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COXETER CATEGORIES AND QUANTUM GROUPS

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Abstract. We define the notion of braided Coxeter category, which is informally a monoidal category carrying compatible, commuting actions of a generalised braid group B_W and Artin's braid groups B_n on the tensor powers of its objects. The data which defines the action of B_W bears a formal similarity to the associativity constraints in a monoidal category, but is related to the coherence of a family of fiber functors. We show that the quantum Weyl group operators of a quantised Kac–Moody algebra $U_\hbar\mathfrak{g}$, together with the universal R–matrices of its Levi subalgebras, give rise to a braided Coxeter category structure on integrable, category $\mathcal{O}$–modules for $U_\hbar\mathfrak{g}$. By relying on the 2–categorical extension of Etingof–Kazhdan quantisation obtained in [3], we then prove that this structure can be transferred to integrable, category $\mathcal{O}$–representations of $\mathfrak{g}$. These results are used in [5] to give a monodromic description of the quantum Weyl group operators of $U_\hbar\mathfrak{g}$, which extends the one obtained by the second author for a semisimple Lie algebra.

Contents

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1. Introduction

1.1. This is the first of a series of three papers the aim of which is to extend the description of the monodromy of the rational Casimir connection of a complex semisimple Lie algebra in terms of quantum Weyl groups obtained in [35, 36, 37, 38] to the case of an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$.

The method we follow is close, in spirit at least, to that of [37]. It relies on the notion of *braided Coxeter category*, the definition of which is the first main contribution of the present article. Informally, such a category is a monoidal category carrying compatible, commuting actions of a given generalised braid group and Artin's braid groups on the tensor products of its objects. This structure arises for example on the category $\mathcal{O}^{\text{int}}_{U_\hbar\mathfrak{g}}$ of integrable, highest weight representations of the quantum group $U_\hbar\mathfrak{g}$, from the quantum Weyl group operators of $U_\hbar\mathfrak{g}$ and the R–matrices of its Levi subalgebras.

A rigidity result, proved in the second paper of this series [4], shows that there is at most one braided Coxeter structure with prescribed restriction functors, R–matrices and local monodromies on the category $\mathcal{O}^{\text{int}}_{\mathfrak{g}}$ of integrable, highest weight representations of $\mathfrak{g}$. It follows that the generalised braid group actions arising from quantum Weyl groups and the monodromy of the Casimir connection [5] are equivalent, provided the braided Coxeter structure underlying the former can be transferred from $\mathcal{O}^{\text{int}}_{U_\hbar\mathfrak{g}}$ to $\mathcal{O}^{\text{int}}_{\mathfrak{g}}$. This result is the second main contribution of this article.

1.2. In the rest of the introduction, we outline the definition of a Coxeter and of a braided Coxeter category. We then focus on two main sources of examples. The first arises from *diagrammatic* Lie bialgebras, and generalises category $\mathcal{O}$ for a symmetrisable Kac–Moody algebra $\mathfrak{g}$. The second arises from diagrammatic Hopf algebras, and generalises category $\mathcal{O}$ for the quantum group $U_\hbar\mathfrak{g}$. Finally, we explain how the Etingof–Kazhdan quantisation of Lie bialgebras [15, 16], and its 2–categorical extension recently obtained in [3] give rise to an equivalence between a canonical deformation of the first class of examples and the second, thus yielding the transfer theorem alluded to above.

1.3. The definition of a Coxeter category bears some formal similarity to that of a braided monoidal category, with Artin's braid groups $\{\mathsf{B}_n\}_{n\geqslant 2}$ replaced by a given generalised braid group B_W of Coxeter type W. If $\mathcal{C}$ is braided monoidal then, for any object $V\in\mathcal{C}$ and $n\geqslant 2$, there is an action

$$\rho_b:\mathsf{B}_n\to\mathsf{Aut}(V_b^{\otimes n})$$

for any bracketing b on the non–associative monomial $x_1\cdots x_n$.[1] The choice of b is in a sense immaterial since, for any two bracketings b,b', the associativity constraint $\Phi_{b'b}:V_b^{\otimes n}\to V_{b'}^{\otimes n}$ of $\mathcal{C}$ intertwines the corresponding actions of B_n. Similarly, if V is an object in a Coxeter category $\mathcal{Q}$, there is an action

$$\lambda_{\mathcal{F}}:\mathsf{B}_W\to\mathsf{Aut}(V_{\mathcal{F}})$$

which depends on a discrete choice $\mathcal{F}$. Moreover, for any two such choices $\mathcal{F},\mathcal{G}$, there is an isomorphism $\Upsilon_{\mathcal{G}\mathcal{F}}:V_{\mathcal{F}}\to V_{\mathcal{G}}$ which intertwines the actions of B_W.

[1] The notation $V_b^{\otimes n}$ indicates that n copies of V have been tensored together according to b. For example, if $b=(x_1x_2)x_3$, $V_b^{\otimes 3}=((V\otimes V)\otimes V)$.

1.4. The relevant discrete choice is that of a *maximal nested set* $\mathcal{F}$ on the Dynkin diagram D of W, a combinatorial notion introduced by De Concini–Procesi [9] which generalises that of a bracketing on $x_1 \cdots x_n$ when W is the symmetric group $\mathfrak{S}_n$ with diagram A_{n-1}. Specifically, to a pair of parentheses $x_1 \cdots (x_i \cdots x_j) \cdots x_n$, one can associate the connected subdiagram of A_{n-1} with nodes $\{i, \ldots, j-1\}$. Under this identification, a (complete) bracketing on $x_1 \cdots x_n$ corresponds to a (maximal) collection $\mathcal{F} = \{B\}$ of connected subdiagrams of A_{n-1} which are pairwise *compatible*, *i.e.,* such that for any $B, B' \in \mathcal{F}$, one has

$$B \subseteq B', \qquad B' \subseteq B \qquad \text{or} \qquad B \perp B'$$

where the latter condition means that B and B' have no vertices in common, and that no edge in A_{n-1} connects a vertex in B to one in B'. Such a collection is called a nested set on A_{n-1}, and may be defined for any Coxeter group, and in fact any diagram D.[2]

1.5. Despite the above formal similarities, there is one significant difference between braided monoidal categories and Coxeter categories. In a Coxeter category $\mathcal{C}$, the braid group B_W does not act by morphism in $\mathcal{C}$. For example, the quantum Weyl group operators do not commute with the action of $U_\hbar\mathfrak{g}$. Thus, B_W does not act through morphism of $\mathcal{C} = \mathrm{Rep}\, U_\hbar\mathfrak{g}$, but rather automorphisms of the forgetful functor $F : \mathrm{Rep}\, U_\hbar\mathfrak{g} \to \mathrm{Vect}$. This is a general feature: in a Coxeter category $\mathcal{C}$, the braid group B_W acts by automorphisms of a fiber functor from $\mathcal{C}$ to a base category $\mathcal{C}_\emptyset$. In fact, $\mathcal{C}$ is endowed with a *collection* of such functors $F_\mathcal{F} : \mathcal{C} \to \mathcal{C}_\emptyset$, labelled by the maximal nested sets on D. For any such $\mathcal{F}$, and object $V \in \mathcal{C}$, there is a homomorphism

$$\lambda_\mathcal{F} : \mathsf{B}_W \to \mathsf{Aut}_{\mathcal{C}_\emptyset}(V_\mathcal{F})$$

where $V_\mathcal{F} = F_\mathcal{F}(V)$. Further, for any $\mathcal{F}, \mathcal{G}$, there is an isomorphism of fiber functors $\Upsilon_{\mathcal{G}\mathcal{F}} : F_\mathcal{F} \Rightarrow F_\mathcal{G}$ which give rise to an identification of B_W–modules $V_\mathcal{F} \to V_\mathcal{G}$.

1.6. In a (braided) Coxeter category, the fiber functors $F_\mathcal{F}$ are additionally required to factorise vertically in the following sense. For any subdiagram $B \subseteq D$, one is given a (braided monoidal) category $\mathcal{C}_B$. In the case of quantum groups, $\mathcal{C}_B$ consists of representations of the subalgebra $U_\hbar\mathfrak{g}_B$ of $U_\hbar\mathfrak{g}$ with generators labelled by the vertices of B. Moreover, for any pair of subdiagrams $B' \subseteq B$, there is a family of (monoidal) functors $F_\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_{B'}$, which can be thought of as restriction functors. These are labelled by maximal nested sets on B relative to B', that is nested sets on B whose elements are compatible with, but not strictly contained in B'.[3] As in the absolute case $B' = \emptyset \subset D = B$ discussed in 1.5, the functors $F_\mathcal{F}$ are related by a transitive family of isomorphisms $\Upsilon_{\mathcal{G}\mathcal{F}} : F_\mathcal{F} \Rightarrow F_\mathcal{G}$. Finally, for any triple of subdiagrams $B'' \subseteq B' \subseteq B$, a maximal nested set $\mathcal{F}$ on B relative to B' and a maximal nested set $\mathcal{F}'$ on B' relative to B'', the composition $F_{\mathcal{F}'} \circ F_\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_{B''}$ is isomorphic to $F_{\mathcal{F}' \cup \mathcal{F}''}$ via a coherent isomorphism.

[2] We use the term diagram to denote an undirected graph, with mo mutiple edges or loops.

[3] If $D = \mathsf{A}_{n-1}$, $B = D$ and B' corresponds to the pair of parentheses $x_1 \cdots x_{i-1} \cdot (x_i \cdots x_j) \cdot x_{j+1} \cdots x_n$, a maximal nested set on B relative to B' consists of a complete bracketing of the monomial $x_1 \cdots x_{i-1} \cdot x_{ij} \cdot x_{j+1} \cdots x_n$.

1.7. Let now $(D, \{m_{ij}\})$ be a labelled diagram with set of vertices $\mathbf{I}$, and B_D the generalised braid group corresponding to D[4] *i.e.*,

$$\mathsf{B}_D = \langle S_i \rangle_{i\in\mathbf{I}} / \underbrace{S_iS_j\ S_i\cdots}_{m_{ij}} = \underbrace{S_jS_iS_j\cdots}_{m_{ij}}$$

For any pair of subdiagrams $B' \subseteq B$ of D, we denote by $\mathsf{Mns}(B, B')$ the collection of maximal nested sets on B relative to B'.

A braided Coxeter category of type D consists of the following five pieces of data.

(1) *Diagrammatic categories.* For any subdiagram $B \subseteq D$, a braided monoidal category $\mathcal{C}_B$.
(2) *Restriction functors.* For any pair of subdiagrams $B' \subseteq B$, and maximal nested set $\mathcal{F}$ on B relative to B', a monoidal functor $F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}$.
(3) *De Concini–Procesi associators.* For any $B' \subseteq B$, and maximal nested sets $\mathcal{F}, \mathcal{G}$ on B relative to B', an isomorphism of monoidal functors

$$\Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}}$$

such that $\Upsilon_{\mathcal{H}\mathcal{G}} \cdot \Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{F}}$ for any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mns}(B, B')$.
(4) *Vertical joins.* For any triple of subdiagrams $B'' \subseteq B' \subseteq B$, and maximal nested sets $\mathcal{F} \in \mathsf{Mns}(B, B')$, $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, an isomorphism of monoidal functors $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}'\cup\mathcal{F}}$, such that the following properties hold
   (a) *Vertical Factorisation.* For any subdiagrams $B'' \subseteq B' \subseteq B$, and maximal nested sets $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}', \mathcal{G}' \in \mathsf{Mns}(B', B'')$

$$\begin{tikzcd}[row sep=huge]
\mathcal{C}_B \arrow[d, bend right=50, "F_{\mathcal{F}}"'] \arrow[d, bend left=50, "F_{\mathcal{G}}"] \arrow[d, phantom, "\Leftarrow\Upsilon_{\mathcal{F}\mathcal{G}}"] \arrow[dd, bend right=90, "F_{\mathcal{F}'\cup\mathcal{F}}"'] \arrow[dd, bend left=90, "F_{\mathcal{G}'\cup\mathcal{G}}"] \\
\mathcal{C}_{B'} \arrow[r, phantom, "\Upsilon_{\mathcal{F}'\cup\mathcal{F}\,\mathcal{G}'\cup\mathcal{G}}"] \arrow[d, bend right=50, "F_{\mathcal{F}'}"'] \arrow[d, bend left=50, "F_{\mathcal{G}'}"] \arrow[d, phantom, "\Leftarrow\Upsilon_{\mathcal{F}'\mathcal{G}'}"] & {} \\
\mathcal{C}_{B''}
\end{tikzcd}$$

   where the triangular faces are given by $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}$ and $\mathsf{a}^{\mathcal{G}}_{\mathcal{G}'}$.
   (b) *Vertical associativity.* For any $B''' \subseteq B'' \subseteq B' \subseteq B$, and maximal nested sets $\mathcal{F} \in \mathsf{Mns}(B, B'), \mathcal{F}' \in \mathsf{Mns}(B', B''), \mathcal{F}'' \in \mathsf{Mns}(B'', B''')$, the following equality holds

$$\mathsf{a}^{\mathcal{F}\cup\mathcal{F}'}_{\mathcal{F}''} \circ \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'\cup\mathcal{F}''} \circ \mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}$$

   as natural transformations $F_{\mathcal{F}''} \circ F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}''}$.

[4]A labelling on D is the additional data of integers $m_{ij} \in \{2, \ldots, \infty\}$ for any two $i \neq j \in \mathbf{I}$ such that $m_{ij} = m_{ji}$ and $m_{ij} = 2$ if $i \perp j$.

(5) *Local monodromies.* For any vertex $i \in D$, an element $S_i^{\mathcal{C}} \in \mathsf{Aut}(F_{\{i\}})$, where $\{i\}$ is the unique element in $\mathsf{Mns}(i, \emptyset)$, satisfying

(a) *Braid relations.* For any $B \subseteq D$, $i \neq j \in B$ and maximal nested sets $\mathcal{F}, \mathcal{G}$ on B with $i \in \mathcal{F}, j \in \mathcal{G}$, the following holds in $\mathsf{Aut}(F_{\mathcal{G}})$

$$\underbrace{\mathsf{Ad}\left(\Upsilon_{\mathcal{GF}}\right)(S_i^{\mathcal{C}}) \cdot S_j^{\mathcal{C}} \cdot \mathsf{Ad}\left(\Upsilon_{\mathcal{GF}}\right)(S_i^{\mathcal{C}}) \cdots}_{m_{ij}} = \underbrace{S_j^{\mathcal{C}} \cdot \mathsf{Ad}\left(\Upsilon_{\mathcal{GF}}\right)(S_i^{\mathcal{C}}) \cdot S_j^{\mathcal{C}} \cdots}_{m_{ij}}$$

where $S_i^{\mathcal{C}}$ is regarded as an automorphism of $F_{\mathcal{F}}$ via the factorisation $\mathsf{a}_{\{i\}}^{\mathcal{F}\setminus\{i\}} : F_{\{i\}} \circ F_{\mathcal{F}\setminus\{i\}} \Rightarrow F_{\mathcal{F}}$, and $S_j^{\mathcal{C}}$ is similarly regarded as an automorphism of $F_{\mathcal{G}}$.

(b) *Coproduct identity.* For any vertex $i \in D$, and $V, W \in \mathcal{C}_i$ the following diagram in $\mathcal{C}_\emptyset$ is commutative

$$\begin{tikzcd}
F_{\{i\}}(V) \otimes F_{\{i\}}(W) \arrow[r, "S_i^{\mathcal{C}} \otimes S_i^{\mathcal{C}}"] \arrow[d, "J_{\{i\}}^{V,W}"'] & F_{\{i\}}(V) \otimes F_{\{i\}}(W) \arrow[r, "c_\emptyset"] & F_{\{i\}}(W) \otimes F_{\{i\}}(V) \arrow[d, "J_{\{i\}}^{W,V}"] \\
F_{\{i\}}(V \otimes W) \arrow[r, "S_i^{\mathcal{C}}"'] & F_{\{i\}}(V \otimes W) \arrow[r, "F_{\{i\}}(c_i)"'] & F_{\{i\}}(W \otimes V)
\end{tikzcd} \tag{1.1}$$

where $J_{\{i\}}$ is the tensor structure on $F_{\{i\}} : \mathcal{C}_i \to \mathcal{C}_\emptyset$, and $c_i, c_\emptyset$ are the opposite braidings in $\mathcal{C}_i$ and $\mathcal{C}_\emptyset$, respectively.[5]

Remarks.

(1) The diagram (1.1) codifies the coproduct identity $\Delta(S_i) = R_i^{21} \cdot S_i \otimes S_i$ satisfied by quantum Weyl group elements [28, Prop. 5.3.4]. It relates the failure of $F_{\{i\}}$ to be a braided monoidal functor and that of $S_i^{\mathcal{C}}$ to be a monoidal isomorphism. That is, if (1.1) is commutative, then $S_i^{\mathcal{C}}$ is monoidal if and only if $F_{\{i\}}$ is braided.

(2) As mentioned in 1.3, the definition of a Coxeter category $\mathcal{C}$ is tailored to produce a family of equivalent representations of B_D. Specifically, there is a collection of homomorphisms $\lambda_{\mathcal{F}} : \mathsf{B}_D \to \mathsf{Aut}(F_{\mathcal{F}})$, labelled by the maximal nested sets on D, which is uniquely determined by
- $\lambda_{\mathcal{F}}(S_i) = S_i^{\mathcal{C}}$ if $i \in \mathcal{F}$.
- $\lambda_{\mathcal{G}} = \mathsf{Ad}(\Upsilon_{\mathcal{GF}}) \circ \lambda_{\mathcal{F}}$, for any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(D)$.

1.8. An important class of braided *pre*–Coxeter categories, that is structures satisfying the axioms (1)–(4) of 1.7 but not necessarily endowed with local monodromies, arises from split *diagrammatic Lie bialgebras*. Recall first that a Lie bialgebra is a triple $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$, where $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ is a Lie algebra and $(\mathfrak{b}, \delta_{\mathfrak{b}})$ a Lie coalgebra such that the cobracket $\delta_{\mathfrak{b}}$ and the bracket $[\cdot,\cdot]_{\mathfrak{b}}$ satisfy an appropriate compatibility condition.

A natural class of representations over a Lie bialgebra $\mathfrak{b}$ is that of *Drinfeld–Yetter modules* [16]. Such a module is a triple (V, π, π^*) such that $\pi : \mathfrak{b} \otimes V \to V$ gives V the structure of a left $\mathfrak{b}$–module, $\pi^* : V \to \mathfrak{b} \otimes V$ that of a right $\mathfrak{b}$–comodule, and π, π^* satisfy a compatibility condition. The latter is designed so as to give rise to a representation of the Drinfeld double $\mathfrak{g}_{\mathfrak{b}} = \mathfrak{b} \oplus \mathfrak{b}^*$ of $\mathfrak{b}$, with $\phi \in \mathfrak{b}^*$ acting on V by $\phi \otimes \mathrm{id}_V \circ \pi^*$.

[5]In a braided monoidal category with braiding β, the opposite braiding is $\beta^{\mathrm{op}}_{X,Y} := \beta^{-1}_{Y,X}$.

If $\mathfrak{b}$ is finite–dimensional, the symmetric monoidal category $\mathsf{DY}_{\mathfrak{b}}$ of such modules coincides in fact with that of $\mathfrak{g}_{\mathfrak{b}}$–modules, with the coaction of $\mathfrak{b}$ on $V \in \mathrm{Rep}(\mathfrak{g}_{\mathfrak{b}})$ given by $\pi^*(v) = \sum_i b_i \otimes b^i v$, where $\{b_i\}, \{b^i\}$ are dual bases of $\mathfrak{b}$ and $\mathfrak{b}^*$. For an arbitrary $\mathfrak{b}$, $\mathsf{DY}_{\mathfrak{b}}$ is isomorphic to the category $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ of *equicontinuous* modules over $\mathfrak{g}_{\mathfrak{b}}$ [15], that is those for which $\mathfrak{b}^*$ acts locally finitely. This makes $\mathsf{DY}_{\mathfrak{b}}$ more convenient to study than $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$.

If $\mathfrak{a} \xrightarrow{i} \mathfrak{b} \xrightarrow{p} \mathfrak{a}$ is a split embedding of Lie bialgebras, there is a tensor restriction functor $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}} : \mathsf{DY}_{\mathfrak{b}} \to \mathsf{DY}_{\mathfrak{a}}$ defined by

$$\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}(V, \pi_V, \pi_V^*) = (V, \pi_V \circ i \otimes \mathrm{id}_V, p \otimes \mathrm{id}_V \circ \pi_V^*)$$

Moreover, if $\mathfrak{a} \hookrightarrow \mathfrak{b} \hookrightarrow \mathfrak{c}$ is a chain of split embeddings, then $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}} \circ \mathrm{Res}_{\mathfrak{b},\mathfrak{c}} = \mathrm{Res}_{\mathfrak{a},\mathfrak{c}}$. In terms of Drinfeld doubles, a split embedding gives rise to an isometric embedding of Lie algebras $j = i \oplus p^t : \mathfrak{g}_{\mathfrak{a}} \to \mathfrak{g}_{\mathfrak{b}}$, and the functor $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}$ corresponds to the pull–back functor j^* from (equicontinuous) modules over $\mathfrak{g}_{\mathfrak{b}}$ to those over $\mathfrak{g}_{\mathfrak{a}}$.

1.9. A *(split) diagrammatic* Lie (bi)algebra $\mathfrak{b}$ over a diagram D is the datum of a family of Lie (bi)algebras $\{\mathfrak{b}_B\}_{B \subseteq D}$ labelled by the subdiagrams of D, together with (split) morphisms $\mathfrak{b}_{B'} \to \mathfrak{b}_B$ for any $B' \subseteq B$. These are assumed to be transitive under compositions $B'' \subseteq B' \subseteq B$, and such that if $B', B'' \subset D$ are orthogonal subdiagrams, $\mathfrak{b}_{B' \sqcup B''}$ is isomorphic to $\mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''}$ as Lie (bi)algebras.

If $\mathfrak{b}$ is a split diagrammatic Lie bialgebra, there is a symmetric pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}}$ defined as follows

(1) For any $B \subseteq D$, $\mathbb{DY}_{\mathfrak{b},B}$ is the symmetric monoidal category $\mathsf{DY}_{\mathfrak{b}_B}$
(2) For any $B' \subseteq B$ and maximal nested set $\mathcal{F}$ on B relative to B', the restriction functor $F_{\mathcal{F}} : \mathbb{DY}_{\mathfrak{b},B} \to \mathbb{DY}_{\mathfrak{b},B'}$ is given by $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$
(3) For any $B' \subseteq B$ and maximal nested sets $\mathcal{F}, \mathcal{G}$ on B relative to B', the associator $\Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}}$ is the identity on $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$
(4) For any $B'' \subseteq B' \subseteq B$, and maximal nested sets $\mathcal{F} \in \mathsf{Mns}(B, B')$, $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, the vertical join $\mathsf{a}_{\mathcal{F}'}^{\mathcal{F}} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}' \cup \mathcal{F}}$ is the equality $\mathrm{Res}_{\mathfrak{b}_{B''},\mathfrak{b}_{B'}} \circ \mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B} = \mathrm{Res}_{\mathfrak{b}_{B''},\mathfrak{b}_B}$.

A deformation of $\mathbb{DY}_{\mathfrak{b}}$, where the restriction functors $F_{\mathcal{F}}$ and associators $\Phi_{\mathcal{G}\mathcal{F}}$ genuinely depend on the choice of maximal nested sets will be outlined in 1.15.

1.10. Semisimple Lie algebras are basic examples of diagrammatic Lie bialgebras. Specifically, let $\mathfrak{g}$ be a complex semisimple Lie algebra, with opposite Borel subalgebras $\mathfrak{b}_{\pm} \subset \mathfrak{g}$, Dynkin diagram D, Serre generators $\{e_i, f_i, h_i\}_{i \in D}$, and standard Lie bialgebra structure determined by $\mathfrak{b}_{\pm}$ and an invariant inner product on $\mathfrak{g}$ (see 11.7). Then, $\mathfrak{g}$ is a diagrammatic Lie bialgebra where, for any $B \subseteq D$, $\mathfrak{g}_B \subseteq \mathfrak{g}$ is the subalgebra generated by $\{e_i, f_i, h_i\}_{i \in B}$.

The diagrammatic structure on $\mathfrak{g}$ determines a split diagrammatic one on $\mathfrak{b}_{\pm}$ as follows. For any $B \subseteq D$, let $\mathfrak{b}_{\pm,B} = \mathfrak{b}_{\pm} \cap \mathfrak{g}_B$ be the subalgebras generated by $\{h_i, e_i\}_{i \in B}$ and $\{h_i, f_i\}_{i \in B}$ respectively. If $B' \subseteq B$, let $i_{\pm,BB'} : \mathfrak{b}_{\pm,B'} \to \mathfrak{b}_{\pm,B}$ be the embedding, and regard its transpose $i_{\pm,BB'}^t$ as a map $p_{\mp,B'B} : \mathfrak{b}_{\mp,B} \to \mathfrak{b}_{\mp,B'}$ via the identifications $\mathfrak{b}_{\mp,C} \cong \mathfrak{b}_{\pm,C}^*$ given by the inner product. Then, $\{i_{\pm,BB'}, p_{\pm,B'B}\}$ give the required splitting of $\mathfrak{b}_{\pm}$.

Taking Drinfeld–Yetter modules gives rise to a symmetric pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}_{\pm}}$, as explained in 1.9. Moreover, the realisation of each $\mathfrak{g}_B$ as a quotient of the Drinfeld double of $\mathfrak{b}_{\pm,B}$ gives rise to an embedding of the pre–Coxeter category of $\mathfrak{g}$–modules with standard restriction functors to $\mathbb{DY}_{\mathfrak{b}_{\pm}}$.

1.11. The above example does not immediately extend to the case of a symmetrisable Kac–Moody algebra $\mathfrak{g}$, however, since $\mathfrak{g}$ need not be diagrammatic (Sect. 12). To remedy this, we introduce the notion of an *extended* Kac–Moody algebra.

Fix an $|\mathbf{I}| \times |\mathbf{I}|$ matrix A with entries in a field k. The extended Kac–Moody algebra $\overline{\mathfrak{g}}(\mathsf{A})$ corresponding to A is the quotient of the Lie algebra generated by $\{e_i, f_i, \alpha_i^\vee, \lambda_i^\vee\}_{i\in\mathbf{I}}$, with relations $[\alpha_i^\vee, \alpha_j^\vee] = 0$, $[\lambda_i^\vee, \lambda_j^\vee] = 0$, $[\alpha_i^\vee, \lambda_j^\vee] = 0$,

$$[\alpha_i^\vee, e_j] = a_{ji} e_j, \qquad [\alpha_i^\vee, f_j] = -a_{ji} f_j, \qquad [\lambda_i^\vee, e_j] = \delta_{ij} e_j, \qquad [\lambda_i^\vee, f_j] = -\delta_{ij} f_j$$

and $[e_i, f_j] = \delta_{ij} h_i$, for any $i, j \in \mathbf{I}$, by the maximal ideal intersecting the span of $\{\alpha_i^\vee, \lambda_i^\vee\}_{i\in\mathbf{I}}$ trivially.

$\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(\mathsf{A})$ is non–canonically a split central extension of the Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(\mathsf{A})$ corresponding to A (12.6). Unlike $\mathfrak{g}$, however, the Lie algebra $\overline{\mathfrak{g}}$ always possesses a diagrammatic structure over the Dynkin diagram D of A, which is given by associating to any $B \subseteq D$ the subalgebra $\overline{\mathfrak{g}}_B \subseteq \overline{\mathfrak{g}}$ generated by $\{e_i, f_i, \alpha_i^\vee, \lambda_i^\vee\}_{i\in B}$. In particular, $\overline{\mathfrak{g}}_B$ is the extended Kac–Moody algebra corresponding to A_B.

If A is symmetrisable, the Borel subalgebras $\overline{\mathfrak{b}}_+, \overline{\mathfrak{b}}_-$ generated by $\{e_i, \alpha_i^\vee, \lambda_i^\vee\}_{i\in\mathbf{I}}$ and $\{f_i, \alpha_i^\vee \lambda_i^\vee\}_{i\in\mathbf{I}}$ respectively, are split diagrammatic Lie bialgebras. Each gives rise to a symmetric pre–Coxeter category $\mathbb{DY}_{\overline{\mathfrak{b}}_\pm}$ with diagrammatic categories $\mathsf{DY}_{\overline{\mathfrak{b}}_{\pm,B}}$, $B \subseteq D$, and, as in 1.10 there is a canonical embedding of the pre–Coxeter category of $\overline{\mathfrak{g}}$–modules with a locally finite $\mathfrak{b}_\mp$–action to $\mathbb{DY}_{\overline{\mathfrak{b}}_\pm}$.

1.12. A quantum analogue of the symmetric pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}}$ can be obtained along similar lines from split diagrammatic Hopf algebras. A Drinfeld–Yetter module over a Hopf algebra $\mathfrak{B}$ is a triple $(\mathcal{V}, \rho, \rho^*)$, where $\rho : \mathfrak{B} \otimes \mathcal{V} \to \mathcal{V}$ is a left $\mathfrak{B}$–module, $\rho^* : \mathcal{V} \to \mathfrak{B} \otimes \mathcal{V}$ a right $\mathfrak{B}$–comodule, and ρ, ρ^* satisfy an appropriate compatibility [39, 16]. Such modules form a braided monoidal category $\mathsf{DY}_{\mathfrak{B}}$, with commutativity constraints $\beta_{\mathcal{U},\mathcal{V}} : \mathcal{U} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{U}$ given by

$$\beta_{\mathcal{U},\mathcal{V}} = (1\,2) \circ \rho_{\mathcal{U}} \otimes \mathrm{id}_{\mathcal{V}} \circ (1\,2) \circ \mathrm{id}_{\mathcal{U}} \otimes \rho^*_{\mathcal{V}}.$$

If $\mathfrak{B}$ is finite–dimensional, the category $\mathsf{DY}_{\mathfrak{B}}$ coincides with that of representations of the quantum double of $D\mathfrak{B}$ of $\mathfrak{B}$ [11]. As a coalgebra, $D\mathfrak{B}$ is the tensor product $\mathfrak{B} \otimes \mathfrak{B}^\circ$, where $\mathfrak{B}^\circ$ is the dual Hopf algebra $\mathfrak{B}^*$ endowed with the opposite coproduct. Moreover, $D\mathfrak{B}$ is endowed with a unique product such that $\mathfrak{B}, \mathfrak{B}^\circ$ are subalgebras, and $D\mathfrak{B}$ is a quasitriangular Hopf algebra, with R–matrix given by the canonical element in $\mathfrak{B} \otimes \mathfrak{B}^\circ$. The isomorphism $\mathsf{DY}_{\mathfrak{B}} \cong \mathrm{Rep}(D\mathfrak{B})$ is obtained by letting $\phi \in \mathfrak{B}^\circ$ act on $\mathcal{V} \in \mathsf{DY}_{\mathfrak{B}}$ by $\phi \otimes \mathrm{id}_{\mathcal{V}} \circ \rho^*$, and conversely defining the coaction of $\mathfrak{B}$ on $\mathcal{V} \in \mathrm{Rep}(D\mathfrak{B})$ by $\rho^* v = R\, 1 \otimes v$.

A similar equivalence holds if $\mathfrak{B}$ is a quantised universal enveloping algebra (QUE), that is a topological Hopf algebra over $\mathsf{k}[\![\hbar]\!]$ such that $\mathfrak{B}/\hbar\mathfrak{B}$ is a universal enveloping algebra $U\mathfrak{b}$. If $\mathfrak{b}$ is finite–dimensional, one can consider the quantised formal group $\mathfrak{B}' \subset \mathfrak{B}$ corresponding to $\mathfrak{B}$ defined in [11, 21], define the dual QUE $\mathfrak{B}^\vee$ as $(\mathfrak{B}')^*$, and the quantum double of $\mathfrak{B}$ as the double crossed product $D\mathfrak{B} = \mathfrak{B} \bowtie \mathfrak{B}^\vee$ introduced in [30]. The latter is a quasitriangular QUE, which quantises the Drinfeld double of $\mathfrak{b}$, with R–matrix given by the canonical element in $\mathfrak{B}' \otimes \mathfrak{B}^\vee \subset D\mathfrak{B}^{\otimes 2}$. The representations of $D\mathfrak{B}$ then coincide, as a braided monoidal category, with the category $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{B}}$ of *admissible* Drinfeld–Yetter modules over $\mathfrak{B}$, that is are those for which the coaction $\rho^* : \mathcal{V} \to \mathfrak{B} \otimes \mathcal{V}$ factors through $\mathfrak{B}' \otimes \mathcal{V}$.

More generally, let $\mathfrak{B} = \bigoplus_{n\geqslant 0} \mathfrak{B}_n$ be an $\mathbb{N}$–graded QUE such that $\mathfrak{B}_0$ deforms $U\mathfrak{b}$ with $\dim \mathfrak{b} < \infty$, and each $\mathfrak{B}_n$ is finitely–generated over $\mathfrak{B}_0$. Then, admissible Drinfeld–Yetter modules over $\mathfrak{B}$ coincide with modules $\mathcal{V}$ over the quantum double of $\mathfrak{B}$ such that the action of $\mathfrak{B}^\vee = \bigoplus_{n\geqslant 0}(\mathfrak{B}' \cap \mathfrak{B}_n)^*$ is locally finite, *i.e.,* such that for any $v \in \mathcal{V}$, $\mathfrak{B}_n^\vee v = 0$ for n large enough (Sect. 6.4).

1.13. As in the case of Lie bialgebras, a split pair $\mathfrak{A} \xrightarrow{i} \mathfrak{B} \xrightarrow{p} \mathfrak{A}$ of Hopf algebras gives rise to a monoidal restriction functor $\mathrm{Res}_{\mathfrak{B},\mathfrak{A}} : \mathsf{DY}_{\mathfrak{B}} \to \mathsf{DY}_{\mathfrak{A}}$ defined by

$$\mathrm{Res}_{\mathfrak{A},\mathfrak{B}}(\mathcal{V}, \rho_{\mathcal{V}}, \rho_{\mathcal{V}}^*) = (\mathcal{V}, \rho_{\mathcal{V}} \circ i \otimes \mathrm{id}_{\mathcal{V}}, p \otimes \mathrm{id}_{\mathcal{V}} \circ \rho_{\mathcal{V}}^*)$$

If both $\mathfrak{A}, \mathfrak{B}$ are finite–dimensional, $\mathrm{Res}_{\mathfrak{A},\mathfrak{B}}$ corresponds to the pullback functor $(i \otimes p^t)^* : \mathrm{Rep}(D\mathfrak{B}) \to \mathrm{Rep}(D\mathfrak{A})$. If both $\mathfrak{A}, \mathfrak{B}$ are QUEs, $\mathrm{Res}_{\mathfrak{A},\mathfrak{B}}$ restricts to a functor $\mathsf{DY}_{\mathfrak{B}}^{\mathrm{adm}} \to \mathsf{DY}_{\mathfrak{A}}^{\mathrm{adm}}$. It follows that if $\mathfrak{B}$ is a diagrammatic QUE, there is a braided pre–Coxeter category $\mathbb{DY}_{\mathfrak{B}}^{\mathrm{adm}}$ with diagrammatic categories $\mathsf{DY}_{\mathfrak{B}_B}^{\mathrm{adm}}$, $B \subseteq D$, restriction functors $\mathrm{Res}_{\mathfrak{B}_{B'},\mathfrak{B}_B}$, $B' \subseteq B$, and trivial associators and vertical joins.

Such an example arises from a quantised extended Kac–Moody algebra algebra $U_\hbar\overline{\mathfrak{g}}$, specifically from the split diagrammatic structure on its quantum Borel subalgebras $U_\hbar\overline{\mathfrak{b}}_\pm$. Moreover, the realisation of $U_\hbar\overline{\mathfrak{g}}$ as a central quotient of the quantum double of $U_\hbar\overline{\mathfrak{b}}_\pm$ yields an embedding of the pre–Coxeter category of $U_\hbar\overline{\mathfrak{g}}$–modules with a locally finite action of $U_\hbar\overline{\mathfrak{b}}_\mp$ into $\mathbb{DY}_{\overline{\mathfrak{b}}_\pm}^{\mathrm{adm}}$. Moreover, once attention is restricted to integrable modules, Lusztig's quantum Weyl group elements extend the structure to that of a braided Coxeter category.

1.14. We now explain how the 2–categorical extension of Etingof–Kazhdan quantisation obtained in [3] yields an equivalence between a deformation of the braided pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}}$ described in 1.9, and its quantum counterpart described in 1.13.

In [15, 16], Etingof and Kazhdan construct a quantisation functor $\mathcal{Q}$ from the category of Lie bialgebras over a field k of characteristic zero to that of QUEs. $\mathcal{Q}$ depends on the choice of an associator Φ, and is compatible with taking Drinfeld–Yetter modules. Specifically, it is endowed with a braided tensor equivalence

$$H_{\mathfrak{b}} : \mathsf{DY}_{\mathfrak{b}}^{\Phi} \longrightarrow \mathsf{DY}_{\mathcal{Q}(\mathfrak{b})}^{\mathrm{adm}} \tag{1.2}$$

where $\mathsf{DY}_{\mathfrak{b}}^{\Phi}$ is the category of deformation Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{b}$, with deformed associativity constraints given by Φ [17, 3]. If $\mathfrak{g}$ is a symmetrisable (extended) Kac–Moody algebra with negative Borel subalgebra $\mathfrak{b}$, this implies in particular the existence of an equivalence between category $\mathcal{O}$ representations of $\mathfrak{g}$ and those of the quantum group $U_\hbar\mathfrak{g}$.

1.15. Assume now that $\mathfrak{b}$ is a split diagrammatic Lie bialgebra. By functoriality, its quantisation $\mathcal{Q}(\mathfrak{b})$ is a split diagrammatic QUE and, by 1.13, there is a braided pre–Coxeter category $\mathbb{DY}_{\mathcal{Q}(\mathfrak{b})}^{\mathrm{adm}}$ with diagrammatic categories $\mathsf{DY}_{\mathcal{Q}(\mathfrak{b}_B)}^{\mathrm{adm}}$, $B \subseteq D$. The equivalence (1.2) then raises the following question: is there a braided pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}}^{\Phi}$ with diagrammatic categories $\mathsf{DY}_{\mathfrak{b}_B}^{\Phi}$, such that the Etingof–Kazhdan equivalences $\{H_{\mathfrak{b}_B}\}_{B\subseteq D}$ fit within an equivalence $H : \mathbb{DY}_{\mathfrak{b}}^{\Phi} \to \mathbb{DY}_{\mathcal{Q}(\mathfrak{b})}^{\mathrm{adm}}$?

This involves in particular constructing, for any $B' \subseteq B$ and maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, a monoidal restriction functor $F_{\mathcal{F}} : \mathsf{DY}_{\mathfrak{b}_B}^{\Phi} \to \mathsf{DY}_{\mathfrak{b}_{B'}}^{\Phi}$, and a natural isomorphism $v_{\mathcal{F}}$ making the following diagram commutative

$$\begin{tikzcd} \mathsf{DY}^{\Phi}_{\mathfrak{b}_B} \arrow[r, "H_{\mathfrak{b}_B}"] \arrow[d, "F_{\mathcal{F}}"'] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_B)} \arrow[d, "\mathrm{Res}_{\mathcal{Q}(\mathfrak{b}_{B'}),\mathcal{Q}(\mathfrak{b}_B)}"] \\ \mathsf{DY}^{\Phi}_{\mathfrak{b}_B} \arrow[r, "H_{\mathfrak{b}_{B'}}"'] \arrow[ur, Rightarrow, "v_{\mathcal{F}}"] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_{B'})} \end{tikzcd} \tag{1.3}$$

In order for the pre–Coxeter category $\mathbb{DY}^{\Phi}_{\mathfrak{b}}$ to fall within the scope of the rigidity theorem proved in [4], we require further that the non–monoidal functor underlying $F_{\mathcal{F}}$ be equal to $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$, which renders the problem non–trivial.[6]

We answer this question in the affermative by relying on the compatibility of Etingof–Kazhdan quantisation with respect to restrictions proved in [3], and get the following (Thms. 10.2 and 10.10)

Theorem. *Let $\mathfrak{b}$ be a split diagrammatic Lie bialgebra, and Φ a Lie associator.*

(1) There is a canonical braided pre–Coxeter category $\mathbb{DY}^{\Phi}_{\mathfrak{b}}$ with the following properties.
- *For any $B \subseteq D$, the diagrammatic category $\mathbb{DY}^{\Phi}_{\mathfrak{b},B}$ is given by $\mathsf{DY}^{\Phi}_{\mathfrak{b}_B}$.*
- *For any $B' \subseteq B$, and $\mathcal{F} \in \mathsf{Mns}(B, B')$, the functor*
$$F_{\mathcal{F}} : \mathsf{DY}^{\Phi}_{\mathfrak{b}_B} \to \mathsf{DY}^{\Phi}_{\mathfrak{b}_{B'}}$$
is of the form $(\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}, J_{\mathcal{F}})$ for some tensor structure $J_{\mathcal{F}}$.
- *For any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathcal{F} \in \mathsf{Mns}(B', B'')$, the composition $F_{\mathcal{F}'} \circ F_{\mathcal{F}}$ is equal to $F_{\mathcal{F}\cup\mathcal{F}'}$ as functors $\mathsf{DY}^{\Phi}_{\mathfrak{b}_B} \to \mathsf{DY}^{\Phi}_{\mathfrak{b}_{B''}}$, and the vertical join $\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}\cup\mathcal{F}'}$ is the identity.*
- *$\mathbb{DY}^{\Phi}_{\mathfrak{b}}$ reduces to $\mathbb{DY}_{\mathfrak{b}}$ mod $\hbar$.*

(2) The Etingof–Kazhdan equivalences $H_{\mathfrak{b}_B} : \mathsf{DY}^{\Phi}_{\mathfrak{b}_B} \to \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_B)}$, fit within an equivalence of braided pre–Coxeter categories $\mathbb{H}_{\mathfrak{b}} : \mathbb{DY}^{\Phi}_{\mathfrak{b}} \to \mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$.

1.16. Recall that the functoriality of Etingof–Kazhdan quantisation is a direct consequence of its realisation in the context of PROPs [16]. Roughly, this consists in obtaining formulae which define a Hopf algebra $\mathcal{Q}([\mathfrak{b}])$ which quantises the *universal Lie bialgebra* $[\mathfrak{b}]$ over k. By definition, the latter is the generating object of a k–linear, symmetric monoidal category $\underline{\mathsf{LBA}}$ endowed with a morphism $[\mathfrak{b}] \otimes [\mathfrak{b}] \to [\mathfrak{b}]$, which is antisymmetric and satisfies the Jacobi identity. The definition of $\underline{\mathsf{LBA}}$ implies that the category of Lie bialgebras over k is equivalent to that of monoidal functors $F : \underline{\mathsf{LBA}} \to \mathrm{Vect}_{\mathsf{k}}$, via the functor mapping F to $F([\mathfrak{b}])$. As a consequence, a quantisation of $[\mathfrak{b}]$ in $\underline{\mathsf{LBA}}$ can be applied to any Lie bialgebra $\mathfrak{b}$, and gives rise to a quantisation functor $\mathfrak{b} \mapsto \mathcal{Q}(\mathfrak{b})$.

An extension of the PROPic definition of Etingof–Kazhdan quantisation plays an even greater role in proving the compatibility of the equivalences $H_{\mathfrak{b}}$ with the restriction functors (cf. (1.3)), as well as proving that the functor $H_{\mathfrak{b}}$ is an equivalence [3] .

[6]Equivalently, we require that the composition $H^{-1}_{\mathfrak{b}_{B'}} \circ \mathrm{Res}_{\mathcal{Q}(\mathfrak{b}_{B'}),\mathcal{Q}(\mathfrak{b}_B)} \circ H_{\mathfrak{b}_B}$ be isomorphic, as a non–monoidal functor, to $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$.

1.17. In a similar vein, the braided pre–Coxeter category $\mathbb{DY}^{\Phi}_{\mathfrak{b}}$ of Theorem 1.15 is constructed through suitable PROPs. To this end, we introduce a universal split diagrammatic Lie bialgebra $[\mathfrak{b}]$ by extending the category $\underline{\mathsf{LBA}}$ by a family of idempotents $\theta_B \in \mathrm{End}([\mathfrak{b}])$ labelled by the subdiagrams of D, which satisfy $\theta_D = \mathrm{id}$,

$$\theta_B \circ \theta_{B'} = \theta_{B'} = \theta_{B'} \circ \theta_B \qquad \text{and} \qquad \theta_{B' \sqcup B''} = \theta_{B'} + \theta_{B''}$$

whenever $B' \subseteq B$ and $B' \perp B''$ respectively. By relying on [3], we then construct a braided pre–Coxeter structure $\mathbb{DY}^{\Phi}_{[\mathfrak{b}]}$ on Drinfeld–Yetter modules over $[\mathfrak{b}]$. This structure gives rise to a braided pre–Coxeter category $\mathbb{DY}^{\Phi}_{\mathfrak{b}}$ for any split diagrammatic Lie bialgebra $\mathfrak{b}$.

Other than its economy, the use of $\mathbb{DY}^{\Phi}_{[\mathfrak{b}]}$ shows that the structure constants of each $\mathbb{DY}^{\Phi}_{\mathfrak{b}}$ are *universal*, that is admit a lift to the algebras of endomorphisms of tensor products of Drinfeld–Yetter modules over $[\mathfrak{b}]$. This feature is a crucial requirement of the rigidity result obtained in [4].

1.18. Finally, we apply these results to an extended symmetrisable Kac–Moody algebra $\overline{\mathfrak{g}}$, with negative Borel subalgebra $\overline{\mathfrak{b}}$ and Dynkin diagram D.

The Drinfeld–Jimbo quantum group $U_\hbar\overline{\mathfrak{b}}$ is a split diagrammatic QUE. As such, it gives rise to a braided pre–Coxeter category $\mathbb{DY}^{\mathrm{adm}}_{U_\hbar\overline{\mathfrak{b}}}$. Consider the subcategories defined as follows.

