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The flavor moonshine hypothesis is formulated to suppose that all particle masses (leptons, quarks, Higgs, and gauge particles—more precisely, their mass ratios) are expressed as coefficients in the Fourier expansion of some modular forms just as, in mathematics, dimensions of representations of a certain group are expressed as coefficients in the Fourier expansion of some modular forms. The mysterious hierarchical structure of the quark and lepton masses is thus attributed to that of the Fourier coefficient matrices of certain modular forms. Our intention here is not to prove this hypothesis starting from some physical assumptions but rather to demonstrate that this hypothesis is experimentally verified and, assuming that the string theory correctly describes the natural law, to calculate the geometry (Kähler potential and the metric) of the moduli space of the Calabi–Yau manifold, thus providing a way to calculate the metric of the Calabi–Yau manifold itself directly from the experimental data.

Subject Index B41, B55

1. Introduction

Some researchers, including one of the authors of this work (H.S.), have been working on flavor physics, assuming that some discrete symmetry plays an important role in its understanding [1–9]; S_3, S_4, A_4, etc. However, the outcome is very limited and so far we have no clear understanding of flavor physics. A topological definition of Higgs Yukawa coupling has also not led to any useful prediction on flavor physics to date [10].

On the mathematical side, a dramatic phenomenon called "moonshine" has been described [11–14], in which a discrete symmetry (specifically, the dimensions of a representation of the monster group) is manifested in a modular form in a rather unexpected manner. When this happens, we may use this fact for the discrete symmetry in flavor physics: We start by assuming that the symmetry of flavor physics is manifested in a certain modular form. Corresponding to each flavor we assume such a modular form. The modular forms must contain all the information about flavor physics with the understanding that all this information is contained in the Higgs coupling to leptons and hadrons.

More precisely, we assume that the particle masses (the mass ratios), rather than the dimensions of a representation of a discrete group, are directly written in the Fourier coefficients of these modular forms—the flavor moonshine hypothesis. The mass ratios are scale-independent quantities [15] and do not vary with energy scale. We observe that at least in the lowest-order perturbation calculations the logarithmic scale dependence cancels out completely both in quantum chromodynamics (QCD) and in electroweak theory, although it does not exclude the renormalization effect proportional to such terms as $\log(m_1/m_2)$. We refer to Ref. [15] here for the non-perturbative calculations. Therefore, the mass ratio is an appropriate quantity to discuss physics even at the highest energy scale. The

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gauge particle masses must also be written as some modular forms but we will not discuss that matter in this work.

In pure mathematics, we anticipate a generalization of conventional "moonshine" from a single-variable modular form to a multi-variable modular form. A certain mathematical "object," perhaps the representation matrix of a certain group rather than the dimension of the monster group, must be written in the Fourier expansion of the multi-variable modular forms. We will identify the mass matrix with this "object."

The question arises: What are those modular forms that manifest the discrete symmetry appearing in flavor physics? For the time being, we postpone the question of justifying our adoption of a certain modular form for each flavor based on a general formalism such as string theory, but rather we proceed backward and investigate instead what the experimentally acceptable modular forms are. We then determine what kind of geometry can yield such a modular form when we consider the compactification of the string theory.

We define the flavor modular form in the following way. Suppose we have a two-variable modular form for each flavor. Then it can be Fourier expanded as

$$J(q,r) = \frac{1}{g}\sum_{i=0,j=-\infty}^{\infty} g_{ij} q^i r^j \tag{1}$$

where g_{ij} for $i \geq 0$ and $g_{i,-j} = g_{ij}$ for the symmetric modular form [16]. The g_{ij} is supposed to correspond to the Higgs coupling of i and j quarks or to the corresponding leptons. By solving Eq. (1) backwards we have

$$g_{ij} = g\int_0^1\int_0^1 J(q,r)q^{-i}r^{-j}d\tau d\sigma = \frac{g}{(2\pi i)^2}\int_C\int_C J(q,r)q^{-i-1}r^{-j-1}dqdr. \tag{2}$$

Here, $q = e^{2\pi i\tau}$ and $r = e^{2\pi i\sigma}$. The integration is done along the circle C of radius 1 with the center at the origin. It is important that we integrate over the modular variables to obtain the coefficient.

If the modular form is based on the ring of integers, the forms are numerous and it is hard to pinpoint the appropriate form. Fortunately, if we generalize the integer ring appropriately to constrain the possible forms, then in the case that we are considering where $g_{i,-j} = g_{ij}$, called the symmetric modular form, it is known that all the modular forms can be constructed rather easily [16].

Specifically, as the simplest generalization, we use $\mathrm{SL}(2,\mathbb{Z}(\sqrt{2}))$ to define the flavor modular group of the two-variable modular form rather than $\mathrm{SL}(2,\mathbb{Z})$.[1] We put

$$q = e^{\pi i(z+z')}, \quad r = e^{\pi i(z-z')/\sqrt{2}}; \quad 2\tau = z+z', \quad 2\sqrt{2}\sigma = z-z'. \tag{3}$$

Then the condition for the modularity is the transformation property:

$$J(e^{2\pi iz}, e^{2\pi iz'}) \;\to\; J(e^{2\pi iz}, e^{2\pi iz'})\left((\gamma z+\delta)(\gamma' z'+\delta')\right)^{2k} \tag{4}$$

[1] When we thought of flavor moonshine, it was clear that the relevant modular form must have more than one variable. It also seemed that $\mathrm{SL}(2,\mathbb{Z})$ is insufficiently constrained, allowing too many choices for the forms. Therefore, we looked for some work enlarging $\mathrm{SL}(2,\mathbb{Z})$ so that the choice becomes manageable, and we encountered a paper by H. Cohn and J. Deutsch [16] where we learned that there are only three generators for the entire $\mathrm{SL}(2,\mathbb{Z}(\sqrt{2}))$, which is the simplest kind of $\mathrm{SL}(2,\mathbb{Z})$ extension. To our great surprise, we found that its modular form in the lowest level ($k = 1$) describes the charged lepton mass ratios correctly (in Sect. 2.1).

under

$$(z,z') \to \left(\frac{\alpha z+\beta}{\gamma z+\delta}, \frac{\alpha' z'+\beta'}{\gamma' z'+\delta'}\right), \quad \alpha,\beta,\gamma,\delta,\ldots \in \mathbb{Z}(\sqrt{2}) \tag{5}$$

and $\alpha = a+b\sqrt{2},\ \alpha' = a-b\sqrt{2},\ldots$ with $a,\ b$: integer. $2k$ is called the "level".

