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© *A. P. Kolinichenko, L. B. Ryashko*

## ANALYSIS OF STOCHASTIC SENSITIVITY OF TURING PATTERNS IN DISTRIBUTED REACTION–DIFFUSION SYSTEMS

In this paper, a distributed stochastic Brusselator model with diffusion is studied. We show that a variety of stable spatially heterogeneous patterns is generated in the Turing instability zone. The effect of random noise on the stochastic dynamics near these patterns is analysed by direct numerical simulation. Noise-induced transitions between coexisting patterns are studied. A stochastic sensitivity of the pattern is quantified as the mean-square deviation from the initial unforced pattern. We show that the stochastic sensitivity is spatially non-homogeneous and significantly differs for coexisting patterns. A dependence of the stochastic sensitivity on the variation of diffusion coefficients and intensity of noise is discussed.

*Keywords*: reaction–diffusion model, Turing instability, self-organization, stochastic sensitivity

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## Introduction

Self-organization [1] is found in many natural phenomena. The mechanisms of ordered and stable states formation are crucial in various processes studied in different fields of science [2–4]. Often, these phenomena are too complex for observation and experiment. In these cases, investigation of self-organization is possible by introducing an appropriate mathematical model [5, 6].

One of the examples of self-organization is connected with Turing pattern formation in distributed reaction-diffusion systems [7, 8]. Here, the expected effect of the diffusion flux on the system is the appearance of a homogeneous state. However, in the parametric zones of Turing instability, a stable non-homogeneous state (pattern) is formed and maintained. Often, for the same set of parameter values, the model exhibits several coexisting stable patterns. This phenomenon of multistability was discussed in [9].

While deterministic analysis of Turing patterns is an interesting field of research by itself, the main concern of this work is the effect of random perturbations on the system dynamics. Indeed, real systems are always subject to disturbances of a different nature: such as Brownian motion, solvent impurities, changing temperature or pressure. Studying stochastic models may allow to better understand the various phenomena that take place in real systems. Noise-induced effects in reaction-diffusion systems attract the attention of many researchers [10–13].

Mathematical modeling of stochastic processes shows the positive role of noise: in distributed reaction-diffusion systems with random perturbations the ordered and stable states can appear. It should be noted that different patterns respond to the perturbations in different ways: some of them may dissipate while others are preserved. Constructive effects of random noise in nonlinear system attract attention of many researchers [14, 15]. For the study of these effects, a new probabilistic approach using stochastic sensitivity analysis was developed in [16–18].

In the present paper, we consider a distributed reaction-diffusion system based on the classic Brusselator model. In this model, various scenaria of deterministic and stochastic pattern generation in this multistable model can occur. In [19], a phenomenon of noise-induced transitions between patterns was revealed and studied on the base of modality analysis.

This work is devoted to the further development of research begun in [9, 19], which introduce the stochastic phenomena and provide the means to identify them. While previous work implies the difference in pattern stability, the main focus of the present paper is the study and comparison of stochastic sensitivity of the coexisting patterns through the means of statistical analysis and numerical methods. For weak noise, the stochastic sensitivity of the pattern is quantitatively defined by the mean-square deviation of random states from the initial unforced pattern. We apply numerical modeling to show the difference in the sensitivity to noise for different patterns. By this analysis, we explain the "preference" of system in noise-induced transitions. Finally, we analyse the dependence of stochastic sensitivity on parameters of diffusion flow, and study the influence of increasing perturbation intensity on the behavior of different patterns.