- $\mathbb{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}} \subset \mathbb{DY}^{\mathrm{adm}}_{U_\hbar\overline{\mathfrak{b}}}$. The diagrammatic category corresponding to $B \subseteq D$ consists of admissible Drinfeld–Yetter modules over $U_\hbar\overline{\mathfrak{b}}_B$ which arise from integrable $U_\hbar\overline{\mathfrak{g}}_B$–modules. Specifically, since $U_\hbar\overline{\mathfrak{g}}_B$ is a quotient of the quantum double of $U_\hbar\overline{\mathfrak{b}}_B$, we require that the action of $D(U_\hbar\overline{\mathfrak{b}}_B)$ factor through an integrable action of $U_\hbar\overline{\mathfrak{g}}_B$.
- $\mathbb{O}^{\mathrm{int}}_{\infty,U_\hbar\overline{\mathfrak{g}}} \subset \mathbb{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}}$. The corresponding diagrammatic categories consist of integrable $U_\hbar\overline{\mathfrak{g}}_B$–modules in category $\mathcal{O}_\infty$.[7]

The quantum Weyl group operators of $U_\hbar\overline{\mathfrak{g}}$ [28] endow $\mathbb{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}}$, and therefore $\mathbb{O}^{\mathrm{int}}_{\infty,U_\hbar\overline{\mathfrak{g}}}$, with the structure of a braided Coxeter category.

The combination of Theorem 1.15 and the isomorphism of diagrammatic Hopf algebras $U_\hbar\overline{\mathfrak{b}} \simeq \mathcal{Q}(\overline{\mathfrak{b}})$ yields our main result.

Theorem. *Let $\overline{\mathfrak{g}}$ be an extended symmetrisable Kac–Moody algebra.*

(1) *There is a universal braided Coxeter category $\mathbb{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}}$ such that*
   - *The diagrammatic category corresponding to $B \subseteq D$ is $\mathcal{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}_B}$.*
   - *The functor $F_{\mathcal{F}} : \mathcal{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}_B} \to \mathcal{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}_{B'}}$ corresponding to $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B,B')$ is the standard restriction functor endowed with an appropriate tensor structure.*
   - *The vertical joins $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}\cup\mathcal{F}'}$ are trivial.*
   - *The underlying braided pre–Coxeter structure is* PROP*ic, and trivial modulo $\hbar$.*

(2) *The Etingof–Kazhdan equivalences $H_{\overline{\mathfrak{b}}_B}$ restrict to equivalences $\mathcal{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}_B} \to \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar\overline{\mathfrak{g}}_B}$, and fit within an equivalence of braided pre–Coxeter categories $\mathbb{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}} \to \mathbb{O}^{\mathrm{int}}_{\infty,U_\hbar\overline{\mathfrak{g}}}$.*

[7]The symbol ∞ refers to the fact that we allow infinite–dimensional weight spaces. This is required by the fact that the restriction corresponding to $\overline{\mathfrak{g}}_{B'} \subset \overline{\mathfrak{g}}_B$ or $U_\hbar\overline{\mathfrak{g}}_{B'} \subset U_\hbar\overline{\mathfrak{g}}_B$ does not preserve the finite–dimensionality of weight spaces if $B' \subsetneq B$.

1.19. **Outline of the paper.** We begin in Section 2 by reviewing a number of combinatorial notions related to diagrams. We lay out the definition of a Coxeter object in an arbitrary 2–category in Section 3, and of a braided Coxeter category in Section 4. In Sections 5 and 6 we produce examples of braided Coxeter categories through Drinfeld–Yetter modules over diagrammatic Lie bialgebras and their quantisations. In Section 7, we introduce a PROP which describes diagrammatic Lie bialgebras. In Sections 8 and 9, we describe in terms of PROPs a universal braided pre–Coxeter structure on the category of Drinfeld–Yetter modules over a diagrammatic Lie bialgebra. In Section 10, we apply the results from [3] to the case of a diagrammatic Lie bialgebra $\mathfrak{b}$. We show in particular that the braided pre–Coxeter structure of the Etingof–Kazhdan quantisation $\mathcal{Q}(\mathfrak{b})$ is equivalent to a universal braided pre–Coxeter structure on the category of Drinfeld–Yetter modules over $\mathfrak{b}$. In Section 11, we review the definition and basic properties of the Kac–Moody algebra associated to an $n\times n$ matrix A. In Section 12, we define extended Kac–Moody algebras which are associated to a (non–minimal) realisation of A of dimension $2n$, and show that they are naturally endowed with a structure of diagrammatic Lie bialgebras. In Section 13, we show that integrable Drinfeld–Yetter modules over an extended quantum group has a natural structure of braided Coxeter category. We then apply the results from Section 10, and obtain the desired transport of the braided Coxeter structure of the quantum group $U_\hbar\mathfrak{g}$ to the category of integrable Drinfeld–Yetter modules for $\mathfrak{g}$. Finally, in Appendix A, we provide an alternative description of the axioms of a Coxeter object in terms of the standard graphical calculus for 2–categories.

1.20. The main results of this paper first appeared in more condensed form in the preprint [2]. The latter is superseded by the present paper, and its companion [3].

2. Diagrams and nested sets

We review in this section a number of combinatorial notions associated to a diagram D, in particular the definition of nested sets on D following [9], and [37, Section 2].

2.1. **Nested sets on diagrams.** A *diagram* is an undirected graph D with no multiple edges or loops. A *subdiagram* $B\subseteq D$ is a full subgraph of D, that is, a graph consisting of a (possibly empty) subset of vertices of D, together with all edges of D joining any two elements of it.

Two subdiagrams $B_1, B_2 \subseteq D$ are *orthogonal* if they have no vertices in common, and no two vertices $i\in B_1$, $j\in B_2$ are joined by an edge in D. We denote by $B_1 \sqcup B_2$ the disjoint union of orthogonal subdiagrams. Two subdiagrams $B_1, B_2\subseteq D$ are *compatible* if either one contains the other or they are orthogonal.

A *nested set* on D is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of D which contains the empty subdiagram and $\mathsf{conn}(D)$, where $\mathsf{conn}(D)$ denotes the set of connected components of D. It is easy to see that the cardinality of any maximal nested set on D is equal to $|D|+1$.

Let $\mathsf{Ns}(D)$ be the set of nested sets on D, and $\mathsf{Mns}(D)$ that of maximal nested sets. Every (maximal) nested set $\mathcal{H}$ on D is uniquely determined by a collection $\{\mathcal{H}_i\}_{i=1}^r$ of (maximal) nested sets on the connected components D_i of D. We therefore obtain canonical identifications

$$\mathsf{Ns}(D) = \prod_{i=1}^{r} \mathsf{Ns}(D_i) \qquad \text{and} \qquad \mathsf{Mns}(D) = \prod_{i=1}^{r} \mathsf{Mns}(D_i).$$

2.2. **Relative nested sets.** If $B' \subseteq B \subseteq D$ are two subdiagrams of D, a nested set on B *relative* to B' is a collection of subdiagrams of B which contains $\mathsf{conn}(B)$ and $\mathsf{conn}(B')$, and in which every element is compatible with, but not properly contained in any of the connected components of B'. We denote by $\mathsf{Ns}(B, B')$ and $\mathsf{Mns}(B, B')$ the collections of nested sets and maximal nested sets on B relative to B'. In particular, $\mathsf{Ns}(B) = \mathsf{Ns}(B, \emptyset)$ and $\mathsf{Mns}(B) = \mathsf{Mns}(B, \emptyset)$.

Relative nested sets are endowed with the following operations, which preserve maximal nested sets.

(1) **Vertical union.** For any $B'' \subseteq B' \subseteq B$, there is an embedding

$$\cup : \mathsf{Ns}(B, B') \times \mathsf{Ns}(B', B'') \to \mathsf{Ns}(B, B''),$$

given by the union of nested sets. Its image is the collection $\mathsf{Ns}_{B'}(B, B'') \subseteq \mathsf{Ns}(B, B'')$ of relative nested sets which contain $\mathsf{conn}(B')$.

(2) **Orthogonal union.** For any $B = B_1 \sqcup B_2$ and $B' = B_1' \sqcup B_2'$ with $B_1' \subseteq B_1$, $B_2' \subseteq B_2$, there is a bijection

$$\mathsf{Ns}(B_1, B_1') \times \mathsf{Ns}(B_2, B_2') \to \mathsf{Ns}(B, B'),$$

mapping $(\mathcal{H}_1, \mathcal{H}_2) \mapsto \mathcal{H}_1 \cup \mathcal{H}_2$.

2.3. **Nested sets and chains of subdiagrams.**

Definition. A *chain* from $B \subseteq D$ to $B' \subseteq B$ is a sequence of subdiagrams

$$\mathbf{C} : B' = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = B.$$

A chain is called *maximal* if $|B_k \setminus B_{k-1}| = 1$ for every k. The sets of chains and maximal chains from B to B' are denoted $\mathsf{Ch}(B, B')$ and $\mathsf{MCh}(B, B')$, respectively.

Note that, unlike the notion of nested set, that of chain is independent of the connectivity of the graph and only depends on the underlying set of vertices. The following is clear.

Lemma. *There is a surjective map* $p : \mathsf{Ch}(B, B') \to \mathsf{Ns}(B, B')$ *given by*

$$p(B' = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = B) = \bigcup_{k=0}^{m} \mathsf{conn}(B_k),$$

Moreover, p *restricts to a surjection* $p : \mathsf{MCh}(B, B') \to \mathsf{Mns}(B, B')$.

The operations defined in 2.2 naturally extend to chains, and it is easy to check that the maps p preserve these operations. In particular,

- **Vertical union.** For any $B'' \subseteq B' \subseteq B$, $\mathbf{C} \in \mathsf{Ch}(B, B')$, and $\mathbf{C}' \in \mathsf{Ch}(B', B'')$, we denote by $\mathbf{C} \cup \mathbf{C}' \in \mathsf{Ch}(B, B'')$ the chain obtained by vertical composition
- **Orthogonal union.** For any $B = B_1 \sqcup B_2$ and $B' = B_1' \sqcup B_2'$ with $B_1' \subseteq B_1$, $B_2' \subseteq B_2$, $\mathbf{C} \in \mathsf{Ch}(B_1 \sqcup B_2, B_1' \sqcup B_2')$, we denote by $\mathbf{C}_{B_k} \in \mathsf{Ch}(B_k, B_k')$, $k = 1, 2$, the chains determined by $\mathbf{C}$ on B_1 and B_2.

Two chains give rise to the same nested set if they differ only at the level of orthogonal subdiagrams. Specifically, if $B_1' \subsetneq B_1 \perp B_2 \supsetneq B_2'$, the chains

$$\begin{aligned} \mathbf{C}_1 &: B_1' \sqcup B_2' \subset B_1 \sqcup B_2' \subset B_1 \sqcup B_2 \\ \mathbf{C} \ &: B_1' \sqcup B_2' \subset B_1 \sqcup B_2 \\ \mathbf{C}_2 &: B_1' \sqcup B_2' \subset B_1' \sqcup B_2 \subset B_1 \sqcup B_2 \end{aligned} \tag{2.1}$$

give rise to the same nested set in $\mathsf{Ns}(B_1 \sqcup B_2, B_1' \sqcup B_2')$. More generally, for any $B' \subseteq B$, we denote by $\mathbf{G}_{B,B'}$ the graph having $\mathsf{Ch}(B, B')$ as set of vertices, and an edge between $\mathbf{C}^1$ and $\mathbf{C}^2$ if and only if their difference is limited to a subchain of the form (2.1) for the same subdiagrams B_1', B_2', B_1, B_2. More precisely, $\mathbf{C}^1$ and $\mathbf{C}^2$ are connected by an edge if and only if $\mathbf{C}^1 \neq \mathbf{C}^2$ and the following holds

- $\mathbf{C}^1 \not\subset \mathbf{C}^2$ and $\mathbf{C}^2 \not\subset \mathbf{C}^1$, $\mathbf{C}^1, \mathbf{C}^2$ are of the same length, there is an index i such that $B_j^1 = B_j^2$, for $j \neq i$, and subdiagrams $B_1' \subsetneq B_1 \perp B_2 \supsetneq B_2'$ such that
$$B_{i+1}^1 = B_1 \sqcup B_2 = B_{i+1}^2$$
$$B_i^1 = B_1 \sqcup B_2' \qquad B_1' \sqcup B_2 = B_i^2$$
$$B_{i-1}^1 = B_1' \sqcup B_2' = B_{i-1}^2$$
- $\mathbf{C}^1 \subset \mathbf{C}^2$, there is an index i such that $B_j^1 = B_j^2$ if $j < i$ and $B_j^1 = B_{j+1}^2$ if $j > i$, $B_1' \subsetneq B_1 \perp B_2 \supsetneq B_2'$ such that
$$\begin{aligned} B_i^1 &= B_1 \sqcup B_2 = B_{i+1}^2 \\ B_1 \sqcup B_2' &= B_i^2 \\ B_{i-1}^1 &= B_1' \sqcup B_2' = B_{i-1}^2 \end{aligned}$$
(and similarly for $\mathbf{C}^2 \subset \mathbf{C}^1$).

The following is straightforward.

Proposition. *The map $p : \mathsf{Ch}(B, B') \to \mathsf{Ns}(B, B')$ descends to a bijection*

$$p : \ \mathsf{Ch}(B, B')/_{\sim} \to \mathsf{Ns}(B, B')$$

where $\sim$ is the equivalence relation defined by the graph $\mathbf{G}_{B,B'}$, i.e., $\mathbf{C} \sim \mathbf{C}'$ if and only if they are connected in $\mathbf{G}_{B,B'}$.

Remark. The map p admits a canonical section $s : \mathsf{Ns}(B, B') \to \mathsf{Ch}(B, B')$ which assigns to a nested set $\mathcal{H}$ the chain $s(\mathcal{H})$ defined recursively as follows

- $s(\mathcal{H})_{\mathrm{top}} = B$
- $s(\mathcal{H})_{k-1}$ is the union of the elements of $\mathcal{H}$ which are properly contained and maximal in $s(\mathcal{H})_k$

Clearly, $p(s(\mathcal{H})) = \mathcal{H}$. Note, however, that s does not preserve the vertical union of nested set. Namely, if $\mathcal{H} \in \mathsf{Ns}(B, B')$ and $\mathcal{H}' \in \mathsf{Ns}(B', B'')$, then in general $s(\mathcal{H}) \cup s(\mathcal{H}') \neq s(\mathcal{H} \cup \mathcal{H}')$. Also, s does not map maximal nested set to maximal chains. Indeed, if $\mathcal{F} \in \mathsf{Mns}(B, B')$, $|s(\mathcal{F})_k \setminus s(\mathcal{F})_{k-1}| = |\mathsf{conn}(s(\mathcal{F})_k)| \geqslant 1$.

3. Coxeter objects

In this section, we define Coxeter objects in an arbitrary 2–category $\mathfrak{X}$.

3.1. **2–Categories.** By definition, a 2–category is a category enriched over Cat, the category of categories, functors and natural transformations [14, 25]. In particular, a 2–category is a special example of a bicategory [6]. The difference between the two notions lies in the composition of 1–morphisms, which is required to be associative up to a prescribed isomorphism in a bicategory, and strictly associative in a 2–category. In particular, a 2–category with one object is a strict (small) monoidal category.

For simplicity, in this section we work with a fixed 2–category $\mathfrak{X}$, though our definitions easily carry over to a bicategory.

3.2. **The diagrammatic 2–category** $\mathsf{Diagr}(\mathfrak{X})$**.** Let $B' \subseteq B$ be two diagrams. If $\mathcal{K} \in \mathsf{Ns}(B, B')$ is a relative nested set, we denote by $\mathsf{Mns}_{\mathcal{K}}(B, B')$ the collection of relative maximal nested sets on B which contain $\mathcal{K}$. If $C_1, \dots, C_m \subseteq B$ are compatible diagrams such that $\mathcal{K} = \mathsf{conn}(C_1) \cup \dots \cup \mathsf{conn}(C_m)$ is a relative nested set in $\mathsf{Ns}(B, B')$, we abbreviate $\mathsf{Mns}_{\mathcal{K}}(B, B')$ to $\mathsf{Mns}_{\{C_1,\dots,C_m\}}(B, B')$.

Definition. The diagrammatic category $\mathsf{Diagr}(\mathfrak{X})$ is the following 2–category

(1) If B is a diagram, a *B–object* is an object $\mathcal{C}_B$ in $\mathfrak{X}$ labelled by B.
(2) If $B' \subseteq B$ are diagrams, $\mathcal{C}_B$ a B–object, $\mathcal{C}_{B'}$ a B'–object, and $\mathcal{K} \in \mathsf{Ns}(B, B')$, a *diagrammatic* 1*–morphism* $\mathcal{C} \to \mathcal{C}'$ *of degree* $\mathcal{K}$ is the datum of
  - for any $\mathcal{F} \in \mathsf{Mns}_{\mathcal{K}}(B, B')$, a 1–morphism $F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}$
  - for any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}_{\mathcal{K}}(B, B')$, a 2–isomorphism $\Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}}$

  such that the morphisms Υ are transitive, *i.e.*, for any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mns}_{\mathcal{K}}(B, B')$,
$$\Upsilon_{\mathcal{H}\mathcal{G}} \circ \Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{F}}$$
  This implies in particular that $\Upsilon_{\mathcal{F}\mathcal{F}} = \mathrm{id}_{F_{\mathcal{F}}}$, and that $\Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon_{\mathcal{F}\mathcal{G}}^{-1}$ for any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}_{\mathcal{K}}(B, B')$. We denote the collection of 1–morphisms $\mathcal{C}_B \to \mathcal{C}_{B'}$ of degree $\mathcal{K}$ by $\mathrm{Hom}(\mathcal{C}_B, \mathcal{C}_{B'})[\mathcal{K}]$, and set[8]
$$\mathsf{Diagr}(\mathfrak{X})(\mathcal{C}_B, \mathcal{C}_{B'}) = \bigsqcup_{\mathcal{K} \in \mathsf{Ns}(B,B')} \mathsf{Hom}(\mathcal{C}_B, \mathcal{C}_{B'})[\mathcal{K}]$$
(3) If $B'' \subseteq B' \subseteq B$ are encased diagrams, $\mathcal{C}_B, \mathcal{C}_{B'}, \mathcal{C}_{B''}$ are B, B', and B''–objects, $\mathcal{K} \in \mathsf{Ns}(B, B')$ and $\mathcal{K}' \in \mathsf{Ns}(B', B'')$, the composition of 1–morphisms
$$F : \mathcal{C}_B \to \mathcal{C}_{B'} \qquad \text{and} \qquad F' : \mathcal{C}_{B'} \to \mathcal{C}_{B''}$$
  of degrees $\mathcal{K}$ and $\mathcal{K}'$ is a 1–morphism $F' \circ F : \mathcal{C}_B \to \mathcal{C}_{B''}$ of degree $\mathcal{K} \cup \mathcal{K}' \in \mathsf{Ns}(B, B'')$. Specifically, if $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}_{\mathcal{K}\cup\mathcal{K}'}(B, B'')$, the 1– and 2–morphisms
$$F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B''} \qquad \text{and} \qquad \Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{G}} \Rightarrow F_{\mathcal{F}}$$
  corresponding to $F' \circ F$ are given by the composition $F'_{\mathcal{F}_{B''B'}} \circ F_{\mathcal{F}_{B'B}}$ and the vertical composition $\begin{array}{c} \Upsilon_{\mathcal{G}_{B'B}\mathcal{F}_{B'B}} \\ \Upsilon'_{\mathcal{G}_{B''B'}\mathcal{F}_{B''B'}} \end{array}$ respectively.[9]

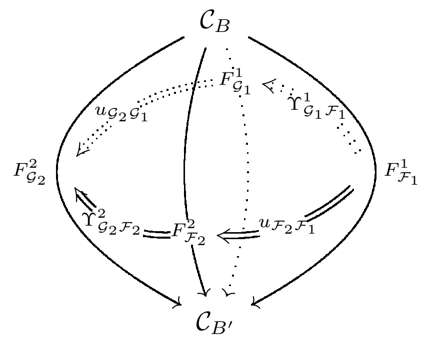
[8]Note that if $\mathcal{K}_1 \subseteq \mathcal{K}_2 \in \mathsf{Ns}(B, B')$ then $\mathsf{Mns}_{\mathcal{K}_1}(B, B') \supseteq \mathsf{Mns}_{\mathcal{K}_2}(B, B')$, and there is a forgetful map $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')[\mathcal{K}_1] \to \mathrm{Hom}(\mathcal{C}, \mathcal{C}')[\mathcal{K}_2]$

[9]Note that the composition $F' \circ F$ forgets some of the data of F, namely the 1–morphisms $F_{\mathcal{F}}$ and 2–morphisms $\Upsilon_{\mathcal{F}\mathcal{G}}$ corresponding to $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}_{\mathcal{K}}(B, B'') \setminus \mathsf{Mns}_{\mathcal{K}\cup\mathcal{K}'}(B, B'')$.

(4) If $F^1, F^2 : \mathcal{C}_B \to \mathcal{C}_{B'}$ are 1–morphisms of degrees $\mathcal{K}_1, \mathcal{K}_2 \in \mathsf{Ns}(B, B')$ respectively, a *diagrammatic 2–morphism* $u : F^1 \Rightarrow F^2$ is the datum, for any $\mathcal{F}_1 \in \mathsf{Mns}_{\mathcal{K}_1}(B, B')$ and $\mathcal{F}_2 \in \mathsf{Mns}_{\mathcal{K}_2}(B, B')$, of a 2–morphism $u_{\mathcal{F}_2\mathcal{F}_1} : F^1_{\mathcal{F}_1} \Rightarrow F^2_{\mathcal{F}_2}$ in $\mathfrak{X}$ such that, for any $\mathcal{F}_1, \mathcal{G}_1 \in \mathsf{Mns}_{\mathcal{K}_1}(B)$ and $\mathcal{F}_2, \mathcal{G}_2 \in \mathsf{Mns}_{\mathcal{K}_2}(B)$,

$$u_{\mathcal{G}_2\mathcal{G}_1} \circ \Upsilon^1_{\mathcal{G}_1\mathcal{F}_1} = \Upsilon^2_{\mathcal{G}_2\mathcal{F}_2} \circ u_{\mathcal{F}_2\mathcal{F}_1} \tag{3.1}$$

as 2–morphisms $F^1_{\mathcal{F}_1} \Rightarrow F^2_{\mathcal{G}_2}$. This amounts to the commutativity of



(3.2)

If D is a fixed diagram, we denote by $\mathsf{Diagr}_D(\mathfrak{X}) \subset \mathsf{Diagr}(\mathfrak{X})$ the full 2–subcategory of B–objects, where $B \subseteq D$.

3.3. **Pre–Coxeter objects.** Let D be a diagram.

Definition. A *pre–Coxeter object* of type D in $\mathfrak{X}$ is the datum of

- for any $B \subseteq D$, a B–object $\mathcal{C}_B$
- for any $B' \subseteq B$, a diagrammatic 1–morphism $F_{B'B} : \mathcal{C}_B \to \mathcal{C}_{B'}$ of minimal degree $\mathcal{K} = \mathsf{conn}(B) \cup \mathsf{conn}(B')$
- for any $B'' \subseteq B' \subseteq B$, a diagrammatic 2–isomorphism

$$\begin{tikzcd} & \mathcal{C}_{B'} \arrow[dl, "F_{B''B'}"'] \arrow[d, phantom, "\Downarrow \alpha^{B'}_{B''B}"] & \\ \mathcal{C}_{B''} & & \mathcal{C}_B \arrow[ll, "F_{B''B}"] \arrow[ul, "F_{B'B}"'] \end{tikzcd} \tag{3.3}$$

such that

- for any $B' \subseteq B$, $F_{BB} = \mathrm{id}_{\mathcal{C}_B}$ and $\alpha^{B'}_{BB'} = \mathrm{id}_{F_{B'B}} = \alpha^{B}_{BB'}$.
- the 2–morphisms α are associative, *i.e.*, for any $B''' \subseteq B'' \subseteq B' \subseteq B$, the following tetrahedron in $\mathsf{Diagr}_D(\mathfrak{X})$ with 2–faces given by the morphisms α is commutative

$$\begin{tikzcd} \mathcal{C}_{B'''} & & \mathcal{C}_{B''} \arrow[ll, "F_{B'''B''}"'] & \\ & & & \mathcal{C}_{B'} \arrow[ul, "F_{B''B'}"'] \arrow[lllu, dotted, "F_{B'''B'}"] \\ & \mathcal{C}_B \arrow[uu, phantom] \arrow[luu, "F_{B'''B}"] \arrow[ruu, "F_{B''B}"] \arrow[rru, "F_{B'B}"'] & & \end{tikzcd} \tag{3.4}$$

In other words, the following equality holds
$$\alpha^{B''}_{B'''B}\circ\alpha^{B'}_{B''B}=\alpha^{B'}_{B'''B}\circ\alpha^{B''}_{B'''B'}$$
as 2–isomorphisms $F_{B'''B''}\circ F_{B''B'}\circ F_{B'B}\Rightarrow F_{B'''B}$.

3.4. **Unfolding the definition.** We give below a more hands–on description of a pre–Coxeter object, which will be used throughout this paper to construct examples.

Proposition. *A pre–Coxeter object of type D in $\mathfrak{X}$ is equivalently described by the datum of*

- *for any $B\subseteq D$, an object $\mathcal{C}_B\in\mathfrak{X}$*
- *for any $B'\subseteq B$ and $\mathcal{F}\in\mathsf{Mns}(B,B')$, a 1–morphism $F_{\mathcal{F}}:\mathcal{C}_B\to\mathcal{C}_{B'}$*
- *for any $B'\subseteq B$ and $\mathcal{F},\mathcal{G}\in\mathsf{Mns}(B,B')$, a 2–isomorphism $\Upsilon_{\mathcal{G}\mathcal{F}}:F_{\mathcal{F}}\Rightarrow F_{\mathcal{G}}$*
- *for any $B''\subseteq B'\subseteq B$, $\mathcal{F}\in\mathsf{Mns}(B,B')$ and $\mathcal{F}'\in\mathsf{Mns}(B',B'')$, a 2–isomorphism $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}:F_{\mathcal{F}'}\circ F_{\mathcal{F}}\Rightarrow F_{\mathcal{F}'\cup\mathcal{F}}$.*

such that

1. *if $\mathcal{F}$ and $\mathcal{F}'$ are the unique elements in $\mathsf{Mns}(B,B)$ and $\mathsf{Mns}(B',B')$, respectively, and $\mathcal{G}\in\mathsf{Mns}(B,B')$, then $F_{\mathcal{F}}=\mathrm{id}_{\mathcal{C}_B}$, $F_{\mathcal{F}'}=\mathrm{id}_{\mathcal{C}_{B'}}$ and $\mathsf{a}^{\mathcal{F}}_{\mathcal{G}}=\mathrm{id}_{F_{\mathcal{G}}}=\mathsf{a}^{\mathcal{G}}_{\mathcal{F}'}$*
2. *the 2–isomorphisms Υ are transitive, i.e., for any $\mathcal{F},\mathcal{G},\mathcal{H}\in\mathsf{Mns}(B,B')$, $\Upsilon_{\mathcal{F}\mathcal{G}}\circ\Upsilon_{\mathcal{G}\mathcal{H}}=\Upsilon_{\mathcal{F}\mathcal{H}}$*
3. *the 2–isomorphism a are associative, i.e., for any $B'''\subseteq B''\subseteq B'\subseteq B$, and maximal nested sets $\mathcal{F}\in\mathsf{Mns}(B,B'),\mathcal{F}'\in\mathsf{Mns}(B',B''),\mathcal{F}''\in\mathsf{Mns}(B'',B''')$, the following holds*
$$\mathsf{a}^{\mathcal{F}\cup\mathcal{F}'}_{\mathcal{F}''}\circ\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}=\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'\cup\mathcal{F}''}\circ\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}$$
as 2–morphisms $F_{\mathcal{F}''}\circ F_{\mathcal{F}'}\circ F_{\mathcal{F}}\Rightarrow F_{\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}''}$.
4. *for any $B''\subseteq B'\subseteq B$, $\mathcal{F},\mathcal{G}\in\mathsf{Mns}(B,B')$, and $\mathcal{F}',\mathcal{G}'\in\mathsf{Mns}(B',B'')$,*
$$\Upsilon_{\mathcal{F}'\cup\mathcal{F},\mathcal{G}'\cup\mathcal{G}}\circ\mathsf{a}^{\mathcal{G}}_{\mathcal{G}'}=\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}\circ\begin{array}{c}\Upsilon_{\mathcal{F}\mathcal{G}}\\ \\ \Upsilon_{\mathcal{F}'\mathcal{G}'}\end{array}$$

Proof. First, we show that any pre–Coxeter object $(\mathcal{C},F,\alpha)$ gives rise to the datum described above.

By definition, $F_{B'B}:\mathcal{C}_B\to\mathcal{C}_{B'}$ is a diagrammatic 1–morphism, *i.e.,* it amounts to a collection of 1–morphisms $F_{\mathcal{F}}:\mathcal{C}_B\to\mathcal{C}_{B'}$ and 2–isomorphisms $\Upsilon_{\mathcal{G}\mathcal{F}}:F_{\mathcal{F}}\to F_{\mathcal{G}}$, labeled by $\mathcal{F},\mathcal{G}\in\mathsf{Mns}(B,B')$ and satisfying (1).

The diagrammatic 2–isomorphism $\alpha^{B'}_{B''B}$ amounts to a collection of 2–isomorphisms

$$\begin{tikzcd}
\mathcal{C}_B \arrow[dr, "F_{\mathcal{G}'}"] \arrow[dd, "F_{\mathcal{F}}"'] & \\
\arrow[r, phantom, "\Leftarrow\alpha^{\mathcal{G}'}_{\mathcal{F}\mathcal{G}''}"] & \mathcal{C}_{B'} \arrow[dl, "F_{\mathcal{G}''}"] \\
\mathcal{C}_{B''} &
\end{tikzcd}$$

labelled by $\mathcal{F}\in\mathsf{Mns}(B,B'')$, $\mathcal{G}'\in\mathsf{Mns}(B,B')$ and $\mathcal{G}''\in\mathsf{Mns}(B',B'')$, satisfying the compatibility condition (3.1) and (3.2). We set $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}\coloneqq a_{\mathcal{F}'\cup\mathcal{F},\mathcal{F}'}^{\mathcal{F}}$. Then, the condition (3.4), encoding the associativity of the morphisms α (3.4), clearly implies (2). Then, it follows from (3.2) (with $\mathcal{F}_1=\mathcal{G}'\cup\mathcal{G}=\mathcal{G}_1$ and $\mathcal{F}_2=\mathcal{G}'\cup\mathcal{G}$, $\mathcal{G}_2=\mathcal{F}$) that $a_{\mathcal{F}\mathcal{G}'}^{\mathcal{G}}=\Upsilon_{\mathcal{F},\mathcal{G}'\cup\mathcal{G}}\circ\mathsf{a}^{\mathcal{G}}_{\mathcal{G}'}$. This implies that α is completely determined by the 2–isomorphisms $\{\mathsf{a}^{\mathcal{G}}_{\mathcal{G}'}\}_{\mathcal{G},\mathcal{G}'}$. Finally, the condition (3) follows directly from (3.2), by choosing $\widetilde{\mathcal{F}} \in \mathsf{Mns}_{B'}(B, B'')$ and setting $\mathcal{F} = \widetilde{\mathcal{F}}_{B'B} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}' = \widetilde{\mathcal{F}}_{B''B'} \in \mathsf{Mns}(B', B'')$. The converse is proved similarly. $\square$

3.5. **The 2–categories $\mathfrak{P}(D)$ and $\mathfrak{Ns}(D)$.** We give below a succinct definition of a pre–Coxeter object as a 2–functor to the diagrammatic category $\mathsf{Diagr}_D(\mathfrak{X})$.

Let $\mathfrak{P}(D)$ be the 2–category where

- the objects are the subdiagrams of D
- the 1–morphisms $B \to B'$ are the inclusions $B' \subseteq B$
- the 2–morphisms are equalities

Consider also the 2–category $\mathfrak{Ns}(D)$ where

- the objects are the subdiagrams of D
- the 1–morphisms $B \to B'$ are the relative nested sets $\mathcal{K} \in \mathsf{Ns}(B, B')$, with composition given by union
- for any $\mathcal{K}_1, \mathcal{K}_2 \in \mathsf{Ns}(B, B')$, there is a unique 2–isomorphism $\mathcal{K}_1 \to \mathcal{K}_2$

There is a forgetful 2–functor $\mathsf{f}_D : \mathfrak{Ns}(D) \to \mathfrak{P}(D)$, which is the identity on objects, maps all 1–morphisms in $\mathsf{Ns}(B, B')$ to the inclusion $B' \subseteq B$, and the 2–morphisms to the identity. f_D has a canonical section $\mathsf{s}_D : \mathfrak{P}(D) \to \mathfrak{Ns}(D)$, which maps the inclusion $B' \subseteq B$ to $\mathcal{K}_{\min} = \mathsf{conn}(B) \cup \mathsf{conn}(B') \in \mathsf{Ns}(B, B')$.[10]

Consider now the 2–functor $\mathsf{f}_{D,\mathfrak{X}} \colon \mathsf{Diagr}_D(\mathfrak{X}) \to \mathfrak{Ns}(D)$, which maps a B–object to the underlying diagram $B \subseteq D$, and a 1–morphism $\mathcal{C}_B \to \mathcal{C}_{B'}$ to its degree in $\mathsf{Ns}(B, B')$. Then, a pre–Coxeter object in $\mathfrak{X}$ is a (pseudo) 2–functor $\mathcal{C} : \mathfrak{P}(D) \to \mathsf{Diagr}_D(\mathfrak{X})$ such that $\mathsf{f}_{D,\mathfrak{X}} \circ \mathcal{C} = \mathsf{s}_D$, that is

$$\begin{tikzcd} \mathfrak{P}(D) \arrow[r, "\mathcal{C}"] \arrow[dr, "\mathsf{s}_D"'] & \mathsf{Diagr}_D(\mathfrak{X}) \arrow[d, "\mathsf{f}_{D,\mathfrak{X}}"] \\ & \mathfrak{Ns}(D) \end{tikzcd} \tag{3.5}$$

3.6. **Morphisms.** A *1–morphism* $\mathcal{C} \to \mathcal{C}'$ of pre–Coxeter objects in $\mathfrak{X}$ is a natural transformation of the corresponding functors $\mathfrak{P}(D) \to \mathsf{Diagr}_D(\mathfrak{X})$, which is compatible with (3.5). Concretely, this consists of the datum of

- for any $B \subseteq D$, a diagrammatic 1–morphism $H_B : \mathcal{C}_B \to \mathcal{C}'_B$
- for any $B' \subseteq B$, a diagrammatic 2–isomorphism

$$\begin{tikzcd} \mathcal{C}_B \arrow[r, "H_B"] \arrow[d, "F_{B'B}"'] & \mathcal{C}'_B \arrow[d, "F'_{B'B}"] \arrow[dl, phantom, "\Swarrow\gamma_{B'B}"] \\ \mathcal{C}_{B'} \arrow[r, "H_{B'}"'] & \mathcal{C}'_{B'} \end{tikzcd}$$

[10]Note that s_D is technically a pseudo 2–functor, since it preserves the composition only up to a coherent 2–isomorphism. Namely, for any $B'' \subseteq B' \subseteq B$, set $\mathcal{K} = \mathsf{conn}(B) \cup \mathsf{conn}(B')$, $\mathcal{K}' = \mathsf{conn}(B') \cup \mathsf{conn}(B'')$ and $\mathcal{K}'' = \mathsf{conn}(B) \cup \mathsf{conn}(B'')$. Then, the 2–isomorphism $\mathcal{K}' \cup \mathcal{K} \to \mathcal{K}''$ in $\mathfrak{Ns}(D)$ gives an identification$\mathsf{s}_D(B' \to B'') \circ \mathsf{s}_D(B \to B') \to \mathsf{s}_D(B \to B'')$.

such that the morphisms γ factorise vertically, *i.e.*, for any $B'' \subseteq B' \subseteq B$, the following prism in $\mathsf{Diagr}_D(\mathfrak{X})$ is commutative

$$
\begin{tikzcd}
\mathcal{C}_B \arrow[rr, "H_B"] \arrow[dr, "F_{B'B}"] \arrow[dd, "F_{B''B}"'] & & \mathcal{C}'_B \arrow[dr, "F_{B'B}"] \arrow[dd, dotted, "F_{B''B}"] & \\
& \mathcal{C}_{B'} \arrow[rr, "H_{B'}"] \arrow[dl, "F_{B''B'}"] & & \mathcal{C}'_{B'} \arrow[dl, "F_{B''B'}"] \\
\mathcal{C}_{B''} \arrow[rr, "H_{B''}"'] & & \mathcal{C}'_{B''} &
\end{tikzcd}
$$

where the rectangular 2–faces are the morphisms γ, and the triangular ones the morphisms α, α'.

Remark. In view of 3.4, a 1–morphism of pre–Coxeter object $\mathcal{C} \to \mathcal{C}'$ is equivalently described as the datum of

- for any $B \subseteq D$, a 1–morphism $H_B : \mathcal{C}_B \to \mathcal{C}'_B$
- for any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, a 2–isomorphism $\gamma_{\mathcal{F}} : F'_{\mathcal{F}} \circ H_B \Rightarrow H_{B'} \circ F_{\mathcal{F}}$

such that, for any $B'' \subseteq B' \subseteq B$ and $\mathcal{F}' \in \mathsf{Mns}(B, B')$, $\mathcal{F}'' \in \mathsf{Mns}(B', B'')$ and $\mathcal{F} \in \mathsf{Mns}(B, B'')$, the following prism

$$
\begin{tikzcd}
\mathcal{C}_B \arrow[rr, "H_B"] \arrow[dr, "F_{\mathcal{F}'}"] \arrow[dd, "F_{\mathcal{F}}"'] & & \mathcal{C}'_B \arrow[dr, "F'_{\mathcal{F}'}"] \arrow[dd, dotted, "F'_{\mathcal{F}}"] & \\
& \mathcal{C}_{B'} \arrow[rr, "H_{B'}"] \arrow[dl, "F_{\mathcal{F}''}"] & & \mathcal{C}'_{B'} \arrow[dl, "F'_{\mathcal{F}''}"] \\
\mathcal{C}_{B''} \arrow[rr, "H_{B''}"'] & & \mathcal{C}'_{B''} &
\end{tikzcd}
$$

where the rectangular 2–faces are the morphisms γ, and the triangular ones the morphisms a, a'. Note also that, if $B' = B''$, for $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, one just get $\Upsilon_{\mathcal{F}\mathcal{G}} \circ \gamma_{\mathcal{G}} = \gamma_{\mathcal{F}} \circ \Upsilon'_{\mathcal{F}\mathcal{G}}$.

If $H^1, H^2 : \mathcal{C} \to \mathcal{C}'$ are 1–morphisms of pre–Coxeter objects in $\mathfrak{X}$, a *2–morphism* $u : H^1 \Rightarrow H^2$ is likewise a morphism of the natural transformations of the corresponding functors $\mathfrak{P}(D) \to \mathsf{Diagr}_D(\mathfrak{X})$. Specifically, u consists of the datum of a diagrammatic 2–morphism $u_B : H^1_B \to H^2_B$ for any $B \subseteq D$ such that, for any $B' \subseteq B$, the following cylinder in $\mathsf{Diagr}_D(\mathfrak{X})$ is commutative

$$
\begin{tikzcd}
& \mathcal{C}'_B \arrow[dd, "F'_{B'B}"] \\
\mathcal{C}_B \arrow[ur, bend left, "H^1_B"] \arrow[ur, bend right, "H^2_B"'] \arrow[dd, "F_{B'B}"'] & \\
& \mathcal{C}'_{B'} \\
\mathcal{C}_{B'} \arrow[ur, bend left, dotted, "H^1_{B'}"] \arrow[ur, bend right, "H^2_{B'}"'] &
\end{tikzcd}
$$

where the rectangular 2–faces are the morphisms γ, γ' and the circular ones the morphisms $u_B, u_{B'}$.

Remark. In view of 3.4, a 2–morphism $u : H^1 \to H^2$ is equivalently described as a collection of 2–morphisms $u_B : H^1_B \to H^2_B$, indexed by $B \subseteq D$, such that, for any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, it holds $\gamma^2_{\mathcal{F}} \circ u_B = u_{B'} \circ \gamma^1_{\mathcal{F}}$.

3.7. **Υ–strict pre–Coxeter objects.** A pre–Coxeter object $\mathcal{C}$ in $\mathfrak{X}$ is Υ–*strict* if, for any $B' \subseteq B$, and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$ the following holds

$$F_{\mathcal{F}} = F_{\mathcal{G}} \qquad \text{and} \qquad \Upsilon_{\mathcal{F}\mathcal{G}} = \mathrm{id}_{F_{\mathcal{G}}}$$

We denote the common value of $\{F_{\mathcal{F}}\}_{\mathcal{F} \in \mathsf{Mns}(B,B')}$ by $F_{B'B} : \mathcal{C}_B \to \mathcal{C}_{B'}$. It follows from condition (4) in Definition 3.4 that, for any $B'' \subseteq B' \subseteq B$, $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, $\mathcal{F}', \mathcal{G}' \in \mathsf{Mns}(B', B'')$, $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{G}}_{\mathcal{G}'}$. We denote the common value of $\{\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}\}_{\mathcal{F}, \mathcal{F}'}$ by $\mathsf{a}_{B''B'B} : F_{B''B'} \circ F_{B'B} \Rightarrow F_{B''B}$.

Proposition.

(1) A Υ–strict pre–Coxeter object of type D in $\mathfrak{X}$ is equivalently described by the datum of

- *for any $B \subseteq D$, an object $\mathcal{C}_B \in \mathfrak{X}$*
- *for any $B' \subseteq B$, a 1–morphism $F_{B'B} : \mathcal{C}_B \to \mathcal{C}_{B'}$*
- *for any $B'' \subseteq B' \subseteq B$, a 2–isomorphism*

$$\mathsf{a}_{B''B'B} : F_{B''B'} \circ F_{B'B} \Rightarrow F_{B''B}$$

such that

- *for any $B' \subseteq B$, $F_{BB} = \mathrm{id}_{\mathcal{C}_B}$ and $\mathsf{a}_{B'BB} = \mathrm{id}_{F_{B'B}} = \mathsf{a}_{B'B'B}$*
- *the 2–isomorphism a are associative, i.e., for any $B''' \subseteq B'' \subseteq B' \subseteq B$,*

$$\mathsf{a}_{B'''B''B} \circ \mathsf{a}_{B''B'B} = \mathsf{a}_{B'''B''B} \circ \mathsf{a}_{B'''B''B'}$$

as 2–morphisms $F_{B'''B''} \circ F_{B''B'} \circ F_{B'B} \Rightarrow F_{B'''B}$

(2) Every pre–Coxeter object $\mathcal{C}$ in $\mathfrak{X}$ is equivalent to a Υ–strict pre–Coxeter object in $\mathfrak{X}$.

Proof. (1) is clear. (2) For any $B' \subseteq B$, choose a maximal nested set $\mathcal{E}(B, B') \in \mathsf{Mns}(B, B')$. We denote by $\overline{\mathcal{C}}$ the Υ–strict pre–Coxeter object with $\overline{\mathcal{C}}_B \coloneqq \mathcal{C}_B$, $F_{B'B} \coloneqq F_{\mathcal{E}(B,B')}$, and

$$\mathsf{a}_{B''B'B} \coloneqq \Upsilon_{\mathcal{E}(B,B''), \mathcal{E}(B,B') \cup \mathcal{E}(B',B'')} \circ \mathsf{a}^{\mathcal{E}(B,B')}_{\mathcal{E}(B',B'')}$$

Then, there is a canonical equivalence of pre–Coxeter objects $\mathcal{C} \to \overline{\mathcal{C}}$ with $H_B \coloneqq \mathrm{id}_{\mathcal{C}_B}$ and $\gamma_{\mathcal{F}} \coloneqq \Upsilon_{\mathcal{E}(B,B'), \mathcal{F}}$ for $\mathcal{F} \in \mathsf{Mns}(B, B')$. □

Remark. We show in Sections 12.9 and 13.3 that Kac–Moody algebras and their quantum groups naturally give rise to Υ–strict pre–Coxeter objects in $\mathsf{Cat}^{\otimes}$. On the other hand, we prove in [5] that the monodromy of the Casimir connection of a symmetrisable Kac–Moody algebra naturally gives rise to a pre–Coxeter structure which is not Υ–strict. The latter, however, is a–strict in the following sense.

3.8. **a–strict pre–Coxeter objects.** A pre–Coxeter object $\mathcal{C}$ in $\mathfrak{X}$ is a–*strict* if, for any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, and

$$F_{\mathcal{F}' \cup \mathcal{F}} = F_{\mathcal{F}'} \circ F_{\mathcal{F}} \qquad \text{and} \qquad \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathrm{id}_{F_{\mathcal{F}'} \circ F_{\mathcal{F}}}$$

In contrast with Proposition 3.7, not every pre–Coxeter object $\mathcal{C}$ is equivalent to an a–strict one. We give, however, a sufficient condition for that to be the case below.

Let $B_1' \subset B_1 \perp B_2 \supset B_2'$, with $|B_k \setminus B_k'| = 1$, and denote by $\mathcal{F}_k$ the unique element in $\mathsf{Mns}(B_k, B_k')$. Consider the diagram

$$\begin{tikzcd}
& \mathcal{C}_{B_1\sqcup B_2} \arrow[dl, "F_{(\mathcal{F}_1,B_2)}"'] \arrow[dr, "F_{(B_1,\mathcal{F}_2)}"] \arrow[dd, "F_{(\mathcal{F}_1,\mathcal{F}_2)}" description] & \\
\mathcal{C}_{B_1'\sqcup B_2} \arrow[dr, "F_{(B_1',\mathcal{F}_2)}"'] & & \mathcal{C}_{B_1\sqcup B_2'} \arrow[dl, "F_{(\mathcal{F}_1,B_2')}"] \\
& \mathcal{C}_{B_1'\sqcup B_2'} &
\end{tikzcd} \tag{3.6}$$

where the triangular 2–faces are given by the vertical joins $\mathsf{a}_{(B_1',\mathcal{F}_2)}^{(\mathcal{F}_1,B_2)}$ and $\mathsf{a}_{(\mathcal{F}_1,B_2')}^{(B_1,\mathcal{F}_2)}$ respectively. We say that (3.6) is *trivial* if

$$F_{(B_1',\mathcal{F}_2)} \circ F_{(\mathcal{F}_1,B_2)} = F_{(\mathcal{F}_1,B_2')} \circ F_{(B_1,\mathcal{F}_2)} \tag{3.7}$$

$$\mathsf{a}_{(B_1',\mathcal{F}_2)}^{(\mathcal{F}_1,B_2)} = \mathsf{a}_{(\mathcal{F}_1,B_2')}^{(B_1,\mathcal{F}_2)} \tag{3.8}$$

as 2–morphisms $F_{(B_1',\mathcal{F}_2)} \circ F_{(\mathcal{F}_1,B_2)} = F_{(\mathcal{F}_1,B_2')} \circ F_{(B_1,\mathcal{F}_2)} \Rightarrow F_{(\mathcal{F}_1,\mathcal{F}_2)}$. Note that this is the case if $\mathcal{C}$ is a–strict.

