Inaugural Lecture
by

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Title: A Survey of the Classification of Fuzzy Subgroups of some Finite Groups

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Programme Director and Dean of Research Prof Gideon de Wet, Vice Chancellor Dr Mvuyo Tom, DVC Academic Prof Larry Obi, Deans of Faculties, Professors, Academics, Students, Family members, Ladies and Gentlemen, I feel really honoured by your presence here this afternoon, thank you very much.
Everthing that I am going to say is written down for the ease of the presentation.

2 Rationale

Any mathematician who is to present a mathematical talk to a mixed audience is always worried about how many people will understand him/her. I am no exception! When I was invited to give an inaugural talk, I thought of pretending to be a mathematics educationist and thus talk about challenges in the learning and teaching of mathematics, particularly in South African schools. However, I remembered that mathematicians and mathematics educationists do not always agree on the causes of the learning problems in schools and how to address them. Indeed at some of the meetings involving mathematicians and mathematics educationists, one observes posturing and point-scoring on both sides. There is a clear dividing line. A mathematics educationist wants to paint a mathematician as someone who knows nothing about school teaching and challenges learners are facing. Thus mathematicians are sometimes despised by mathematics educationists and regarded as too elitist.
On the other hand, mathematicians tend to look down upon mathematics educationists and regard them as people who pretend to know mathematics but actually don't know it. Their level of mathematics is thus despised by mathematicians. Thank God I despise no one. In fact I am a school teacher who has also taught school mathematics in the past. I understand both sides and to a certain degree I agree with both sides' theories and concerns.
Since I am a coward who tries to avoid controversy, I decided to stay clear of mathematics education, politics of education and related learning challenges in school mathematics. This is not running away from doing something for the schools, I am avoiding unproductive talk and blame game. My Department of Mathematics is trying to help school learners in a small way.
I came to my senses and realised that I am primarily a pure mathematician, not even an applied mathematician. In mathematics I postulate and conjecture. I syllogise and

prove. I am a logician and a philosopher in mathematics. I also disprove certain theories by providing counter-examples. So I am also a fault-finder! However, mathematicians are usually very polite. Their aim in fault-finding is never to hurt or embarrass but to build up and enrich. They merely want the truth and nothing else but the truth. Proving and disproving gives a sense of joy and peace of mind. I wanted that peace of mind in deciding to present my research work instead of opening a can of worms elsewhere. I will attempt to keep it interesting but that cannot be guaranteed.

3 Abstract

In this lecture we survey the classification of fuzzy subgroups of finite groups as studied by BB Makamba and V Murali. We present the impact of the research on our postgraduate students. The classification is focusing on finite abelian p-groups and dihedral groups, giving a mixture of abelian and non-abelian groups. We show some highlights and what still needs to be done in the classification of fuzzy subgroups. We also touch on what other researchers have achieved in the classification of fuzzy subgroups and how our work is related to theirs. We begin with a historical background of fuzzy logic.

4 Introduction

The first important person to introduce fuzzy thinking was Buddha who lived in India about 500 BC and founded a religion called Buddhism. His philosophy was based on the thought that the world is filled with contradictions, that almost everything contains some of its opposite, i.e. things can be right and wrong at the same time. About 200 years later, the Greek philosopher and scientist Aristotle developed binary logic which was contrary to Buddha's. Aristotle thought that the world was made up of opposites, for example good versus bad, right versus wrong, dry versus wet, active versus passive, etc. Everything has to be A or not-A, it can't be both. Aristotle's logic was accepted by the Greek scholars and later got spread all over Europe; first by the Romans and then through to the Christian world. Aristotle's binary logic became the base and cornerstone of science; if something got proven with logic, it was and still is accepted as scientifically correct. Buddha's logic was rejected by most scholars, particularly those who subscribed to Christianity.

