologous modulo p to $k[S^2 \times \{(0,0)\}]$ for some $k \neq 0 \mod p$.

The problem would be solved this way if we could prove that under the antipodal map $\sigma : S^2 \to S^2$, extended to $S^2 \times \mathbb{R}^2$ by the trivial action of σ on $\mathbb{R}^2$, some point of the support of Z would go to some other point of the support of Z. But below we build an example of a modulo p cycle Z that satisfies all the assumptions that we know it must satisfy in the problem, but has disjoint Z and $\sigma(Z)$.

Let us build Z inside $S^2 \times D$, where D is the unit disk in the plane $\mathbb{R}^2$. Let us split S^2 by its equator S^1 into closed hemispheres D_+ and D_-. Start by building the part of Z that lies over S^1: Let L be the graph of

$$z \mapsto z^n,$$

where we identify D with the unit disk in the complex plane and S^1 with the unit complex numbers. For odd n the circles L and σL do not intersect and their linking number (if we consider the solid torus $S^1 \times D$ lying standardly in $\mathbb{R}^3$) is $\operatorname{lk}(L, \sigma L) = n$, since for odd n the circle σL is the graph of

$$z \mapsto -z^n,$$

and the linking number of two circles, close to each other, equals the winding number of their difference vector when we pass along the circles.

Letting this n be equal to the prime number p from the formula $m = 2p^s$, we thus have that L and σL are non-linked 1-dimensional modulo p cycles. Now we pass from the torus $S^1 \times D$ to the topological 4-dimensional ball $B^4 = D_+ \times D$. The torus $S^1 \times D$ is a part of its boundary $S^3 = \partial B^4$ and the cycles L and σL are non-linked modulo p cycles in S^3, since the torus embeds into S^3 without a twist. It follows that we may choose two 2-dimensional modulo p cycles in B^4 relative to S^3, M and N, so that $\partial M = L$, $\partial N = \sigma L$, and the supports of M and N are disjoint.

Indeed, choose M as any topologycally embedded disk in B^4, whose boundary maps homeomorphically to the circle L. By Alexander duality for the pair (B^4, S^3), we have

$$H_1(B^4 \setminus M; \mathbb{Z}/p\mathbb{Z}) = H^2(M, L; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}.$$

Hence the homology class $[\sigma(L)] \in H_1(B^4 \setminus M; \mathbb{Z}/p\mathbb{Z})$ is fully determined by an element of $\mathbb{Z}/p\mathbb{Z}$, which is in fact the linking number $\operatorname{lk}(L, \sigma L)$. Having this linking number 0 modulo p, we may conclude that σL is a modulo p boundary of a 2-dimensional modulo p chain N in $B^4 \setminus M$. The chains N and M thus have disjoint supports.

Now we pass to $S^2 \times D$ from $D_+ \times D$ and take the 2-dimensional modulo p cycle $Z = M - \sigma(N)$, this is indeed a cycle, since

$$\partial Z = \partial M - \sigma(\partial N) = L - \sigma(\sigma(L)) = 0.$$

From the construction of M and N we may conclude that Z and σZ are disjoint. At the same time, Z is homologous modulo p to $[S^2 \times \{(0,0)\}]$, which is equivalent to saying that it intersects $\{x\} \times D$, for generic $x \in S^2$, 1 modulo p number of times, counted with signs. The last claim is evidently true for $x \in S^1$, where

$$(\{x\} \times D) \cap Z = (\{x\} \times D) \cap L.$$

Our construction of M and N allows them to have collars near $S^3 \subset B^4$ that allows us to keep the uniqueness of such an intersection for x in a neighborhood of S^1 in S^2. If we want Z to be homologous to a multiple $k[S^2 \times \{(0,0)\}]$ modulo p then we may just repeat this construction in k smaller disks $D_1, \dots, D_s$ embedded in D and take the sum of the obtained cycles.

Thus we have checked that Z has the properties that a graph of the multivalued function from our attempted proof must have, but does not allow to make the final step of the proof.