Proposition. *Let $\mathcal{C}$ be a pre–Coxeter object in $\mathfrak{X}$. If the diagrams* (3.6) *are trivial, then $\mathcal{C}$ is canonically equivalent to an a–strict pre–Coxeter object.*

PROOF. Retain the notation from 2.3. Let $B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathbf{C}$: $B' = B_0 \subsetneq B_1 \cdots \subsetneq B_\ell = B$ a maximal chain corresponding to $\mathcal{F}$. Denote by $\overline{F}_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}$ the composition $F_{\mathcal{F}_1} \circ \cdots \circ F_{\mathcal{F}_\ell}$, where $\mathcal{F}_k$ is the unique element in $\mathsf{Mns}(B_k, B_{k-1})$. By (3.7), $\overline{F}_{\mathcal{F}}$ does not depend upon the choice of $\mathbf{C} \in p^{-1}(\mathcal{F})$. Moreover, for any $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, one has $\overline{F}_{\mathcal{F}'} \circ \overline{F}_{\mathcal{F}} = \overline{F}_{\mathcal{F}' \cup \mathcal{F}}$.

For any $\mathcal{F} \in \mathsf{Mns}(B, B')$, let $u_{\mathcal{F}} : \overline{F}_{\mathcal{F}} \Rightarrow F_{\mathcal{F}}$ be the 2–morphism obtained as the composition of vertical joins $\mathsf{a}_{\mathcal{F}_1}^{\mathcal{F}_2 \cup \cdots \cup \mathcal{F}_\ell} \circ \cdots \circ \mathsf{a}_{\mathcal{F}_{\ell-2}}^{\mathcal{F}_{\ell-1} \cup \mathcal{F}_\ell} \circ \mathsf{a}_{\mathcal{F}_{\ell-1}}^{\mathcal{F}_\ell}$. By (3.8), $u_{\mathcal{F}}$ is independent of the choice of a maximal chain $\mathbf{C} \in p^{-1}(\mathcal{F})$. For any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, set

$$\overline{\Upsilon}_{\mathcal{F}\mathcal{G}} \coloneqq u_{\mathcal{F}}^{-1} \circ \Upsilon_{\mathcal{F}\mathcal{G}} \circ u_{\mathcal{G}} : \overline{F}_{\mathcal{G}} \Rightarrow \overline{F}_{\mathcal{F}}$$

Then, the datum of the objects $\overline{\mathcal{C}}_B = \mathcal{C}_B$, 1–morphisms $\overline{F}_{\mathcal{F}}$, and 2–morphisms $\overline{\Upsilon}_{\mathcal{F}\mathcal{G}}$ gives rise to an a–strict pre–Coxeter object $\overline{\mathcal{C}}$. Moreover, there is a canonical equivalence $\mathcal{C} \to \overline{\mathcal{C}}$ with $H_B = \mathrm{id}_{\mathcal{C}_B}$ and $\gamma_{\mathcal{F}} = u_{\mathcal{F}}^{-1}$. $\square$

3.9. Generalised braid groups.

Definition. A *labelling* $\underline{m}$ of a diagram D is the assignment of an integer $m_{ij} \in \{2, 3, \ldots, \infty\}$ to any pair i, j of distinct vertices of D such that $m_{ij} = m_{ji}$ and $m_{ij} = 2$ if i and j are orthogonal.

The *generalised* braid group corresponding to D and a labelling $\underline{m}$ is the group $\mathcal{B}_D^{\underline{m}}$ with generators $\{\mathsf{S}_i\}_{i \in D}$ and relations

$$\underbrace{\mathsf{S}_i \cdot \mathsf{S}_j \cdot \mathsf{S}_i \ \cdots}_{m_{ij}} = \underbrace{\mathsf{S}_j \cdot \mathsf{S}_i \cdot \mathsf{S}_j \ \cdots}_{m_{ij}} \tag{3.9}$$

If $B \subseteq D$ is a subdiagram, we denote by $\mathcal{B}_B^{\underline{m}} \subseteq \mathcal{B}_D^{\underline{m}}$ the subgroup generated by the elements S_i, $i \in B$, which is isomorphic to the generalised braid group corresponding to B and the labelling $\underline{m}$ restricted to B.

3.10. **Coxeter objects.** Let $(D, \underline{m})$ be a labelled diagram.

Definition. A *Coxeter object* of type $(D, \underline{m})$ in $\mathfrak{X}$ is the datum of

- a pre–Coxeter object $\left(\mathcal{C}_B, F_{B'B}, \alpha_{BB''}^{B'}\right)$ of type D in $\mathfrak{X}$
- for any $i \in D$, a diagrammatic 2–isomorphism $S_i : F_{\emptyset i} \Rightarrow F_{\emptyset i}$

such that for any subdiagram $B \subseteq D$, and $i, j \in B$ with $i \neq j$

$$\underbrace{S_i^B \cdot S_j^B \cdot S_i^B \ \cdots}_{m_{ij}} = \underbrace{S_j^B \cdot S_i^B \cdot S_j^B \ \cdots}_{m_{ij}} \tag{3.10}$$

where $S_i^B : F_{\emptyset B} \Rightarrow F_{\emptyset B}$ is the diagrammatic 2–morphism

$$F_{\emptyset B} \xRightarrow{(\alpha_{B\emptyset}^i)^{-1}} F_{\emptyset i} \circ F_{iB} \xRightarrow{S_i} F_{\emptyset i} \circ F_{iB} \xRightarrow{\alpha_{B\emptyset}^i} F_{\emptyset B}$$

Remark. More explicitly, the equation (3.10) reads as follows. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B)$ be two maximal nested sets on B such that $\{i\} \in \mathcal{F}$, $\{j\} \in \mathcal{G}$, so that $\mathcal{G} = \mathcal{G}_j \cup \mathcal{G}'$, with $\mathcal{G}_j = \{\emptyset, \{j\}\}$. Let $\xi_{\mathcal{G}}^j : \mathsf{End}\left(F_{\emptyset j}\right) \to \mathsf{End}\left(F_{\mathcal{G}}\right)$ be the natural isomorphism induced by the map $\mathsf{a}_{\mathcal{G}_j}^{\mathcal{G}'} : F_{\mathcal{G}_j} \circ F_{\mathcal{G}'} \Rightarrow F_{\mathcal{G}}$, and set $\xi_{\mathcal{G}\mathcal{F}}^i := \mathsf{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}}) \circ \xi_{\mathcal{F}}^i$, so that $\xi_{\mathcal{G}\mathcal{F}}^i : \mathsf{End}\left(F_{\emptyset i}\right) \to \mathsf{End}\left(F_{\mathcal{G}}\right)$. Then, (3.10) reads

$$\underbrace{\xi_{\mathcal{G}\mathcal{F}}^i(S_i) \cdot \xi_{\mathcal{G}}^j(S_j) \cdot \xi_{\mathcal{G}\mathcal{F}}^i(S_i) \ \cdots}_{m_{ij}} = \underbrace{\xi_{\mathcal{G}}^j(S_j) \cdot \xi_{\mathcal{G}\mathcal{F}}^i(S_i) \cdot \xi_{\mathcal{G}}^j(S_j) \ \cdots}_{m_{ij}},$$

as an identity in $\mathsf{End}\left(F_{\mathcal{G}}\right)$.

A 1–morphism $\mathcal{C} \to \mathcal{C}'$ of Coxeter objects in $\mathfrak{X}$ is one of the underlying pre–Coxeter objects, which preserves the braid group operators S. That is, it consists of a datum $(H_B, \gamma_{B'B})$ defined as in 3.6 such that, for any $i \in D$,

$$\mathsf{Ad}(\gamma_{\emptyset i})(H_\emptyset(S_i)) = S_i'|_{H_i}$$

in $\mathsf{Diagr}_D(\mathfrak{X})(F'_{\emptyset i} \circ H_i, F'_{\emptyset i} \circ H_i)$. A 2–morphism is defined as in 3.6.

3.11. **Braid group actions.** Let $(D, \underline{m})$ be a labelled diagram. , and $H : \mathcal{C} \to \mathcal{D}$ a 1–isomorphism of Coxeter objects.

Proposition.

(1) *Let $\mathcal{C}$ be Coxeter object of type $(D, \underline{m})$ in $\mathfrak{X}$. For any subdiagram $B \subseteq D$, there is a unique homomorphism $\rho_B^{\mathcal{C}} : \mathcal{B}_B^{\underline{m}} \to \mathsf{Diagr}_D(\mathfrak{X})(F_{\emptyset B}, F_{\emptyset B})$, such that, for any $i \in B$, $\rho_B^{\mathcal{C}}(S_i) = S_i^B$. Moreover, for any $B' \subseteq B$, the following diagram is commutative*

$$\begin{tikzcd}
\mathcal{B}_B^{\underline{m}} \arrow[r, "\rho_B"] & \mathsf{Diagr}_D(\mathfrak{X})(F_{\emptyset B}, F_{\emptyset B}) \\
\mathcal{B}_{B'}^{\underline{m}} \arrow[u] \arrow[r, "\rho_{B'}"'] & \mathsf{Diagr}_D(\mathfrak{X})(F_{\emptyset B'}, F_{\emptyset B'}) \arrow[u]
\end{tikzcd}$$

where the vertical right arrow is induced by the 2–isomorphism $\alpha_{\emptyset B}^{B'} : F_{\emptyset B'} \circ F_{B'B} \Rightarrow F_{\emptyset B}$.

(2) *Let $\mathcal{C}, \mathcal{D}$ be Coxeter objects of type $(D, \underline{m})$ in $\mathfrak{X}$ and $H : \mathcal{C} \to \mathcal{D}$ a 1–isomorphism of Coxeter objects. For any subdiagram $B \subseteq D$, the representations $\rho_B^{\mathcal{C}}$ and $\rho_B^{\mathcal{D}}$ of $\mathcal{B}_B^{\underline{m}}$ are equivalent, i.e., the following diagram is commutative*

$$\begin{tikzcd}
& \mathsf{Diagr}_D(\mathfrak{X})(F^{\mathcal{C}}_{\emptyset B}, F^{\mathcal{C}}_{\emptyset B}) \\
\mathcal{B}_B^{\underline{m}} \arrow[ur, "\rho_B^{\mathcal{C}}"] \arrow[dr, "\rho_B^{\mathcal{D}}"'] & \\
& \mathsf{Diagr}_D(\mathfrak{X})(F^{\mathcal{D}}_{\emptyset B}, F^{\mathcal{D}}_{\emptyset B}) \arrow[uu]
\end{tikzcd}$$

where the vertical arrow is induced by the 2–isomorphism $\gamma_B : F^{\mathcal{D}}_{\emptyset B} \circ H_B \Rightarrow F^{\mathcal{C}}_{\emptyset B}$.

Proof. (1) The existence of the homomorphisms ρ_B, $B \subseteq D$, follows by construction. For the commutativity of the diagram, it is enough to observe that the map $\mathsf{Diagr}_D(\mathfrak{X})(F_{\emptyset B'}, F_{\emptyset B'}) \to \mathsf{Diagr}_D(\mathfrak{X})(F_{\emptyset B}, F_{\emptyset B})$ sends a 2–endomorphism ϕ to $(\alpha^{B'}_{\emptyset B}) \circ \phi|_{F_{B'B}} \circ (\alpha^{B'}_{\emptyset B})^{-1}$. Therefore, for any $i \in B'$, one has

$$\begin{aligned}
(\alpha^{B'}_{\emptyset B}) \circ S_i^{B'} \circ (\alpha^{B'}_{\emptyset B})^{-1} &= (\alpha^{B'}_{\emptyset B}) \circ \left((\alpha^{i}_{\emptyset B'}) \circ S_i \circ (\alpha^{i}_{\emptyset B'})^{-1}\right)|_{F_{B'B}} \circ (\alpha^{B'}_{\emptyset B})^{-1} \\
&= (\alpha^{i}_{\emptyset B}) \circ S_i \circ (\alpha^{i}_{\emptyset B})^{-1} \\
&= S_i^B
\end{aligned}$$

where the second equality follows from the associativity of α. (2) follows immediately from the definition of 1–morphism of Coxeter objects (cf. 3.10). □

Remark. In the 2–category $\mathfrak{X}$, the representations ρ_B are described as follows. For any $B \subseteq D$ and $\mathcal{F} \in \mathsf{Mns}(B)$, there is a collection of homomorphisms $\rho_{\mathcal{F}} : \mathcal{B}_B^{\underline{m}} \to \mathsf{Aut}_{\mathfrak{X}}(F_{\mathcal{F}})$, $\mathcal{F} \in \mathsf{Mns}(B)$, uniquely determined by the conditions

- $\rho_{\mathcal{F}}(\mathsf{S}_i) = S_i^{\mathcal{F}}$, if $\{i\} \in \mathcal{F}$
- $\rho_{\mathcal{G}} = \mathsf{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}}) \circ \rho_{\mathcal{F}}$

3.12. **Lax diagrammatic algebras** [37, Sec. 3]**.** A *lax diagrammatic algebra*[11] is the datum of

- for any $B \subseteq D$, a k–algebra A_B
- for any $B' \subseteq B$, a homomorphism $i_{BB'} : A_{B'} \to A_B$

such that

- for any $B \subseteq D$, $i_{BB} = \mathrm{id}_{A_B}$
- for any $B'' \subseteq B' \subseteq B$, $i_{BB'} \circ i_{B'B''} = i_{BB''}$
- for any $B = B' \sqcup B''$, with $B' \perp B''$, $m_B \circ i_{BB'} \otimes i_{BB''}$ is a morphism of algebras $A_{B'} \otimes A_{B''} \to A_B$, where m_B denotes the multiplication in A_B.

A morphism of lax diagrammatic algebras $\varphi : A \to A'$ is a collection of homomorphisms $\varphi_B : A_B \to A'_B$ such that $\varphi_B \circ i_{BB'} = i'_{BB'} \circ \varphi_{B'}$ for any $B' \subseteq B$.[12]

[11]The terminology adopted here differs from the one in [37], where the adjective lax is not used in particular. In the present paper, we reserve the term diagrammatic algebra for a lax diagrammatic algebra such that $m_B \circ i_{BB'} \otimes i_{BB''} : A_{B'} \otimes A_{B''} \to A_B$ is an isomorphism for any $B = B' \sqcup B''$, which implies in particular that $A_\emptyset = \mathsf{k}$ (see Remark 5.14).

[12]In [37], a morphism of lax diagrammatic algebras is referred to as a *strict* morphism.

3.13. **Pre–Coxeter categories from lax diagrammatic algebras.** A lax diagrammatic algebra A gives rise to an (a,Υ)–strict pre–Coxeter object $\mathcal{C} = \mathrm{Rep}(A)$ in $\mathfrak{X} = \mathsf{Cat}$ given by[13]

- For any $B \subseteq D$, $\mathcal{C}_B = \mathrm{Rep}(A_B)$
- For any $B' \subseteq B$, $F_{B'B} : \mathcal{C}_B \to \mathcal{C}_{B'}$ is the pullback functor $i^*_{BB'}$

Moreover, a morphism of lax diagrammatic algebras $\varphi : A \to A'$ gives rise to a morphism of pre–Coxeter objects $\mathrm{Rep}(A') \to \mathrm{Rep}(A)$.

If $(D, \underline{m})$ is a labelled diagram, the group algebra $\mathsf{k}\mathcal{B}^{\underline{m}}_D$ is naturally endowed with a lax diagrammatic algebra structure. If a lax diagrammatic algebra A is further endowed with a morphism of lax diagrammatic algebras $\rho_B : \mathsf{k}\mathcal{B}^{\underline{m}}_B \to A_B$, $B \subseteq D$, then the elements $\rho(S_i) \in A_i = \mathsf{End}\,(F_{\emptyset i})$ give rise to the structure of Coxeter object on $\mathrm{Rep}(A)$.

This construction can be generalised by replacing the categories $\mathrm{Rep}(A_B)$ by a collection of subcategories $\mathcal{C}_B \subseteq \mathrm{Rep}(A_B)$ stable under restrictions, and ρ by a morphism of lax diagrammatic algebras $\mathsf{k}\mathcal{B}^{\underline{m}}_D \to \mathsf{End}\,(F_{\emptyset D}) =: \widehat{A}$. We show in Section 13 that an example of such Coxeter objects is provided by quantum Weyl groups of quantised Kac–Moody algebras.

3.14. **Topological definition.** In [19], Finkelberg and Schechtman propose an alternative definition of a (pre–)Coxeter object in Cat for Dynkin diagrams of finite type, which is akin to Deligne's topological definition of a braided monoidal category [10]. This is given by a category $\mathcal{C}_B$ for every diagram $B \subseteq D$, together with

- for any $B' \subseteq B$, a Weyl group equivariant local system of restriction functors $\mathfrak{F}_{B'B} : \mathcal{C}_B \to \mathcal{C}_{B'}$, defined over $(\mathfrak{h}_{B/B'})_{\mathrm{reg}}$ [14]
- for any $B'' \subseteq B' \subseteq B$, a suitable analogue of the factorisation isomorphism $\alpha^{B'}_{B''B}$.

This gives rise to a Coxeter object in the sense of 3.3, where, for each $\mathcal{F} \in \mathsf{Mns}(B, B')$, the functor $F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, is the limit of $\mathfrak{F}_{B'B}$ at the point at infinity $p_{\mathcal{F}}$ in the De Concini–Procesi compactification of $(\mathfrak{h}_{B/B'})_{\mathrm{reg}}$ [9].

3.15. **Example: rational Cherednik algebras.** Let $\mathfrak{h}$ be a finite–dimensional complex vector space, and $W \subset GL(\mathfrak{h})$ a finite complex reflection group. Let c be a conjugation invariant function on the set S of reflections in W, and $H_c(W, \mathfrak{h})$ the corresponding rational Cherednik algebra. Let $\mathcal{O}(W, \mathfrak{h})$ be the category of highest weight $H_c(W, \mathfrak{h})$–modules, $W' \subset W$ a parabolic subgroup, $\mathfrak{h}' = \mathfrak{h}/\mathfrak{h}^{W'}$, c' the restriction of c to $S \cap W'$.

In [7] Bezrukavnikov and Etingof construct a parabolic restriction functor

$$\mathrm{Res}_b : \mathcal{O}(W, \mathfrak{h}) \to \mathcal{O}(W', \mathfrak{h}')$$

where $b \in \mathfrak{h}^{W'}_{\mathrm{reg}}$. In [32, Cor. 2.5], Shan shows that the composition of two parabolic restriction functors is isomorphic to a parabolic restriction functor, compatibly with

[13] Note that the commutativity of $A_{B'}, A_{B''}$ in A_B, for any $B', B'' \subseteq B$ with $B' \perp B''$, has no relevance in the above construction of pre–Coxeter structure on $\mathrm{Rep}(A)$. On the other hand, this feature is particularly convenient in the construction of examples arising from the quantisation of Lie bialgebras (cf. Section 10, in particular Lemma 10.9).

[14] Here, $\mathfrak{h}_B$ is the Cartan subalgebra of $\mathfrak{g}_B \subseteq \mathfrak{g}_D$, $\mathfrak{h}_{B/B'} \subseteq \mathfrak{h}_B$ is the orthogonal complement of $\mathfrak{h}_{B'}$, and $(\mathfrak{h}_{B/B'})_{\mathrm{reg}}$ is the complement in $\mathfrak{h}_{B/B'}$ to the root hyperplanes in $\mathfrak{h}_B$ not containing $\mathfrak{h}_{B/B'}$.

the parameter b. If W is a Weyl group with Dynkin diagram D, these functors and their factorisation isomorphisms give rise to topological Coxeter object in Cat, in the sense sketched in 3.14.

4. Braided Coxeter categories

4.1. Denote by $\mathsf{Cat}^{\otimes}$ (resp. $\mathsf{Cat}^{\otimes,\beta}$) the 2–category of monoidal (resp. braided monoidal) categories.

Definition. Let D be a diagram.

(1) A *braided pre–Coxeter category of type* D is a tuple $(\mathcal{C}_B, F_{B'B}, \alpha^{B'}_{BB''})$ such that
- $\mathcal{C}_B$ is a B–object in $\mathsf{Cat}^{\otimes,\beta}$
- $(\mathcal{C}_B, F_{B'B}, \alpha^{B'}_{BB''})$ is a pre–Coxeter object in $\mathsf{Cat}^{\otimes}$

(2) If $\underline{m}$ is a labelling on D, a *braided Coxeter category of type* $(D,\underline{m})$ is a tuple $(\mathcal{C}_B, F_{B'B}, \alpha^{B'}_{BB''}, S_i)$ such that
- $\mathcal{C}_B$ is a B–object in $\mathsf{Cat}^{\otimes,\beta}$
- $(\mathcal{C}_B, F_{B'B}, \alpha^{B'}_{BB''})$ is a pre–Coxeter object in $\mathsf{Cat}^{\otimes}$
- $(\mathcal{C}_B, F_{B'B}, \alpha^{B'}_{BB''}, S_i)$ is a Coxeter object in Cat

and, for any $i \in D$, the following holds in $\mathsf{Aut}(F_i \otimes F_i)$

$$J_i^{-1} \circ F_i(c_i) \circ \Delta(S_i) \circ J_i = c_\emptyset \circ S_i \otimes S_i \tag{4.1}$$

where $F_i = F_{\emptyset i}$, J_i is the tensor structure on F_i and $c_i, c_\emptyset$ are the opposite braidings in $\mathcal{C}_i$ and $\mathcal{C}_\emptyset$, respectively.[15] In other words, the following diagram is commutative for any $V, W \in \mathcal{C}_i$,

$$\begin{tikzcd}
F_i(V)\otimes F_i(W) \arrow[r, "S_i^V \otimes S_i^W"] \arrow[d, "J_i^{V,W}"'] & F_i(V)\otimes F_i(W) \arrow[r, "c_\emptyset"] & F_i(W)\otimes F_i(V) \arrow[d, "J_i^{W,V}"] \\
F_i(V\otimes W) \arrow[r, "S_i^{V\otimes W}"'] & F_i(V\otimes W) \arrow[r, "F_i(c_i)"'] & F_i(W\otimes V)
\end{tikzcd}$$

(3) A functor of braided Coxeter categories $\mathcal{C} \to \mathcal{C}'$ is a tuple $(H_B, \gamma_{B'B})$ such that
- $H_B : \mathcal{C}_B \to \mathcal{C}'_B$ is a 1–morphism of B–objects in $\mathsf{Cat}^{\otimes,\beta}$;
- $(H_B, \gamma_{B'B})$ is a 1–morphism of pre–Coxeter objects in $\mathsf{Cat}^{\otimes}$.

Finally, a natural transformation $u : H \Rightarrow H'$ is a 2–morphism of B–objects in $\mathsf{Cat}^{\otimes,\beta}$.

Remarks.

- The identity (4.1) relates the failure of (F_i, J_i) to be a braided monoidal functor and that of S_i to be a monoidal isomorphism. That is, if (4.1) holds, then S_i is monoidal if and only if J_i is braided. Conversely, if S_i is monoidal and J_i is braided, then (4.1) automatically holds. In particular, every Coxeter object in $\mathsf{Cat}^{\otimes,\beta}$ is a braided Coxeter category.
- The main examples of braided Coxeter categories arise as representations of a *quasi–Coxeter quasitriangular quasibialgebra*, as defined in [37, Sec. 3].

[15] In a braided monoidal category with braiding β, the opposite braiding is $\beta^{\mathrm{op}}_{X,Y} := \beta^{-1}_{Y,X}$.

- In [4], we only consider a–strict braided Coxeter categories and, for simplicity, refer to them as braided Coxeter categories.

4.2. **Unfolded definition.** In view of 3.4, braided Coxeter category of type $(D, \underline{m})$ is equivalently described by the datum of

- for any $B \subseteq D$, a braided monoidal category $\mathcal{C}_B \in \mathfrak{X}$
- for any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, a (not necessarily braided) monoidal functor $F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}$
- for any $B' \subseteq B$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, an isomorphism of monoidal functors $\Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}}$
- for any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, an isomorphism of monoidal functors $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}' \cup \mathcal{F}}$
- for any $i \in D$, an isomorphism of functors $S_i : F_{\emptyset\{i\}} \Rightarrow F_{\emptyset\{i\}}$ (not necessarily preserving the tensor structure)

satisfying the properties listed in 3.4, 3.10, and the coproduct identity (4.1).

4.3. **Balanced categories.** In [19], the coproduct identity (4.1) is replaced by the assumption that the categories $\mathcal{C}_i$ are balanced categories (in fact, that $\mathcal{C}_B$ is balanced for any $B \subseteq D$). We point out below that, in general, this assumption is stronger than (4.1).

Recall that a braided monoidal category $(\mathcal{C}, \otimes, b, \Phi)$ is *balanced* if there is a $\theta \in \mathsf{Aut}(\mathrm{id}_{\mathcal{C}})$ such that

$$\theta_{V \otimes W} = b_{W,V} \circ b_{V,W} \circ \theta_V \otimes \theta_W \tag{4.2}$$

for any $V, W \in \mathcal{C}$.

Proposition. *Let $\mathcal{C}$ be a braided Coxeter category such that*

(1) $\mathcal{C}_\emptyset$ is symmetric
(2) $S_i^2 = F_i(\theta_i)$ for some $\theta_i \in \mathsf{Aut}(\mathrm{id}_{\mathcal{C}_i})$
(3) $F_i : \mathcal{C}_i \to \mathcal{C}_\emptyset$ is faithful

Then $\mathcal{C}_i$ is a balanced monoidal category with balance θ_i.

Proof. Squaring the right–hand side of (4.1) yields

$$(c_\emptyset \circ S_i \otimes S_i)^2 = c_\emptyset^2 \circ S_i^2 \otimes S_i^2 = F_i(\theta_i) \otimes F_i(\theta_i)$$

where we used the binaturality of $c_\emptyset$ and the assumptions (1) and (2). On the other hand, the square of the right–hand side of (4.1) is equal to

$$\begin{aligned} J_i^{-1} \circ F_i(c_i) \circ \Delta(S_i) \circ F_i(c_i) \circ \Delta(S_i) \circ J_i &= J_i^{-1} \circ F_i(c_i^2) \circ \Delta(S_i^2) \circ J_i \\ &= J_i^{-1} \circ F_i(c_i^2) \circ F_i(\theta_i \circ \otimes) \circ J_i \end{aligned}$$

where we used the naturality of S_i. Since $J_i \circ F_i(\theta_i) \otimes F_i(\theta_i) \circ J_i^{-1} = F_i(\theta_i \otimes \theta_i)$ by naturality of J_i, we get

$$F_i(c_i^2 \circ \theta_i \circ \otimes) = F_i(\theta_i \otimes \theta_i)$$

hence the required result since F_i is faithful. □

Remark.

- The converse of Proposition 4.3 does not hold in general. That is, the existence of a balance does not imply (4.1). Instead, the correct categorical interpretation of (4.1) corresponds to the braided monoidal categories $\mathcal{C}_i$ (with the tensor functors F_i) being *half–balanced* (cf. [33, Sec. 4]).

- Finally, we note that the coproduct identity (4.1) cannot in general be extended to subdiagrams with more than one vertex. Specifically, in the examples of braided Coxeter structures described in Sections 10 and 13, the categories $\mathcal{C}_B$, with $|B| > 1$, do not in general admit a half–balanced structure.

5. Diagrammatic Lie bialgebras

In this section, we introduce the notion of a diagrammatic Lie bialgebra $\mathfrak{b}$. We then show that Drinfeld–Yetter modules over the canonical subalgebras of $\mathfrak{b}$ give rise to a symmetric pre–Coxeter category.

5.1. **Lie bialgebras** [11]**.** A Lie bialgebra is a triple $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ where $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}})$ is a Lie algebra, $(\mathfrak{b}, \delta_{\mathfrak{b}})$ a Lie coalgebra, and the cobracket $\delta_{\mathfrak{b}} : \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{b}$ satisfies the cocycle condition

$$\delta_{\mathfrak{b}}([X, Y]_{\mathfrak{b}}) = \mathrm{ad}(X)\,\delta_{\mathfrak{b}}(Y) - \mathrm{ad}(Y)\,\delta_b(X)$$

5.2. **Manin triples** [11, 15]**.** A Manin triple is the data of a Lie algebra $\mathfrak{g}$ with

- a nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$
- isotropic Lie subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$

such that

- $\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{b}_+$ as vector spaces
- the inner product defines an isomorphism $\mathfrak{b}_+ \to \mathfrak{b}_-^*$
- the Lie bracket of $\mathfrak{g}$ is continuous with respect to the topology obtained by putting the discrete and the weak topologies on $\mathfrak{b}_-$ and $\mathfrak{b}_+$ respectively. Equivalently, the bracket on $\mathfrak{b}_+$ is continuous with respect to the weak topology.

Under these assumptions, the commutator on $\mathfrak{b}_+ \simeq \mathfrak{b}_-^*$ induces a cobracket $\delta : \mathfrak{b}_- \to \mathfrak{b}_- \otimes \mathfrak{b}_-$ which satisfies the cocycle condition, thus endowing $\mathfrak{b}_-$ with a Lie bialgebra structure. In general, however, $\mathfrak{b}_+$ is only a topological Lie bialgebra.

One can similarly consider *restricted* Manin triples, where

- $\mathfrak{g}$ is $\mathbb{Z}$–graded as a Lie algebra, with finite–dimensional components $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$
- the inner product satisfies $\langle \mathfrak{g}_n, \mathfrak{g}_m \rangle = 0$ unless $n + m = d$, for a given $d \in \mathbb{Z}$
- $\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{b}_+$ as vector spaces, with the isotropic subalgebra $\mathfrak{b}_-$ (resp. $\mathfrak{b}_+$) concentrated in non–negative (resp. non–positive) degrees

In this case, the inner product induces an isomorphism $\mathfrak{b}_\pm \to \mathfrak{b}_\mp^\star$, where $\mathfrak{b}_\mp^\star = \bigoplus_n (\mathfrak{b}_{\mp,n})^*$ is the restricted dual of $\mathfrak{b}_\mp$. The joint continuity of the bracket on $\mathfrak{g}$ is automatic, and both $\mathfrak{b}_-$ and $\mathfrak{b}_+$ are Lie bialgebras with a cobracket of degree d.[16]

5.3. **Example.** A finite–dimensional Lie algebra $\mathfrak{l}$ with an invariant inner product $(-,-)$ gives rise to a restricted Manin triple as follows.

$$\mathfrak{g} = \mathfrak{l}[t, t^{-1}] \qquad \mathfrak{b}_- = \mathfrak{g}[t] \qquad \mathfrak{b}_+ = t^{-1}\mathfrak{l}[t^{-1}]$$

[16] Note that the Lie algebra grading on $\mathfrak{b}_+$ inherited from $\mathfrak{g}$ differs from that induced by the identification $\mathfrak{b}_+ \cong \mathfrak{b}_-^\star$ by a shift since the inner product yields an isomorphism $(\mathfrak{b}_{-,n})^* \cong \mathfrak{b}_{+,-n+d}$. Note also that the isotropy of $\mathfrak{b}_\pm$ implies that $\mathfrak{b}_{-,n} = 0$ if $n \leqslant d - 1$ and $\mathfrak{b}_{+,n} = 0$ if $n \geqslant d + 1$.

with the standard grading $\deg(\mathfrak{l}\otimes t^m) = m$, and inner product given by the residue pairing $\langle f, g\rangle = \mathrm{Res}_{t=0}(f(t), g(t))$, so that $\langle X\otimes t^m, Y\otimes t^n\rangle = (X,Y)\delta_{m+n,-1}$. In this case, $\mathfrak{b}_-$ has a degree $d=-1$ cobracket given by

$$\delta(f)(t,s) = \left[f(t)\otimes 1 + 1\otimes f(s), \frac{\Omega}{s-t}\right]$$

where $\Omega\in(\mathfrak{l}\otimes\mathfrak{l})^{\mathfrak{l}}$ corresponds to $\langle\cdot,\cdot\rangle$.

The corresponding Manin triple is $\big(\mathfrak{l}((t^{-1})), \mathfrak{l}[t], t^{-1}\mathfrak{l}[[t^{-1}]]\big)$.

5.4. **Drinfeld double** [11]**.** The Drinfeld double of a Lie bialgebra $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ is the Lie algebra $\mathfrak{g}_{\mathfrak{b}}$ defined as follows. As a vector space, $\mathfrak{g}_{\mathfrak{b}} = \mathfrak{b}\oplus\mathfrak{b}^*$. The duality pairing $\mathfrak{b}^*\otimes\mathfrak{b}\to\mathsf{k}$ extends uniquely to a symmetric, non–degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_{\mathfrak{b}}$, with respect to which both $\mathfrak{b}$ and $\mathfrak{b}^*$ are isotropic subspaces. The Lie bracket on $\mathfrak{g}_{\mathfrak{b}}$ is defined as the unique bracket which coincides with $[\,,\,]_{\mathfrak{b}}$ on $\mathfrak{b}$, with $\delta_{\mathfrak{b}}^t$ on $\mathfrak{b}^*$, and is compatible with $\langle\cdot,\cdot\rangle$, *i.e.,* satisfies $\langle[x,y],z\rangle = \langle x,[y,z]\rangle$ for all $x,y,z\in\mathfrak{g}_{\mathfrak{b}}$. The mixed bracket of $x\in\mathfrak{b}$ and $\phi\in\mathfrak{b}^*$ is then given by

$$[x,\phi] = \mathsf{ad}^*(x)(\phi) + \phi\otimes\mathrm{id}_{\mathfrak{b}}\circ\delta(x)$$

where ad^* is the coadjoint action of $\mathfrak{b}$ on $\mathfrak{b}^*$. $(\mathfrak{g}_{\mathfrak{b}}, \mathfrak{b}, \mathfrak{b}^*)$ is a Manin triple, and any such triple arises this way.

Similarly, if $\mathfrak{b}$ is a Lie bialgebra which is $\mathbb{N}$–graded with finite–dimensional components, and such that the bracket and cobracket are homogeneous of degrees 0 and $d\in\mathbb{Z}$ respectively,[17] the restricted double of $\mathfrak{b}$ is defined as $\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}} = \mathfrak{b}\oplus\mathfrak{b}^\star[d]$, where $\mathfrak{b}^\star[d]_n = (\mathfrak{b}_{-n+d})^*$, and is a restricted Manin triple.

5.5. **Drinfeld–Yetter modules** [16]**.** A Drinfeld–Yetter module over a Lie bialgebra $\mathfrak{b}$ is a triple (V,π_V,π_V^*), where (V,π_V) is a left $\mathfrak{b}$–module, (V,π_V^*) a right $\mathfrak{b}$–comodule, and the maps $\pi_V:\mathfrak{b}\otimes V\to V$ and $\pi_V^*: V\to\mathfrak{b}\otimes V$ satisfy the following relation in $\mathrm{End}(\mathfrak{b}\otimes V)$

$$\mathrm{id}_{\mathfrak{b}}\otimes\pi_V\circ(12)\circ\mathrm{id}_{\mathfrak{b}}\otimes\pi_V^* - \pi_V^*\circ\pi_V = -[\cdot,\cdot]_{\mathfrak{b}}\otimes\mathrm{id}_V\circ\mathrm{id}_{\mathfrak{b}}\otimes\pi_V^* + \mathrm{id}_{\mathfrak{b}}\otimes\pi_V\circ\delta_{\mathfrak{b}}\otimes\mathrm{id}_V$$

The category $\mathsf{DY}_{\mathfrak{b}}$ of Drinfeld–Yetter modules over $\mathfrak{b}$ is a symmetric tensor category. For any $V,W\in\mathsf{DY}_{\mathfrak{b}}$, the action and coaction on the tensor product $V\otimes W$ are defined, respectively, by

$$\pi_{V\otimes W} = \pi_V\otimes\mathrm{id}_W + \mathrm{id}_V\otimes\pi_W\circ(1\,2)\otimes\mathrm{id}_W$$
$$\pi_{V\otimes W}^* = \pi_V^*\otimes\mathrm{id}_W + (1\,2)\otimes\mathrm{id}_W\circ\mathrm{id}_V\otimes\pi_W^*$$

The associativity constraints are trivial, and the braiding is defined by $\beta_{VW} = (1\,2)$.

5.6. **Representations of the Drinfeld double.** The category $\mathsf{DY}_{\mathfrak{b}}$ is canonically isomorphic to the category $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ of *equicontinuous* $\mathfrak{g}_{\mathfrak{b}}$–modules [15], *i.e.,* those endowed with a locally finite $\mathfrak{b}^*$–action. This condition yields a functor $E:\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}\to\mathsf{DY}_{\mathfrak{b}}$, which assigns to any $V\in\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ the Drinfeld–Yetter $\mathfrak{b}$–module (V,π,π^*), where π is the restriction of the action of $\mathfrak{g}_{\mathfrak{b}}$ to $\mathfrak{b}$, and the coaction π^* is given by

$$\pi^*(v) = \sum_i b_i\otimes b^i\,v\in\mathfrak{b}\otimes V$$

where $\{b_i\},\{b^i\}$ are dual bases of $\mathfrak{b}$ and $\mathfrak{b}^*$. The inverse functor is obtained by letting $\phi\in\mathfrak{b}^*\subset\mathfrak{g}_{\mathfrak{b}}$ act on $V\in\mathsf{DY}_{\mathfrak{b}}$ by $\phi\otimes\mathrm{id}_V\circ\pi^*$.

[17] In the sequel, we shall abusively refer to such a $\mathfrak{b}$ as an $\mathbb{N}$–graded Lie bialgebra.

If $\mathfrak{b}$ is $\mathbb{N}$–graded with finite–dimensional homogeneous components, the formulae defining E similarly give rise to an isomorphism E^{res} between the category $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}}}$ of equicontinuous modules over the restricted double of $\mathfrak{b}$ and $\mathsf{DY}_{\mathfrak{b}}$. Moreover, the categories $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ and $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}}}$ are isomorphic, since any locally finite action of $\mathfrak{b}^\star$ extends uniquely to one of $\mathfrak{b}^*$, and the following diagram is commutative

$$\begin{tikzcd} \mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}} \arrow[rr] \arrow[dr, "E"'] & & \mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}}} \arrow[dl, "E^{\mathrm{res}}"] \\ & \mathsf{DY}_{\mathfrak{b}} & \end{tikzcd}$$

5.7. **Split pairs of Lie bialgebras** [3]**.** A *split pair* of Lie bialgebras $(\mathfrak{b},\mathfrak{a})$ is the datum of two Lie bialgebras $\mathfrak{a},\mathfrak{b}$, together with Lie bialgebra morphisms $i:\mathfrak{a}\to\mathfrak{b}$ and $p:\mathfrak{b}\to\mathfrak{a}$ such that $p\circ i=\mathrm{id}_{\mathfrak{a}}$.

As mentioned in 5.4, the assignment $\mathfrak{b}\mapsto\mathfrak{g}_{\mathfrak{b}}$ gives rise to a one–to–one correspondence between Lie bialgebras and Manin triples. Similarly, there is a one–to–one correspondence between split pairs of Lie bialgebras and *split morphisms* of Manin triples. A morphism of Manin triples $i:(\mathfrak{g}_{\mathfrak{a}},\mathfrak{a}_-,\mathfrak{a}_+)\to(\mathfrak{g}_{\mathfrak{b}},\mathfrak{b}_-,\mathfrak{b}_+)$ is a morphism of Lie algebras $i:\mathfrak{g}_{\mathfrak{a}}\to\mathfrak{g}_{\mathfrak{b}}$ which is continuous, preserves inner products, and is such that $i(\mathfrak{a}_\pm)\subset\mathfrak{b}_\pm$.[18] Set

$$i_\pm = i|_{\mathfrak{a}_\pm}:\mathfrak{a}_\pm\to\mathfrak{b}_\pm \qquad\text{and}\qquad p_\pm = i^t_\mp:\mathfrak{b}_\pm\to\mathfrak{a}_\pm$$

i is *split* if the projections $p_\pm$ are morphisms of Lie algebras. The following holds [3, Prop. 3.3]

- If $i:(\mathfrak{g}_{\mathfrak{a}},\mathfrak{a}_-,\mathfrak{a}_+)\longrightarrow(\mathfrak{g}_{\mathfrak{b}},\mathfrak{b}_-,\mathfrak{b}_+)$ is a split inclusion of Manin triples, then $(\mathfrak{a}_-,\mathfrak{b}_-,i_-,p_-)$ is a split pair of Lie bialgebras.
- Conversely, if $(\mathfrak{a},\mathfrak{b},i,p)$ is a split pair of Lie bialgebras, then $i\oplus p^t$: $(\mathfrak{g}_{\mathfrak{a}},\mathfrak{a},\mathfrak{a}^*)\longrightarrow(\mathfrak{g}_{\mathfrak{b}},\mathfrak{b},\mathfrak{b}^*)$ is a split inclusion of Manin triples.

This correspondence may be reformulated as follows. Let $\mathsf{sLBA}(\mathsf{k})$ be the category of split Lie bialgebras. The objects of $\mathsf{sLBA}(\mathsf{k})$ are the same as those of $\mathsf{LBA}(\mathsf{k})$, and the morphisms are given by

$$\mathrm{Hom}_{\mathsf{sLBA}(\mathsf{k})}(\mathfrak{a},\mathfrak{b}) = \{(i,p)\in\mathrm{Hom}_{\mathsf{LBA}(\mathsf{k})}(\mathfrak{a},\mathfrak{b})\times\mathrm{Hom}_{\mathsf{LBA}(\mathsf{k})}(\mathfrak{b},\mathfrak{a})\mid p\circ i=\mathrm{id}_{\mathfrak{a}}\} \tag{5.1}$$

Let $\mathsf{sMT}(\mathsf{k})$ be the category of Manin triples and split morphisms. Then, the assignment $\mathfrak{b}\to\mathfrak{g}_{\mathfrak{b}}$, $(i,p)\to i\oplus p^t$ is an isomorphism of categories $\mathsf{sLBA}(\mathsf{k})\to\mathsf{sMT}(\mathsf{k})$.

5.8. **Split pairs and restriction functors** [3]**.** For any split pair of Lie bialgebras $(\mathfrak{b},\mathfrak{a})$, there is a monoidal restriction functor $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}:\mathsf{DY}_{\mathfrak{b}}\to\mathsf{DY}_{\mathfrak{a}}$ defined by

$$\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}(V,\pi_V,\pi_V^*) = (V,\pi_V\circ i\otimes\mathrm{id}_V, p\otimes\mathrm{id}_V\circ\pi_V^*)$$

Moreover, if $\mathfrak{a}\hookrightarrow\mathfrak{b}\hookrightarrow\mathfrak{c}$ is a chain of split embeddings, then $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}\circ\mathrm{Res}_{\mathfrak{b},\mathfrak{c}}=\mathrm{Res}_{\mathfrak{a},\mathfrak{c}}$. Under the identification of $\mathsf{DY}_{\mathfrak{b}},\mathsf{DY}_{\mathfrak{a}}$ with the categories of equicontinuous modules over the doubles $\mathfrak{g}_{\mathfrak{b}}$ and $\mathfrak{g}_{\mathfrak{a}}$ respectively, $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}$ coincides with the pullback functor corresponding to the morphism $i\oplus p^t:\mathfrak{g}_{\mathfrak{a}}\to\mathfrak{g}_{\mathfrak{b}}$.

[18]Note that such an i is necessarily an embedding.

5.9. **Diagrammatic Lie bialgebras.** A *diagrammatic Lie (bi)algebra* $\mathfrak{b}$ is the datum of

- a diagram D
- for any $B \subseteq D$, a Lie (bi)algebra $\mathfrak{b}_B$
- for any $B' \subseteq B$, a Lie (bi)algebra morphism $i_{BB'} : \mathfrak{b}_{B'} \to \mathfrak{b}_B$

such that

- for any $B \subseteq D$, $i_{BB} = \mathrm{id}_{\mathfrak{b}_B}$
- for any $B'' \subseteq B' \subseteq B$, $i_{BB'} \circ i_{B'B''} = i_{BB''}$
- for any $B = B' \sqcup B''$
$$i_{BB'} + i_{BB''} : \mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''} \to \mathfrak{b}_B$$
is an isomorphism of Lie (bi)algebras.

The above properties imply in particular that $\mathfrak{b}_\emptyset = 0$, and that $U\mathfrak{b}$ is a diagrammatic algebra, with $(U\mathfrak{b})_B = U\mathfrak{b}_B$ (cf. 3.12).

A morphism $\varphi : \mathfrak{b} \to \mathfrak{c}$ of diagrammatic Lie (bi)algebras with the same underlying diagram D is a collection of Lie (bi)algebra morphisms $\varphi_B : \mathfrak{b}_B \to \mathfrak{c}_B$ labelled by the subdiagrams $B \subseteq D$ such that, for any $B' \subseteq B$, $\varphi_B \circ i^{\mathfrak{b}}_{BB'} = i^{\mathfrak{c}}_{BB'} \circ \varphi_{B'}$.

5.10. **Split diagrammatic Lie bialgebras and Manin triples.** A diagrammatic Lie (bi)algebra $\mathfrak{b}$ is *split* if there are Lie (bi)algebra morphisms $p_{B'B} : \mathfrak{b}_B \to \mathfrak{b}_{B'}$ for any $B' \subseteq B$, such that $p_{B'B} \circ i_{BB'} = \mathrm{id}_{\mathfrak{b}_{B'}}$, and

- for any $B \subseteq D$, $p_{BB} = \mathrm{id}_{\mathfrak{b}_B}$
- for any $B'' \subseteq B' \subseteq B$, $p_{B''B'} \circ p_{B'B} = p_{B''B}$
- for any $B = B' \sqcup B''$
$$p_{B'B} \oplus p_{B''B} : \mathfrak{b}_B \to \mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''}$$
is an isomorphism of Lie (bi)algebras, and is the inverse of $i_{BB'} + i_{BB''}$.[19]

A morphism $\varphi : \mathfrak{b} \to \mathfrak{c}$ of split diagrammatic Lie (bi)algebras with the same underlying diagram is one of the underlying diagrammatic Lie (bi)algebras such that, for any $B' \subseteq B$, $p^{\mathfrak{c}}_{B'B} \circ \varphi_B = \varphi_{B'} \circ p^{\mathfrak{b}}_{B'B}$.