In 1964, professor Lotfi A. Zadeh, a US mathematician, electrical engineer, computer scientist, artificial intelligence researcher, started wondering, if there wasn't a better logic to use in machinery. He had the idea that if you could tell an air-conditioner to work a little faster when it gets hotter, or similar problems, it would be much more efficient than having to give a rule for each temperature. **That was the day fuzzy logic, the way we know it today, was born**. Zadeh introduced fuzzy logic in a way that appealed to computer scientists and mathematicians. Many mathematicians in the US and Western Europe started to accept fuzziness in mathematics. Indeed nowadays, with fuzzy logic you can tell an air-conditioner to slow down as soon as it gets chilly or to work faster when it gets hotter. In this way fuzzy logic has application in many more machineries such as washing machines, microwave ovens, generally in robotics and computer programming. Engineers, philosophers, psychologists, and sociologists soon became interested in applying fuzzy logic into their sciences.

In 1987, the first subway system was built which worked with a fuzzy logic-based automatic train operation control system in Japan. It was a great success and resulted in a fuzzy boom. Universities as well as industries got interested in developing the new ideas using fuzzy logic. Many mathematicians began fuzzifying the classical mathematics and extending it. Thus nowadays we have Fuzzy Topology, Fuzzy Group Theory, Fuzzy Ring Theory, Fuzzy Vector Spaces, in fact almost any mathematical concept can be fuzzified.

5 The Mathematics of Fuzzy Sets

If A is a set, then A can be characterized by a function $f : A \to \{0, 1\}$ such that x is an element of A iff $f(x) = 1$. Thus contrapositively and equivalently, x is not an element of A iff $f(x) = 0$. The image f(x) is usually called the degree of membership of x and f is the membership function of the set. So the study of set A becomes the study of the characteristic (or membership) function of A. Thus we may call f a set. Now suppose we replace the co-domain $\{0, 1\}$ by the interval $[0, 1]$ and characterize a set B by $f : B \to [0, 1]$ such that for some x, $f(x) = \frac{1}{3}$. Since $f(x) \neq 0$ and $\neq 1$, we cannot say x is in B or x is not in B. In this case we say x is in B to the degree $\frac{1}{3}$. The set B is an example of a fuzzy set since some elements are partially (not fully) in B. Other examples of fuzzy sets: $A = \{tall\ people\}$, $B = \{blind\ people\}$, $C = \{intelligent\ people\}$. So fuzzy sets are characterised by imprecision.

GROUP: A group is a non-empty set G on which a binary relation (operation) $*$ is defined such that (i) $*$ is associative, (ii) there is a unique element e in G such that $e * a = a = a * e$ for every a in G, (iii) for each a in G there exists a^{-1} in G such that $a^{-1} * a = e = a * a^{-1}$. A subset H of G that is a group under the binary operation $*$ of G is a subgroup of G. When G is a finite set, then G is a finite group.
If the binary operation $*$ is commutative, then G is said to be an abelian group. (Named after the Norwegian mathematician Abel who died at the age of 26).

6 Introduction of Fuzzy Groups

In 1971, US mathematician and computer scientist, Azriel Rosenfeld introduced the notion of a fuzzy group as follows:

Definition 6.1 *Let G be a group. A fuzzy subgroup of G is a mapping $\mu : G \to [0, 1]$ satisfying, for any x, $y \in G$, (i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and (ii) $\mu(x) = \mu(x^{-1})$.*

He showed that many concepts of group theory can be extended in an elementary manner to develop the theory of fuzzy groups. In particular, he characterized all the fuzzy subgroups of cyclic groups of prime order. Fuzzy groups were further investigated and characterised by P S Das of Madras University in India. He introduced the notion of level subgroups of a fuzzy subgroup which is based on the notion of a level subset introduced earlier by L Zadeh. For a finite group Das showed that these level subgroups of a fuzzy subgroup form a chain.

After Rosenfeld's first paper on fuzzy groups, various other mathematicians and researchers started to do further research in fuzzy subgroups. We mention a few names: P. Bhattacharya and N P Mukherjee who published a few papers in the 1980s on group-theoretic analogues; John Mordeson and DS Malik published some papers jointly on fuzzy ideals. In 1987, my co-researcher, Venkat Murali completed a Ph D thesis titled *A Study of Universal Algebras in Fuzzy Set Theory*. In the early 1990's, M Mashinchi and M Mukaidono began to look at the classification of fuzzy subgroups and published papers related to that problem.

