

Evolution of f_{NL} to the adiabatic limit

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Abstract. We study inflationary perturbations in multiple-field models, for which ζ typically evolves until all isocurvature modes decay—the “adiabatic limit”. We use numerical methods to explore the sensitivity of the local-shape bispectrum to the process by which this limit is achieved, finding an appreciable dependence on model-specific data such as the time at which slow-roll breaks down or the timescale of reheating. In models with a sum-separable potential where the isocurvature modes decay before the end of the slow-roll phase we give an analytic criterion for the asymptotic value of f_{NL} to be large. Other examples can be constructed using a waterfall field to terminate inflation while f_{NL} is transiently large, caused by descent from a ridge or convergence into a valley. We show that these two types of evolution are distinguished by the sign of the bispectrum, and give approximate expressions for the peak f_{NL} .

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1 Introduction

In single-field inflation with canonical kinetic terms, the curvature perturbation produced at horizon crossing is conserved with nearly Gaussian statistics [1, 2]. Multiple-field models support a richer phenomenology, driven by a flow of power from isocurvature modes into the curvature perturbation. This flow is sourced dynamically, and where only canonical kinetic terms are present the dynamics are determined by the potential. Therefore, there is some hope that we may use one to learn about the other.

How much information could be extracted? A general potential is a complicated landscape, and it is unlikely that observations will be sufficient to single out a specific shape. But by piecing together clues from dynamical evolution it may be possible to obtain information about the topography of the landscape in our local neighbourhood. This is a form of potential reconstruction [3–5].

Sensitivity to dynamical effects is helpful when distinguishing observational outcomes. Unfortunately, it complicates the task of extracting predictions. In principle, the statistics of the curvature perturbation should be tracked until the time of last scattering—where the microwave background anisotropy was imprinted—and in our present state of ignorance this is an impossible undertaking. Therefore, to connect the physics of inflation with observation, we must rely on *conservation*: if the isocurvature modes are exhausted, quenching the flow of power into the curvature perturbation, it will cease to evolve. It is the statistics which apply at the onset of conservation which will be inherited by observable quantities.

This point of view was developed soon after multiple-field models entered the literature [6]. For practical purposes we require a characterization of the conditions under which ζ becomes constant. In the absence of isocurvature modes, conservation of ζ was demonstrated by Rigopoulos & Shellard [7], Lyth, Malik & Sasaki [1] and later Langlois & Vernizzi [8–12] using a gradient expansion. Christopherson & Malik extended these results to models in which the Lagrangian can be an arbitrary Lorentz-invariant function of the scalar field and its first derivatives [13]. More recently, Naruko & Sasaki and Gao [14, 15] applied similar arguments to higher-derivative models which preserve second-derivative field equations, where conservation can be subtle [16, 17]. Weinberg developed a different approach [18–21], adapting the techniques of Goldstone’s theorem to show that ζ would become massless on superhorizon scales, admitting a time-independent solution. Whether this solution is selected is a model-dependent question.

The statistics of ζ are fossilized in the radiation fluid at last scattering, and its two-point correlations have been studied since their presence was confirmed by COBE [22, 23]. More recently, sophisticated Cosmic Microwave Background (CMB) experiments have raised the possibility that three- and higher n -point correlations may be detectable [24, 25]. These correlations are interesting because, in principle, they are sensitive probes of physical processes and dynamical conditions in the early universe [2, 26–29]. But precisely because of these desirable properties, such observables carry an unavoidable risk: they may be equally sensitive to *subsequent* dynamics, including the process by which ζ becomes conserved. We may learn important physics from studying the details of this process—but it need have little to do with our theories of the very early universe, and if we wish to use n -point correlations to

study these theories then we should proceed with caution. For this reason it is important to understand which predictions of early-universe models can be connected reliably to late-time observations.

Our imprecise knowledge of physics above the TeV frontier means it is not possible to give a complete answer. In this paper we pursue a more modest objective. Focusing on three-point correlations—with amplitude measured by f_{NL} [24]—and considering models where the flow of power from isocurvature modes is quenched at or near the end of inflation, we study how the asymptotic value of f_{NL} depends on the process by which the isocurvature modes become exhausted. We refer to this exhausted state as the *adiabatic limit*. Our arguments are phrased in terms of three-point correlations, but many of our conclusions are general and apply to arbitrary n -point functions including the two-point function.

Classification of models. We restrict attention to models where a significant f_{NL} is generated dynamically by inflation. This excludes examples such as the curvaton [30, 31] or modulated reheating [32, 33] which rely on an inflationary seed perturbation but amplify it by a noninflationary mechanism. If we disallow noncanonical kinetic terms, Maldacena’s result guarantees that the models of interest must include two or more dynamically relevant fields [2].

Under these conditions, field fluctuations are generated at horizon crossing with almost Gaussian statistics [34–36]. However, unlike single-field inflation, these perturbations may cause spatially separated regions of the universe to experience different expansion histories. The set of phase space trajectories associated with an ensemble of such regions is initially a narrowly collimated bundle whose ‘width’ is set by quantum scatter. (We give details in §2.) The curvature perturbation is a precise measure of the relative expansion between spatial patches. Therefore its evolution is a consequence of focusing or defocusing of the bundle: as nearby patches of the universe evolve towards or away from each other in phase space, they experience varying expansion rates. In the adiabatic limit, the bundle degenerates to a caustic. Its width shrinks to zero, and all trajectories converge to a single line.

When does convergence occur? The answer is model-dependent, but we can recognize broad classes of behaviour.

- The potential may contain a focusing region, which enables trajectories to converge ‘naturally.’ If inflation ends in the vicinity of this region, an adiabatic limit is automatic. Examples include Nflation and related models [37–43]. However, one is always free to build models in which the adiabatic limit is achieved differently, perhaps by a waterfall transition. If the ‘natural’ limit is employed, there are two relevant questions.

First, is the limit achieved before the end of inflation? If so, it is not necessary to specify details of the subsequent phases. Otherwise, we must choose among the various scenarios for reheating and later dynamics, and the predictions of the model may depend on our choice.

Second, in the case where an adiabatic limit is reached during inflation, does this occur before the slow-roll approximation fails? This raises no issues of principle, but may

influence how one chooses to study the model; for example, numerical methods may be required. We will return to this question in §5.

- Alternatively, there may be no natural means by which trajectories converge. In such models there is no alternative: the inflationary model does not make unambiguous predictions by itself, but only when embedded in a larger scenario which determines at least the mechanism by which inflation ends and the universe reheats.

None of these observations are new, but their application to non-Gaussian statistics has yet to be studied in detail.

Accepting the separate universe principle, to be discussed in §2 below, one obtains explicit expressions for the n -point functions of the curvature perturbation [28] which automatically respect these conclusions. Therefore, concrete predictions can be obtained whenever it is possible to calculate these expressions accurately until the onset of conservation. However, in many cases this ideal procedure is impossible or impractical. Working in a special class of models where the potential is separable, Meyers & Sivanandam [44, 45] argued that the connected n -point correlation functions would be damped towards slow-roll suppressed values. Our analysis is closely related, but we argue that the value achieved in the adiabatic limit need not be especially small [43]. Indeed, in some cases, the adiabatic limit is associated with growth towards the asymptotic value, rather than decay.

Objectives. In this paper we apply these ideas to the primordial bispectrum, and its amplitude f_{NL} in the squeezed limit. There are three principal objectives. First, we illustrate that local f_{NL} can be sensitive to details of when and how the adiabatic limit is reached. Using examples, we demonstrate that—even in models where convergence occurs naturally— f_{NL} may depend on the details of this process. Second, the calculations necessary to obtain a precise estimate of f_{NL} can be technical, perhaps requiring recourse to numerical methods. For some models, simple techniques exist which allow a qualitative estimate of the evolution and asymptotic value of f_{NL} . We outline these methods and apply them to simple examples. Third, we use numerical methods to perform a detailed study of the evolution of f_{NL} in a selection of models. By themselves these calculations already reveal interesting patterns of behaviour, but also provide guidance regarding the asymptotics of models where an adiabatic limit is reached only through the intervention of post-inflationary dynamics.

Outline. The plan of the paper is as follows. In §2 we discuss the separate universe approach to perturbation theory in phase space, and the δN formalism. In §3 we discuss mechanisms for generating large evolving local non-Gaussianity, and estimates for the maximum value and its dependence on initial conditions. §4 includes a brief account of analytic expressions for estimating f_{NL} , and discusses conditions necessary for this value to be large. §5 reports a detailed numerical study of f_{NL} in a selection of models with two or more fields. We conclude in §6. An Appendix contains details of some analytic calculations.

2 Phase space description of slow-roll inflation

Consider inflation driven by multiple canonical scalar fields ϕ_i (where $i = 1, 2, \dots, N_f$), self-interacting through a potential $W(\phi_i)$. Defining $W_{,i} = \partial W / \partial \phi_i$, the scalar equations of motion are

$$\ddot{\phi}_i + 3H\dot{\phi}_i + W_{,i} = 0. \quad (2.1)$$

Inflation occurs when $\epsilon \equiv -\dot{H}/H^2 < 1$. Eq. (2.1) generates a $2N_f$ -dimensional phase space Π . In the “slow-roll” limit where $\epsilon \ll 1$ there is a dynamical attractor, allowing the decaying mode to be discarded and restricting evolution to an N_f -dimensional submanifold Π' on which (for example) the $\dot{\phi}_i$ are determined in terms of the ϕ_i . The growing mode on Π' satisfies $3H\dot{\phi}_i + W_{,i} = 0$. In the slow-roll limit it is possible to write ϵ as a sum of independent contributions from each field, yielding $\epsilon = \sum_i \epsilon_i + O(\epsilon_i^2)$, where the ϵ_i satisfy

$$\epsilon_i \equiv \frac{M_{\text{P}}^2}{2} \left(\frac{W_{,i}}{W} \right)^2. \quad (2.2)$$

In simple models it may happen that the matrix $\eta_{ij} \equiv M_{\text{P}}^2 W_{,ij} / W$ also has small components $|\eta_{ij}| \ll 1$.

Density fluctuations are generated by the inflationary background and can be measured by the curvature perturbation on uniform density spatial hypersurfaces, denoted ζ . For cosmological purposes its statistical properties are characterized by low-order correlation functions, of which the first two are

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle \equiv (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) P(k_1), \quad (2.3a)$$

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(k_1, k_2, k_3). \quad (2.3b)$$

The amplitude of the three-point function is usually measured in terms of a momentum-dependent parameter f_{NL} , satisfying

$$\frac{6}{5} f_{\text{NL}}(k_1, k_2, k_3) = \frac{B(k_1, k_2, k_3)}{\sum_{i < j} P(k_i) P(k_j)}. \quad (2.4)$$

where $i, j \in \{1, 2, 3\}$. The problem at hand is to calculate f_{NL} .

Bundles of trajectories. A comoving scale k is outside the horizon when $k/aH < 1$, where a is the scale factor and $H \equiv \dot{a}/a$ is the Hubble parameter. An overdot denotes a derivative with respect to time. During inflation H is approximately constant, whereas a is growing rapidly. Therefore k/aH rapidly becomes negligible a few e-folds after horizon crossing. Smoothing over a comoving scale somewhat larger than $(aH)^{-1}$, widely separated spatial patches will locally evolve like an unperturbed universe up to small corrections. This is the separate universe picture [46–48].