One can define similarly a diagrammatic Manin triple as a diagrammatic Lie algebra $\mathfrak{g} = \{\mathfrak{g}_B\}_{B \subseteq D}$, where each $\mathfrak{g}_B$ is a Manin triple, and the maps $i_{BB'} : \mathfrak{g}_{B'} \to \mathfrak{g}_B$ are split morphisms of Manin triples (see 5.7). The equivalence of categories $\mathsf{sLBA}(\mathsf{k}) \cong \mathsf{sMT}(\mathsf{k})$ implies that a split diagrammatic Lie bialgebra $\mathfrak{b} = \{b_B\}_{B \subseteq D}$ gives rise to a diagrammatic Manin triple $\mathfrak{g}_{\mathfrak{b}} = \{\mathfrak{g}_{\mathfrak{b}_B}\}_{B \subseteq D}$, which will be referred to as the double of $\mathfrak{b}$, and that any such triple arises this way.

Similarly, if $\mathfrak{b}$ is an $\mathbb{N}$–graded split diagrammatic Lie bialgebra with finite–dimensional homogeneous components (*i.e.,* for any $B \subseteq D$, $\mathfrak{b}_B$ is $\mathbb{N}$–graded, with finite–dimensional homogeneous components and, for any $B' \subseteq B$, the morphisms $i_{B'B}$ and $p_{B'B}$ are homogeneous of degree 0), one can similarly define a diagrammatic Lie bialgebra $\mathfrak{g}^{\mathrm{res}}_{\mathfrak{b}}$, with $(\mathfrak{g}^{\mathrm{res}}_{\mathfrak{b}})_B \coloneqq \mathfrak{g}^{\mathrm{res}}_{\mathfrak{b}_B}$, endowed with a canonical morphism of diagrammatic Lie bialgebras $\mathfrak{b} \to \mathfrak{g}^{\mathrm{res}}_{\mathfrak{b}}$.

[19] The requirements on $p_{B'B}$ are formulated so as to mirror those in 5.9. Note, however, that 1) $p_{BB} = \mathrm{id}_{\mathfrak{b}_B}$ follows from $p_{BB} \circ i_{BB} = \mathrm{id}_{b_B}$ and $i_{BB} = \mathrm{id}_{\mathfrak{b}_B}$ and 2) the fact that $p_{B'B} \oplus p_{B''B}$ is the inverse of $i_{BB'} + i_{BB''}$ implies that it is a Lie (bi)algebra morphism. Note also that since $p_{CB} \circ i_{BC} = \mathrm{id}_{b_C}$ for $C = B', B''$, the requirement that $p_{B'B} \oplus p_{B''B} = (i_{BB'} + i_{BB''})^{-1}$ is equivalent to $p_{B'B} \circ i_{BB''} = 0$ for any $B' \perp B''$.

5.11. **Example.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with opposite Borel subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$, Dynkin diagram D, Serre generators $\{e_i, f_i, h_i\}_{i\in D}$, and standard Lie bialgebra structure determined by $\mathfrak{b}_\pm$ and an invariant inner product on $\mathfrak{g}$ (see 11.7). Then $\mathfrak{g}$ is a diagrammatic Lie bialgebra where, for any $B \subseteq D$, $\mathfrak{g}_B \subseteq \mathfrak{g}$ is the subalgebra generated by $\{e_i, f_i, h_i\}_{i\in B}$.

The diagrammatic structure on $\mathfrak{g}$ determines a split diagrammatic one on $\mathfrak{b}_\pm$ as follows. For any $B \subseteq D$, let $\mathfrak{b}_{\pm,B} = \mathfrak{b}_\pm \cap \mathfrak{g}_B$ be the subalgebras generated by $\{h_i, e_i\}_{i\in B}$ and $\{h_i, f_i\}_{i\in B}$ respectively. If $B' \subseteq B$, let $i_{\pm,BB'} : \mathfrak{b}_{\pm,B'} \to \mathfrak{b}_{\pm,B}$ be the standard embedding, and regard $p_{\pm,B'B} = i^t_{\mp,BB'}$ as a map $\mathfrak{b}_{\pm,B} \to \mathfrak{b}_{\pm,B'}$ via the identifications $\mathfrak{b}^*_{\mp,C} \cong \mathfrak{b}_{\pm,C}$ given by the inner product. Then, $\ker(p_{\pm,B'B})$ is a Lie subalgebra in $\mathfrak{b}_{\pm,B}$, and therefore $\{p_{\pm,B'B}\}$ give the required splitting of the Lie bialgebra $\mathfrak{b}_\pm$.

5.12. **Drinfeld–Yetter modules over diagrammatic Lie bialgebras.** The following is straightforward.

Proposition. *Let $\mathfrak{b}$ be a split diagrammatic Lie bialgebra. Then, $\mathfrak{b}$ gives rise to an (a, Υ)–strict symmetric pre–Coxeter category $\mathbb{DY}_\mathfrak{b}$, which is defined as follows.*

- *For any $B \subseteq D$, $\mathbb{DY}_{\mathfrak{b},B}$ is the symmetric monoidal category $\mathsf{DY}_{\mathfrak{b}_B}$.*
- *For any $B' \subseteq B$, the functor $F_{B'B} : \mathbb{DY}_{\mathfrak{b},B} \to \mathbb{DY}_{\mathfrak{b},B'}$ is the restriction functor $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B} : \mathsf{DY}_{\mathfrak{b}_B} \to \mathsf{DY}_{\mathfrak{b}_{B'}}$.*

Note that the orthogonality condition $\mathfrak{b}_{B'\sqcup B''} \simeq \mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''}$ is not needed to define the pre–Coxeter category $\mathbb{DY}_\mathfrak{b}$. However, it is convenient to construct its deformations as we explain in Sections 9–10.

5.13. **Partial monoidal categories.** The notion of diagrammatic Lie bialgebra may be reformulated in terms of monoidal functors between *partial* monoidal categories. A partial monoidal category generalises a monoidal category, in that the tensor product is only assumed to be defined on a full subcategory $\mathcal{C}^{(2)} \subseteq \mathcal{C}\times\mathcal{C}$. A monoidal functor

$$(F, J) : (\mathcal{C}, \mathcal{C}^{(2)}, \otimes_\mathcal{C}, \Phi_\mathcal{C}) \to (\mathcal{D}, \mathcal{D}^{(2)}, \otimes_\mathcal{D}, \Phi_\mathcal{D})$$

between two such categories is the datum of

- a functor $F : \mathcal{C} \to \mathcal{D}$ which preserves the unit, and is such that $F\times F$ maps $\mathcal{C}^{(2)}$ to $\mathcal{D}^{(2)}$
- an isomorphism over $\mathcal{C}^{(2)}$
$$J : \otimes_\mathcal{D} \circ F^2 \to F \circ \otimes_\mathcal{C}$$
which is compatible with the unit and the associativity constraint.

5.14. **Functorial description of diagrammatic Lie bialgebras.** Let $\mathcal{P}(D)$ be the category whose objects are the subdiagrams of D, and the morphisms $B' \to B$ are given by inclusions $B' \subseteq B$. The union $\sqcup$ of orthogonal diagrams is a (symmetric, strict) partial tensor product on $\mathcal{P}(D)$, with $\emptyset$ as unit object.[20] Let $(\mathsf{LBA}(\mathsf{k}), \oplus)$ be the category of Lie bialgebras, with monoidal structure given by the direct sum, and 0 as unit object.

Proposition. *The category of diagrammatic Lie bialgebras is isomorphic to that of monoidal functors $\mathcal{P}(D) \to \mathsf{LBA}(\mathsf{k})$. Specifically,*

[20]Note that $\mathcal{P}(D)$ is the opposite category to the category $\mathfrak{P}(D)$ introduced in 3.5.

(1) A monoidal functor

$$(F, J) : (\mathcal{P}(D), \sqcup) \to (\mathsf{LBA}(\mathsf{k}), \oplus)$$

gives rise to a diagrammatic Lie bialgebra $\mathfrak{b}$ *defined as follows*
- *for any* $B \subseteq D$, $\mathfrak{b}_B = F(B)$
- *for any* $B' \subseteq B$, $i_{BB'} = F(B' \to B)$

Conversely, any diagrammatic Lie bialgebra arises this way for a unique monoidal functor (F, J).

(2) A natural transformation of monoidal functors $(\mathcal{P}(D), \sqcup) \to (\mathsf{LBA}(\mathsf{k}), \oplus)$ *gives rise to a morphism of the corresponding diagrammatic Lie bialgebras, and any such natural transformation arises this way.*

Proof. (1) It is clear that $i_{BB} = \mathrm{id}_{\mathfrak{b}_B}$, and that $i_{BB'} \circ i_{B'B''} = i_{BB''}$ for any $B'' \subseteq B' \subseteq B$. The key point is to observe that the existence of the natural isomorphism $J_{B',B''} : F(B') \oplus F(B'') \to F(B' \sqcup B'')$ for $B' \perp B''$ is equivalent to the requirement that $i_{BB'} + i_{BB''} : \mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''} \to \mathfrak{b}_{B' \sqcup B''}$ be an isomorphism of Lie bialgebras.

To this end, note that the naturality of J implies the commutativity of the following diagram

$$\begin{tikzcd}
F(B') \oplus F(\emptyset) \arrow[r, "F(\mathrm{id}_{B'}) \oplus F(\emptyset \to B'')"] \arrow[d, "J_{B',\emptyset}"'] & F(B') \oplus F(B'') \arrow[d, "J_{B',B''}"] & F(\emptyset) \oplus F(B'') \arrow[l, "F(B' \leftarrow \emptyset) \oplus F(\mathrm{id}_{B''})"'] \arrow[d, "J_{\emptyset,B''}"] \\
F(B') \arrow[r, "F(B' \to B' \sqcup B'')"'] & F(B' \sqcup B'') & F(B'') \arrow[l, "F(B' \sqcup B'' \leftarrow B'')"]
\end{tikzcd}$$

Since $F(\emptyset) = 0$, it follows that $F(\emptyset \to B'') = 0 = F(B' \leftarrow \emptyset)$. Moreover, the compatibility of J with the unit, that is $J_{C,\emptyset} = \mathrm{id}_{F(C)} = J_{\emptyset,C}$, implies that the above diagram reduces to

$$\begin{tikzcd}
& F(B') \oplus F(B'') \arrow[d, "J_{B',B''}"] & \\
F(B') \arrow[ur, "\mathrm{id} \oplus 0"] \arrow[r, "F(B' \to B' \sqcup B'')"'] & F(B' \sqcup B'') & F(B'') \arrow[ul, "0 \oplus \mathrm{id}"'] \arrow[l, "F(B' \sqcup B'' \leftarrow B'')"]
\end{tikzcd}$$

so that $J_{B',B''} = i_{BB'} + i_{BB''}$.

(2) If $(F, J), (G, K)$ are monoidal functors, a natural transformation $F \Rightarrow G$ of the underlying functors is clearly the same as a morphism $\varphi : \mathfrak{b} \to \mathfrak{c}$ of the corresponding diagrammatic Lie bialgebras. The only point is to observe that φ is automatically compatible with the tensor structures, which follows from the commutativity of the following diagram for any $B = B' \sqcup B''$

$$\begin{tikzcd}
\mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''} \arrow[r, "i^{\mathfrak{b}}_{BB'} \oplus i^{\mathfrak{b}}_{BB''}"] \arrow[d, "\varphi_{B'} \oplus \varphi_{B''}"'] & \mathfrak{b}_B \oplus \mathfrak{b}_B \arrow[r, "+"] \arrow[d, "\varphi_B \oplus \varphi_B"] & \mathfrak{b}_B \arrow[d, "\varphi_B"] \\
\mathfrak{c}_{B'} \oplus \mathfrak{c}_{B''} \arrow[r, "i^{\mathfrak{c}}_{BB'} \oplus i^{\mathfrak{c}}_{BB''}"'] & \mathfrak{c}_B \oplus \mathfrak{c}_B \arrow[r, "+"'] & \mathfrak{c}_B
\end{tikzcd}$$

$\square$

Split diagrammatic Lie bialgebras can be described in similar terms. Let $\mathsf{sLBA}(\mathsf{k})$ be the category of split Lie bialgebras (5.1). Then, the category of monoidal functors $(F,J):(\mathcal{P}(D),\sqcup)\to(\mathsf{sLBA}(\mathsf{k}),\oplus)$ is canonically isomorphic to that of split diagrammatic Lie bialgebras. Note also that any such functor is automatically symmetric.

Remark. In view of Proposition 5.14, it is natural to define a diagrammatic object in a monoidal category $(\mathcal{C},\otimes)$ as a monoidal functor $(\mathcal{P}(D),\sqcup)\to(\mathcal{C},\otimes)$, and a morphism of such objects as a natural transformation of the corresponding functors.

6. Diagrammatic Hopf algebras

In this section, we introduce the notion of diagrammatic Hopf algebra and quantised universal enveloping algebra (QUE). We then point out that the quantisation $\mathcal{Q}(\mathfrak{b})$ of a diagrammatic Lie bialgebra $\mathfrak{b}$ is a diagrammatic QUE, and that admissible Drinfeld–Yetter modules over $\mathcal{Q}(\mathfrak{b})$ and its canonical subalgebras give rise to a braided pre–Coxeter category.

6.1. **Drinfeld–Yetter modules over a Hopf algebra** [16, 39]**.** A Drinfeld–Yetter module over a Hopf algebra $\mathfrak{B}$ is a triple $(\mathcal{V},\pi_\mathcal{V},\pi_\mathcal{V}^*)$, where $(\mathcal{V},\pi_\mathcal{V})$ is a left $\mathfrak{B}$–module, $(\mathcal{V},\pi_\mathcal{V}^*)$ a right $\mathfrak{B}$–comodule, and the maps $\pi_\mathcal{V}:\mathfrak{B}\otimes\mathcal{V}\to\mathcal{V}$ and $\pi_\mathcal{V}^*:\mathcal{V}\to\mathfrak{B}\otimes\mathcal{V}$ satisfy the following compatibility condition in $\mathrm{End}(\mathfrak{B}\otimes\mathcal{V})$

$$\pi_\mathcal{V}^*\circ\pi_\mathcal{V}=m^{(3)}\otimes\pi_\mathcal{V}\circ(1\,3)(2\,4)\circ S^{-1}\otimes\mathrm{id}^{\otimes 4}\circ\Delta^{(3)}\otimes\pi_\mathcal{V}^*$$

where $m^{(3)}:\mathfrak{B}^{\otimes 3}\to\mathfrak{B}$ and $\Delta^{(3)}:\mathfrak{B}\to\mathfrak{B}^{\otimes 3}$ are the iterated multiplication and comultiplication respectively, and $S:\mathfrak{B}\to\mathfrak{B}$ is the antipode.

The category $\mathsf{DY}_\mathfrak{B}$ of such modules is a braided monoidal category. For any $\mathcal{V},\mathcal{W}\in\mathsf{DY}_\mathfrak{B}$, the action and coaction on the tensor product $\mathcal{V}\otimes\mathcal{W}$ are defined by

$$\pi_{\mathcal{V}\otimes\mathcal{W}}=\pi_\mathcal{V}\otimes\pi_\mathcal{W}\circ(2\,3)\circ\Delta\otimes\mathrm{id}_{\mathcal{V}\otimes\mathcal{W}}\quad\text{and}\quad\pi_{\mathcal{V}\otimes\mathcal{W}}^*=m^{21}\otimes\mathrm{id}_{\mathcal{V}\otimes\mathcal{W}}\circ(2\,3)\circ\pi_\mathcal{V}^*\otimes\pi_\mathcal{W}^*$$

The associativity constraints are trivial, and the braiding is defined by $\beta_{\mathcal{V}\mathcal{W}}=(1\,2)\circ R_{\mathcal{V}\mathcal{W}}$, where the R–matrix $R_{\mathcal{V}\mathcal{W}}\in\mathrm{End}(\mathcal{V}\otimes\mathcal{W})$ is defined by

$$R_{\mathcal{V}\mathcal{W}}=\pi_\mathcal{V}\otimes\mathrm{id}_\mathcal{W}\circ(1\,2)\circ\mathrm{id}_\mathcal{V}\otimes\pi_\mathcal{W}^*$$

The linear map $R_{\mathcal{V}\mathcal{W}}$ is invertible, with inverse

$$R_{\mathcal{V}\mathcal{W}}^{-1}=\pi_\mathcal{V}\otimes\mathrm{id}_\mathcal{W}\circ S\otimes\mathrm{id}_{\mathcal{V}\otimes\mathcal{W}}\circ(1\,2)\circ\mathrm{id}_\mathcal{V}\otimes\pi_\mathcal{W}^*$$

The braiding $\beta_{\mathcal{V}\mathcal{W}}$ is therefore invertible, with inverse $R_{\mathcal{V}\mathcal{W}}^{-1}\circ(1\,2)$.

6.2. **The finite quantum double** [11]**.** Let $\mathfrak{B}$ be a finite–dimensional Hopf algebra, and $\mathfrak{B}^\circ$ the dual Hopf algebra $\mathfrak{B}^*$ with opposite coproduct. The quantum double of $\mathfrak{B}$ is the unique quasitriangular Hopf algebra $(D\mathfrak{B},R)$ such that 1) $D\mathfrak{B}=\mathfrak{B}\otimes\mathfrak{B}^\circ$ as vector spaces 2) $\mathfrak{B}$ and $\mathfrak{B}^\circ$ are Hopf subalgebras of $D\mathfrak{B}$ and 3) R is the canonical element in $\mathfrak{B}\otimes\mathfrak{B}^\circ\subset D\mathfrak{B}\otimes D\mathfrak{B}$. The multiplication in $D\mathfrak{B}$ is given in Sweedler's notation by

$$b\otimes f\cdot b'\otimes f'=\langle S^{-1}(b'_1),f_1\rangle\langle b'_3,f_3\rangle\, b\cdot b'_2\otimes f_2\cdot f'\tag{6.1}$$

where $b,b'\in\mathfrak{B}$, $f,f'\in\mathfrak{B}^\circ$, and $\langle\cdot,\cdot\rangle:\mathfrak{B}\otimes\mathfrak{B}^\circ\to\mathsf{k}$ is the duality pairing [11, Sec. 13]. The quantum double can also be realised as the double cross product Hopf algebra $\mathfrak{B}\bowtie\mathfrak{B}^*$ associated to a matched pair of Hopf algebras. given by the coadjoint actions of $\mathfrak{B}$ on $\mathfrak{B}^*$ and of $\mathfrak{B}^*$ on $\mathfrak{B}$ [30] (see also [3, Appendix A]).

The category Rep $D\mathfrak{B}$ is canonically isomorphic, as a braided monoidal category, to $\mathsf{DY}_{\mathfrak{B}}$. Namely, there are two braided monoidal functors

$$\mathsf{DY}_{\mathfrak{B}} \underset{\Theta}{\overset{\Xi}{\rightleftarrows}} \mathrm{Rep}\, D\mathfrak{B} \tag{6.2}$$

which are defined as follows

- For any $D\mathfrak{B}$–module $(\mathcal{V}, \xi_{\mathcal{V}})$, $\Theta(\mathcal{V}, \xi_{\mathcal{V}}) = (\mathcal{V}, \pi_{\mathcal{V}}, \pi_{\mathcal{V}}^*)$ is the Drinfeld–Yetter $\mathfrak{B}$–module whose action $\pi_{\mathcal{V}}$ is given by restricting $\xi_{\mathcal{V}}$ to $\mathfrak{B}$, and coaction iby the formula $\pi_{\mathcal{V}}^*(v) = R\,1 \otimes v$.
- For any Drinfeld–Yetter $\mathfrak{B}$–module $(\mathcal{V}, \pi_{\mathcal{V}}, \pi_{\mathcal{V}}^*)$, $\Xi(\mathcal{V}, \pi_{\mathcal{V}}, \pi_{\mathcal{V}}^*) = (\mathcal{V}, \xi_{\mathcal{V}})$ is the $D\mathfrak{B}$–module such that $\mathfrak{B}$ acts by $\pi_{\mathcal{V}}$, and $\phi \in \mathfrak{B}^\circ$ by $\phi \otimes \mathrm{id}_{\mathcal{V}} \circ \pi_{\mathcal{V}}^*$.

One checks easily that the two functors are well–defined, and are each other's inverses [3, Prop. A.4].

6.3. **Quantum double for QUEs.** The construction of the quantum double can be adapted for quantised universal enveloping algebras (QUE). Recall that a QUE is a Hopf algebra $\mathfrak{B}$ over $\mathsf{K} = \mathsf{k}[\![\hbar]\!]$ which reduces modulo $\hbar$ to an enveloping algebra $U\mathfrak{b}$ for some Lie bialgebra $\mathfrak{b}$, and is such that, for any $x \in \mathfrak{b}$,

$$\delta(x) = \frac{\Delta(\widetilde{x}) - \Delta^{21}(\widetilde{x})}{\hbar} \mod \hbar$$

where $\widetilde{x} \in \mathfrak{B}$ is any lift of x. A QUE is of finite type if the underlying Lie bialgebra $\mathfrak{b}$ is finite–dimensional. In this case, the dual $\mathfrak{B}^* = \mathrm{Hom}_{\mathsf{K}}(\mathfrak{B}, \mathsf{K})$ is a quantised formal series Hopf algebra (QFSH), *i.e.,* a topological Hopf algebra over K which reduces modulo $\hbar$ to $\widehat{S\mathfrak{b}} = \prod_n S^n\mathfrak{b}$. Conversely, the dual of a QFSH of finite type is a QUE (cf. [11, 21] or [3, Sec. 2.19]).

If $\mathfrak{B}$ is a QUE, set

$$\mathfrak{B}' = \{b \in \mathfrak{B} \mid (\mathrm{id} - \iota \circ \varepsilon)^{\otimes n} \circ \Delta^{(n)}(b) \in \hbar^n \mathfrak{B}^{\otimes n} \text{ for any } n \geqslant 0\}$$

where $\Delta^{(n)} : \mathfrak{B} \to \mathfrak{B}^{\otimes n}$ is the iterated coproduct. Then, $\mathfrak{B}'$ is a Hopf subalgebra of $\mathfrak{B}$, and a QFSH [11, 21]. In particular, if $\mathfrak{B}$ is of finite type, $\mathfrak{B}^\vee := (\mathfrak{B}')^*$ is a QUE. As in 6.2, $(\mathfrak{B}, \mathfrak{B}^\vee)$ is a matched pair of Hopf algebras [3, A.5]. The double cross product $D\mathfrak{B} = \mathfrak{B} \bowtie \mathfrak{B}^\vee$ is a quasitriangular QUE, whose R–matrix is the canonical element $R \in \mathfrak{B}' \otimes \mathfrak{B}^\vee$ and underlying Lie bialgebra is the Drinfeld double $\mathfrak{g}_{\mathfrak{b}} = \mathfrak{b} \oplus \mathfrak{b}^*$.

This construction extends to the case of *finitely* $\mathbb{N}$*–graded* QUEs, *i.e.,* $\mathbb{N}$–graded Hopf algebras $\mathfrak{B} = \bigoplus_{n \geqslant 0} \mathfrak{B}_n$ such that $\mathfrak{B}_0$ is a QUE of finite type, and each $\mathfrak{B}_n$ is a finitely generated $\mathfrak{B}_0$–module. Note that such a QUE is a quantisation of an $\mathbb{N}$–graded Lie bialgebra with finite–dimensional components and cobracket of degree $d = 0$ (cf. 5.4). Moreover, $\mathfrak{B}' = \bigoplus_{n \geqslant 0}(\mathfrak{B}' \cap \mathfrak{B}_n)$ is also graded, and its *restricted dual* $\mathfrak{B}^\star := \bigoplus_{n \geqslant 0}(\mathfrak{B}' \cap \mathfrak{B}_n)^*$ is a finitely $\mathbb{N}$–graded QUE quantising the restricted dual Lie bialgebra $\mathfrak{b}^\star$. The double cross product $(D\mathfrak{B})^{\mathrm{res}} := \mathfrak{B} \bowtie \mathfrak{B}^\star$ is called the *restricted* quantum double of $\mathfrak{B}$. $(D\mathfrak{B})^{\mathrm{res}}$ is a quasitriangular, finitely $\mathbb{Z}$–graded QUE whose R–matrix is the canonical element in the graded completion of $\mathfrak{B}' \otimes \mathfrak{B}^\star$, and underlying Lie bialgebra is the restricted Drinfeld double $\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}} = \mathfrak{b} \oplus \mathfrak{b}^\star$.

6.4. **Admissible Drinfeld–Yetter modules over a QUE.** The isomorphism (6.2) between the categories of modules over the quantum double and Drinfeld–Yetter modules does not hold as is for a QUE and needs to be corrected.

An *admissible* Drinfeld–Yetter module over a QUE $\mathfrak{B}$ is a Drinfeld–Yetter module $(\mathcal{V}, \pi_\mathcal{V}, \pi_\mathcal{V}^*)$ for which the coaction $\pi_\mathcal{V}^* : \mathcal{V} \to \mathfrak{B} \otimes \mathcal{V}$ factors through $\mathfrak{B}' \otimes \mathcal{V}$. We denote the category of such modules by $\mathsf{DY}_\mathfrak{B}^{\mathrm{adm}}$.[21] We show in [3, Prop. 2.22] that $\mathsf{DY}_\mathfrak{B}^{\mathrm{adm}}$ reduces modulo $\hbar$ to $\mathsf{DY}_\mathfrak{b}$.

The following holds.

- If $\mathfrak{B}$ is a QUE of finite type, since $R \in \mathfrak{B}' \otimes \mathfrak{B}^\vee$, the functors Ξ, Θ from (6.2) define an isomorphism of braided monoidal categories between $\mathsf{DY}_\mathfrak{B}^{\mathrm{adm}}$ and $\mathrm{Rep}\, D\mathfrak{B}$. Moreover, this reduces modulo $\hbar$ to the isomorphism between $\mathsf{DY}_\mathfrak{b}$ and $\mathrm{Rep}\, U\mathfrak{g}_\mathfrak{b}$.
- If $\mathfrak{B}$ is a finitely $\mathbb{N}$–graded QUE, since R belongs to the grading completion of $\mathfrak{B}' \otimes \mathfrak{B}^\star$, the functors Ξ, Θ define an isomorphism of braided monoidal categories between $\mathsf{DY}_\mathfrak{B}^{\mathrm{adm}}$ and the category of $D\mathfrak{B}$–modules whose action of $\mathfrak{B}^\star$ is locally finite (*i.e.,* for any $v \in \mathcal{V}$, $(\mathfrak{B}' \cap \mathfrak{B}_n)^* v = 0$ for $n \gg 0$). Moreover, this reduces modulo $\hbar$ to the isomorphism E^{res} between $\mathsf{DY}_\mathfrak{b}$ and $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}}$ (cf. 5.6).

6.5. **Diagrammatic Hopf algebras.** Let D be a diagram. A *diagrammatic* Hopf algebra with underyling diagram D is a monoidal functor

$$(F, J) : (\mathcal{P}(D), \sqcup) \to (\mathsf{HA}(\mathsf{k}), \otimes)$$

where $\mathsf{HA}(\mathsf{k})$ is the category of Hopf algebras over k (cf. Remark 5.14). Concretely, this consists of the datum of

- for any $B \subseteq D$, a Hopf algebra $\mathfrak{B}_B$
- for any $B' \subseteq B$, a morphism of Hopf algebras $i_{BB'} : \mathfrak{B}_{B'} \to \mathfrak{B}_B$

such that

- for any $B \subseteq D$, $i_{BB} = \mathrm{id}_{\mathfrak{B}_B}$
- for any $B'' \subseteq B' \subseteq B$, $i_{BB'} \circ i_{B'B''} = i_{BB''}$
- for any $B = B' \sqcup B''$,
$$m_B \circ i_{BB'} \otimes i_{BB''} : \mathfrak{B}_{B'} \otimes \mathfrak{B}_{B''} \to \mathfrak{B}_B$$
is an isomorphism of Hopf algebras, where m_B is the multiplication of $\mathfrak{B}_B$.

The above properties imply in particular that $\mathfrak{B}_\emptyset$ is equal to k. Diagrammatic QUEs are defined similarly.

A morphism $\varphi : \mathfrak{B} \to \mathfrak{B}'$ of diagrammatic Hopf algebras (resp. QUEs) is a collection of Hopf algebra morphisms $\varphi_B : \mathfrak{B}_B \to \mathfrak{B}'_B$ labelled by the subdiagrams $B \subseteq D$ such that, for any $B' \subseteq B$, $\varphi_B \circ i_{BB'}^{\mathfrak{B}} = i_{BB'}^{\mathfrak{B}'} \circ \varphi_{B'}$.

6.6. **Split diagrammatic Hopf algebras.** Recall that a split pair of Hopf algebras is the datum of two Hopf algebras $\mathfrak{A}, \mathfrak{B}$ together with Hopf algebra morphisms $\mathfrak{A} \xrightarrow{i} \mathfrak{B} \xrightarrow{p} \mathfrak{A}$ such that $p \circ i = \mathrm{id}_\mathfrak{A}$ [3, Sec. 4.6]. We denote by $(\mathsf{sHA}(\mathsf{k}), \otimes)$ the monoidal category of split Hopf algebras. The objects in $\mathsf{sHA}(\mathsf{k})$ are the same as those in $\mathsf{HA}(\mathsf{k})$, and the morphisms are

$$\mathrm{Hom}_{\mathsf{sHA}(\mathsf{k})}(\mathfrak{A}, \mathfrak{B}) = \{(i, p) \in \mathrm{Hom}_{\mathsf{HA}(\mathsf{k})}(\mathfrak{A}, \mathfrak{B}) \times \mathrm{Hom}_{\mathsf{HA}(\mathsf{k})}(\mathfrak{A}, \mathfrak{B}) \mid p \circ i = \mathrm{id}_\mathfrak{A}\}$$

A *split diagrammatic* Hopf algebra is a monoidal functor $(\mathcal{P}(D), \sqcup) \to (\mathsf{sHA}(\mathsf{k}), \otimes)$. Concretely, this consists of a diagrammatic Hopf algebra $\mathfrak{B} = \{\mathfrak{B}_B\}_{B \subseteq D}$, together

[21] The notion of admissible Drinfeld–Yetter module is due to P. Etingof (private communication), and is studied in detail in [3, 2.20–2.22].

with Hopf algebra morphisms $p_{B'B} : \mathfrak{B}_B \to \mathfrak{B}_{B'}$ for any $B' \subseteq B$, such that $p_{B'B} \circ i_{BB'} = \mathrm{id}_{\mathfrak{B}_{B'}}$ and

- for any B, $p_{BB} = \mathrm{id}_{\mathfrak{b}_B}$
- for any $B'' \subseteq B' \subseteq B$, $p_{B''B'} \circ p_{B'B} = p_{B''B}$
- for any $B = B' \sqcup B''$, $p_{B'B} \otimes p_{B''B} \circ \Delta_B : \mathfrak{B}_B \to \mathfrak{B}_{B'} \otimes \mathfrak{B}_{B''}$ is a morphism of Hopf algebras, and the inverse of $m_B \circ i_{BB'} \otimes i_{BB''}$.

Split diagrammatic QUEs are defined similarly. A morphism $\varphi : \mathfrak{B} \to \mathfrak{B}'$ of split diagrammatic Hopf algebras (resp. QUEs) is one of the underlying diagrammatic Hopf algebras (resp. QUEs) such that, for any $B' \subseteq B$, $p^{\mathfrak{B}}_{B'B} \circ \varphi_B = \varphi_{B'} \circ p^{\mathfrak{B}'}_{B'B}$.

Remark. One can formulate in this context a quantum analogue of the Drinfeld double of a diagrammatic Lie bialgebra defined in 5.10. If $\mathfrak{B}$ is a split diagrammatic Hopf algebra, where $\mathfrak{B}_B$ are finite–dimensional Hopf algebras (resp. finitely $\mathbb{N}$–graded QUE), there is a diagrammatic Hopf algebra $D\mathfrak{B}$ with $(D\mathfrak{B})_B = D\mathfrak{B}_B$ (resp. $(D\mathfrak{B})^{\mathrm{res}}$ with $(D\mathfrak{B})^{\mathrm{res}}_B = (D\mathfrak{B}_B)^{\mathrm{res}}$), endowed with a canonical embedding of diagrammatic Hopf algebras $\mathfrak{B} \to D\mathfrak{B}$ (resp. $\mathfrak{B} \to (D\mathfrak{B})^{\mathrm{res}}$).

6.7. **Drinfeld–Yetter modules over split diagrammatic Hopf algebras.** If $\mathfrak{A} \leftrightarrows \mathfrak{B}$ is a split pair of Hopf algebras, there is a monoidal restriction functor $\mathrm{Res}_{\mathfrak{A},\mathfrak{B}} : \mathsf{DY}_{\mathfrak{B}} \to \mathsf{DY}_{\mathfrak{A}}$ given by

$$\mathrm{Res}_{\mathfrak{A},\mathfrak{B}}(\mathcal{V}, \pi_{\mathcal{V}}, \pi^*_{\mathcal{V}}) = (\mathcal{V}, \pi_{\mathcal{V}} \circ i \otimes \mathrm{id}_{\mathcal{V}}, p \otimes \mathrm{id}_{\mathcal{V}} \circ \pi^*_{\mathcal{V}})$$

If $\mathfrak{A}, \mathfrak{B}$ are QUEs, $\mathrm{Res}_{\mathfrak{A},\mathfrak{B}}$ restricts to a functor $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{B}} \to \mathsf{DY}^{\mathrm{adm}}_{\mathfrak{A}}$.

Proposition. *Let $\mathfrak{B}$ be a split diagrammatic Hopf algebra. Then, $\mathfrak{B}$ gives rise to an (a, Υ)–strict braided pre–Coxeter category $\mathbb{DY}_{\mathfrak{B}}$, which is defined as follows.*

- *For any $B \subseteq D$, $\mathbb{DY}_{\mathfrak{B},B}$ is the braided monoidal category $\mathsf{DY}_{\mathfrak{B}_B}$.*
- *For any $B' \subseteq B$, the functor $F_{B'B} : \mathbb{DY}_{\mathfrak{B},B} \to \mathbb{DY}_{\mathfrak{B},B'}$ is the restriction functor $\mathrm{Res}_{\mathfrak{B}_{B'},\mathfrak{B}_B} : \mathsf{DY}_{\mathfrak{B}_B} \to \mathsf{DY}_{\mathfrak{B}_{B'}}$.*

Similarly, a split diagrammatic QUE $\mathfrak{B}$ gives rise to a braided pre–Coxeter category $\mathbb{DY}^{\mathrm{adm}}_{\mathfrak{B}}$ given by $\mathbb{DY}^{\mathrm{adm}}_{\mathfrak{B},B} = \mathsf{DY}^{\mathrm{adm}}_{\mathfrak{B}_B}$.

6.8. **Quantisation of diagrammatic Lie bialgebras.** In [15, 16], Etingof and Kazhdan construct a quantisation functor $\mathcal{Q}$ from the category of Lie bialgebras over k to the category of quantised universal enveloping algebras over $\mathsf{K} = \mathsf{k}[\![\hbar]\!]$. One checks easily that $\mathcal{Q}$ respects direct sums, *i.e.,* for any Lie bialgebras $\mathfrak{a}, \mathfrak{b}$, there is an isomorphism of Hopf algebras $J_{\mathfrak{a},\mathfrak{b}} : \mathcal{Q}(\mathfrak{a}) \otimes \mathcal{Q}(\mathfrak{b}) \to \mathcal{Q}(\mathfrak{a} \oplus \mathfrak{b})$. In fact, this holds for any quantisation functor.

Proposition. *Every quantisation functor $\mathcal{Q}$ is canonically endowed with a monoidal structure $(\mathcal{Q}, J) : (\mathsf{LBA}(\mathsf{k}), \oplus) \to (\mathsf{QUE}(\mathsf{K}), \otimes)$.*

Proof. The result is an easy consequence of Radford's theorem [31]. Namely, let $i_{\mathfrak{a}} : \mathfrak{a} \to \mathfrak{a} \oplus \mathfrak{b}$ and $p_{\mathfrak{a}} : \mathfrak{a} \oplus \mathfrak{b} \to \mathfrak{a}$ be the canonical injection of and projection to $\mathfrak{a}$ and set $\pi_{\mathfrak{a}} = i_{\mathfrak{a}} \circ p_{\mathfrak{a}}$. Then, $\mathcal{Q}(\mathfrak{a} \oplus \mathfrak{b})$ projects onto $\mathcal{Q}(\mathfrak{a})$ through $\mathcal{Q}(i_{\mathfrak{a}})$ and $\mathcal{Q}(p_{\mathfrak{a}})$. By Radford's theorem, $\mathcal{Q}(\mathfrak{a} \oplus \mathfrak{b})$ is canonically isomorphic, as a Hopf algebra, to the *Radford product* $\mathcal{Q}(\mathfrak{a}) \star L$, where $L = \{x \in \mathcal{Q}(\mathfrak{a} \oplus \mathfrak{b}) \mid \mathcal{Q}(\pi_{\mathfrak{a}}) \otimes \mathrm{id} \circ \Delta(x) = 1 \otimes x\}$. It is easy to show that, in this case, $L = \mathcal{Q}(\mathfrak{b})$ and $\mathcal{Q}(\mathfrak{a}) \star \mathcal{Q}(\mathfrak{b}) = \mathcal{Q}(\mathfrak{a}) \otimes \mathcal{Q}(\mathfrak{b})$. The isomorphism $J_{\mathfrak{a},\mathfrak{b}} : \mathcal{Q}(\mathfrak{a}) \otimes \mathcal{Q}(\mathfrak{b}) \to \mathcal{Q}(\mathfrak{a} \oplus \mathfrak{b})$ is given by $J_{\mathfrak{a},\mathfrak{b}} = m_{\mathcal{Q}(\mathfrak{a} \oplus \mathfrak{b})} \circ \mathcal{Q}(i_{\mathfrak{a}}) \otimes \mathcal{Q}(i_{\mathfrak{b}})$, it is natural and defines a monoidal structure on $\mathcal{Q}$. $\square$

The same holds for $\mathsf{sLBA}(\mathsf{k})$ and $\mathsf{sQUE}(\mathsf{K})$, since the quantisation of a split pair of Lie bialgebras is a split pair of QUEs.

Corollary. *The quantisation of a (split) diagrammatic Lie bialgebra is a (split) diagrammatic QUE.*

PROOF. A (split) diagrammatic Lie bialgebra is a monoidal functor $(\mathcal{P}(D),\sqcup)\to((\mathsf{s})\mathsf{LBA}(\mathsf{k}),\oplus)$. By composition with the quantisation functor, we obtain a monoidal functor $(\mathcal{P}(D),\sqcup)\to((\mathsf{s})\mathsf{QUE}(\mathsf{K}),\oplus)$, *i.e.,* a (split) diagrammatic QUE. $\square$

6.9. **Drinfeld–Yetter $\mathcal{Q}(\mathfrak{b})$–modules.** The following is a direct consequence of Propositions 6.8 and 6.7.

Corollary. *Let $\mathcal{Q}:\mathsf{LBA}(\mathsf{k})\to\mathsf{QUE}(\mathsf{K})$ be a quantisation functor, and $\mathfrak{b}$ a split diagrammatic Lie bialgebra. Then, there is an (a,Υ)–strict braided pre–Coxeter category $\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ defined by the following data*

- *For any $B\subseteq D$, $\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}),B}$ is the braided monoidal category $\mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_B)}$*
- *For any $B'\subseteq B$ and $\mathcal{F}\in\mathsf{Mns}(B,B')$, the functor*
$$F_{\mathcal{F}}:\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}),B}\to\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}),B'}$$
is the restriction functor $\mathrm{Res}_{\mathcal{Q}(\mathfrak{b}_{B'}),\mathcal{Q}(\mathfrak{b}_B)}$.

One checks easily that $\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ reduces modulo $\hbar$ to the braided pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}}$defined in 5.12. In 10.10, we construct an equivalence of pre–Coxeter categories between $\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ and a (non a–strict) deformation of $\mathbb{DY}_{\mathfrak{b}}$.

7. Diagrammatic PROPs

We review in this section the definition of PROPs, and introduce a PROP which governs split diagrammatic Lie bialgebras.

7.1. PROPs [27, 29, 12, 3]. A PROP is a k–linear, strict, symmetric monoidal category P whose objects are the non–negative integers, and such that $[n]\otimes[m]=[n+m]$. In particular, $[0]$ is the unit object and $[1]^{\otimes n}=[n]$. A morphism of PROPs is a symmetric monoidal functor $\mathcal{G}:\mathsf{P}\to\mathsf{Q}$ which is the identity on objects, and is endowed with the trivial tensor structure
$$\mathrm{id}:\mathcal{G}([m]_{\mathsf{P}})\otimes\mathcal{G}([n]_{\mathsf{P}})=[m]_{\mathsf{Q}}\otimes[n]_{\mathsf{Q}}=[m+n]_{\mathsf{Q}}=\mathcal{G}([m+n]_{\mathsf{P}})$$

Fix henceforth a complete bracketing b_n on n letters for any $n\geqslant 2$, and set $\mathbf{b}=\{b_n\}_{n\geqslant 2}$. A *module* over P in a symmetric monoidal category $\mathcal{N}$ is a symmetric monoidal functor $(\mathcal{G},J):\mathsf{P}\to\mathcal{N}$ such that[22]
$$\mathcal{G}([n])=\mathcal{G}([1])^{\otimes n}_{b_n}$$
and the following diagram is commutative
$$\begin{tikzcd}
\mathcal{G}([m])\otimes\mathcal{G}([n]) \arrow[r, "J_{[m],[n]}"] \arrow[d, equal] & \mathcal{G}([m+n]) \arrow[d, equal] \\
\mathcal{G}([1])^{\otimes m}_{b_m}\otimes\mathcal{G}([1])^{\otimes n}_{b_n} \arrow[r, "\Phi"'] & \mathcal{G}([1])^{\otimes(m+n)}_{b_{m+n}}
\end{tikzcd}\tag{7.1}$$

[22]In a monoidal category $(\mathcal{C},\otimes)$, $V^{\otimes n}_{b_n}$ denotes the n–fold tensor product of $V\in\mathcal{C}$ bracketed according to b_n. For example $V^{\otimes 3}_{(\bullet\bullet)\bullet}=(V\otimes V)\otimes V$.

where Φ is the associativity constraint in $\mathcal{N}$.[23] A *morphism* of modules over P is a natural transformation of functors. The category of P–modules in $\mathcal{N}$ is denoted by $\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\mathsf{P},\mathcal{N})$.

7.2. **The PROPs LA, LCA and LBA.** Let LA be the PROP generated by a morphism $\mu:[2]\to[1]$, subject to the relations

$$\mu\circ(\mathrm{id}_{[2]}+(1\,2))=0 \qquad\text{and}\qquad \mu\circ(\mu\otimes\mathrm{id}_{[1]})\circ(\mathrm{id}_{[3]}+(1\,2\,3)+(3\,1\,2))=0$$

as morphisms $[2]\to[1]$ and $[3]\to[1]$ respectively. Then, there is a canonical isomorphism of categories $\mathsf{Fun}_{\mathbf{b}}(\mathsf{LA},\mathrm{Vect}_{\mathsf{k}})\simeq\mathsf{LA}(\mathsf{k})$, where $\mathsf{LA}(\mathsf{k})$ is the category of Lie algebras over k. We denote by LCA and LBA the PROPs corresponding to the notions of Lie coalgebras and Lie bialgebras.

7.3. **The Karoubi envelope.** Recall that the Karoubi envelope of a category $\mathcal{C}$ is the category $\mathsf{Kar}(\mathcal{C})$ whose objects are pairs (X,π), where $X\in\mathcal{C}$ and $\pi:X\to X$ is an idempotent. The morphisms in $\mathsf{Kar}(\mathcal{C})$ are defined as

$$\mathsf{Kar}(\mathcal{C})((X,\pi),(Y,\rho))=\{f\in\mathcal{C}(X,Y)\mid\rho\circ f=f=f\circ\pi\}$$

with $\mathrm{id}_{(X,\pi)}=\pi$. In particular, $\mathsf{Kar}(\mathcal{C})((X,\mathrm{id}),(Y,\mathrm{id}))=\mathcal{C}(X,Y)$, so that the functor $\mathcal{C}\to\mathsf{Kar}(\mathcal{C})$ which maps $X\mapsto(X,\mathrm{id})$ and $f\mapsto f$ is fully faithful.

Every idempotent in $\mathsf{Kar}(\mathcal{C})$ splits canonically. Namely, if $q\in\mathsf{Kar}(\mathcal{C})((X,\pi),(X,\pi))$ satisfies $q^2=q$, the maps

$$i=q:(X,q)\to(X,\pi) \qquad\text{and}\qquad p=q:(X,\pi)\to(X,q)$$

satisfy $i\circ p=q$ and $p\circ i=\mathrm{id}_{(X,q)}$.

If P is a PROP, we denote by $\underline{\mathsf{P}}$ the closure under infinite direct sums of the Karoubi completion of P. If $\mathcal{N}$ is a symmetric monoidal category, a *module* over $\underline{\mathsf{P}}$ in $\mathcal{N}$ is a symmetric monoidal functor $\underline{\mathsf{P}}\to\mathcal{N}$ such that the composition $\mathsf{P}\to\underline{\mathsf{P}}\to\mathcal{N}$ is a module over P. We denote the category of such modules by $\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\underline{\mathsf{P}},\mathcal{N})$. It is clear that, if $\mathcal{N}$ is Karoubi complete and closed under infinite direct sums, the pull–back functor

$$\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\underline{\mathsf{P}},\mathcal{N})\to\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\mathsf{P},\mathcal{N})$$

is an equivalence of categories.

7.4. **Diagrammatic PROPs.** Let D be a non–empty diagram. We denote by P_D the PROP generated by an idempotent $\theta_B:[1]\to[1]$ for any $B\subseteq D$ subject to the relations

- $\theta_D=\mathrm{id}_{[1]}$
- for any $B'\subseteq B$, $\theta_{B'}\circ\theta_B=\theta_{B'}=\theta_B\circ\theta_{B'}$
- for any $B'\perp B''$, $\theta_{B'\sqcup B''}=\theta_{B'}+\theta_{B''}$.

[23] Note that the requirement (7.1) determines J uniquely. In fact, given any functor $\mathcal{G}:\mathsf{P}\to\mathcal{N}$ such that $\mathcal{G}([n])=\mathcal{G}([1])^{\otimes n}_{b_n}$, (7.1) defines a family of isomorphisms $J_{m,n}:\mathcal{G}([m])\otimes\mathcal{G}([n])\to\mathcal{G}([m+n])$, which is easily seen to be compatible with the commutativity and associativity constraints in P and $\mathcal{N}$. Such a J, however, need not be natural with respect to morphisms in P, that is satisfy $\mathcal{G}(f\otimes g)=J_{m_2,n_2}\cdot G(f)\otimes G(g)\cdot J^{-1}_{m_1,n_1}$ for any $f\in\mathsf{P}([m_1],[m_2])$ and $g\in\mathsf{P}([n_1],[n_2])$. For example, if $\mathcal{N}$ is strict, then $J=\mathrm{id}$, and J is natural if and only if $\mathcal{G}$ is multiplicative with respect to tensor products of morphisms.