Cohn–Deutsch [16] shows that there are only three generator modular forms in this case. They are given by G_2, G_4, G_6 with $k=1,2,3$. What we use are the coefficients in Fourier expansion of these modular forms. We may also choose different combinations H_2, H_4, H_6, which are given by

$$H_2 = G_2, \quad H_4 = \frac{11G_2^2 - G_4}{576}, \quad H_6 = \frac{361G_2^3 - G_6 - 50\,976 G_2 H_4}{224\,640}. \tag{6}$$

We have only one modular form for $k=1$: G_2; two forms for $k=2$: G_4 and G_2^2; and three forms for $k=3$: $G_6,\ G_2^3$, and G_2G_4. A linear combination of forms of the same level $2k$ is again a modular form. Therefore all modular forms up to level 6 ($k=3$) are given by

$$k=1: \quad G_2 \tag{7}$$

$$k=2: \quad G_4 + a_4 G_2^2 \tag{8}$$

$$k=3: \quad G_6 + a_6 G_2^3 + b_6 G_2 G_4 \tag{9}$$

where $a_4,\ a_6$, and b_6 are complex numbers.

In order to write down the Higgs coupling of quarks and leptons, we define the following: First we define, for the Higgs coupling of a certain flavor,

$$F(q^{-1}, r^{-1}) \equiv gH \lim_{G\to\infty} \sum_{i,j=0}^{G-1} \overline{\psi}_{Rj}\psi_{Li} q^{-i} r^{-j} \tag{10}$$

where H is the Higgs field and ψ_L, ψ_R are quark or lepton fields. The Yukawa coupling is given by

$$\begin{aligned} Y &= \int J(q,r)F(q^{-1},r^{-1})d\tau d\sigma = \frac{1}{(2\pi i)^2}\int J(q,r)F(q^{-1},r^{-1})\frac{dqdr}{qr} \\ &= H\sum_{i,j=0}^{\infty} g_{ij}\overline{\psi}_{Rj}\psi_{Li} = gH \sum_{i,j,k=0}^{\infty} U^{\dagger}_{Lik}\lambda_k U_{Rkj}\overline{\psi}_{Rj}\psi_{Li} \end{aligned} \tag{11}$$

where U_L, U_R are unitary matrices and λ denotes elements of the diagonalized g_{ij} matrix, i.e.,

$$g_{ij} = g \lim_{G\to\infty} \sum_{k=0}^{G-1} U^{\dagger}_{Lik}\lambda_k U_{Rkj} \tag{12}$$

for $i,j = 0,1,\ldots,G-1$. Then we have

$$Y = gH \lim_{G\to\infty}\sum_{k=0}^{G-1} \lambda_k \overline{\chi}_{Rk}\chi_{Lk} \tag{13}$$

where $\chi_{Lk} = U^{\dagger}_{Lik}\psi_{Li}$ and $\chi_{Rk} = \psi_{Rj}U^{\dagger}_{Rjk}$. To maintain the modular invariance of the Yukawa coupling, we assume the transformation property:

$$F(q^{-1},r^{-1}) = F(z,z') \ \to \ \left((\gamma z+\delta)(\gamma' z'+\delta')\right)^{-2k+2} F(z,z') \tag{14}$$

under the modular transformation (5). The level $-2k+2$ is to take care of the transformation property of $dqdr/qr$:

$$\frac{dqdr}{qr} = \frac{d\tau d\sigma}{(2\pi i)^2} = \frac{dzdz'}{2\sqrt{2}(2\pi i)^2} \rightarrow (\gamma z + \delta)^{-2}(\gamma' z' + \delta')^{-2}\frac{dzdz'}{2\sqrt{2}(2\pi i)^2}. \tag{15}$$

If the original τ, σ are real, so are the transformed τ, σ. Therefore, the unit circle goes to the unit circle and the modular invariance is maintained.

Some remarks are in order:

(1) This construction suggests the definition of the fields:

$$\psi_L(x,q) = \lim_{G\to\infty}\sum_{i=0}^{G-1}\psi_{Li}q^{-i}, \quad \psi_R(x,r) = \lim_{G\to\infty}\sum_{j=0}^{G-1}\psi_{Ri}r^{-j}. \tag{16}$$

We do not need to assume any specific transformation property of the individual field under modular transformation, while the bilinear form expressed in Eq. (10) must transform covariantly under the modular transformation. We also note that the transformation property (10) is consistent only when the number of generations G is infinite. Finite G violates the modular invariance of the Yukawa coupling.

(2) We treat here, just for simplicity, a pristine Higgs field H. However, in Sect. 4, we will define and use the modular form corresponding to the Higgs field:

$$J_H(w) = \sum_k h_k w^k \tag{17}$$

corresponding to $J(q,r)$. We can also define the field

$$H(w^{-1}) = \sum_k H_k w^{-k} \tag{18}$$

with the Higgs field $H = H_0$.

(3) Our modular variables q, r eventually become the moduli of the Calabi–Yau manifold as will be shown later in Sect. 4. The usual treatment of these variables is to regard them as a scalar field in the 4D space-time and to try to find a way to stabilize them. We regard them as variables to distinguish different vacua, and we integrate over them as in Eq. (13) to obtain the Yukawa coupling. This roughly corresponds to superposing all possible equivalent vacua. The Yukawa interaction resolves this degeneracy, so that each value of generation G corresponds to a different vacuum. We have $G = 3$ in this work as it concerns the low-energy experimental data. It may happen that a phase transition occurs at high energy, in which case the particle masses would change suddenly at that energy scale.

(4) Our definition of the "generation" is not the same as the usual one in string theory. It corresponds to the expansion coefficient of the modulus-dependent fields defined in Eqs. (16) and (18).

2. Numerical results

Equation (12) shows that g_{ij} is a mass matrix, and Eq. (1) shows that it is just the Fourier coefficient of the modular form $J(q,r)$. In this section we consider each case of Eqs. (7), (8), and (9) separately.

2.1. Case of $k = 1$

The modular form $J(q,r) = G_2$ in this case. From a table given by Cohn and Deutsch [16], we have

$$g_{ij} = g \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 48 & 0 & 0 & 0 & \cdots \\ 720 & 384 & 336 & 0 & 0 & \cdots \\ 1440 & 864 & 1152 & 480 & 144 & \cdots \\ 3024 & 1536 & 2688 & 1152 & 1488 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{19}$$

From now on we restrict ourselves to the $G = 3$ case:

$$g_{ij} = g \begin{pmatrix} 1 & 0 & 0 \\ 144 & 48 & 0 \\ 720 & 384 & 336 \end{pmatrix} =: gM_3. \tag{20}$$

The mass square matrix is given by $gg^\dagger$ and it will be diagonalized as

$$(gg^\dagger)_{ij} = g^2 U^\dagger_{Lik} |\lambda_k|^2 U_{Rkj} \tag{21}$$

with sum over the indices k. We diagonalize the mass square matrix M_3^2 and find that its square root is

$$\sqrt{M_3 M_3^T} = \begin{pmatrix} 0.2929 & 0 & 0 \\ 0 & 61.63 & 0 \\ 0 & 0 & 893.3 \end{pmatrix}. \tag{22}$$

By normalizing the lowest mass to be the electron mass of 0.5110 MeV, we obtain

$$\left(\sqrt{M_3 M_3^T}\right)_{\text{normalized}} = \begin{pmatrix} 0.5110 & 0 & 0 \\ 0 & 107.5 & 0 \\ 0 & 0 & 1558 \end{pmatrix}. \tag{23}$$

This shows that the modular form G_2 embodies the charged lepton masses in its Fourier coefficients. There is no free parameter in this case except for the entire normalization, which is of course scale dependent, unlike the mass ratios [15].