An ensemble of smoothed regions picks out a collection of trajectories in phase space. If the ensemble is drawn from a spacetime region of finite comoving extent L then we can

expect this collection to be narrowly collimated provided L is not too large.¹ We refer to this ensemble of clustered trajectories as a bundle. When making predictions for microwave background scales it may contain of order 10^6 trajectories or more [55]. In practice the analysis is simplified by working in a thermodynamic limit where the bundle formally contains an infinite number of trajectories. To avoid spurious infrared problems we should demand that these reheat almost surely in the same vacuum.

The isocurvature modes label Fermi normal coordinates on Π' , adapted to the bundle. Each isocurvature field s has equation of motion $\dot{s} = 0$ and constitutes a conserved quantity [6, 56]. Together, these conserved quantities identify a trajectory. A particular location on each trajectory is specified by the integrated expansion $N \sim \ln a(t)$. This ‘trajectory’ approach can be traced to Hawking’s formulation of perturbation theory [57], and was applied to inflation by several authors [46, 58–60]. An explicit description in terms of trajectories on Π' was given by Salopek [56] and García-Bellido & Wands [6]. Locally, N and the isocurvature fields generate a coordinate chart on Π' . Fluctuations along the *same* trajectory generate the adiabatic mode, $\zeta = \delta N$. Isocurvature modes represent fluctuations *between* trajectories. In an N_f -field model, the reduced phase space supports $N_f - 1$ isocurvature modes.

Bundle sections. Consider a foliation of Π' by submanifolds which are nowhere tangent to the bundle. An important example is foliation by surfaces of fixed energy density. We may use any such foliation to replace N as a label for length along the trajectories. Intersecting the bundle with an individual hypersurface generates a cross-section with coordinates inherited from the isocurvature modes. For example, working on uniform density hypersurfaces, the scalar fields take values ϕ_i^c . These “coordinates” are not independent but are subject to the constraint $dW(\phi_i^c) = 0$, leaving the expected $N_f - 1$ isocurvature labels.

If the bundle has degenerated to a caustic then each hypersurface intersects the bundle at a unique point ϕ_i^c . Making a small change of trajectory $\delta\phi_j^*$ earlier in the bundle’s evolution, one will observe no change in ϕ_i^c . Therefore $\delta\phi_i^c = 0$ and we conclude $\partial\phi_i^c/\partial\phi_j^* = 0$. One can regard this as a requirement that physical predictions computed from statistical properties of the bundle become independent of its initial conditions. Speaking loosely, we describe this behaviour as an “attractor.” The attracting trajectory is the caustic, and we will sometimes refer to it as the *limiting trajectory*. An ensemble of smoothed patches traversing the limiting trajectory differ only by their relative position within it, making ζ conserved.

This argument identifies regions where $\partial\phi_i^c/\partial\phi_j^* \rightarrow 0$ with regions where ζ is conserved.² We describe it as the *adiabatic limit*. In what follows we will see that estimates of the decay rate of $\partial\phi_i^c/\partial\phi_j^*$ play an important role in analysing the adiabatic limit. As an example, consider the purely scalar dynamics associated with slow-roll inflation. A common type of limiting trajectory lies on a valley floor in the landscape generated by the potential.

¹Comoving quantities such as L are not physical, and can not appear in predictions for observable quantities [49]. The discussion in this paper is independent of L , but as a point of principle L should be removed from physical quantities by supplying a distribution function for the large-scale modes [50–52]. After doing so, all predictions depend only on physical scales. Similar conclusions have been obtained by a number of different methods [53, 54].

²The condition that $\partial\phi_i^c/\partial\phi_j^* \rightarrow 0$ is sufficient to show that that ζ becomes conserved within the separate universe picture, but this argument does not demonstrate that it is necessary.

Descending into the valley, the mass-squared matrix associated with perturbations orthogonal to the direction of motion should be nondegenerate, with all eigenvalues large and positive. Taking the smallest eigenvalue of magnitude $\sim m_\perp$ one will generically find $\partial\phi_i^c/\partial\phi_j^* \sim e^{-m_\perp^2 N/3H^2}$. We will discuss this example more carefully in §4.

Bundle averages. These principles give a procedure to determine the statistics of ζ in an adiabatic limit. One uses the bundle of trajectories to determine any required n -point correlation functions, and then imposes the requirement $\partial\phi_i^c/\partial\phi_j^* \rightarrow 0$. Unfortunately there is no unique way to do so. Different approaches to the limit correspond to different focusing mechanisms. As we have explained, focusing regions may occur naturally in some models; the examples studied by Meyers & Sivanandam [44, 45] are of this type. But whether or not a model naturally contains an adiabatic limit we may usually elect to impose a different one, perhaps by enlarging the field content to include a waterfall transition. We will study some possibilities below.

It is first necessary to obtain the relevant correlation functions. When the slow-roll attractor is operative, the e-foldings along each trajectory can be expressed as a function of its initial conditions. Therefore $N = N(\phi_i^*)$. Expanding in the neighbourhood of a fiducial trajectory yields

$$\zeta \equiv \delta N = \sum_i N_{,i} \delta\phi_i^* + \frac{1}{2} \sum_{i,j} N_{,ij} \delta\phi_i^* \delta\phi_j^* + \dots, \quad (2.5)$$

where $N_{,i} \equiv \partial N/\partial\phi_i^*$ and $N_{,ij} \equiv \partial^2 N/\partial\phi_i^* \partial\phi_j^*$. The $\delta\phi_i^*$ measure deviations from the fiducial trajectory, and will typically be of order the quantum scatter. After restriction to connected correlation functions there is no dependence on the arbitrary choice of fiducial trajectory.

Eq. (2.5) enables the low-order correlation functions to be expressed in terms of the data $N_{,i}$, $N_{,ij}$, which can be computed in some models [39, 44, 61]. One finds $\langle\zeta\zeta\rangle = N_{,i} N_{,j} \langle\delta\phi_i \delta\phi_j\rangle_*$, which determines the power spectrum (2.3a). Similarly, f_{NL} can be written³ [28]

$$f_{\text{NL}} = \frac{5}{6} \frac{\sum_{i,j} N_{,i} N_{,j} N_{,ij}}{\left(\sum_i N_{,i}^2\right)^2}. \quad (2.6)$$

3 Transitory behaviour of f_{NL}

It was explained in §§1–2 that our interest lies in adiabatic regions where all isocurvature modes are exhausted, preventing further evolution of ζ . This limit need not be achieved smoothly. For example, in hybrid scenarios the inflationary phase is suddenly destabilized by a waterfall transition, leading to abrupt convergence of trajectories. Although a convincing demonstration has not yet been given, in some circumstances the subsequent dynamics may preserve the value of f_{NL} at the waterfall. This strategy has been invoked by various authors [63–66]. Some fine-tuning would be required to arrange $|f_{\text{NL}}| \gg 1$ at the transition point. Nevertheless, in these scenarios and others it may be misleading to focus exclusively on regions where phase space trajectories naturally converge. For this reason we pause to study

³We are neglecting intrinsic non-Gaussianities of the $\delta\phi_i^*$. For slow-roll inflation with canonical kinetic terms these are negligible whenever f_{NL} is large enough to be observable [39, 62].

the qualitative evolution of f_{NL} , whether or not we are close to an adiabatic region. We focus on scenarios where its value changes rapidly, before returning to focusing regions in §4.

Under which circumstances should we expect the moments of the distribution function to change significantly? The distribution function describes how trajectories cluster around the core of the bundle. It is conserved under linear evolution, but is sheared or distorted on curved paths [67, 68]. These effects reshape the distribution function: even when it is initially Gaussian we expect probability to be relocated from the core to the outer layers of the bundle. This is associated with the generation of significant third- or higher n^{th} -order moments.

Curved paths can be generated by many choices of microphysics. We study only an especially simple mechanism. Where the potential’s topography includes a ridge or valley we will encounter curved trajectories diverging from the ridge or converging into the valley floor. Examples of such trajectories have been studied by a number of authors [40, 65, 66, 69].

3.1 Ridges: Diverging trajectories

We restrict attention to two-field models, which already capture the principal dynamical features, and label Π' by coordinates ϕ and χ . We assume a “ridge” or separatrix at $\chi = 0$. In the neighbourhood of an arbitrary point $(\phi_0, 0)$ on the ridge the potential will generically have the form $W \approx W_0 + g_0(\phi - \phi_0) - \frac{1}{2}m_\chi^2\chi^2$. The mass-squared m_χ^2 is positive, and omitted terms are higher-order in $\phi - \phi_0$ and χ . These become relevant at some point after the trajectory has been ejected from the vicinity of $\chi = 0$. The trajectory $\chi = 0$ is classically stable, although depopulated by quantum fluctuations [70–74].

Trajectories. Measuring length along each trajectory by the energy density, the evolution equations are

$$\frac{1}{3M_{\text{P}}^2} \frac{d\phi}{d(H^2)} = \frac{g_0}{g_0^2 + (m_\chi^2\chi)^2}, \quad (3.1a)$$

$$-\frac{1}{3M_{\text{P}}^2} \frac{d\chi}{d(H^2)} = \frac{m_\chi^2\chi}{g_0^2 + (m_\chi^2\chi)^2}. \quad (3.1b)$$

According to (3.1b), a trajectory emanating from (H_*, χ_*) and evolving to (H_c, χ_c) satisfies

$$\frac{m_\chi^2}{2}(\chi_c^2 - \chi_*^2) + \frac{g_0^2}{m_\chi^2} \ln \frac{\chi_c}{\chi_*} = 3M_{\text{P}}^2(H_*^2 - H_c^2). \quad (3.2)$$

If $|\chi_c| \lesssim |g_0|/m_\chi^2$ then the logarithm dominates and the trajectories disperse linearly in the sense $\chi_c = \chi_*D$, where the growth factor D satisfies $D \equiv e^{\beta(H_*^2 - H_c^2)}$ and $\beta \equiv 3(M_{\text{P}}m_\chi/g_0)^2$. Nonlinear dispersion occurs in the region $|\chi_c| \gtrsim |g_0|/m_\chi^2$ where the quadratic term dominates. The transition between the two is the “turn,” beyond which each trajectory is ejected from the ridge and rapidly evolves to $|\chi_c| \gg |g_0|/m_\chi^2$. At the turn we have

$$|\chi_{\text{turn}}| \sim \frac{|g_0|}{m_\chi^2}, \quad (3.3)$$

which makes the kinetic energy in each field roughly equal, $|\dot{\phi}_{\text{turn}}| \approx |\dot{\chi}_{\text{turn}}|$.

This leads to the following physical picture. Trajectories which are still close to the ridge preserve their initial Gaussian profile. Trajectories populating the downhill-edge of the bundle quickly slide away, generating a heavy tail at large $|\chi_c|$. In this region kinetic energy has greater relative importance, slowing the expansion rate and enhancing the frequency of excursions to large negative δN . Therefore this mechanism will generate negative f_{NL} from a Gaussian distribution.

Whether a large negative amplitude is achieved in practice depends on the initial distribution of trajectories within the bundle, the nonlinear relation between the fields and ζ , and for how long the mechanism operates. Sufficiently far down the ridge the trajectories depend on the completion of W . Therefore, the approach to an adiabatic limit cannot be described by the techniques of this section.