BB Makamba joined the fuzzy band-wagon in 1990 while doing a Ph D in Mathematics at Rhodes University. He fuzzified and developed various group-theoretic concepts such as normality, solvability, nilpotency, fuzzy direct products, fuzzy isomorphism, fuzzy equivalence, the well-known Jordan-Holder Theorem, The Basis Theorem and the Remak-Krull-Schmidt Theorem, inter alia.

Although my thesis was completed in 1992 and we published a few papers arising from it, it is only in 1998 that I began rigorous and serious research that is based on some of the concepts done rather superficially in my Ph D thesis. One such concept is equivalence of fuzzy subgroups. At the time of my Ph D research there was no paper I could refer to for equivalence of fuzzy subgroups, so I came up with my own definition of equivalence. However, this concept was not developed in my thesis because the thesis was primarily about various fuzzy concepts. The initial objective was to fuzzify crisp concepts and then look at properties that are unique to fuzzy subgroups, hoping to develop a new way of looking at mathematical problems.

7 Classification of Fuzzy Subgroups

My research in the classification of fuzzy subgroups has been a joint venture with Prof V Murali of Rhodes University. The objective was to classify fuzzy subgroups according to the algebraic properties they possess. Such work is usually cumbersome when working alone, hence my collaboration with V Murali and others at Rhodes.
The classification of fuzzy subgroups was motivated by the fact that a finite group G has a finite number of subgroups, whereas the same G always has an infinite number of fuzzy subgroups. We wanted to count fuzzy subgroups to a finite number.
Indeed: If μ is a fuzzy subgroup of G then so is $\alpha\mu$ for any $\alpha \in (0, 1)$. Hence G has infinitely many fuzzy subgroups. However, fuzzy subgroups having the same behaviour may be lumped together and regarded as one (class).

All the fuzzy subgroups that are in the same class are said to be equivalent (or preferentially equal). So we seek an equivalence relation that will give a finite number of

equivalence classes of fuzzy subgroups. This classification allows us to loosely say that a finite group G has a finite number of fuzzy subgroups.
Our first paper on equivalence, published in 2001 and titled On an equivalence of fuzzy subgroups I, presents the notion of equivalence of fuzzy subgroups and further characterisation of equivalence.

Definition 7.1 *Let G be a finite group. Two fuzzy subgroups μ and ν of G are equivalent and we write $\mu \sim \nu$ if for $x, y \in G$, $\mu(x) > \mu(y)$ iff $\nu(x) > \nu(y)$ and $\mu(x) = 0$ iff $\nu(x) = 0$.*

Further, the same paper presents formulae for the number of equivalence classes of fuzzy subgroups of the groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^n} + \mathbb{Z}_q$ where p and q are distinct primes and n is a positive integer. The notion of equivalence is shown to lead to a notion of isomorphism of fuzzy subgroups.

Equivalent fuzzy subgroups display many similar properties. This equivalence partitions the set F_G of all fuzzy subgroups into a finite family of classes of fuzzy subgroups. Two equivalence classes are $distinct$ iff they are disjoint. Having classified fuzzy subgroups into cells, we want to count the number of these cells in each finite group G.