## § 1. Pattern formation

The distributed Brusselator model with diffusion is defined by the following system of differential equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= a - (b+1)u + u^2 v + D_u \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} &= bu - u^2 v + D_v \frac{\partial^2 v}{\partial x^2}. \end{aligned} \quad (1.1)$$

Here, the variables $u(t,x)$ and $v(t,x)$ stand for the concentration of the reactants, parameters $a$ and $b$ are positive, and $D_u$, $D_v$ are diffusion coefficients. The spatial scalar variable $x$ varies in the interval $[0, L]$. The following boundary conditions are assumed:

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) = \frac{\partial v}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,L) = 0. \quad (1.2)$$

The non-distributed system (when $D_u = D_v = 0$) has a fixed point at $\bar{u} = a$, $\bar{v} = \frac{b}{a}$. In the distributed system, this fixed point defines the homogeneous equilibrium – a state in which $u(x) = \bar{u}$, $v(x) = \bar{v}$ for every $x$. A parametric zone where the fixed point of the non-distributed system is stable and the homogeneous equilibrium is unstable is called the Turing instability zone. Instability of the homogeneous equilibrium causes formation of spatially heterogeneous stable structures, namely Turing patterns.

In this paper, we fix $a = 3$, $b = 9$, $D_v = 10$, $L = 40$. In this case, the Turing bifurcation value is $D_u^* = \frac{40}{9}$, and patterns are observed for $0 < D_u < D_u^*$.

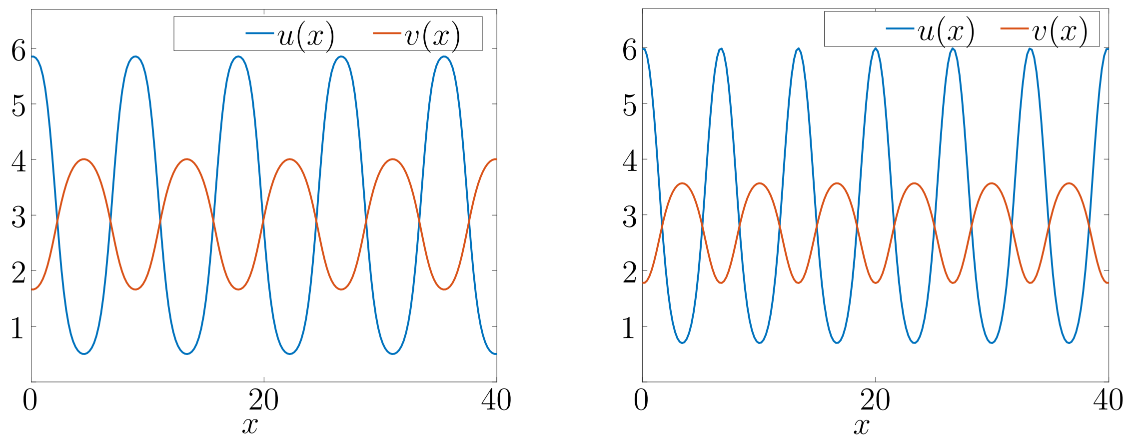


**Figure 1.** Examples of spatial patterns for $D_u = 2$: 4.5-peak pattern (left), 6-peak pattern (right)

The patterns are wave-like structures, that can be distinguished by periodicity (number of peaks) and tendency on the left edge of the spatial interval (ascending or descending). When assigning type, we consider the $u(x)$ component of the resulting state. Fig. 1 shows examples of two patterns: 4.5-down pattern (left) and 6-down pattern (right). Both examples are obtained for $D_u = 2$.

For numerical simulation of $u_{j,i} = u(t_j, x_i), v_{j,i} = v(t_j, x_i)$, we use the following difference scheme:

$$\begin{cases} u_{j+1,i} = u_{j,i} + \tau f_{j,i} + \tau D_u \dfrac{u_{j,i-1} - 2u_{j,i} + u_{j,i+1}}{h^2}, \\ v_{j+1,i} = v_{j,i} + \tau g_{j,i} + \tau D_v \dfrac{v_{j,i-1} - 2v_{j,i} + v_{j,i+1}}{h^2}. \end{cases} \tag{1.3}$$