The corresponding experimental data are in Appendix A: the central values of the μ and τ masses are $(m_\mu, m_\tau) = (105.7, 1776)$ MeV. Deviations of our results are at most 12.32%, so we may say that our calculations reproduce the experimental data well. In the following, we mainly use the central values of the experimental results, i.e., we neglect the errors just for simplicity.

2.2. Case of $k = 2$

In this case, we have the modular form

$$J(q,r) = G_4 + a_4 G_2^2. \tag{24}$$

For the time being we ignore the second term (i.e., put $a_4 = 0$). Then we have, for the three-generation case,

$$G_4 \to M_3 = \begin{pmatrix} 11 & 0 & 0 \\ 4320 & 480 & 0 \\ 280\,800 & 165\,120 & 35\,040 \end{pmatrix}. \tag{25}$$

The normalized and diagonalized mass matrix becomes

$$\left(\sqrt{M_3 M_3^T}\right)_{\text{normalized}} = \begin{pmatrix} 0.000163 & 0 & 0 \\ 0 & 0.964 & 0 \\ 0 & 0 & 173 \end{pmatrix}. \tag{26}$$

Here we used a top quark mass of 173 GeV as the input mass. Then the charm quark mass is obtained as 0.964 GeV, which is a little smaller than the actual mass 1.27 GeV (by 24.1%). The up quark mass turns out to be 0.163 MeV, which is too small compared with the QCD calculations. We have one complex parameter a_4 in this case and we must work out its effect: The detailed fit to the quark masses and also the CKM matrix will be given in Appendix A. This discussion justifies that the modular form of $k = 2$ (level 4) gives the charge $+2/3$ quark masses in its Fourier coefficients.

2.3. *Case of* $k = 3$

In this case, we have

$$J(q, r) = G_6 + a_6 G_2^3 + b_6 G_2 G_4. \tag{27}$$

Suppose that for the sake of argument we take

$$J(q, r) = H_6 = \frac{361 G_2^3 - G_6 - 50\,976 G_2 H_4}{224\,640}, \tag{28}$$

then we find

$$H_6 \to M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 12 & -16 & -2 \end{pmatrix}. \tag{29}$$

We regard the modular form of $k = 3$ as an expression of the charge $-1/3$ quark masses. With the QCD calculated bottom quark mass of 4.18 GeV as an input mass, we obtain

$$\left(\sqrt{M_3 M_3^T}\right)_{\text{normalized}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.167 & 0 \\ 0 & 0 & 4.18 \end{pmatrix}. \tag{30}$$

The down quark mass is zero and the strange/bottom mass ratio is off by a factor of 1.6. Of course we have two complex parameters a_6, b_6 to be fixed in this case, and we must adjust these parameters to get a more precise fit to the experimental data.

As shown above, in the case of $k = 1$ where there is no adjustable parameter the fit is almost perfect, and the other two cases require refinement but it is amazing that the values obtained in these cases are also not that distant from the experimental data. Now we need to choose appropriate values for a_4, a_6, and b_6. In fact, these complex parameters are needed to fit the CKM matrix that contains some phase factor to explain the CP violation. See Appendix A for the concrete calculation.

2.4. *Case of* $k = 4$

In this case, we assume that the modular form describes the neutrino masses. The neutrino has two possibilities: 1. Dirac neutrino and 2. Majorana neutrino.

In the case of the pure Dirac neutrino, the mass matrix becomes

$$M_D = a_8 G_2^4 + b_8 G_4^2 + c_8 G_4 G_2^2 + G_6 G_2 \tag{31}$$

where

$$G_2^4 = \begin{pmatrix} 1 & 0 & 0 \\ 576 & 192 & 0 \\ 154\,944 & 84\,480 & 15\,168 \end{pmatrix}, \quad G_4^2 = \begin{pmatrix} 121 & 0 & 0 \\ 95\,040 & 10\,560 & 0 \\ 25\,300\,800 & 7779\,840 & 1001\,280 \end{pmatrix}$$
$$G_4 G_2^2 = \begin{pmatrix} 11 & 0 & 0 \\ 7488 & 1536 & 0 \\ 1911\,744 & 878\,592 & 113\,856 \end{pmatrix}, \quad G_6 G_2 = \begin{pmatrix} 361 & 0 & 0 \\ 85\,248 & 18\,336 & 0 \\ 39\,242\,304 & 18\,822\,912 & 1235\,136 \end{pmatrix}. \tag{32}$$

In the case of the Majorana neutrino with the seesaw approximation, the mass matrix is given as

$$M_M = M_D M_R^{-1} M_D^T. \tag{33}$$

Although the right-handed Majorana mass M_R has the same form as in Eq. (31), it turns out that it has the following unique form since it must be a symmetric matrix:

$$M_R = d_8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -720 & 0 \\ 0 & 0 & -82\,080 \end{pmatrix}. \tag{34}$$

In this work we discuss these two limiting cases: one is the pure Dirac case corresponding to Majorana mass $= 0$ and the other is the seesaw case where the Majorana mass is much larger than the Dirac mass. The actual data fitting is done in Appendix A.

2.5. *Case of $k \geq 5$*

For $k = 5$, for instance, we have the modular forms $G_2^5, G_2^3 G_4, G_2 G_4^2, G_2^2 G_6, G_4 G_6$. This sort of new flavor particle presumably has neither charges nor color charges, but may have some weak interactions in addition to gravitational interactions. Therefore, it may be a good candidate for dark matter.