Example 7.2 *Consider a two-element group $H = \{e, a\}$. Then H has two subgroups viz $\{e\}$ and $H = \{e, a\}$. However, H has 3 distinct fuzzy subgroups viz*

$$\mu_o(x) = \begin{cases} 1 & \text{if} \quad x = e \\ 0 & \text{if} \quad x \neq e \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \lambda & \text{if} \quad x \neq e \end{cases} \quad \text{for } 0 < \lambda < 1$$

$$\mu_2(x) = 1 \, \forall \, x \in G$$

Example 7.3 *Let G be a finite group and $\{e\} \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$ be a chain of subgroups of G such that no new subgroups can be inserted in the chain. Such a chain is a maximal chain, also called a flag. Any flag such as above may be associated with a fuzzy subgroup μ by assigning a membership value (called a pin) to each subgroup in the chain as follows:*

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \lambda_1 & \text{if} \quad x \in H_1 \setminus \{e\} \\ \lambda_2 & \text{if} \quad x \in H_2 \setminus H_1 \\ \lambda_3 & \text{if} \quad x \in H_3 \setminus H_2 \\ \vdots & \\ \lambda_n & \text{if} \quad x \in G \setminus H_{n-1} \end{cases}$$

for $0 < \lambda_n < \lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_1 < 1$.
The fuzzy subgroup μ may be expressed as a pinned-flag as follows:
$\{e\} \subseteq H_1^{\lambda_1} \subseteq H_2^{\lambda_2} \subseteq \cdots \subseteq H_n^{\lambda_n} = G^{\lambda_n}$

A variation of μ above could be

$$\mu_1(x) = \begin{cases} 1 & \text{if} \quad x \in H_1 \\ \lambda_1 & \text{if} \quad x \in H_2 \setminus H_1 \\ \lambda_2 & \text{if} \quad x \in H_3 \setminus H_2 \\ \vdots & \\ \lambda_{n-1} & \text{if} \quad x \in G \setminus H_{n-1} \end{cases}$$

There are more variations of μ. Such variations are not equivalent. So they constitute the full set of distinct fuzzy subgroups of G.

Now how many distinct fuzzy subgroups are there for the above flag? In the paper On an equivalence of fuzzy subgroups I, we established that there are $2^{n+1} - 1$ such distinct fuzzy subgroups.
In the same paper, we looked at $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$ and established that there are $n+1$ maximal chains and that each maximal chain has a distinguishing factor. If we pick any chain and label it (1), then this chain contributes $2^{n+1} - 1$ distinct fuzzy subgroups since any maximal chain of G is of length $n+1$. By our counting technique, each of the remaining chains contributes 2^n distinct fuzzy subgroups. Adding, we obtain $2^{n+2} - 1$ distinct fuzzy subgroups of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$.

Following the work in our first paper on equivalence, we made further developments in a second paper on equivalence titled On an equivalence of fuzzy subgroups II in which we characterized fuzzy subgroups of $G = \mathbb{Z}_{p_1} + ... + \mathbb{Z}_{p_n}$ where the p_i are distinct primes, and established formulae for the number of equivalence classes of fuzzy subgroups using the notions of keychains, n-pad, padidity, index and n-chain. Here is the best result achieved:

Theorem 7.4 *Let $n = k_1 + k_2 + \cdots + k_m + s + 1$ where s is the number of non-repeating pins in an (n-1)-chain of pins $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$ and k_i is the number of times a pin λ_i appears. In the group $\mathbb{Z}_{p_1} + \cdots + \mathbb{Z}_{p_{n-1}}$, the number of distinct fuzzy subgroups represented by all (n-1)-chains each of length n with index $(k_1, k_2, \cdots, k_m)$ is $4\frac{(s+m)!(n-1)!}{s!k_1!k_2!\cdots k_m!}$ if all the k_i are distinct. If some k_i's are identical, divide the first expression by the factorials of the numbers of the identical k_i's.*

Example 7.5 *Compute the number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p_1} + \cdots + \mathbb{Z}_{p_n}$ represented by all the 6-chains of length 7 with index $(2,3)$.*
Solution. Using the above theorem, $n - 1 = 6$, thus $n = 7 = 2+2+2+0+1$. So $s = 1$ since only 1 pin (λ_6) is not repeating (remember that here we have 6 pins other that 1. $k_1 = 2$ and $k_2 = 3$ implies the pins λ_1 and $_2$ are identical and the pins λ_3, λ_4 and $_5$ are identical while λ_6 is not repeating. m, the number of classes of repeating pins k_i's, is equal to 2. All the k_i are distinct, hence the required number is $4\frac{(1+2)!(7-1)!}{1!2!3!} = 1440$.