Here,

$$f_{j,i} = f(u_{j,i}, v_{j,i}), \quad g_{j,i} = g(u_{j,i}, v_{j,i}),$$

$$f(u,v) = a - (b+1)u + u^2 v, \quad g(u,v) = bu - u^2 v,$$

$$\tau = 10^{-4}, \quad h = 0.2, \quad t_j = j\tau, \quad x_i = ih.$$

The system demonstrates multistable behavior: for the same parameter values, several different patterns may be obtained depending on the initial state. For lower values of $D_u$, there can be up to 20 different patterns. Shape of the structure also depends on the diffusion intensity. With decreasing $D_u$, deviations of the system variables from the homogeneous equilibrium increase. Moreover, every pattern has its own stability zone. Outside this zone, it cannot be obtained as attractor, and will most likely appear only as a transient state. Fig. 2 summarizes these observations by showing the extrema of $u$-variable of patterns in corresponding stability intervals.

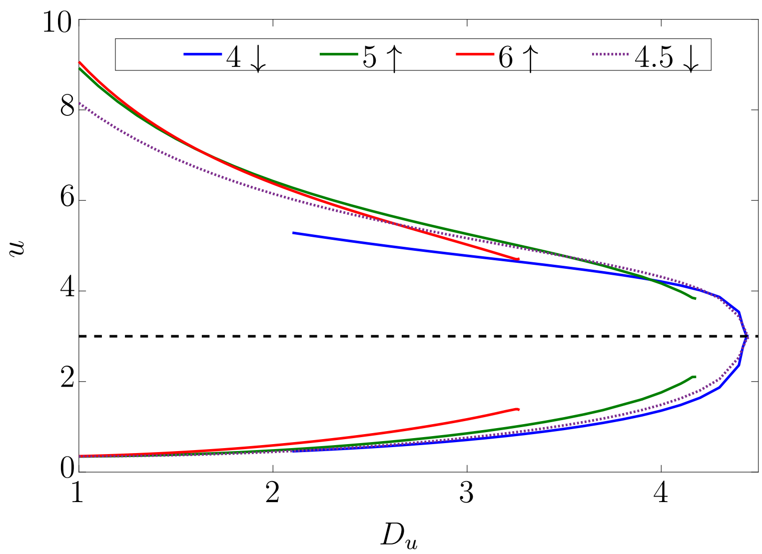


**Figure 2.** Extrema of several pattern-attractors on their stability intervals

Temporal dynamics of pattern formation is another matter worth looking into. As a mechanism of self-organization, the phenomenon of Turing instability contributes to the establishment of organized state from a disorganized one. If we take randomly generated state as the initial condition, then the system will quickly organize itself to an ordered state. In the Turing instability zone, this state is a stable spatially-heterogeneous structure.

Fig. 3 shows an example of pattern generation from a random initial state. The initial condition is generated using uniform distribution on the interval $[0, 6]$ around the homogeneous equilibrium $(\bar{u}, \bar{v}) = (3, 3)$.

The top panel shows dynamics of a process of pattern formation started from a randomly distributed initial condition. The horizontal axis is the temporal axis, while the vertical axis is