3. Some additional considerations

3.1. *Lagrangian*

We may write down the kinetic energy part of the Lagrangian using the fields defined in Eq. (16). We have

$$K_R(z) = \sum_{i,j=-\infty}^{\infty} \overline{\psi}_{Rj}^{a,\alpha} \left(D_{a,\alpha}^{b,\beta}\right)_\mu \gamma^\mu \psi_{b,\beta,Ri} e^{-2\pi i z} e^{2\pi j z} \tag{35}$$

$$K_L(z') = \sum_{i,j=-\infty}^{\infty} \overline{\psi}_{Lj}^{a,\alpha} \left(D_{a,\alpha}^{b,\beta}\right)_\mu \gamma^\mu \psi_{b,\beta,Li} e^{-2\pi i z'} e^{2\pi j z'} \tag{36}$$

where the indices a, b indicate flavor type and α, β are indices for the gauge group representation. The right and left modes can belong to different representations. The covariant derivative includes the gauge field A_μ:

$$\left(D_{a,\alpha}^{b,\beta}\right)_\mu = i\delta_a^b \delta_\alpha^\beta \partial_\mu + \left(A_{a,\alpha}^{b,\beta}\right)_\mu. \tag{37}$$

Then the kinetic part of the Lagrangian density is given by

$$\int K_R(z)dz + \int K_L(z')dz'. \tag{38}$$

To maintain the modular invariance we must impose the modular transformation:

$$K_R(z) \ \rightarrow \ (\gamma z+\delta)^2 K_R(z), \quad K_L(z') \ \rightarrow \ (\gamma' z'+\delta')^2 K_L(z'), \tag{39}$$

which means that the kinetic term is a single-variable modular form of level 2 in contrast to the Yukawa coupling.

3.2. *Supersymmetrization*

We may trivially write the Lagrangian in a supersymmetric form. Corresponding to Eq. (10), we define

$$F(q^{-1}, r^{-1}) = gH \sum_{i,j=0}^{G-1} \Phi_{Rj}\Phi_{Li} q^{-i} r^{-j}. \tag{40}$$

Corresponding to Eq. (13), we obtain

$$Y = \sum_{i,j=0}^{G-1} g_{ij}\Phi_{Rj}\Phi_{Li}H\big|_{\theta\theta} = g\sum_{i,j=0}^{G-1} U^{\dagger}_{Lik}\lambda_k U_{Rkj}\Phi_{Rj}\Phi_{Li}H\big|_{\theta\theta} \tag{41}$$

where Φ_{Rj} and Φ_{Li} are the chiral fields corresponding to a certain flavor. Then we have

$$g_{ij} = gU^{\dagger}_{Lik}\lambda_k U_{Rkj} \tag{42}$$

for $i,j = 0, 1, \ldots, G-1$. Using a standard form for the chiral field $\Phi = A + \sqrt{2}\theta\psi + \theta\theta F$ [17], we get

$$\Phi_{Rj}\Phi_{Li}H\big|_{\theta\theta} = \left(F_{Rj}A_{Li} + A_{Rj}F_{Li}\right)H + A_{Rj}A_{Li}F_H - \left(A_{Rj}\psi_{Li} - \psi_{Rj}A_{Li}\right)\psi_H - \psi_{Rj}\psi_{Li}H. \tag{43}$$

Then the Yukawa coupling (41) can be written as

$$Y = g\sum_{k=0}^{G-1}\lambda_k\left[\chi_{Lk}\chi_{Rk}H + (B_{Lk}\chi_{Rk} - \chi_{Lk}B_{Rk})\psi_H + (G_{Lk}B_{Rk} + B_{Lk}G_{Rk})H + B_{Lk}B_{Rk}F_H\right] \tag{44}$$

where

$$\begin{aligned} \chi_{Lk} &= U^{\dagger}_{Lik}\psi_{Li}, \quad & \chi_{Rk} &= \psi_{Rj}U_{Rjk}, \\ B_{Lk} &= U^{\dagger}_{Lik}A_{Li}, \quad & B_{Rk} &= A_{Rj}U_{Rjk}, \\ G_{Lk} &= U^{\dagger}_{Lik}F_{Li}, \quad & G_{Rk} &= F_{Rj}U_{Rjk}. \end{aligned} \tag{45}$$

The kinetic energy part is given by

$$K_R = \sum_{i=0}^{\infty}\Phi^{\dagger}_{Ri}\Phi_{Ri}\big|_{\theta\theta\overline{\theta\theta}},$$

$$K_L = \sum_{i=0}^{\infty} \Phi_{Li}^{\dagger}\Phi_{Li}\big|_{\theta\theta\overline{\theta\theta}},$$

$$K_H = \sum_{i=0}^{\infty} \Phi_{H}^{\dagger}\Phi_{H}\big|_{\theta\theta\overline{\theta\theta}}, \tag{46}$$

with

$$\Phi_{Ri}^{\dagger}\Phi_{Ri}\big|_{\theta\theta\overline{\theta\theta}} = G_{Ri}^{\dagger}G_{Ri} + B_{Ri}^{\dagger}\Box B_{Ri} + i\partial_m \overline{\chi}_{Ri}\overline{\sigma}^m \chi_{Ri} \tag{47}$$

and similar forms for Φ_{Li}, Φ_H.

4. Calculation of the geometry of the moduli space of the Calabi–Yau manifold

In superstring theory, the generation number is customarily explained as the number of zero modes determined by a topological quantity. Our approach is different from this interpretation as explained in the introduction. We integrate over the modular variables when we define the low-energy Lagrangian that includes Yukawa couplings among Higgs and fermions. Identifying the modular variables with the Calabi–Yau moduli, this means that we superpose vacuum states defined by each modulus. Yukawa couplings resolve the degeneracy of the vacua and each vacuum is defined by the number of generations G. At low energy we know that $G = 3$, but there may be phase transitions when we go to high energy. At the highest energy we may even reach $G \to \infty$.

Another observation if we want to interpret our result in the context of string theory is that our case may not be consistent with grand unification. In fact, each flavor corresponds to a modular form of a different level: level 2 for charged leptons, level 4 for $+2/3$ quarks, level 6 for $-1/3$ quarks, and level 8 for neutrinos. It is not entirely excluded that it is consistent with grand unification, because we may have a finite number of generations G even at the grand unified scale, and we may not worry about maintaining the modular invariance anyway.

With these conceptual modifications, our Yukawa coupling before the modular variable integration may be interpreted as coming from the compactification of the superstring theory.

First, we assume that the following formula first derived by Strominger and Witten [10] is correct in spite of the above conceptual modifications:

$$J(q,r,w) = J(q,r)J_H(w) = \frac{1}{g}\sum_{i,j,k} g_{ij}h_k q^i r^j w^k = \int_K a^{\mu} \wedge b^{\nu} \wedge c^{\rho} \wedge \Omega_{\mu\nu\rho} \tag{48}$$

where K is a certain Calabi–Yau manifold and Ω is a holomorphic 3-form. The a, b, c originate from gauge fields (principal or vector bundle) in the compactified Calabi–Yau space and are interpreted as harmonic (massless) (0, 1)-forms. If we restrict ourselves to the case of moduli corresponding to the complex structure deformation, rather than the Kähler structure deformation, the (0, 1)-form a, b, c must originate in the (2, 1)-form. The gauge group A is the maximal subgroup such that, e.g.,

$$E_8 \otimes E_8 \supset A \otimes SU(3) \otimes SU(2) \otimes U(1). \tag{49}$$

We restrict ourselves to this case, and then it is shown by Candelas and de la Ossa [18] that the rightmost side of Eq. (48) can be written as

$$\int_K a^{\alpha\mu} \wedge b^{\beta\nu} \wedge c^{\gamma\rho} \wedge \Omega_{\mu\nu\rho} = \frac{\partial^3 \mathcal{G}}{\partial z^{\alpha}\partial z^{\beta}\partial z^{\gamma}}. \tag{50}$$

Here the moduli variables z^α ($\alpha = 1, 2, \ldots,$ Betti number $b_{2,1}$) are chosen to be the periods themselves:

$$z^\alpha = \int_{A^\alpha} \Omega \tag{51}$$

where A^α is an appropriate homology basis.