Remark 6.11. Using several circles L_i, given by $z \mapsto c_i + \varepsilon z^{n_i}$ for different $c_i \in D$, odd integers n_i, and sufficiently small $\varepsilon > 0$, it is possible to replace L in the above argument with an algebraic combination $L' = \sum_i L_i$, such that

$$\mathrm{lk}(L', \sigma L') = \sum_i n_i = 0$$

as an integer. We may also make L' modulo p (but not integrally!) homologous to $k[S^1 \times \{(0,0)\}]$, by choosing the number of the L_i to equal k modulo p. Then we choose M as an oriented surface in $B^4 = D_+ \times D$ with boundary L', N as a integral chain in $B^4 \setminus M$ with boundary $\sigma(L')$. The integral chain $Z = M - \sigma(N)$ then becomes an integral cycle, modulo p (but not integrally!) equivalent to $k[S^2 \times \{(0,0)\}]$. And Z is disjoint from $\sigma(Z)$, that is the Borsuk–Ulam theorem cannot be generalized to the corresponding multivalued map $S^2 \to \mathbb{R}^2$.

7 Inscribed quadrilaterals

7.1 Introduction

To **inscribe** a quadrilateral Q in a Jordan curve is to find a non-degenerate scaling, rotation, and translation which maps all of the vertices of Q to the curve.

Which quadrilaterals can be inscribed in any closed convex curve? Obviously, the quadrilateral must be **cyclic**, that is inscribed in a circle. Such quadrilaterals are characterized by a remarkable property that the sum of their opposite angles is π. It turns out that for C^1-curves this condition is also sufficient.[1]

Theorem 7.1. *Any cyclic quadrilateral can be inscribed in any closed convex C^1-curve.*

The C^1-smoothness requirement is necessary in Theorem 7.1. For example, the kite with angles $\pi/2$ and $2\pi/3$ cannot be inscribed in the thin triangle with angles $\pi/10$, $\pi/10$, $4\pi/5$ (Figure 7.1). However, if Q is a rectangle the smoothness condition can be relaxed.

Theorem 7.2. *Any rectangle can be inscribed in any closed convex curve.*

[1]The results of this chapter were independently obtained by B. Matschke, whose preprint [67] appeared just a few days later. In addition it contains a stronger version of Theorem 7.2, which is proved not only for rectangles but also for any cyclic trapezoids.

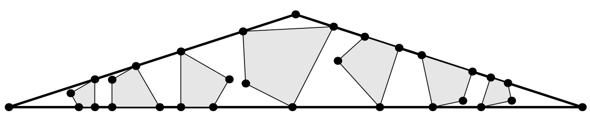

Figure 7.1

In [62] V. Makeev conjectured that any cyclic quadrilateral can be inscribed in any Jordan curve. He proved the conjecture for the case of star-shaped C^2-curves intersecting any circle at no more than 4 points. Theorem 7.1 proves it for convex C^1-curves. The example above shows that the conjecture fails without the smoothness assumption. This example can be generalized to any cyclic quadrilateral except for trapezoids. So, I. Pak [76] conjectured that Makeev's conjecture still holds for cyclic trapezoids even without the smoothness assumption.

Makeev's conjecture is a part of a substantial topic originating from the famous question in geometry known as the Square Peg problem or Toeplitz' conjecture: Does every Jordan curve contain all the vertices of a square? In its general form the Square Peg problem is still open. In the last hundred years it was positively solved, however, for a wide variety of classes of curves. For instance, for convex curves and later for piecewise analytic curves by A. Emch [28; 29], for C^2-curves by L. Schnirelman [109], for locally monotone curves by W. Stromquist [102], for curves without special trapezoids, for curves inscribed in a certain annulus, and for centrally symmetric curves [74]. There were also high-dimension extensions of these results [39; 47; 63; 49]. For more details we refer the reader to the survey [66] by B. Matschke.

Similar to the Square Peg problem, there exists the Rectangular Peg conjecture stating that every Jordan curve contains the vertices of a rectangle with the prescribed aspect ratio. A proof was claimed by H. Griffiths [36], but an error was found later. For a discussion see [66, Conjecture 8]. The specific case of aspect ratio $\sqrt{3}$ was first solved by B. Matschke [65] only for "close to convex" curves, and later by C. Hugelmeyer [48] for all smooth Jordan curves. Theorem 7.2 proves the Rectangular Peg conjecture for the case of convex curves. In a recent preprint [35] the conjecture was finally proved (for smooth curves).

