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STUDENTS' THINKING PROCESS IN SOLVING MATHEMATICAL PROOF PROBLEM

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Abstrak

Penelitian ini bertujuan mendeskripsikan proses berpikir mahasiswa dalam menyelesaikan masalah pembuktian matematis. Subjek dalam penelitian ini adalah mahasiswa Program Studi Pendidikan Matematika Universitas Sulawesi Barat yang sedang menempuh mata kuliah Struktur Aljabar tahun akademik 2019/2020 dan dikelompokkan ke dalam 3 kategori kemampuan pembuktian, yakni tinggi, sedang, dan rendah. Untuk setiap kategori, dipilih 2 subjek untuk diwawancarai guna mendapatkan data deskripsi proses berpikir mahasiswa. Dari hasil tes dan wawancara diperoleh bahwa pada entry phase, subjek kategori tinggi dan sedang mampu menemukan prosedur awal pembuktian yang tepat, sedangkan subjek kategori rendah belum mampu. Pada attack phase, subjek kategori tinggi mampu menyelesaikan proses pembuktian sampai tahap akhir dengan sangat sistematis, subjek kategori sedang mampu menggunakan sifat-sifat dan teorema namun masih kesulitan mencapai hasil akhir yang tepat, sedangkan subjek kategori rendah berusaha menggunakan sifat-sifat dan teorema namun tidak berkaitan dengan proses pembuktian. Pada review phase, subjek kategori tinggi mampu menjelaskan kembali garis besar prosedur pembuktian, subjek kategori sedang berusaha melengkapi kekurangan prosedurnya, dan subjek kategori rendah menyadari pentingnya proses awal pembuktian.
Kata kunci: pembuktian matematis; pemecahan masalah; proses berpikir.

Abstract

This research aimed to describe students' thinking process in solving mathematical proof problem. The subjects in this research were Mathematics Education Study Program students in Universitas Sulawesi Barat who were taking Abstract Algebra course on academic year of 2019/2020 and were grouped into 3 proving ability categories, high, medium, and low. For each category, two subjects were chosen to be interviewed in order to obtain a description of students' thinking processes. From the results of tests and interviews, it was found that in the entry phase, high and medium category subjects were able to find the right initial verification procedure, whereas low category subjects were not yet able. In the attack phase, high category subjects were able to complete the process of proof to the final stage very systematically, medium category subjects were able to use the properties and theorems but still have difficulty achieving the right end result, while low category subjects try to use the properties and theorems but are not related to the verification process. In the review phase, high category subjects were able to explain the procedure outline again, category subjects were trying to supplement the shortcomings of the procedure, and low category subjects realized the importance of the initial proof process.
Keywords: mathematical proof; problem solving; thinking process.

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INTRODUCTION

Mathematical proof is one of the most fundamental components in learning mathematics at the undergraduate level. Some courses such as set theory, number theory, discrete mathematics, abstract algebra, and real analysis include basic logic and methods of proof in the materials. Stefanowicz, et al. (2014) revealed that proof is the essence of mathematics learning. Lecturers at universities will be very thorough with their explanations and will present notations clearly and every theorem will be proven. The importance of mathematical proof was stated by Lesseig, et al. (2019), "the centrality of proof of mathematics is indisputable" and previous expert who argued that one needs to prove a mathematical proposition to ensure that what has been considered true is true (Hernadi, 2008).

Proof becomes very important for students of mathematics because mathematics in undergraduate level requires quite high reasoning and thinking, in contrast to mathematics at the secondary level where learning does not have to be preceded by definitions or theorems. Several descriptions of mathematical proof have been provided by previous researchers (Abdussakir, 2014; Hanna, G; Barbeau, 2010; Kartini & Suanto, 2016; Nurrahmah & Karim, 2018). According to Hanna & Barbeau (2010), proof is a determinant of the truth of mathematical claims, the truth of a mathematical proposition is determined after proven true. Abdussakir (2014) concluded that activities to produce mathematical proofs mean a series of activities assembling and logically connecting true statements to prove and explain the truth of a mathematical statement. Kartini (2016) stated that mathematical proof is a fundamental part because the truth value of a mathematical proposition depends on its proof. Nurrahmah & Karim (2018) argued that mathematical proof is a demonstration of using logic and mathematics to ascertain the truth of formulas and theorems.