By identifying our modular variables with the complex structure variables z^α [18], we can explicitly calculate $\mathcal{G}$ and, therefore, the Kähler potential K is

$$e^K = -i\left(z^\alpha \frac{\partial}{\partial \bar{z}^\alpha}\overline{\mathcal{G}} - \bar{z}^\alpha \frac{\partial}{\partial z^\alpha}\mathcal{G}\right) \tag{52}$$

and the Kähler metric of the moduli space of the Calabi–Yau manifold is

$$G_{\alpha\beta} = \frac{\partial^2}{\partial z^\alpha \partial z^\beta}\mathcal{G}. \tag{53}$$

The precise relation between our modular variables q, r, and w and the period z^α must respect the scaling behavior under $z \to \lambda z$:

$$\mathcal{G}(\lambda z) = \lambda^2 \mathcal{G}(z), \quad \frac{\partial^3 \mathcal{G}}{\partial z^\alpha \partial z^\beta \partial z^\gamma} \to \lambda^{-1} \frac{\partial^3 \mathcal{G}}{\partial z^\alpha \partial z^\beta \partial z^\gamma}, \tag{54}$$

whereas the scaling behavior of a modular form depends on its level. Here we consider the $\mathrm{SL}(2, \mathbb{Z}(\sqrt{2}))$ transformation (5) with $\beta, \gamma, \beta', \gamma' = 0$:

$$z \to \frac{\alpha}{\delta} z = \alpha^2 z, \quad z' \to \frac{\alpha'}{\delta'} z' = \alpha'^2 z'. \tag{55}$$

With $q = e^{2\pi i \tau}$, $r = e^{2\pi i \sigma}$, and $w = e^{2\pi i \rho}$, we have

$$\tau \to \alpha^2 \tau, \quad \sigma \to \alpha^2 \sigma, \tag{56}$$

since α must be equal to α' so that τ and σ have the same scaling factor. This means that the $\sqrt{2}$ term in $\alpha = a + b\sqrt{2}$ must be zero and the scaling is guaranteed only for integers. If one allows this, then we obtain

$$J(q, r) \to \left(\delta\delta'\right)^{2k} J(q, r) = \left(\alpha\alpha'\right)^{-2k} J(q, r) = \alpha^{-4k} J(q, r) \tag{57}$$

and

$$\rho \to \alpha^2 \rho, \quad J(w) \to \alpha^{-h} J(w) \tag{58}$$

where h is the level of the Higgs modular form. Therefore,

$$J(q, r) J(w) \to \alpha^{-h-4k} J(q, r) J(w), \tag{59}$$

and we can put

$$\alpha^{-h-4k} =: \lambda^{-1} \tag{60}$$

and

$$\tau \to \lambda^{\frac{2}{h+4k}} \tau, \quad \sigma \to \lambda^{\frac{2}{h+4k}} \sigma, \quad \rho \to \lambda^{\frac{2}{h+4k}} \rho. \tag{61}$$

This shows that the period variables z^α are given by our modular variables:

$$z^\alpha = \left(\tau^{\frac{h+4k}{2}}, \sigma^{\frac{h+4k}{2}}, \rho^{\frac{h+4k}{2}}\right). \tag{62}$$

There are four of these combinations corresponding to charged leptons ($k = 1$), charge $+2/3$ quarks ($k = 2$), charge $-1/3$ quarks ($k = 3$), and neutrinos ($k = 4$):

$$\mathcal{G} = \sum_{f=1}^{4} \mathcal{G}_f. \tag{63}$$

Although ρ corresponds to the Higgs field, each combination has a different relation between ρ and z^α as in Eq. (62) because each combination has its own value of k. This means that there are multiple modular variables corresponding to the Higgs particle, which is acceptable because these variables turn out just to be integration variables. We obtain

$$\begin{aligned} J(q,r,w) = J(q,r)J_H(w) &= \frac{1}{g}\sum_{i,j,k} g_{ij} h_k q^i r^j w^k \\ &= \frac{\partial^3 \mathcal{G}_f}{\partial z^\alpha \partial z^\beta \partial z^\gamma} = \frac{\left(\frac{2}{h+4k}\right)^3}{\sqrt{(\tau\sigma\rho)^{h+4k-2}}} \frac{\partial^3 \mathcal{G}_f}{\partial\tau\partial\sigma\partial\rho}. \end{aligned} \tag{64}$$

Therefore,

$$\mathcal{G}_f = \left(\frac{h+4k}{2}\right)^3 \int^{\tau_f}\int^{\sigma_f}\int^{\rho_f} \sqrt{(\tau\sigma\rho)^{h+4k-2}} J(q,r)J_H(w) d\tau d\sigma d\rho. \tag{65}$$

Then the metric of the moduli space of the Calabi–Yau manifold is given by

$$G_{\alpha\beta} = \sum_{f=1}^{4} G_{\alpha\beta,f} = \sum_{f=1}^{4} \frac{\partial^2}{\partial z^\alpha \partial z^\beta} \mathcal{G}_f. \tag{66}$$

For example,

$$G_{\tau\sigma,f} = \frac{\left(\frac{2}{h+4k}\right)^2}{\sqrt{(\tau_f\sigma_f)^{h+4k-2}}} \frac{\partial^2 \mathcal{G}_f}{\partial\tau_f \partial\sigma_f} = \frac{h+4k}{2} J(q_f, r_f) \int^{\rho_f} \sqrt{\rho_f^{h+4k-2}} J_H(w) d\rho. \tag{67}$$

We remark that the other derivatives such as $\frac{\partial^3\mathcal{G}_f}{\partial\tau^3}$, $\frac{\partial^3\mathcal{G}_f}{\partial\tau^2\partial\sigma}$, etc. can correspond to some Yukawa couplings, but all these seem not to appear in physics because of the gauge symmetry of the theory. For example, $\frac{\partial^3\mathcal{G}_f}{\partial\tau^3}$ could potentially correspond to the triple Higgs coupling, but it is forbidden by the standard model symmetry.