Example 7.6 . *Compute the number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p_1} + \cdots + \mathbb{Z}_{p_n}$ represented by all the 6-chains of length 7 with index $(2,2,2)$.*
Solution. Using the above theorem, $n - 1 = 6$, and $s = 0$ since all the 6 pins are repeating. Thus $n = 7 = 2+2+2+0+1$ and $k_1 = 2 = k_2 = k_3$. m, the number

of $k_i's$, is equal to 3. There are 3 identical $k_i's$, hence the required number of fuzzy subgroups is $4\frac{(0+3)!(7-1)!}{0!2!2!2!3!} = 360$.

Improving on our first paper on equivalence, we considered $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$. Counting the maximal chains of G, we obtained that G has $\sum_{i=-1}^{m-1} r_i(m-i)$ maximal chains where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$ for $n \geq 2$. This result was established inductively.
This enabled us to count the number of distinct fuzzy subgroups, and we established the formula $2^{n+m+1} \sum_{r=0}^{m} 2^{-r} \binom{n}{n-r} \binom{m}{r} - 1$ where $m \leq n$ for this number. This was achieved by first looking at specific cases till a conjecture was possible. Details appear in our paper titled *Counting the number of fuzzy subgroups of an abelian group of order* $p^n q^m$.

8 Techniques of Counting Fuzzy Subgroups

We established two techniques of counting fuzzy subgroups: $cross - cut$ counting and $criss - cut$ counting.
Cross-cut: Start off with a keychain $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and associate it with distinct flags (maximal chains) such as $0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$. Count the number of distinct fuzzy subgroups obtained.
Next take another preferential keychain and repeat the process until all keychains have been exhausted. Find the grand total.

Criss-cut: Start off with a flag (maximal chain) such as $0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$. Associate it with all preferential keychains $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \cdots \subseteq G_n = G$. Count the number of distinct fuzzy subgroups obtained.
Next take another flag and repeat the process until all flags have been exhausted. Find the grand total.

Example 8.1 *Let* $G = \{e, a, b, c\}$ *be the Klein 4-group. Thus* $a^2 = e = b^2 = c^2$. *The subgroups of G are* $\{e\}$, $H_1 = \{e, a\}$; $H_2 = \{e, b\}$; $H_3 = \{e, c\}$. *Thus the maximal chains are*
$\{e\} \subseteq H_1 \subseteq G$
$\{e\} \subseteq H_2 \subseteq G$
$\{e\} \subseteq H_3 \subseteq G$
The keychains are 111, 11λ, 110, $1\lambda\lambda$, $1\lambda\beta$, $1\lambda 0$, 100.
Using the cross-cut method, we start with 111 *and associate it with the 3 flags. This gives only one fuzzy group. Similarly* $1\lambda\lambda$ *and* 100 *yield only one fuzzy group each. Each of the remaining 4 keychains gives one fuzzy subgroup on each flag, thus effectively 3 fuzzy subgroups on the 3 flags. Thus the total number of distinct fuzzy subgroups is* $1 + 1 + 1 + 3 + 3 + 3 + 3 = 15$.
Using the criss-cut method, we start off with a flag and associate it with the 7 keychains, obtaining $7 = 2^3 - 1$ *distinct fuzzy subgroups. We repeat the process on the 2nd flag and 3rd flags and each gives* 2^2 *distinct fuzzy subgroups. Thus the grand total is* $2^3 - 1 + 2^2 + 2^2 = 15$.

