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Original
Homogenization of multivalued monotone operators with variable growth exponent / Pastukhova, Svetlana; Chiadò Piat, Valeria. - In: NETWORKS AND HETEROGENEOUS MEDIA. - ISSN 1556-1801. - ELETTRONICO. - 15:2(2020), pp. 281-305.

Availability:
This version is available at: 11583/2833412 since: 2020-06-07T09:54:45Z

Publisher:
AMER INST MATHEMATICAL SCIENCES-AIMS

Published
DOI:10.3934/nhm.2020013

Homogenization of multivalued monotone operators with variable growth exponent

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Abstract We consider the Dirichlet boundary value problem for an elliptic multivalued maximal monotone operator $\mathcal{A}_\varepsilon$ satisfying growth estimates of power type with a variable exponent. This exponent $p_\varepsilon(x)$ and also the symbol of the operator $\mathcal{A}_\varepsilon$ oscillate with a small period ε with respect to the space variable x. We prove a homogenization result for this problem, thus, obtaining, in the limit as $\varepsilon \to 0$, a multivalued maximal monotone operator $\mathcal{A}$, much simpler than the original one. Namely, $\mathcal{A}$ does not depend on the space variable and satisfies new type growth estimates which are formulated in terms of a convex function $f(\xi)$ instead of the power function $|\xi|^{p_\varepsilon(x)}$ used in growth conditions for $\mathcal{A}_\varepsilon$. The function $f(\xi)$ as well as the symbol of $\mathcal{A}$ are found via auxiliary problems stated on the unit cell of periodicity. The Dirichlet problem for the operator $\mathcal{A}_\varepsilon$ is posed in the variable order Sobolev space $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, while the limit problem with the operator $\mathcal{A}$ is posed in the Sobolev space $W_0^f(\Omega)$ with generalized Orlicz integrability defined by the function $f(\xi)$. We extend here the homogenization result, obtained previously by V. Zhikov and S. Pastukhova in the case of single-valued strictly monotone operators.

Keywods Nonlinear homogenization; Maximal monotone operators; Non-standard growth conditions; Variable exponent spaces; Sobolev spaces with generalized Orlicz integrability; Compensated compactness

1 Introduction

This paper relates to the intersection of two fields in homogenization theory. The first one studies problems under non-standard coerciveness and boundedness conditions; the second one is connected with multivalued mappings corresponding to maximal monotone operators. The paper is motivated with the survey article by A.Pankov [15] where some open problems are formulated for elliptic monotone operators under nonstandard growth conditions. Our exposition concerns one of the problems posed in [15].

Non-standard coerciveness and boundedness conditions. Integral functionals subordinate to non-standard boundedness conditions were studied from the 70's-80's by many mathematicians (see the overview [22] on this topic and references there). The model functional of this type is

$$F(u) = \int_\Omega \frac{|Du(x)|^{p(x)}}{p(x)}\, dx, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$,

$$p \in L^\infty(\Omega), \quad 1 < \alpha \le p(\cdot) \le \beta < +\infty, \tag{1.2}$$

and the corresponding integrand

$$f(x,\xi) = |\xi|^{p(x)},\ \xi \in \mathbb{R}^n,$$

satisfies coercivity and boundedness conditions

$$c_1|\xi|^\alpha - c_0 \le f(x,\xi) \le c_0 + c_2|\xi|^\beta, \quad \forall \xi \in \mathbb{R}^n, \quad c_1, c_2 > 0,\ c_0 \ge 0. \tag{1.3}$$

Here, the different constant exponents α, β stand in the lower and the upper bounds, while, for the functionals of the classical calculus of variations, the constant exponent should be the same in both sides of the bilateral estimate of the type (1.3), i.e. $\alpha = \beta$.

In the contest of homogenization theory, a strongly nonhomogeneous functional of the form (1.1) emerges if the variable exponent $p(x)$ is replaced by an ε-periodic exponent $p(x/\varepsilon)$, highly oscillating as the positive parameter ε tends to zero. Namely,

$$F_\varepsilon(u) = \int_\Omega \frac{|Du(x)|^{p(x/\varepsilon)}}{p(x/\varepsilon)}\, dx, \quad \varepsilon \in (0,1], \tag{1.4}$$

the corresponding integrand is $f_\varepsilon(x,\xi) = |\xi|^{p(x/\varepsilon)}$. In numerous papers by Zhikov, beginning from the 80's, various aspects of the theory of nonstandard functionals satisfying (1.3) were studied. Among them, the so-called Lavrentiev phenomenon and Γ-convergence of integrands (or functionals) with the account of the Lavrentiev phenomenon (see [22] for details). In the framework of Γ-convergence theory, we single out the homogenization result [20] concerning the functional (1.4). In this case, the homogenized functional

$$F_0(u) = \int_\Omega f_0(Du(x))\, dx$$

appear in the limit as $\varepsilon \to 0$. Its integrand $f_0(\xi)$ is a convex function (not power-like) subordinate to the estimate (1.3), and $f_0(\xi)$ can be different in minimization problems considered on different spaces, for example, on $W^{1,\alpha}(\Omega)$ or $W^{1,\beta}(\Omega)$, α and β being from (1.3).

The Euler-Lagrange equations corresponding to minimization problems with the functional (1.1) contain the $p(\cdot)$-Laplacian acting as $\Delta_{p(\cdot)}u = -\operatorname{div}\big(|Du|^{p(x)-2}Du\big)$. Note that the $p(\cdot)$-Laplacian is a monotone operator, it arises as a result of the differentiation of the appropriate convex functional. The next step is to study monotone operators of variable nonlinearity order which, in contrast with $\Delta_{p(\cdot)}$, do not relate to any convex functional via differentiation. The model example is the anisotropic $p(\cdot)$-Laplace operator

$$L_{p(\cdot)}u = -\operatorname{div}\Big(|Du|^{p(x)-2}A(x)Du\Big), \tag{1.5}$$

where $A = A(x)$ is a measurable symmetric matrix such that

$$\exists\, \nu > 0 :\ \nu|\xi|^2 \leqslant A\xi\cdot\xi \leqslant \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

As for monotonicity, this property is not provided here for an arbitrary matrix A even if A is constant and diagonal (see an example in [19]). A Cordes-type condition should be imposed on the matrix A to guarantee monotonicity properties of the operator $L_{p(\cdot)}$ (see details in [19] and [18]).

Now, we proceed to monotone operators of more general form

$$\mathcal{A}u = -\operatorname{div} a(x, Du), \tag{1.6}$$

where $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathèodory function (with continuity in ξ) such that $a(x,\cdot)$ is monotone and subordinate to the following coercivity and boundedness estimates:

$$a(x,\xi)\cdot\xi \ge c_1|\xi|^{p(x)} - c_0, \tag{1.7}$$

$$|a(x,\xi)|^{p'(x)} \le c_2|\xi|^{p(x)} + c_0, \quad p'(x) = p(x)(p(x)-1)^{-1}, \tag{1.8}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$ with the exponent $p(x)$ as in (1.2) and the constants $c_1, c_2 > 0$, $c_0 \geq 0$. Clearly, the anisotropic $p(\cdot)$-Laplacian (1.5) satisfies these conditions.

Because of properties (1.7), (1.8), it is natural to define the operator (1.6) on the Sobolev-Orlicz space $W^{1,p(\cdot)}(\Omega)$ that is the extension of the classical Sobolev space $W^{1,p}(\Omega)$ with the constant exponent p to the case of the variable exponent $p(x)$. This allows the adequate description and study of the operator $\mathcal{A}$. By the way, the latter remark is valid also for functionals of the type (1.1).

Suppose that functions $p(\cdot)$ and $a(\cdot,\xi)$ are 1-periodic and such that estimates of the type (1.7), (1.8) are valid. Then, coercivity and boundedness conditions for $a(x/\varepsilon,\xi)$ are formulated with the exponent $p(x/\varepsilon)$ highly oscillating as $\varepsilon \to 0$. Consequently, operators

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(x/\varepsilon, Du), \quad \varepsilon \in (0,1], \tag{1.9}$$

are adequately defined on the Sobolev-Orlicz spaces $W^{1,p_\varepsilon(\cdot)}(\Omega)$, $p_\varepsilon(x) = p(x/\varepsilon)$. Therefore, in homogenization problems related to operators (1.9), one should "pass to the limit" not only in the operator $\mathcal{A}_\varepsilon$ but also in the energy space $W^{1,p_\varepsilon(\cdot)}(\Omega)$ varying as $\varepsilon \to 0$. The homogenization results for operators of the type (1.9), both scalar and vector, were established in [23] and [25].

Multi-valued operators. The homogenization of nonlinear monotone operators of the form $-\mathrm{div}a\,(x/\varepsilon, Du)$, $u \in W_0^{1,p}(\Omega)$, where p is constant and $a(y,\xi)$ is allowed to be multivalued, was considered in [5], as a natural generalization of the variational setting. In fact, the homogenization of integral functionals

$$\int_\Omega f\left(\frac{x}{\varepsilon}, Du\right)\,dx,$$

which are correlated with boundary-value problems if $a(y,\xi) = \nabla_\xi f(y,\xi)$, was achieved gradually removing the differentiability assumption for $f(y,\cdot)$ (see [14], [7], [3]) The (possibly) multivalued map $a(y,\xi) = \partial_\xi f(y,\xi)$ represents a model for homogenization problems in the multivalued setting.

Another reason to treat multivalued monotone operators in homogenization theory stems from this theory itself. Assume that we deal with a family of non-strictly monotone single-valued operators $\mathcal{A}_\varepsilon$ ($\varepsilon> 0$ is a small parameter characterizing their heterogeneity) for which there is no, in general, unique solvability of operator equations $\mathcal{A}_\varepsilon u_\varepsilon = h$. Then it may happen that the limit, or homogenized, operator $\mathcal{A}_0$, as $\varepsilon \to 0$, is multivalued. One can easily believe in this (rather strange only at first sight) phenomenon following the lines of the homogenization procedure. In fact, the so-called cell problem generates the symbol of the homogenized operator via its solutions according to the certain rule. Thus, the multiplicity of cell problem solutions may be a prerequisite for the emergence of the multi-valued homogenized operator. The example of a single-valued operator whose associated homogenized operator turns out to be multivalued is given in [5].

Meanwhile, it is interesting to look at homogenization of multi-scale problems with non-strictly monotone operators where reiterated homogenization procedure should be used. Homogenization of such kind multi-scale operators, but only strictly monotone, was studied, e.g. in [17]. Under assumption of the mere monotonicity (when we drop the strict form of it), we may encounter naturally, as it is explained above, multivalued operators even at intermediate stages of reiterated homogenization before we come to the final homogenized operator.