It is noteworthy that all of the proofs mentioned above use topological obstruction theory (except for [48] and [35]). Unfortunately, this approach fails in the more general cases. The proof of Theorem 7.1 and 7.2 is based on a "non-topological" observation first made by R. Karasev, [50]. It allowed him, in particular, to prove the infinitesimal version of Theorem 7.1. R. Karasev noticed and proved that during the rotation of any three out of four vertices of a quadrilateral Q along a curve γ the fourth vertex travels along a path bounding the same signed area as the original curve γ. A similar idea was recently independently discovered by T. Tao who used it to prove the Toeplitz' conjecture for new types of curves [103].

7.2 The case of strictly convex C^∞-curves.

In this section we prove the following theorem.

Theorem 7.3. *Let Q be a cyclic quadrilateral and let γ be a closed strictly convex C^∞-curve. Then for any $\varepsilon > 0$ there is a cyclic quadrilateral Q_ε which is ε-close to Q and can be inscribed in γ.*

The proofs of Theorems 7.1 and 7.2 are obtained from Theorem 7.3 by "going to the limit" type argument in the next section.

Until the end of the section let us fix a closed strictly convex C^∞-curve γ.

For a quadrilateral Q its **drawing** is the image of Q under some non-degenerate scaling, rotation, and translation. If the angle of the rotation is α we also call it an **α-drawing**.

Lemma 7.4. *Pick a vertex of a quadrilateral Q. For any angle α there is a unique α-drawing of Q with all of the remaining 3 vertices being on γ.*

Proof. The existence of a drawing follows by a simple continuity argument similar to the argument in the proof of the following Lemma 7.6. The drawing is unique because the vertices of two distinct homothetic triangles cannot lie on a strictly convex curve. □

For a vertex d of a quadrilateral Q denote by $d(\alpha)$ the position of d in the α-drawing of Q with the remaining 3 vertices being on γ.

Lemma 7.5. *Let Q be a cyclic quadrilateral. Then for any $\varepsilon > 0$ there is a cyclic quadrilateral Q_ε which is ε-close to Q and such that $d(\alpha)$ is a closed C^∞-curve for any vertex d of Q_ε.*

Proof. Let U be the space of ordered triples of pairwise distinct points of γ. Consider the map $f : U \to S^1 \times S^1$ which sends a triple (x, y, z) to the pair of angles $(\angle xyz, \angle yzx)$. Clearly, f is C^∞.

Let $abcd$ be the vertices of Q in the counterclockwise order. Then the curve $d(\alpha)$ corresponds to the f-preimage of the pair of angles $(\angle abc, \angle bca)$. By Sard's lemma the set of the critical values of f has Lebesgue measure 0, i.e., the set of cyclic quadrilaterals Q such that $d(\alpha)$ is *not* a C^∞-curve also has Lebesgue measure 0. Applying this argument to every vertex of Q we get the statement of the lemma. $\square$

Lemma 7.6. *Let Q be a cyclic quadrilateral. Then there is a vertex d of Q such that the angle at d is non-acute and $d(\alpha)$ contains a point either on γ or in the exterior of γ.*

Proof. Let $abcd$ be the vertices of Q in the counterclockwise order.

There are two adjacent vertices of Q with non-acute angles because Q is cyclic. Without the loss of generality let us assume that the angles at c and d are non-acute. We may also assume that γ is tangent to the lines $y = 0$ and $y = 1$.

For $t \in (0, 1)$ denote by a_t and b_t the leftmost and the rightmost, respectively, of the two intersections of γ with $y = t$. Denote by c_t and d_t the points such that $a_t b_t c_t d_t$ is a drawing of Q. Note that c_t and d_t are above the line $y = t$, see Figure 7.2.

Consider the case when t is very close to 1. Then $\angle d_t a_t b_t$ is greater than the angle between $a_t b_t$ and the tangent to γ at a_t. Which places d_t is in the exterior of γ. Likewise, c_t is also in the exterior of γ.

Consider now the opposite case of t being very close to 0. Then $\angle d_t a_t b_t$ is less than the angle between $a_t b_t$ and the tangent to γ at a_t. Also, the segment $a_t d_t$ is "short", i.e., much shorter than the intersection of the interior of γ with the line parallel to $a_t d_t$ and going through the common point of γ and $y = 0$. Which means that d_t is in the interior of γ. Likewise, c_t is also in the interior of γ.