δN analysis. We now translate to ζ and repeat the analysis in the language of the δN method [28]. Consider two trajectories originating well before the critical turning point, but initially separated by a distance $(\delta\phi_*, \delta\chi_*)$. It is useful to define $\delta \equiv m_\chi^2 |\chi/g_0|$, where $\delta_* \ll 1$ indicates the initial point is very close to the ridge. In this region surfaces of constant energy density in Π' practically coincide with surfaces of constant ϕ . Therefore, to bring this pair of trajectories to a common energy density $H = H_*$ requires a small excess expansion $\delta N \approx (2\epsilon_\phi^*)^{-1/2} \delta\phi_*/M_{\text{P}}$. The subsequent expansion history, measured to a surface $H = H_c$, can be written $N = N(H_c; H_*, \chi_*)$.

In what follows we work without loss of generality on the positive branch $\chi > 0$, and suppose $\phi - \phi_0$ and χ remain sufficiently small that higher-order terms in the potential do not become relevant. Once this assumption fails, $\dot{\phi}$ may acquire a nonnegligible dependence on χ_* , potentially invalidating our conclusions. Passing to the limit where $\delta\phi_*$ and $\delta\chi_*$ become infinitesimal, we conclude that on arrival at $H = H^c$ the trajectories have experienced expansion histories which differ by

$$dN \approx \frac{3H_*^2}{g_0} d\phi_* + 18m_\chi^4 M_{\text{P}}^2 d\chi_* \int_{H_*^2}^{H_c^2} \frac{H^2 d(H^2)}{[g_0^2 + (m_\chi^2 \chi)^2]^2} \chi \left(\frac{\partial \chi}{\partial \chi_*} \right)_{H_*}, \quad (3.4)$$

where the partial derivative is to be evaluated at constant H_* and $\chi = \chi(H)$.

Invoking the chain rule, Eq. (3.4) determines all derivatives of N . We find

$$N_{,\chi\chi} = 18m_\chi^4 M_{\text{P}}^2 \int_{H_*^2}^{H_c^2} \frac{H^2 d(H^2)}{[g_0^2 + (m_\chi^2 \chi)^2]^2} \left[\frac{g_0^2 - 3(m_\chi^2 \chi)^2}{g_0^2 + (m_\chi^2 \chi)^2} \left(\frac{\partial \chi}{\partial \chi_*} \right)_{H_*}^2 + \chi \left(\frac{\partial^2 \chi}{\partial \chi_*^2} \right)_{H_*} \right]. \quad (3.5)$$

So far our considerations have been general. Prior to the turn, Eq. (3.2) makes $\partial^2 \chi / \partial \chi_*^2$ negligible whereas $\partial \chi / \partial \chi_* \approx \chi / \chi_*$ is exponentially growing. In this region our assumptions make the integrands of Eqs. (3.4) and (3.5) positive, and therefore both $N_{,\chi}$ and $N_{,\chi\chi}$ are negative and decreasing.

If m_χ is not too small, the integrals of (3.4) and (3.5) are dominated by their upper limits—where the exponential growth is maximized. Taking the initial evolution in χ to be almost negligible, this requires

$$m_\chi^2 \gg \frac{3\epsilon_* H_*^2}{1 - (H_c/H_*)^2} \approx 3\epsilon_* H_*^2, \quad (3.6)$$

where the approximate equality applies if H_c is at least a little smaller than H_* . If χ is to be sufficiently light to acquire a quantum fluctuation then $m_\chi \ll H_*$, and if both conditions are to be compatible we must require $\epsilon_* \ll 1$. A short calculation yields

$$N_{,\chi} \approx -\frac{3m_\chi^2 H_c^2}{g_0^2} \chi_* \left(\frac{\chi_c}{\chi_*}\right)^2 \quad (3.7a)$$

$$N_{,\chi\chi} \approx \frac{N_{,\chi}}{\chi_*}. \quad (3.7b)$$

This relation between $N_{,\chi}$ and $N_{,\chi\chi}$ is a consequence of the exponential growth of χ prior to the turn.

Initially, $N_{,\chi}$ and $N_{,\chi\chi}$ are small in comparison with $N_{,\phi}$ and $N_{,\phi\phi}$. In addition, $N_{,\phi\chi} \approx \delta_*/M_{\text{P}}^2$ is constant and can safely be neglected. Therefore ζ is dominated by the fluctuation in ϕ , which is practically Gaussian. Using (2.6), we find

$$\frac{6}{5}f_{\text{NL}} \approx \left[2\epsilon_* + \left(\frac{N_{,\chi}}{N_{,\phi}}\right)^3 \frac{m_\chi^2}{3H_*^2} \frac{1}{\delta_*} + \text{O}\left(\frac{N_{,\chi}}{N_{,\phi}}\delta_*\right) \right] \left[1 + \frac{N_{,\chi}^2}{N_{,\phi}^2} \right]^{-2}. \quad (3.8)$$

While $|N_{,\chi}| \ll |N_{,\phi}|$, the first term dominates and (3.8) gives $|f_{\text{NL}}| \approx \epsilon_* < 1$. As the trajectory moves away from the ridge the χ_* -derivatives become increasingly important whereas the ϕ_* -derivatives are constant. When $|N_{,\chi}|$ and $|N_{,\phi}|$ are comparable, f_{NL} is dominated by the second term in (3.8). In virtue of (3.6) and the initial condition $\delta_* \ll 1$, this is much larger than ϵ_* and causes a spike in f_{NL} . Estimating the peak to occur when $N_{,\phi} \approx -N_{,\chi}$, we find

$$f_{\text{NL}}|_{\text{peak}} \approx \frac{\eta_{\chi_*}}{\delta_*} \approx -0.3\epsilon_*^{1/2} \frac{M_{\text{P}}}{|\chi_*|}, \quad (3.9)$$

where $\eta_\chi \approx M_{\text{P}}^2 W_{,\chi\chi}/W$ is the standard η -parameter associated with χ . In this expression and similar ones below, including Eq. (3.18), the numerical prefactor is uncertain by an $\text{O}(1)$ quantity which depends on the precise balance between $N_{,\phi}$ and $N_{,\chi}$ at the peak.

On approach to the spike, Eq. (3.8) predicts that f_{NL} is negative and growing like $(\chi_c/\chi_*)^6$. Subsequently, χ continues to increase and $|N_{,\chi}|$ eventually dominates $|N_{,\phi}|$. In this region ζ is composed almost entirely of the χ fluctuation. The non-Gaussianity becomes practically independent of δ_* and decays like $(\chi_c/\chi_*)^{-2}$. These estimates of the growth rate and decay rate are valid before the turn, where χ_c is growing exponentially as described below Eq. (3.2).

Dropping numerical factors of order unity and using (3.3) to estimate f_{NL} when the fiducial trajectory passes the turn, we find

$$f_{\text{NL}}|_{\text{turn}} \sim -\frac{m_\chi^2}{H_c^2} \approx \eta_\chi|_c. \quad (3.10)$$

This is much less than Eq. (3.9) and therefore occurs some time after the peak f_{NL} is achieved. If Eq. (3.10) is not invalidated by higher-order terms in the potential, it implies that the height of the spike is adjustable independently of f_{NL} on entry or exit. Since the peak f_{NL} occurs *before* most trajectories in the bundle reach the turning point (3.3), our analysis will apply provided these higher-order terms become relevant only after the turn.

Scaling relations. Eqs. (3.8) and (3.9) give interesting scaling relations for the peak $|f_{\text{NL}}|$, and for its growth and decay near the spike. Eq. (3.9) suggests that the maximum $|f_{\text{NL}}|$ attained during the spike has a practically universal power-law scaling for any potential which can be approximated by the coefficients g_0 and m_χ up to the turn of the trajectories: for such potentials we should expect $|f_{\text{NL}}| \propto |\chi_*|^{-\nu}$ with exponent $\nu \approx 1$. In §5 we will use numerical methods to study models which exhibit this scaling behaviour.

3.2 Valleys: Converging trajectories

The converse process occurs when a trajectory approaches a valley, where a bundle of trajectories is nonlinearly focused rather than defocused. As above, we specialize to a two-dimensional field space labelled by coordinates (ϕ, χ) and suppose there exists a valley aligned with the χ direction. In the neighbourhood of the valley we write $W \approx W_0 + W_\phi + W_\chi$, where W_0 is a constant and

$$W_\phi = \frac{1}{2}m_\phi^2\phi^2 \quad (3.11a)$$

$$W_\chi = g_0\chi + \frac{1}{2}m_\chi^2\chi^2. \quad (3.11b)$$

If $m_\phi \gtrsim m_\chi$ then the slopes will be relatively steep in comparison with the valley floor. Omitted terms are higher order in ϕ and χ , but become increasingly irrelevant as $\phi, \chi \rightarrow 0$. Sufficiently far from $\phi = 0$ the motion is almost entirely in the ϕ direction.

During descent into the valley, trajectories populating the uphill edge of the bundle experience a larger velocity in the orthogonal χ direction compared to those lower down the slope. Therefore the uphill edge is compressed towards the centroid, generating a nonlinear distribution. The tail of the distribution is again on the downhill side, but in this case the tail has *lower* kinetic energy and enhances the frequency of excursions to a large *positive* δN . Therefore this mechanism generates a positive f_{NL} .

δN analysis. The evolution equations are

$$\frac{1}{3M_{\text{P}}^2} \frac{d\phi}{d(H^2)} = \frac{W'_\phi}{(W'_\phi)^2 + (W'_\chi)^2}, \quad (3.12a)$$

$$\frac{1}{3M_{\text{P}}^2} \frac{d\chi}{d(H^2)} = \frac{W'_\chi}{(W'_\phi)^2 + (W'_\chi)^2}. \quad (3.12b)$$

In analogy with the ridge case, it is helpful to define a dimensionless measure of distance, δ , from the valley floor. We choose $\delta \equiv W'_\phi/W'_\chi$, which measures the relative partition of kinetic energy between the fields. Our analysis applies when the trajectories begin from an initial position sufficiently high above the valley, where $\delta_* \gg 1$ and only the ϕ -field is in motion. In this regime, Eqs. (3.12a)–(3.12b) can be integrated to find

$$\frac{\phi_c^2}{\phi_*^2} = 1 + \frac{6M_{\text{P}}^2}{m_\phi^2} \frac{H_c^2 - H_*^2}{\phi_*^2} \quad (3.13a)$$

$$\chi_c + \frac{g_0}{m_\chi^2} = \left(\chi_* + \frac{g_0}{m_\chi^2} \right) \left(\frac{\phi_c}{\phi_*} \right)^{m_\chi^2/m_\phi^2} \quad (3.13b)$$

up to corrections of relative magnitude $1/\delta^2$. Eqs. (3.13a)–(3.13b) cease to be a good approximation no later than $\delta \sim 1$, when $\dot{\phi} \sim \dot{\chi}$ and the kinetic energy in each field is approximately equal. In typical models this occurs at the turn.