However, mathematical proof for students is still a pretty serious problem. Some research results (Imamoglu & Togrol, 2015; Muliawati, 2018; Ozdemir &

Ovez, 2012) revealed that there were still many shortcomings faced by students in solving mathematical proof problems. Ozdemir & Ovez (2012) found as many as 55% of 67 elementary mathematics prospective teachers knew about the importance of formal proof in mathematical proof but preferred to use informal proof because they were not familiar with formal proof. According to Imamoglu & Togrol (2015), higher semester students also still had difficulty in constructing proofs and conducting evaluations. Muliawati (2018) revealed that the majority of students only memorized the concept of proof of Group Theory but the understanding of the concepts inherent in their cognition was still very low.

Based on Stevanowicz, et al. (2014), common mistakes that students make when trying to present the proofs are misunderstanding of definition, not enough words, lack of understanding, and incorrect steps. The result of Stylianou, Blanton & Rotou (2015) research showed that one of the causes of the low ability to prove is a passive classroom environment. In addition, the cause of the low ability of proof of students is that many students try to ignore the problem of proof and avoid it (Hasan, 2016).

The process of mathematical proofing requires sufficient understanding and experience. Mathematical proof is also closely related to thinking skills in choosing strategies and extracting knowledge in memory that has been obtained previously. The thinking process is one of the factors that need special attention when students are conducting mathematical proofing activities. Suryana (2015) revealed that in constructing mathematical proofs the ability to think creatively is needed. Likewise according to Ozdemir & Ovez (2016) that logic and logical thinking skills are important in writing mathematical proofs.

According to Yohanie, Sujadi & Usodo (2016), the thinking process is a process or way of thinking. Mason, et al. (2010) suggested that the mathematical thinking process in completing problems is divided into three phases called the entry phase, attack phase, and review phase. The entry phase starts when first meet with a question. The entry phase is done to overcome a question that is when it first confronts a question and ends when it has begun to try to solve it. The attack phase should be the most important part because it covers the largest part of the

mathematical activities undertaken. The attack phase can be said to be complete if the problem is abandoned or resolved. The attack phase is done by taking several approaches that can be used as well as formulating and trying out plans. If the plan has been carried out, there will be good progress in working to resolve the problem. The review phase is done when a satisfactory solution has been reached or when it is about to give up, so it is important to review the work that has been done. The review phase is useful in reflecting on the previous phases. In this phase it will help to check whether the mathematical thinking process in problem solving is correct and whether the problem has been solved. Activities in the review phase are ways of solving problems and reflecting on what has been done and why.

The importance of the ability to solve the mathematical proof problem for students and the many obstacles of students in mathematical proof as stated above make researchers interested in describing students' thinking processes in solving mathematical proof problem. Research that aims to see the relationship of thinking ability with the ability to solve mathematical problems has been done by many researchers, as well as research to observe the thinking process of students in solving mathematical problems, has been conducted by several researchers. This research aims to provide a description of students' thinking processes in solving mathematical problems, but is focused on the proof problems.

METHOD

This research was a case study research with a qualitative approach. The selection of research subjects was done by purposeful sampling. According to Creswell (2015), the term of research used for qualitative sampling is purposeful sampling. The subjects of this research were Mathematics Education Study Program students in Universitas Sulawesi Barat who were taking Abstract Algebra courses on academic year of 2019/2020. Considerations in the selection of this subject were (1) the students are certain to be taking the topic of Group Theory which is used as material in the mathematical proof test instrument, (2) the subjects' ability to communicate or express their thoughts, and (3) the subject's willingness to participate in data collection during research. The data collection process began

with the provision of mathematical proof tests to all Mathematics Education Study Program students in Universitas Sulawesi Barat who were taking Abstract Algebra courses on academic year of 2019/2020, then from these results the researchers chose 6 students to be interviewed related to their works, two of each people for 3 ability levels: high, medium, and low. The categorization of these subjects aims to obtain a broad description of the student's thinking process in this research. To check the validity of the data in this research, persons triangulation and method triangulation were carried out. The data obtained will then be analyzed through the stages of data reduction, data presentation, and drawing conclusions.