Let us now continuously decrease t from 1 to 0. At some moment one of the vertices d or c is going to intersect γ while another one is still in the exterior of γ, or on γ. Without the loss of generality we may assume that the latter vertex is d. Then $d(\alpha)$ contains a point either on γ or in the exterior of γ. $\square$

Proof of Theorem 7.3. Choose a cyclic quadrilateral Q_ε as in the statement of Lemma 7.5.

By Lemma 7.6, there is a vertex d of Q_ε with a non-acute angle and such that $d(\alpha)$ contains a point either on γ or in the exterior of γ.

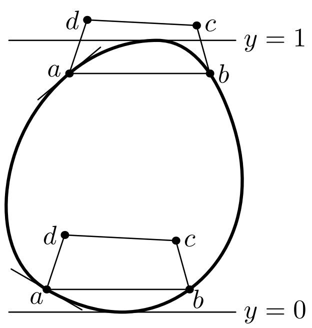

Figure 7.2

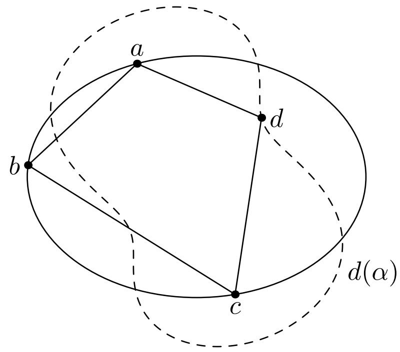

Figure 7.3

If $d(\alpha)$ intersects γ then we are done, so we may assume that $d(\alpha)$ and γ are disjoint.

By the Jordan curve theorem, $d(\alpha)$ lies in the exterior of γ. On the other hand, γ lies in the interior of $d(\alpha)$. Indeed, as we revolve once along $d(\alpha)$, the diagonal bd of the corresponding α-drawing of Q_ε must also complete a 2π rotation. This would be impossible if the interiors of γ and $d(\alpha)$ were disjoint.

By the following lemma, the curve $d(\alpha)$ has no self-intersections in the exterior of γ, i.e., no self-intersections at all.

Lemma 7.7. *Let $abcd$ and $a'b'c'd$ be two distinct drawings of the same cyclic quadrilateral with a non-acute $\angle d$. Suppose that d is outside of the convex hull of the points a,b,c,a',b',and c'. Then six points a, b, c and a', b', c' cannot be in strictly convex position.*

It is left to note that the area of $d(\alpha)$ must then be greater than the area of γ which contradicts to the following lemma proven in [50].

Lemma 7.8. *Let Q be a cyclic quadrilateral and let $d(\alpha)$ be a C^∞-curve for some vertex d of Q. Then the area of $d(\alpha)$ is equal to the area of γ.*

□

7.3 Proofs of Theorem 7.1 and Theorem 7.2.

Proof of Theorem 7.1. Let γ be a closed convex C^1-curve and let Q be a cyclic quadrilateral. Choose a sequence γ_i of closed strictly convex C^∞-curves converging to γ pointwise. By Theorem 7.3, there is a sequence Q_i of cyclic quadrilaterals converging to Q and such that Q_i can be inscribed in γ_i for each i.

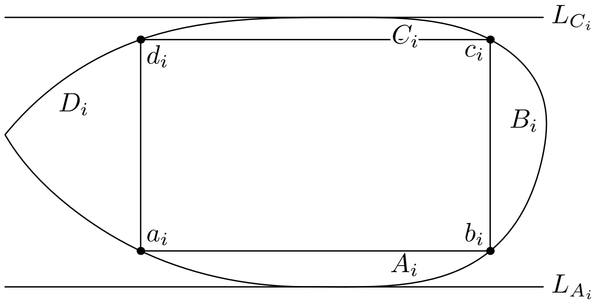

Figure 7.4

Let $a_i, b_i, c_i, d_i \in \gamma_i$ be the vertices of Q_i in the counterclockwise order. By passing to the subsequences, we may assume that the sequences a_i, b_i, c_i, and d_i have limits, a, b, c, and d, respectively.