For $\delta \gg 1$, surfaces of constant energy density are practically aligned with surfaces of constant ϕ . Adopting the methods of §3.1, we bring a pair of nearby trajectories separated by the displacement $(\delta\phi_*, \delta\chi_*)$ to a common value of H , and write the number of e-foldings to a subsequent surface of constant energy density H_c as $N = N(H_c; H_*, \chi_*)$. Passing to the limit of infinitesimal $\delta\phi_*$ and $\delta\chi_*$, and using Eqs. (3.13a)–(3.13b), we conclude that on arrival at $H = H_c$, the trajectories have experienced expansion histories which differ by

$$dN \approx \frac{3H_*^2}{m_\phi^2 \phi_*} d\phi_* - 18 \frac{M_{\text{P}}^2 \phi_*}{m_\phi^4 \delta_*} d\chi_* \int_{H_*^2}^{H_c^2} \frac{H^2 d(H^2)}{\phi^4} \left\{ 1 - \mu \left(\frac{\phi^2}{\phi_*^2} \right)^\mu \right\}, \quad (3.14)$$

where ϕ is to be understood as a function of H and we have introduced the mass ratio $\mu \equiv m_\chi^2/m_\phi^2 < 1$. Corrections to Eq. (3.14) are suppressed by $1/\delta^2$.

Eq. (3.14) reproduces many features of the ridge analysis. The derivative $N_{,\phi}$ is constant, whereas $|N_{,\chi}|$ is initially zero but growing. Performing the integral, we find

$$N_{,\chi} = \frac{1}{2\delta_*} \frac{\phi_*}{M_{\text{P}}^2} \Phi \left(\frac{\phi^2}{\phi_*^2} \right), \quad (3.15)$$

where the “growth factor” $\Phi(x)$ satisfies

$$\Phi(x) \equiv -\ln x + (x^\mu - 1) + \frac{W_0}{W_{\phi_*}} (x^{-1} - 1) + \frac{W_0}{W_{\phi_*}} \frac{\mu}{\mu - 1} (x^{\mu-1} - 1). \quad (3.16)$$

We have assumed W_0 dominates W_* , but the generalization to other cases is straightforward. At $x = 1$ we have $\Phi(1) = 0$. For $x < 1$ the dominant growing term depends on microphysical details of the model. Under our assumption $W_0 \gg W_{\phi_*}$ the dominant growth is initially from x^{-1} . Inflation will not end naturally in a model of this type, so some other exit mechanism must be invoked. We will see examples of this kind in §5. On the other hand, if $W_0/W_{\phi_*} \lesssim 1$ the logarithm will initially dominate. In either case, the asymptotic growth in the limit $x \ll 1$ is from x^{-1} . Therefore, for a typical model $\Phi(x)$ is a complicated function determined by a competition for dominance between the various terms. However, remarkably, in many cases the behaviour of f_{NL} is almost independent of these complicated microscopic details.

Differentiating (3.15), we find

$$N_{,\chi\chi} = \left\{ \mu\delta_* + \frac{2}{\delta_*} \Delta \left(\frac{\phi^2}{\phi_*^2} \right) \right\} \frac{N_{,\chi}}{\phi_*}, \quad (3.17)$$

where $\Delta(x) \equiv (x - 1) d \ln \Phi(x) / dx$. Eq. (3.16) implies that Δ is growing as ϕ decreases to zero. While $|N_{,\chi}|$ is increasing towards $|N_{,\phi}|$ we find f_{NL} increases, achieving a maximum value when $|N_{,\chi}| \sim |N_{,\phi}|$. First, suppose the Δ -dependent term is subdominant at this time, which implies $\Delta \lesssim \mu\delta_*^2$. We find

$$f_{\text{NL}}|_{\text{peak}} \sim \eta_{\chi_*} \delta_* \approx 0.3 \epsilon_*^{1/2} \frac{m_\chi^2 M_{\text{P}}}{g + m_\chi^2 |\chi_*|}. \quad (3.18)$$

which is independent of the growth rate (3.16) and the mass ratio μ . In this sense, the maximum value (3.18) is a “universal” phenomenon. When $|N_{,\chi}| > |N_{,\phi}|$ we find that f_{NL} decays like Φ^{-1} , at least until $\Delta \sim \mu\delta_*^2$, when it may stabilize as we will explain below. In analogy with the ridge, this sequence of growth and decay gives rise to a spike in f_{NL} . Ultimately ϕ/ϕ_* will decrease until $\delta \sim 1$, and the subsequent behaviour of f_{NL} must be determined by different methods, such as those described in §4. Written in terms of the dimensionless measure δ , Eq. (3.18) coincides with (3.9) with the identification $\delta_{\text{valley}} = 1/\delta_{\text{ridge}}$. In this language, the sign of f_{NL} is inherited from the sign of η_χ .

Second, consider the “nonuniversal” case where the Δ -dependent term dominates (3.17). In this case, f_{NL} increases until $|N_{,\chi}| \sim |N_{,\phi}|$, achieving a value *larger* than (3.18). Its precise value is set by the ratio Δ/δ_* , and may depend on details of the potential, including the mass ratio. When $|N_{,\chi}| > |N_{,\phi}|$ the time dependence of f_{NL} is set by Δ/Φ . Its precise scaling depends on the dominant term in Φ . In particular, if $\Phi \sim x^{-1}$ then Δ/Φ is approximately constant and f_{NL} does not decay. In such cases, f_{NL} exhibits a plateau and it may no longer make sense to speak of a spike at all. After the turn is completed, the nonlinear deformation of the bundle will partially relax, leading to decay of $|N_{,\chi\chi}|$. The precise details, including the decay rate, are model-dependent. Eventually the fields reach equipartition of kinetic energy and this analysis breaks down.

4 Asymptotic behaviour of f_{NL}

Whether a large $|f_{\text{NL}}|$ can be generated during an epoch of slow-roll inflation—perhaps from the “spike” mechanisms described above—is irrelevant unless it can be preserved in some adiabatic limit. The methods of §3 are insufficient to resolve this question.

The potential may be such that a focusing region is naturally available. If inflation terminates in this region then the transitory evolution of f_{NL} studied in §3 has no necessary connection with its final asymptotic value. In certain circumstances, where the focusing region can be analysed in detail, a relatively simple statement is possible. These are the scenarios studied by Meyers & Sivanandam [44, 45]. One might have thought that the final f_{NL} would depend only on the local shape of the potential in the focusing region, which will typically be a stable parabolic minimum. If so, the asymptotic value of f_{NL} would be universal among all potentials sharing a similarly-shaped minimum. However, this is not the case. As we will explain, the asymptotic value of f_{NL} generally depends on properties of the potential far from the focusing region.

If multiple focusing regions are available, one must be selected by a combination of dynamics and initial conditions. To determine which possibility should be expected by late-time observers who map the anisotropy of the CMB requires an understanding of the infrared structure of the entire inflating volume [50]. This difficult “measure problem” remains unsolved.

If a natural focusing region is not available, or is not selected, then one must be imposed and the entire analysis becomes significantly more complicated. In this case the transitory evolution studied in §3 may become relevant. We will have little to say about this possibility, although we investigate some numerical cases in §5.

Natural focusing. In this section, we study models where inflation ends in a region of the potential where the trajectories are naturally focused. Broadly speaking, two possibilities exist.

- The asymptotic value of f_{NL} generated during focusing may be unobservably small, erasing any transiently large non-Gaussianity generated by spikes or other features. This possibility was emphasized by Meyers & Sivanandam [44, 45], who worked with a particular class of separable N_f -field models to be discussed below. However, other possibilities exist.
- It may be possible to make the focusing process itself generate a large f_{NL} by suitable choice of W . An example of such a model was given by Kim et al. [43]. (Indeed, in this model, f_{NL} grows sharply during approach to the adiabatic limit.)

In principle, the behaviour of f_{NL} in a focusing region could be determined from (2.5)–(2.6) by imposing the limit $\partial\phi_i^c/\partial\phi_j^* \rightarrow 0$. Unfortunately, it is not known how to compute the “ δN coefficients” $N_{,i}$ and $N_{,ij}$ for an arbitrary model. Therefore a systematic discussion of this limit must apparently await future analytic developments.

Explicit expressions for the δN coefficients are known only in very restricted circumstances. Formulae for quadratic potentials were discussed by Lyth & Rodríguez [28], Lyth & Alabidi [38] and Alabidi [40]. Later, Vernizzi & Wands [39] and Battefeld & Easther [75] gave expressions for an arbitrary sum-separable potential. Taking $W = \sum_i V_i(\phi_i)$, one finds

$$N_{,i} = \frac{1}{M_{\text{P}}^2} \left(\frac{V_i}{V_i'} \Big|_* - \sum_k \frac{V_k}{V_k'} \Big|_c \frac{\partial\phi_k^c}{\partial\phi_i^*} \right), \quad (4.1)$$

where $\partial\phi_k^c/\partial\phi_i^*$ satisfies

$$\frac{\partial\phi_k^c}{\partial\phi_i^*} = -\frac{W_c}{W_*} \sqrt{\frac{\epsilon_k^c}{\epsilon_i^*}} \left(\frac{\epsilon_i^c}{\epsilon^c} - \delta_{ik} \right). \quad (4.2)$$

A similar expression for a product-separable potential $W = \prod_i V_i(\phi_i)$ was obtained by Choi et al. [76]. Comparable results for a general class of sum- and product-type potentials were given by Wang [61] and are summarized in the Appendix. It is also possible to take the Hubble rate to be separable rather than the potential [77, 78].

Focusing in a valley. A typical example of a focusing region is a valley of the potential landscape, perhaps terminating in a local minimum. For N_f fields, there are at least $N_f - 1$ heavy directions with masses greater than the Hubble rate. Quantum fluctuations are suppressed in these directions, which prevent the bundle from diffusing up the sides of the valley. The steep slopes cause exponential convergence, and rapidly focus the bundle to a line.

In the neighbourhood of the valley floor, we assume it is possible to choose coordinates on field space for which the potential approximately separates

$$W \approx V_\varphi(\varphi) + \sum_\alpha V_\alpha(s_\alpha) \approx V_\varphi(\varphi) + \frac{1}{2} \sum_\alpha m_\alpha^2 s_\alpha^2 \quad (4.3)$$

where φ labels distance along the valley floor—which may be a light direction—and the $N_f - 1$ fields s_α are stabilized with masses $m_\alpha = V''_\alpha \gtrsim H$. By a suitable choice of coordinates we can arrange that $\langle s_\alpha \rangle = 0$. To describe a complicated valley it may be necessary to glue several such regions together. Focusing on the particular region described by (4.3), we denote the field values on entry to its domain of validity $\bar{\varphi}$ and \bar{s}_α . These will be functions of the initial fields ϕ_i^* . This initial point could generically occur far from the valley, where (4.3) need not be a good approximation.

The heavy fields s_α evolve according to $3H\dot{s}_\alpha = -m_\alpha^2 s_\alpha$. After N e-foldings from the point of entry, one finds

$$s_\alpha = \bar{s}_\alpha(\phi_i^*) e^{-\int_0^N \eta_\alpha(N') dN'}. \quad (4.4)$$

The total number of e-folds available within the valley is model-dependent. In a long valley the focusing may practically go to completion, making s_α effectively zero. Alternatively, if the valley rapidly terminates in a local minimum there may be insufficient time to focus the bundle completely.

The fields ϕ_k can be written as linear combinations of φ and the s_α , giving $\phi_k = \gamma_k \varphi + \sum_\alpha \beta_k^\alpha s_\alpha$. The γ_k and β_k^α are constants, which depend only on the choice of separable coordinates used in (4.3). They are independent of the entry point $(\bar{\varphi}, \bar{s}_\alpha)$, which implies

$$\frac{\partial \phi_k^c}{\partial \phi_j^*} = \sum_\alpha \left(\beta_k^\alpha - \gamma_k \frac{V_\alpha'^c}{V_\varphi'^c} \right) \frac{\partial s_\alpha^c}{\partial \phi_j^*}. \quad (4.5)$$

Therefore $\partial \phi_k^c / \partial \phi_j^*$ behaves like a linear combination of derivatives $\partial s_\alpha^c / \partial \phi_j^*$.