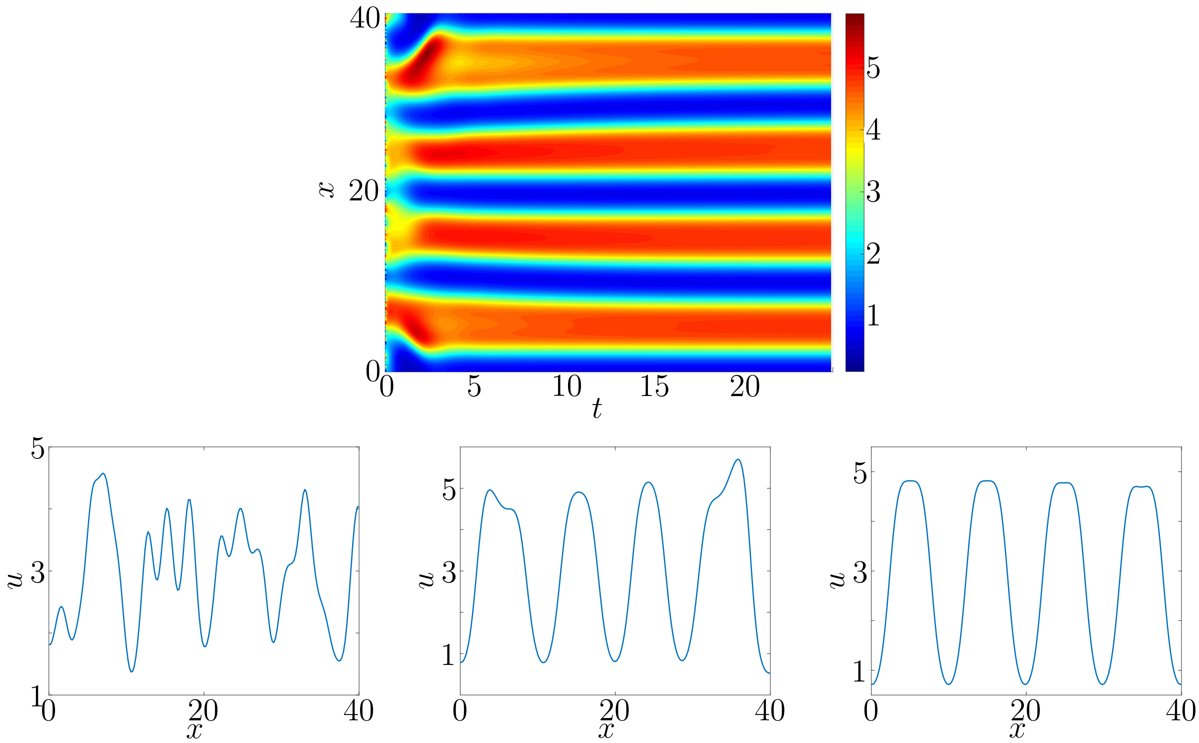


**Figure 3.** Pattern formation dynamics for $a = 3$, $b = 9$, $D_u = 3$, $D_v = 10$

spatial. Values of $u(t, x)$ are displayed by color. The bottom figures show snapshots of $u(t, x)$ for three values of $t$. From left to right: the initial state at $t = 0$, a transient state at $t = 2$, and almost formed pattern at $t = 10$. As the figure shows, the transition to the organized state occurs quite rapidly.

## § 2. Stochastic analysis

In order to study noise-induced phenomena, the stochastic variant of the distributed Brusselator model is introduced:

$$\begin{aligned} \frac{\partial u}{\partial t} &= a - (b+1)u + u^2 v + D_u \frac{\partial^2 u}{\partial x^2} + \gamma_1 \xi(t, x), \\ \frac{\partial v}{\partial t} &= bu - u^2 v + D_v \frac{\partial^2 v}{\partial x^2} + \gamma_2 \eta(t, x). \end{aligned} \tag{2.1}$$

Here, $\xi(t, x)$ and $\eta(t, x)$ are uncorrelated Gaussian random noises with intensities $\gamma_1, \gamma_2$ and parameters $\langle \xi(t, x) \rangle = \langle \eta(t, x) \rangle = 0$, $\langle \xi(t, x)\xi(s, y) \rangle = \delta(s - t)\delta(y - x)$, $\langle \eta(t, x)\eta(s, y) \rangle = \delta(s - t)\delta(y - x)$.

For computer simulations, we use the Euler–Maruyama scheme [20,21] for time discretization

$$\begin{cases} u_{j+1,i} = u_{j,i} + \tau f_{j,i} + \tau D_u \dfrac{u_{j,i-1} - 2u_{j,i} + u_{j,i+1}}{h^2} + \gamma_1 r_{j,i} \sqrt{\tau}, \\ v_{j+1,i} = v_{j,i} + \tau g_{j,i} + \tau D_v \dfrac{v_{j,i-1} - 2v_{j,i} + v_{j,i+1}}{h^2} + \gamma_2 q_{j,i} \sqrt{\tau}, \end{cases} \tag{2.2}$$