9 Research with departmental staff and students

Here I present the work I did with a colleague in my Department, Mr O Ndiweni, during his research for a doctoral degree.
So far all our classifications of fuzzy subgroups have been using finite abelian groups. Ndiweni and I therefore decided to deviate for a while and consider non-abelian groups. One such group is a dihedral group D_n of order $2n$.
What is a dihedral group? It is a finite group whose elements are symmetries of a regular n-gon. For example D_3 consists of the symmetries of an equilateral triangle. In this case there are 3 rotational symmetries and 3 diagonal flips for a total of $6 = 3!$ elements. Similarly D_4 consists of symmetries of a square which is a regular 4-gon. Dihedral and symmetric groups are quite useful in mathematics as they sometimes provide counter-examples. Our first paper on the dihedral group focused only on the case when $n = p^m$ for any prime p. So now let $G = D_{p^n}$.
First we established subgroups of G and thereafter we computed the number of maximal chains. This enabled us to establish that $G = D_{p^n}$ has $\sum_{i=0}^{n} p^i$ maximal chains.

As observed before, maximal chains help us to count distinct fuzzy subgroups. Hence we established that $G = D_{p^n}$ has $2^{n+2} - 1 + 2^{n+1} \sum_{i=1}^{n} p^i$ distinct fuzzy subgroups. We started small by picking specific primes and specific exponents. We used our ingenuity to observe patterns that are somewhat concealed. Thereafter we came up with these nice formulae.

Next we used the dihedral group $G = D_{pq}$, for distinct primes p, q. When dealing with a dihedral group in the classification problem, it is always useful to give the dihedral group its purely algebraic flavour as opposed to geometric flavour. Thus $D_n =< a, b : a^n = e = b^2 = (ab)^2 >$. We will illustrate shortly how to compute subgroups of D_n. For $G = D_{pq}$, we established that there are $2pq + (p + q) + 2$ maximal chains and $12pq + 8(p + q) + 23$ distinct fuzzy subgroups.

Example 9.1 *Let $G = D_6 = D_{2\times 3} =< a, b : a^6 = e = b^2 = (ab)^2 >$, so G has 12 elements. Here are the subgroups of G.*
Cyclic ones: $\{e\}$; $< a >$; $< a^2 >$; $< a^3 >$; $< b >$; $< ab >$; $< a^2b >$; $< a^3b >$; $< a^4b >$; $< a^5b >$.
Dihedral Ones: D_6; $D_3^b =< a^2, b : (a^2)^3 = e = b^2 = (a^2b)^2 >$; $D_2^b =< a^3, b : (a^3)^2 = e = b^2 = (a^3b)^2 >$;
$D_3^{ab} =< a^2, ab : (a^2)^3 = e = (ab)^2 = (a^2(ab))^2 >$;
$D_3^{a^2b} =< a^2, a^2b : (a^2)^3 = e = (a^2b)^2 = (a^2(a^2b))^2 >$;
$D_2^{ab} =< a^3, ab : (a^3)^2 = e = (ab)^2 = (a^3ab)^2 >$;
Computing the number of maximal chains and the number of distinct fuzzy subgroups manually, we obtain the numbers 40 and 135 respectively. These agree with the numbers obtained by using our above formulae.

Next, for $G = D_{pqr}$, for distinct primes p, q, r, we obtained the following formulae: $6 + 2(p + q + r + pq + qr + pr) + 6pqr$ for maximal chains and $103 + (2^4 + 2^3)(p + q + r + pq + pr + qr) + (2^4 + 4 \times 2^3 + 2^2)pqr$ for distinct fuzzy subgroups.

For $G = D_{pqrs}$, for distinct primes p, q, r, s, we obtained the following formulae: $4!+3!(p+q+r+s)4!(pq+pr+ps+qr+qs+rs)+3!(pqr+pqs+prs+qrs)+4!pqrs$ for maximal chains and $2^6-1+2^5[11+(p+q+r+s)+(pq+qr+ps+qs+pr+rs)+(pqr+pqs+prs+qrs)+pqrs]+2^4[11+4(p+q+r+s)+2(pq+pr+qr+ps+qs+rs)+4(pqr+pqs+prs+qrs)+11pqrs]+2^3[1+(p+q+r+s)+(pq+pr+qr+ps+rs+qs)+(pqr+pqs+prs+qrs)+12pqrs]$ for distinct fuzzy subgroups.