RESULTS AND DISCUSSION

The recapitulation of the Mathematical Proof Test scores from 67 students as a basis for determining the research subjects is presented in Table 1.

Table 1. Mathematical Proof Test Scores

Score	Category	Number of Students	Percentage
≥ 80	High	3	4%
$60 - 80$	Medium	2	3%
< 60	Low	62	93%
Total		67	100%

Based on Table 1, the initial conclusion is that students who have high mathematical proof ability are 4%, medium ability 3%, and low ability 93%. For each category 2 subjects have been chosen to be interviewed in order to obtain a description of the thinking process of each subject. The results of mathematical proofs of high category subjects are presented in Figure 1 below.

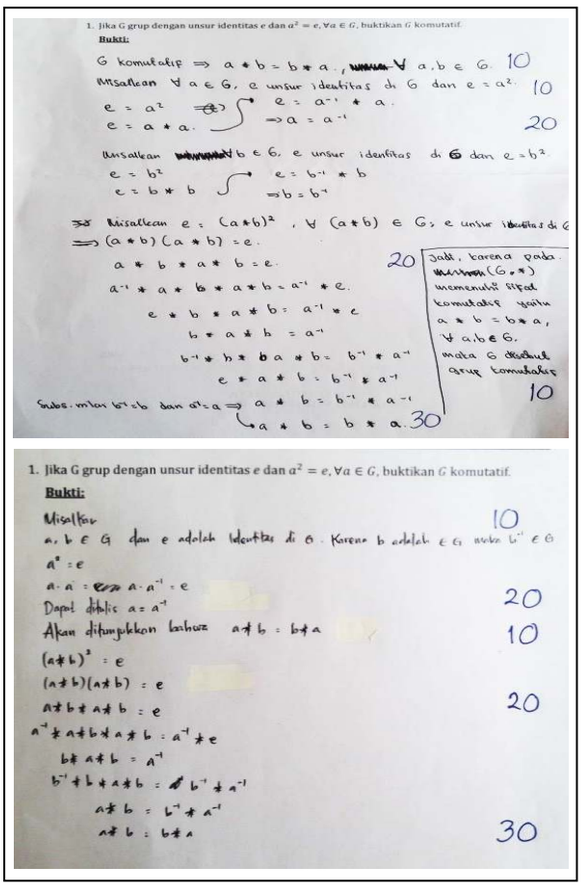

Figure 1. The Results of Mathematical Proof Test of High Category Subjects

The results of high category subjects interviews related to the results of mathematical proof test are presented in Table 2.

Table 2. The Results of High Category Subjects Interviews

Subject	Interview Result
HCS1 (High Category Subject 1)	Entry Phase • Subject explained the initial steps of the proofing process carried out, namely how to prove consequent. • Mentioned what elements were given to the problem and other elements that were not given to the problem but will be used in the proving process. • Explained the concepts of all terms contained in the problem. • Described the elements that were given to the problem to get a new element. Attack Phase • Subject used the group's basic axioms to prove.