The above relations imply that $\theta_\emptyset = 0$, and that $\theta_{B'} \circ \theta_{B''} = 0 = \theta_{B''} \circ \theta_{B'}$ for any $B' \perp B''$ since if p, q are idempotents, $p + q$ is an idempotent if and only if $pq = 0 = qp$.[24]

Let Q be a PROP, and consider the PROP Q_D generated by the morphisms in Q and P_D subject to the relation

$$\theta_B^{\otimes m} \circ f = f \circ \theta_B^{\otimes n}$$

for any $f \in \mathsf{Q}([n], [m])$ and $B \subseteq D$.

7.5. **The PROP LBA_D.** By definition, LBA_D is generated by a Lie bialgebra object $([1], \mu, \delta)$, and idempotents $\theta_B \in \mathrm{End}([1])$, $B \subseteq D$, which are Lie bialgebra maps.

For any category $\mathcal{C}$, denote by $\mathsf{s}\mathcal{C}$ the category with the same objects as $\mathcal{C}$, and with a morphism $X \to Y$ in $\mathsf{s}\mathcal{C}$ given by a pair of morphisms $i : X \to Y$, $p : Y \to X$ in $\mathcal{C}$ such that $p \circ i = \mathrm{id}_X$.

Proposition. *Let $\mathcal{N}$ be a k–linear, symmetric monoidal category, and $\mathsf{LBA}(\mathcal{N})$ the category of Lie bialgebras in $\mathcal{N}$. Let $(\mathcal{P}(D), \sqcup)$ be the partial monoidal category of subdiagrams of D introduced in 5.14. Then, there is a canonical isomorphism of categories*

$$\mathsf{Fun}_{\mathfrak{b}}(\underline{\mathsf{LBA}}_D, \mathcal{N}) \simeq \mathsf{Fun}_{\otimes}\left((\mathcal{P}(D), \sqcup), (\mathsf{sLBA}(\mathcal{N}), \oplus)\right)$$

In particular, the notions of module over $\underline{\mathsf{LBA}}_D$ and split diagrammatic Lie bialgebra in $\mathcal{N}$ coincide.

Proof. Let $\mathcal{T} : \mathcal{P}(D) \to \underline{\mathsf{sLBA}}_D$ be the functor given by

- $\mathcal{T}(B) = ([1], \theta_B)$
- $\mathcal{T}(B' \subseteq B) = (i = \theta_{B'} : ([1], \theta_{B'}) \to ([1], \theta_B), p = \theta_{B'} : ([1], \theta_B) \to ([1], \theta_{B'}))$

$\mathcal{T}$ is a tensor functor $(\mathcal{P}(D), \sqcup) \to (\underline{\mathsf{sLBA}}_D, \oplus)$ with the (iso)morphism $\mathcal{T}(B') \oplus \mathcal{T}(B'') \to \mathcal{T}(B' \sqcup B'')$ given by the pair of morphisms

$$i = \theta_{B'} + \theta_{B''} : ([1] \oplus [1], \theta_{B'} \oplus \theta_{B''}) \to ([1], \theta_{B' \sqcup B''})$$

$$p = \theta_{B'} \oplus \theta_{B''} : ([1], \theta_{B' \sqcup B''}) \to ([1] \oplus [1], \theta_{B'} \oplus \theta_{B''})$$

which are each other's inverses because $\theta_{B' \sqcup B''} = \theta_{B'} + \theta_{B''}$.

The functor $\mathsf{Fun}_{\mathfrak{b}}(\underline{\mathsf{LBA}}_D, \mathcal{N}) \to \mathsf{Fun}_{\otimes}(\mathcal{P}(D), \mathsf{sLBA}(\mathcal{N}))$ is defined by precomposition with $\mathcal{T}$, and is easily seen to be an isomorphism. $\square$

8. Universal algebras

In this section, we define a family of algebras which are universal analogues of the tensor powers $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$ of the enveloping algebra of the double of a diagrammatic Lie bialgebra.

8.1. **Colored PROPs.** A *colored* PROP P is a k–linear, strict, symmetric monoidal category whose objects are finite sequences over a set A, *i.e.,*

$$\mathsf{Obj}(\mathsf{P}) = \coprod_{n \geqslant 0} \mathsf{A}^n$$

with tensor product given by the concatenation of sequences, and tensor unit given by the empty sequence. Modules over a colored PROP P and its closure $\underline{\mathsf{P}}$ are defined as in 7.1 and 7.3, respectively.

[24]If p, q are idempotents, $(p+q)^2 = p + q$ is equivalent to $pq = -qp$. This implies $pq = pq^2 = -qpq = q^2p = qp$, and therefore $pq = 0$.

8.2. **Universal Drinfeld–Yetter modules.** Given a diagram D and $n \geqslant 0$, the category $\underline{\mathsf{DY}}^n_{\mathrm{D}}$ is the colored PROP generated by $n+1$ objects, $[1]$ and $\{\underline{\mathsf{V}}_k\}_{k=1}^n$, and morphisms

- $\theta_B : [1] \to [1]$, $B \subseteq D$
- $\mu : [2] \to [1]$, $\delta : [1] \to [2]$
- $\pi_k : [1] \otimes \underline{\mathsf{V}}_k \to \underline{\mathsf{V}}_k$ and $\pi_k^* : \underline{\mathsf{V}}_k \to [1] \otimes \underline{\mathsf{V}}_k$

such that

- $([1], \{\theta_B\}_{B\subseteq D}, \mu, \delta)$ is an $\underline{\mathsf{LBA}}_{\mathrm{D}}$–module in $\underline{\mathsf{DY}}^n_{\mathrm{D}}$
- every $(\underline{\mathsf{V}}_k, \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over $[1]$

In particular, $\underline{\mathsf{DY}}^0_{\mathrm{D}} = \underline{\mathsf{LBA}}_{\mathrm{D}}$.

8.3. **Modules over $\underline{\mathsf{DY}}^n_{\mathbf{D}}$.** If $\mathcal{N}$ is a k–linear symmetric monoidal category, $\underline{\mathsf{DY}}^n_{\mathrm{D}}$–modules in $\mathcal{N}$ are isomorphic to the category whose objects are tuples $(\mathfrak{b}; V_1, \dots, V_n)$ consisting of a diagrammatic Lie bialgebra $\mathfrak{b}$ in $\mathcal{N}$, and n Drinfeld–Yetter modules $V_1, \dots, V_n \in \mathcal{N}$ over $\mathfrak{b}_D$. A morphism $(\mathfrak{b}; V_1, \dots, V_n) \mapsto (\mathfrak{c}; W_1, \dots, W_n)$ is a tuple $(\phi; f_1, \dots, f_n)$, where $\phi : \mathfrak{b} \to \mathfrak{c}$ is a morphism of diagrammatic Lie bialgebras, and $f_i : V_i \to W_i$ are such that the following diagrams are commutative

$$
\begin{tikzcd}
\mathfrak{b}_D \otimes V_i \arrow[r, "\pi_{V_i}"] \arrow[d, "\phi_D \otimes f_i"'] & V_i \arrow[d, "f_i"] \\
\mathfrak{c}_D \otimes W_i \arrow[r, "\pi_{W_i}"'] & W_i
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V_i \arrow[r, "\pi^*_{V_i}"] \arrow[d, "f_i"'] & \mathfrak{b}_D \otimes V_i \arrow[d, "\phi_D \otimes f_i"] \\
W_i \arrow[r, "\pi^*_{W_i}"'] & \mathfrak{c}_D \otimes W_i
\end{tikzcd}
$$

so that f_i is a morphism of $\mathfrak{b}_D$–modules $V_i \to \phi_D^* W_i$ as well as a morphism of $\mathfrak{c}_D$–comodules $(\phi_D)_* V_i \to W_i$.

8.4. **Universal algebras.** Let $\mathfrak{U}^n_{\mathrm{D}}$ be the algebra defined by

$$\mathfrak{U}^n_{\mathrm{D}} = \mathsf{End}_{\underline{\mathsf{DY}}^n_{\mathrm{D}}}(\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n)$$

Let $\mathcal{N}$ be a symmetric tensor category and $(\mathfrak{b}; V_1, \dots, V_n)$ a $\underline{\mathsf{DY}}^n_{\mathrm{D}}$–module in $\mathcal{N}$. The corresponding realisation functor $\mathcal{G}_{\mathfrak{b};V} : \underline{\mathsf{DY}}^n_{\mathrm{D}} \to \mathcal{N}$ yields a homomorphism $\mathfrak{U}^n_{\mathrm{D}} \to \mathrm{End}_{\mathcal{N}}(V_1 \otimes \cdots \otimes V_n)$. We shall need the following.

Lemma. *Let $(\mathfrak{b}; V_1, \dots, V_n)$ and $(\mathfrak{c}; W_1, \dots, W_n)$ be two $\underline{\mathsf{DY}}^n_D$–modules in $\mathcal{N}$, $\phi : \mathfrak{b} \to \mathfrak{c}$ a morphism of split diagrammatic Lie bialgebras, and*

$$f : V_1 \otimes \cdots \otimes V_n \longrightarrow W_1 \otimes \cdots \otimes W_n$$

a morphism which intertwines the action of $\mathfrak{b}_D$ and the coaction of $\mathfrak{c}_D$ on each tensor factor. Then, f intertwines the action of $\mathfrak{U}^n_D$ on $V_1 \otimes \cdots \otimes V_n$ and $W_1 \otimes \cdots \otimes W_n$.

Proof. Let $\mathcal{G}_{\mathfrak{b};V}, \mathcal{G}_{\mathfrak{c};W} : \underline{\mathsf{DY}}^n_{\mathrm{D}} \to \mathcal{N}$ be the realisation functors corresponding to $(\mathfrak{b}; V_1, \dots, V_n)$ and $(\mathfrak{c}; W_1, \dots, W_n)$.

By 8.3, the result holds if f is of the form $f_1 \otimes \cdots \otimes f_n$, where each $f_k : V_k \to W_k$ intertwines the action of $\mathfrak{b}_D$ and coaction of $\mathfrak{c}_D$. Indeed, in that case $(\phi; f_1, \dots, f_n)$ gives rise to a morphism $\mathcal{G}_{\mathfrak{b};V} \to \mathcal{G}_{\mathfrak{c};W}$, whose value on $V_1 \otimes \cdots \otimes V_n$ is f.

More generally, consider the colored PROP $\underline{\mathsf{DY}}^{1,n}_{\mathrm{D}}$ generated by an $\underline{\mathsf{LBA}}_{\mathrm{D}}$–module $([1], \{\theta_B\}_{B\subseteq D}, \mu, \delta)$, together with an object $\underline{\mathsf{V}}$ endowed with n commuting actions

$\pi_k : [1]\otimes\underline{\mathsf{V}} \to \underline{\mathsf{V}}$, and n commuting coactions $\pi_k^* : \underline{\mathsf{V}} \to [1]\otimes\underline{\mathsf{V}}$ such that $(\underline{\mathsf{V}}, \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over [1] for any $1 \le k \le n$. There is a natural tensor functor $\Delta : \underline{\mathsf{DY}}^{1,n}_{\mathrm{D}} \to \underline{\mathsf{DY}}^n_{\mathrm{D}}$ which maps [1] to [1] and $\underline{\mathsf{V}}$ to $\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$.

The pair $(\phi; f)$ gives rise to a morphism of functors $\mathcal{G}_{\mathfrak{b};V} \circ \Delta \to \mathcal{G}_{\mathfrak{c};W} \circ \Delta$, so that f intertwines the action of $\mathrm{End}_{\underline{\mathsf{DY}}^{1,n}_{\mathrm{D}}}(\underline{\mathsf{V}})$ on $V_1 \otimes \cdots \otimes V_n$ and $W_1 \otimes \cdots \otimes W_n$. The result now follows because the functor Δ is full. □

8.5. **Diagrammatic structure on universal algebras.** For any $B' \subseteq B$, there is a canonical realisation functor $\underline{\mathsf{DY}}^n_{B'} \to \underline{\mathsf{DY}}^n_B$ which sends the object $[1]_{B'}$ in $\underline{\mathsf{DY}}^n_{B'}$ to the Lie bialgebra $\theta_{B'}([1]_B) = ([1]_B, \theta_{B'})$ in $\underline{\mathsf{DY}}^n_B$, and each $(\underline{\mathsf{V}}_{B',k}, \pi_{B',k}, \pi^*_{B',k})$ to

$$\mathrm{Res}_{\theta_{B'}([1]_B),[1]_B}(\underline{\mathsf{V}}_{B,k}, \pi_{B,k}, \pi^*_{B,k}) = (\underline{\mathsf{V}}_{B,k}, \pi_{B,k} \circ \theta_{B'} \otimes \mathrm{id}, \theta_{B'} \otimes \mathrm{id} \circ \pi^*_{B,k})$$

where $\theta_{B'}$ is regarded both as the split injection $([1]_B, \theta_{B'}) \to [1]_B$ and projection $[1]_B \to ([1]_B, \theta_{B'})$ (cf. 7.3). The functor induces a homomorphism $\mathsf{i}_{BB'} : \mathfrak{U}^n_{B'} \to \mathfrak{U}^n_B$, and it is clear that $\mathsf{i}_{BB} = \mathrm{id}_{\mathfrak{U}^n_B}$, and $\mathsf{i}_{BB'} \circ \mathsf{i}_{B'B''} = \mathsf{i}_{BB''}$ for any $B'' \subseteq B' \subseteq B$.

Proposition. *The algebras $\{\mathfrak{U}^n_B\}_{B\subseteq D}$ and maps $\{\mathsf{i}_{BB'}\}_{B'\subseteq B}$ give rise to a lax diagrammatic algebra, which we denote by $\mathfrak{U}^n$.*

Proof. We need to prove that if $B' \perp B''$, the images of $\mathsf{i}_{DB'}$ and $\mathsf{i}_{DB''}$ commute in $\mathfrak{U}^n_D$. This can be proved by a direct computation [4, Prop. 10.6]. We give a more conceptual proof below.

By Lemma 8.4, it suffices to show that the action of $\mathfrak{U}^n_{B''}$ on $\underline{\mathsf{V}}_1 \otimes\cdots\otimes \underline{\mathsf{V}}_n \in \underline{\mathsf{DY}}^n_{\mathrm{D}}$ commutes with the action and coaction of $[1]_{B'}$ on each $\underline{\mathsf{V}}_k$. It is easy to check that each of these commutes with both the action and the coaction of $[1]_{B''}$ on $\underline{\mathsf{V}}_k$. This implies that the maps

$$\pi_{B',k} : \underline{\mathsf{V}}_1 \otimes \cdots \otimes ([1]_{B'} \otimes \underline{\mathsf{V}}_k) \otimes \cdots \otimes \underline{\mathsf{V}}_n \longrightarrow \underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$$
$$\pi^*_{B',k} : \underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n \longrightarrow \underline{\mathsf{V}}_1 \otimes \cdots \otimes ([1]_{B'} \otimes \underline{\mathsf{V}}_k) \otimes \cdots \otimes \underline{\mathsf{V}}_n$$

commute with the action and coaction of $[1]_{B''}$ on each tensor factor, where $[1]_{B'}$ is given the structure of trivial Drinfeld–Yetter module over $[1]_{B''}$.

By Lemma 8.4, if $x'' \in \mathfrak{U}^n_{B''}$, and $x''_{\underline{\mathsf{V}}_1,\ldots,\underline{\mathsf{V}}_n}$ (resp. $x''_{\underline{\mathsf{V}}_1,\ldots,[1]_{B'}\otimes\underline{\mathsf{V}}_k,\ldots,\underline{\mathsf{V}}_n}$) denote its action on $\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$ (resp. $\underline{\mathsf{V}}_1 \otimes \cdots \otimes ([1]_{B'} \otimes \underline{\mathsf{V}}_k) \otimes \cdots \otimes \underline{\mathsf{V}}_n$), then

$$x''_{\underline{\mathsf{V}}_1,\ldots,\underline{\mathsf{V}}_n} \cdot \pi_{B',k} = \pi_{B',k} \cdot x''_{\underline{\mathsf{V}}_1,\ldots,[1]_{B'}\otimes\underline{\mathsf{V}}_k,\ldots,\underline{\mathsf{V}}_n}$$
$$\pi^*_{B',k} \cdot x''_{\underline{\mathsf{V}}_1,\ldots,\underline{\mathsf{V}}_n} = x''_{\underline{\mathsf{V}}_1,\ldots,[1]_{B'}\otimes\underline{\mathsf{V}}_k,\ldots,\underline{\mathsf{V}}_n} \cdot \pi^*_{B',k}$$

The conclusion now follows from the fact that, since $[1]_{B'}$ is regarded as a trivial Drinfeld–Yetter module over $[1]_{B''}$,

$$x''_{\underline{\mathsf{V}}_1,\ldots,[1]_{B'}\otimes\underline{\mathsf{V}}_k,\ldots,\underline{\mathsf{V}}_n} = \mathrm{id}_{[1]_{B'}} \otimes x''_{\underline{\mathsf{V}}_1,\ldots,\underline{\mathsf{V}}_n}$$

under the identification $\underline{\mathsf{V}}_1 \otimes\cdots\otimes ([1]_{B'} \otimes \underline{\mathsf{V}}_k) \otimes\cdots\otimes \underline{\mathsf{V}}_n \cong [1]_{B'} \otimes \underline{\mathsf{V}}_1 \otimes\cdots\otimes \underline{\mathsf{V}}_n$. □

Remark. We show in [4] that, for any $B' \subseteq B$, the homomorphism $\mathsf{i}_{BB'} : \mathfrak{U}^n_{B'} \to \mathfrak{U}^n_B$ is injective. We shall therefore regard $\mathfrak{U}^n_{B'}$ as a subalgebra of $\mathfrak{U}^n_B$ and, for $x \in \mathfrak{U}^n_{B'}$, write $x \in \mathfrak{U}^n_B$ instead of $\mathsf{i}_{BB'}(x) \in \mathfrak{U}^n_B$. Moreover, $\{\mathfrak{U}^n_B\}_{B\subseteq D}$ is a diagrammatic algebra, since multiplication induces an isomorphism $\mathfrak{U}^n_{B_1\sqcup B_2} \cong \mathfrak{U}^n_{B_1} \otimes \mathfrak{U}^n_{B_2}$ [4, Prop. 10.6 (4)].

8.6. **Fiber functors and diagrammatic structures.** Let now $\mathfrak{b}$ be a split diagrammatic Lie bialgebra. For any $B \subseteq D$, let

$$\mathsf{f}_{\mathfrak{b}_B} : \mathsf{DY}_{\mathfrak{b}_B} \to \mathrm{Vect}_{\mathsf{k}} \qquad \text{and} \qquad \mathcal{U}^n_{\mathfrak{b}_B} = \mathsf{End}\left(\mathsf{f}^{\boxtimes n}_{\mathfrak{b}_B}\right)$$

be the forgetful functor and algebra of endomorphisms of $\mathsf{f}^{\boxtimes n}_{\mathfrak{b}_B}$. By definition, an element of $\mathcal{U}^n_{\mathfrak{b}_B}$ is a collection $x_{V_1,\dots,V_n} \in \mathrm{End}_{\mathsf{k}}(V_1\otimes\cdots\otimes V_n)$ labelled by $V_1,\dots,V_n \in \mathsf{DY}_{\mathfrak{b}_B}$ such that if $f_k \in \mathrm{End}_{\mathsf{DY}_{\mathfrak{b}_B}}(V_k, W_k)$, $1 \leq k \leq n$, then

$$f_1 \otimes \cdots \otimes f_n \circ x_{V_1,\dots,V_n} = x_{W_1,\dots,W_n} \circ f_1 \otimes \cdots \otimes f_n$$

The equivalence between $\mathsf{DY}_{\mathfrak{b}_B}$ and equicontinuous modules over $\mathfrak{g}_{\mathfrak{b}_B}$ (Section 5.6) gives rise to a map $U\mathfrak{g}^{\otimes n}_{\mathfrak{b}_B} \to \mathcal{U}^n_{\mathfrak{b}_B}$, which is an isomorphism if $\dim \mathfrak{b}_B < \infty$.

For any $B' \subseteq B$, $(\mathfrak{b}_{B'}, \mathfrak{b}_B)$ is a split pair of Lie bialgebras. The corresponding restriction functor $\mathsf{DY}_{\mathfrak{b}_B} \to \mathsf{DY}_{\mathfrak{b}_{B'}}$ induces a homomorphism $i_{BB'} : \mathcal{U}^n_{\mathfrak{b}_{B'}} \to \mathcal{U}^n_{\mathfrak{b}_B}$, which clearly satisfies $i_{BB} = \mathrm{id}_{\mathcal{U}^n_{\mathfrak{b}_B}}$ and $i_{BB'} \circ i_{B'B''} = i_{BB''}$ for any $B'' \subseteq B' \subseteq B$.

Proposition. *The algebras $\{\mathcal{U}^n_{\mathfrak{b}_B}\}_{B\subseteq D}$ and maps $\{i_{BB'}\}_{B'\subseteq B}$ give rise to a lax diagrammatic algebra, which we denote by $\mathcal{U}^n_{\mathfrak{b}}$.*

Proof. We need to prove that if $B' \perp B''$, the images of $i_{DB'}$ and $i_{DB''}$ commute in $\mathcal{U}^n_{\mathfrak{b}_D}$. It is easy to check that the action and coaction of $\mathfrak{b}_{B'}$ commute with those of $\mathfrak{b}_{B''}$ on any $V \in \mathsf{DY}_{\mathfrak{b}_D}$. Thus, $\mathfrak{b}_{B'}$ acts and coacts on each tensor factor of $V_1\otimes\cdots\otimes V_n$, $V_k \in \mathsf{DY}_{\mathfrak{b}_D}$, through morphisms in $\mathsf{DY}_{\mathfrak{b}_{B''}}$. By definition of $\mathcal{U}^n_{\mathfrak{b}_{B''}}$, the action of the latter on $V_1\otimes\cdots\otimes V_n$ therefore commutes with the action and coaction of $b_{B'}$ on each tensor factor. Thus, $\mathcal{U}^n_{\mathfrak{b}_{B''}}$ acts by tensor products of morphisms in $\mathsf{DY}_{\mathfrak{b}_{B'}}$ and thefore commutes with $\mathcal{U}^n_{\mathfrak{b}_{B'}}$. □

8.7. **Universal algebras as endomorphisms of fiber functors.** The following shows that the lax diagrammatic algebra $\mathfrak{U}^n$ obtained in 8.5 is a universal analogue of the lax diagrammatic algebra $\mathcal{U}^n_{\mathfrak{b}}$ obtained in 8.6.

Let $B \subseteq D$. For any n–tuple $\{V_k, \pi_k, \pi_k^*\}_{k=1}^n$ of Drinfeld–Yetter modules over $\mathfrak{b}_B$, let

$$\mathcal{G}_{(\mathfrak{b}_B;V_1,\dots,V_n)} : \underline{\mathsf{DY}}^n_B \longrightarrow \mathrm{Vect}_{\mathsf{k}}$$

be the corresponding realisation functor.

Proposition.

(1) There is an algebra homomorphism

$$\rho^n_{\mathfrak{b}_B} : \mathfrak{U}^n_B \to \mathcal{U}^n_{\mathfrak{b}_B}$$

which assigns to any $T \in \mathfrak{U}^n_B$, and any $V_1,\dots,V_n \in \mathsf{DY}_{\mathfrak{b}_B}$ the endomorphism $\mathcal{G}_{(\mathfrak{b}_B;V_1,\dots,V_n)}(T) \in \mathrm{End}_{\mathsf{k}}(V_1\otimes\cdots\otimes V_n)$.

(2) The collection of homomorphisms $\{\rho^n_{\mathfrak{b}_B}\}_{B\subseteq D}$ is a morphism of lax diagrammatic algebras $\rho^n_{\mathfrak{b}} : \mathfrak{U}^n \to \mathcal{U}^n_{\mathfrak{b}}$.

Proof. (1) follows from Lemma 8.4. (2) is clear. □

8.8. **Cosimplicial structure of $\mathcal{U}^\bullet_{\mathfrak{b}}$.** For any $B \subseteq D$, the tensor structure on $\mathsf{DY}_{\mathfrak{b}_B}$ endows the tower $\{\mathcal{U}^n_{\mathfrak{b}_B}\}_{n\geqslant 0}$, with the structure of a cosimplicial complex of algebras

$$\mathsf{k} \rightrightarrows \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}\right) \rightrightarrows\!\!\!\!\!\!\rightarrow \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes 2}\right) \rightrightarrows\!\!\!\!\!\!\rightrightarrows \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes 3}\right) \quad \cdots$$

which is compatible with the cosimplicial structure on $\{U\mathfrak{g}_{\mathfrak{b}_B}^{\otimes n}\}_{n\geq 0}$ induced by the coproduct, via the maps $U\mathfrak{g}_{\mathfrak{b}_B}^{\otimes n} \to \mathcal{U}^n_{\mathfrak{b}_B}$.

The corresponding face morphisms $d^n_i : \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n}\right) \to \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n+1}\right)$, $i = 0, \dots, n+1$ are given by $(d^0_0\varphi)_V = (d^0_1\varphi)_V = \varphi\cdot \mathrm{id}_V$, for $\varphi \in \mathsf{k}$ and $V \in \mathsf{DY}_{\mathfrak{b}_B}$, and, for $n \geqslant 1$, $\varphi \in \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n}\right)$, and $V_i \in \mathsf{DY}_{\mathfrak{b}_B}$, $1 \leqslant i \leqslant n+1$,

$$(d^n_i\varphi)_{V_1,\dots,V_{n+1}} = \begin{cases} \mathrm{id}_{V_1} \otimes \varphi_{V_2,\dots,V_{n+1}} & i = 0 \\ \varphi_{V_1,\dots,V_i\otimes V_{i+1},\dots V_{n+1}} & 1 \leqslant i \leqslant n \\ \varphi_{V_1,\dots,V_n} \otimes \mathrm{id}_{V_{n+1}} & i = n+1 \end{cases}$$

The degeneration homomorphisms $\varepsilon^i_n : \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n}\right) \to \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n-1}\right)$, $i = 1, \dots, n$, are

$$(\varepsilon^i_n\varphi)_{X_1,\dots,X_{n-1}} = \varphi_{X_1,\dots,X_{i-1},\mathbf{1},X_i,\dots,X_{n-1}}$$

where $\mathbf{1}$ is the trivial Drinfeld–Yetter module. The morphisms ε^i_n, d^n_i satisfy the standard relations

$$\begin{aligned} d^j_{n+1}d^i_n &= d^i_{n+1}d^{j-1}_n & i < j \\ \varepsilon^j_n\varepsilon^i_{n+1} &= \varepsilon^i_n\varepsilon^{j+1}_{n+1} & i \leqslant j \end{aligned}$$

$$\varepsilon^j_{n+1}d^n_i = \begin{cases} d^i_{n-1}\varepsilon^{j-1}_n & i < j \\ \mathrm{id} & i = j, j+1 \\ d^{i-1}_{n-1}\varepsilon^j_n & i > j+1 \end{cases}$$

and give in particular rise to the Hochschild differential

$$d^n = \sum_{i=0}^{n+1} (-1)^i d^n_i : \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n}\right) \to \mathsf{End}\left(\mathsf{f}_{\mathfrak{b}_B}^{\boxtimes n+1}\right)$$

The cosimplicial structure is compatible with the maps $\{i_{BB'}\}_{B'\subseteq B\subseteq D}$ and therefore determines a cosimplicial lax diagrammatic algebra $\mathcal{U}^\bullet_{\mathfrak{b}}$.

8.9. **Cosimplicial structure of $\mathfrak{U}^\bullet$.** The above construction can be lifted to the PROPs $\underline{\mathsf{DY}}^n_B$. For every $B \subseteq D$, $n \geqslant 1$ and $i = 0, \dots, n+1$, there are faithful functors

$$\mathcal{D}^n_i : \underline{\mathsf{DY}}^n_B \to \underline{\mathsf{DY}}^{n+1}_B$$

mapping [1] to [1], and given by

$$\mathcal{D}^n_0(\underline{\mathsf{V}}_k) = \underline{\mathsf{V}}_{k+1} \qquad \text{and} \qquad \mathcal{D}^n_{n+1}(\underline{\mathsf{V}}_k) = \underline{\mathsf{V}}_k$$

for $1 \leqslant k \leqslant n$, and, for $1 \leqslant i \leqslant n$,

$$\mathcal{D}^n_i(\underline{\mathsf{V}}_k) = \begin{cases} \underline{\mathsf{V}}_k & 1 \leqslant k \leqslant i-1 \\ \underline{\mathsf{V}}_i \otimes \underline{\mathsf{V}}_{i+1} & k = i \\ \underline{\mathsf{V}}_{k+1} & i+1 \leqslant k \leqslant n \end{cases}$$

and $\mathcal{E}^{(i)}_n : \underline{\mathsf{DY}}^n_B \to \underline{\mathsf{DY}}^{n-1}_B$

$$\mathcal{E}^{(i)}_n = \mathcal{G}_{([1],\underline{\mathsf{V}}_1,\dots,\underline{\mathsf{V}}_{i-1},\mathbf{1},\underline{\mathsf{V}}_{i+1},\dots,\underline{\mathsf{V}}_{n-1})}$$

where $\mathbf{1}$ is the tensor unit in $\underline{\mathsf{DY}}^n_B$, regarded as trivial Drinfeld–Yetter module. These induce algebra homomorphisms

$$\Delta^n_i : \mathfrak{U}^n_B \to \mathfrak{U}^{n+1}_B$$

which are universal analogues of the insertion/coproduct maps on $U\mathfrak{g}_{\mathfrak{b}_B}^{\otimes n}$. They endow the tower $\{\mathfrak{U}^n_B\}_{n\geqslant 0}$ with the structure of a cosimplicial algebra, with Hochschild differential $d^n = \sum_{i=0}^{n+1}(-1)^i\Delta^n_i : \mathfrak{U}^n_B \to \mathfrak{U}^{n+1}_B$. This structure is compatible with the maps $\{\mathsf{i}_{BB'}\}_{B'\subseteq B\subseteq D}$ and therefore it extends to a cosimplicial structure on the lax diagrammatic algebras $\mathfrak{U}^n$.

The following is straightforward.

Proposition. *The morphism of lax diagrammatic algebras $\rho^\bullet_\mathfrak{b} : \mathfrak{U}^\bullet \to \mathcal{U}^\bullet_\mathfrak{b}$ obtained in 8.7 is compatible with the cosimplicial structures.*

9. Universal braided pre–Coxeter structures

We introduce in this section a class of braided pre–Coxeter categories related to split diagrammatic Lie bialgebras. They are universal, in that they are arise from the PROPs $\underline{\mathsf{DY}}^n_D$ defined in Section 8.

9.1. **Gradings.** Let $B \subseteq D$. The PROP $\underline{\mathsf{DY}}^n_B$ has a natural $\mathbb{N}$–bigrading given by $\deg(\sigma) = (0,0) = \deg(\theta_{B'})$ for any $\sigma \in \mathfrak{S}_N$ and $B' \subseteq B$,

$$\deg(\mu) = (1,0) = \deg(\pi_{\underline{\mathsf{V}}_k}) \qquad \text{and} \qquad \deg(\delta) = (0,1) = \deg(\pi^*_{\underline{\mathsf{V}}_k})$$

for any $1 \leqslant k \leqslant n$.

The algebra $\mathfrak{U}^n_B$ inherits this bigrading and is concentrated in bidegrees (N,N), since a degree (p,q) morphism with source $\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$ is easily seen to map to $[1]^{\otimes(q-p)} \otimes \underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$. For any $a, b \in \mathbb{N}$, the corresponding $\mathbb{N}$–grading determined by mapping $(1,0),(0,1)$ to a, b respectively yields the same graded completion $\widehat{\mathfrak{U}}^n_B$ of $\mathfrak{U}^n_B$, so long as $a + b > 0$. For definiteness, we set $a = 0$ and $b = 1$. The morphisms $\{i_{BB'}\}_{B'\subseteq B}$ naturally extends to the graded completions and induce on the algebras $\widehat{\mathfrak{U}}^n_B$, $B \subseteq D$, a lax diagrammatic algebra structure $\widehat{\mathfrak{U}}^n$, which extends $\mathfrak{U}^n$.

9.2. **Invariants.** For any pair of subdiagrams $B' \subseteq B$, denote by $\widehat{\mathfrak{U}}^n_{B,B'} \subseteq \widehat{\mathfrak{U}}^n_B$ the subalgebra of elements which commute with the diagonal action and coaction of $[\mathfrak{b}_{B'}] = ([1], \theta_{B'})$ on $\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$. Note that, by Lemma 8.4, $\widehat{\mathfrak{U}}^n_{B,B'}$ commutes with the diagonal action of $\widehat{\mathfrak{U}}_{B'}$ on $\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$, which is given by

$$\widehat{\mathfrak{U}}_{B'} \ni x \longrightarrow x_{1,2,\dots,n} = \Delta^{n-1}_1 \circ \cdots \circ \Delta^2_1 \circ \Delta^1_1(x)$$

9.3. **Associators.** Fix $B \subseteq D$. Define the r–matrix $r = r_{\underline{\mathsf{V}}_1,\underline{\mathsf{V}}_2} \in \widehat{\mathfrak{U}}^2_B$ as the composition

$$r_{\underline{\mathsf{V}}_1,\underline{\mathsf{V}}_2} = \pi_{\underline{\mathsf{V}}_1} \otimes \mathrm{id}_{\underline{\mathsf{V}}_2} \circ (1\,2) \circ \mathrm{id}_{\underline{\mathsf{V}}_1} \otimes \pi^*_{\underline{\mathsf{V}}_2}$$

(resp. $r^{21}_{\underline{\mathsf{V}}_1,\underline{\mathsf{V}}_2} = \mathrm{id}_{\underline{\mathsf{V}}_1} \otimes \pi_{\underline{\mathsf{V}}_2} \circ (1\,2) \circ \pi^*_{\underline{\mathsf{V}}_1} \otimes \mathrm{id}_{\underline{\mathsf{V}}_2}$), and set $\Omega = r^{12} + r^{21}$. An invertible, invariant element $\Phi \in \widehat{\mathfrak{U}}^3_{B,B}$ is called an *associator* if the following relations are satisfied (in $\widehat{\mathfrak{U}}^4_B$ and $\widehat{\mathfrak{U}}^3_B$ respectively).

- **Pentagon relation**

$$\Phi_{1,2,34}\Phi_{12,3,4} = \Phi_{2,3,4}\Phi_{1,23,4}\Phi_{1,2,3}$$

- **Hexagon relations**
$$e^{\Omega_{12,3}/2} = \Phi_{3,1,2} e^{\Omega_{13}/2} \Phi_{1,3,2}^{-1} e^{\Omega_{23}/2} \Phi_{1,2,3}$$
$$e^{\Omega_{1,23}/2} = \Phi_{2,3,1}^{-1} e^{\Omega_{13}/2} \Phi_{2,1,3} e^{\Omega_{12}/2} \Phi_{1,2,3}^{-1}$$
- **Duality**
$$\Phi_{3,2,1} = \Phi_{1,2,3}^{-1}$$
- 2–**jet**
$$\Phi = 1 + \frac{1}{24}[\Omega_{12}, \Omega_{23}] \qquad \mod (\widehat{\mathfrak{U}}^3_B)_{\geqslant 3}$$

9.4. **Braided pre–Coxeter structures on $\widehat{\mathfrak{U}}^\bullet$.**

Definition. A *braided pre–Coxeter structure* $(\Phi_B, J_\mathcal{F}, \Upsilon_{\mathcal{F}\mathcal{G}}, \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})$ on $\widehat{\mathfrak{U}}^\bullet$ consists of the following data.

(1) **Associators.** For any $B \subseteq D$, an associator $\Phi_B \in \widehat{\mathfrak{U}}^3_{B,B}$. We set $R_B = \exp(\Omega_B/2) \in \widehat{\mathfrak{U}}^2_{B,B}$.

(2) **Relative twists.** For any $B' \subseteq B$ and maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, an invertible element $J_\mathcal{F} \in \widehat{\mathfrak{U}}^2_{B,B'}$ such that $(J_\mathcal{F})_0 = 1$ and $\varepsilon^1_2(J_\mathcal{F}) = 1 = \varepsilon^2_2(J_\mathcal{F})$, where $\varepsilon^1_2, \varepsilon^2_2 : \widehat{\mathfrak{U}}^2_B \to \widehat{\mathfrak{U}}_B$ are the degeneration homomorphisms, and satisfying the following properties.

- **Compatibility with associators.** The relative twist equation holds,
$$J_{\mathcal{F},1,23} \cdot J_{\mathcal{F},23} \cdot \Phi_{B'} = \Phi_B \cdot J_{\mathcal{F},12,3} \cdot J_{\mathcal{F},12} \tag{9.1}$$
- **Normalisation.** For any $B \subseteq D$, $J_B = 1$.[25]

(3) **De Concini–Procesi associators.** For any $B' \subseteq B$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, an invertible element $\Upsilon_{\mathcal{G}\mathcal{F}} \in \widehat{\mathfrak{U}}_{B,B'}$ such that $(\Upsilon_{\mathcal{G}\mathcal{F}})_0 = 1$, $\varepsilon(\Upsilon_{\mathcal{G}\mathcal{F}}) = 1$, and satisfying the following properties.

- **Compatibility with J.** For any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$,
$$J_\mathcal{G} = (\Upsilon_{\mathcal{G}\mathcal{F}})_{12} \cdot J_\mathcal{F} \cdot (\Upsilon_{\mathcal{G}\mathcal{F}})_1^{-1} \cdot (\Upsilon_{\mathcal{G}\mathcal{F}})_2^{-1}$$
- **Horizontal factorisation.** For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mns}(B, B')$,
$$\Upsilon_{\mathcal{H}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{G}} \cdot \Upsilon_{\mathcal{G}\mathcal{F}}$$
In particular, $\Upsilon_{\mathcal{F}\mathcal{F}} = 1$ and $\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{G}\mathcal{F}}^{-1}$.

(4) **Vertical joins.** For any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, an invertible element $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} \in \widehat{\mathfrak{U}}_{B,B''}$ such that $(\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})_0 = 1$, $\varepsilon(\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'}) = 1$, and satisfying the following properties.

- **Compatibility with J (*vertical J–factorisation*).**
$$J_{\mathcal{F}' \cup \mathcal{F}} = (\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})_{12} \cdot J_\mathcal{F} \cdot J_{\mathcal{F}'} \cdot (\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})_1^{-1} \cdot (\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})_2^{-1}$$
- **Compatibility with Υ (*vertical Υ–factorisation*).** For any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}', \mathcal{G}' \in \mathsf{Mns}(B', B'')$,
$$\Upsilon_{(\mathcal{G} \cup \mathcal{G}')(\mathcal{F} \cup \mathcal{F}')} \cdot \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{G}}_{\mathcal{G}'} \cdot \Upsilon_{\mathcal{G}\mathcal{F}} \cdot \Upsilon_{\mathcal{G}'\mathcal{F}'}$$

[25]Here B is identified with the unique element in $\mathsf{Mns}(B, B)$.

- **Associativity.** For any $B''' \subseteq B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B,B')$, $\mathcal{F}' \in \mathsf{Mns}(B',B'')$, and $\mathcal{F}'' \in \mathsf{Mns}(B'',B''')$,
$$\mathsf{a}^{\mathcal{F}'\cup\mathcal{F}}_{\mathcal{F}''} \cdot \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{F}}_{\mathcal{F}''\cup\mathcal{F}'} \cdot \mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}$$
- **Normalisation.** For any $\mathcal{F} \in \mathsf{Mns}(B,B')$,
$$\mathsf{a}^{\mathcal{F}}_{B'} = 1 = \mathsf{a}^{B}_{\mathcal{F}}$$

Consistently with the diagrammatic algebra structure on $\widehat{\mathfrak{U}}^{\bullet}$, specifically the fact that $\widehat{\mathfrak{U}}^n_{B_1\sqcup B_2} \cong \widehat{\mathfrak{U}}^n_{B_1} \otimes \widehat{\mathfrak{U}}^n_{B_2}$ (Remark 8.5), we further require that J, Υ and a satisfy the following property.

- **Orthogonal factorisation.** If $B_1'' \subseteq B_1' \subseteq B_1 \perp B_2 \supseteq B_2' \supseteq B_2''$, $(\mathcal{F}_1,\mathcal{F}_2) \in \mathsf{Mns}(B_1 \sqcup B_2, B_1' \sqcup B_2')$, $(\mathcal{F}_1',\mathcal{F}_2') \in \mathsf{Mns}(B_1' \sqcup B_2', B_1'' \sqcup B_2'')$,
$$\begin{aligned} \Phi_{B_1\sqcup B_2} &= \Phi_{B_1} \cdot \Phi_{B_2} \\ J_{(\mathcal{F}_1,\mathcal{F}_2)} &= J_{\mathcal{F}_1} \cdot J_{\mathcal{F}_2} \\ \Upsilon_{(\mathcal{G}_1,\mathcal{G}_2)(\mathcal{F}_1,\mathcal{F}_2)} &= \Upsilon_{\mathcal{G}_1\mathcal{F}_1} \cdot \Upsilon_{\mathcal{G}_2\mathcal{F}_2} \\ \mathsf{a}^{(\mathcal{F}_1,\mathcal{F}_2)}_{(\mathcal{F}_1',\mathcal{F}_2')} &= \mathsf{a}^{\mathcal{F}_1}_{\mathcal{F}_1'} \cdot \mathsf{a}^{\mathcal{F}_2}_{\mathcal{F}_2'} \end{aligned}$$

Note that $R_{B_1\sqcup B_2} = R_{B_1} \cdot R_{B_2}$. Moreover, since $\widehat{\mathfrak{U}}^n_{B_1}$ and $\widehat{\mathfrak{U}}^n_{B_2}$ commute, the order of the products in the above identities is irrelevant. The following is a direct consequence of the orthogonal factorisation and normalisation of J, Υ, and a.

Lemma.

(1) *For any $B' \subseteq B \perp B''$, and $\mathcal{F} \in \mathsf{Mns}(B,B')$, $J_{(\mathcal{F},B'')} = J_{\mathcal{F}}$.*
(2) *For any $B' \subseteq B \perp B''$, and $\mathcal{F},\mathcal{G} \in \mathsf{Mns}(B,B')$, $\Upsilon_{(\mathcal{F},B'')(\mathcal{G},B'')} = \Upsilon_{\mathcal{F}\mathcal{G}}$.*
(3) *For any $B_1' \subseteq B_1 \perp B_2 \supseteq B_2'$, $\mathcal{F}_1 \in \mathsf{Mns}(B_1,B_1')$, and $\mathcal{F}_2 \in \mathsf{Mns}(B_2,B_2')$, $\mathsf{a}^{(\mathcal{F}_1,B_2)}_{(B_1',\mathcal{F}_2)} = 1 = \mathsf{a}^{(B_1,\mathcal{F}_2)}_{(\mathcal{F}_1,B_2')}$.*

9.5. **Twisting of braided pre–Coxeter structures on $\widehat{\mathfrak{U}}^{\bullet}$.**

Definition.

(1) A *twist* in $\widehat{\mathfrak{U}}^{\bullet}$ is a pair $\mathbb{T} = (u,K)$ where
   (a) $u = \{u_{\mathcal{F}}\}$ is a collection of invertible elements in $\widehat{\mathfrak{U}}_{B,B'}$, indexed by a maximal nested set $\mathcal{F} \in \mathsf{Mns}(B,B')$, which satisfy $\varepsilon(u_{\mathcal{F}}) = 1$ and orthogonal factorisation, *i.e.,* for any $B_1' \subseteq B_1 \perp B_2 \supseteq B_2'$, and $(\mathcal{F}_1,\mathcal{F}_2)$ in $\mathsf{Mns}(B_1,B_1') \times \mathsf{Mns}(B_2,B_2') = \mathsf{Mns}(B_1 \sqcup B_2, B_1' \sqcup B_2')$,
$$u_{(\mathcal{F}_1,\mathcal{F}_2)} = u_{\mathcal{F}_1} \cdot u_{\mathcal{F}_2} = u_{\mathcal{F}_2} \cdot u_{\mathcal{F}_1}$$
   (b) $K = \{K_B\}$ is a collection of invertible elements of $\widehat{\mathfrak{U}}^2_{B,B}$, indexed by subdiagrams $B \subseteq D$, which satisfy $\varepsilon_2^1(K_B) = 1 = \varepsilon_2^2(K_B)$, are symmetric,*i.e.,* $(K_B)_{21} = K_B$, [26] $d(K_B)_1 = 0$, and such that $K_{B_1\sqcup B_2} = K_{B_1} \cdot K_{B_2}$.
(2) The *twisting* of a braided pre–Coxeter structure $\mathfrak{C} = (\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}})$ by a twist $\mathbb{T} = (u,K)$ is the braided pre–Coxeter structure
$$\mathfrak{C}_{\mathbb{T}} = ((\Phi_B)_{F_B}, (J_{\mathcal{F}})_{(u,K)}, (\Upsilon_{\mathcal{F}\mathcal{G}})_u, (\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})_u)$$

[26] There is a natural action of $\mathfrak{S}_n$ on $\mathfrak{U}^n_B$ given by permutations of $\underline{\mathsf{V}}_1 \otimes \cdots \otimes \underline{\mathsf{V}}_n$ (cf. [4, Sec. 7.2]), which is a propic version of the action of $\mathfrak{S}_n$ on $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$.

given by

$$\begin{aligned}(\Phi_B)_K &= (K_B)_{23}^{-1}\cdot (K_B)_{1,23}^{-1}\cdot \Phi_B\cdot (K_B)_{12,3}\cdot (K_B)_{12}\\ (J_{\mathcal{F}})_{(u,K)} &= (u_{\mathcal{F}})_{12}^{-1}\cdot K_B^{-1}\cdot J_{\mathcal{F}}\cdot K_{B'}\cdot (u_{\mathcal{F}})_1\cdot (u_{\mathcal{F}})_2\\ (\Upsilon_{\mathcal{F}\mathcal{G}})_u &= u_{\mathcal{F}}^{-1}\cdot \Upsilon_{\mathcal{F}\mathcal{G}}\cdot u_{\mathcal{G}}\\ (\mathsf{a}_{\mathcal{F}'}^{\mathcal{F}})_u &= u_{\mathcal{F}'\cup\mathcal{F}}^{-1}\cdot \mathsf{a}_{\mathcal{F}'}^{\mathcal{F}}\cdot u_{\mathcal{F}'}\cdot u_{\mathcal{F}}\end{aligned}$$

Remark. The twisting of a braided pre–Coxeter structure does not affect the R–matrix $R_B = \exp(\Omega_B/2)$ (cf. [4, Sec. 13.2]).