The paper is organized as follows. In Section 2, we formulate the homogenization problem under consideration after presenting, in a short form, the necessary background material on variable exponent spaces and on multivalued-monotone maps. Section 3 contains some known results for the case of single-valued operators, both in the variational and non-variational setting, from which we proceed. Here, we introduce, incidentally, the function f as well as the Sobolev space with generalized Orlicz integrability $W_0^{1,f}(\Omega)$ that appears in the formulation of the homogenized problem. Section 4 is dedicated to the study of the, so-called, cell problem, its solvability and the main properties of the solutions. The homogenized operator and the homogenized problem are defined and studied in Section 5. The main homogenization result, that is Theorem 6.2, is stated and proved in Section 6. A crucial tool in the proof of the convergence of the solutions of the given problem to those of the homogenized one is a peculiar version of the Compensated Compactness Lemma (see below, Lemma 6.1), which is due to Zhikov and Pastukhova [25].

2 Description of the Problem

In this section, we introduce the notation, recall some standard definitions, collect necessary theorems and facts that will be used to establish our main result, and eventually give a problem set-up.

2.1 Variable exponent spaces

Good overview on variable exponent spaces appeared in the long-standing publication [12]. For a more detailed study see also the recent book [8]. In what follows, we present some necessary explications.

Given a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and a real-valued measurable bounded function $p : \Omega \to \mathbb{R}$ with values in the diapason $[\alpha, \beta]$, where $1 < \alpha < \beta < \infty$, we recall the definition of the Lebesgue-Orlicz and Sobolev-Orlicz spaces

$$L^{p(\cdot)}(\Omega) = \{v \in L^1(\Omega) : \int_\Omega |v(x)|^{p(x)}\, dx < \infty\}, \tag{2.1}$$

$$W_0^{1,p(\cdot)}(\Omega) = \{u \in W_0^{1,1}(\Omega) : \int_\Omega |Du(x)|^{p(x)}\, dx < \infty\}. \tag{2.2}$$

Here and hereafter, we make no difference in notation for spaces of scalar and vector-valued functions. For instance, in (2.1) any function $v \in L^1(\Omega, \mathbb{R}^m)$, $m \in \mathbb{N}$, is meant.

Equipped with the following norms

$$||v||_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{v(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}, \tag{2.3}$$

called the *Luxemburg norm*, and

$$||u||_{W_0^{1,p(\cdot)}(\Omega)} = ||Du||_{L^{p(\cdot)}(\Omega)}, \tag{2.4}$$

the sets (2.1), (2.2) become reflexive, separable Banach spaces.

The conjugate space to $L^{p(\cdot)}(\Omega)$ coincides with $L^{p'(\cdot)}(\Omega)$, where $p'(x) = p(x)(p(x) - 1)^{-1}$ is the Hölder conjugate exponent.

From the definition of the Luxemburg norm, the bilateral estimate follows:

$$||v||^\alpha_{L^{p(\cdot)}(\Omega)} - 1 \le \int_\Omega |v(x)|^{p(x)}\, dx \le ||v||^\beta_{L^{p(\cdot)}(\Omega)} + 1. \tag{2.5}$$

It means that boundedness property in $L^{p(\cdot)}(\Omega)$ may be equivalently considered in the sense of the norm $|| \cdot ||_{L^{p(\cdot)}(\Omega)}$ or in the sense of the integral functional standing in (2.5) usually called a modular. Further, we will deal mostly with modular boundedness.

Evidently,

$$L^\beta(\Omega) \subset L^{p(\cdot)}(\Omega) \subset L^\alpha(\Omega), \quad W_0^{1,\beta}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,\alpha}(\Omega), \tag{2.6}$$

and these embeddings are continuous (see, e.g. [12] and [8]).

In the framework of the Lebesgue-Orlicz spaces, the Hölder inequality acquires the form

$$\int_\Omega |u \cdot v|\, dx \le 2||u||_{L^{p(\cdot)}(\Omega)}||v||_{L^{p'(\cdot)}(\Omega)}, \tag{2.7}$$

where the constant 2 is not sharp and can be replaced with $c_p = \frac{1}{\alpha} + \frac{1}{\beta'} < 2$.

The peculiar feature of the Sobolev-Orlicz spaces is that the set $\mathcal{C}_0^\infty(\Omega)$ may be not dense in $W_0^{1,p(\cdot)}(\Omega)$, and its closure, denoted by $H_0^{1,p(\cdot)}(\Omega)$, is generally a proper subspace of $W_0^{1,p(\cdot)}(\Omega)$. The non-coincidence of these two spaces, i.e.

$$W_0^{1,p(\cdot)}(\Omega) \neq H_0^{1,p(\cdot)}(\Omega), \tag{2.8}$$

is often referred to as the *Lavrentiev phenomenon*. In the case (2.8), the exponent $p(\cdot)$ is called *nonregular*; thus, the set $\mathcal{C}_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ for a *regular* exponent $p(\cdot)$.

It is known that $p(\cdot)$ is regular under the so-called log-condition (see, e.g. [22] or [8])

$$\exists k > 0 \quad |p(x) - p(y)| \leq \frac{k}{\log(1/|x-y|)}, \quad \forall x, y \in \Omega, |x-y| < \frac{1}{2}. \tag{2.9}$$

Another sufficient condition of the type (2.9) with the double logarithm in the numerator of the right-hand side can be found in [21], [22].

The examples show (see, e.g. [24]) that, in order to avoid (2.8), the exponent $p(\cdot)$ should not be merely continuous but with a proper modulus of continuity. On the other hand, discontinuous exponents can be regular if they are subordinate to certain monotonicity condition (see, e.g. [9]).

Only for simplification of our exposition, which will be already rather cumbersome because of the multivalued setting, we avoid the Lavrentiev phenomenon and consider only the Sobolev-Orlicz spaces with regular exponents. In other words, *we assume further that the exponent* $p(\cdot)$ *is regular.*

2.2 Multivalued monotone maps

Now, we introduce notation and recall some definitions together with several fundamental results connected with multivalued mappings and measurability.

Multivalued maps. A multivalued function F from a set X to a set Y is a map that associates to any $x \in X$ a subset Fx of Y. A selection of a multi-valued map F is a function $f : X \to Y$ such that for any $x \in X$ $f(x) \in Fx$. Let $(X, \mathcal{T})$ be a measurable space.

Definition 2.1 *We say that a multivalued function* $F : X \to \mathbb{R}^n$ *is measurable if*

$$F^{-1}(C) = \{x \in X : Fx \cap C \neq \emptyset\} \in \mathcal{T}$$

for each closed set $C \subset \mathbb{R}^n$.

Remark 2.2 It is known that if $(X, \mathcal{T}, \mu)$ is a measurable space with a σ-finite complete measure μ defined on $\mathcal{T}$, and $F : X \to \mathbb{R}^n$ is a multivalued function with non-empty closed values, then F is measurable if and only if its graph

$$Gr(F) = \{(x, y) \in X \times \mathbb{R}^n : y \in Fx\}$$

belongs to the σ-algebra $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$ (see Chapter III, Section 2, in [4]), where $\mathcal{B}(\mathbb{R}^n)$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^n$.

It is useful to recall the so-called Projection Theorem (see Theorem III.23, in [4]).

Theorem 2.3 *If* $(X, \mathcal{T}, \mu)$ *is a measurable space with a* σ*-finite complete measure* μ *defined on* $\mathcal{T}$*, and* $Gr(F) \in \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$*, then the projection of* $Gr(F)$ *on the first factor* X *belongs to* $\mathcal{T}$*.*

We will deal with measurable selections of multivalued functions. A condition for the existence of measurable selections is due to Aumann-von Neumann (see Theorem III.22, in [4]).

Theorem 2.4 *Let* $(X, \mathcal{T})$ *be a measurable space and* $F : X \to \mathbb{R}^n$ *be a multivalued map with non-empty values. If* $Gr(F) \in \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$ *and there exists a complete* σ*-finite measure defined on* $\mathcal{T}$*, then* F *has a measurable selection.*

Definition 2.5 *A multivalued map* $F : X \to Y$ *is said upper-semicontinuous if for every* $x \in X$ *and for every open neighbourhood* V *of* Fx *in* Y *there exists a neighbourhood* U *of* x *in* X *such that* $Fz \subset V$ *for all* $z \in U$*.*

Multivalued monotone maps. Let X be a Banach space, X' be its dual, and $\langle \cdot, \cdot \rangle$ denotes the corresponding duality product. For $X = \mathbb{R}^n$ the duality product will be simply denoted by (the scalar product) $x \cdot y$. We recall the main definitions and results concerning multivalued monotone maps from X to X'.

Definition 2.6 *A (possibly) multivalued map $F : X \to X'$ is monotone if its graph $Gr(F)$ is monotone in the following sense:*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \forall (x_i, y_i) \in Gr(F). \tag{2.10}$$

Moreover, F is said maximal monotone if its graph $Gr(F)$ is monotone and it is the maximal monotone subset of $X \times Y$, i.e. from

$$\langle y - \eta, x - \xi \rangle \geq 0 \quad \forall (\xi, \eta) \in Gr(F) \tag{2.11}$$

it follows that $y \in Fx$.

Remark 2.7 We note that, since $(x, y) \in Gr(F)$ iff $(y, x) \in Gr(F^{-1})$, then F is (maximal) monotone iff F^{-1} has the same property. Moreover, if F is maximal monotone and $Fx \neq \emptyset$, then Fx is closed and convex (see, for example, [16], Chapter III.2).

Lastly, we formulate sufficient conditions for multivalued operators to be, first, maximal monotone and, second, surjective.

Theorem 2.8 *(see [2], Theorem 3.18) Let $F : X \to X'$ be a (multivalued) monotone map with nonempty weakly* closed convex values, such that, for each line segment in X, F is upper-semicontinuous from the line segment to X' with the weakly* topology. Then F is maximal monotone.*

Theorem 2.9 *(see [16], Chapter III, Theorem 2.10) Let $F : X \to X'$ be a (multivalued) maximal monotone map, where X is a reflexive Banach space. If F is coercive, i.e.*

$$\lim_{||x|| \to +\infty} \frac{\langle Fx, x \rangle}{||x||}, \tag{2.12}$$

then $F(X) = X'$.