The Kähler metric of the moduli space (66) is related to the Calabi–Yau metric through the equation

$$G_{\alpha\beta} = \frac{1}{2V}\int_M g^{\kappa\overline{\mu}} g^{\lambda\overline{\nu}} \left(\frac{\partial g_{\kappa\lambda}}{\partial z^\alpha}\frac{\partial g_{\overline{\mu\nu}}}{\partial z^\beta} + \frac{\partial B_{\kappa\lambda}}{\partial z^\alpha}\frac{\partial B_{\overline{\mu\nu}}}{\partial z^\beta}\right) d^6x \tag{68}$$

where $g_{\mu\nu}$ is the Calabi–Yau metric, $B_{\mu\nu}$ is a 2-form related to $g_{\mu\nu}$ by supersymmetry, and V is the volume of the Calabi–Yau manifold. For example, we obtain

$$G_{\tau\sigma,f} = \frac{1}{2V}\frac{\left(\frac{2}{h+4k}\right)^2}{\sqrt{(\tau_f\sigma_f)^{h+4k-2}}}\int_M g^{\kappa\overline{\mu}} g^{\lambda\overline{\nu}} \left(\frac{\partial g_{\kappa\lambda}}{\partial \tau_f}\frac{\partial g_{\overline{\mu\nu}}}{\partial \sigma_f} + \frac{\partial B_{\kappa\lambda}}{\partial \tau_f}\frac{\partial B_{\overline{\mu\nu}}}{\partial \sigma_f}\right) d^6x. \tag{69}$$

Equation (68) must be supplemented with the Calabi–Yau condition; the Ricci flatness and Kähler constraints. Only when there is a solution to this equation are we justified in our claim that there exists a Calabi–Yau manifold with $J(q, r, w)$ modularity. The existence of the solution is not a priori guaranteed because the moduli space metric (66) may not give us the genuine Calabi–Yau metric. Furthermore, if we restrict ourselves to the minimum Calabi–Yau manifold, meaning that all of its moduli are directly determined by the experiments as above, we may be able to determine its metric $g_{\mu\nu}$ by solving Eq. (68) together with the Ricci flat and Kähler constraints. We would like to come back to this issue in a future publication.

5. Concluding remarks

(1) As we have shown above, the hypothesis of flavor moonshine is at least correctly realized experimentally to some extent. We need to use multi-variable modular forms for this purpose. These forms are well studied in mathematics as a branch of number theory and they constitute part of more general forms called Hilbert modular forms [19].

(2) We use only the Fourier coefficients of these forms to define the Yukawa coupling and the modular invariance of the total Lagrangian is assumed.[2] As such, it corresponds to the procedure of integrating over the modular variables that are identified as Calabi–Yau moduli if we combine our model with string theory. We do not regard these moduli as scalar fields to be stabilized. Insofar as we can see, the moduli are not related to any physical quantities in the low-energy theory. Thus they seem to have no physical degrees of freedom as scalar fields, although we are fully aware that this interpretation is different from the conventional idea of regarding the Calabi–Yau moduli as some scalar fields. Therefore, our treatment of them as moduli to be integrated out when we define the low-energy action seems to be a natural process.

(3) Of course, there are many mysteries to be solved. Why does nature seem to choose a very specific form such as the one we used that is based on $\mathrm{SL}(2, \mathbb{Z}(\sqrt{2}))$? Why do we have $k = 1$ for charged leptons, $k = 2$ for charge $+2/3$ quarks, $k = 3$ for charge $-1/3$ quarks, and $k = 4$ for neutrinos? There remain a lot of work to be done: How good or bad are the other modular groups like $\mathrm{SL}(2, \mathbb{Z}(\sqrt{N}))$, $\mathrm{SL}(2, \mathbb{Z}(i))$ etc.? Can we extend the modular form to be more than two variables? What exactly is the mathematical moonshine for the modular form of two variables? If we understand the mathematical implication of the matrices that appear in the Fourier coefficients of two-variable modular forms, we will be able to prove flavor moonshine by understanding the physical principle that identifies mass matrices with these matrices.

(4) Probably more urgent work from the string theory standpoint is to find out the specific Calabi–Yau metric by solving Eq. (68) and to elucidate its other physical consequences. Further questions arise such as: Do we have a grand unified scale? Do we have a phase transition from $G = 3$ to $G \geq 4$ at some point at higher energy?

(5) Experimentally, we need to explore the properties of the Higgs particle in more detail, especially its coupling to low-mass particles such as u, d, e, μ, and even neutrinos. Construction of the International Linear Collider (ILC), therefore, is urgent. A good neutrino facility is also highly

[2] During the preparation of this paper, we encountered work by Criado and Feruglio [20] and Feruglio [21]. Their work has nothing to do with moonshine, but, since they assume the modular invariance of the low-energy action, although the way they impose it is very different from ours, their Yukawa coupling depends on modular forms (not their Fourier coefficients) and it may be possible to calculate the Calabi–Yau moduli space geometry in this case, too.

desirable. The Higgs particle is indeed the "God particle", the term coined by Leon Lederman [22], in the sense that its Yukawa couplings determine the highest-energy physics without the need to perform the highest-energy experiments.

(6) It is possible that the whole idea of flavor moonshine is just nonsense,[3] although the agreement with the experimental data seems to us too good to be just an accident.

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Appendix A. Numerical fitting for experimental data

We calculated numerically the CKM and PMNS matrices and fit the experimental data to them. In the former case we have three complex parameters a_4, a_6, and b_6 as shown in Eqs. (8) and (9). For the PMNS matrix we have two choices of pure Dirac neutrino or Majorana neutrino (with the seesaw approximation). Either way, we have again three complex parameters a_8, b_8, and c_8 shown in Eq. (31). Since the parameter d_8 in Eq. (34) is an overall factor, we need not consider it in our discussion.

Let us briefly explain how we get the CKM matrix, which is parallel to the PMNS matrix. Now we have the mass matrix M_3 for u, c, t quarks, as in Sect. 2.2, with the complex parameters.

First we calculate the squared mass matrix as in Eq. (21): $M_3M_3^\dagger$ or $M_3^\dagger M_3$. Here we have two choices that give us the same eigenvalues but different eigenvectors. To obtain its eigenvalues and eigenvectors, we compute

$$U^\dagger(M_3M_3^\dagger)U = D \quad \text{or} \quad U^\dagger(M_3^\dagger M_3)U = D \tag{A.1}$$

where U is a unitary matrix and D is a diagonal matrix. The masses of u, c, t quarks are given by the square root of the eigenvalues:

$$D = \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix}. \tag{A.2}$$

Here we have swapped the columns of U and D so that $m_u < m_c < m_t$. Then the eigenvectors are regarded as the quark mass states

$$\begin{pmatrix} u & c & t \end{pmatrix}_{\text{mass}} = U \begin{pmatrix} u & c & t \end{pmatrix}_{\text{current}} = U \tag{A.3}$$