9.6. Gauging of twists transformation.

Definition.

(1) A *gauge* is a collection $a = \{a_B\}$ of invertible elements $a_B \in \widehat{\mathfrak{U}}_{B,B}$ indexed by subdiagrams $B \subseteq D$ and satisfying

$$a_{B_1\sqcup B_2} = a_{B_1}\cdot a_{B_2}$$

(2) The *gauging* of a twist $\mathbb{T} = (u, K)$ by a is the twist $\mathbb{T}_a = (u_a, K_a)$ given by

$$\begin{aligned}(u_{\mathcal{F}})_a &= a_{B'}\cdot u_{\mathcal{F}}\cdot a_B^{-1}\\ (K_a)_B &= (a_B)_{12}\cdot K_B\cdot (a_B)_1^{-1}\cdot (a_B)_2^{-1}\end{aligned}$$

Remark. It is easy to see that if (u, F) is a twist, and a a gauge, the twist of a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^{\bullet}$ by (u, F) is the same as that by (u_a, F_a).

9.7. **Deformation Drinfeld–Yetter modules.** Let $\mathfrak{b}$ be a split diagrammatic Lie bialgebra and $\mathfrak{g}_{\mathfrak{b}}$ its Drinfeld double. We explained in 8.4 that $\mathfrak{U}^n$ is a universal analogue of the diagrammatic algebra $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$. In a similar vein, we now show that the completion $\widehat{\mathfrak{U}}^n$ introduced in 9.1 is a universal analogue of the trivially deformed diagrammatic algebra $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}[\![\hbar]\!]$.

Let for this purpose $\mathfrak{c}$ be a Lie bialgebra and $\mathsf{DY}_{\mathfrak{c}}^{\hbar}$ the category of Drinfeld–Yetter $\mathfrak{c}$–modules in topologically free $\mathsf{k}[\![\hbar]\!]$–modules. Scaling the coaction on $V \in \mathsf{DY}_{\mathfrak{c}}^{\hbar}$ by $\hbar$ yields an isomorphism between $\mathsf{DY}_{\mathfrak{c}}^{\hbar}$ and the category $\mathsf{DY}_{\mathfrak{c}^{\hbar}}^{\mathrm{adm}}$ of Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{c}^{\hbar} = (\mathfrak{c}[\![\hbar]\!], [\cdot,\cdot], \hbar\delta)$, whose coaction is divisible by $\hbar$. We denote by $\widehat{\mathcal{U}}_{\mathfrak{c}}^n$ the algebra of endomorphisms of the n–fold tensor power of the forgetful functor $\mathsf{f}_{\mathfrak{c}} : \mathsf{DY}_{\mathfrak{c}}^{\hbar} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$. $\widehat{\mathcal{U}}_{\mathfrak{c}}^n$ identifies canonically with the analogous completion defined for $\mathsf{DY}_{\mathfrak{c}_{\hbar}}^{\mathrm{adm}}$.

In the case of the diagrammatic Lie bialgebra $\mathfrak{b}$, the realisation functors

$$\mathcal{G}_{(\mathfrak{b}_B^{\hbar};V_1,\dots,V_n)} : \underline{\mathsf{DY}}_B^n \longrightarrow \mathrm{Vect}_{\mathsf{k}[\![\hbar]\!]}$$

corresponding to $V_1,\dots,V_n \in \mathsf{DY}_{\mathfrak{b}_B^{\hbar}}^{\mathrm{adm}} \cong \mathsf{DY}_{\mathfrak{b}_B}^{\hbar}$ induce a homomorphism $\widehat{\rho}_{\mathfrak{b}}^n : \mathfrak{U}^n \to \widehat{\mathcal{U}}_{\mathfrak{b}}^n$ which naturally extends to $\widehat{\mathfrak{U}}^n$.[27] In particular,

$$\widehat{\rho}_{\mathfrak{b}_B}^1(\pi_{\underline{\mathsf{V}}_1}\circ\pi_{\underline{\mathsf{V}}_1}^*) = \hbar\sum_i b_i b^i \qquad\text{and}\qquad \widehat{\rho}_{\mathfrak{b}_B}^2(r_{\underline{\mathsf{V}}_1,\underline{\mathsf{V}}_2}) = \hbar\sum_i b_i\otimes b^i$$

[27] Note that $\mathsf{DY}_{\mathfrak{c}}^{\hbar}$ can also be identified with the category of Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{c}_{\hbar} = (\mathfrak{c}[\![\hbar]\!], \hbar[\cdot,\cdot], \delta)$ whose action is divisible by $\hbar$. The corresponding realisation functors for $\mathfrak{b}_{B,\hbar}$ yield the same homomorphism $\widehat{\rho}_{\mathfrak{b}}^n : \widehat{\mathfrak{U}}^n \to \widehat{\mathcal{U}}_{\mathfrak{b}}^n$.

where $\{b_i\},\{b^i\}$ are dual bases of $\mathfrak{b}_B$ and $\mathfrak{b}_B^*$.[28] Note also that if $B'\subseteq B$, the definition of the subalgebra of $[\mathfrak{b}_{B'}]$–invariants in $\widehat{\mathfrak{U}}^n_B$ (§9.3) implies that $\widehat{\mathfrak{U}}^n_{B,B'}$ is mapped by $\widehat{\rho}^n_{\mathfrak{b}_B}$ to elements of $\widehat{\mathcal{U}}^n_{\mathfrak{b}_B}$ commuting with the diagonal (co)action of $\mathfrak{b}_{B'}$.

9.8. **From universal algebras to Drinfeld–Yetter modules.** We shall make use of the following standard construction due to Drinfeld. Let $\mathfrak{b}$ be a diagrammatic Lie bialgebra, $B\subseteq D$, and $\Phi_B\in\widehat{\mathfrak{U}}^3_B$ an associator. Then, $\mathsf{DY}^{\Phi}_{\mathfrak{b}_B}$ is the braided monoidal category with the same objects as $\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}$, and commutativity and associativity constraints given respectively by

$$\beta_{\mathfrak{b}_B}=(1\,2)\circ\widehat{\rho}^2_{\mathfrak{b}_B}(e^{\Omega_B/2})\qquad\text{and}\qquad\Phi_{\mathfrak{b}_B}=\widehat{\rho}^3_{\mathfrak{b}_B}(\Phi_B)$$

Proposition. *Let $\mathfrak{b}$ be a diagrammatic Lie bialgebra.*

(1) A braided pre–Coxeter structure $\mathfrak{C}$ on $\widehat{\mathfrak{U}}^\bullet$ gives rise to a braided pre–Coxeter category $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b}}$ with
  - *diagrammatic categories $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b},B}=\mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B}$*
  - *restriction functors $F^{\mathfrak{C}}_{\mathcal{F}}:\mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B}\to\mathsf{DY}^{\Phi_{B'}}_{\mathfrak{b}_{B'}}$ of the form $\left(\mathrm{Res}_{\mathfrak{b}_{B'}\mathfrak{b}_B},J^{\mathfrak{C}}_{\mathcal{F}}\right)$ for some tensor structure $J^{\mathfrak{C}}_{\mathcal{F}}$ on $\mathrm{Res}_{\mathfrak{b}_{B'}\mathfrak{b}_B}$*

  Moreover, $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b}}$ is a deformation of $\mathbb{DY}_{\mathfrak{b}}$ (cf. 5.12).

(2) A twist $\mathbb{T}$ in $\widehat{\mathfrak{U}}^\bullet$ gives rise to a 1–isomorphism $\mathbb{T}_{\mathfrak{b}}:\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b}}\to\mathbb{DY}^{\mathfrak{C}_{\mathbb{T}}}_{\mathfrak{b}}$.

(3) A gauge $\mathbf{g}$ in $\widehat{\mathfrak{U}}^\bullet$ gives rise to a 2–isomorphism $\mathbf{g}_{\mathfrak{b}}:\mathbb{T}_{\mathfrak{b}}\Rightarrow(\mathbb{T}_{\mathbf{g}})_{\mathfrak{b}}$.

Proof. (1) Consider the following data.

- *Diagrammatic categories.* For any $B\subseteq D$, set $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b},B}\coloneqq\mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B}$.
- *Restriction functors.* For any $B'\subseteq B$ and $\mathcal{F}\in\mathsf{Mns}(B,B')$, the action of $J^{\mathfrak{C}}_{\mathcal{F}}\coloneqq\widehat{\rho}^2_{\mathfrak{b}_B}(J_{\mathcal{F}})$ defines a linear automorphism of $V\otimes W$, for any $V,W\in\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b},B}$. By the properties of $J_{\mathcal{F}}$, this defines a tensor structure on the standard restriction functor $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$. Then, we set $F^{\mathfrak{C}}_{\mathcal{F}}\coloneqq\left(\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B},J^{\mathfrak{C}}_{\mathcal{F}}\right)$, where the tensor structure is given by the natural isomorphism
$$(J^{\mathfrak{C}}_{\mathcal{F}})_{V,W}:\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}(V)\otimes\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}(W)\to\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}(V\otimes W)$$
- *De Concini–Procesi associators.* For any $B'\subseteq B$, and $\mathcal{F},\mathcal{G}\in\mathsf{Mns}(B,B')$, the action of $\Upsilon^{\mathfrak{C}}_{\mathcal{G}\mathcal{F}}\coloneqq\widehat{\rho}_{\mathfrak{b}_B}(\Upsilon_{\mathcal{G}\mathcal{F}})$ defines a linear automorphism of $V\in\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b},B}$. By the properties of $\Upsilon_{\mathcal{G}\mathcal{F}}$, this defines an isomorphism of tensor functors $F^{\mathfrak{C}}_{\mathcal{F}}\Rightarrow F^{\mathfrak{C}}_{\mathcal{G}}$.
- *Vertical joins.* For any $B''\subseteq B'\subseteq B$, $\mathcal{F}''\in\mathsf{Mns}(B',B'')$, $\mathcal{F}'\in\mathsf{Mns}(B,B')$, let ${\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}}^{\mathfrak{C}}:F^{\mathfrak{C}}_{\mathcal{F}''}\circ F^{\mathfrak{C}}_{\mathcal{F}'}\Rightarrow F^{\mathfrak{C}}_{\mathcal{F}'\cup\mathcal{F}''}$ be the tensor isomorphism defined by $\widehat{\rho}_{\mathfrak{b}_B}(\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''})$, together with the equality $\mathrm{Res}_{\mathfrak{b}_{B''},\mathfrak{b}_{B'}}\circ\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}=\mathrm{Res}_{\mathfrak{b}_{B''},\mathfrak{b}_B}$.

These satisfy the conditions of Proposition 3.4, so that $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b}}=(\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b},B},F^{\mathfrak{C}}_{\mathcal{F}},\,J^{\mathfrak{C}}_{\mathcal{F}},\,\Upsilon^{\mathfrak{C}}_{\mathcal{F}\mathcal{G}},{\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}}^{\mathfrak{C}})$ is a braided pre–Coxeter category.

[28] Note that the realisation functors corresponding to the tuples $(\mathfrak{b}^{\hbar}_B;V_1,\dots,V_n)$ and $(\mathfrak{b}_B;V_1,\dots,V_n)$, where $V_1,\dots,V_n\in\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{b}^{\hbar}_B}\cong\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}$ do not lead to the same homomorphism $\mathfrak{U}^n\to\widehat{\mathcal{U}}^n_{\mathfrak{b}}$ because $\mathfrak{b}^{\hbar}_B$ is not isomorphic to $\mathfrak{b}_B[\![\hbar]\!]$ as Lie bialgebras.

(2) Let $\mathbb{T}=(u,F)$ be a twist in $\widehat{\mathfrak{U}}^\bullet$ and $\mathfrak{C}_\mathbb{T}$ the twisted braided pre–Coxeter structure (cf. 9.5). Define a 1–isomorphism $\mathbb{T}_\mathfrak{b}=(H_B^\mathbb{T},\gamma_\mathcal{F}^\mathbb{T}):\mathbb{DY}_\mathfrak{b}^\mathfrak{C}\to\mathbb{DY}_\mathfrak{b}^{\mathfrak{C}_\mathbb{T}}$ as follows.

- For any $B\subseteq D$, we denote by $H_B^\mathbb{T}$ the identity functor on $\mathbb{DY}_{\mathfrak{b},B}^\mathfrak{C}$ endowed with the tensor structure $\widehat{\rho}_B^2(F_B)$. It follows from Definition 9.5 that $H_B^\mathbb{T}$ is a braided tensor equivalence $\mathbb{DY}_{\mathfrak{b},B}^\mathfrak{C}\to\mathbb{DY}_{\mathfrak{b},B}^{\mathfrak{C}_\mathbb{T}}$.
- For any $B'\subseteq B\subseteq D$ and $\mathcal{F}\in\mathsf{Mns}(B,B')$, we denote by $\gamma_\mathcal{F}^\mathbb{T}$ the natural isomorphism $F_\mathcal{F}^{\mathfrak{C}_\mathbb{T}}\circ H_B^\mathbb{T}\Rightarrow H_{B'}^\mathbb{T}\circ F_\mathcal{F}^\mathfrak{C}$ induced by $\widehat{\rho}_{BB'}(u_\mathcal{F})$. $\gamma_\mathcal{F}^\mathbb{T}$ is a well–defined isomorphism of tensor functors satisfying the vertical factorisation property.

(3) Finally, let $\mathbf{g}$ be a gauge in $\widehat{\mathfrak{U}}^\bullet$ and $\mathbb{T}_\mathbf{g}$ the gauged twist (cf. 9.6). Define a 2–isomorphism $\mathbf{g}_\mathfrak{b}:\mathbb{T}_\mathfrak{b}\Rightarrow(\mathbb{T}_\mathbf{g})_\mathfrak{b}$ as follows. For any $B\subseteq D$, denote by $v_B^\mathbf{g}$ the isomorphism of braided tensor functors $H_B^\mathbb{T}\Rightarrow H_B^{\mathbb{T}_\mathbf{g}}$ given by $\widehat{\rho}_B(\mathbf{g}_B)$. It follows from the definition of $\mathbf{g}$ that $\gamma_\mathcal{F}^{\mathbb{T}_\mathbf{g}}\circ v_B^\mathbf{g}=v_{B'}^\mathbf{g}\circ\gamma_\mathcal{F}^\mathbb{T}$. □

Definition. Let $\mathfrak{b}$ be a diagrammatic Lie bialgebra, and $\{\Phi_B\}_{B\subseteq D}$ a collection of associators. A braided pre–Coxeter category with diagrammatic categories $\mathsf{DY}_{\mathfrak{b}_B}^{\Phi_B}$ is called *universal* if it is of the form $\mathbb{DY}_\mathfrak{b}^\mathfrak{C}$, for some braided pre–Coxeter structure $\mathfrak{C}$ on $\widehat{\mathfrak{U}}^\bullet$.

9.9. **Coherence and minimal data.** Let $\mathfrak{C}=(\Phi_B,J_\mathcal{F},\Upsilon_{\mathcal{F}\mathcal{G}},\mathsf{a}_{\mathcal{F}'}^\mathcal{F})$ be a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^\bullet$. We show in this section that $\mathfrak{C}$ is determined by its vertical joins, together with a minimal collection of associators, relative twists, De Concini–Procesi associators, vertical joins. We shall need two preliminary results.

9.9.1. Let $B'\subseteq B$, $\mathcal{F}\in\mathsf{Mns}(B,B')$ and $\mathbf{C}:B'=B_0\subsetneq B_1\subsetneq\cdots\subsetneq B_\ell=B$ a maximal chain from B to B' corresponding to $\mathcal{F}$ (cf. 2.3). For any $1\leq k\leq\ell$, denote by $\mathcal{F}_k\in\mathsf{Mns}(B_k,B_0)$ the restriction of $\mathcal{F}$ to B_k, and note that $\mathcal{F}_k=\mathcal{F}_{k-1}\cup\mathcal{E}_k$, where $\mathcal{E}_k$ is the unique element in $\mathsf{Mns}(B_k,B_{k-1})$.

Lemma. *Define* $\mathsf{b}_\mathbf{C}\in\widehat{\mathfrak{U}}_{B,B'}$ *by*

$$\mathsf{b}_\mathbf{C}=\mathsf{a}_{\mathcal{F}_{\ell-1}}^{\mathcal{E}_\ell}\cdot\mathsf{a}_{\mathcal{F}_{\ell-2}}^{\mathcal{E}_{\ell-1}}\cdots\mathsf{a}_{\mathcal{F}_1}^{\mathcal{E}_2}\tag{9.2}$$

1. *$\mathsf{b}_\mathbf{C}$ is independent of the choice of $\mathbf{C}$, and will be denoted $\mathsf{b}_\mathcal{F}$.*
2. *For any $B''\subseteq B'\subseteq B$, $\mathcal{F}'\in\mathsf{Mns}(B,B')$ and $\mathcal{F}''\in\mathsf{Mns}(B',B'')$,*

$$\mathsf{a}_{\mathcal{F}''}^{\mathcal{F}'}=\mathsf{b}_{\mathcal{F}'\cup\mathcal{F}''}\cdot\mathsf{b}_{\mathcal{F}'}^{-1}\cdot\mathsf{b}_{\mathcal{F}''}^{-1}$$

Proof. (1) Lemma 9.4 (3) implies that b is constant on the connected components of the graph $\mathbf{G}_{B,B'}$ (cf. 2.3). (2) Let $\mathbf{C}:B''=B_0\subsetneq B_1\subsetneq\cdots\subsetneq B_\ell=B$ be a maximal chain such that $B'=B_p$, for some $1\leq p\leq\ell-1$, and the restriction of $\mathbf{C}$ to a chain from B'' to B' (resp. B' to B) corresponds to $\mathcal{F}''$ (resp. $\mathcal{F}'$). Note that, with respect to the notation established above, we have

$$\mathcal{F}''=\mathcal{F}_p=\mathcal{E}_p\cup\cdots\cup\mathcal{E}_1\qquad\text{and}\qquad\mathcal{F}'=\mathcal{E}_\ell\cup\mathcal{E}_{\ell-1}\cup\cdots\cup\mathcal{E}_{p+1}$$

For $1<k\leqslant\ell-p$, we set $\mathcal{F}'_k:=\mathcal{E}_{p+k}\cup\cdots\cup\mathcal{E}_{p+1}$. By definition,

$$\begin{aligned}\mathsf{b}_{\mathcal{F}'\cup\mathcal{F}''}&=\mathsf{a}_{\mathcal{F}_{\ell-1}}^{\mathcal{E}_\ell}\cdots\mathsf{a}_{\mathcal{F}_{p+1}}^{\mathcal{E}_{p+2}}\cdot\mathsf{a}_{\mathcal{F}_p}^{\mathcal{E}_{p+1}}\cdot\mathsf{a}_{\mathcal{F}_{p-1}}^{\mathcal{E}_p}\cdots\mathsf{a}_{\mathcal{E}_1}^{\mathcal{E}_2}\\&=\mathsf{a}_{\mathcal{F}_{\ell-1}}^{\mathcal{E}_\ell}\cdots\mathsf{a}_{\mathcal{F}_{p+1}}^{\mathcal{E}_{p+2}}\cdot\mathsf{a}_{\mathcal{F}_p}^{\mathcal{E}_{p+1}}\cdot\mathsf{b}_{\mathcal{F}''}\end{aligned}$$

Note that, by the associativity of the vertical joins,

$$\mathsf{a}_{\mathcal{F}_{p+1}}^{\mathcal{E}_{p+2}} \cdot \mathsf{a}_{\mathcal{F}_p}^{\mathcal{E}_{p+1}} = \mathsf{a}_{\mathcal{F}_p}^{\mathcal{E}_{p+2}\cup\mathcal{E}_{p+1}} \cdot \mathsf{a}_{\mathcal{E}_{p+1}}^{\mathcal{E}_{p+2}}$$

More in general, for any $1 < k < \ell - p$, we have $\mathcal{F}_{p+k} = \mathcal{F}'_k \cup \mathcal{F}_p$ and

$$\mathsf{a}_{\mathcal{F}_{p+k}}^{\mathcal{E}_{p+k+1}} \cdot \mathsf{a}_{\mathcal{F}_p}^{\mathcal{F}'_k} = \mathsf{a}_{\mathcal{F}_p}^{\mathcal{F}'_{k+1}} \cdot \mathsf{a}_{\mathcal{F}'_k}^{\mathcal{E}_{p+k+1}}$$

Therefore, we get

$$\begin{aligned}
\mathsf{b}_{\mathcal{F}'\cup\mathcal{F}''} &= \mathsf{a}_{\mathcal{F}_{\ell-1}}^{\mathcal{E}_\ell} \cdots \mathsf{a}_{\mathcal{F}_{p+1}}^{\mathcal{E}_{p+2}} \cdot \mathsf{a}_{\mathcal{F}_p}^{\mathcal{E}_{p+1}} \cdot \mathsf{b}_{\mathcal{F}''} \\
&= \mathsf{a}_{\mathcal{F}_p}^{\mathcal{F}'_{\ell-p}} \cdot \mathsf{a}_{\mathcal{F}'_{\ell-p-1}}^{\mathcal{E}_\ell} \cdots \mathsf{a}_{\mathcal{E}_{p+1}}^{\mathcal{E}_{p+2}} \cdot \mathsf{b}_{\mathcal{F}''} \\
&= \mathsf{a}_{\mathcal{F}''}^{\mathcal{F}'} \cdot \mathsf{b}_{\mathcal{F}'} \cdot \mathsf{b}_{\mathcal{F}''}
\end{aligned}$$

where the second identity follows from iterated applications of the associativity of the vertical joins. □

9.9.2. Let now $B' \subseteq B$, and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$. Assume there is a chain of inclusions $B' = B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 = B$, and $\mathcal{F}_k, \mathcal{G}_k \in \mathsf{Mns}(B_k, B_{k-1})$, $1 \leq k \leq 3$, such that

$$\begin{gathered}
\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \qquad \mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \\
\mathcal{F}_1 = \mathcal{G}_1 \qquad \text{and} \qquad \mathcal{F}_3 = \mathcal{G}_3
\end{gathered}$$

so that $\mathcal{F}, \mathcal{G}$ only differ in the choice of an element in $\mathsf{Mns}(B_2, B_1)$.

Lemma. *The following holds*

$$\mathsf{b}_{\mathcal{G}}^{-1} \cdot \Upsilon_{\mathcal{G}\mathcal{F}} \cdot \mathsf{b}_{\mathcal{F}} = \mathsf{b}_{\mathcal{G}_2}^{-1} \cdot \Upsilon_{\mathcal{G}_2\mathcal{F}_2} \cdot \mathsf{b}_{\mathcal{F}_2}$$

PROOF. The compatibility of the associators Υ with the vertical joins yields

$$\begin{aligned}
\Upsilon_{\mathcal{G}\mathcal{F}} &= \mathsf{a}_{\mathcal{G}_2\cup\mathcal{G}_1}^{\mathcal{G}_3} \cdot \mathsf{a}_{\mathcal{G}_1}^{\mathcal{G}_2} \cdot \Upsilon_{\mathcal{G}_1\mathcal{F}_1} \cdot \Upsilon_{\mathcal{G}_2\mathcal{F}_2} \cdot \Upsilon_{\mathcal{G}_3\mathcal{F}_3} \cdot \left(\mathsf{a}_{\mathcal{F}_2\cup\mathcal{F}_1}^{\mathcal{F}_3} \cdot \mathsf{a}_{\mathcal{F}_1}^{\mathcal{F}_2}\right)^{-1} \\
&= \mathsf{b}_{\mathcal{G}} \cdot \mathsf{b}_{\mathcal{G}_3}^{-1} \cdot \mathsf{b}_{\mathcal{G}_2}^{-1} \cdot \mathsf{b}_{\mathcal{G}_1}^{-1} \cdot \Upsilon_{\mathcal{G}_2\mathcal{F}_2} \cdot \mathsf{b}_{\mathcal{F}_1} \cdot \mathsf{b}_{\mathcal{F}_2} \cdot \mathsf{b}_{\mathcal{F}_3} \cdot \mathsf{b}_{\mathcal{F}}^{-1} \\
&= \mathsf{b}_{\mathcal{G}} \cdot \mathsf{b}_{\mathcal{G}_2}^{-1} \cdot \Upsilon_{\mathcal{G}_2\mathcal{F}_2} \cdot \mathsf{b}_{\mathcal{F}_2} \cdot \mathsf{b}_{\mathcal{F}}^{-1}
\end{aligned}$$

where the second identity follows from Lemma 9.9.1, $\mathcal{F}_1 = \mathcal{G}_1$, and $\mathcal{F}_3 = \mathcal{G}_3$, and the third from the invariance of $\Upsilon_{\mathcal{G}_2\mathcal{F}_2}$ under $[\mathfrak{b}_{B_1}]$ and that of $\mathsf{b}_{\mathcal{G}_3}$ under $[\mathfrak{b}_{B_2}]$. □

Remark. Recall that if $B' \subseteq B$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, there is a sequence $\mathcal{F} = \mathcal{H}_1, \ldots, \mathcal{H}_m = \mathcal{G}$ in $\mathsf{Mns}(B, B')$ such that $\mathcal{H}_k$ and $\mathcal{H}_{k-1}$ differ by one element [37, Prop. 3.26]. We term such a sequence an *elementary sequence*. Moreover, if $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$ differ by one element, there are a unique $\overline{B} \in \mathcal{F} \cap \mathcal{G}$, vertices $i \neq j \in \overline{B}$, and maximal nested sets $\overline{\mathcal{F}}, \overline{\mathcal{G}} \in \mathsf{Mns}(\overline{B}, \overline{B} \setminus \{i, j\})$ such that

$$\mathcal{F} = \mathcal{H}' \cup \overline{\mathcal{F}} \cup \mathcal{H}'' \qquad \text{and} \qquad \mathcal{G} = \mathcal{H}' \cup \overline{\mathcal{G}} \cup \mathcal{H}''$$

for some $\mathcal{H}' \in \mathsf{Mns}(B, \widetilde{B})$, $\mathcal{H}'' \in \mathsf{Mns}(\widetilde{B} \setminus \{i, j\}, B')$, where

$$\widetilde{B} = \overline{B} \cup \bigcup_{\substack{B'' \in \mathsf{conn}(B') \\ B'' \not\subseteq \overline{B}}} B''$$

Then, it follows from the result above and Lemma 9.4 (2), that

$$\mathsf{b}_{\mathcal{G}}^{-1} \cdot \Upsilon_{\mathcal{G}\mathcal{F}} \cdot \mathsf{b}_{\mathcal{F}} = \mathsf{b}_{\overline{\mathcal{G}}}^{-1} \cdot \Upsilon_{\overline{\mathcal{G}\mathcal{F}}} \cdot \mathsf{b}_{\overline{F}}$$

9.9.3. We show below that $\mathfrak{C}$ is determined by the elements $\mathsf{b}_{\mathcal{H}}$, where $\mathcal{H}$ is any maximal nested set, and $(\Phi_B, J_{B,B'}, \Upsilon_{\mathcal{F}\mathcal{G}})$, where B is connected, $B' \subsetneq B$ is a 1–step maximal chain with B connected, and $\mathcal{F}, \mathcal{G}$ are maximal 2–step chains of the form $B'' \subsetneq B'_1 \subsetneq B$ and $B'' \subsetneq B'_2 \subsetneq B$ respectively, with B connected.

Proposition. *Let $\mathfrak{C} = (\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}}, \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})$ be a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^{\bullet}$. Then,*

(1) *For any $B \subseteq D$,*
$$\Phi_B = \prod \Phi_{B_i}$$
where the product is over the connected components of B.

(2) *For any $B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$*
$$J_{\mathcal{F}} = (\mathsf{b}_{\mathcal{F}})_{12} \cdot J_{B_\ell, B_{\ell-1}} \cdots J_{B_1, B_0} \cdot (\mathsf{b}_{\mathcal{F}})_1^{-1} \cdot (\mathsf{b}_{\mathcal{F}})_2^{-1}$$
where $B' = B_0 \subsetneq \cdots \subsetneq B_\ell = B$ is a maximal chain corresponding to $\mathcal{F}$.

(3) *For any $B' \subseteq B$, and elementary sequence $\mathcal{H}_1, \ldots, \mathcal{H}_m$ in $\mathsf{Mns}(B, B')$,*
$$\Upsilon_{\mathcal{H}_m\mathcal{H}_1} = \mathsf{b}_{\mathcal{H}_m} \cdot \left(\mathsf{b}^{-1}_{\overline{\mathcal{H}}_m} \cdot \Upsilon_{\overline{\mathcal{H}}_m\overline{\mathcal{H}}_{m-1}} \cdot \mathsf{b}_{\overline{\mathcal{H}}_{m-1}}\right) \cdots \left(\mathsf{b}^{-1}_{\overline{\mathcal{H}}_2} \cdot \Upsilon_{\overline{\mathcal{H}}_2\overline{\mathcal{H}}_1} \cdot \mathsf{b}_{\overline{\mathcal{H}}_1}\right) \mathsf{b}^{-1}_{\mathcal{H}_1}$$

Proof. (1) is the orthogonal factorisation property of the associators Φ.

(2) For $k = 1, \ldots, \ell$, let $\mathcal{F}_k$ be the restriction of $\mathcal{F}$ to B_k and $\mathcal{E}_k$ the unique element in $\mathsf{Mns}(B_k, B_{k-1})$, so that $\mathcal{F}_k = \mathcal{E}_k \cup \mathcal{F}_{k-1}$, and $\mathcal{F} = \mathcal{F}_\ell$. Then,

$$\begin{aligned}
J_{\mathcal{F}} &= (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}})_{12} \cdot J_{\mathcal{E}_\ell} \cdot J_{\mathcal{F}_{\ell-1}} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}})_1^{-1} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}})_2^{-1} \\
&= (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}})_{12} \cdot J_{\mathcal{E}_\ell} \cdot (\mathsf{a}^{\mathcal{E}_{\ell-1}}_{\mathcal{F}_{\ell-2}})_{12} \cdot J_{\mathcal{E}_{\ell-1}} \cdot J_{\mathcal{F}_{\ell-2}} \cdot (\mathsf{a}^{\mathcal{E}_{\ell-1}}_{\mathcal{F}_{\ell-2}})_1^{-1} \cdot (\mathsf{a}^{\mathcal{E}_{\ell-1}}_{\mathcal{F}_{\ell-2}})_2^{-1} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}})_1^{-1} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}})_2^{-1} \\
&= (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}} \cdot \mathsf{a}^{\mathcal{E}_{\ell-1}}_{\mathcal{F}_{\ell-2}})_{12} \cdot J_{\mathcal{E}_\ell} \cdot J_{\mathcal{E}_{\ell-1}} \cdot J_{\mathcal{F}_{\ell-2}} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}} \cdot \mathsf{a}^{\mathcal{E}_{\ell-1}}_{\mathcal{F}_{\ell-2}})_1^{-1} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}} \cdot \mathsf{a}^{\mathcal{E}_{\ell-1}}_{\mathcal{F}_{\ell-2}})_2^{-1} \\
&\;\;\vdots \\
&= (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}} \cdots \mathsf{a}^{\mathcal{E}_2}_{\mathcal{E}_1})_{12} \cdot J_{\mathcal{E}_\ell} \cdots J_{\mathcal{E}_1} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}} \cdots \mathsf{a}^{\mathcal{E}_2}_{\mathcal{E}_1})_1^{-1} \cdot (\mathsf{a}^{\mathcal{E}_\ell}_{\mathcal{F}_{\ell-1}} \cdots \mathsf{a}^{\mathcal{E}_2}_{\mathcal{E}_1})_2^{-1} \\
&= (\mathsf{b}_{\mathcal{F}})_{12} \cdot J_{\mathcal{E}_\ell} \cdots J_{\mathcal{E}_1} \cdot (\mathsf{b}_{\mathcal{F}})_1^{-1} \cdot (\mathsf{b}_{\mathcal{F}})_2^{-1}
\end{aligned}$$

where the second identity follows from the invariance of $J_{\mathcal{E}_\ell}$ under $[\mathfrak{b}_{B_{\ell-1}}]$.

(3) follows from Lemma 9.9.2 and the subsequent remark. $\square$

9.10. **Strict pre–Coxeter structures.** By Proposition 9.8, a braided pre–Coxeter structure $\mathfrak{C} = (\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}}, \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'})$ on $\widehat{\mathfrak{U}}^{\bullet}$ gives rise to a braided pre–Coxeter category $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b}}$. The following conditions ensure that $\mathbb{DY}^{\mathfrak{C}}_{\mathfrak{b}}$ is Υ–*strict* or a–*strict* (cf. 3.7 and 3.8).

We say that

- $\mathfrak{C}$ is Υ–*strict* if $\Upsilon_{\mathcal{F}\mathcal{G}} = 1$ for any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$
- $\mathfrak{C}$ is a–*strict* if $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = 1$ for any $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$ [29]

The following result shows that we can always restrict to either of these cases.

Proposition. *Let $\mathfrak{C}$ be a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^{\bullet}$.*

(1) *$\mathfrak{C}$ is twist equivalent to a Υ–strict braided pre–Coxeter structure.*

[29] In [4] we only consider a–strict braided pre–Coxeter structures and for simplicity refer to them as braided pre–Coxeter structures. Note also that such a structure is essentially a quasi–Coxeter quasitriangular quasi–Hopf algebra structure on the diagrammatic algebra $\widehat{\mathfrak{U}}^{\bullet}$, as defined in [37].

(2) $\mathfrak{C}$ *is canonically twist equivalent to an* a*–strict braided pre–Coxeter structure.*

Proof. (1) The trivialisation of the associators $\Upsilon_{\mathcal{F}\mathcal{G}}$ follows as in Proposition 3.7, and can be thought of as a universal lift of the fact that every pre–Coxeter category is equivalent to a Υ–strict one. Equivalently, it is enough to observe that, for any choice of maximal nested sets $\mathcal{E}=\{\mathcal{E}(B,B')\}_{B'\subseteq B}$, $\mathbb{T}_{\mathcal{E}}=(u,F)$ with

$$u_{\mathcal{F}}=\Upsilon_{\mathcal{E}(B,B')\mathcal{F}} \qquad \text{and} \qquad F_B=1_B$$

is a twist in $\widehat{\mathfrak{U}}^{\bullet}$, and $\mathfrak{C}_{\mathbb{T}_{\mathcal{E}}}$ is a Υ–strict braided pre–Coxeter structure.

(2) The trivialisation of the vertical joins $\mathsf{a}_{\mathcal{F}'}^{\mathcal{F}}$ follows as in Proposition 3.8. Indeed, the result of Lemma 9.4 (3) implies that the propic analogues of the diagrams (3.6) are trivial in $\widehat{\mathfrak{U}}^{\bullet}$. Equivalently, it is enough to observe that $\mathbb{T}=(u,F)$ with

$$u_{\mathcal{F}}=\mathsf{b}_{\mathcal{F}}^{-1} \qquad \text{and} \qquad F_B=1_B$$

is a twist in $\widehat{\mathfrak{U}}^{\bullet}$, and $\mathfrak{C}_{\mathbb{T}}$ is an a–strict braided pre–Coxeter structure (cf. Proposition 9.9 (4)). $\square$

Remark. It is easy to see that Proposition 9.10 cannot be used to obtain a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^{\bullet}$ which is both Υ–strict and a–strict.

10. An equivalence of braided pre–Coxeter categories

In this section, we rely on the results of [3] to prove the existence of a braided pre–Coxeter structure $\mathfrak{C}$ on $\widehat{\mathfrak{U}}^{\bullet}$. We then show that, for any diagrammatic Lie bialgebra $\mathfrak{b}$, the braided pre–Coxeter category $\mathbb{DY}_{\mathfrak{b}}^{\mathfrak{C}}$ determined by $\mathfrak{C}$ and $\mathfrak{b}$ is equivalent to that of admissible Drinfeld–Yetter modules over the Etingof–Kazhdan quantisation $\mathcal{Q}(\mathfrak{b})$ of $\mathfrak{b}$.

10.1. **Factorisable associators.** Let $\underline{\mathsf{LBA}}$ be the PROP describing Lie bialgebras, and $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{LBA}}$ the corresponding universal algebra.[30] Let $\underline{\mathsf{LBA}_{\circ}}$ be the PROP describing a Lie bialgebra $[\mathfrak{b}]$, which decomposes as the direct sum $[\mathfrak{b}]=[\mathfrak{b}_1]\oplus[\mathfrak{b}_2]$ of two Lie bialgebras, and $\widehat{\mathfrak{U}}^{\bullet}_{\circ}$ the corresponding universal algebra. Equivalently, $\underline{\mathsf{LBA}_{\circ}}$ is the PROP generated by a Lie bialgebra object $[\mathfrak{b}]$, together with two Lie bialgebra idempotents $\theta_1,\theta_2\in\mathrm{End}([\mathfrak{b}])$ satisfying $\theta_1\cdot\theta_2=0=\theta_2\cdot\theta_1$ and $\theta_1+\theta_2=\mathrm{id}_{[\mathfrak{b}]}$. It therefore coincides with the PROP $\underline{\mathsf{LBA}}_D$ for the diagram $\overset{1}{\circ}\ \ \overset{2}{\circ}$ consisting of two disconnected vertices.

Let $\Phi\in\widehat{\mathfrak{U}}^{3}_{\mathsf{LBA}}$ be an associator, and $\Phi_{[\mathfrak{b}]},\Phi_{[\mathfrak{b}_1]},\Phi_{[\mathfrak{b}_2]}\in\widehat{\mathfrak{U}}^{3}_{\circ}$ its images under the homomorphisms $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{LBA}}\to\widehat{\mathfrak{U}}^{\bullet}_{\circ}$ corresponding to the Lie bialgebras $[\mathfrak{b}],[\mathfrak{b}_1]$ and $[\mathfrak{b}_2]$ respectively. Φ is said to be *factorisable* if the following holds in $\widehat{\mathfrak{U}}^{3}_{\circ}$ [31]

$$\Phi_{[\mathfrak{b}]}=\Phi_{[\mathfrak{b}_1]}\cdot\Phi_{[\mathfrak{b}_2]}$$

This is the case for example if Φ is a *Lie associator*, that is the exponential of a Lie series in Ω_{12} and Ω_{23}.

[30] Note that $\underline{\mathsf{LBA}}$ (resp. $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{LBA}}$) coincides with the PROP (resp. universal algebra) $\underline{\mathsf{LBA}}_D$ (resp. $\widehat{\mathfrak{U}}^{\bullet}_D$) for a diagram D consisting of a single vertex.

[31] The order of the factors is irrelevant, since the images of $\widehat{\mathfrak{U}}^{n}_{\overset{1}{\circ}}$ and $\widehat{\mathfrak{U}}^{n}_{\overset{2}{\circ}}$ commute in $\widehat{\mathfrak{U}}^{n}_{\overset{1}{\circ}\,\overset{2}{\circ}}=\widehat{\mathfrak{U}}^{n}_{\circ}$ by Proposition 8.5.

10.2. **A pre–Coxeter structure on $\widehat{\mathfrak{U}}^\bullet$.** Let now D be a fixed diagram. By construction, the generating object in $\underline{\mathsf{LBA}}_{\mathrm{D}}$ is a split diagrammatic Lie bialgebra. In particular, for any $B \subseteq D$, the subobject $[\mathfrak{b}_B] = ([1], \theta_B)$ is a Lie bialgebra in $\underline{\mathsf{LBA}}_{\mathrm{D}}$. This induces a functor $\underline{\mathsf{LBA}} \to \underline{\mathsf{LBA}}_{\mathrm{D}}$ which factors through $\underline{\mathsf{LBA}}_B$, and a homomorphism $\widehat{\rho}^n_B : \widehat{\mathfrak{U}}^n_{\mathsf{LBA}} \to \widehat{\mathfrak{U}}^n_B$.

Theorem. *For any factorisable associator Φ in $\widehat{\mathfrak{U}}^3_{\mathsf{LBA}}$, there is a Υ–strict braided pre–Coxeter structure $\mathfrak{C}^{\Upsilon\text{-str}}_\Phi$ on $\widehat{\mathfrak{U}}^\bullet$ which is trivial modulo $\widehat{\mathfrak{U}}^\bullet_{\geqslant 1}$, and such that $\Phi_B = \widehat{\rho}^3_B(\Phi)$ for any $B \subseteq D$.*

The proof of Theorem 10.2 is given in 10.9. It relies on our earlier results in [3], which are reviewed in 10.3–10.7.

Remarks.

- Theorem 10.2 and Proposition 9.10 imply the existence of an a–strict braided pre–Coxeter structure $\mathfrak{C}^{\mathsf{a}\text{-str}}_\Phi$ on $\widehat{\mathfrak{U}}^\bullet$ with associators $\Phi_B = \widehat{\rho}^3_B(\Phi)$, which is canonically twist equivalent to $\mathfrak{C}^{\Upsilon\text{-str}}_\Phi$.
- As mentioned in 3.7, and proved in [5], the monodromy of the Casimir connection of a Kac–Moody algebra is encoded by an a–strict pre–Coxeter structure, which is more naturally compared with $\mathfrak{C}^{\mathsf{a}\text{-str}}_\Phi$.

Corollary. *Let $\Phi \in \widehat{\mathfrak{U}}^3_{\mathsf{LBA}}$ be a factorisable associator. Then, for any split diagrammatic Lie bialgebra $\mathfrak{b}$, there is a Υ–strict (resp. a–strict) braided pre–Coxeter category $\mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}}$ (resp. $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str}}_{\mathfrak{b}}$) with*

- *diagrammatic categories $(\mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}})_B = \mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B} = (\mathbb{DY}^{\Phi,\mathsf{a}\text{-str}}_{\mathfrak{b}})_B$*
- *restriction functors $\mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B} \to \mathsf{DY}^{\Phi_{B'}}_{\mathfrak{b}_{B'}}$ of the form $\left(\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}, J_{\mathcal{F}}\right)$ for some tensor structure $J_{\mathcal{F}}$ on $\mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$*

Moreover, $\mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}}$ and $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str}}_{\mathfrak{b}}$ are canonically equivalent braided pre–Coxeter categories.

Proof. This follows by applying Proposition 9.8 to the braided pre–Coxeter structures $\mathfrak{C}^{\Upsilon\text{-str}}_\Phi$, $\mathfrak{C}^{\mathsf{a}\text{-str}}_\Phi$, and setting

$$\mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}} := \mathbb{DY}^{\mathfrak{C}^{\Upsilon\text{-str}}_\Phi}_{\mathfrak{b}} \qquad \text{and} \qquad \mathbb{DY}^{\Phi,\mathsf{a}\text{-str}}_{\mathfrak{b}} := \mathbb{DY}^{\mathfrak{C}^{\mathsf{a}\text{-str}}_\Phi}_{\mathfrak{b}}$$

By the remark above, $\mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}}$ and $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str}}_{\mathfrak{b}}$ are canonically equivalent. $\square$

10.3. **A relative fiber functor.** Let $\mathsf{sLBA}(\mathsf{k})$ be the category whose objects are Lie bialgebras, and morphisms are split embeddings (cf. (5.1)). Fix an associator Φ in $\widehat{\mathfrak{U}}^3_{\mathsf{LBA}}$. In [3], we construct a 2–functor

$$\mathsf{DY}^\Phi : \mathsf{sLBA}(\mathsf{k}) \longrightarrow \mathsf{Cat}^{\otimes}_{\mathsf{K}}$$

which assigns

- to any Lie bialgebra $\mathfrak{b}$, the monoidal category $\mathsf{DY}^\Phi_{\mathfrak{b}}$ of deformation Drinfeld–Yetter $\mathfrak{b}$–modules with associativity constraint $\Phi_{\mathfrak{b}} = \widehat{\rho}^3_{\mathfrak{b}}(\Phi)$
- to any split embedding $\mathfrak{a} \hookrightarrow \mathfrak{b}$, a monoidal structure $J_{\mathfrak{a},\mathfrak{b}}$ on the restriction functor $\mathrm{Res}_{\mathfrak{a},\mathfrak{b}} : \mathsf{DY}^\Phi_{\mathfrak{b}} \to \mathsf{DY}^\Phi_{\mathfrak{a}}$

- to any chain of split embeddings (a *split triple*) $\mathfrak{a} \hookrightarrow \mathfrak{b} \hookrightarrow \mathfrak{c}$, an isomorphism of monoidal functors

$$u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} : (\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}, J_{\mathfrak{a},\mathfrak{b}}) \circ (\mathrm{Res}_{\mathfrak{b},\mathfrak{c}}, J_{\mathfrak{b},\mathfrak{c}}) \longrightarrow (\mathrm{Res}_{\mathfrak{a},\mathfrak{c}}, J_{\mathfrak{a},\mathfrak{c}})$$

in such a way that, for any chain $\mathfrak{a} \hookrightarrow \mathfrak{b} \hookrightarrow \mathfrak{c} \hookrightarrow \mathfrak{d}$, one has

$$u_{\mathfrak{a},\mathfrak{b},\mathfrak{d}} \circ u_{\mathfrak{b},\mathfrak{c},\mathfrak{d}} = u_{\mathfrak{a},\mathfrak{c},\mathfrak{d}} \circ u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} \tag{10.1}$$

as isomorphisms

$$(\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}, J_{\mathfrak{a},\mathfrak{b}}) \circ (\mathrm{Res}_{\mathfrak{b},\mathfrak{c}}, J_{\mathfrak{b},\mathfrak{c}}) \circ (\mathrm{Res}_{\mathfrak{c},\mathfrak{d}}, J_{\mathfrak{c},\mathfrak{d}}) \longrightarrow (\mathrm{Res}_{\mathfrak{a},\mathfrak{d}}, J_{\mathfrak{a},\mathfrak{d}})$$

Moreover, $J_{\mathfrak{a},\mathfrak{a}}$, $u_{\mathfrak{a},\mathfrak{a},\mathfrak{b}}$, and $u_{\mathfrak{a},\mathfrak{b},\mathfrak{b}}$ are trivial and, if Φ is factorisable, then

$$J_{\mathfrak{a}_1\oplus\mathfrak{a}_2,\mathfrak{b}_1\oplus\mathfrak{b}_2} = J_{\mathfrak{a}_1,\mathfrak{b}_1} \cdot J_{\mathfrak{a}_2,\mathfrak{b}_2} \qquad \text{and} \qquad u_{\mathfrak{a}_1\oplus\mathfrak{a}_2,\mathfrak{b}_1\oplus\mathfrak{b}_2,\mathfrak{c}_1\oplus\mathfrak{c}_2} = u_{\mathfrak{a}_1,\mathfrak{b}_1,\mathfrak{c}_1} \cdot u_{\mathfrak{a}_2,\mathfrak{b}_2,\mathfrak{c}_2}$$

Remark. When $\mathfrak{a} = 0$, $J_{\mathfrak{a},\mathfrak{b}}$ is gauge equivalent to the monoidal structure on the forgetful functor $\mathsf{DY}^{\Phi}_{\mathfrak{b}} \to \mathrm{Vect}_K$ constructed by Etingof–Kazhdan [15].