2.3 Multivalued monotone maps depending on the space variable

Let us fix a bounded domain $\Omega \subset \mathbb{R}^n$. We consider now multivalued maps $a = a(x, \cdot)$ that depend on the space variable $x \in \Omega$ and satisfy some further conditions. Let $p \in L^\infty(\Omega)$ satisfy (1.2), and let p' be its Hölder conjugate. We denote by $M_\Omega^{p(\cdot)}$ the set of all *multivalued* maps with closed values $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ subordinate to the following conditions:

(i) $a(x, \cdot)$ is maximal monotone for a.e. $x \in \Omega$;

(ii) the coercivity and boundedness estimates

$$c_1 |\xi|^{p(x)} \leq \xi \cdot \eta + m_1, \tag{2.13}$$

$$c_2 |\eta|^{p'(x)} \leq \xi \cdot \eta + m_2 \tag{2.14}$$

hold for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, $\eta \in a(x, \xi)$ with fixed constants $c_1, c_2, > 0$ and $m_1, m_2 \geq 0$;

(iii) a is measurable with respect to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$, i.e.

$$a^{-1}(C) = \{(x, \xi) \in \Omega \times \mathbb{R}^n : a(x, \xi) \cap C \neq \emptyset\} \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$$

for every closed set $C \subseteq \mathbb{R}^n$;

(iv) $a(x,0)=0$ for a.e. $x \in \Omega$.

In condition (iii), $\mathcal{L}(\Omega)$ is the σ-algebra of Lebesgue measurable subsets of Ω.

Remark 2.10 In condition (ii), estimate (2.14) may be replaced with the following one:

$$|\eta|^{p'(x)} \leq c_3|\xi|^{p(x)} + m_3 \tag{2.15}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, $\eta \in a(x,\xi)$. The pair of estimates (2.13) and (2.14) is equivalent to the pair of estimates (2.13) and (2.15). The latter pair is usually called in literature as coercivity and boundedness estimates. In the same fashion, we refer to (2.13), (2.14).

Let $Y = (0,1)^n$ be the unit cube in $\mathbb{R}^n$. Assume that $p(\cdot)$ and $a(\cdot,\xi)$ are Y-periodic on $\mathbb{R}^n$ and $a : Y \times \mathbb{R}^n \to \mathbb{R}^n$ is of class $M_Y^{p(\cdot)}$. The set of functions a of this type will be denoted by $M_{\mathrm{per}}^{p(\cdot)}$. Setting

$$p_\varepsilon(x) = p(\frac{x}{\varepsilon}), \quad a_\varepsilon(x,\xi) = a(\frac{x}{\varepsilon},\xi), \quad \varepsilon \in (0,1], \tag{2.16}$$

we obtain the ε-periodic mappings $a_\varepsilon(x,\xi)$ for $x \in \Omega$ and $\varepsilon \in (0,1]$. Then, condition (ii) in the definition of the class $M_Y^{p(\cdot)}$ implies estimates for $a_\varepsilon(x,\xi)$ with highly oscillating exponent $p_\varepsilon(x)$ as $\varepsilon \to 0$, but with fixed constants c_i, m_i. Namely,

$$c_1|\xi|^{p_\varepsilon(x)} \leq \xi \cdot \eta + m_1, \tag{2.17}$$

$$c_2|\eta|^{p'_\varepsilon(x)} \leq \xi \cdot \eta + m_2 \tag{2.18}$$

hold for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, $\eta \in a_\varepsilon(x,\xi)$. In a similar way, from (2.15) it follows that

$$|\eta|^{p'_\varepsilon(x)} \leq c_3|\xi|^{p_\varepsilon(x)} + m_3 \tag{2.19}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, $\eta \in a_\varepsilon(x,\xi)$.

2.4 Statement of the problem

From now on, we fix $p(\cdot)$ that is Y-periodic on $\mathbb{R}^n$ and then take $a(\cdot,\xi) \in M_{\mathrm{per}}^{p(\cdot)}$. Given a vector function $h \in L^\infty(\Omega)$, we consider the differential inclusion with Dirichlet boundary condition

$$\begin{cases} u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega), \\ g_\varepsilon(x) \in a_\varepsilon(x, Du_\varepsilon) \quad \text{for a.e. } x \in \Omega, \\ \displaystyle\int_\Omega g_\varepsilon \cdot Dv\, dx = \int_\Omega h \cdot Dv\, dx \quad \text{for all } v \in W_0^{1,p_\varepsilon(\cdot)}(\Omega), \end{cases} \tag{2.20}$$

where $p_\varepsilon(x)$ and $a_\varepsilon(x,\xi)$ are defined in (2.16).

In the sequel, we use the fact that the flow $g_\varepsilon - h$, corresponding to (2.20), is a solenoidal vector. A vector $z \in L^1(\Omega)$ is said to be solenoidal if $\operatorname{div} z = 0$ in the sense of distributions on Ω (or shortly, in $\mathcal{D}'(\Omega)$). Denote by $L_{\mathrm{sol}}^{p'_\varepsilon(\cdot)}(\Omega)$ the set of all vectors from $L^{p'_\varepsilon(\cdot)}(\Omega)$ that are solenoidal. Then $g_\varepsilon - h \in L_{\mathrm{sol}}^{p'_\varepsilon(\cdot)}(\Omega)$, due to (2.20) and (2.19).

Theorem 2.11 *Under the above assumptions, for every $h \in L^\infty(\Omega)$, problem* (2.20) *has at least one solution $(u_\varepsilon, g_\varepsilon)$. Moreover, the following estimates hold true*

$$\int_\Omega |Du_\varepsilon|^{p_\varepsilon(x)}\, dx \leq c, \quad \int_\Omega |g_\varepsilon|^{p'_\varepsilon(x)}\, dx \leq c, \tag{2.21}$$

where $c > 0$ does not depend on ε.

Proof: The existence of $(u_\varepsilon, g_\varepsilon)$ is a consequence of a more general result that can be found in [1]. To prove estimates (2.21), we insert $v = u_\varepsilon$ in the integral identity in (2.20) and get

$$\int_\Omega g_\varepsilon \cdot Du_\varepsilon \, dx = \int_\Omega h \cdot Du_\varepsilon \, dx,$$

$$c_1 \int_\Omega |Du_\varepsilon|^{p_\varepsilon} \, dx \le \int_\Omega g_\varepsilon \cdot Du_\varepsilon \, dx + m_1|\Omega| =$$

$$= \int_\Omega h \cdot Du_\varepsilon \, dx + m_1|\Omega| \le$$

$$\le ||h||_{L^\infty(\Omega)} \int_\Omega |Du_\varepsilon| \, dx \, + m_1|\Omega|,$$

where (2.17) is used.

Now, due to Young inequality, for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$\int_\Omega |Du_\varepsilon| \, dx \le \int_\Omega (\delta|Du_\varepsilon|^{p_\varepsilon} + c_\delta) \, dx.$$

For sufficiently small δ, the above inequalities imply estimate (2.21) for u_ε. Due to inequality (2.19), the estimate for g_ε follows directly from that for u_ε. □

We are interested in the asymptotic behaviour, as $\varepsilon \to 0$, of the solutions of problem (2.20). For constant p in the case of multivalued mapping, the result can be found in [5]), while, for single-valued setting with a variable exponent $p(\cdot)$, the result is contained in [25].

According to (2.20), the function u_ε belongs to the energy space $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ depending on ε (varying space). Consequently, due to the estimate (2.19), the momentum g_ε belongs to the space $L^{p'_\varepsilon(\cdot)}(\Omega)$ also depending on ε. Hence, using embeddings of the type (2.6), we consider $\{u_\varepsilon\}$ as a family from the fixed space $W_0^{1,\alpha}(\Omega)$ and, similarly, $\{g_\varepsilon\}$ as a family from the fixed space $L^{\beta'}(\Omega)$. Note for the latter that the Hölder conjugate exponent $p'(\cdot)$ satisfies an estimate of the type (1.2), namely

$$1 < \beta' \le p'(\cdot) \le \alpha' < +\infty$$

(by prime the Hölder conjugation is always denoted, e.g. $\alpha'=\alpha/(\alpha-1)$). Thereby, $L^{p'_\varepsilon(\cdot)}(\Omega) \subset L^{\beta'}(\Omega)$.

Thanks to estimates (2.21) and continuity of embeddings (2.6), we have weak convergence (up to subsequence)

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,\alpha}(\Omega), \tag{2.22}$$

$$g_\varepsilon \rightharpoonup g \quad \text{in } L^{\beta'}(\Omega). \tag{2.23}$$

Passing to the limit in the integral identity of problem (2.20), we obtain

$$\int_\Omega g \cdot Dv \, dx = \int_\Omega h \cdot Dv \, dx \tag{2.24}$$

for all $v \in \mathcal{C}_0^\infty(\Omega)$. The main problem is then to establish the connection between the limit functions u and g. It will be a relation of the same type as in (2.20), i.e. a differential inclusion, but much simpler in a certain sense. Namely, $g \in b(Du)$, where the multivalued maximal monotone mapping $b : \mathbb{R}^n \to \mathbb{R}^n$ does not depend on the space variable x, which is quite customary in homogenization theory. On the other hand, the coercivity and boundedness estimates (2.17), (2.18) for the initial multivalued mapping $a_\varepsilon(x, \xi)$ are subjected to the homogenization process. As a result, there appear estimates of a new type for the map b (see (5.2), (5.3)), which are formulated in terms of the convex function $f^{\text{hom}}(\xi) = f(\xi)$ (see Section 3 for the exact definition of $f(\xi)$). We underline here that the function $f(\xi)$ is not power-like, but it inherits some important properties from the power-like functions $f_\varepsilon(x, \xi) = |\xi|^{p_\varepsilon(x)}$ involved in (2.17), (2.18). The reason lays in the fact that $f(\xi)$ is the so-called Γ-limit of the family $f_\varepsilon(x, \xi)$ in the sense of Γ-convergence of integrands (see [20], [22] and references there). Simultaneously, the condition $u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, subjected to the homogenization process, gives rise to a similar condition for the limit function, namely $u \in W_0^f(\Omega)$. The function f that has been already discussed above turns to be a, so-called, N-function, according to the traditional terminology [8]. Thus, $W_0^f(\Omega)$ is the Sobolev space with generalized Orlicz integrability defined by the function $f(\xi)$.

The homogenization result concerning problem (2.20) is formulated exactly in Theorem 6.2 and proved in Section 6. Some preliminary material is collected in Sections 3-5. Meanwhile, in Section 3 we make a survey of some important results for single-valued monotone operators, from which we proceed.