Subject	Interview Result
	• In the process of completion, the subject was always based on the consequences and used the elements that have been obtained in the previous step for reuse, so the proving process was carried out very systematically. • Concluded the proving process correctly and related it with the consequences of the problem. Review Phase • Abled to explain in a coherent proof of the process carried out starting from the initial step until drawing conclusions.
HCS1 (High Category Subject 2)	Entry Phase • The subject explained the initial steps of the proving process carried out, i.e. mentioning all the elements that are given to the problem, for example a new element not mentioned in the problem. • The new elements were mentioned in the previous step will be used for the proofing process. • Described the elements that were given to the problem to get a new element. Attack Phase • Subject described what will be shown in the problem • Used the new elements that appear in the entry phase to start the proving process. • Used the group's basic axioms to process the proof so that it results in accordance with what will be shown. Review Phase • Subject explained the lack of proof that is done that is not writing down the reasons on each line of proof. • Although successfully completed the element to be demonstrated, the subject has not written down conclusions and related them with consequent problems. • Abled to explain the process of proof in a coherent and clear manner.

The results of mathematical proofs of medium category subjects are presented in Figure 2 below.

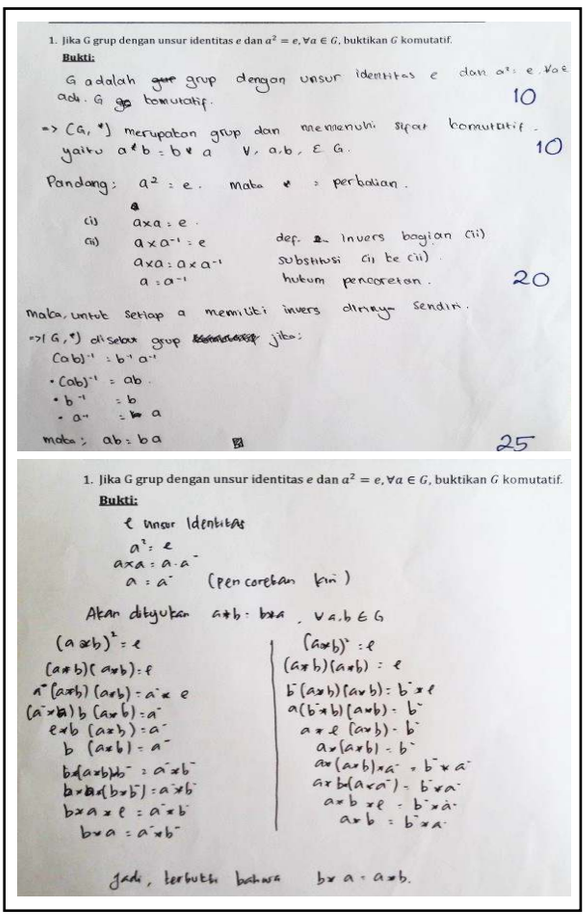

Figure 2. The Results of Mathematical Proof Test of Medium Category Subjects

The results of medium category subjects interviews related to the results of mathematical proof tests are presented in Table 3.

Table 3. The Results of Medium Category Subjects Interviews

Subject	Interview Result
MCS1 (Medium Category Subject 1)	Entry Phase • Subject started the proving process by paying attention to the consequences and then proceeded by taking note of the given elements • Abled to decipher consequently into what will be shown. Attack Phase • Subject used other theorems to help the proving process. • Tried to relate all new elements obtained in the previous steps but has not been able to write and explain systematically the procedure towards the end of the proof.

	Review Phase • Subject stated that the lack of work is less systematic in concluding the results of the proof. • Abled to review the proving process by explaining the outline of the proving process, especially how the initial steps of the proving process.
MCS2 (Medium Category Subject 2)	Entry Phase • Subject started the proving process by re-mentioning what elements were given. • Brought up new elements obtained from given elements. • Abled to explain the consequences to be the part that will be shown. Attack Phase • Subject described the proving process by a method of two. • Used basic group axioms to carry out the verification process even though there was still a mistake in one of the parts so that it gets an incorrect result at the end. • During the interview, the subject was aware of the mistakes made when working on the matter of proof and tried to improve the work even though there were still errors in the process. Review Phase • Subject explained the outline of the proving process, which was to see what will be addressed to the problem, solve one by one, and relate the elements that were given to complete the proving process.

The results of mathematical proofs of low category subjects are presented in Figure 3 below.