The quadrilateral $abcd$ is inscribed in γ and is obtained from Q by a composition of scaling, rotation, and translation. It remains to prove that the scaling is non-degenerate.

Assume to the contrary that $a = b = c = d$. Then both a_ib_i and b_ic_i converge to the tangent to γ at $a = b = c = d$. I.e., $\angle a_ib_ic_i$ converges to either 0 or π. On the other hand, $\angle a_ib_ic_i$ must converge to the corresponding angle of Q. Angles of Q are neither 0 nor π, which leads to a contradiction. $\square$

Proof of Theorem 7.2. Let γ be a closed convex curve and let Q be a rectangle. Let γ_i, Q_i, a_i, b_i, c_i, d_i, a, b, c, and d be as in the proof of Theorem 7.1.

Again, it remains to prove that the rectangle $abcd$ is non-degenerate. Assume to the contrary, that $a = b = c = d$.

The vertices of Q_i divide γ_i into 4 arcs which we denote in the counterclockwise order by A_i, B_i, C_i, and D_i, see Figure 7.4. The lengths of 3 out of 4 arcs converge to 0. Without the loss of generality we assume that the lengths $l(A_i)$ and $l(C_i)$ of A_i and C_i, respectively, converge to 0.

Denote by L_{A_i} and L_{C_i} the tangents to A_i and C_i parallel to a_ib_i, see Figure 7.4. The distance between L_{A_i} and L_{C_i} is less than $l(A_i) + l(C_i) + |b_ic_i|$, i.e., it converges to 0. Which contradicts to the fact that the curve γ_i lies between L_{A_i} and L_{C_i} for each i. $\square$

7.4 Proof of Lemma 7.7

Assume to the contrary of the statement of the lemma that the points a, b, c, a', b', and c' lie in strictly convex position.

Draw the lines containing the sides of the triangle abc, and denote the angular regions of the plane formed by them as shown in Figure 7.5. At first, let us note that the point a' cannot belong to the region C_a, because in that case a is covered by the triangle $a'bc$. Analogously, the points b' and c' cannot belong to the regions C_b and C_c, respectively.

Denote by Ω the circumcircle of $abcd$ and let ℓ be the tangent line to Ω at a. Together with the lines ab and ac the line ℓ forms two additional angular regions denoted by C'_b and C'_c, respectively, see Figure 7.5.

It is easy to see that the composition of a homothety and a rotation around d which sends a to b and a' to b' also sends C'_b to C_b. So, if a' lies in C'_b then b' lies in C_b, which we already showed to be impossible. Therefore a' cannot lie in C'_b. Analogously, a' cannot lie in C'_c.

We proved that a' cannot lie in the union of C_a, C'_b, and C'_c, which means that a' and d lie on the same side of the line ab. Similarly, a and d lie on the same side of the line $a'b'$. From the latter fact we conclude that a' lies in the exterior of the circle ω passing through d and tangent to ab at a (Figure 7.6). To see this the reader might consider that $a'b'$ passes through a iff a' belongs to ω.

We know that d lies outside of the convex hull of a, b, c, a', b', and c' and that $\angle cda = \angle c'da' \geq \pi/2$, so $\angle ada' < \pi/2$. Thus, without the loss of generality we may assume that the counterclockwise rotation sending da to da' is at most $\pi/2$. Therefore, a' lies in the angular region formed by $\angle cda$.

This region is covered by the four grey zones in Figures 7.5, 7.6, 7.7, 7.8. Let us give a verbal description of the zones and prove that a' cannot lie in them. This will conclude the proof of the lemma.

- The grey zone in Figure 7.5 is the halfplane bounded by ab and not containing d.

- The grey zone in Figure 7.6 is the interior of ω.

We have already proved that a' cannot lie in the grey zones in Figures 7.5 and 7.6.