3 Problems with single-valued operators

3.1 Variational setting

The special case of single-valued monotone operator with symbol

$$a(y,\xi) = |\xi|^{p(y)-2}\xi, \tag{3.1}$$

where $p(\cdot)$ is 1-periodic, which corresponds to the $p(\cdot)$-Laplacian with the variable exponent (see Introduction), was studied by Zhikov in the early 90's, e.g. in [20]. The symbol (3.1) is related, after substitutions (2.16), to the minimum problem stated as: find

$$E_\varepsilon = \min\left\{\int_\Omega \left(\frac{|Du|^{p_\varepsilon(x)}}{p_\varepsilon(x)} - h\cdot Du\right) dx : u \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)\right\}. \tag{3.2}$$

The Euler-Lagrange equation for this problem gets actually a form simpler than (2.20), namely:

$$\begin{cases} u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega), \\ g_\varepsilon(x) = a_\varepsilon(x, Du_\varepsilon) \quad \text{for a.e. } x\in\Omega, \\ \displaystyle\int_\Omega g_\varepsilon \cdot Dv\, dx = \int_\Omega h\cdot Dv\, dx \quad \text{ for all } v \in W_0^{1,p_\varepsilon(\cdot)}(\Omega), \end{cases} \tag{3.3}$$

where $a_\varepsilon(x, Du_\varepsilon) = |Du_\varepsilon|^{p_\varepsilon(x)-2}Du_\varepsilon$.

Zhikov proved that

$$\lim_{\varepsilon\to 0} E_\varepsilon = E, \quad E = \min\left\{\int_\Omega (f(Du) - h\cdot Du)\, dx : u \in W_0^{1,1}(\Omega)\right\}, \tag{3.4}$$

where

$$f(\xi) = \min\left\{\int_Y \frac{|\xi + Du|^{p(y)}}{p(y)}\, dy :\ u \in W^{1,p(\cdot)}_{\mathrm{per}}(Y)\right\}, \tag{3.5}$$

and the minimization is taken over

$$W^{1,p(\cdot)}_{\mathrm{per}}(Y) = \{w \in W^{1,1}_{\mathrm{per}}(Y) : \int_Y w\, dy = 0, \int_Y |Dw(y)|^{p(y)}\, dy < \infty\}, \tag{3.6}$$

which is the Sobolev-Orlicz space on the cell of periodicity $Y = (0,1)^n$.

The proof of Zhikov's result relies, first, on duality arguments and, second, on the following lower-semicontinuity result for power-like integral functionals with oscillating exponents $p_\varepsilon(\cdot)$ or $p'_\varepsilon(\cdot)$ considered on the space of potential vectors $\{Du_\varepsilon : u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)\}$ or on the space of solenoidal vectors $L^{p'_\varepsilon(\cdot)}_{\mathrm{sol}}(\Omega)$, respectively.

Lemma 3.1 *(i) Assume* $u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ *and* $Du_\varepsilon \rightharpoonup Du$ *in* $L^\alpha(\Omega)$*. Then*

$$\liminf_{\varepsilon\to 0} \int_\Omega |Du_\varepsilon|^{p_\varepsilon(x)}\,dx \geq \int_\Omega f(Du)\,dx. \tag{3.7}$$

(ii) Assume $w_\varepsilon \in L_{\rm sol}^{p'_\varepsilon(\cdot)}(\Omega)$ *and* $w_\varepsilon \rightharpoonup w$ *in* $L^{\beta'}(\Omega)$*. Then*

$$\liminf_{\varepsilon\to 0} \int_\Omega |w_\varepsilon|^{p'_\varepsilon(x)}\,dx \geq \int_\Omega f^*(w)\,dx, \tag{3.8}$$

where $f^*(\eta)$ *is the Fenchel conjugate of* $f(\xi)$*, i.e.*

$$f^*(\eta) = \sup_\xi \left(\eta\cdot\xi - f(\xi)\right). \tag{3.9}$$

In [20] a deeper result was actually proved, taking into account the Lavrentiev phenomenon in the Sobolev-Orlicz spaces, provided that the exponent $p(\cdot)$ is not necessarily regular. All the above statements can differ by the choice of the set over which the functional in (3.2) is minimized: the set of all admissible functions, that is the Sobolev-Orlicz space $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, or the set $C_0^\infty(\Omega)$, the same as $H_0^{1,p_\varepsilon(\cdot)}(\Omega)$. It can happen that the minimum over a narrower set is greater than the minimum over a wider set. For this gap, the Lavrentiev phenomenon is responsible. This difference will be preserved in the limit (3.4): two different limit functionals appear in (3.4) provided that the minimization in the initial problem is taken over the Sobolev-Orlicz spaces $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ or $H_0^{1,p_\varepsilon(\cdot)}(\Omega)$ and these two spaces fail to coincide. Accordingly, in [20] the lower-semicontinuity properties (3.7), (3.8) are proved in a sharper form with account of the Lavrentiev phenomenon. For example, the property (i) in Lemma 3.1 may have two different functionals as a lower bound if the sequence u_ε is taken from $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ or from its proper subspace $H_0^{1,p_\varepsilon(\cdot)}(\Omega)$.

The results of [20] were extended in [26] and [27] to the case of functionals with integrands $f_\varepsilon(x,u,Du)$, which are ε-periodic with respect to x and convex with respect to Du.

Finally, we list properties of the homogenized integrand $f(\xi)$ defined in (3.5) (for the proof see [20], [25]). The function $f:\mathbb{R}^n\to\mathbb{R}$ is non-negative, convex and subordinate to the estimate

$$C_1|\xi|^\alpha - 1 \leq f(\xi) \leq C_2|\xi|^\beta + 1, \quad \forall\xi\in\mathbb{R}^n, \tag{3.10}$$

where C_1, C_2 are positive constants. Besides, $f(\xi)=0$ iff $\xi=0$; $f(\xi)$ is even; $f(\xi)$ and its Fenchel conjugate $f^*(\eta)$ satisfy the Δ_2-condition that means

$$f(2\xi) \leq cf(\xi) \quad \forall\xi\in\mathbb{R}^n, \quad c>0. \tag{3.11}$$

More precisely, the following estimate holds:

$$f(\xi/\lambda)(\lambda^\alpha - 1) \leq f(\xi) \leq f(\xi/\lambda)(\lambda^\beta + 1), \quad \lambda>0, \tag{3.12}$$

fromwhere, in particular, the Δ_2-condition follows for f and f^*.

The properties of $f(\xi)$ permit to define the Orlicz class

$$L^f(\Omega) = \left\{v\in L^1(\Omega) : f(v)\in L^1(\Omega)\right\} \tag{3.13}$$

and endow it with the Luxemburg norm

$$||v||_{L^f(\Omega)} = \inf\left\{\lambda>0 : \int_\Omega f(\frac{v}{\lambda})\,dx \leq 1\right\}. \tag{3.14}$$

Thanks to the properties of f, $L^f(\Omega)$ is a reflexive Banach space. Its dual space coincides with $L^{f^*}(\Omega)$, and the following Hölder inequality holds true:

$$\left|\int_\Omega u\cdot v\,dx\right| \leq 2||u||_{L^f(\Omega)}||v||_{L^{f^*}(\Omega)}. \tag{3.15}$$

The estimates (3.12) together with the definition of the norm (3.14) imply that

$$||v||^{\alpha}_{L^f(\Omega)} - 1 \leq \int_{\Omega} f(v)\, dx \leq ||v||^{\beta}_{L^f(\Omega)} + 1. \tag{3.16}$$

This is a counterpart of (2.5). Clearly, $L^{\beta}(\Omega) \subset L^f(\Omega) \subset L^{\alpha}(\Omega)$, as a corollary of (3.10).

Next, we introduce the Sobolev space $W^f_0(\Omega)$ with generalized Orlicz integrability, defined by the function $f(\xi)$, as follows:

$$W^f_0(\Omega) = \left\{ v \in W^{1,1}_0(\Omega) : Dv \in L^f(\Omega) \right\}, \tag{3.17}$$

and endow it with the norm $||v||_{W^f_0(\Omega)} = ||Dv||_{L^f(\Omega)}$. Thanks to the properties of f and f^*, $W^f_0(\Omega)$ is a reflexive Banach space. It is known that the set $\mathcal{C}^{\infty}_0(\Omega)$ is dense in $W^f_0(\Omega)$ (see, e.g. [10]).

According to (3.7), the limit function u in Lemma 3.1(i) belongs to the space $W^f_0(\Omega)$. This motivates the introduction of this space in our exposition.

3.2 Non-variational settings

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, let us consider the Dirichlet boundary value problem for the operator $\mathcal{A}_{\varepsilon} : W_{\varepsilon} \to (W_{\varepsilon})'$ defined in (1.6)–(1.9), where $W_{\varepsilon} = W^{1,p_{\varepsilon}(\cdot)}_0(\Omega)$, for brevity. We state the problem in a weak form and say that u_{ε} is its solution if

$$u_{\varepsilon} \in W_{\varepsilon}, \quad \int_{\Omega} a_{\varepsilon}(x, Du_{\varepsilon}) \cdot Dv\, dx = \int_{\Omega} h \cdot Dv\, dx \quad \text{for all } v \in W_{\varepsilon}, \tag{3.18}$$

where $h \in L^{\infty}(\Omega)$ is a given function. Additionally, let $a(x, \cdot)$ be a strictly monotone vector.

For every fixed $\varepsilon \in (0, 1]$, there exists a unique solution to (3.18). The solvability result is established in a similar fashion as in the case of a monotone operator with a constant exponent p in growth conditions (see [13]). Now, we describe the asymptotic behaviour, as $\varepsilon \to 0$, of the solutions u_{ε} and the momenta $g_{\varepsilon} = a_{\varepsilon}(x, Du_{\varepsilon})$. According to [25], these families converge in a certain sense, specified a little bit later, to a solution u and a momentum $g = a(Du)$ of the problem

$$u \in W, \quad \int_{\Omega} a(Du) \cdot Dv\, dx = \int_{\Omega} h \cdot Dv\, dx \quad \text{for all } v \in W, \tag{3.19}$$

where the space $W = W^f_0(\Omega)$ is defined in (3.17). The symbol $a(\xi)$ in (3.19) is constructed with the help of solutions to the cell problem

$$w_{\xi} \in W_{\text{per}}, \quad \int_{Y} a(y, \xi + Dw_{\xi}) \cdot Dv\, dy = 0 \quad \text{for all } v \in W_{\text{per}}, \tag{3.20}$$

where $\xi \in \mathbb{R}^n$ is a vector parameter and the space $W_{\text{per}} = W^{1,p(\cdot)}_{\text{per}}(Y)$ is the same as in (3.6). More precisely,

$$a(\xi) = \int_{Y} a(y, \xi + Dw_{\xi})\, dy. \tag{3.21}$$

This function $a : \mathbb{R}^n \to \mathbb{R}^n$ turns to be continuous, strictly monotone and subordinate to the estimates

$$a(\xi) \cdot \xi \geq c(f(\xi) - 1),\ c > 0, \tag{3.22}$$

$$f^*(a(\xi)) \leq C(f(\xi) + 1) \tag{3.23}$$

with the convex functions f and f^* defined in (3.5) and (3.9). We see that coercivity and boundedness conditions for the symbol (3.21) are of a new form, not power-like. Nevertheless, the limit problem (3.19) is well posed.