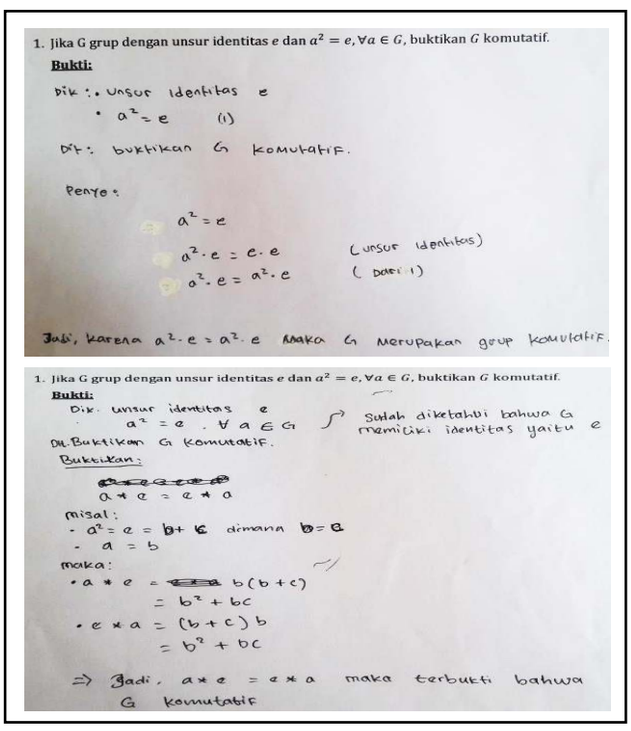

Figure 3. The Results of Mathematical Proof Test of Low Category Subjects

The results of low category subjects interviews related to the results of mathematical proof tests are presented in Table 4.

Table 4. The Results of Low Category Subjects Interviews

Subject	Interview Result
LCS1 (Low Category Subject 1)	Entry Phase • Subject started the explanation of the proving process by remembering a theorem that has been proven before. The subject tried to link the theorem with the given elements in the problem. • Knew what will be proven even though he/she has not been able to explain correctly what should be shown in this problem. • Mentioned the elements that were given and understood the concepts of these elements. Attack Phase • Subject tried to use and describe given elements even though the process did not lead to what will be shown in the problem. • At the end of the completion step the subject got an incorrect result despite tried to use the group's basic axioms. Review Phase • In the interview process, the subject was aware of the inaccurate final results obtained and started thinking about the proving process that should be done.
LCS2 (Low Category Subject 2)	Entry Phase • Subject mentions the elements that are given in the problem. • Described what will be proven in the problem. • Subject was able to write what will be shown to solve the problem but was still not quite right. Attack Phase • Subject tried to use elements that were given in the problem but in the next step were still not quite right. • Intended to prove using the method for two, but it was still not quite correct despite tried to use the elements obtained in the previous step. • Concluded the proving process even though what was shown was not right to prove the consequences. Review Phase • Subject recounted the proving process and realized that showing what will be proven was an important process in the proving process.

Based on the triangulation results obtained from the mathematical proof problem test and interviews to two subjects for each category of mathematical proof ability obtained that at the entry phase, high category subjects were able to know the initial steps of the proving process to be carried out, namely breaking down the consequences into the form to be shown then the subject brings up a new element that was not mentioned in the matter obtained through the decomposition of the

given elements, the new element would be used to help the process of proof, besides that subjects were also able to explain the concept of all the elements given to the problem. In the attack phase, the subjects were able to use basic axioms for the proving process and used new elements obtained in the entry phase to go to the consequences that will be demonstrated so that the proving process was very systematic. The end result of proof was linked back by the subjects to the consequences, during the process of proof the consequence was used as a basic benchmark by the subjects in the process of proof. In the last phase, the review phase, the subjects were able to explain again the outline of the proving process carried out. Even so, the HCS2 subject did not write their final conclusions on the answer sheet.