10.4. **Functoriality of the Etingof–Kazhdan equivalence.** In [17], Etingof and Kazhdan define an equivalence of braided monoidal categories $H_{\mathfrak{b}} : \mathsf{DY}^{\Phi}_{\mathfrak{b}} \to \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$, where $\mathfrak{b}$ is a Lie bialgebra and $\mathcal{Q}(\mathfrak{b})$ its Etingof–Kazhdan quantisation. We prove in [3] that the equivalence $H_{\mathfrak{b}}$ is functorial with respect to split embeddings. Specifically, let $\mathsf{s}\mathcal{Q} : \mathsf{sLBA}(\mathsf{k}) \to \mathsf{sQUE}(\mathsf{K})$ be the Etingof–Kazhdan quantisation functor between the categories of split Lie bialgebras and split QUEs. We show that there is an isomorphism of 2–functors

$$\begin{tikzcd}
\mathsf{sLBA}(\mathsf{k}) \arrow[rr, "\mathsf{s}\mathcal{Q}"] \arrow[dr, "\mathsf{DY}^{\Phi}"'] & \arrow[d, phantom, "\Leftarrow"] & \mathsf{sQUE}(\mathsf{k}) \arrow[dl, "\mathsf{DY}^{\mathrm{adm}}"] \\
& \mathsf{Cat}^{\otimes}_{\mathsf{K}} &
\end{tikzcd}$$

which assigns to a Lie bialgebra $\mathfrak{b}$ the equivalence $H_{\mathfrak{b}}$. In particular,

- For any split embedding $\mathfrak{a} \hookrightarrow \mathfrak{b}$, there is a natural isomorphism $v_{\mathfrak{a},\mathfrak{b}}$ making the following diagram commute

$$\begin{tikzcd}
\mathsf{DY}^{\Phi}_{\mathfrak{b}} \arrow[r, "H_{\mathfrak{b}}"] \arrow[d, "{(\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}, J_{\mathfrak{a},\mathfrak{b}})}"'] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})} \arrow[d, "{(\mathrm{Res}_{\mathcal{Q}(\mathfrak{a}),\mathcal{Q}(\mathfrak{b})}, \mathrm{id})}"] \\
\mathsf{DY}^{\Phi}_{\mathfrak{a}} \arrow[r, "H_{\mathfrak{a}}"'] \arrow[ur, Rightarrow, "v_{\mathfrak{a},\mathfrak{b}}"] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{a})}
\end{tikzcd} \tag{10.2}$$

where $(\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}, J_{\mathfrak{a},\mathfrak{b}})$ is the monoidal functor from 10.3, and the functor $\mathrm{Res}_{\mathcal{Q}(\mathfrak{a}),\mathcal{Q}(\mathfrak{b})}$ is induced by the split embedding $\mathcal{Q}(\mathfrak{a}) \hookrightarrow \mathcal{Q}(\mathfrak{b})$.

- For any chain of split embeddings $\mathfrak{a} \hookrightarrow \mathfrak{b} \hookrightarrow \mathfrak{c}$, the following prism is commutative

$$
\begin{tikzcd}
 & & \mathsf{DY}^{\mathrm{adm}}_{U_\hbar\mathfrak{c}} \arrow[ddr, "\mathrm{Res}_{\mathcal{Q}(\mathfrak{b}),\mathcal{Q}(\mathfrak{c})}"] \arrow[dddd, dotted, "\mathrm{Res}_{\mathcal{Q}(\mathfrak{a}),\mathcal{Q}(\mathfrak{c})}"'] & \\
\mathsf{DY}^{\Phi}_{\mathfrak{c}} \arrow[urr, "H_{\mathfrak{c}}"] \arrow[ddr, "\mathrm{Res}_{\mathfrak{b},\mathfrak{c}}"] \arrow[dddd, bend right, "\mathrm{Res}_{\mathfrak{a},\mathfrak{c}}"'] & & & \\
 & & & \mathsf{DY}^{\mathrm{adm}}_{U_\hbar\mathfrak{b}} \arrow[ddl, "\mathrm{Res}_{\mathcal{Q}(\mathfrak{a}),\mathcal{Q}(\mathfrak{b})}"] \\
 {} & \mathsf{DY}^{\Phi}_{\mathfrak{b}} \arrow[l, Rightarrow, "u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}"'] \arrow[urr, "H_{\mathfrak{b}}"] \arrow[ddl, "\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}"] & & \\
 & & \mathsf{DY}^{\mathrm{adm}}_{U_\hbar\mathfrak{a}} & \\
\mathsf{DY}^{\Phi}_{\mathfrak{a}} \arrow[urr, "H_{\mathfrak{a}}"'] & & &
\end{tikzcd}
\tag{10.3}
$$

where $u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ is the isomorphism from 10.3, the back 2–face is the identity, and the lateral 2–faces are the isomorphisms $v_{\mathfrak{a},\mathfrak{c}}, v_{\mathfrak{b},\mathfrak{c}}, v_{\mathfrak{a},\mathfrak{b}}$.[32]

Remarks.

- The natural isomorphism $v_{\mathfrak{a},\mathfrak{b}}$ is not trivial in general. Indeed, the strict commutativity of (10.2), even as a diagram of non–monoidal functors, would contradict Prop. 3.2 in [37] (see [1, Sec. 1.10]). Namely, let $\mathcal{U}^{\hbar}_{\mathfrak{b}}$ and $\mathcal{U}_{\mathcal{Q}(\mathfrak{b})}$ be the algebras of endomorphisms of the forgetful functors $\mathsf{DY}^{\hbar}_{\mathfrak{b}} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$ and $\mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$, respectively. The Etingof–Kazhdan equivalence $H_{\mathfrak{b}} : \mathsf{DY}^{\hbar}_{\mathfrak{b}} \to \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ intertwines the forgetful functors and gives rise to an isomorphism of algebras $\mathcal{U}^{\hbar}_{\mathfrak{b}} \to \mathcal{U}_{\mathcal{Q}(\mathfrak{b})}$. Through the classical and quantum restriction functors, we get canonical inclusions $\mathcal{U}^{\hbar}_{\mathfrak{a}} \hookrightarrow \mathcal{U}^{\hbar}_{\mathfrak{b}}$ and $\mathcal{U}_{\mathcal{Q}(\mathfrak{a})} \hookrightarrow \mathcal{U}_{\mathcal{Q}(\mathfrak{b})}$. Therefore, the strict commutativity of (10.2) is equivalent to the commutativity of the diagram

$$
\begin{tikzcd}
\mathcal{U}^{\hbar}_{\mathfrak{b}} \arrow[r] & \mathcal{U}_{\mathcal{Q}(\mathfrak{b})} \\
\mathcal{U}^{\hbar}_{\mathfrak{a}} \arrow[r] \arrow[u, hook] & \mathcal{U}_{\mathcal{Q}(\mathfrak{a})} \arrow[u, hook]
\end{tikzcd}
$$

In the case of a semisimple Lie algebra $\mathfrak{g}$, $\mathcal{Q}(\mathfrak{g})$ is isomorphic to the Drinfeld–Jimbo quantum groups $U_\hbar\mathfrak{g}$ as a diagrammatic QUE (cf. 13 and Proposition 13.6). Thus, we obtain a diagrammatic isomorphism $U\mathfrak{g}[\![\hbar]\!] \to \mathcal{Q}(\mathfrak{g}) \simeq U_\hbar\mathfrak{g}$, which contradicts [37, Pro. 3.2].

- The natural transformation $u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ described in 10.3 is in fact *defined* so as to make (10.3) commutative. Namely, since $H_{\mathfrak{a}}$ is invertible, $v_{\mathfrak{a},\mathfrak{b}}$ induces a natural isomorphism $w_{\mathfrak{a},\mathfrak{b}}$

$$
\begin{tikzcd}
\mathsf{DY}^{\Phi}_{\mathfrak{b}} \arrow[r, "H_{\mathfrak{b}}"] \arrow[d, "{(\mathrm{Res}_{\mathfrak{a},\mathfrak{b}}, J_{\mathfrak{a},\mathfrak{b}})}"'] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})} \arrow[d, "{(\mathrm{Res}_{\mathcal{Q}(\mathfrak{a}),\mathcal{Q}(\mathfrak{b})}, \mathrm{id})}"] \\
\mathsf{DY}^{\Phi}_{\mathfrak{a}} \arrow[ur, Rightarrow, "w_{\mathfrak{a},\mathfrak{b}}"] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{a})} \arrow[l, "H_{\mathfrak{a}}^{-1}"]
\end{tikzcd}
$$

[32]To alleviate the notation, tensor structures are suppressed from the diagram (10.3).

The natural transformation $u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ is then defined as

$$u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} = w_{\mathfrak{a},\mathfrak{c}}^{-1} \circ w_{\mathfrak{a},\mathfrak{b}} \circ w_{\mathfrak{b},\mathfrak{c}}$$

In particular, this makes the associativity (10.1) of u manifest. Finally, one observes that $w_{\mathfrak{a},\mathfrak{a}}$ is trivial and $w_{\mathfrak{a}_1\oplus\mathfrak{a}_2,\mathfrak{b}_1\oplus\mathfrak{b}_2} = w_{\mathfrak{a}_1,\mathfrak{b}_1} \cdot w_{\mathfrak{a}_2,\mathfrak{b}_2}$, so that the normalisation and factorisation properties of u follow.

10.5. **Auxiliary PROPs.** The constructions described in 10.3–10.4 are universal, in that the relative twist $J_{\mathfrak{a},\mathfrak{b}}$, the natural transformation $u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ and their properties are induced by analogous elements and relations in the universal algebras associated to the following PROPs.

Let $\underline{\mathsf{LBA}}_{\mathsf{sp}}$ be the PROP generated by a Lie bialgebra object $([1],\mu,\delta)$, together with a Lie bialgebra idempotent $\theta : [1] \to [1]$. We denote by $\mathfrak{U}_{\mathsf{sp}}^{\bullet}$ the corresponding universal algebras. A split embedding of Lie bialgebras $(i,p) : \mathfrak{a} \to \mathfrak{b}$ is equivalent to a realisation functor $\mathcal{G}_{\mathfrak{b}} : \underline{\mathsf{LBA}}_{\mathsf{sp}} \to \mathrm{Vect}_{\mathsf{k}}$ given by

$$\mathcal{G}_{\mathfrak{b}}[1] = \mathfrak{b} \qquad \text{and} \qquad \mathcal{G}_{\mathfrak{b}}\theta = i \circ p$$

It therefore gives rise to a map $\rho_{\mathfrak{a},\mathfrak{b}}^{\bullet} : \mathfrak{U}_{\mathsf{sp}}^{\bullet} \to \mathcal{U}_{\mathfrak{b}}^{\bullet}$. We denote the Lie bialgebra objects $[1], \theta[1]$ by $[\mathfrak{b}], [\mathfrak{a}]$, respectively.

Let $\underline{\mathsf{LBA}}_{\mathsf{st}}$ be the PROP generated by a Lie bialgebra object $([1],\mu,\delta)$ with idempotents $\theta,\theta' : [1] \to [1]$ such that $\theta\theta' = \theta' = \theta'\theta$. We denote by $\mathfrak{U}_{\mathsf{st}}^{\bullet}$ the corresponding universal algebras. A split triple of Lie bialgebras $(i,p)\circ(i',p') : \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c}$ is equivalent to a realisation functor $\mathcal{G}_{\mathfrak{c}} : \underline{\mathsf{LBA}}_{\mathsf{st}} \to \mathrm{Vect}_{\mathsf{k}}$ given by

$$\mathcal{G}_{\mathfrak{c}}[1] = \mathfrak{c}, \qquad \mathcal{G}_{\mathfrak{c}}\theta = i \circ p \qquad \text{and} \qquad \mathcal{G}_{\mathfrak{c}}\theta' = i \circ i' \circ p' \circ p$$

It therefore gives rise to a map $\rho_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}^{\bullet} : \mathfrak{U}_{\mathsf{st}}^{\bullet} \to \mathcal{U}_{\mathfrak{c}}^{\bullet}$. We denote the Lie bialgebra objects $[1], \theta[1], \theta'[1]$ by $[\mathfrak{c}], [\mathfrak{b}], [\mathfrak{a}]$, respectively. The PROP $\underline{\mathsf{LBA}}_{\mathsf{sq}}$ and its universal algebra $\mathfrak{U}_{\mathsf{sq}}^{\bullet}$, corresponding to split quadruples, are similarly defined.

Let $\underline{\mathsf{LBA}}_{\mathsf{osp}}$ (resp. $\mathfrak{U}_{\mathsf{osp}}^{\bullet}$) be the PROP (resp. its universal algebra) consisting of a split pair $[\mathfrak{a}] \hookrightarrow [\mathfrak{b}]$ which decomposes as the direct sum of two split pairs $[\mathfrak{a}_1] \hookrightarrow [\mathfrak{b}_1]$ and $[\mathfrak{a}_2] \hookrightarrow [\mathfrak{b}_2]$. The PROPs $\underline{\mathsf{LBA}}_{\mathsf{ost}}$, $\underline{\mathsf{LBA}}_{\mathsf{osq}}$ and their universal algebras $\mathfrak{U}_{\mathsf{ost}}^{\bullet}$, $\mathfrak{U}_{\mathsf{osq}}^{\bullet}$, corresponding, respectively, to a split triple and a split quadruple with a direct sum decomposition, are similarly defined.

10.6. **Universal relative twists and joins.** Let $\Phi \in \widehat{\mathfrak{U}}_{\mathsf{LBA}}^{3}$ be an associator. An element $J \in \widehat{\mathfrak{U}}_{\mathsf{sp}}^{2}$ is a *relative twist* if it is such that $(J)_0 = 1$, $\varepsilon_2^1(J) = 1 = \varepsilon_2^2(J)$, it commutes with the diagonal action and coaction of $[\mathfrak{a}]$, and satisfies the relative twist equation with respect to Φ

$$J_{1,23} \cdot J_{23} \cdot \Phi_{[\mathfrak{a}]} = \Phi_{[\mathfrak{b}]} \cdot J_{12,3} \cdot J_{12}$$

J is said to be

- *normalised* if $J_{[\mathfrak{a}],[\mathfrak{a}]} = 1$, where $J_{[\mathfrak{a}],[\mathfrak{a}]}$ is the image of J under the map $\widehat{\mathfrak{U}}_{\mathsf{sp}}^{\bullet} \to \widehat{\mathfrak{U}}_{\mathsf{LBA}}^{\bullet}$, corresponding to the split pair $([1],[1])$ in $\underline{\mathsf{LBA}}$
- *factorisable* if Φ is a factorisable associator, and

$$J_{[\mathfrak{a}_1]\oplus[\mathfrak{a}_2],[\mathfrak{b}_1]\oplus[\mathfrak{b}_2]} = J_{[\mathfrak{a}_1],[\mathfrak{b}_1]} \cdot J_{[\mathfrak{a}_2],[\mathfrak{b}_2]}$$

  where $J_{[\mathfrak{a}_1],[\mathfrak{b}_1]}$, $J_{[\mathfrak{a}_2],[\mathfrak{b}_2]}$, $J_{[\mathfrak{a}_1]\oplus[\mathfrak{a}_2],[\mathfrak{b}_1]\oplus[\mathfrak{b}_2]}$ are the images of J under the maps $\widehat{\mathfrak{U}}_{\mathsf{sp}}^{\bullet} \to \widehat{\mathfrak{U}}_{\mathsf{osp}}^{\bullet}$ induced by the corresponding split pairs in $\underline{\mathsf{LBA}}_{\mathsf{osp}}$.

Let J be a relative twist, and denote by $J_{[\mathfrak{a}],[\mathfrak{b}]}$, $J_{[\mathfrak{b}],[\mathfrak{c}]}$, $J_{[\mathfrak{a}],[\mathfrak{c}]}$ the images of J under the homomorphisms $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{sp}} \to \widehat{\mathfrak{U}}^{\bullet}_{\mathsf{st}}$ induced by the corresponding to split pairs in $\underline{\mathsf{LBA}}_{\mathsf{st}}$. An element $u \in \widehat{\mathfrak{U}}_{\mathsf{st}}$ is a *vertical join* if is such that $(u)_0 = 1$, $\varepsilon(u) = 1$, it commutes with the action and coaction of $[\mathfrak{a}]$, and satisfies

$$J_{[\mathfrak{a}],[\mathfrak{c}]} = u_{12} \cdot J_{[\mathfrak{b}],[\mathfrak{c}]} \cdot J_{[\mathfrak{a}],[\mathfrak{b}]} \cdot u_1^{-1} \cdot u_2^{-1}$$

u is said to be

- *normalised* if $u_{[\mathfrak{a}],[\mathfrak{a}],[\mathfrak{b}]} = 1 = u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{b}]}$, where $u_{[\mathfrak{a}],[\mathfrak{a}],[\mathfrak{b}]}$ and $u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{b}]}$ are the images of u under the homomorphism $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{st}} \to \widehat{\mathfrak{U}}^{\bullet}_{\mathsf{sp}}$, induced by the split triples $([\mathfrak{a}],[\mathfrak{a}],[\mathfrak{b}])$ and $([\mathfrak{a}],[\mathfrak{b}],[\mathfrak{b}])$ in $\underline{\mathsf{LBA}}_{\mathsf{sp}}$
- *associative* if

$$u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{d}]} \cdot u_{[\mathfrak{b}],[\mathfrak{c}],[\mathfrak{d}]} = u_{[\mathfrak{a}],[\mathfrak{c}],[\mathfrak{d}]} \cdot u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{c}]}$$

  where $u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{d}]}$, $u_{[\mathfrak{b}],[\mathfrak{c}],[\mathfrak{d}]}$, $u_{[\mathfrak{a}],[\mathfrak{c}],[\mathfrak{d}]}$ and $u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{c}]}$ are the images of u under the homomorphisms $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{st}} \to \widehat{\mathfrak{U}}^{\bullet}_{\mathsf{osq}}$, induced by the corresponding split triples in $\underline{\mathsf{LBA}}_{\mathsf{sq}}$
- *factorisable* if Φ and J are factorisable, and

$$u_{[\mathfrak{a}_1]\oplus[\mathfrak{a}_2],[\mathfrak{b}_1]\oplus[\mathfrak{b}_2],[\mathfrak{c}_1]\oplus[\mathfrak{c}_2]} = u_{[\mathfrak{a}_1],[\mathfrak{b}_1],[\mathfrak{c}_1]} \cdot u_{[\mathfrak{a}_2],[\mathfrak{b}_2],[\mathfrak{c}_2]}$$

  where $u_{[\mathfrak{a}_1]\oplus[\mathfrak{a}_2],[\mathfrak{b}_1]\oplus[\mathfrak{b}_2],[\mathfrak{c}_1]\oplus[\mathfrak{c}_2]}$, $u_{[\mathfrak{a}_1],[\mathfrak{b}_1],[\mathfrak{c}_1]}$ and $u_{[\mathfrak{a}_2],[\mathfrak{b}_2],[\mathfrak{c}_2]}$ are the images of u under the homomorphisms $\widehat{\mathfrak{U}}^{\bullet}_{\mathsf{st}} \to \widehat{\mathfrak{U}}^{\bullet}_{\mathsf{ost}}$, induced by the corresponding split triples in $\underline{\mathsf{LBA}}_{\mathsf{ost}}$

10.7. **Existence of a universal relative twist and join.**

Theorem. *Let $\Phi \in \widehat{\mathfrak{U}}^{3}_{\mathsf{LBA}}$ be an associator.*

(1) There is a relative twist $J \in \widehat{\mathfrak{U}}^{2}_{\mathsf{sp}}$, which is normalised and such that, for any split pair $\mathfrak{a} \hookrightarrow \mathfrak{b}$, $J_{\mathfrak{a},\mathfrak{b}} = \widehat{\rho}^{2}_{\mathfrak{a},\mathfrak{b}}(J)$.

(2) There is a vertical join $u \in \widehat{\mathfrak{U}}_{\mathsf{st}}$, which is normalised, associative and such that, for any split triple $\mathfrak{a} \hookrightarrow \mathfrak{b} \hookrightarrow \mathfrak{c}$, $u_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} = \widehat{\rho}_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(u)$.

Moreover, if Φ is factorisable, then so are J and u.

Proof. (1) The existence of a relative twist $J \in \widehat{\mathfrak{U}}^{2}_{\mathsf{sp}}$ is proved in [3, Prop. 7.7, 8.2.2]. By construction, J satisfies $J_{\mathfrak{a},\mathfrak{b}} = \widehat{\rho}^{2}_{\mathfrak{a},\mathfrak{b}}(J)$ and, by direct inspection, it is normalised and factorisable (for the latter property, see also [3, Prop. 2.25]).

(2) We show in [3, Sec. 6.17] that the Etingof–Kazhdan equivalence $H_{\mathfrak{b}} : \mathsf{DY}^{\Phi}_{\mathfrak{b}} \to \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ is PROPic. Specifically, let $\underline{\mathsf{DY}}_{\mathsf{UE}_{\mathsf{cP}}}$ (resp. $\underline{\mathsf{DY}}_{\mathsf{QUE}}$) be the PROP describing an admissible Drinfeld–Yetter module over a co–Poisson universal enveloping algebra (resp. over a QUE). Then, the category $\mathsf{DY}^{\Phi}_{\mathfrak{b}}$ (resp. $\mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$) is equivalent to that of realisation functors from $\underline{\mathsf{DY}}_{\mathsf{UE}_{\mathsf{cP}}}$ (resp. $\underline{\mathsf{DY}}_{\mathsf{QUE}}$) to $\mathrm{Vect}_{\mathsf{k}[\![\hbar]\!]}$. Under these identifications, $H_{\mathfrak{b}}$ arises as the *pullback* of an isomorphism of PROPs $H : \underline{\mathsf{DY}}_{\mathsf{QUE}} \to \underline{\mathsf{DY}}_{\mathsf{UE}_{\mathsf{cP}}}$. Similarly, one shows that the natural isomorphism $v_{\mathfrak{a},\mathfrak{b}}$ is PROPic, *i.e.,* it is induced by

$$\begin{tikzcd}
\underline{\mathsf{DY}}_{\mathsf{QUE}} \arrow[r] \arrow[d] & \underline{\mathsf{DY}}_{\mathsf{UE}_{\mathsf{cP}}} \arrow[d] \arrow[dl, Rightarrow, "v_{[\mathfrak{a}],[\mathfrak{b}]}"'] \\
\underline{\mathsf{DY}}_{\mathsf{QUE},\mathsf{sp}} \arrow[r] & \underline{\mathsf{DY}}_{\mathsf{UE}_{\mathsf{cP}},\mathsf{sp}}
\end{tikzcd}$$

where

- $\underline{\mathsf{DY}}_{\mathsf{UE_{cP}},\mathsf{sp}}$ (resp. $\underline{\mathsf{DY}}_{\mathsf{QUE},\mathsf{sp}}$) denote the PROPs describing a Drinfeld–Yetter module over a split pair of co–Poisson universal enveloping algebras $[A_0] \to [B_0]$ (resp. over a split pair of QUEs $[A] \to [B]$);
- the vertical arrows are the canonical functors mapping the generating objects of $\mathsf{UE_{cP}}$ and QUE to $[A_0]$ and $[A]$, respectively
- the horizontal arrows are PROPic Etingof–Kazhdan equivalences

The natural transformation $v_{[\mathfrak{a}],[\mathfrak{b}]}$ is normalised and factorisable, *i.e.,* $v_{[\mathfrak{a}],[\mathfrak{a}]}$ is trivial and $v_{[\mathfrak{a}_1]\oplus[\mathfrak{a}_2],[\mathfrak{b}_1]\oplus[\mathfrak{b}_2]} = v_{[\mathfrak{a}_1],[\mathfrak{b}_1]} \cdot v_{[\mathfrak{a}_2],[\mathfrak{b}_2]}$. The construction of $u_{[\mathfrak{a}],[\mathfrak{b}],[\mathfrak{c}]}$, and its normalisation, associativity and factorisability follow as in 10.4, by considering the PROPic analogue of the diagram (10.3). $\square$

10.8. **Υ–strict braided pre–Coxeter structures.** It is useful to observe, in analogy with Proposition 3.7, that a Υ–strict braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^{\bullet}$ is described by the datum of

- for any $B \subseteq D$, an associator $\Phi_B \in \widehat{\mathfrak{U}}^3_{B,B}$
- for any $B' \subseteq B$, a relative twist $J_{B'B} \in \widehat{\mathfrak{U}}^2_{B,B'}$ satisfying
$$J_{B'B,1,23} \cdot J_{B'B,23} \cdot \Phi_{B'} = \Phi_B \cdot J_{B'B,12,3} \cdot J_{B'B,12}$$
together with the normalisation $J_{BB} = 1$
- for any $B'' \subseteq B' \subseteq B$, a vertical join $\mathsf{a}_{B''B'B} \in \widehat{\mathfrak{U}}_{B,B''}$ satisfying
$$J_{B''B} = (\mathsf{a}_{B''B'B})_{12} \cdot J_{B''B'} \cdot J_{B'B} \cdot (\mathsf{a}_{B''B'B})_1^{-1} \cdot (\mathsf{a}_{B''B'B})_2^{-1}$$
together with the associativity
$$\mathsf{a}_{B'''B''B} \cdot \mathsf{a}_{B''B'B} = \mathsf{a}_{B'''B''B} \cdot \mathsf{a}_{B'''B''B'}$$
for any $B''' \subseteq B'' \subseteq B' \subseteq B$, and the normalisation $\mathsf{a}_{B'B'B} = 1 = \mathsf{a}_{B'BB}$

Moreover, for any $B_1'' \subseteq B_1' \subseteq B_1 \perp B_2 \supseteq B_2' \supseteq B_2''$, the following holds
$$\begin{aligned} \Phi_{B_1 \sqcup B_2} &= \Phi_{B_1} \cdot \Phi_{B_2} \\ J_{B_1' \sqcup B_2', B_1 \sqcup B_2} &= J_{B_1'B_1} \cdot J_{B_2'B_2} \\ \mathsf{a}_{B_1'' \sqcup B_2'', B_1' \sqcup B_2', B_1 \sqcup B_2} &= \mathsf{a}_{B_1''B_1'B_1} \cdot \mathsf{a}_{B_2''B_2'B_2} \end{aligned}$$

10.9. **Proof of Theorem 10.2.** We now construct a Υ–strict braided pre–Coxeter structure $\mathfrak{C}_{\Phi}^{\Upsilon\text{-str}} = (\Phi_B, J_{B'B}, \mathsf{a}_{B''B'B})$ in $\widehat{\mathfrak{U}}^{\bullet}$.

Associators. We use the notation from Section 10.2. For any $B \subseteq D$, set $\Phi_B = \widehat{\rho}^3_B(\Phi) \in \widehat{\mathfrak{U}}^3_B$. Since Φ is a factorisable associator, $\Phi_{B_1 \sqcup B_2} = \Phi_{B_1} \cdot \Phi_{B_2}$.

Relative twists. For any $B' \subseteq B \subseteq D$, the Lie bialgebra objects $[\mathfrak{b}_B]$ and $[\mathfrak{b}_{B'}]$ a split pair in $\underline{\mathsf{LBA}}_{\mathrm{D}}$. This induces a functor $\mathcal{G}_{[\mathfrak{b}_{B'}],[\mathfrak{b}_B]} : \underline{\mathsf{LBA}}_{\mathsf{sp}} \to \underline{\mathsf{LBA}}_{\mathrm{D}}$, and a homomorphism $\widehat{\rho}^n_{B'B} : \widehat{\mathfrak{U}}^n_{\mathsf{sp}} \to \widehat{\mathfrak{U}}^n_B$. Set $J_{B'B} = \widehat{\rho}^2_{B'B}(J) \in \widehat{\mathfrak{U}}^2_{B,B'}$. By Theorem 10.6, the relative twists $J_{B'B}$ satisfy the required properties of normalisation and orthogonal factorisation.

Vertical joins. For any chain of subdiagrams $B'' \subseteq B' \subseteq B \subseteq D$, the Lie bialgebra objects $[\mathfrak{b}_B]$, $[\mathfrak{b}_{B'}]$, and $[\mathfrak{b}_{B''}]$ induce a functor $\mathcal{G}_{[\mathfrak{b}_{B''}],[\mathfrak{b}_{B'}],[\mathfrak{b}_B]} : \underline{\mathsf{LBA}}_{\mathsf{st}} \to \underline{\mathsf{LBA}}_{\mathrm{D}}$, and a homomorphism $\widehat{\rho}^n_{B'',B',B} : \widehat{\mathfrak{U}}^n_{\mathsf{st}} \to \widehat{\mathfrak{U}}^n_B$. Then, we set $\mathsf{a}_{B''B'B} = \widehat{\rho}_{B'',B',B}(u) \in \widehat{\mathfrak{U}}_{B,B''}$. By Theorem 10.6, the vertical joins $\mathsf{a}_{B''B'B}$ satisfy the required properties of associativity, normalisation, and orthogonal factorisation. □

10.10. **An equivalence of braided pre–Coxeter categories.** We now show that the braided pre–Coxeter structures associated to a diagrammatic Lie bialgebra $\mathfrak{b}$ and to its Etingof–Kazhdan quantisation $\mathcal{Q}(\mathfrak{b})$ are equivalent.

Theorem. *Let $\mathfrak{b}$ be a split diagrammatic Lie bialgebra. For any factorisable associator $\Phi \in \widehat{\mathfrak{U}}^3_{\mathsf{LBA}}$, there is an equivalence of braided pre–Coxeter categories*

$$\mathbb{H}_{\mathfrak{b}} : \mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}} \longrightarrow \mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$$

where $\mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}}$ and $\mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ are defined in 10.2 and 6.9, respectively, and whose diagrammatic equivalences are given by the Etingof–Kazhdan functors $H_{\mathfrak{b}_B} : \mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B} \to \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_B)}$, $B \subseteq D$.

Proof. By definition, an equivalence $\mathbb{H}_{\mathfrak{b}} : \mathbb{DY}^{\Phi,\Upsilon\text{-str}}_{\mathfrak{b}} \longrightarrow \mathbb{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b})}$ of braided pre–Coxeter categories is the datum of

- For any $B \subseteq D$, an equivalence of braided monoidal categories $H_B : \mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B} \longrightarrow \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_B)}$
- For any $B' \subseteq B$, a natural transformation of monoidal functors

$$\begin{tikzcd}
\mathsf{DY}^{\Phi_B}_{\mathfrak{b}_B} \arrow[r, "H_B"] \arrow[d, "(\mathrm{Res}_{\mathfrak{b}_{B'}\mathfrak{b}_B}, J_{B'B})"'] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_B)} \arrow[d, "(\mathrm{Res}_{\mathcal{Q}(\mathfrak{b}_{B'}),\mathcal{Q}(\mathfrak{b}_B)}, \mathrm{id})"] \\
\mathsf{DY}^{\Phi_{B'}}_{\mathfrak{b}_{B'}} \arrow[r, "H_{B'}"'] \arrow[ur, Rightarrow, "\gamma_{B'B}"] & \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_{B'})}
\end{tikzcd}$$

satisfying the properties 3.10. Then, it is enough to set set $H_B = H_{\mathfrak{b}_B}$ and $\gamma_{B'B} = v_{\mathfrak{b}_{B'},\mathfrak{b}_B}$. The required properties are easily verified and the result follows. □

11. Kac–Moody algebras

Let k be a field of characteristic zero, $\mathbf{I}$ a finite set, and $\mathsf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ a fixed $|\mathbf{I}| \times |\mathbf{I}|$ matrix with entries in k. We review in this section the definition and basic properties of the Kac–Moody algebra associated to A. Our treatment is a little more general than [24], in that we consider realisations of A whose dimension is not assumed to be minimal. Such realisations will be used in Section 12 to endow a Kac–Moody algebra and its Borel subalgebras with a diagrammatic structure.

11.1. **Realisations.** Departing slightly from the terminology in [24], we define a *realisation* of A to be a triple $(V, \Pi, \Pi^\vee)$, where[33]

- V is a finite–dimensional vector space over k
- $\Pi = \{\alpha_i\}_{i\in\mathbf{I}}$ is a linearly independent subset of V^*
- $\Pi^\vee = \{\alpha_i^\vee\}_{i\in\mathbf{I}}$ is a linearly independent subset of V
- $\alpha_i(\alpha_j^\vee) = a_{ji}$ for any $i, j \in \mathbf{I}$

[33] In [24], V is additionally required to be of dimension $2|\mathbf{I}| - \mathrm{rank}(\mathsf{A})$.

Given a realisation $(V, \Pi, \Pi^\vee)$, we denote by $V' \subset V$ the $|\mathbf{I}|$–dimensional subspace spanned by $\Pi^\vee$, and by $\Pi^\perp \subset V$ the $|\mathbf{I}|$–codimensional subspace given by the annihilator of Π.

Lemma. *Let $(V, \Pi, \Pi^\vee)$ be a realisation of A. Then*

(1) $\dim V \geqslant 2|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$.
(2) $\Pi^\perp \subset V'$ *if, and only if V is of minimal dimension* $2|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$.

Proof. (1) Let $\langle\Pi\rangle \subset V^*$ and $\langle\Pi^\vee\rangle \subset V$ be the subspaces spanned by Π and $\Pi^\vee$. Restriction to $\langle\Pi^\vee\rangle$ gives rise to a surjection $V^* \to \langle\Pi^\vee\rangle^*$ which maps $\langle\Pi\rangle$ to a subspace V^*_{A} of dimension $\operatorname{rank}(\mathsf{A})$. Thus,

$$\dim V - |\mathbf{I}| = \dim\left(V^*/\langle\Pi\rangle\right) \geq \dim\left(\langle\Pi^\vee\rangle^*/V^*_{\mathsf{A}}\right) = |\mathbf{I}| - r$$

(2) $\Pi^\perp$ is of dimension $\dim V - |\mathbf{I}|$, while $\Pi^\perp \cap V_1$ is of dimension $|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$. □

11.2. **Subrealisations.** If $(V, \Pi, \Pi^\vee)$ is a realisation of A, a *subrealisation* of V is a subspace $U \subseteq V$ such that $\Pi^\vee \subset U$ and the restriction of the linear forms $\{\alpha_i\}_{i\in\mathbf{I}}$ to U are linearly independent, so that $(U, \Pi|_U, \Pi^\vee)$ is a realisation of A.

If $(U, \Pi, \Pi^\vee)$ is a realisation of A, and U^0 a finite–dimensional vector space, then $(V = U \oplus U^0, \Pi, \Pi^\vee)$ is a realisation of A, U a subrealisation and U^0 a *null subspace* that is a subspace of V contained in $\Pi^\perp$.

Lemma. *If $(V, \Pi, \Pi^\vee)$ is a realisation of A, there is a subrepresentation $U \subseteq V$ of minimal dimension $2|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$ and a null subspace $U^0 \subseteq V$ such that V is equal to the realisation $U \oplus U^0$.*

Proof. Note first that $U \subseteq V$ is a subrepresentation iff $V' \subseteq U$, and $U^\perp \cap \langle\Pi\rangle = 0$ or equivalently $U + \Pi^\perp = V$. Let now $q : V \to V/V'$ be the quotient map. Since $\Pi^\perp \cap V'$ is of dimension $|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$, $q(\Pi^\perp) = \Pi^\perp/\Pi^\perp \cap V'$ is of dimension $\dim V - (2|\mathbf{I}| - \operatorname{rank}(\mathsf{A}))$. Thus, if $\overline{U} \subset V/V'$ is a complementary subspace to $q(\Pi^\perp)$, then $U = q^{-1}(\overline{U})$ is a subrepresentation of V of dimension $2|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$. Note also that $U \cap \Pi^\perp = V' \cap \Pi^\perp$ since the right–hand side is contained in the left–hand side and their dimensions agree. Let now $U^0 \subset V$ be a complementary subspace to $V' \cap \Pi^\perp$ in $\Pi^\perp$. Then U^0 is a null subspace of V such that $U \oplus U^0 = V$. □

11.3. **Morphisms of realisations.** A *morphism* $(V_1, \Pi_1, \Pi_1^\vee) \to (V_2, \Pi_2, \Pi_2^\vee)$ of realisations is a linear map $T : V_1 \to V_2$ such that $T\alpha^\vee_{1,i} = \alpha^\vee_{2,i}$ and $T^t\alpha_{2,i} = \alpha_{1,i}$ for any $i \in \mathbf{I}$. We denote the set of such morphisms by $\operatorname{Hom}_{\mathsf{A}}(V_1, V_2)$.

Proposition.

(1) Let $T \in \operatorname{Hom}_{\mathsf{A}}(V_1, V_2)$ be a morphism of realisations.
   (a) If V_1 is of minimal dimension, T is injective.
   (b) If V_2 is of minimal dimension, T is surjective.

(2) Given two realisations $\{(V_i, \Pi_i, \Pi_i^\vee)\}_{i=1,2}$ of A, the set $\operatorname{Hom}_{\mathsf{A}}(V_1, V_2)$ is non–empty. Moreover, the map

$$\operatorname{Hom}_{\mathsf{k}}(V_1/V_1', \Pi_2^\perp) \times \operatorname{Hom}_{\mathsf{A}}(V_1, V_2) \to \operatorname{Hom}_{\mathsf{A}}(V_1, V_2)$$

defined by $(\delta, T) \to T + \delta$ gives $\operatorname{Hom}_{\mathsf{A}}(V_1, V_2)$ the structure of a torsor over the abelian group $\operatorname{Hom}_{\mathsf{k}}(V_1/V_1', \Pi_2^\perp)$.

(3) There is, up to (non–unique) isomorphism, a unique realisation of A of minimal dimension $2|\mathbf{I}| - \operatorname{rank}(\mathsf{A})$.

Proof. (1a) Since $\alpha_{2,i}\circ T=\alpha_{i,1}$ for any $i\in\mathbf{I}$, $\mathrm{Ker}(T)\subset\Pi_1^\perp\subseteq V_1'$, where the last inclusion holds by (2) of Lemma 11.1. Since the restriction of T to V_1' is injective, it follows that so is T. (1b) follows from (1a) since $T^t:(V_2^*,\Pi_2^\vee,\Pi_2)\to(V_1^*,\Pi_1^\vee,\Pi_1)$ is a morphism of realisations of A^t.

(2) The second part of the claim is clear, once the non–emptyness of $\mathrm{Hom}_{\mathsf{A}}(V_1,V_2)$ is proved. A linear map $T\in\mathrm{Hom}_{\mathsf{k}}(V_1,V_2)$ satisfies $T\alpha_{i,1}^\vee=\alpha_{i,2}^\vee$ for any $i\in\mathbf{I}$ iff in a(ny) decomposition $V_1=V_1'\oplus V_1''$, T has the block form $T=\begin{pmatrix}\imath & *\end{pmatrix}$, where $\imath$ is the map $V_1'=V_2'\hookrightarrow V_2$, $\alpha_{1,i}^\vee\to\alpha_{2,i}^\vee$. Similarly, given a decomposition $V_2=\Pi_2^\perp\oplus\widetilde{V}_2$, let p be the map $V_1\to\langle\Pi_1\rangle^*=\langle\Pi_2\rangle^*=V_2/\Pi_2^\perp\cong\widetilde{V}_2$ given by assigning to $v_1\in V_1$ the unique $v_2\in\widetilde{V}_2$ such that $\alpha_{2,i}(v_2)=\alpha_{1,i}(v_1)$ for any $i\in\mathbf{I}$. Then, $\alpha_{2,i}\circ T=\alpha_{i,1}$ holds for any $i\in\mathbf{I}$ iff T has the block form $T=\begin{pmatrix}*\\ p\end{pmatrix}$. Combining, we see that T is a morphism of realisations iff it has the form

$$T=\begin{pmatrix}\imath_{\Pi_2^\vee} & *\\ \imath_{\widetilde{V}_2}=p_{V_1'} & p_{V_2'}\end{pmatrix}$$

where the equality $\imath_{\widetilde{V}_2}=p_{V_1'}$ follows because $\alpha_{2,i}(\alpha_{2,j}^\vee)=a_{ij}=\alpha_{1,i}(\alpha_{1,j}^\vee)$. In particular, $\mathrm{Hom}_{\mathsf{A}}(V_1,V_2)$ is non–empty.

(3) It is easy to see that there is a realisation of A of minimal dimension. Its uniqueness then follows from (2) and (1). □

Abusing language slightly, we shall refer to a realisation of A of minimal dimension $2|\mathbf{I}|-\mathrm{rank}(\mathsf{A})$ as *the* realisation of A, and denote the underlying vector space by $\mathfrak{h}$.

11.4. **Invariant forms.** Recall that A is symmetrisable if there is an invertible diagonal matrix $\mathsf{D}=\mathrm{Diag}(d_i)_{i\in\mathbf{I}}$ such that DA is symmetric, that is such that $d_ia_{ij}=d_ja_{ji}$ for any $i,j\in\mathbf{I}$.

If A is symmetrisable, an *invariant form* on a realisation $(V,\Pi,\Pi^\vee)$ is a non–degenerate, symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V such that $\langle\alpha_i^\vee,\cdot\rangle=d_i^{-1}\alpha_i$.

Proposition. *Assume that* A *is symmetrisable. Then*

1. *If V is a realisation of minimal dimension, then any symmetric bilinear form on V such that $\langle\alpha_i^\vee,-\rangle=d_i^{-1}\alpha_i$ is non–degenerate, and therefore an invariant form.*
2. *Any realisation $(V,\Pi,\Pi^\vee)$ of A possesses an invariant form.*

Proof. (1) If $v\in V$ is such that $\langle v,\cdot\rangle=0$, then $v\in\Pi^\perp\subset V'$, where the last inclusion follows by part (2) of Lemma 11.3. The result then follows from the fact the map $\nu:V'\to V^*$ given by $\alpha_i^\vee\to d_i^{-1}\alpha_i=\langle\alpha_i^\vee,\cdot\rangle$ is an injection.

(2) By Lemma 11.2, there is a subrepresentation $U\subseteq V$ of minimal dimension, and a null subspace $U^0\subset V$ such that $V=U\oplus U^0$. By (1), U admits an invariant form $\langle\cdot,\cdot\rangle$. If $\langle\cdot,\cdot\rangle^0$ is a non–degenerate symmetric bilinear form on U^0, $\langle\cdot,\cdot\rangle\oplus\langle\cdot,\cdot\rangle^0$ is an invariant form on V. □

11.5. **Kac–Moody algebras.** Let $(V,\Pi,\Pi^\vee)$ be a realisation of A, and $\widetilde{\mathfrak{g}}=\widetilde{\mathfrak{g}}(V)$ the Lie algebra generated by V and elements $\{e_i,f_i\}_{i\in\mathbf{I}}$, with relations $[h,h']=0$ for any $h,h'\in V$, and

$$[h,e_i]=\alpha_i(h)e_i\qquad[h,f_i]=-\alpha_i(h)f_i\qquad[e_i,f_j]=\delta_{ij}\alpha_i^\vee$$

The Lie algebra $\widetilde{\mathfrak{g}}$ is graded by the root lattice $\mathsf{Q} = \bigoplus_i \mathbb{Z}\alpha_i \subset V^*$, that is $\widetilde{\mathfrak{g}} = \bigoplus_{\alpha\in\mathsf{Q}} \widetilde{\mathfrak{g}}_\alpha$, where $\widetilde{\mathfrak{g}}_\alpha = \{X \in \widetilde{\mathfrak{g}} | [h, X] = \alpha(h)X,\ h \in V\}$ is finite–dimensional. In fact, if $\mathsf{Q}_+ = \bigoplus_{i\in\mathbf{I}} \mathbb{Z}_{\geqslant 0}\alpha_i$, then $\widetilde{\mathfrak{g}}$ has the triangular decomposition

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{n}}_+$$

where $\widetilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha\in\mathsf{Q}_+\setminus\{0\}} \widetilde{\mathfrak{g}}_{\pm\alpha}$, and $\widetilde{\mathfrak{g}}_0 = V$.

The Kac–Moody algebra corresponding to $(V, \Pi, \Pi^\vee)$ is the quotient $\mathfrak{g} = \mathfrak{g}(V) = \widetilde{\mathfrak{g}}/\widetilde{I}$, where $\widetilde{I}$ is the sum of all (graded) ideals in $\widetilde{\mathfrak{g}}$ having trivial intersection with $\widetilde{\mathfrak{g}}_0$. $\mathfrak{g}$ inherits the Q–grading and triangular decomposition of $\widetilde{\mathfrak{g}}$, and $\mathfrak{g}_0 = V$.[34]

Lemma. *Let $T \in \mathrm{Hom}_{\mathsf{A}}(V_1, V_2)$ be a morphism of realisations of A. Then*

(1) The assignments $v_1 \to T(v_1)$, $e_i \to e_i$, $f_i \to f_i$ extend uniquely to a Lie algebra homomorphism $\mathfrak{g}(T) : \mathfrak{g}(V_1) \to \mathfrak{g}(V_2)$.

(2) $\mathfrak{g}(T)$ is homogeneous with respect to the Q–grading. Its restriction to

$$V_1 = \mathfrak{g}(V_1)_0 \to \mathfrak{g}(V_2)_0 = V_2$$

is equal to T, and its restriction to $\mathfrak{g}(V_1)_\alpha \to \mathfrak{g}(V_2)_\alpha$ is an isomorphism for any $\alpha \in \mathsf{Q} \setminus \{0\}$.

(3) If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are morphisms of realisations, then

$$\mathfrak{g}(T_2 \circ T_1) = \mathfrak{g}(T_2) \circ \mathfrak{g}(T_1) \qquad \textit{and} \qquad \mathfrak{g}(\mathrm{id}_{V_1}) = \mathrm{id}_{\mathfrak{g}(V_1)}$$

PROOF. (1) The given assignments clearly uniquely determine a Lie algebra homomorphism $\widetilde{\mathfrak{g}}(T) : \widetilde{\mathfrak{g}}(V_1) \to \widetilde{\mathfrak{g}}(V_2)$. If $I_1 \subset \widetilde{\mathfrak{g}}_1$ is an ideal, then $\widetilde{\mathfrak{g}}(T)(I_1)$ is stable under the adjoint action of V_2 since the latter factors through $V_2/\Pi_2^\perp \cong \langle\Pi_1\rangle^* = \langle\Pi_2\rangle^* \cong V_1/\Pi_1^\perp$. Since $\widetilde{\mathfrak{g}}(T)(I_1)$ is also stable under the adjoint action of $e_i = \widetilde{\mathfrak{g}}(T)(e_i)$ and $f_i = \widetilde{\mathfrak{g}}(T)(f_i)$, it is an ideal in $\widetilde{\mathfrak{g}}_2$ and $\widetilde{\mathfrak{g}}(T)$ descends to $\widetilde{\mathfrak{g}}(V_1)/\widetilde{I}_1 \to \widetilde{\mathfrak{g}}(V_2)/\widetilde{I}_2$.