Now we specify the convergence mentioned above. Pay attention, first, that the solution u_{ε} is found in the space $W_{\varepsilon} = W^{1,p_{\varepsilon}(\cdot)}_0(\Omega)$ varying as ε tends to zero. On the other hand, u_{ε} belongs to $W_0^{1,\alpha}(\Omega)$ and is uniformly bounded in this fixed space for all ε. A similar fact is valid for the momenta $g_\varepsilon = a_\varepsilon(x, Du_\varepsilon) \in L^{p'_\varepsilon(\cdot)}(\Omega) \subset L^{\beta'}(\Omega)$. Therefore, weak convergences (2.22) and (2.23) were considered in the formulation of homogenization result in [25].

In [25] a deeper result was actually proved assuming the exponent $p(\cdot)$ be not necessarily regular. In this case, the Dirichlet boundary value problem for the operator $\mathcal{A}_\varepsilon$ has non-unique setting. There are the Dirichlet problems of at least two types: stated in the spaces $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ and $H_0^{1,p_\varepsilon(\cdot)}(\Omega)$ which do not not coincide in the presence of the Lavrentiev phenomenon. The Dirichlet problem of the first type is formulated in (3.18). Replacing W_ε with the space $H_\varepsilon = H_0^{1,p_\varepsilon(\cdot)}(\Omega)$ everywhere in (3.18), we come to the setting of the Dirichlet problem of the second type. Both Dirichlet problems are well posed; their solutions, in general, do not coincide; moreover, homogenization procedures are different for them and lead to the different homogenized problems.

We mention here also the interesting result by Zhikov [23] related to homogenization of a Navier-Stokes type system for electrorheological fluids. The constitutive equation for the motion of highly non-homogeneous electrorheological fluids contains the operator similar to (1.9) satisfying the growth condition with oscillating ε-periodic exponent. After homogenization, a new type constitutive equation emerges in which there stands the operator satisfying coercivity and boundedness conditions, as in (3.22), (3.23).

Remark 3.2 Being in multivalued setting, the problem (2.20) is a direct extension of the problem (3.18). Thus, in the process of its homogenization, it is quite natural to expect the counterparts relating to the problems (3.19), (3.20), the definition (3.21), and the inequalities (3.22), (3.23). Certainly, all of them may acquire a more complicated form. This is indeed the case we have, which is shown in subsequent sections.

4 The cell problem

4.1 Set-up and solvability

In this section, we deal with a boundary-value problem defined on the unit cube Y, with periodic boundary conditions. To state this problem, we need the Sobolev-Orlicz space of periodic functions (3.6) endowed with the norm

$$||w||_{W^{1,p(\cdot)}_{\rm per}(Y)} = ||Dw||_{L^{p(\cdot)}(Y)}.$$

For any fixed $\xi \in \mathbb{R}^n$, we consider the problem

$$\begin{cases} w \in W^{1,p(\cdot)}_{\rm per}(Y), \\ k(y) \in a(y, \xi + Dw) \quad \text{for a.e. } y \in Y, \\ \displaystyle\int_\Omega k \cdot D\varphi \, dy = 0 \qquad \text{for all } \varphi \in W^{1,p(\cdot)}_{\rm per}(Y). \end{cases} \tag{4.1}$$

Evidently, introducing the space of potential vectors

$$V = \{v = Dw : w \in W^{1,p(\cdot)}_{\rm per}(Y)\} \subset L^{p(\cdot)}(Y) \tag{4.2}$$

endowed with the strong topology of $L^{p(\cdot)}(Y)$, we can rewrite the problem (4.1) in the following form: find (v, k) such that

$$\begin{cases} v \in V, \\ k(y) \in a(y, \xi + v(y)) \quad \text{for a.e. } y \in Y, \\ \displaystyle\int_Y k \cdot z \, dy = 0 \qquad \text{for all } z \in V. \end{cases} \tag{4.3}$$

Theorem 4.1 *For any fixed $\xi \in \mathbb{R}^n$, there exists a solution (v, k) of the cell problem* (4.3)*.*

To prove this existence result, we follow the lines of the proof of Theorem 2.7 in [6] adapting it to our case. We introduce the multivalued operator $\mathcal{A} : V \to V'$ defined for every $v \in V$ as follows: $\Phi \in \mathcal{A}v$ if and only if there exists a measurable selection $k(y) \in a(y, \xi + v(y))$ such that

$$\langle \Phi, z \rangle = \int_Y k \cdot z \, dy \quad \forall z \in V. \tag{4.4}$$

The solvability of (4.3) means that there exists $v \in V$ such that $0 \in \mathcal{A}v$, i.e. $0 \in R(\mathcal{A})$. In what follows, we prove more than needed for solving problem (4.3).

Theorem 4.2 *Let $\mathcal{A}$ be the operator from V to V' defined in* (4.4)*. Then $\mathcal{A}$ is maximal monotone and $R(\mathcal{A}) = V'$.*

Proof: By using Theorem 2.8 we first prove that $\mathcal{A}$ is maximal monotone.

(a) The proof of the monotonicity of $\mathcal{A}$ is a direct consequence of its definition and of the monotonicity of $a(x, \cdot)$ itself (see, e.g. Step 2 in the proof of Theorem 5.1).

(b) Given any $v \in V$, the image $\mathcal{A}v$ is non empty. In fact, by assumptions on a, the set $a(y, \xi + v(y))$ is non empty and closed, as a subset of $\mathbb{R}^n$, for a.e. $y \in Y$. Moreover, the multivalued map $F : Y \to \mathbb{R}^n$ defined by $Fy = a(y, \xi + v(y))$ is measurable in the sense of Definition 2.1. Hence, by Remark 2.2 and Theorem 2.4, F has a measurable selection $k : Y \to \mathbb{R}^n$. By estimate (2.15), k belongs to $L^{p'(\cdot)}(Y)$, and the corresponding functional Φ defined by (4.4) turns to be from the image $\mathcal{A}v$, so (b) is proved.

(c) For every $v \in V$, $\mathcal{A}v$ is a convex set in V'. This follows from the fact that, by Remark 2.7, $a(y, \xi + v(y))$ is a convex subset of $\mathbb{R}^n$ for a.e. $y \in Y$.

(d) For every $v \in V$, the set $\mathcal{A}v$ is weakly closed in V' and the operator $\mathcal{A}$ is upper-semicontinuous from the strong topology of V to the weak topology of V'. In order to prove this assertion, thanks to the boundedness condition (2.15), it is enough to prove the following: given v_j, Φ_j such that $v_j \to v$ strongly in V, $\Phi_j \to \Phi$ weakly in V', and $\Phi_j \in \mathcal{A}v_j$, then $\Phi \in \mathcal{A}v$. Since $\Phi_j \in \mathcal{A}v_j$, by the definition of $\mathcal{A}$ there exists $k_j(y) \in a(y, \xi + v_j(y))$ a.e.in Y such that

$$\langle \Phi_j, z \rangle = \int_Y k_j \cdot z \, dy \quad \forall z \in V.$$

Due to (2.15), k_j is bounded in $L^{p'(\cdot)}(Y)$ and hence (up to a subsequence) is weakly converging to some $k \in L^{p'(\cdot)}(Y)$. In order to conclude the proof, we have to show that $k(y) \in a(y, \xi + v(y))$ a.e.in Y. To this end, let us set

$$E = \{y \in Y : \exists \zeta \in \mathbb{R}^n, \exists \eta \in a(y, \zeta) \text{ such that } (k(y) - \eta) \cdot (\xi + v(y) - \zeta) < 0\}. \tag{4.5}$$

If we prove that E has Lebesgue measure zero, i.e. $|E| = 0$, it follows that $k(y) \in a(y, \xi + v(y))$ a.e.in Y, by the maximal monotonicity of a. In order to show that E is Lebesgue measurable, let us rewrite (4.5) in the form $E = \{y \in Y : Gy \neq \emptyset\}$, where $G : Y \to \mathbb{R}^n \times \mathbb{R}^n$ is a multi-valued function defined for each $y \in E$ as

$$Gy = \{(\zeta, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : \eta \in a(y, \zeta), (k(y) - \eta) \cdot (\xi + v(y) - \zeta) < 0\}.$$

Now, by the measurability assumptions for a and Remark 2.2 it follows that the graph of G belongs to $\mathcal{L}(Y) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$, thus $E \in \mathcal{L}(Y)$ due to the projection Theorem 2.3. By the Aumann-von Neumann Theorem 2.4, there exists a measurable selection (ζ, η) of G defined on E. Therefore $\eta(y) \in a(y, \zeta(y))$ and

$$(k(y) - \eta(y)) \cdot (\xi + v(y) - \zeta(y)) < 0 \tag{4.6}$$

for every $y \in E$. On the other hand, the monotonicity of a implies that

$$(k_j(y) - \eta(y)) \cdot (\xi + v_j(y) - \zeta(y)) \geq 0 \quad \text{a.e. } y \in E \tag{4.7}$$

for every j. If $|E| > 0$, there exists a measurable subset E' of E with $|E'| > 0$ such that $(\zeta(y), \eta(y))$ is bounded on E'. By integrating (4.7) on E' and passing to the limit as $j \to +\infty$ we get

$$\int_{E'} (k(y) - \eta(y)) \cdot (\xi + v(y) - \zeta(y))\, dy \geq 0$$

which contradicts (4.6) under assumption $|E'| > 0$. Therefore, we conclude that $|E| = 0$ which is required. This proves (d) and completes the proof of maximal monotonicity of $\mathcal{A}$.