where $r_{j,i}$ and $q_{j,i}$ are uncorrelated Gaussian random noises on the grid with parameters $\langle r_{j,i} \rangle = \langle q_{j,i} \rangle = 0$, $\langle r_{j,i} r_{k,l} \rangle = \langle q_{j,i} q_{k,l} \rangle = \delta_{j,k}\delta_{i,l}$, where $\delta_{j,k}$ equals one if $j = k$ and zero otherwise. Here, we suppose $\gamma_1 = \gamma_2 = \gamma$. The method steps are $\tau = 10^{-4}$, $h = 0.2$.

For studying stochastic sensitivity of the pattern, the deviation of a perturbed structure from its deterministic counterpart is considered. First, we obtain the pattern by modeling without random perturbations. Then, this pattern is used as the initial condition for the stochastic model. The stochastic modeling is repeated a large amount of times. With this statistical data the deviation of a pattern can be evaluated for every $x$ as follows:

$$\mathcal{S}_u(x,\gamma) = E(u^\gamma(x) - u^*(x))^2, \quad \mathcal{S}_v(x,\gamma) = E(v^\gamma(x) - v^*(x))^2, \tag{2.3}$$

where $u^*(x)$, $u^*(x)$ are coordinates of the initial unforced deterministic pattern, and $u^\gamma(x)$, $v^\gamma(x)$ are results of stochastic modeling with noise intensity $\gamma$. Note, that these sensitivity functions are obtained numerically.