Attempting a general Formula:
A general formula for the number of maximal chains of the group $G = D_{p_1p_2\cdots p_n}$, where the p_i are all distinct primes and n any positive integer, was always looming. First we established the following result:

Proposition 9.2 *Let $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$, $n \geq 2$. Then the number of cyclic maximal chains of G is $n!$, where the p_i are distinct primes.*

A cyclic maximal chain is one whose proper subgroups are all cyclic.
Second, we have

Proposition 9.3 *The number of d-cyclic maximal chains of subgroups of $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$ is $(n-1)![p_1+p_2+\cdots+p_n]$, $n \geq 2$.*

A d-cyclic maximal chain is one whose proper subgroups are all cyclic except the maximal one which is dihedral.

Proposition 9.4 *The number of 2d-cyclic maximal chains of subgroups of $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$ is $2(n-2)![p_1p_2+p_1p_3+\cdots+p_1p_n+p_2p_3+p_2p_4+\cdots+p_2p_n+\cdots+p_{n-1}p_n] = 2(n-2)!\sum_{i<j} p_ip_j$, $i, j \in \{1, 2, \cdots, n\}$, $n > 2$.*

A 2d-cyclic maximal chain is one whose proper subgroups are all cyclic except the top 2 ones which are dihedral. An md-cyclic maximal chain is similarly defined.

Generally,

Conjecture 9.5 *The number of md-cyclic maximal chains of subgroups of $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$ is*
$2(n-m)!\sum_{k_1<k_2<\cdots<k_m}[p_{k_1}p_{k_2}\cdots p_{k_m}]$, $k_i \in \{1, 2, \cdots, n\}$, for $m > 1$ and $i \in \{1, 2, \cdots, m\}$.

The proof is similar to the proofs of the preceding propositions.

Proposition 9.6 *The number of b-cyclic maximal chains of subgroups of $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$ is $n![p_1p_2p_3\cdots p_n]$.*

A b-cyclic maximal chain is one whose proper subgroups are all dihedral except two viz. the trivial subgroup and the one generated by a^kb where k is a non-negative integer depending on the nature of the dihedral group.

Theorem 9.7 *The number of maximal chains of subgroups of* $G = D_{p_1p_2\cdots p_n} =< a,\ b\ :\ a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$*, is equal to* $n! + (n-1)!\sum_{i=1}^{n}[p_i] + 2(n-2)!\sum_{i<j}[p_ip_j] + 2(n-3)!\sum_{i<j<k}[p_ip_jp_k] + \cdots + 2(n-3)!\sum_{i_1<i_2<\cdots<i_{n-3}}[p_{i_1}p_{i_2}\cdots p_{i_{n-3}}] + 2(n-2)!\sum_{i_1<i_2<\cdots<i_{n-2}}[p_{i_1}p_{i_2}\cdots p_{i_{n-2}}] + (n-1)!\sum_{i_1<i_2<\cdots<i_{n-1}}[p_{i_1}p_{i_2}\cdots p_{i_{n-1}}] + n![p_1p_2p_3\cdots p_n]$.

NOTE: (1) We do have a conjecture for the number of distinct fuzzy subgroups but it still needs some fine-tuning.
(2) Ndiweni and Makamba have already published (jointly) 3 papers on the classification of fuzzy subgroups of dihedral groups. A 4th paper has been accepted for publication.

10 Work with my M Sc student, Isaac Appiah:

Appiah and I, after hard labour and sweat, have finally established the following results, which are good enough to be published:

Theorem 10.1 *Let* $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \mathbb{Z}_p$*. Then* G *has*
(1) $p(2pn + n + 1) + n + 3$ *subgroups for* $p > 2$
(2) $(p+1) + (p^2+p)(\frac{n(n+1)}{2}p + n)$ *maximal chains*
(3) $2^{n+3} - 1 + 2^{n+2}[p(2pn+n+1)] + 2^{n+1}[p+1+(p^2+p)(\frac{n(n+1)}{2}p+n) - p(2pn+n+1) - 1]$ *distinct fuzzy subgroups.*