The number of e-foldings, $N^c(\phi^*)$, which occur between the entry point $(\bar{\varphi}, \bar{s}_\alpha)$ and the surface c will usually depend on the initial point ϕ^* . Assuming $N^c(\phi^*)$ does not exhibit a dramatic sensitivity to these initial conditions, Eq. (4.4) shows that $\partial s_\alpha^c / \partial \phi_j^*$ will decay exponentially as the trajectory settles into the valley. Potentials may exist which violate this condition, but we believe it will be satisfied for a majority of trajectories which flow over reasonably smooth potential landscapes. Where it is satisfied, this estimate of the decay rate applies once a trajectory has been captured by the focusing region, no matter what form the potential takes globally.

In Eq. (4.5) the γ_k term will typically decay exponentially, because $V_\alpha'^c \sim s_\alpha$ whereas $V_\varphi'^c$ decays less rapidly. Therefore Eq. (4.4) implies the derivatives $\partial \phi_k^c / \partial \phi_j^*$ decay at least as fast as the lightest isocurvature field. We conclude⁴

$$\frac{\partial \phi_k^c}{\partial \phi_j^*} \lesssim e^{-\int_0^N \eta_s(N') dN'} \approx e^{-\eta_s N}, \quad (4.6)$$

where $\eta_s = \min\{\eta_\alpha\}$ and N is the same quantity occurring in Eq. (4.4). The final equality applies if η_s is approximately constant during the focusing process.

Separable potentials. In a globally sum-separable model, for which $N_{,i}$ satisfies (4.1), it may happen that (4.6) is sufficiently powerful to make the final “ c -term” irrelevant. In these circumstances the correlation functions of ζ , including the spectrum and bispectrum, can be

⁴The asymptotic notation $x \asymp y$ indicates that x and y share a common decay rate.

determined from the remaining term of (4.1), which depends only on boundary data at the initial time. For correlation functions among fields carrying comparable momenta of order k this is often taken to be the horizon-crossing time $|\eta| \sim 1/k$, where η is the conformal time. For this reason, the scheme has sometimes been called the *horizon-crossing approximation* [41–43]. Despite the name, we caution that this approximation does not consist of assuming that the perturbations are constant after horizon-crossing, but rather that their values in the adiabatic limit can be determined in terms of the shape of the potential there. A similar procedure can be applied in product-separable cases.

It is less straightforward to estimate the minimum number of e-folds required to make the c -terms of (4.1) negligible. Although Eq. (4.6) gives information concerning the decay rate, the number of e-folds required to damp any contribution from the c -terms depends on their amplitude on entry to the valley. This is a function of each species’ relative contribution to the energy density of the universe on the initial and final slices c and $*$, from which it does not appear straightforward to draw general conclusions. However, since the isocurvature masses should be comfortably heavier than the Hubble scale, the parameter η_s will typically be much larger than unity. In these circumstances, rather less than $O(10)$ e-foldings are usually required to accumulate a very substantial suppression of the c -terms.

In the language of Meyers & Sivanandam [44, 45], this damping of the c -terms is precisely the exponential suppression which they suggested would drive the bi- and trispectrum to slow-roll suppressed values. In the language of §2 it represents focusing of the bundle to a caustic. Our analyses are entirely consistent, but it is helpful to recall that the $*$ -term in (4.1) need not be especially small. We briefly comment on this possibility at the end of this section. If that is the case, suppression of the c -terms can cause the correlation functions to experience a short phase of exponential growth as they approach their asymptotic values. Note that all these conclusions depend on the existence of a globally separable potential. We are not aware of a systematic study of the asymptotics of f_{NL} in more general cases.

One might harbour some reservations that the c -terms do not decay if the fields settle into a stable minimum, for which V_k/V'_k diverges. Near an arbitrary point, which can be chosen as the origin without loss of generality, V_k can be written $V_k \approx A + B\phi_k$ and V_k/V'_k approaches a constant. Near a minimum, one finds instead $V_k \approx A + B\phi_k^2$. Therefore

$$\frac{V_k}{V'_k} \frac{\partial \phi_k^c}{\partial \phi_j^*} \approx \frac{A + B(\gamma_k \varphi^c + \sum_{\alpha} \beta_k^{\alpha} s_{\alpha}^c)^2}{2B(\gamma_k \varphi^c + \sum_{\alpha} \beta_k^{\alpha} s_{\alpha}^c)} \sum_{\rho} \left(\beta_k^{\rho} - \gamma_k \frac{V_{\rho}{}^{c'}}{V_{\varphi}{}^{c'}} \right) \frac{\partial s_{\rho}^c}{\partial \phi_j^*}. \quad (4.7)$$

If $A = 0$ the prefactor decays. Since the potential is sum-separable, we may always redefine all but one V_k to satisfy this condition. However, if the potential is not zero at the minimum then the remaining V_k must have nonzero A . Eq. (4.7) shows that we should choose the field ϕ_k to have nonzero overlap with the direction of the valley floor, i.e., $\gamma_k \neq 0$. Under these circumstances the right-hand side of (4.7) still decays (although perhaps at a reduced rate), because by assumption φ decays strictly more slowly than any isocurvature mode.

Large non-Gaussianity after natural focusing. Is it possible to obtain large f_{NL} at the adiabatic limit? Working in a sum-separable potential, the foregoing discussion implies that the c -dependent terms in (4.1) may be discarded provided enough focusing can be achieved

before the end of inflation. In general, the conditions required to achieve large f_{NL} may still be complicated. However, a relatively simple picture emerges if we assume that $N_{,i}$ is large for one field ϕ (or at most a few such fields) [43]. Therefore, V_ϕ/V'_ϕ at horizon crossing dominates the analogous terms for all other fields and f_{NL} can be written

$$f_{\text{NL}} \approx -\frac{5}{6}M_{\text{P}}^2 \left. \frac{V''_\phi}{V_\phi} \right|_* . \quad (4.8)$$

In a single-field model the quantity V''_ϕ/V_ϕ would be the inflationary η -parameter. But in an assisted inflation the total potential may be much larger than V_ϕ [79, 80]. Therefore η_ϕ can remain small, making ϕ light at horizon crossing and causing it to acquire a quantum fluctuation by the usual mechanism, while V''_ϕ/V_ϕ can be appreciable. We study an example of this type in §5.1.2. In such models the sign of f_{NL} is inherited from an “enhanced” η -parameter, as in Eqs. (3.9) and (3.18), but unlike these cases the enhancement is measured by the initial share of the energy density contributed by ϕ , rather than the parameters δ_{ridge} , δ_{valley} . In Eq. (4.8) this enhancement factor is $W/V_\phi \gg 1$.

If several fields have comparable $N_{,i}$, their perturbations contribute equally to ζ at the adiabatic limit and dilute any non-Gaussianity by the same interference effect which leads to the central limit theorem. Therefore the largest values of f_{NL} will be achieved where a single field has a dominant $N_{,i}$.

A similar discussion can be given for product-separable potentials. In this case the formulas depend solely on one field, labeled ϕ_k , which must be the field still evolving at the adiabatic limit. Therefore $f_{\text{NL}} \approx 2\epsilon_k^* - \eta_{kk}^*$, and a large f_{NL} would require a violation of slow-roll. We conclude that large $|f_{\text{NL}}|$ is not possible at the natural adiabatic limit in this class of models.

5 Models

The results of §4 show that, even for models where an adiabatic limit can be approached analytically, numerical calculations may be necessary to determine the degree of focusing which occurs near the end of inflation. In other cases there is simply no alternative.

In this section, we report the results of numerical simulations and compare the outcome to the analytic theory developed in §§3–4. In appropriate circumstances we show that the simplified description of “spikes” obtained in §3 is an accurate match for full numerical simulations. We give examples where the focusing described in §4 goes to completion—making the “horizon crossing approximation” highly accurate—and others where it does not. A case of special interest occurs when the bundle would focus only slightly after the end of inflation. One might expect that the error in analytic predictions based on the strict adiabatic limit would be small, but it transpires that f_{NL} can be rather sensitive to the details of the model. In models where there is no natural adiabatic region, reheating must occur before an adiabatic limit is reached. In these cases we perform a qualitative study of the dependence of f_{NL} on the details of the reheating phase.

Evolution after slow-roll using δN . The phase space description of inflationary trajectories was discussed in §2. During multiple-field slow-roll inflation, each trajectory lies on a submanifold Π' of the full phase space.

In a model more general than multiple-field slow-roll inflation, extra coordinates will typically be required. First, if the slow-roll approximation fails then one must work on the full phase space Π rather than the attractive submanifold Π' . Therefore new isocurvature modes are typically required to label the conjugate momenta $\pi_i \sim \dot{\phi}_i$. Second, matter species other than scalar fields may be included, perhaps to describe a phase of reheating. In such cases, the full phase space splits into a Cartesian product constructed from the phase space for each species, and suitable isocurvature modes labelling all these coordinates will be required. For thermalized radiation, a common choice is the temperature, T .

Several numerical approaches exist to compute the statistics of the density fluctuation [67–69]. Here we take the simple approach of calculating the derivatives of N using a finite difference scheme. This requires the slow-roll approximation at horizon crossing, where initial conditions are set, but not subsequently. We have verified that our results are insensitive to changes in the step size of the finite difference scheme. Although slower than other approaches [67, 68], direct δN has the advantage of straightforward comparison with our analytic methods. Moreover, it requires only the evolution of an unperturbed universe, making a simple description of reheating—assuming thermal equilibrium and a single radiation fluid—easy to implement. These assumptions are at best quasi-realistic, but serve to indicate a plausible phenomenology.

5.1 Two-field models

5.1.1 Transitory models with interruption

All two-field models documented in the literature which produce large, transient f_{NL} exploit the spikes described in §3 [40, 65, 66]. Eqs. (3.8) and (3.9) show that to obtain large $|f_{\text{NL}}|$ from a ridge, one must tune the initial conditions so that $\chi_*/M_{\text{P}} \ll 1$. Also, if this large f_{NL} is to be preserved in the adiabatic limit, Eq. (3.10) implies that some mechanism must operate to end inflation before the majority of trajectories in the bundle encounter the turn. In two-field models with separable potentials, the parameter combinations required to ensure these conditions were given by Byrnes et al. [65]. The observables predicted in such models depend strongly on the choice of exit mechanism. In certain cases, such as two-field hybrid inflation, it is possible that f_{NL} is not erased. In other cases this outcome appears unlikely.

Two-field hybrid inflation. This model was studied by Alabidi & Lyth [40] and later by Byrnes et al. [65, 66]. The potential is

$$W = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}(g_\phi^2\phi^2\sigma^2 + g_\chi^2\chi^2\sigma^2) + \frac{1}{4}\lambda(\sigma^2 - v^2)^2 \quad (5.1)$$

where ϕ and χ are slowly-rolling fields, and σ is a waterfall field which becomes destabilized when $g_\phi^2\phi^2 + g_\chi^2\chi^2 = \lambda v^2$. We take the masses m_ϕ and m_χ to be positive, and assume $g_\phi^2/g_\chi^2 = m_\phi^2/m_\chi^2$. This ensures that the waterfall occurs at fixed energy density, making it unnecessary to account for the effect of inflation ending on different hypersurfaces [63, 64, 81,

82]. If the masses are not equal, there is a steep slope in the direction of the more massive field. The trajectories evolve along this steep direction and then turn towards the global minimum. We expect some non-Gaussianity to be generated during this process.