(e) Now, we prove the coerciveness of $\mathcal{A}$, that is

$$\frac{\langle \Phi, v \rangle}{||v||_{L^{p(\cdot)}(Y)}} \to +\infty \tag{4.8}$$

as $||v||_{L^{p(\cdot)}(Y)} \to +\infty$, where $\Phi \in \mathcal{A}v$. To this end, let us fix $v \in V$ and $k(y) \in a(y, \xi + v(y))$ such that (4.4) is valid. Therefore,

$$\langle \Phi, v \rangle = \int_Y k \cdot v(y)\, dy = \int_Y k \cdot (v(y) + \xi)\, dy - \int_Y k \cdot \xi\, dy. \tag{4.9}$$

By estimate (2.13),

$$\int_Y k \cdot (v + \xi)\, dy \geq -m_1 + c_1 \int_Y |v + \xi|^{p(y)}\, dy. \tag{4.10}$$

On the other hand, by the Young inequality,

$$\int_Y k \cdot \xi\, dy \leq \delta \int_Y |k(y)|^{p'(y)}\, dy + C_\delta \int_Y |\xi|^{p(y)}\, dy, \tag{4.11}$$

where, due to (2.15),

$$\int_Y |k(y)|^{p'(y)}\, dy \leq c_3 \int_Y |v + \xi|^{p(y)}\, dy + m_3. \tag{4.12}$$

Besides, by convexity of the power function $|\xi|^{p(y)}$ and the estimate (1.2),

$$\int_Y |v + \xi|^{p(y)}\, dy \geq 2^{1-\beta} \int_Y |v|^{p(y)}\, dy - \int_Y |\xi|^{p(y)}\, dy. \tag{4.13}$$

Collecting the above inequalities (4.10)-(4.13) and choosing $\delta > 0$ sufficiently small, we can find contants $c > 0$ and $d, m \geq 0$ such that

$$\langle \Phi, v \rangle \geq c \int_Y |v|^{p(y)}\, dy - d \int_Y |\xi|^{p(y)}\, dy - m. \tag{4.14}$$

Dividing (4.14) by $||v||_{L^{p(\cdot)}(Y)}$, using the bilateral estimate (2.5), and letting $||v||_{L^{p(\cdot)}(Y)} \to +\infty$, we easily deduce (4.8).

At this point, by Theorem 2.9 we conclude that $R(\mathcal{A}) = V'$, since $\mathcal{A}$ is maximal monotone and coercive. $\square$

Remark 4.3 The integral identity in (4.1) implies that $k \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a solenoidal vector, i.e. $\operatorname{div} k = 0\, (\text{in}\, \mathcal{D}'(\mathbb{R}^n))$. Besides, $k \in L^{p'_\varepsilon(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ owing to estimate (2.15) and first two relations in (4.1).

4.2 Boundedness and continuity properties

The cell problem depends on a vector parameter $\xi \in \mathbb{R}^n$. Below, we establish properties of cell problem solutions with respect to ξ.

Lemma 4.4 *Let (v, k) be a solution to problem* (4.3) *for a given $\xi \in \mathbb{R}^n$. Then*

$$\int_Y |v|^{p(y)}\,dy + \int_Y |k|^{p'(y)}\,dy \le d_1|\xi|^\beta + d_2, \tag{4.15}$$

where constants $d_1, d_2 > 0$ depend only on c_1, c_2, m_1, m_2 that appear in estimates (2.13), (2.14), *and β is the upper bound of the exponent $p(\cdot)$ in* (1.2).

Proof: From (2.14) it follows

$$c_2 \int_Y |k|^{p'(y)}\,dy \le \int_Y (v + \xi)\cdot k\,dy + m_2 =$$

$$= \int_Y \xi \cdot k\,dy + m_2,$$

where we use the fact that

$$\int_Y k \cdot v\,dy = 0$$

due to the integral identity in (4.3). By the Young inequality, for any $\delta > 0$, there exists $c_\delta > 0$ such that

$$\int_Y \xi \cdot k\,dy \le \delta \int_Y |k|^{p'(y)}\,dy + c_\delta \int_Y |\xi|^{p(y)}\,dy,$$

where obviously $|\xi|^{p(y)} \le |\xi|^\beta + 1$. Hence,

$$\int_Y |k|^{p'(y)}\,dy \le d_1|\xi|^\beta + d_2.$$

Using now (2.13) and similar arguments as above, we get the estimate also for v, which completes the proof. □

Lemma 4.5 *Let $\xi_j, \xi \in \mathbb{R}^n$ such that $\xi_j \to \xi$. Let (v_j, k_j) be a solution to the cell problem* (4.3) *corresponding to the parameter ξ_j. Then (up to a subsequence)*

$$v_j \rightharpoonup v \quad \text{weakly in } L^{p(\cdot)}(Y), \tag{4.16}$$

$$k_j \rightharpoonup k \quad \text{weakly in } L^{p'(\cdot)}(Y), \tag{4.17}$$

and (v, k) is a solution of the cell problem (4.3) *corresponding to the parameter ξ.*

Proof: By estimate (4.15), v_j and k_j are bounded in $L^{p(\cdot)}(Y)$ and $L^{p'(\cdot)}(Y)$, respectively. Therefore, we can assume (4.16), (4.17). Clearly $v \in V$ and $k \in V^\perp$, in other words, k satisfies the integral identity in the cell problem (4.3). It remains to prove only the differential inclusion

$$k(y) \in a(y, \xi + v(y)) \quad \text{a.e. in } Y. \tag{4.18}$$

To this end, we introduce the multivalued operator $A : L^{p(\cdot)}(Y) \to L^{p'(\cdot)}(Y)$ defined by

$$A\eta = \{\zeta \in L^{p'(\cdot)}(Y) : \zeta(y) \in a(y, \eta(y)) \text{ a.e. in } Y\}, \tag{4.19}$$

It is possible to prove that A is a maximal monotone operator similarly to what has been done previously in the proof of Theorem 4.1 for operator $\mathcal{A}$. Clearly (v_j, k_j) belongs to the graph $Gr(A)$ of A. So, by the maximal monotonicity of A, for any pair $(\eta, \zeta) \in Gr(A)$ we have

$$\int_Y (v_j + \xi_j - \eta)\cdot(k_j - \zeta)\,dy \ge 0. \tag{4.20}$$

Passing to the limit as $j \to +\infty$ gives

$$\int_Y (v + \xi - \eta)\cdot(k - \zeta)\,dy \ge 0 \quad \forall(\eta, \zeta) \in Gr(A),$$

and the maximal monotonicity of A implies the inclusion (4.18).

Let us explain the above passage to the limit. The left-hand side of inequality (4.20) consists actually of four integral terms, containing products $(v_j+\xi_j)\cdot k_j$, $\eta\cdot\xi$, $(v_j+\xi_j)\cdot\zeta$, $\eta\cdot k_j$. In the last two of them, the passage to the limit is done merely due to the weak convergences (4.16), (4.17). The only integral term that requires care is this one:

$$\int_Y (v_j+\xi_j)\cdot k_j\,dy.$$

We can pass to the limit in it, rewriting it thanks to the integral identity in the cell problem, as follows:

$$\int_Y (v_j+\xi_j)\cdot k_j\,dy = \int_Y \xi_j\cdot k_j\,dy \to \int_Y \xi\cdot k\,dy = \int_Y (v+\xi)\cdot k\,dy.$$

□

5 The limit problem

We define a multivalued map $b:\mathbb{R}^n\to\mathbb{R}^n$ as

$$b(\xi)=\left\{\int_Y k(y)\,dy : k\in L^{p'(\cdot)}(Y),\ \exists v\in V \text{ such that } (v,k) \text{ satisfies (4.3)}\right\}. \tag{5.1}$$

Theorem 5.1 *The map b is maximal monotone and satisfies the estimates*

$$c_1 f(\xi)\le \xi\cdot\nu+m_1, \tag{5.2}$$

$$c_2 f^*(\nu)\le \xi\cdot\nu+m_2 \tag{5.3}$$

for every $\xi\in\mathbb{R}^n$, $\nu\in b(\xi)$, where f, f^* are defined in (3.5), (3.9), *and the constants c_1,m_1,c_2,m_2 are the same as in* (2.13), (2.14).

Proof: We prove the estimates (5.2), (5.3) and then verify assumptions of Theorem 2.8 to conclude maximal monotonicity of the map b.

Step 1 Estimates (5.2), (5.3) are deduced from the definition of b and assumptions (2.13), (2.14) for the map a by applying the minimization procedure (3.5). In fact, given $\xi\in\mathbb{R}^n$ and $\nu\in b(\xi)$, there exists a pair (v,k) satisfying (4.3), such that $\int_Y k(y)\,dy=\nu$. By (2.13),

$$c_1|\xi+v(y)|^{p(y)}\le(\xi+v(y))\cdot k(y)+m_1 \quad \text{a.e. in } Y.$$

Integrating over Y and using properties of the solution (v,k), we get

$$\nu\cdot\xi=\int_Y k(y)\cdot\xi\,dy=\int_Y k(y)\cdot(\xi+v(y))\,dy\ge$$

$$c_1\int_Y|\xi+v(y)|^{p(y)}\,dy-m_1\ge c_1\int_Y\frac{|\xi+v(y)|^{p(y)}}{p(y)}\,dy-m_1\ge c_1 f(\xi)-m_1,$$

where at the last step we use the definition of the symbol $f(\xi)$ in (3.5).

To prove (5.3) we use the dual formulation of the minimum problem (3.5), that is,

$$f^*(\nu)=\min\left\{\int_Y\frac{|z|^{p'(y)}}{p'(y)}\,dy:\quad z\in L^{p'(\cdot)}_{\text{per}}(Y),\ \operatorname{div} z=0,\ \int_Y z\,dy=\nu\right\}, \tag{5.4}$$

given in [20]. Then, similarly to the above manipulations, we deduce

$$\nu\cdot\xi=\int_Y k(y)\cdot\xi\,dy=\int_Y k(y)\cdot(\xi+v(y))\,dy\ge$$

$$\geq c_2 \int_Y |k(y)|^{p'(y)}\, dy - m_2 \geq c_2 \int_Y \frac{|k(y)|^{p'(y)}}{p'(y)}\, dy - m_2 \geq c_2 f^*(\nu) - m_2.$$

We use here (2.14), (5.4) and the properties of k such that $\mathrm{div} k = 0$ and $\int_Y k\, dy = \nu$.

Step 2 The map b is monotone. In fact, take any $\xi_i \in \mathbb{R}^n$, $\nu_i \in b(\xi_i)$, $i = 1, 2$, and any solution $(v_i, k_i) \in V \times L^{p'(\cdot)}$ to problem (4.3) such that $\int_Y k_i(y)\, dy = \nu_i$. Then

$$(\nu_1 - \nu_2) \cdot (\xi_1 - \xi_2) = \int_Y (k_1(y) - k_2(y)) \cdot (\xi_1 - \xi_2)\, dy =$$

$$= \int_Y (k_1(y) - k_2(y)) \cdot (v_1 + \xi_1 - (v_2 + \xi_2))\, dy,$$

where the last equality is due to the integral identity in the cell problem (4.3). The last integral is then non negative, by the monotonicity of a.