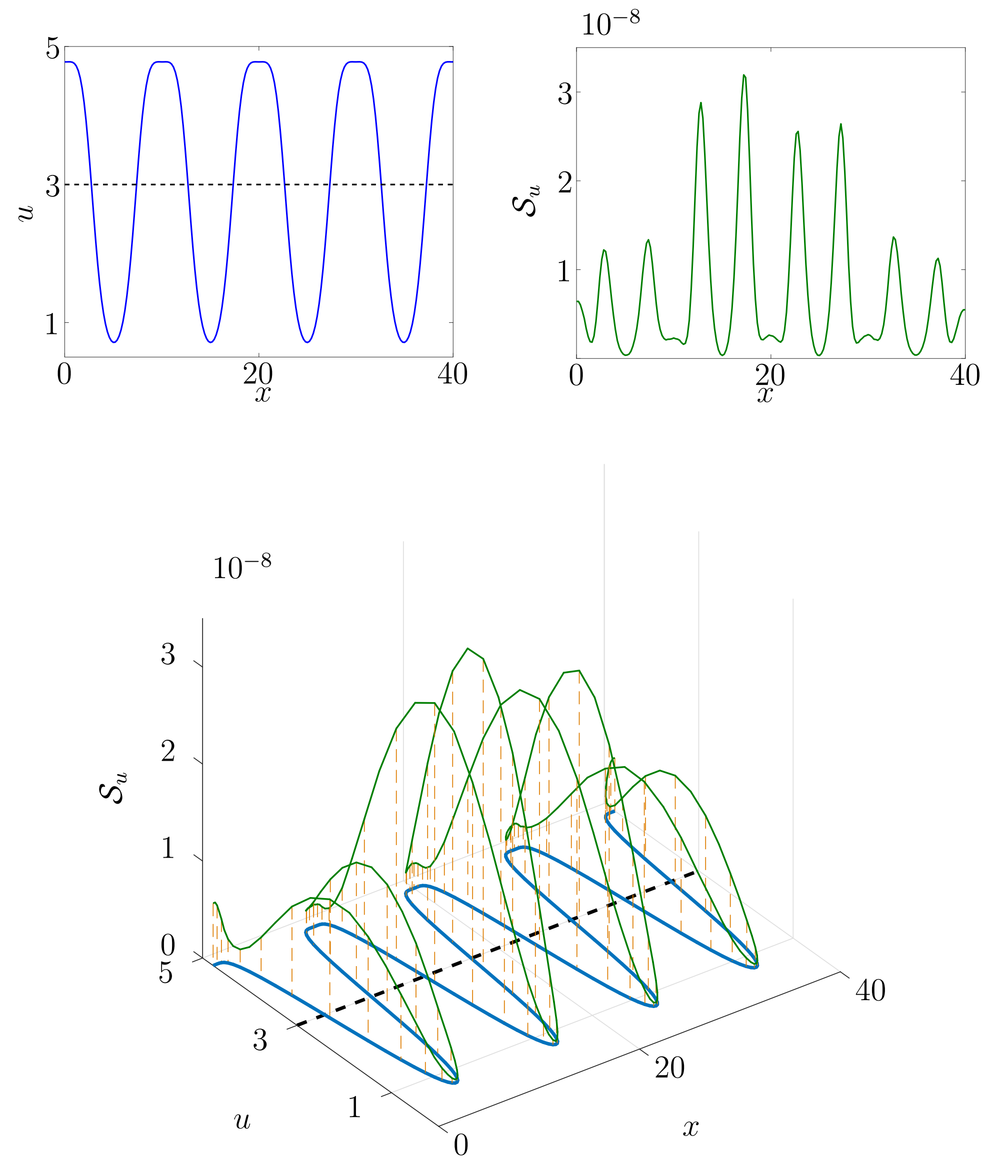


**Figure 4.** Analysis of stochastic sensitivity of $4\downarrow$ pattern for $a = 3$, $b = 9$, $D_u = 3$, $D_v = 10$, $\gamma = 10^{-4}$

Fig. 4 shows that stochastic sensitivity $\mathcal{S}_u(x,\gamma)$ is non-homogeneous: some points of the pattern are more sensitive to noise than others. The least sensitive fragments are localised near the pattern extrema, while the highest sensitivity is observed near the homogeneous equilibrium (dashed line).

The next step is to study how this deviation will change depending on system parameters. For measuring, comparing, and displaying results, the following functions are introduced:

$$\bar{\mathcal{S}}_u(\gamma) = \max_{x\in[0,L]} \mathcal{S}_u(x,\gamma), \quad \bar{\mathcal{S}}_v(\gamma) = \max_{x\in[0,L]} \mathcal{S}_v(x,\gamma). \tag{2.4}$$

Fig. 5 shows plots of $\bar{\mathcal{S}}_u(\gamma)$ and $\bar{\mathcal{S}}_v(\gamma)$ for changing noise intensity $\gamma$ for three patterns: $5\uparrow$, $6\uparrow$, and $7\uparrow$. Observations imply that the dependence on the noise intensity is similar to quadratic growth. Moreover, the response of different patterns to perturbation intensity differs. While there is only slight difference in the growth for $5\uparrow$ and $6\uparrow$, the $7\uparrow$ rises sharper than the others. This, in order, implies higher sensitivity to noise of higher intensity. As a result, the shape of this pattern will be affected more by random perturbations.