In Fig. 1 we plot the evolution of f_{NL} for $m_\phi/m_\chi = 5$, $\eta_\phi = 4M_{\text{P}}^2 m_\phi^2 / (\lambda v^4) = 0.08$ and initial conditions $\chi_* = 0.001M_{\text{P}}$, $\phi_* = 0.5M_{\text{P}}$. We adjust the remaining parameters so that the waterfall occurs when $|f_{\text{NL}}| > 1$ and takes much less than a Hubble time to complete. The blue dotted line represents the f_{NL} generated by the slow-roll fields, ignoring the waterfall. If inflation fails to terminate on the spike this leads to a negligible asymptotic bispectrum. The potential is of ‘valley’ type with $g = 0$, for which the analysis of §3 explains the positive spike. Beginning from $f_{\text{NL}} \sim \epsilon_*$ there is rapid growth to a positive peak, followed by a softer decay. The peak value is well approximated by Eq. (3.18) which yields $f_{\text{NL}} \approx 8$. By varying the initial conditions, we have confirmed that the peak value scales approximately as $1/\chi_*$, as predicted by (3.18).

The solid red line represents a numerical evolution, terminated by a waterfall transition on the growing arm of the spike. In the early stages, the numerical results follow the analytic prediction. We continue the calculation past the end of inflation by allowing the waterfall field σ to become operative. Care must be taken in modelling this transition. We give the waterfall field a small value consistent with the typical RMS value expected from quantum mechanical excitations of a massive field in de Sitter, $\sigma_{\text{RMS}} \approx H^3/M$, where M is representative of the waterfall mass before the transition. The results are extremely insensitive to its precise value.⁵

The hybrid field is heavy at horizon crossing and is therefore unperturbed. Hence, we need not differentiate N with respect to σ . Using these assumptions, we find that f_{NL} appears to be conserved through the hybrid transition (see Fig. 1). The numerical evolution is only continued for a fraction of an e-fold after the transition, during which time f_{NL} *does* evolve due to oscillation of the primary fields. However, the resulting oscillations in f_{NL} are decaying and are centred around a fixed value. This behaviour is apparently generic for a range of parameter values, provided the transition happens sufficiently rapidly—in less than an e-fold. One must already impose this “rapid transition” condition to avoid issues with primordial black holes [83].⁶

5.1.2 Large non-Gaussianity at the natural adiabatic limit

We illustrate this case using a model closely related to the N -axion model of Kim et al. [43], in which the potential is taken to be $V = \sum_i \Lambda_i^4 (1 - \cos 2\pi f_i^{-1} \phi_i)$. The sum is taken over a large number of uncoupled axions, and f_i is the decay constant for the i^{th} axion. We will study further examples of this type in §5.2. In Ref. [43], many axions were invoked to generate a phase of assisted inflation. Because the potential is sum-separable, the perturbations can

⁵In reality, this RMS value is made up of many inhomogeneous short scale modes. Their collective evolution approximates that of a homogeneous mode, at least in the initial stages before the minimum is reached [83]. We expect this approximation captures at least some of the physics which occurs at the hybrid transition.

⁶Note that when the hybrid transition does not occur on a uniform density hypersurface a significant extra contribution to f_{NL} can be generated [81, 82]. We have checked a small number of these cases numerically and find agreement with the formulae given in Ref. [66], although we have only considered examples of positive curvature rather than ridges.

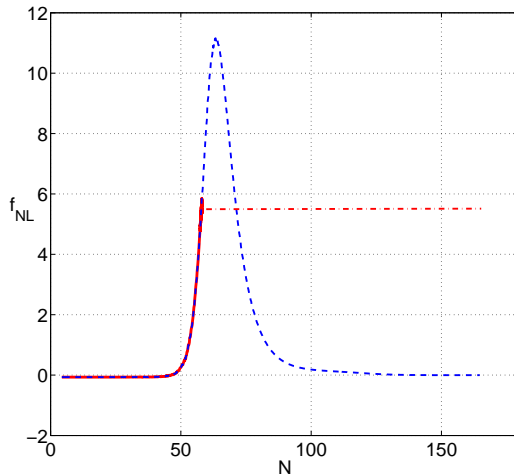


Figure 1. Evolution of f_{NL} for the model (5.1) calculated numerically (solid red line) with the hybrid transition included for the parameter values in the text, the (red) dot-dashed line is added to illustrate the nearly constant final level of f_{NL} . The (blue) dashed line, represents the analytical evolution, with no hybrid transition included.

be calculated at the adiabatic limit using (4.1) after dropping the c -term *provided* inflation ends when the final field gracefully exits from slow-roll. Whether this occurs depends on the number of fields and the choice of f_i . Taking $f_i = f < M_{\text{P}}$ for all i and supposing that the initial conditions are chosen so that only a small number of fields populate the hilltop region near $\phi_i = 0$, the asymptotic f_{NL} can be calculated using (4.8). It will typically be moderate or large.

In this section we study a related two-field model. Dynamically, the large number of axions which begin away from the hilltop region serve only to source the Hubble rate. The single field closest to the hilltop sources the non-Gaussianity. (This model has some similarity to the scenario of Boubekur & Lyth [84].) Therefore, most of the axions can be replaced by a single effective field with a quadratic potential, retaining the full cosine only for the axion closest to the hilltop,

$$V = \frac{1}{2}m^2\phi^2 + \Lambda^4 \left(1 - \cos \frac{2\pi\chi}{f} \right), \quad (5.2)$$

where Λ and f are constants.

Near the hilltop, the axion potential approximately satisfies $V(\chi) = 2\Lambda^4(1 - \pi^2\chi^2/f^2)$. This yields a tachyonic mass $2\pi\Lambda^2/f$. Adjusting the ϕ potential if necessary to ensure that χ remains light at horizon crossing, the mass induces a large f_{NL} via (4.8).

In Figs. 2 and 3 we show a numerical evolution for two choices of parameters. In Fig. 2 we take $f = M_{\text{P}}$ and $\Lambda^4 = 25m^2f^2/(4\pi^2)$, which makes the mass of the axion five times greater than the mass of ϕ . The initial conditions are $\phi_* = 16M_{\text{P}}$ and $\chi_* = (f/2 - 0.001)M_{\text{P}}$. As a consequence of its large mass, the axion rolls off the ridge quite early. Therefore the system evolves to the limiting trajectory long before the end of inflation. According to §3, departure from the ridge should produce a large negative spike. Later, convergence into the minimum

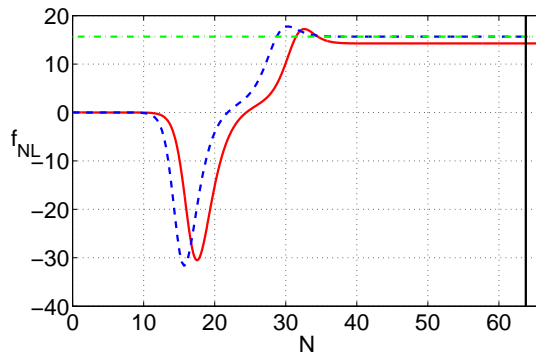


Figure 2. Evolution of f_{NL} for the model of Eq. (5.2) (first set of parameter choices). The solid red line is a numerical calculation. The blue dashed line is an analytic prediction. The horizontal green dashed line represents the analytically calculated adiabatic limiting value.

should produce a positive spike, perhaps followed by a plateau. Finally, as the isocurvature modes are exhausted, the system should evolve to the adiabatic limit (4.8). These features are clearly visible in Fig. 2. Matching the potential (5.2) to the analysis of §3 and using Eq. (3.9), we expect the negative peak of f_{NL} to occur at $f_{\text{NL}} \approx -0.3M_{\text{P}}\epsilon_*^{1/2}(f/2 - \chi_*)^{-1} \approx -26$. This gives good agreement with the observed value. By varying the initial conditions we have verified that scaling with $1/(f/2 - \chi_*)^{-1}$ is reproduced to a good approximation. In this case the difference between our slow-roll analysis and the full numerical calculation is at the level of a few percent, consistent with the accuracy of the slow-roll approximation.

In the second example we take $f = M_{\text{P}}$ and $\Lambda^4 = m^2 f^2 / (4\pi^2)$, giving both fields the

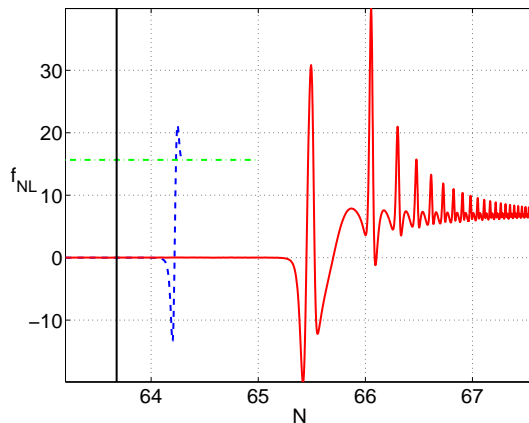


Figure 3. Evolution of f_{NL} for Eq. (5.2) (second set of parameter choices). The solid red line is a numerical calculation. The blue dashed line is an analytic prediction. Only the final few e-folds are shown. The horizontal green dashed line represents the analytically calculated adiabatic limiting value. The solid vertical line indicates when inflation ends, computed using the exact equations of motion. As the axion rolls, inflation momentarily restarts and the slow-roll expressions cease to be a good approximation.

same mass. In this case the axion starts to evolve only near the end of inflation, where ϕ is approaching the minimum. Indeed, much of its evolution takes place while ϕ is oscillating. In these circumstances the adiabatic limit cannot be calculated analytically using (4.1). Nevertheless, the important features can still be understood. While ϕ is oscillating in the minimum, its potential energy contributes to the energy density in a way not accounted for by the slow-roll approximation. If we suppose the ϕ oscillations do not lead to rapid reheating or preheating, we may expect ζ to approach a constant as the trajectories settle in the minimum and Hubble friction drains their energy. The results are given in Fig. 3. In this simple example, f_{NL} oscillates around an asymptotic value which is lower than would be expected if the adiabatic limit were reached during inflation. In more sophisticated examples, where complex dynamical behaviour can occur during the oscillating phase, it would be necessary to follow their decay in precise detail [85–91].

This example is representative of a class of model where natural focusing occurs—in this case, caused simply by Hubble damping—but does so only after the slow-roll assumption is violated. There are other models in this class which lead to a large non-Gaussianity at the natural adiabatic limit, such as models which possess an inflection point in their potential with a very slight gradient. We intend to return to these cases in future work [92].