For the medium category subjects, at the entry phase, the subjects started the proving process by paying attention to the consequences and given elements. Next outline the consequences to what will be shown. Before beginning proof of subjects the emergence of new elements not mentioned in the questions obtained through the breakdown of the given elements, the new elements would be used to assist the proving process. Entering the attack phase, the subjects used basic axioms and theorems to try to solve the proving problem and used the new elements obtained in the entry phase to go to the consequences to be demonstrated. Even so, the subjects still had difficulty in the final step of proof to be consistent. The subject tried to link between the elements he obtained but was not able to reach the final conclusions to be addressed. The steps and process of explanation by the subjects were still less systematic. In the review phase, the subjects were able to explain the evidentiary procedures carried out but only focus on the initial steps, while for the main proving step, the subjects seemed to have difficulty in explaining the thinking process carried out.

As for the low category subjects, at the entry phase, the subjects recounted the elements that were given to the problem. Furthermore, the subjects mentioned the consequences of the problem but did not know what to show from the consequence. The subjects wrote what will be indicated on the answer sheet but was not quite right. In the attack phase, the subjects used and described elements

that were given even though the use is still not quite right, used properties and theorems that had no relationship with the process of proof. At the end of the settlement process, the subjects concluded but were not quite right and were unable to prove the consequences. As for the review phase, the subjects realized that the most important process to be carried out in the proving process was to describe what will be proven to be what will be demonstrated.

Knuth (2002) and Stylianides (2007) revealed that the growing attention to the important role of proof in mathematics made many attempts to identify mathematical learners' thinking processes and the development of these thinking processes. Based on the theory of Mason, et al. (2010), there are 3 stages of thinking in solving mathematical problems, namely entry phase, attack phase, and review phase. The results of this research indicate that the thinking process of high category subjects is classified as very good, the stages of the thinking process are passed very smoothly. In general, the two subjects in the high category were able to explain very well their thinking processes at each step they wrote. As for the medium category subject, they are also able to complete and explain the three stages of the thinking process even though the process requires a little complexity and has not been able to be arranged properly. While the low category subjects felt very difficult in working on the problem of proof, this is illustrated by the inability of the two subjects to start the verification process at the entry phase. The difference in the thinking process is in accordance with the results of Suryana's (2015) research, namely that students' thinking ability has a significant correlation to the ability to construct mathematical evidence. The results of this research are also in line with what was found by Netty (2018) in her research which also examines students' thinking processes in constructing mathematical evidence and find five stages of thinking, namely understanding the evidentiary problem, making connections and selecting, finding main ideas, compiling evidence and concluding, and reflect.

CONCLUSION

Based on the results and discussion presented in the previous section, it can be concluded that subjects with three categories of ability to solve the mathematical proof problem through a different thinking process in each phase. In the entry phase, subjects in the high category were able to think and plan well for the initial procedure of proof, while subjects in the low category were not able. Furthermore, in the attack phase, high category subjects were able to complete the proving process to the final stage by very systematically using properties and theorems, medium category subjects were able to use properties and theorems but still have difficulty achieving what they want to show, while low category subjects tried to use the properties and theorems but not related to the proving process. In the final phase, namely the review phase, high category subjects were able to fully explain again the outline of the proofing procedure carried out, the category subject was tried to complete and understood the shortcomings of the procedure, and the low category subject realized the importance of the initial proving process. One of the side findings in this research is the students' low ability to solve mathematical proof problem. Based on this finding, it is recommended for further studies to be able to examine specifically about the causes of the low mathematical proof ability of students, especially on Group Theory material. Furthermore, researchers also recommend the development of tools or learning models that can improve students' mathematical proof-solving abilities.

REFERENCES

Abdussakir. (2014). Proses Berpikir Mahasiswa dalam Menyusun Bukti Matematis dengan Strategi Semantik. *Jurnal Pendidikan Sains*, *2*(3), 132–140. http://journal.um.ac.id/index.php/jps/

Cresswell, J. W. (2015). *Riset Pendidikan: Perencanaan, Pelaksanaan, dan Evaluasi Riset Kualitatif & Kuantitatif*. Pustaka Pelajar.