[3] Reference [23] says, "The term 'monstrous moonshine' was coined by Conway, who, when told by John McKay in the late 1970s that the coefficient of q (namely 196 884) was precisely one more than the degree of the smallest faithful complex representation of the monster group (namely 196 883), replied that this was 'moonshine' (in the sense of being a crazy or foolish idea). Thus, the term not only refers to the monster group M; it also refers to the perceived craziness of the intricate relationship between M and the theory of modular functions." M in the present work refers to "mass" rather than "monster group".

where we set the quark current states as

$$u_{\text{current}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_{\text{current}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad t_{\text{current}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.4}$$

We repeat similar calculations for d, s, b quarks (see Sect. 2.3) and obtain

$$\begin{pmatrix} d & s & b \end{pmatrix}_{\text{mass}} = V \begin{pmatrix} d & s & b \end{pmatrix}_{\text{current}} = V \tag{A.5}$$

where V is a unitary matrix including the eigenvectors of the squared mass matrix for d, s, b quarks. Note that, by definition, the current quarks should satisfy

$$\begin{pmatrix} u^\dagger \\ c^\dagger \\ t^\dagger \end{pmatrix}_{\text{current}} \begin{pmatrix} d & s & b \end{pmatrix}_{\text{current}} = I. \tag{A.6}$$

Therefore, the CKM matrix can be calculated as

$$\text{CKM} = \begin{pmatrix} u^\dagger d & u^\dagger s & u^\dagger b \\ c^\dagger d & c^\dagger s & c^\dagger b \\ t^\dagger d & t^\dagger s & t^\dagger b \end{pmatrix}_{\text{mass}} = U^\dagger V. \tag{A.7}$$

For calculation of the PMNS matrix, we use the mass matrix M_3 for charged leptons in Sect. 2.1 and $M_3 = M_D$ or M_M for neutrinos in Sect. 2.4.

Appendix A.1. Methods

Our goal is to find a set of complex parameters that best fit the experimental results. The experimental results that we use here are

- the absolute values of the elements of the mixing (CKM or PMNS) matrix ζ_{ij}
- the ratios of masses ξ_k.

The mixing matrices in both cases have $3 \times 3 = 9$ elements. Note that the CP violation phases are not used for our fittings. For quark masses, we choose the parameters $\xi_k = (m_t/m_c, m_b/m_s)$. This means that we do not fit u and d quark masses: In all the results that we obtained they are much smaller than the experimental results, just as we saw in Sect. 2.2. For lepton masses, we choose $\xi_k = \Delta m_{21}^2/\Delta m_{32}^2$, i.e., a ratio of difference of squared neutrino masses. Since the masses of e, μ, and τ are already fixed, as in Sect. 2.1, we have no parameters to fit them.

Then we define the loss function to measure a "difference" between our results and the experimental results:

$$\text{Loss} = \sum_{i,j=1}^{3} \left| \log \frac{\zeta_{ij}^{\text{cal}}}{\zeta_{ij}^{\text{exp}}} \right| + 2 \sum_{k} \left| \log \frac{\xi_k^{\text{cal}}}{\xi_k^{\text{exp}}} \right| \tag{A.8}$$

where ζ_{ij}^{exp} and ξ_k^{exp} are the experimental results, while ζ_{ij}^{cal} and ξ_k^{cal} are the results of our numerical calculations (which depend on the three complex parameters). The factor of 2 existing in the second term ensures that the contribution from this term cannot be much smaller than that from the first term: the ratios of masses have only 2 (or 1) parameters in the quark (or lepton) case, while the mixing matrix has 9 parameters.

Now let us search the complex parameters at the minimum of the loss function (A.8). First we divide the 3 complex parameters into 6 real parameters x_i. Since the following discussion includes calculating eigenvectors of matrices, the iterative approximation with gradient descent is not suitable to be used. Instead, we choose 11 lattice points for each real parameter:

$$x_i = x_i^0 - 5\delta x,\ x_i^0 - 4\delta x,\ \ldots,\ x_i^0 + 5\delta x \tag{A.9}$$

where for simplicity the lattice spacing δx is the same for all i, and at first we set $x_i^0 = 0$ for all i. Then we have 11^6 lattice sites in total.

After calculating the loss function (A.8) at all the lattice sites, we find a set of parameters $x_i^{\min}$ with the minimum loss among them. Next we set $x_i^0 = x_i^{\min}$ and $\delta x \to \delta x/6$, and repeat this procedure six times. Finally the lattice spacing becomes $\delta x/6^6$.

We tried several cases satisfying $10^{-3} \leq \delta x/6^6 \leq 10^{-2}$, and calculated both cases of the squared mass matrix (A.1). Then we obtain a certain set of parameters with the minimum loss among all the results that we obtained. In our discussion we regard it as the best fit for the experimental results.

Appendix A.2. CKM matrix

The best fit that we obtained for the CKM matrix is

$$\mathrm{CKM} = \begin{pmatrix} 0.974 & 0.226 & 0.004e^{-1.17i} \\ -0.226 & 0.973 & 0.043 \\ 0.009e^{-0.435i} & -0.042 & 0.999 \end{pmatrix} \tag{A.10}$$

with quark masses

$$\begin{aligned} (m_u, m_c, m_t) &= (5.30 \times 10^{-5}, 1.30, 173)\ \mathrm{GeV} \\ (m_d, m_s, m_b) &= (1.18 \times 10^{-6}, 0.013, 4.18)\ \mathrm{GeV}. \end{aligned} \tag{A.11}$$

Here we input m_t and m_b for normalization. The CKM can be expressed in terms of Wolfenstein parameters:

$$\begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{A.12}$$

and we obtain

$$\lambda = 0.226,\ A = 0.839,\ \rho = 0.161,\ \eta = 0.382. \tag{A.13}$$

The experimental values for these are [24]

$$\lambda = 0.226,\ A = 0.836,\ \rho = 0.125,\ \eta = 0.364 \tag{A.14}$$

with quark masses

$$\begin{aligned} (m_u, m_c, m_t) &= (2.2 \times 10^{-3}, 1.27, 173)\ \mathrm{GeV} \\ (m_d, m_s, m_b) &= (4.7 \times 10^{-3}, 0.093, 4.18)\ \mathrm{GeV}. \end{aligned} \tag{A.15}$$

Note that, again, we look at only the central values of the experimental data.

Some comments are in order for these results:

(1) The agreement is generally excellent.
(2) The masses of u, d, s quarks come out rather small. This is due to the large hierarchical property of the mass matrices. The lattice QCD mass is somewhat different from the Higgs coupling, especially its renormalization corrections, but it is not clear at this time whether this fact can account for the difference.
(3) The CKM matrix also has renormalization corrections [25]. The fact that our result is not far from the experimental value may indicate that our theory is indeed a low-energy theory rather than a very short distance theory.