5.1.3 Models with no reconvergence in field space

The third possibility discussed in §1 occurs when the trajectories disperse in field-space but the potential provides no region which would enable them to refocus. An example is provided by the model

$$V = V_0 \phi^2 e^{-\lambda \chi^2}, \quad (5.3)$$

which was introduced by Byrnes et al. [65]. The dynamics were followed only until the end of slow-roll, at which time the non-Gaussianity was indeed large. However, at this point, the isocurvature modes were not exhausted and the curvature perturbation was still evolving. To study this model, we make the the same parameter choices as Byrnes et al., setting $\lambda = 0.05/M_{\text{P}}^2$, $\phi_i = 16M_{\text{P}}$ and $\chi_i = 0.001M_{\text{P}}$ [65].

The initial stage is descent from a ridge, and therefore we expect f_{NL} to approach negative values. This is confirmed in Figs. 4 and 5. As the bundle rolls along the ridge (defined by $\chi = 0$), we find that Eq. (3.9) gives $f_{\text{NL}} \approx -26$, in good agreement with the first negative peak in f_{NL} . We have confirmed that the expected scaling with initial conditions is approximately respected. With this choice of parameters, a large f_{NL} is still present as slow-roll breaks down. But because no limiting trajectory is available, f_{NL} continues to evolve—and subsequently oscillates wildly, as the fields oscillate about the line $\phi = 0$. This does not represent a stable attractor: $\phi = 0$ is a degenerate vacuum, and the field evolves only along the χ direction during the oscillations. To reach an adiabatic limit we must apply a prescription for reheating. Here, we adopt a very simple perturbative model in which energy

is transferred from the field into a radiation component. The dynamical equations are

$$\ddot{\phi}_i + 3H\dot{\phi}_i = -\Gamma_i\dot{\phi}_i - \frac{\partial W}{\partial \phi_i} \quad (5.4a)$$

$$\dot{\rho} = -4H\rho + \sum_i \Gamma_i \dot{\phi}_i^2, \quad (5.4b)$$

where ρ is energy density of radiation, and the Γ_i represent the decay rate from species i . We illustrate the effect of reheating in Fig. 5. The final value of f_{NL} is sensitive to the choice of Γ_i , and hence the time-scale of reheating. We take $\Gamma_i = \Gamma$ for all i , making reheating begin approximately when $H = \Gamma$ and take place on a uniform density hypersurface. A more complicated prescription leads to strong secondary effects which radically alter the value or sign of f_{NL} . After reheating, if the radiation is the only contribution to the energy density, then the statistics of ζ at this time will be the ones relevant for observation. Fig. 5 indicates that these will depend on microphysical details of the reheating phase, at least through Γ , but a systematic understanding is not yet in place.

Our aim has not been to present a realistic model. Rather, we wish to demonstrate that, if no attractor exists within the inflationary regime, we must follow the dynamics until all observable quantities stop evolving at the adiabatic limit. We can expect the asymptotic value of each observable to be sensitive to this evolution, including the time scale and details of reheating.

5.2 N_f -field models

Similar results naturally apply in models with a larger number of fields. In this section we study the model of Kim et al. [43] involving many axion fields self-interacting through the potentials

$$V_i = \Lambda_i^4 \left(1 - \cos \frac{2\pi\phi_i}{f_i} \right). \quad (5.5)$$

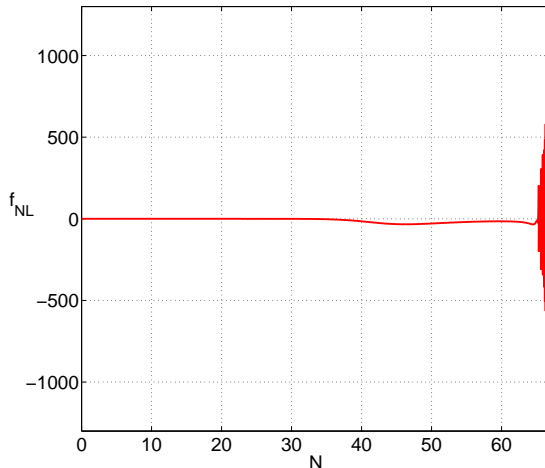


Figure 4. Evolution of f_{NL} for the model (5.3). The solid red line is a numerical calculation for the parameter values quoted in the text and $\Gamma = 0$.

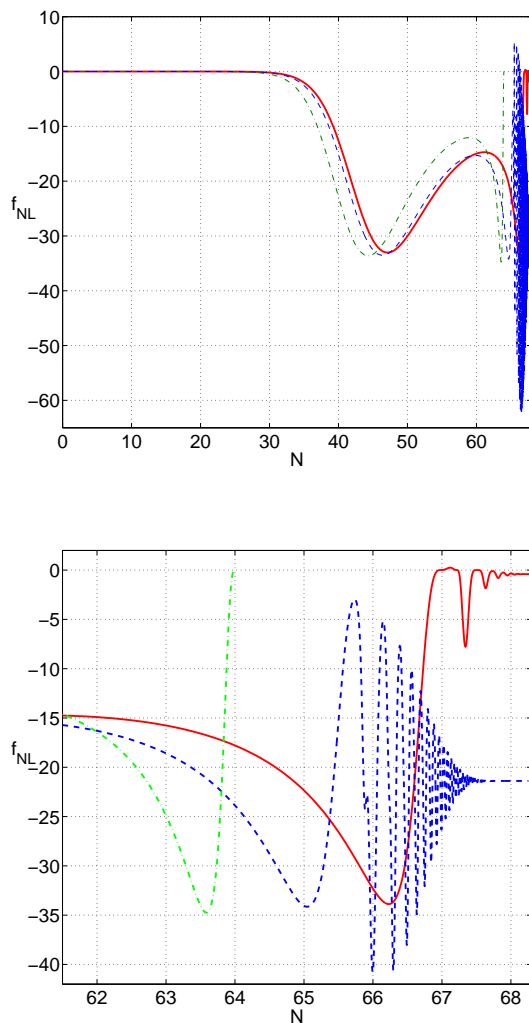


Figure 5. Upper panel: numerical evolution of f_{NL} (solid red line) for the parameter values quoted and $\Gamma = (V_0/10)^{1/2}/M_{\text{P}}$. The blue dashed line represents the corresponding plot with $\Gamma = (V_0/100)^{1/2}/M_{\text{P}}$, and the green dot-dashed line represents the analytical evolution. The analytic evolution terminates when the χ field reaches zero, because the slow-roll expressions can't evolve past this point. Lower panel: magnified in the vicinity of the end of inflation. The asymptotic value of f_{NL} depends on Γ , and therefore on microphysical details of the reheating phase.

Where the parameters Λ_i and f_i take common values Λ and f for each species, and $f \lesssim M_{\text{P}}$, this generates naturally large f_{NL} at the adiabatic limit. Although larger f_{NL} can naïvely be obtained by decreasing the f_i , it is necessary to simultaneously increase the number of fields in order to obtain sufficient inflation. There is another difficulty. As the f_i decrease, the approach of f_{NL} to its asymptotic limit occurs later in the evolution. In Fig. 6 we show one realization of this behaviour for $N_{\text{f}} = 1800$ and $f_i = M_{\text{P}}$, with initial conditions for the fields randomly distributed in the range $0 < \phi_i < \pi M_{\text{P}}$. The slow-roll phase ends at latest when $\epsilon = 1$, marked by the vertical black line. The evolution to the right of this line is not

trustworthy and should be replaced by a numerical calculation. Unfortunately, owing to the large number of fields we have not been able to perform a non-slow roll analysis due to the prohibitive running time of the computation.

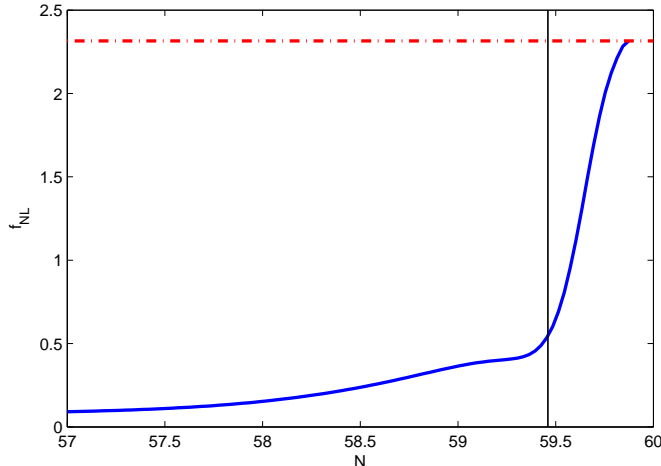


Figure 6. Evolution of f_{NL} for the N_f -axion model, calculated analytically (solid blue line) under the slow-roll approximation. The horizontal dot-dashed red line is the asymptotic value computed using the horizon-crossing approximation. (This is unreliable in the present case.) The vertical black line corresponds to $\epsilon = 1$. Since f_{NL} has not reached the adiabatic limit at this point, this model is an example in which the adiabatic limit is reached after slow-roll ends. Therefore, numerical analysis is required to obtain a reliable value for f_{NL} .

There is a specific case where this model can be related exactly to the two-field axion-plus-quadratic model of the previous section: when $N_f - 1$ fields are initially close to the minimum of the axion potential ($\phi_i \ll f/2$ for $i = 1, \dots, N_f - 1$) with identical initial conditions, and one field is initially close to its maximum, $\phi_{N_f} \approx f/2$. In this case the $N_f - 1$ fields act like a large number of fields with a quadratic potential. When they all evolve from an identical initial condition, the dynamics of the many fields is completely identical to the dynamics of a *single* field $\Phi^2 = \sum_{i=1}^{N_f-1} \phi_i^2$, with a quadratic potential of the same mass as the individual ϕ_i fields. For $f = M_{\text{P}}$, this reduces identically to the second of the two-field axion-plus-quadratic cases studied in §5.1.2. (See Fig. 3.)

Finally, it is interesting to note that the N_f -field axion model—for which a large f_{NL} follows from relatively generic initial conditions—is closely related to a two-field model in which generation of large f_{NL} apparently requires significant fine-tuning. The tuning appears less dramatic in the original N_f -field model. It is interesting to conjecture that the fine-tuning of initial conditions required to give large f_{NL} in two-field models may be reduced in models with many fields.

6 Conclusions

We have studied the evolution of non-Gaussianity in multiple-field models of inflation. Unless all isocurvature modes become exhausted before the end of inflation, we find that there need

not be a unique prediction for f_{NL} . Instead, the final value can depend on independent details, such as the microphysics of a reheating or preheating phase. Where the trajectories naturally focus—for example, if inflation ends with all fields settling into a minimum of the potential—numerical calculations are typically required to determine the precise asymptotic value for f_{NL} . If there is no natural focusing region then numerical calculations *and* a prescription for reheating will be required. This confirms the natural expectation that analytic predictions [38–40, 75] are reliable only if the flow of power from isocurvature to curvature modes is quenched before the end of the slow-roll phase.

If an adiabatic limit is reached without passage through a natural focusing region, perhaps by invoking a waterfall transition, then this may occur when the value of f_{NL} is transiently large. However, we caution that although our numerical calculations indicate that f_{NL} can sometimes be preserved through a hybrid transition, there does not yet appear to be a precise characterization of the conditions required for this to occur. Also, whether the end-point of the waterfall is an adiabatic limit may be model dependent. Temporarily ignoring these subtleties, we have shown that descent from a ridge or convergence into a valley can result in a significant, dynamical but transient enhancement of f_{NL} . The two cases are distinguished by a different sign of the resulting f_{NL} , which is inherited from the local η parameter.