Hanna, G; Barbeau, E. (2010). Proofs as Bearers of Mathematical Knowledge. In *Explanation and Proof in Mathematics: Philosophical and Educational Perspectives* (pp. 85–100).

Hasan, B. (2016). APOTEMA: Jurnal Pendidikan Matematika. *APOTEMA: Jurnal Pendidikan Matematika*, *2*, 33–40.

Hernadi, J. (2008). Metoda Pembuktian dalam Matematika. *Jurnal Pendidikan Matematika*, *2*(1), 1–13. https://doi.org/10.22342/jpm.2.1.295.

Imamoglu, Y., & Togrol, A. Y. (2015). Proof Construction and Evaluation Practices of Prospective Mathematics Educators. *European Journal of Science and Mathematics Education*, *3*(2), 130–144.

Kartini, & Suanto, E. (2016). Analisa Kesulitan Pembuktian Matematis Mahasiswa Pada Mata Kuliah Analisis Real. *SEMIRATA 2015 Bidang MIPA BKS-PTN Barat*, *1*(1), 189–199. http://jurnal.untan.ac.id/index.php/semirata2015/article/view/14066/13275

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405. https://doi.org/10.2307/4149959

Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. *Mathematics Education Research Journal*, *31*(4), 393–418. https://doi.org/10.1007/s13394-019-00260-7

Mason, J; Burton, L; Stacey, K. (2010). *Thinking Mathematically* (2nd Editio). Pearson Education Limited.

Muliawati, N. E. (2018). Proses Berpikir Mahasiswa Dalam Memecahkan Masalah Pembuktian Grup Berdasarkan Langkah Polya. *Jurnal Pendidikan Dan Pembelajaran Matematika*, *4*(2), 32–42.

Netti, S. (2018). Tahapan Berpikir Mahasiswa dalam Mengonstruksi Bukti Matematis. *Matematika Dan Pembelajaran*, *6*(1), 1. https://doi.org/10.33477/mp.v6i1.437

Nurrahmah, A., & Karim, A. (2018). Analisis Kemampuan Pembuktian Matematis Pada Matakuliah Teori Bilangan. *JURNAL E-DuMath*, *4*(2), 21. https://doi.org/10.26638/je.753.2064

Ozdemir, Emine; Ovez, F. T. D. (2016). Eighth Scien 1 The Ohio Sta Corresponden. *Journal of Education and Training Studies*, *5*(1), 10–20. https://doi.org/10.11114/j

Ozdemir, E., & Ovez, F. T. D. (2012). A Research on Proof Perceptions and Attitudes Towards Proof and Proving: Some Implications for Elementary Mathematics Prospective Teachers. *Procedia - Social and Behavioral Sciences*, *46*, 2121–2125. https://doi.org/10.1016/j.sbspro.2012.05.439

Stefanowicz, A., Kyle, J., & Grove, M. (2014). Proofs and Mathematical Reasoning. *University of Birmingham*, *September*.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

Stylianou, Despina A.;Blanton, Maria L; Rotou, O. (2015). Undergraduate Students' Understanding of Proof: Relationships Between Proof Conceptions, Beliefs, and Classroom Experiences with Learning Proof. *International Journal of Research in Undergraduate Mathematics Education*, *1*, 91–134. https://doi.org/https://doi.org/10.1007/s40753-015-0003-0

Suryana, A. (2015). Peran Kemampuan Berpikir Kreatif Dalam Mengkonstruksi Bukti Matematis. *Semnas Matematika Dan Pendidikan Matematika. Universitas Indraprasta PGRI*.

Yohanie, D. D., Sujadi, I., & Usodo, B. (2016). Proses Berpikir Mahasiswa Pendidikan Matematika Dalam Pemecahan Masalah Pembuktian Tahun Akademik 2014/2015. *Journal of Mathematics and Mathematics Education*, *6*(1), 79–90. https://doi.org/10.20961/jmme.v6i1.10048