11 Other Versions of Equivalence

Among researchers in the classification of fuzzy groups, there are a few notable ones with versions of equivalence different from ours. These include, inter alia, Volf [32], Branimir and Tepavcevic [10], Degang et al [13] and Tarnauceanu and Bentea [30], [31]. We attempt to relate their notions of equivalence to ours. Tarnauceanu and Bentea define equivalence as follows:
$\mu \approx \nu \Longleftrightarrow \forall x, y \in G, \mu(x) > \mu(y) \Longleftrightarrow \nu(x) > \nu(y)$.
So their equivalence is ours minus the second property about supports, see our definition.
$\mu \approx \nu \Longleftrightarrow$ (i) $\forall x, y \in G,\ \mu(x) > \mu(y) \Longleftrightarrow \nu(x) > \nu(y)$ and (ii) $\mu(x) = 0 \Longleftrightarrow \mu(y) = 0$.

We usually say their definition is weaker than ours in the sense that if fuzzy subgroups are equivalent in terms of our definition, they must be equivalent in terms of Tarnauceanu's definition but not conversely. This definition gives fewer equivalence classes as it is easier for more fuzzy subgroups to be equivalent. These researchers have justified their choice of equivalence. They have obtained results similar to ours and a few more as their equivalence is less demanding.
According to Volf, two fuzzy subgroups μ and ν of G are equivalent if they have the same set of level subgroups. This definition is equivalent to Tarnauceanu's, hence

weaker than ours.
Branimir and Tepavcevic give the following version of equivalence: Let $\mu, \nu : X \rightarrow L$. Then μ is equivalent to ν iff μ and ν have equal families of cuts.
This equivalence is stronger than ours, i.e. If μ and ν are Branimir-equivalent, then they are Makamba-equivalent. Hence they are also Tarnauceanu-equivalent.

Degang C et al introduced the analysis method. They defined the equivalence of two fuzzy subsets as follows: let μ and ν be fuzzy sets of X, then μ and ν are strong equivalent if $\mu^R = \nu^R$, where μ^R denotes the collection of all $a \in X$ such that $\mu(a)$ is a right limited point of $Im(\mu)$. In addition to strong equivalence, they define two fuzzy subgroups μ and ν of a group X to be S^*-equivalent, denoted $\mu \cong \nu$ if $Im(\mu) = Im(\nu)$, $sup(\mu) \cong sup(\mu)$ and for any $t \in [0,1]$, $\mu^t \neq \emptyset$ implies that there exists an $s \in [0,1]$ such that $\mu^t \cong \nu^s$ and for any $s \in [0,1]$, $\nu^s \neq \emptyset$ implies there exist a $t \in [0,1]$ such that $\nu^s \cong \nu^t$. We observe that if we replace $sup(\mu) \cong sup(\nu)$ by $sup(\mu) = sup(\nu)$ and $\mu^t \cong \nu^s$ by $\mu^t = \nu^s$ then S^*-equivalence relation becomes the strong equivalence relation.
It seems clear that Degang's S^*-equivalence is weaker that the Makamba equivalence, i.e. If μ and ν are Makamba-equivalent then they are S^*-equivalent but not conversely.

12 Further research

On classification of fuzzy subgroups of (1) any dihedral group
(2) any finite abelian p-group
(3) any finite p-group
(4) any finite abelian group
(5) any finite group

13 Citation

Finally, the first three papers published by Murali and Makamba on equivalent fuzzy subgroups have been cited 58, 41 and 32 times respectively by international researchers. The later papers have not done well in terms of citation. However, we are satisfied that we did lay a good foundation for the study of equivalence of fuzzy subgroups.

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*****$\epsilon\mu\chi\alpha\rho\acute{\iota}\sigma\tau\omega$ $\pi o\lambda\nu$ (Efcharisto poly)*****

*****Merci beaucoup*****

*****Enkosi kakhulu*****

*****Baie dankie*****

*****Thank you very much*****

**********END***********