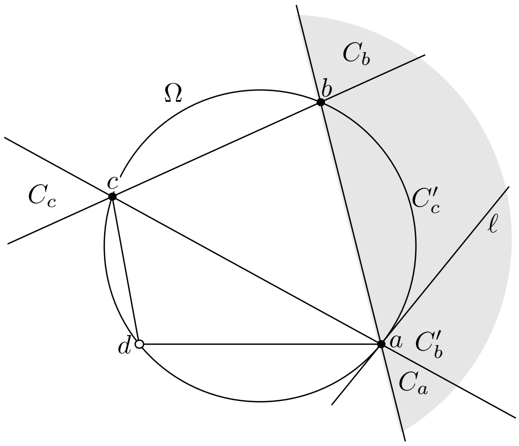

Figure 7.5

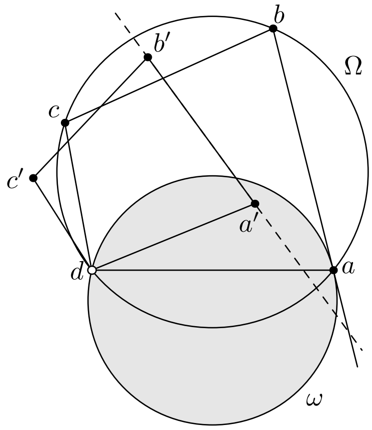

Figure 7.6

- The grey zone in Figure 7.7 is the intersection of the halfplane bounded by ab and containing d, the exterior of ω, and the interior of Ω. Suppose that a' is in that grey zone.

  Denote by x the point of intersection of the line ab with the circle going through a, a', and d. Point x is inside of the segment ab. From the angular property of an iscribed quadrilateral it follows that b', a', and x lie on the same line. So, the intersection point x of ab and $a'b'$ lie inside of the segment ab and outside of the segment $a'b'$ which contradicts to the convex position of a, b, a', and b'.

- The grey zone in Figure 7.8 is the intersection of the halfplane bounded by ℓ and containing c, the halfplane bounded by ac and not containing d, and the exterior of Ω. Suppose that a' is in that grey zone.

  Denote by y the point of intersection of the line ac with the circle going through a, a', and d. Point y is outside of the segment ac. From the angular property of an iscribed quadrilateral it follows that c', a', and y lie on the same line. So, the intersection point y of ac and $a'c'$ lie outside of the segment ac and inside of the segment $a'c'$ which contradicts to the convex position of a, c, a', and c'.

Remark 7.9. Lemma 7.7 does not hold if the quadrilateral is not assumed to be cyclic (Figure 7.9) or if the angle at d is allowed to be acute (Figure 7.10).

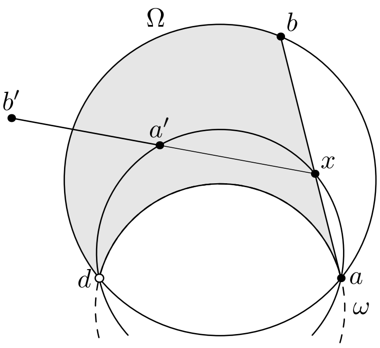

Figure 7.7

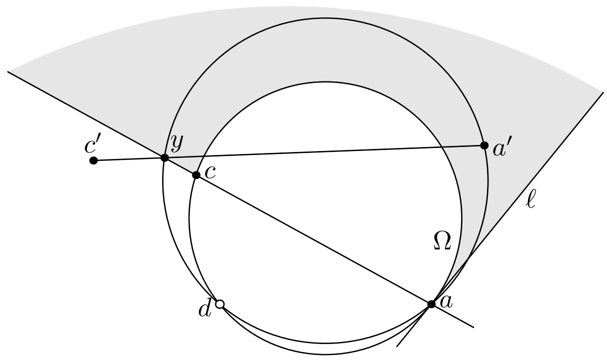

Figure 7.8

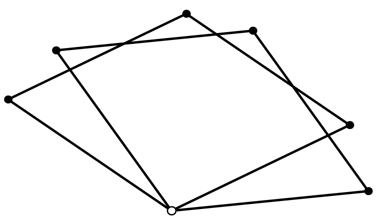

Figure 7.9

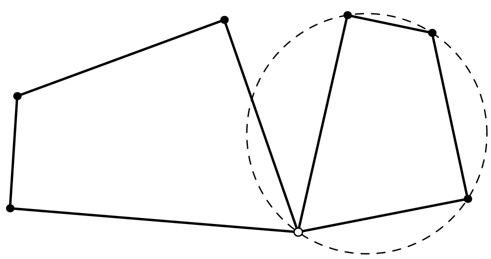

Figure 7.10

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