We have verified that it is possible to construct sum-separable models which exhibit large f_{NL} even when the adiabatic limit is reached during slow-roll inflation. Therefore there is no correlation between large f_{NL} at the end of inflation and the presence of an inexhausted isocurvature perturbation, as has occasionally been suggested. On the other hand, we have demonstrated that this is impossible for product separable cases, where f_{NL} is always of order the slow-roll parameters at horizon crossing.

Among the models we have studied is a new two-field model related to the N_f -field axion model [43]. This exhibits large f_{NL} at the adiabatic limit when this limit is reached before the breakdown of slow-roll. In addition, inflation can end gracefully rather than through a sudden transition. As far as we are aware, this is the first example of such behaviour in the two-field context. This model explicitly illustrates that predictions of the N_f -axion model may be modified if a full numerical calculation for a sufficiently large number of fields could be performed.

Acknowledgments

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A Detailed calculations for separable potentials

A.1 Calculating $N_{,i}$

Under the assumption of slow-roll and monotonicity ($\dot{\phi}_k < 0$), the number of e-folds can be written with the field ϕ_k as a time variable. Taking the functional derivative of this integral generates three components—two “boundary terms” evaluated on the initial (‘*’) and final (‘c’) slices, and a “path term”,

$$N_{,i}M_{\text{P}}^2 = \frac{W}{W_{,k}} \Big|_* \delta_{ik} - \frac{W}{W_{,k}} \Big|_c \frac{\partial \phi_k^c}{\partial \phi_i^*} - \int_*^c \frac{\partial}{\partial \phi_i^*} \left(\frac{W}{W_{,k}} \right) d\phi_k. \quad (\text{A.1})$$

The summation convention is not used. Physical quantities are independent of k , which may be chosen arbitrarily. Employing the notation $S = \sum_{i=1}^{N_f} V_i$ and $P = \prod_{i=1}^{N_f} V_i$, where $V_1 = V_1(\phi_1)$ are *dimensionless*, we restrict attention to potentials of the form $W = M_{\text{P}}^4 F(S)$ and $W = M_{\text{P}}^4 G(P)$, where F and G are arbitrary functions. We refer to these as sum- and product-separable potentials respectively.

For the product-separable potential $W = M_{\text{P}}^4 G(P)$ we evaluate the path term and the final boundary term in Eq. (A.1). Using the slow-roll parameters $\sqrt{2\epsilon_i} = M_{\text{P}} G' P V_i' / G V_i$ and also defining $u_i = \epsilon_i / \epsilon$ we find

$$N_{,i} = \frac{1}{M_{\text{P}}^2} \frac{V_i}{V_i'} \Big|_* \left(\frac{G u_i}{G' P} \Big|_c - \int_*^c \frac{\partial}{\partial \phi_i} \left(\frac{G}{G' P} \right) d\phi_i \right). \quad (\text{A.2})$$

This applies for arbitrary G . Note that any dependence on k has disappeared. If the ratio $G/G'P$ depends on P , then the integral requires knowledge of the variation of V_j with ϕ_i . There are at least two cases in which the P -dependence is lost: if the integrand is either zero or constant. If the integrand is zero then we have $G/G'P = A$ for some constant A . This gives the general solution $G = AP^B$, where the constants A and B can be absorbed into a redefinition of the potential. This yields

$$N_{,i} = \frac{1}{M_{\text{P}}} \frac{u_i^c}{\sqrt{2\epsilon_i^*}}, \quad W = M_{\text{P}}^4 P. \quad (\text{A.3})$$

If the integrand is constant this implies $G = (B \ln P + D)^{1/A}$, where A, B, D are constants of which B and D can be eliminated by a further redefinition. We conclude that a general potential yielding a P -independent integrand can be expressed as $W = M_{\text{P}}^4 (\ln P)^{1/A}$ and gives

$$N_{,i} = \frac{A}{M_{\text{P}}^2} \frac{V_i}{V_i'} \Big|_* (\ln V_i^* - \ln V_i^c + (\ln P)^c u_i^c), \quad W = M_{\text{P}}^4 (\ln P)^{1/A}. \quad (\text{A.4})$$

An identical procedure applies to the sum-separable potential, leading to

$$N_{,i} = \frac{1}{M_{\text{P}}} \frac{u_i^c}{\sqrt{2\epsilon_i^*}}, \quad W = M_{\text{P}}^4 e^S \quad (\text{A.5})$$

$$N_{,i} = \frac{A}{M_{\text{P}}^2 V_i'^*} (V_i^* - V_i^c + S^c u_i^c), \quad W = M_{\text{P}}^4 S^{1/A}. \quad (\text{A.6})$$

A.2 Correspondence between different separable potentials

There is a correspondence between Eqs. (A.5) and (A.3), and between Eqs. (A.6) and (A.4) which was first pointed out by Wang [61] for two field potentials. Redefining $\ln V_i \rightarrow V_i$ turns a $W = M_{\text{P}}^4 P$ potential into a $W = M_{\text{P}}^4 e^S$ potential, and redefining $e^{V_i} \rightarrow V_i$ transforms the potential $W = M_{\text{P}}^4 S^{1/A}$ into the form $W = M_{\text{P}}^4 (\ln P)^{1/A}$. Therefore, it is unnecessary to proceed with all four classes of potential. As an independent pair of potentials, we choose $W = M_{\text{P}}^4 P$ and $W = M_{\text{P}}^4 S^{1/A}$.

For $W = M_{\text{P}}^4 P$:

$$\frac{\partial \phi_k^c}{\partial \phi_i^*} = \sqrt{\frac{\epsilon_k^c}{\epsilon_i^*}} (\delta_{ik} - u_i^c) \quad \text{and} \quad N_{,i} = \frac{1}{M_{\text{P}}} \frac{u_i^c}{\sqrt{2\epsilon_i^*}}. \quad (\text{A.7})$$

For $W = M_{\text{P}}^4 S^{1/A}$:

$$\frac{\partial \phi_k^c}{\partial \phi_i^*} = \frac{S^c}{S^*} \sqrt{\frac{\epsilon_k^c}{\epsilon_i^*}} (\delta_{ik} - u_i^c) \quad \text{and} \quad N_{,i} = \frac{1}{M_{\text{P}} S^* \sqrt{2\epsilon_i^*}} (V_i^* - V_i^c + S^c u_i^c). \quad (\text{A.8})$$

These formulae have previously appeared in Refs. [61, 65, 75].

A.3 $N_{,ij}$ and f_{NL}

To calculate higher-order statistics we require the second derivatives $N_{,ij}$. It proves convenient to introduce new dimensionless slow-roll parameters

$$a_i = M_{\text{P}} \frac{\partial}{\partial \phi_i} \ln \left(\frac{W}{M_{\text{P}}^4} \right) = M_{\text{P}} \frac{W_{,i}}{W} = \sqrt{2\epsilon_i} \quad (\text{A.9})$$

$$b_{ij} = M_{\text{P}}^2 \frac{\partial^2}{\partial \phi_i \partial \phi_j} \ln \left(\frac{W}{M_{\text{P}}^4} \right) = M_{\text{P}}^2 \left(\frac{W_{,ij}}{W} - \frac{W_{,i} W_{,j}}{W^2} \right) = \eta_{ij} - 2\sqrt{\epsilon_i \epsilon_j}. \quad (\text{A.10})$$

These are elements of a vector \mathbf{a} and matrix \mathbf{b} respectively. Note that \mathbf{b} is diagonal for the product-separable potential $W = M_{\text{P}}^4 P$ and so we rewrite $N_{,i} = u_i^c / M_{\text{P}} a_i^*$ with $u_i = a_i^2 / |\mathbf{a}|^2$. A simplification occurs if the limiting trajectory is a straight line in field space, which implies

$$f_{\text{NL}}|_{\text{straight}} = -\frac{5}{6} \frac{1}{|\mathbf{q}^c|^4} \sum_{i=1}^{N_f} (q_i^c)^3 \frac{b_{ii}^*}{a_i^*} \quad (\text{A.11})$$

where the vector \mathbf{q}^c has elements $q_i^c = N_{,i}$. If the limiting trajectory lies along the ϕ axis, then $f_{\text{NL}} = -\frac{5}{6} b_{\phi\phi}^*$. In this case, the sign of f_{NL} is given by the mass of the field at horizon crossing but the magnitude is slow-roll suppressed. If we do not make the simplifying assumption that the limiting trajectory is a straight line, then

$$M_{\text{P}}^2 N_{,ij} = \frac{q_i^c}{a_i^*} (2b_{ii}^c - b_{ii}^*) \delta_{ij} - 2q_i^c q_j^c \left(b_{ii} + b_{jj} - \frac{\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2} \right) \Big|_c \quad (\text{A.12})$$

$$\frac{6}{5} f_{\text{NL}} = \frac{1}{|\mathbf{q}^c|^4} \sum_{i=1}^{N_f} \left(\frac{q_i^c{}^3}{a_i^*} (2b_{ii}^c - b_{ii}^*) \right) - 4 \frac{\mathbf{q} \cdot \mathbf{b} \cdot \mathbf{q}}{|\mathbf{q}|^2} \Big|_c + 2 \frac{\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2} \Big|_c. \quad (\text{A.13})$$

For the potential $W = M_{\text{P}}^4 S^{1/4}$ we have $N_{,i} = (V_i^* - V_i^c + S^c u_i^c) / M_{\text{P}} a_i^* S^*$. In the simpler situation where u_i^c is a constant, we find

$$M_{\text{P}}^2 N_{,ij}|_{\text{straight}} = A\delta_{ij} - q_i^c \left(Aa_j^* + \frac{b_{ij}^*}{a_i^*} \right) - A \frac{S_c^2}{S_*^2} \frac{a_i^{c2}}{a_i^* a_j^*} (\delta_{ij} - u_j^c) \quad (\text{A.14})$$

$$f_{\text{NL}}|_{\text{straight}} = \frac{5}{6} \frac{A}{|\mathbf{q}^c|^2} - \frac{5}{6} \frac{1}{|\mathbf{q}^c|^4} \sum_{i,j=1}^{N_f} \left(q_i^{c2} q_j^c \left(Aa_i^* + \frac{b_{ij}^*}{a_i^*} \right) + A \frac{S_c^2}{S_*^2} \frac{a_i^{c2} q_i^c q_j^c}{a_i^* a_j^*} (\delta_{ij} - u_j^c) \right) \quad (\text{A.15})$$

Dropping the assumption that u_i^c is constant leads to additional terms of the form

$$N_{,ij} = N_{,ij}|_{\text{straight}} + \frac{2}{M_{\text{P}}^2} \frac{S_c^2}{S_*^2} \frac{M_{ij}^c}{a_i^* a_j^* |\mathbf{a}^c|^2} \quad (\text{A.16})$$

$$f_{\text{NL}} = f_{\text{NL}}|_{\text{straight}} + \frac{5}{3} \frac{S_c^2}{S_*^2} \frac{1}{|\mathbf{q}^c|^4 |\mathbf{a}^c|^2} \sum_{i,j=1}^{N_f} \frac{M_{ij}^c q_i^c q_j^c}{a_i^* a_j^*} \quad (\text{A.17})$$

$$M_{ij} = a_i b_{ij} a_j - a_i u_j (\mathbf{b} \cdot \mathbf{a})_i - a_j u_i (\mathbf{b} \cdot \mathbf{a})_j + u_i u_j (\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{a}). \quad (\text{A.18})$$

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