Step 3 For every $\xi \in \mathbb{R}^n$, $b(\xi) \neq \emptyset$. This is a consequence of Theorem 4.1.

Step 4 For every $\xi \in \mathbb{R}^n$, $b(\xi)$ is convex. This assertion follows from maximal monotonicity of the mapping $A : L^{p(\cdot)}(Y) \to L^{p'(\cdot)}(Y)$ defined in (4.19).

Step 5 For every $\xi \in \mathbb{R}^n$, $b(\xi)$ is closed and the map $\xi \mapsto b(\xi)$ is upper-semicontinuous. Thanks to estimates (5.2), (5.3), it is enough to show that for every $\xi_j \to \xi$ and $\nu_j \in b(\xi_j)$ such that $\nu_j \to \nu$ it follows that $\nu \in b(\xi)$. But this fact is ensured by Lemma 4.5. $\square$

As an immediate corollary of Lemma 3.1 and uniform estimates (2.21), we obtain

Lemma 5.2 *Let $(u_\varepsilon, g_\varepsilon)$ be a sequence of solutions to problem* (2.20) *and (u, g) be the limit pair for convergences* (2.22) *and* (2.23). *Then $u \in W_0^f(\Omega)$ and $g \in L^{f^*}(\Omega)$.*

Now we are in position to introduce the homogenized problem

$$\begin{cases} u \in W_0^f(\Omega), \\ g(x) \in b(Du(x)) \quad \text{for a.e. } x \in \Omega, \\ \displaystyle\int_\Omega g \cdot Dv\, dx = \int_\Omega h \cdot Dv\, dx \qquad \text{for all } v \in W_0^f(\Omega). \end{cases} \tag{5.5}$$

Theorem 5.3 *For every $h \in L^{f^*}(\Omega)$, problem* (5.5) *has at least one solution (u, g).*

This is a consequence of a general result from [1] concerning the multivalued operator $\mathcal{A}_0 : W_0^f(\Omega) \to (W_0^f(\Omega))'$ defined as $\mathcal{A}_0 u = -\mathrm{div}\, b(Du)$ which is maximal monotone and coercive. Note only that every $h \in L^{f^*}(\Omega)$ defines a functional

$$\langle h, \varphi \rangle = \int_\Omega h \cdot Dv\, dx$$

on $W_0^f(\Omega)$ thanks to the Hölder inequality (3.15). In our case, $h \in L^\infty(\Omega)$ is inherited from the initial problem (2.20), and clearly $h \in L^{f^*}(\Omega)$.

6 Homogenization

In this section we formulate the homogenization result for problem (2.20) and prove it. The main tool for us will be the following variant of Compensated Compactness Lemma proved in [25].

Lemma 6.1 *Let u_ε, w_ε satisfy the following conditions:*

(i) $u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, $w_\varepsilon \in L^{p'_\varepsilon(\cdot)}(\Omega)$ and $\mathrm{div}\ w_\varepsilon = 0$;

(ii) $\int_\Omega |Du_\varepsilon|^{p_\varepsilon}\,dx \le c$, $\int_\Omega |w_\varepsilon|^{p'_\varepsilon}\,dx \le c$ *for some constant* $c > 0$*;*

(iii) $Du_\varepsilon \rightharpoonup Du$ *in* $L^\alpha(\Omega)$, $w_\varepsilon \rightharpoonup w$ *in* $L^{\beta'}\Omega)$*;*

(iv) $\{u_\varepsilon\}$ *is compact in* $L^\beta(\Omega)$.

Then (up to a subsequence)

$$\int_\Omega Du_\varepsilon \cdot w_\varepsilon \varphi\,dx \to \int_\Omega Du \cdot w\varphi\,dx \quad \forall\varphi \in \mathcal{C}_0^\infty(\Omega). \tag{6.1}$$

Both integrals in (6.1) make sense due to the Hölder inequalities either (2.7) or (3.15). For the latter one, note that the limit elements Du amd w belong to the mutually conjugate spaces $L^f(\Omega)$ and $L^{f^*}(\Omega)$, respectively, thanks to Lemma 3.1.

In the sequel, we will use systematically the *mean value property* of periodic functions: if $z(y)$ is an Y-periodic function that belongs to $L^s(Y)$, $s \in [1, +\infty)$, then

$$z\left(\frac{x}{\varepsilon}\right) \rightharpoonup \int_Y z(y)\,dy \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^n)$$

(see the proof, e.g. in [11], Chapter I, §1).

Our main result is

Theorem 6.2 *Let* $(u_\varepsilon, g_\varepsilon)$ *be a sequence of solutions to problem* (2.20) *and let* (u, g) *be the limit pair from* (2.22) *and* (2.23)*. Then,* $u \in W_0^f(\Omega)$ *and* $g \in L^{f^*}(\Omega)$ *satisfy problem* (5.5).

Proof: We follow the lines of the proof of Theorem 3.1 in [25] adapting it to our case.

1. Given $\xi \in \mathbb{R}^n$, let us take a pair (w, k) that is a solution to problem (4.1) and set

$$w_\varepsilon(x) = \xi \cdot x + \varepsilon w\left(\frac{x}{\varepsilon}\right), \quad k_\varepsilon(x) = k\left(\frac{x}{\varepsilon}\right).$$

Clearly, we have an ε-periodic function $Dw_\varepsilon(x) = \xi + (D_y w(y))|_{y=\frac{x}{\varepsilon}}$ and

$$Dw_\varepsilon \rightharpoonup \xi \quad \text{in } L^\alpha(\Omega), \tag{6.2}$$

by the mean value property. For the same reason,

$$k_\varepsilon(x) \rightharpoonup \nu = \int_Y k(y)\,dy \quad \text{in } L^{\beta'}(\Omega). \tag{6.3}$$

Besides, under our assumptions,

$$Du_\varepsilon(x) \rightharpoonup Du(x) \quad \text{in } L^\alpha(\Omega), \quad g_\varepsilon(x) \rightharpoonup g(x) \quad \text{in } L^{\beta'}(\Omega). \tag{6.4}$$

By the monotonicity of $a(x, \cdot)$, we can write

$$\int_\Omega (g_\varepsilon(x) - k_\varepsilon(x)) \cdot (Du_\varepsilon(x) - Dw_\varepsilon(x))\varphi(x)\,dx \ge 0 \tag{6.5}$$

for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \ge 0$.

Let us assume that the following convergences take place:

$$k_\varepsilon(x) \cdot Dw_\varepsilon(x) \rightharpoonup \nu \cdot \xi \quad \text{in } L^1(\Omega), \tag{6.6}$$

$$g_\varepsilon(x) \cdot Dw_\varepsilon(x) \rightharpoonup g(x) \cdot \xi \quad \text{in } L^1(\Omega), \tag{6.7}$$

$$k_\varepsilon(x) \cdot Du_\varepsilon(x) \rightharpoonup \nu \cdot Du(x) \quad \text{in } L^1(\Omega), \tag{6.8}$$

$$\int_\Omega g_\varepsilon(x) \cdot Du_\varepsilon(x)\varphi(x)\,dx \to \int_\Omega g(x) \cdot Du(x)\varphi(x)\,dx \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\Omega)(\Omega). \tag{6.9}$$

Then, passing to the limit in (6.5) yields

$$\int_\Omega (g(x)-\nu)\cdot(Du(x)-\xi)\varphi(x)\,dx \geq 0$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$. Therefore, for every ξ and $\nu \in b(\xi)$ we have

$$(g(x)-\nu)\cdot(Du(x)-\xi) \geq 0 \quad \text{ for a.e. } x \in \Omega.$$

In particular, the same is true for a dense subset (ξ_j, ν_j) in the graph of b

$$(g(x)-\nu_j)\cdot(Du(x)-\xi_j) \geq 0 \quad \text{ for a.e. } x \in \Omega \text{ and } \forall j \in \mathbb{N}.$$

This implies easily that

$$(g(x)-\nu)\cdot(Du(x)-\xi) \geq 0$$

for a.e. $x \in \Omega$ and for all pairs (ξ, ν) in the graph of b. From here, the maximal monotonicty of b yields that

$$g(x) \in b(Du(x)) \quad \text{ a.e. in } \Omega,$$

which concludes the proof.

2. It remains to justify the passage to the limit in (6.5), in other words, to obtain the convergences (6.6)-(6.9). As for (6.6), it is a direct corollary of the mean value property.:

$$k_\varepsilon(x)\cdot Dw_\varepsilon(x) \rightharpoonup \int_Y k(y)\cdot(\xi + Dw(y))\,dy = \int_Y k(y)\cdot\xi\,dy = \nu\cdot\xi,$$

where we use also the integral identity in (4.1).

What concerns (6.7)-(6.9), the identification of the corresponding limits is not so simple. First, we consider cases (6.7), (6.8), where we deal with families that are weakly convergent in $L^1(\Omega)$. The weak convergence in $L^1(\Omega)$ is verified below. To this end, we recall the well-known result about the weak convergence in $L^1(\Omega)$) (see, e.g. [10]).

Lemma 6.3 (Criterion for weak convergence in $L^1(\Omega)$)
The following assertions are equivalent:

(1) the family v_ε is weakly compact in $L^1(\Omega)$;

(2) the family v_ε is equiintegrable in $L^1(\Omega)$, i.e., $\forall \tau > 0\ \exists \delta > 0$ such that $\int_M |v_\varepsilon|\,dx < \tau$ for any measurable set $M \subset \Omega$ such that the Lebesgue measure $|M| < \delta$.