Appendix A.3. PMNS matrix

Our best fit for the PMNS matrix is obtained as follows. We discuss the two cases of pure Dirac neutrino and Majorana neutrino with the seesaw approximation. In each case, neutrino masses can be in the normal order ($m_1 < m_2 < m_3$) or the inverted order ($m_3 < m_1 < m_2$).

Appendix A.3.1. Case of the pure Dirac neutrino

When neutrino masses are in the normal order, the best fit is

$$\mathrm{PMNS} = \begin{pmatrix} 0.919 & 0.183 & 0.349e^{-1.49i} \\ 0.304e^{2.05i} & 0.598e^{0.09i} & 0.742 \\ 0.250e^{0.98i} & 0.780e^{3.09i} & 0.573 \end{pmatrix} \tag{A.16}$$

with neutrino mass differences

$$\left(\Delta m_{21}^2, \Delta m_{32}^2\right) = \left(m_2^2 - m_1^2, m_3^2 - m_2^2\right) = (7.53 \times 10^{-5}, 3.32 \times 10^{-1})\ \mathrm{eV}^2. \tag{A.17}$$

Here Δm_{21}^2 is our input for normalization, which is the same for all the fittings below.

If neutrino masses are in the inverted order, the best fit becomes

$$\mathrm{PMNS} = \begin{pmatrix} 0.586 & 0.483 & 0.651e^{-2.79i} \\ 0.150e^{2.12i} & 0.814e^{0.13i} & 0.561 \\ 0.796e^{0.15i} & 0.323e^{2.84i} & 0.511 \end{pmatrix} \tag{A.18}$$

with neutrino mass differences

$$\left(\Delta m_{21}^2, \Delta m_{32}^2\right) = (7.53 \times 10^{-5}, -7.53 \times 10^{-5})\ \mathrm{eV}^2. \tag{A.19}$$

The PMNS matrix is can be written as

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{A.20}$$

where $c_{12} = \cos\theta_{12}, s_{12} = \sin\theta_{12}, \ldots$. Then we get

$$\begin{aligned} &s_{12}^2 = 0.0381,\ s_{13}^2 = 0.122,\ s_{23}^2 = 0.626,\ \delta = 1.49 \quad \text{(normal order)} \\ &s_{12}^2 = 0.404,\ s_{13}^2 = 0.424,\ s_{23}^2 = 0.547,\ \delta = 2.79 \quad \text{(inverted order)}. \end{aligned} \tag{A.21}$$

Lepton masses in both cases of normal and inverted orders are the same as in Sect. 2.1:

$$\left(m_e, m_\mu, m_\tau\right) = (0.5110, 107.5, 1558)\ \text{MeV}. \tag{A.22}$$

Appendix A.3.2. Case of the Majorana neutrino with the seesaw approximation

The best fit in the normal order of neutrino masses is

$$\text{PMNS} = \begin{pmatrix} 0.291 & 0.753^{1.96i} & 0.590e^{1.12i} \\ 0.489e^{-2.96i} & 0.527e^{-2.66i} & 0.695 \\ 0.822e^{0.70i} & 0.394e^{-1.40i} & 0.411 \end{pmatrix} \tag{A.23}$$

with neutrino mass differences

$$\left(\Delta m^2_{21}, \Delta m^2_{32}\right) = (7.53 \times 10^{-5}, 2.44 \times 10^{-3})\ \text{eV}^2. \tag{A.24}$$

In the inverted order of neutrino mass, the best fit is

$$\text{PMNS} = \begin{pmatrix} 0.294 & 0.880^{3.14i} & 0.373e^{3.14i} \\ 0.490e^{0.00i} & 0.197e^{3.14i} & 0.849 \\ 0.821e^{3.14i} & 0.433e^{3.14i} & 0.373 \end{pmatrix} \tag{A.25}$$

with neutrino mass differences

$$\left(\Delta m^2_{21}, \Delta m^2_{32}\right) = (7.53 \times 10^{-5}, -7.53 \times 10^{-5})\ \text{eV}^2. \tag{A.26}$$

The PMNS matrix in this case can be written as

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}; \tag{A.27}$$

then we get

$$\begin{aligned} s^2_{12} = 0.870,\ s^2_{13} = 0.348,\ s^2_{23} = 0.741 \quad &(\text{normal order}) \\ s^2_{12} = 0.900,\ s^2_{13} = 0.139,\ s^2_{23} = 0.838 \quad &(\text{inverted order}) \end{aligned} \tag{A.28}$$

and the CP violation phases are

$$\begin{aligned} \delta = -1.68,\ \alpha_{21} = 3.92,\ \alpha_{31} = -1.13 \quad &(\text{normal order}) \\ \delta = 0.00,\ \alpha_{21} = 0.01,\ \alpha_{31} = 0.00 \quad &(\text{inverted order}) \end{aligned} \tag{A.29}$$

modulo 2π. Lepton masses are the same as in the case of the pure Dirac neutrino.

Appendix A.3.3. Experimental data

The current experimental values (its central values) of the PMNS matrix are [24,26]

$$|\text{PMNS}| = \begin{pmatrix} 0.821 & 0.550 & 0.150 \\ 0.304 & 0.598 & 0.742 \\ 0.483 & 0.583 & 0.654 \end{pmatrix}, \tag{A.30}$$

the angles in the expression (A.20) are

$$s_{12}^2 = 0.307,\ s_{13}^2 = 0.0218,\ s_{23}^2 = \begin{cases} 0.512 & \text{(normal order)} \\ 0.536 & \text{(inverted order)} \end{cases}, \tag{A.31}$$

and the CP violation phase is

$$\delta = 1.37\pi = -1.98 \quad \text{(modulo } 2\pi). \tag{A.32}$$

The lepton masses are

$$(m_e, m_\mu, m_\tau) = (0.5110, 105.6, 1777)\ \text{MeV} \tag{A.33}$$

and the neutrino mass differences are

$$(\Delta m_{21}^2, \Delta m_{32}^2) = \begin{cases} (7.53 \times 10^{-5}, 2.44 \times 10^{-3})\ \text{eV}^2 & \text{(normal order)} \\ (7.53 \times 10^{-5}, -2.55 \times 10^{-3})\ \text{eV}^2 & \text{(inverted order)} \end{cases}. \tag{A.34}$$

We see that

(1) The agreement seems the best for the Majorana neutrino case in the normal order, especially at the CP violation phase and the neutrino mass difference.
(2) In that case, $\sin\theta_{23}$ matches well and $\sin\theta_{12}$ agrees within a factor of 3. However, we obtain too large a value for $\sin\theta_{13}$. The discrepancy with the experimental data could be attributed to the renormalization effect or inadequacy of our assignment. Further study is required.
(3) In the inverted order, we obtain no good agreements and the neutrino mass differences in particular completely fail to agree. Since the masses have a large hierarchical property in our calculations, as a consequence $|\Delta m_{32}^2|$ never exceeds $|\Delta m_{21}^2|$.

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