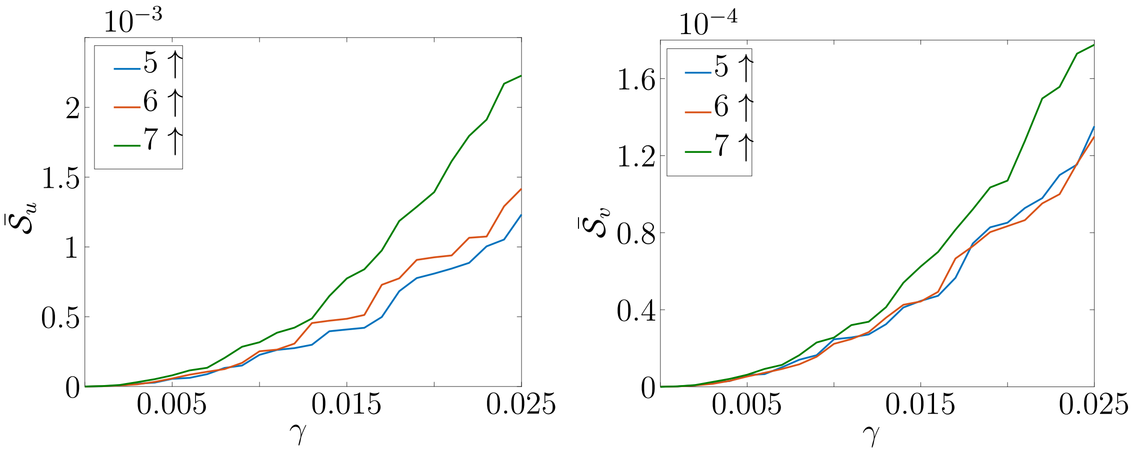


**Figure 5.** Mean square deviation for $a = 3$, $b = 9$, $D_u = 2$, $D_v = 10$ versus noise intensity $\gamma$

Next, consider the dependence of stochastic sensitivity of patterns on the variation of diffusion coefficient $D_u$. For each pattern, there is a range of $D_u$, for which it remains stable in the system without noise. Therefore, varying the diffusion coefficient should affect stochastic sensitivity as well. The results are shown in Fig. 6.

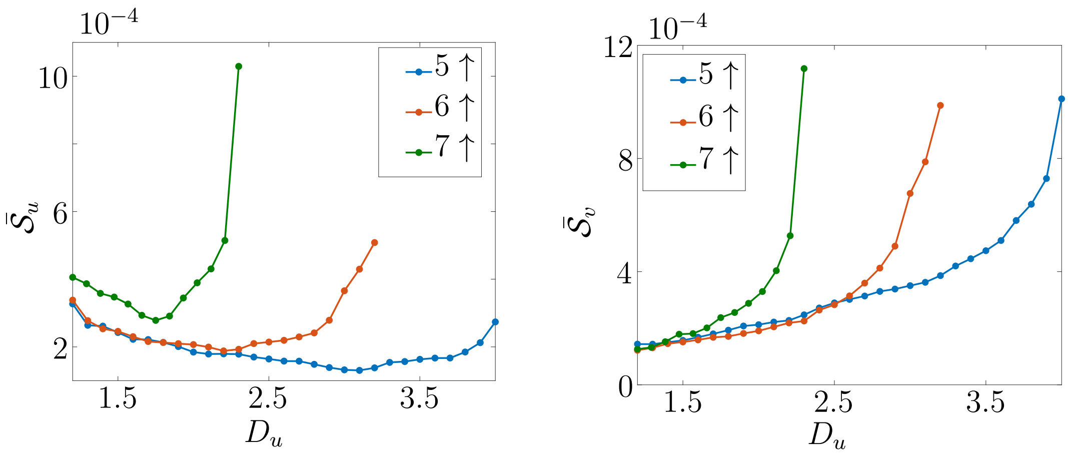


**Figure 6.** Mean square deviation for $a = 3$, $b = 9$, $D_v = 10$, $\gamma = 0.01$ versus $D_u$

All considered patterns have similar tendency of changing deviation depending on the intensity of diffusion flow. As the $D_u$ value increases, the patterns tend to become less sensitive. Further increase shows growth of sensitivity rate. For example, $7\uparrow$ can not be observed in the deterministic system for $D_u \approx 2.35$, thus near this point a sharp growth is expected. In the same manner, such incline is anticipated at $D_u \approx 3.3$ for $6\uparrow$ and $D_u \approx 4.1$ for $5\uparrow$.

Based on the study of these functions it may be possible to predict the emergence of noise-induced transition phenomenon. Fig. 7 shows an example of $7\uparrow$ pattern changing into the $6\uparrow$ pattern, which can be explained by significant difference in stochastic sensitivity.