Proposition 6.4 *The families $W_\varepsilon(x) = |Dw_\varepsilon(x)|^{p_\varepsilon(x)}$, $\quad K_\varepsilon(x) = |k_\varepsilon(x)|^{p'_\varepsilon(x)}$ are equiintegrable.*

Proof: By the mean value property, we have weak convergence

$$K_\varepsilon(x) \rightharpoonup \int_Y |k(y)|^{p'(y)}\,dy \quad \text{ in } L^1(\Omega),$$

and the equiintegrability property is ensured by the Criterion for weak convergence in $L^1(\Omega)$. The same argument is valid for the family $W_\varepsilon(x)$. □

Proposition 6.5 *The families $k_\varepsilon(x)\cdot Du_\varepsilon(x)$, $\quad g_\varepsilon(x)\cdot Dw_\varepsilon(x)$ are equiintegrable.*

Proof: By the Young inequality,

$$\int_M k_\varepsilon(x)\cdot Du_\varepsilon(x)\,dx \leq \delta \int_M |Du_\varepsilon(x)|^{p_\varepsilon(x)}\,dx + C_\delta \int_M |k_\varepsilon(x)|^{p'_\varepsilon(x)}\,dx \leq$$

$$\leq c\delta + C_\delta \int_M |k_\varepsilon(x)|^{p'_\varepsilon(x)}\,dx$$

for any measurable set $M \subset \Omega$, where we use estimate (2.21). For arbitrary $\delta > 0$, the last integral is less than δ / C_δ, provided the Lebesgue measure $|M|$ is sufficiently small, since, $|k_\varepsilon(x)|^{p'_\varepsilon(x)}$ is equiintegrable according to Proposition 6.4. Similar argument gives the equiintegrability of the product $g_\varepsilon(x) \cdot Dw_\varepsilon(x)$. $\square$

3. To find the limits in (6.7)-(6.9), we use Lemma 6.1. However, this Lemma cannot be applied directly to the vectors we deal with, because of the lack of necessary compactness property (see, (iv) in Lemma 6.1). We gain this compactness property using a truncation procedure for the potential vectors. Therefore, we act further according to the following scheme. First, we obtain the limit relations of the type (6.7)-(6.9) with the truncated potential vectors. Then, we explain why the truncation may be omitted everywhere. To this end, the certain properties of the truncation are helpful. In the sequel, we use

$$u_{\varepsilon,m}(x) = \begin{cases} u_\varepsilon(x) & \text{if } \ |u_\varepsilon(x)| \leq m, \\ \pm m & \text{if } \ |u_\varepsilon(x)| > m, \end{cases}$$

for $m \in \mathbb{N}$ and call it a truncated function.

Proposition 6.6 *For the solution $(u_\varepsilon, g_\varepsilon)$ of problem (2.20), the following assertions are valid:*

$$0 \leq g_\varepsilon(x) \cdot Du_{\varepsilon,m}(x) \leq g_\varepsilon(x) \cdot Du_\varepsilon(x) \quad \text{a.e. in } \Omega; \tag{6.10}$$

$$\lim_{m \to +\infty} \int_\Omega g_\varepsilon(x) \cdot (Du_\varepsilon(x) - Du_{\varepsilon,m}(x))\, dx = 0 \tag{6.11}$$

uniformly with respect to ε.

Proof: Due to the monotonicity of $a(x, \cdot)$, and the fact that $a(x, 0) = 0$, we get

$$0 \leq a(x, Du_\varepsilon(x)) \cdot Du_\varepsilon(x)$$

a.e. in Ω. Besides, $a(x, Du_\varepsilon(x)) \cdot Du_{\varepsilon,m}(x)$ is either zero or equal to $a(x, Du_\varepsilon(x)) \cdot Du_\varepsilon(x)$. Hence, we conclude the proof of (6.10).

Letting

$$T_{\varepsilon,m} = \{x \in \Omega : |u_\varepsilon(x)| > m\},$$

we easily obtain

$$|T_{\varepsilon,m}| \leq m^{-1} ||u_\varepsilon||_{L^1(\Omega)} \leq c m^{-1}, \tag{6.12}$$

by boundedness property of the famlily u_ε.

Substituting $v = u_\varepsilon$ and $v = u_{\varepsilon,m}$ in the integral identity of problem (2.20) we derive

$$\begin{aligned} 0 \leq \int_\Omega g_\varepsilon(x) \cdot (Du_\varepsilon(x) - Du_{\varepsilon,m}(x))\, dx &= \int_\Omega h(x) \cdot (Du_\varepsilon(x) - Du_{\varepsilon,m}(x))\, dx = \\ &= \int_{T_{\varepsilon,m}} h(x) \cdot Du_\varepsilon(x)\, dx \leq \\ \leq \left(\int_{T_{\varepsilon,m}} |Du_\varepsilon(x)|^\alpha\, dx \right)^{1/\alpha} \left(\int_{T_{\varepsilon,m}} |h(x)|^{\alpha'}\, dx \right)^{1/\alpha'} &\leq c_1 ||h||_{L^\infty(\Omega)} |T_{\varepsilon,m}|^{1/\alpha'}, \end{aligned} \tag{6.13}$$

where the first inequality in this chain is due to (6.10) and the last one employes the boundedness property of the famlily u_ε. From (6.12) and (6.13) we derive (6.11). $\square$

4. We prove now the convergence (6.9). Obviously, it is enough to show that

$$z_\varepsilon \cdot Du_\varepsilon \rightharpoonup z \cdot Du \quad \text{in } \mathcal{D}'(\Omega), \tag{6.14}$$

where $z_\varepsilon = g_\varepsilon - h$, $z = g - h$. Taking the truncated functions $u_{\varepsilon,m}$ and u_m, we write

$$z_\varepsilon \cdot Du_\varepsilon - z \cdot Du = (z_\varepsilon \cdot Du_{\varepsilon,m} - z \cdot Du_m) + z \cdot (Du_m - Du) + r_\varepsilon^m. \tag{6.15}$$

Here $r_\varepsilon^m = z_\varepsilon \cdot (Du_\varepsilon - Du_{\varepsilon,m})$ for which (6.11) ensures that

$$||r_\varepsilon^m||_{L^1(\Omega)} \le \delta \quad \forall m > m_1(\delta)$$

uniformly with respect to ε.

Setting $T_m = \{x \in \Omega : |x| > m\}$, we easily derive

$$||z \cdot (Du_m - Du)||_{L^1(\Omega)} = ||z \cdot Du||_{L^1(T_m)} \le \int_{T_m} f^*(z)\, dx + \int_{T_m} f(Du)\, dx,$$

where we employ the Young inequality

$$\eta \cdot \zeta \le f(\eta) + f^*(\zeta) \quad \forall \eta, \zeta \in \mathbb{R}^n$$

and the fact that the function f is even. Hence,

$$||z \cdot (Du_m - Du)||_{L^1(\Omega)} \le \delta \quad \forall m > m_2(\delta)$$

since $f^*(z), f(Du) \in L^1(\Omega)$ by Lemma 5.2 and $|T_m| < cm^{-1}$, that is a counterpart of estimate (6.12).

Taking $m > \max\{m_1, m_2\}$ and any $\varphi \in C_0^\infty(\Omega)$, we deduce from (6.15) that

$$\left| \int_\Omega \varphi z_\varepsilon \cdot Du_\varepsilon\, dx - \int_\Omega \varphi z \cdot Du\, dx \right| \le \left| \int_\Omega \varphi z_\varepsilon \cdot Du_{\varepsilon,m}\, dx - \int_\Omega \varphi z \cdot Du_m\, dx \right| + 2\delta, \tag{6.16}$$

where the difference of integrals in the right-hand side vanishes as $\varepsilon \to 0$ by Lemma 6.1. In this case, the assumptions of this lemma are obviously fulfilled, with the exception of condition (iv), which should be verified. To this end, we invoke condition (iii) and properties of truncation, whence $u_{\varepsilon,m} \rightharpoonup u_m$ in $W^{1,\alpha}(\Omega)$ and, therefore,

$$u_{\varepsilon,m} \to u_m \quad \text{ in } L^s(\Omega) \ \ \forall s \ge 1,$$

since $u_{\varepsilon,m} \in L^\infty(\Omega)$. In particular, $s = \beta$ is possible.

Eventually, by the arbitrary smallness of δ, inequality (6.16) implies (6.14).

5. To prove (6.8), we engage again the truncated functions $u_{\varepsilon,m}$, u_m and write

$$k_\varepsilon \cdot Du_\varepsilon - k \cdot Du = (k_\varepsilon \cdot Du_{\varepsilon,m} - \nu \cdot Du_m) + \nu \cdot (Du_m - Du) + r_\varepsilon^m, \tag{6.17}$$

where $r_\varepsilon^m = k_\varepsilon \cdot (Du_\varepsilon - Du_{\varepsilon,m})$. Note that k_ε is the solenoidal vector with the proper integrability property (see Remark 4.3). We apply Lemma 6.1 to the first difference in the right-hand side of (6.17) in a similar way as before, when we justify passing to the limit in (6.15). As for the remaining terms in the right-hand side of (6.17), they are arbitrarily small if the truncation parameter m is large enough. To this end, we rely on the equiintegrability of the family $k_\varepsilon \cdot Du_\varepsilon$ (see Proposition 6.5). In fact, for arbitrary $\delta > 0$ we have

$$||r_\varepsilon^m||_{L^1(\Omega)} = ||k_\varepsilon \cdot D(u_\varepsilon - u_{\varepsilon,m})||_{L^1(\Omega)} = ||k_\varepsilon \cdot Du_\varepsilon||_{L^1(T_{\varepsilon,m})} \le \delta$$

uniformly with respect to ε, provided that m is properly chosen to make the Lebesgue measure $|T_{\varepsilon,m}|$ sufficiently small (see (6.12)).

To prove (6.7), we introduce the truncated function

$$v_\varepsilon(x) = \xi + (D\overline{w})(\frac{x}{\varepsilon}), \quad \overline{w}(y) = \begin{cases} w(y) & \text{if } |w(y)| \le \varepsilon^{-1}, \\ \pm\varepsilon^{-1} & \text{if } |w(y)| > \varepsilon^{-1}. \end{cases}$$

Clearly, $v_\varepsilon \rightharpoonup \xi$ in $L^\alpha(\Omega)$. Setting $T_\varepsilon = \{x \in \Omega : |w(\frac{x}{\varepsilon})| > \varepsilon^{-1}\}$, we have $|T_\varepsilon| < c\varepsilon$. Due to Lemma 6.1,

$$g_\varepsilon(x) \cdot v_\varepsilon(x) \rightharpoonup g \cdot \xi \quad \text{ in } L^1(\Omega),$$

and the truncation can be removed here, thanks to the equiintegrability of the family $g_\varepsilon(x) \cdot Dw_\varepsilon$ (see Proposition 6.5).

Theorem 6.2 is proved.

Remark 6.7 The above homogenization result was proved with a help of the suitable version of Compensated Compactness Lemma. But it can be obtained also by another approach based on two-scale convergence technique developed in [28]. The advantage of the latter one lays in its applicability to vector problems. This may be the subject of a particular article in future.

Acknowledgments

The authors gratefully acknowledge the hospitality and support of the Department of Mathematical Sciences “G.L. Lagrange”, Dipartimento di Eccellenza 2018-2022, of Politecnico di Torino.

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