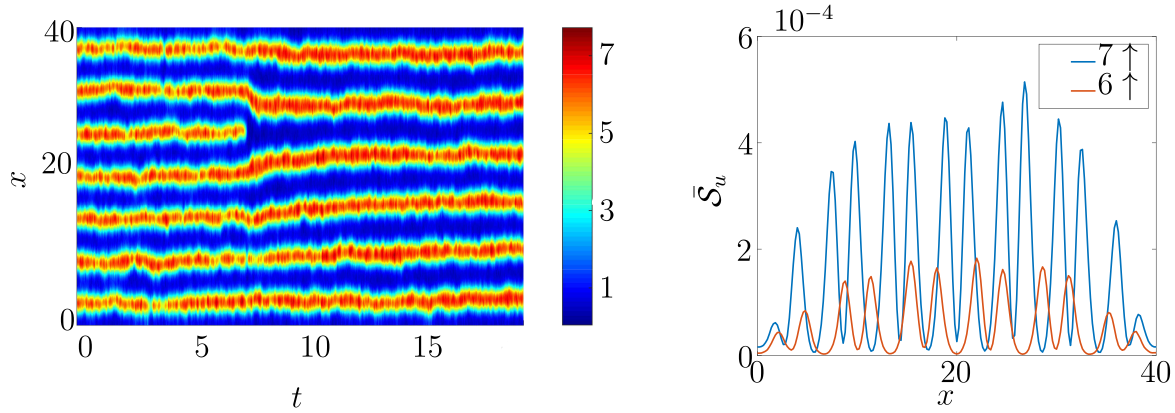


**Figure 7.** Noise-induced transition for $a = 3$, $b = 9$, $D_u = 2.3$, $D_v = 10$: transition from $7\uparrow$ pattern to $6\uparrow$, $\gamma = 1$ (left), deviation functions for $\gamma = 0.01$ (right)

## § 3. Conclusion

In this paper we studied the stochastic sensitivity of spatial patterns in the distributed Brusselator system. We show that the system is multistable and several patterns can coexist for the same set of parameter values. Next, the stochastic phenomenon of noise-induced pattern transition was investigated. A scenario of transition under the influence of random perturbation was demonstrated as an example. Mean square analysis for deviations of randomly forced patterns was performed. It was shown that based on this analysis, one can distinguish stable and sensitive to noise patterns and predict transitions between them.

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Kolinichenko Aleksandr Pavlovich, Student, Ural Federal University, ul. Lenina, 51, Yekaterinburg, 620075, Russia.
E-mail: kolinichenko.ale@gmail.com

Ryashko Lev Borisovich, Doctor of Physics and Mathematics, Professor, Ural Federal University, ul. Lenina, 51, Yekaterinburg, Russia.
E-mail: lev.ryashko@urfu.ru

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*А. П. Колиниченко, Л. Б. Ряшко*

**Анализ стохастической чувствительности тьюринговских паттернов в распределенных системах реакции–диффузии**

*Ключевые слова:* модель реакции–диффузии, неустойчивость Тьюринга, самоорганизация, стохастическая чувствительность

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В данной работе исследуется распределенная стохастическая модель Брюсселятора с диффузией. Мы показываем, что в зоне неустойчивости Тьюринга генерируется множество устойчивых пространственно неоднородных структур. Влияние случайного шума на стохастическую динамику вблизи этих структур анализируется прямым численным моделированием. Изучены шумовые переходы между сосуществующими паттернами. Стохастическая чувствительность модели определяется как среднеквадратичное отклонение от исходной неискаженной модели. Показано, что стохастическая чувствительность пространственно неоднородна и существенно отличается для сосуществующих структур. Обсуждается зависимость стохастической чувствительности от изменения коэффициентов диффузии и интенсивности шума.

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Колиниченко Александр Павлович, студент, Уральский федеральный университет, 620075, Россия, г. Екатеринбург, ул. Ленина, 51.
E-mail: kolinichenko.ale@gmail.com

Ряшко Лев Борисович, д. ф.-м. н., профессор, Уральский федеральный университет, 620075, Россия, г. Екатеринбург, ул. Ленина, 51.
E-mail: lev.ryashko@urfu.ru

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