(2) The homogeneity of $\mathfrak{g}(T)$ is clear, as is the fact that the restriction of $\mathfrak{g}(T)$ to $V_1 \to V_2$ is equal to T. $\mathfrak{g}(T)$ is surjective in degrees $\alpha \neq 0$ since $\widetilde{\mathfrak{g}}(T)$ is. If $K \subset \mathfrak{g}(T_1)$ is the kernel of $\mathfrak{g}(T)$, then $K = \bigoplus_{\alpha\in\mathsf{Q}} K_\alpha$, where $K_\alpha = K \cap \widetilde{\mathfrak{g}}(V_1)_\alpha$. It is easy to check that $K^\times = \bigoplus_{\alpha\in\mathsf{Q}\setminus 0} K_\alpha$ is an ideal in $\widetilde{\mathfrak{g}}(V_1)$ with trivial intersection with V hence it is equal to zero.

(3) is clear. $\square$

Let $\mathsf{Lie}_{\mathsf{Q}}$ be the category of Q–graded Lie algebras $\mathfrak{g}$ over k which are generated by $\mathfrak{g}_0$ and elements $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$, $i \in \mathbf{I}$, with morphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$ which are homogeneous with respect to Q and map e_i^1, f_i^1 to e_i^2, f_i^2. By Lemma 11.5, $\mathfrak{g}(-)$ is a faithful functor from the category of realisations of A to $\mathsf{Lie}_{\mathsf{Q}}$. It is easy to see that $\mathfrak{g}(-)$ is also full.

11.6. **The derived subalgebra $\mathfrak{g}(V)'$.** Lemma 11.5 implies in particular that the derived subalgebras $\mathfrak{g}(V_1)'$ and $\mathfrak{g}(V_2)'$ corresponding to any two realisations of A are canonically isomorphic. Indeed, as vector spaces, each $\mathfrak{g}(V_i)$ is easily seen to be $\mathfrak{n}_- \oplus V_i' \oplus \mathfrak{n}_+$, and any morphism $T \in \mathrm{Hom}_{\mathsf{A}}(V_1, V_2)$ restricts to the canonical identification $V_1' = V_2'$.

[34] If A is a symmetrisable generalised Cartan matrix (*i.e.*, $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leqslant 0}$, $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$), the ideal $\widetilde{I}$ is generated by the Serre relations $\mathsf{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \mathsf{ad}(f_i)^{1-a_{ij}}(f_j)$ for any $i \neq j$ [20]. Note that our terminology differs slightly from the one given in [24] where $\mathfrak{g}(\mathsf{A})$ is called a Kac–Moody algebra only if A is a generalised Cartan matrix.

Moreover, the derived subalgebra $\mathfrak{g}(V)'$ admits a presentation similar to that of $\mathfrak{g}(V)$. Namely, let $\widetilde{\mathfrak{g}}'$ the Lie algebra generated by elements $\{e_i, f_i, \alpha_i^\vee\}$ with relations

$$[\alpha_j^\vee, e_i] = a_{ji} e_i \qquad [\alpha_j^\vee, f_i] = -a_{ji} f_i \qquad [e_i, f_j] = \delta_{ij} \alpha_i^\vee$$

$\widetilde{\mathfrak{g}}'$ is graded by Q, with $\widetilde{\mathfrak{g}}'_0 = \mathfrak{h}'$, where the latter is the $|\mathbf{I}|$–dimensional span of $\{\alpha_i^\vee\}_{i,\in\mathbf{I}}$. The quotient of $\widetilde{\mathfrak{g}}'$ by the sum $\widetilde{I}'$ of its graded ideals with trivial intersection with $\widetilde{\mathfrak{g}}'_0$ is easily seen to be canonically isomorphic to $\mathfrak{g}(V)'$.

11.7. **Symmetrisable Kac–Moody algebras.** Assume that A is symmetrisable, and fix an invertible diagonal matrix $\mathsf{D} = \mathrm{Diag}(d_i)$ such that DA is symmetric. Let $(V, \Pi, \Pi^\vee)$ be a realisation of A endowed with an invariant form $\langle\cdot,\cdot\rangle$. Then, $\langle\cdot,\cdot\rangle$ uniquely extends to a symmetric, invariant, non–degenerate bilinear form on $\mathfrak{g} = \mathfrak{g}(V)$, which satisfies $\langle e_i, f_j\rangle = \delta_{ij} d_i^{-1}$ [24, Thm. 2.2].

Recall that $\mathfrak{g}$ has a standard $\mathbb{Z}$–grading with finite–dimensional homogeneous components, given by $\deg(f_i) = 1 = -\deg(e_i)$ and $\deg(V) = 0$. Set $\mathfrak{b}_\pm = V \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{\pm\alpha} \subset \mathfrak{g}$. Then, $\mathfrak{b}_\pm$ are $\mp\mathbb{N}$–graded Lie algebras with finite–dimensional components. Moreover, the bilinear form induces a canonical isomorphisms $\mathfrak{b}_\pm^\star \simeq \mathfrak{b}_\mp$, where $\mathfrak{b}_\pm^\star$ is the restricted dual of $\mathfrak{b}_\pm$, and is equal to

$$\mathfrak{b}_\pm^\star \coloneqq V^* \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{\pm\alpha}^*$$

These identifications allows to determine on $\mathfrak{b}_\pm$, and therefore on $\mathfrak{g}$, a natural structure of Lie bialgebra compatible with the grading.

More precisely, consider the Lie algebra $\mathfrak{g}^{(2)} = \mathfrak{g} \oplus V$, and endow it with the inner product $\langle\cdot,\cdot\rangle \oplus -\langle\cdot,\cdot\rangle|_{V\times V}$. Let $\pi_0 : \mathfrak{g} \to \mathfrak{g}_0 = V$ be the projection, and $\mathfrak{b}_\pm^{(2)} \subset \mathfrak{g}^{(2)}$ the subalgebra

$$\mathfrak{b}_\pm^{(2)} = \{(X, v) \in \mathfrak{b}_\pm \oplus V \,|\, \pi(X) = \pm v\}$$

Note that the projection $\mathfrak{g}^{(2)} \to \mathfrak{g}$ onto the first component restricts to an isomorphism $\mathfrak{b}_\pm^{(2)} \to \mathfrak{b}_\pm$ with inverse $\mathfrak{b}_\pm \ni X \to (X, \pm\pi_0(X)) \in \mathfrak{b}_\pm^{(2)}$.

Then, the following is easily seen to hold (cf. [11, Ex. 3.2], [17, Prop. 2.1]).

Proposition.

(1) $(\mathfrak{g}^{(2)}, \mathfrak{b}_-^{(2)}, \mathfrak{b}_+^{(2)})$ *is a restricted Manin triple. In particular,* $\mathfrak{b}_\mp^{(2)}$ *and* $\mathfrak{g}^{(2)}$ *are Lie bialgebras, with cobracket* $\delta_{\mathfrak{b}_\mp^{(2)}} = [\cdot,\cdot]^t_{\mathfrak{b}_\pm^{(2)}}$ *and* $\delta_{\mathfrak{g}^{(2)}} = \delta_{\mathfrak{b}_-^{(2)}} - \delta_{\mathfrak{b}_+^{(2)}}$.

(2) The central subalgebra $0 \oplus V \subset \mathfrak{g}^{(2)}$ *is a coideal, so that the projection* $\mathfrak{g}^{(2)} \to \mathfrak{g}$ *induces a Lie bialgebra structure on* $\mathfrak{g}$ *and* $\mathfrak{b}_\mp$.

(3) The Lie bialgebra structure on $\mathfrak{g}$ *is given by*

$$\delta|_V = 0 \qquad \delta(e_i) = d_i \alpha_i^\vee \wedge e_i \qquad \delta(f_i) = d_i \alpha_i^\vee \wedge f_i$$

12. Diagrammatic Kac–Moody algebras

As pointed out in 5.11, a complex semisimple Lie algebra $\mathfrak{g}$ and its positive Borel subalgebra are diagrammatic Lie bialgebras with respect to the Dynkin diagram of $\mathfrak{g}$. The extension of this result to an arbitrary Kac–Moody algebra requires the introduction of *extended* Kac–Moody algebras which correspond to non–minimal realisations of the underlying matrices. These realisations are defined in this section, together with a natural braided Coxeter structure on integrable Drinfeld–Yetter modules over the corresponding Borel subalgebras.

12.1. Fix an $|\mathbf{I}|\times|\mathbf{I}|$ matrix A with entries in k, and let D be the diagram having $\mathbf{I}$ as its vertex set and an edge between $i\neq j$ unless $a_{ij}=a_{ji}=0$. For any $B\subseteq D$, let A_B be the $|B|\times|B|$ matrix $(a_{ij})_{i,j\in B}$, $\mathfrak{g}(\mathsf{A}_B)$ the Kac–Moody algebra corresponding to its minimal realisation, and $\mathfrak{h}(\mathsf{A}_B)$ its Cartan subalgebra.

As pointed out in 11.6, the derived subalgebra $\mathfrak{g}(\mathsf{A})$ is generated by $\{e_i,f_i,\alpha_i^\vee\}_{i\in D}$. It possesses a diagrammatic structure over D which is given by associating to any subdiagram $B\subseteq D$ the derived algebra $\mathfrak{g}(\mathsf{A}_B)'$, and to each inclusion $B'\subseteq B$ the morphism $\imath'_{BB'}:\mathfrak{g}(\mathsf{A}_{B'})'\to\mathfrak{g}(\mathsf{A}_B)'$ mapping $e_i^{B'},f_i^{B'},{\alpha_i^\vee}^{B'}$ to $e_i^B,f_i^B,{\alpha_i^\vee}^B$, $i\in B'$. This is a diagrammatic structure since, if $i\perp j$, e_i (resp. f_i) commutes with e_j (resp. f_j) [24, Lemma 1.6].

We say that $\mathfrak{g}(\mathsf{A})$ is *Cartan diagrammatic* if it is endowed with a diagrammatic structure such that $\mathfrak{g}_B=\mathfrak{g}(\mathsf{A}_B)$ for any $B\subseteq D$, and the following diagram commutes for any $B'\subseteq B$

$$\begin{tikzcd} \mathfrak{g}(\mathsf{A}_{B'}) \arrow[r, "\imath_{BB'}"] & \mathfrak{g}(\mathsf{A}_B) \\ \mathfrak{g}(\mathsf{A}_{B'})' \arrow[u] \arrow[r, "\imath'_{BB'}"'] & \mathfrak{g}(\mathsf{A}_B)' \arrow[u] \end{tikzcd}$$

where the vertial arrows are the natural inclusions.

For any $B\subseteq D$, set $\Pi_B=\{\alpha_i\}_{i\in B}$, $\Pi_B^\vee=\{\alpha_i^\vee\}_{i\in B}$, and let $\langle\Pi_B\rangle\subset\mathfrak{h}(\mathsf{A})^*$ and $\mathfrak{h}'_B=\langle\Pi_B^\vee\rangle\subset\mathfrak{h}(\mathsf{A})$ the subspaces they span respectively.

Proposition.

(1) If $\mathfrak{g}(\mathsf{A})$ is Cartan diagrammatic, each morphism $\imath_{BB'}:\mathfrak{g}(\mathsf{A}_{B'})\to\mathfrak{g}(\mathsf{A}_B)$, $B'\subseteq B$, is an embedding.

(2) $\mathfrak{g}(\mathsf{A})$ is Cartan diagrammatic iff, for any $B\subseteq D$, there is a subspace $\mathfrak{h}_B\subseteq\mathfrak{h}(\mathsf{A})$ such that $\left(\mathfrak{h}_B,\Pi_B|_{\mathfrak{h}_B},\Pi_B^\vee\right)$ is a minimal realisation of A_B, that is

(a) $\mathfrak{h}'_B\subseteq\mathfrak{h}_B$

(b) $\langle\Pi_B\rangle\cap\mathfrak{h}_B^\perp=0$

(c) $\dim\mathfrak{h}_B=2|B|-\mathrm{rank}(\mathsf{A}_B)$

and, for any $B,B'\subseteq D$

(d) $\mathfrak{h}_{B'}\subseteq\mathfrak{h}_B$ if $B'\subseteq B$

(e) $\mathfrak{h}_B\subseteq\Pi_{B'}^\perp$ and $\mathfrak{h}_{B'}\subseteq\Pi_B^\perp$ if $B\perp B'$

Proof. (1) It suffices to show that the restriction $\imath^{\mathfrak{h}}_{BB'}$ of $\imath_{BB'}$ to a map $\mathfrak{h}(\mathsf{A}_{B'})\to\mathfrak{h}(\mathsf{A}_B)$ is injective for any $B'\subseteq B$. Applying $\imath_{BB'}$ to the relation $[h,e_i^{B'}]=\alpha_i^{B'}(h)e_i^{B'}$ shows that $\alpha_i^B\circ\imath^{\mathfrak{h}}_{BB'}=\alpha_i^{B'}$ for any $i\in B'$. It follows that $\mathrm{Ker}\,\imath^{\mathfrak{h}}_{BB'}$ is contained in $\Pi_{B'}^\perp\subseteq\mathfrak{h}(\mathsf{A}_{B'})'$, where the inclusion holds by Lemma 11.1. Since the restriction of $\imath^{\mathfrak{h}}_{BB'}$ to $\mathfrak{h}(\mathsf{A}_{B'})'$ is injective by assumption, the conclusion follows.

(2) Assume that $\mathfrak{g}(\mathsf{A})$ is diagrammatic, and set $\mathfrak{h}_B=\imath_{DB}(\mathfrak{h}(\mathsf{A}_B))$. Since $\imath_{DB}(\alpha_i^\vee)=\alpha_i^\vee$ and $\alpha_i^D\circ\imath_{DB}|_{\mathfrak{h}_B}=\alpha_i^B$ for any $i\in B$, $\mathfrak{h}_B$ contains $\mathfrak{h}'_B$ and the restrictions of the linear forms α_i^D to $\mathfrak{h}_B$ are linearly independent. Moreover, $\mathfrak{h}_B$ has the claimed dimension since $\imath_{DB}$ is injective by (1). The remaining properties are clear.

Conversely, assume given subspaces $\mathfrak{h}_B$ satisfying the above properties. For any B, the triple $(\mathfrak{h}_B,\Pi_B|_{\mathfrak{h}_B},\Pi_B^\vee)$ is a minimal realisation of A_B, which determines a morphism of realisations $\imath^{\mathfrak{h}}_{DB} : \mathfrak{h}(\mathsf{A}_B) \to \mathfrak{h}$ with image $\mathfrak{h}_B$. Since, for any $B' \subseteq B$ the image of $\imath^{\mathfrak{h}}_{DB'}$ is contained in the image of $\imath^{\mathfrak{h}}_{DB}$, there is a uniquely defined morphism of realisations of $\mathsf{A}_{B'}$ such that $\imath^{\mathfrak{h}}_{BB'} : \mathfrak{h}_{B'} \to \mathfrak{h}_B$ such that $\imath^{\mathfrak{h}}_{DB} \circ \imath^{\mathfrak{h}}_{BB'} = \imath^{\mathfrak{h}}_{DB'}$. Let now $B'' \subseteq B' \subseteq B$. We wish to show that $\imath^{\mathfrak{h}}_{BB'} \circ \imath^{\mathfrak{h}}_{B'B''} = \imath^{\mathfrak{h}}_{BB''}$. It suffices to show that this holds after composition with $\imath^{\mathfrak{h}}_{DB}$ since the latter is injective. However,

$$\imath^{\mathfrak{h}}_{DB} \circ \imath^{\mathfrak{h}}_{BB'} \circ \imath^{\mathfrak{h}}_{B'B''} = \imath^{\mathfrak{h}}_{DB'} \circ \imath^{\mathfrak{h}}_{B'B''} = \imath^{\mathfrak{h}}_{DB''} = \imath^{\mathfrak{h}}_{DB} \circ \imath^{\mathfrak{h}}_{BB''}$$

The morphisms of realisations $\imath^{\mathfrak{h}}_{BB'}$ canonically induce Lie algebra homomorphisms $\imath_{BB'} : \mathfrak{g}(\mathsf{A}_{B'}) \to \mathfrak{g}(\mathsf{A}_B)$ which give rise to a Cartan diagrammatic structure on $\mathfrak{g}(\mathsf{A}_B)$. □

In 12.2–12.3 we give sufficient conditions for $\mathfrak{g}(\mathsf{A})$ to be Cartan diagrammatic, together with a counterexample which show that $\mathfrak{g}(\mathsf{A})$ is not Cartan diagrammatic in general.

12.2.

Lemma. *If* $\det(\mathsf{A}_B) \neq 0$ *for any* $B \subset D$ *with* $|D \setminus B| \geqslant 2$*, then* $\mathfrak{g}(\mathsf{A})$ *is Cartan diagrammatic.*

Proof. We rely on part (2) of Proposition 12.1. For any B such that $|D \setminus B| \geqslant 2$, set $\mathfrak{h}_B = \mathfrak{h}'_B$. If $|D \setminus B| = 1$, Lemma 11.2 implies that $\mathfrak{h}(\mathsf{A})$ contains a subspace $\mathfrak{h}_B$ such that $(\mathfrak{h}_B, \Pi_B|_{\mathfrak{h}_B}, \Pi_B^\vee)$ is a minimal realisation of A_B. If B is perpendicular to the single vertex i in $D \setminus B$, we require additionally that $\mathfrak{h}_B$ be chosen in $\mathrm{Ker}(\alpha_i)$. Finally, if $B = D$, set $\mathfrak{h}_B = \mathfrak{h}(\mathsf{A})$. It is easy to see that the subspaces $\mathfrak{h}_B$ satisfy the conditions of Proposition 12.1 except possibly the orthogonality condition (d) when B is such that $|D \setminus B| = 1$. If i is the single vertex in $D \setminus B$ and $a_{ii} \neq 0$, then (d) holds with $B' = i$ by construction. If $a_{ii} = 0$ then, by assumption, A must be the diagonal matrix $\mathrm{Diag}(*, 0)$, and $\mathfrak{g}(\mathsf{A})$ is readily seen to be diagrammatic in this case. □

Remark. The converse of Lemma 12.2 does not hold. Indeed, let A be the zero matrix, which for $n \geqslant 3$ does not satisfy the above condition. Its minimal realisation can be taken to be the $2|\mathbf{I}|$–dimensional vector space $\mathfrak{h}$ with basis $\{\alpha_i^\vee\}_{i\in\mathbf{I}} \cup \{\partial_i\}_{i\in\mathbf{I}}$, and $\{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$ the last $|\mathbf{I}|$ elements of the corresponding dual basis, so that $\alpha_i(\alpha_j^\vee) = 0$ and $\alpha_i(\partial_j) = \delta_{ij}$ for any $i, j \in \mathbf{I}$. The corresponding Kac–Moody algebra $\mathfrak{g}(\mathsf{A})$ is Cartan diagrammatic, with $\mathfrak{g}_B$ the Lie subalgebra of $\mathfrak{g}(\mathsf{A})$ generated by $\{e_i, f_i, \alpha_i^\vee, \partial_i\}_{i\in B}$, $B \subseteq D$.

12.3. Assume in this paragraph that $\mathsf{k} = \mathbb{Q}$, and that A is such that $a_{ij} \leqslant 0$ for $i \neq j$ and that $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. Recall that if A is indecomposable, it is called *finite* if $\mathrm{rank}(\mathsf{A}) = |\mathbf{I}|$, *affine* if $\mathrm{rank}(\mathsf{A}) = |\mathbf{I}| - 1$, and *indefinite* otherwise. A is *hyperbolic* if it is indefinite, but the irreducible components of any A_B, with $B \subsetneq D$, are all of finite or affine type. In A is finite or affine, then any submatrix A_B, with $B \subsetneq D$ decomposes into a direct sum of matrices of finite type [24, Chap. 4].

If A is a direct sum of indecomposable matrices $\mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_m$. Then $\mathfrak{g}(\mathsf{A}) \cong \mathfrak{g}(\mathsf{A}_1) \oplus \cdots \oplus \mathfrak{g}(\mathsf{A}_m)$ is Cartan diagrammatic iff each $\mathfrak{g}(\mathsf{A}_i)$ is.

Proposition. *Assume that* A *is indecomposable. Then*

(1) $\mathfrak{g}(\mathsf{A})$ *is Cartan diagrammatic if* A *is of finite, affine or hyperbolic type.*
(2) $\mathfrak{g}(\mathsf{A})$ *is not Cartan diagrammatic in general.*

Proof. (1) is an immediate consequence of Lemma 12.2. To prove (2), we consider the following example. Let A be the generalised Cartan matrix

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Note that A_B is of full rank if $|B| = 3$, so that $\mathfrak{h}_B = \mathfrak{h}'_B$ for any such B. Then, $\dim \mathfrak{h}_{23} = 3$, while $\mathfrak{h}_{123} \cap \mathfrak{h}_{234} = \mathfrak{h}'_{123} \cap \mathfrak{h}'_{234} = \mathfrak{h}'_{23}$ is of dimension 2. Therefore the condition $\mathfrak{h}_{23} \subseteq \mathfrak{h}_{123} \cap \mathfrak{h}_{234}$ cannot be satisfied. □

12.4. **The canonical realisation.** To remedy the fact that $\mathfrak{g}(\mathsf{A})$ is not diagrammatic in general, we follow a suggestion of P. Etingof, and give in 12.5 a modified definition of $\mathfrak{g}(\mathsf{A})$ along the lines of [18]. The corresponding Cartan subalgebra is given by the following (non–minimal) realisation of A.

Let $(\overline{\mathfrak{h}}, \overline{\Pi}, \overline{\Pi}^\vee)$ be the realisation given by $\overline{\mathfrak{h}} \cong \mathsf{k}^{2|\mathbf{I}|}$ with basis $\{\alpha_i^\vee\}_{i\in\mathbf{I}} \cup \{\lambda_i^\vee\}_{i\in\mathbf{I}}$, $\overline{\Pi}^\vee = \{\alpha_i^\vee\}_{i\in\mathbf{I}}$ and $\overline{\Pi} = \{\alpha_i\}_{i\in\mathbf{I}}$, where α_i is given by

$$\alpha_i(\alpha_j^\vee) = a_{ji} \qquad \text{and} \qquad \alpha_i(\lambda_j^\vee) = \delta_{ij}$$

We refer to $(\overline{\mathfrak{h}}, \overline{\Pi}, \overline{\Pi}^\vee)$ as the *canonical* realisation of A, and denote by $\Lambda^\vee \subset \overline{\mathfrak{h}}$ the $|\mathbf{I}|$–dimensional subspace spanned by $\{\lambda_i^\vee\}_{i\in\mathbf{I}}$.

Proposition. *Let $(V, \Pi, \Pi^\vee)$ be a realisation of A.*

(1) If $p \in \mathrm{Hom}_{\mathsf{A}}(\overline{\mathfrak{h}}, V)$, then $p(\Lambda^\vee) \subset V$ is a complementary subspace to $\Pi^\perp$. Moreover, the map

$$\mathrm{Hom}_{\mathsf{A}}(\overline{\mathfrak{h}}, V) \to \{\widetilde{V} \subseteq V \,|\, \Pi^\perp \oplus \widetilde{V} = V\}, \qquad p \to p(\Lambda^\vee)$$

is a bijection.

(2) If $\imath \in \mathrm{Hom}_{\mathsf{A}}(V, \overline{\mathfrak{h}})$, then $\imath^{-1}(\Lambda^\vee) \subset V$ is a complementary subspace to V'. Moreover, the map

$$\mathrm{Hom}_{\mathsf{A}}(V, \overline{\mathfrak{h}}) \to \{V'' \subseteq V \,|\, V' \oplus V'' = V\}, \qquad \imath \to \imath^{-1}(\Lambda^\vee)$$

is a bijection.

(3) If $\imath \in \mathrm{Hom}_{\mathsf{A}}(V, \overline{\mathfrak{h}})$ and $p \in \mathrm{Hom}_{\mathsf{A}}(\overline{\mathfrak{h}}, V)$ correspond to the subspaces $V'', \widetilde{V} \subset V$ respectively, then $p \circ \imath = \mathrm{id}_V$ if, and only if, $V'' \subset \widetilde{V}$.

Proof. (1) Since p is a morphism, $\mathrm{Ker}(p) \subset p^{-1}(\Pi^\perp) \subseteq \overline{\Pi}^\perp$. It follows in particular that $p(\Lambda^\vee) \subset V$ is an $|\mathbf{I}|$–dimensional subspace with trivial intersection with $\Pi^\perp$ since $\Lambda^\vee \cap \overline{\Pi}^\perp = 0$. Let now $\widetilde{V} \subset V$ be a complementary subspace to $\Pi^\perp$. Then, $\widetilde{V} \cong \Pi^* = \overline{\Pi}^* \cong \Lambda^\vee$ so there is a unique map $q : \Lambda^\vee \to \widetilde{V}$ such that $\alpha_i \circ q = \alpha_i$, and therefore a unique morphism of realisations $p = \mathrm{id}_{\overline{\mathfrak{h}}'} \oplus q : \overline{\mathfrak{h}} \to V$ such that $p(\Lambda^\vee) = \widetilde{V}$.

(2) $\imath^{-1}(\Lambda^\vee)$ has trivial intersection with V' since $\imath(V') \subseteq \overline{\mathfrak{h}}'$. Moreover, $V = V' + \imath^{-1}(V)$. Indeed, let $\imath', \imath''$ be the components of $\imath$ corresponding to the decomposition $\overline{\mathfrak{h}} = \overline{\mathfrak{h}}' \oplus \Lambda^\vee$. Then, for any $v \in V$,

$$\imath(v) = \imath'(v) + \imath''(v) = \imath(\imath|_{V'}^{-1} \circ \imath'(v)) + \imath''(v)$$

so that $v - \imath|_{V'}^{-1} \circ \imath'(v) \in \imath^{-1}(\Lambda^\vee)$. Finally, note that the restriction of $\imath$ to $\imath^{-1}(\Lambda^\vee)$ is necessarily given by $\imath(v) = \sum_i \alpha_i \circ \imath(v)\lambda_i^\vee = \sum_i \alpha_i(v)\lambda_i^\vee$, so that $\imath$ is uniquely

determined by the subspace $\imath^{-1}(\Lambda^\vee)$. Conversely, given a decomposition $V = V' \oplus V''$, then $\imath = \imath' \oplus \imath'' : V \to \overline{\mathfrak{h}}$, where $\imath'$ is the canonical identification $V' \to \overline{\mathfrak{h}}'$, and $\imath'' : V'' \to \Lambda^\vee$ is given by $v \to \sum_i \alpha_i(v)\lambda_i^\vee$ is easily seen to be the unique morphism of realisations such that $V'' = \imath^{-1}(\Lambda^\vee)$.
(3) If $p \circ \imath = \mathrm{id}_V$, then $V'' = p \circ \imath(V'') \subseteq p(\Lambda^\vee) = \widetilde{V}$ since $V'' = \imath^{-1}(\Lambda^\vee)$. To prove the converse, it suffices to show that the restriction of $p \circ \imath$ to V'' is the identity. This follows from the fact that a) for any $v'' \in V''$, $\imath(v'')$ is the unique $\lambda^\vee \in \Lambda^\vee$ such that $\alpha_i(v'') = \alpha_i(\lambda^\vee)$ for any $i \in \mathbf{I}$, b) for any $\lambda^\vee \in \Lambda^\vee$, $p(\lambda)$ is the unique element $\widetilde{v} \in \widetilde{V}$ such that $\alpha_i(\lambda^\vee) = \alpha_i(\widetilde{v})$ for any $i \in \mathbf{I}$ and c) $V'' \subseteq \widetilde{V}$. □

12.5. **Extended Kac–Moody algebras.** We denote by $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(\mathsf{A})$ the *extended* Kac–Moody algebra corresponding to A, that is the Lie algebra associated to the canonical realisation of A. In particular, $\overline{\mathfrak{g}}$ is generated by $\{e_i, f_i, \alpha_i^\vee, \lambda_i^\vee\}_{i\in\mathbf{I}}$, with relations $[\alpha_i^\vee, \alpha_j^\vee] = 0$, $[\lambda_i^\vee, \lambda_j^\vee] = 0$, $[\alpha_i^\vee, \lambda_j^\vee] = 0$,

$$[\alpha_i^\vee, e_j] = a_{ji}e_j, \qquad [\alpha_{ji}^\vee, f_j] = -a_{ji}f_j, \qquad [\lambda_i^\vee, e_j] = \delta_{ij}e_j, \qquad [\lambda_i^\vee, f_j] = -\delta_{ij}f_j,$$

and $[e_i, f_j] = \delta_{ij}h_i$, for any $i, j \in \mathbf{I}$. Unlike $\mathfrak{g}(\mathsf{A})$, $\overline{\mathfrak{g}}(\mathsf{A})$ always possesses a diagrammatic structure over the Dynkin diagram D of A.

Proposition. *The extended Kac–Moody algebra $\overline{\mathfrak{g}}$ is a diagrammatic Lie algebra, with diagrammatic Lie subalgebras $\overline{\mathfrak{g}}_B \coloneqq \langle e_i, f_i, \alpha_i^\vee, \lambda_i^\vee \mid i \in B\rangle = \overline{\mathfrak{g}}(\mathsf{A}_B)$, $B \subseteq D$.*

12.6. **Relation between $\mathfrak{g}$ and $\overline{\mathfrak{g}}$.** The following shows that $\overline{\mathfrak{g}}$ is non–canonically a split central extension of $\mathfrak{g}$, with a $\mathrm{rank}(\mathsf{A})$–dimensional kernel. Let $\mathsf{Lie}_{\mathsf{Q}}$ be the category of Q–graded Lie algebras defined in 11.5.

Proposition.

(1) *Any $p \in \mathrm{Hom}_{\mathsf{Lie}_{\mathsf{Q}}}(\overline{\mathfrak{g}}, \mathfrak{g})$ is surjective, and $\mathrm{Ker}(p)$ is a $\mathrm{rank}(\mathsf{A})$–dimensional subspace of $\overline{\Pi}^\perp = \mathfrak{z}(\overline{\mathfrak{g}})$ which is complementary to $\overline{\Pi}^\perp \cap \overline{\mathfrak{h}}'$.*
(2) *There is a bijection between $\mathrm{Hom}_{\mathsf{Lie}_{\mathsf{Q}}}(\overline{\mathfrak{g}}, \mathfrak{g})$ and the set of subspaces $\widetilde{\mathfrak{h}} \subset \mathfrak{h}$ which are complementary to $\Pi^\perp$, given by mapping $p : \overline{\mathfrak{g}} \to \mathfrak{g}$ to $p(\Lambda^\vee)$.*
(3) *Any $i \in \mathrm{Hom}_{\mathsf{Lie}_{\mathsf{Q}}}(\mathfrak{g}, \overline{\mathfrak{g}})$ is injective.*
(4) *There is a bijection between $\mathrm{Hom}_{\mathsf{Lie}_{\mathsf{Q}}}(\mathfrak{g}, \overline{\mathfrak{g}})$ and the set of subspaces $\widetilde{\mathfrak{h}} \subset \mathfrak{h}$ which are complementary to $\mathfrak{h}'$, given by mapping $i : \mathfrak{g} \to \overline{\mathfrak{g}}$ to $i^{-1}(\Lambda^\vee)$.*
(5) *If $p \in \mathrm{Hom}_{\mathsf{Lie}_{\mathsf{Q}}}(\overline{\mathfrak{g}}, \mathfrak{g})$ and $i \in \mathrm{Hom}_{\mathsf{Lie}_{\mathsf{Q}}}(\mathfrak{g}, \overline{\mathfrak{g}})$ correspond to the subspaces $\widetilde{\mathfrak{h}}$ and $\mathfrak{h}'' \subset \mathfrak{h}$ respectively, then $p \circ i = \mathrm{id}_{\mathfrak{g}}$ if, and only if $\widetilde{\mathfrak{h}} \subset \mathfrak{h}''$.*

Proof. (1) By 11.5, p is of the form $\mathfrak{g}(p_0)$ for a unique $p_0 \in \mathrm{Hom}_A(\overline{\mathfrak{h}}, \mathfrak{h})$. p is surjective by part (2) of Lemma 11.5 and part (1b) of Proposition 11.3. Moreover, $\mathrm{Ker}(p) = \mathrm{Ker}(p_0)$ is a $\mathrm{rank}(\mathsf{A})$ dimensional subspace of $\overline{\Pi}^\perp$ since $\alpha_i \circ p_0 = \alpha_i$. Since p_0 is injective on $\overline{\mathfrak{h}}'$, $\mathrm{Ker}(p_0) \cap (\overline{\Pi}^\perp \cap \overline{\mathfrak{h}}') = 0$ and it follows that the two spaces are in direct sum since their dimensions add up to $|\mathbf{I}| = \dim \overline{\Pi}^\perp$.
(3) The injectivity of i follows from 11.5 and part (1a) of Proposition 11.3.
(2), (4) and (5) Follow from 11.5 and Proposition 12.4. □

12.7. **Split diagrammatic structure.** Assume henceforth that A is symmetrisable. Fix $\mathsf{D} = \mathrm{Diag}(d_i)$ such that DA is symmetric, and an invariant form $\langle\cdot,\cdot\rangle$ on $\overline{\mathfrak{h}}$. Then, by Proposition 11.7, there is a standard Lie bialgebra structure on $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(\mathsf{A})$ given by

$$\delta(\alpha_i^\vee) = 0 = \delta(\lambda_i^\vee) \qquad \delta(e_i) = d_i\alpha_i^\vee \wedge e_i \qquad \delta(f_i) = d_i\alpha_i^\vee \wedge f_i$$

It follows as in 12.5 that $\overline{\mathfrak{g}}$ is a diagrammatic Lie bialgebra with Lie subbialgebras $\overline{\mathfrak{g}}_B = \langle e_i, f_i, \alpha_i^\vee, \lambda_i^\vee \mid i \in B\rangle$, $B \subseteq D$.

As in the finite–dimensional case described in Example 5.11, the diagrammatic structure on $\overline{\mathfrak{g}}$ determines a split diagrammatic one on $\overline{\mathfrak{b}}_\pm$. For any $B \subseteq D$, let $\overline{\mathfrak{b}}_{\pm,B} = \overline{\mathfrak{b}}_\pm \cap \overline{\mathfrak{g}}_B$ be the Lie subbialgebras generated by $\{\alpha_i^\vee, \lambda_i^\vee, e_i\}_{i\in B}$ and $\{\alpha_i^\vee, \lambda_i^\vee, f_i\}_{i\in B}$ respectively. If $B' \subseteq B$, let $i_{\pm,BB'} : \overline{\mathfrak{b}}_{\pm,B'} \to \overline{\mathfrak{b}}_{\pm,B}$ be the standard embedding, and regard $p_{\pm,B'B} = i^t_{\mp,BB'}$ as a map $\overline{\mathfrak{b}}_{\pm,B} \to \overline{\mathfrak{b}}_{\pm,B'}$ via the identifications $\overline{\mathfrak{b}}^\star_{\mp,C} \cong \overline{\mathfrak{b}}_{\pm,C}$ given by the inner product, where as usual $\overline{\mathfrak{b}}^\star_\pm$ is the restricted dual of $\overline{\mathfrak{b}}_\pm$, and is equal to

$$\overline{\mathfrak{b}}^\star_\pm \coloneqq \overline{\mathfrak{h}}^* \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \overline{\mathfrak{g}}^*_{\pm\alpha}$$

Then, $\ker(p_{\pm,B'B})$ is a Lie subalgebra in $\overline{\mathfrak{b}}_{\pm,B}$, and therefore $\{p_{\pm,B'B}\}$ give the required splitting of the Lie bialgebra $\overline{\mathfrak{b}}_\pm$ (cf. 5.10). The splitting can also be explicitly described as follows. Set $\overline{\mathfrak{n}}_{B,\pm} = \bigoplus_{\alpha\in\mathsf{R}_{B,+}} \overline{\mathfrak{g}}_{\pm\alpha} \subset \overline{\mathfrak{b}}_{B,\pm}$.

Lemma. *The projection $p_{\pm,B'B} : \overline{\mathfrak{b}}_{\pm,B} \to \overline{\mathfrak{b}}_{\pm,B'}$ corresponds to the splitting*

$$\overline{\mathfrak{n}}_{B,\pm} = \overline{\mathfrak{n}}_{B',\pm} \oplus \overline{\mathfrak{n}}_{B'B,\pm} \qquad \textit{where} \qquad \overline{\mathfrak{n}}_{B'B,\pm} = \bigoplus_{\alpha\in\mathsf{R}_{B,+}\setminus\mathsf{R}_{B',+}} \overline{\mathfrak{g}}_{\pm\alpha}$$

together with the orthogonal splitting

$$\overline{\mathfrak{h}}_B = \overline{\mathfrak{h}}_{B'} \oplus \overline{\mathfrak{h}}_{B'B} \qquad \textit{where} \qquad \overline{\mathfrak{h}}_{B'B} = \bigoplus_{j\in B\setminus B'} \mathsf{k}\cdot\lambda_j^\vee \oplus \mathsf{k}\cdot\omega^\vee_{B'B,j}$$

and $\omega^\vee_{B'B,j}$ is given by $\alpha_j^\vee - \sum_{i\in B'} \alpha_i(\alpha_j^\vee)\lambda_i^\vee$. In particular, $\overline{\mathfrak{h}}_{B'B} \subset \bigcap_{i\in B'} \mathrm{Ker}(\alpha_i)$.

Proof. It is enough to observe that for any $i \in B'$ and $j \in B \setminus B'$,

$$\langle\alpha_i^\vee, \lambda_j^\vee\rangle = 0 = \langle\lambda_i^\vee, \lambda_j^\vee\rangle \qquad \text{and} \qquad \langle\alpha_i^\vee, \omega^\vee_{B'B,j}\rangle = 0 = \langle\lambda_i^\vee, \omega^\vee_{B'B,j}\rangle$$

□

12.8. **The category $\mathcal{O}_{\overline{\mathfrak{g}}}$.** A $\overline{\mathfrak{g}}$–module V is in category $\mathcal{O}_{\overline{\mathfrak{g}}}$ if the following holds.

($\mathcal{O}$1) $V = \bigoplus_{\lambda\in\overline{\mathfrak{h}}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid h\,v = \lambda(h)v,\ h \in \overline{\mathfrak{h}}\}$

($\mathcal{O}$2) $\dim V_\lambda < \infty$ for any $\lambda \in \mathsf{P}(V) = \{\lambda \in \overline{\mathfrak{h}}^* \mid V_\lambda \neq 0\}$

($\mathcal{O}$3) $\mathsf{P}(V) \subseteq D(\lambda_1) \cup \cdots \cup D(\lambda_m)$, for some $\lambda_1, \ldots, \lambda_m \in \overline{\mathfrak{h}}^*$

where $D(\lambda) = \{\mu \in \overline{\mathfrak{h}}^* \mid \mu \leqslant \lambda\}$, with $\mu \leqslant \lambda$ iff $\lambda - \mu \in \mathsf{Q}_+$. The category $\mathcal{O}_{\overline{\mathfrak{g}}}$ has a natural symmetric tensor structure inherited from $\mathrm{Rep}\,\overline{\mathfrak{g}}$.

We observed in 11.7 that the restricted Drinfeld double of the negative Borel subalgebra $\overline{\mathfrak{b}}_-$ of $\overline{\mathfrak{g}}$ is isomorphic to the trivial central extension $\overline{\mathfrak{g}}^{(2)} = \overline{\mathfrak{g}} \oplus \overline{\mathfrak{h}}^c$ of $\overline{\mathfrak{g}}$ by $\overline{\mathfrak{h}}^c = \overline{\mathfrak{h}}$. It follows by 5.5–5.6 that the category of Drinfeld–Yetter modules over

$\overline{\mathfrak{b}}_-$ is equivalent to the category $\mathcal{E}_{\overline{\mathfrak{g}}^{(2)}}$ of $\overline{\mathfrak{g}}^{(2)}$–modules, where $\overline{\mathfrak{g}}^{(2)} = \overline{\mathfrak{g}} \oplus \overline{\mathfrak{h}}^c$, which carry a locally finite action of $\overline{\mathfrak{b}}_+^{(2)} \subset \overline{\mathfrak{g}}^{(2)}$. This implies the following.

Proposition.

(1) *Category $\mathcal{O}_{\overline{\mathfrak{g}}}$ is isomorphic to the full tensor subcategory of $\mathcal{E}_{\overline{\mathfrak{g}}^{(2)}}$ consisting of those modules carrying a trivial action of $\overline{\mathfrak{h}}^c$ and satisfying, as a module over $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}} \subset \overline{\mathfrak{g}}^{(2)}$, the conditions $(\mathcal{O}1)$–$(\mathcal{O}3)$ above.*

(2) *Under the equivalence $\mathcal{E}_{\overline{\mathfrak{g}}^{(2)}} \simeq \mathsf{DY}_{\overline{\mathfrak{b}}_-}$, $\mathcal{O}_{\overline{\mathfrak{g}}}$ is isomorphic to the full tensor subcategory of $\mathsf{DY}_{\overline{\mathfrak{b}}_-}$ consisting of those modules V such that the action ρ_V and coaction ρ_V^* of $\overline{\mathfrak{h}}$ on V[35] coincide under $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{h}}}$, i.e.,*

$$\rho_V = \langle\cdot,\cdot\rangle_{\overline{\mathfrak{h}}} \otimes \mathrm{id}_V \circ \mathrm{id}_{\overline{\mathfrak{h}}} \otimes \rho_V^* \tag{12.1}$$

as maps $\overline{\mathfrak{h}} \otimes V \to V$ and, as a module over $\overline{\mathfrak{h}} \subset \overline{\mathfrak{b}}_-$, V satisfies the conditions $(\mathcal{O}1)$–$(\mathcal{O}3)$ above.

12.9. **Pre–Coxeter structures and category $\mathcal{O}_\infty$.** Condition $(\mathcal{O}2)$ on the finite–dimensionality of weight spaces in 12.8 is not stable under restriction from $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_D$ to $\overline{\mathfrak{g}}_B$ if $B \subsetneq D$, which makes category $\mathcal{O}_{\overline{\mathfrak{g}}}$ unsuitable to the axiomatic framework of braided pre–Coxeter structures. We therefore omit it, and denote by $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}$ the category of $\overline{\mathfrak{g}}$–modules satisfying conditions $(\mathcal{O}1)$ and $(\mathcal{O}3)$. Proposition 12.8 shows that $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}$ is a full subcategory of $\mathsf{DY}_{\overline{\mathfrak{b}}_-}$. Moreover, the universal braided pre–Coxeter structure on $\{\mathsf{DY}^{\hbar}_{\overline{\mathfrak{b}}_{B,-}}\}_{B\subseteq D}$ restricts to one on $\{\mathcal{O}^{\hbar}_{\infty,\overline{\mathfrak{g}}_B}\}_{B\subseteq D}$.

12.10. **Braid group actions.** Assume now that A is a symmetrisable generalised Cartan matrix, let W be the corresponding Weyl group with set of simple reflections $\{s_i\}_{i\in\mathbf{I}}$, and set $\underline{m} = (m_{ij})$, where m_{ij} is the order of s_is_j in W.

Let $\mathcal{M}^{\mathrm{int}}_{\overline{\mathfrak{g}}}$ be the category of integrable $\overline{\mathfrak{g}}$–modules, *i.e.*, $\overline{\mathfrak{h}}$–semisimple modules endowed with a locally nilpotent action of the elements $\{e_i, f_i\}_{i\in\mathbf{I}}$. For any $i \in D$, let $\widetilde{s}_i \in \mathsf{End}\left(\mathcal{M}^{\mathrm{int}}_{\overline{\mathfrak{g}}} \to \mathrm{Vect}\right)$ be the triple exponential

$$\widetilde{s}_i = \exp(e_i)\cdot\exp(-f_i)\cdot\exp(e_i)$$

It is well–known (cf. [34]) that these satisfy the generalised braid relations (3.9).

Let $\mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ be the category of integrable Drinfeld–Yetter $\overline{\mathfrak{b}}_-$–modules in $\mathsf{DY}_{\overline{\mathfrak{b}}_-}$, *i.e.*, $\overline{\mathfrak{h}}$–diagonalisable, endowed with a locally nilpotent action of the elements $\{f_i\}_{i\in D} \subseteq \overline{\mathfrak{b}}_-$, and satisfying (12.1), so as to give rise to integrable modules over $\overline{\mathfrak{g}}$. In particular, the triple exponential $\widetilde{s}_i$ acts on the objects in $\mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ and the subcategory of integrable modules in $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}$, denoted $\mathcal{O}^{\mathrm{int}}_{\infty,\overline{\mathfrak{g}}}$, is isomorphic to a braided tensor subcategory of $\mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-}$. The following is straightforward.

Proposition. *There is a canonical (a,Υ)–strict symmetric Coxeter category $\mathbb{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ of type $(D,\underline{m})$, defined as follows*

- *For any $B \subseteq D$, $\mathbb{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-,B}$ is the symmetric monoidal category $\mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_{B,-}}$*
- *For any $B' \subseteq B$, the functor $F_{B'B} : \mathbb{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-,B} \to \mathbb{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-,B'}$ is the restriction $\mathrm{Res}_{\overline{\mathfrak{b}}_{B'},\overline{\mathfrak{b}}_B} : \mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_{B,-}} \to \mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_{B',-}}$*
- *for any $i \in D$, $S_i = \widetilde{s}_i$*

[35]The (co)action of $\overline{\mathfrak{h}}$ is defined by restricting that of $\overline{\mathfrak{b}}_-$ as in 5.12, since the inclusion $i_0 : \overline{\mathfrak{h}} \to \overline{\mathfrak{b}}_-$ is a split embedding with left inverse p_0: $\rho_V = \pi_V \circ i_0 \otimes \mathrm{id}_V$, $\rho_V^* = p_0 \otimes \mathrm{id}_V \circ \pi_V^*$.

There is a natural symmetric Coxeter category $\mathbb{O}^{\mathrm{int}}_{\infty,\overline{\mathfrak{g}}}$ obtained from $\mathbb{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ by restriction to the subcategories $\mathcal{O}^{\mathrm{int}}_{\infty,\overline{\mathfrak{g}}_B}$, $B\subseteq D$.

PROOF. It is enough to observe that $\widetilde{s}_i$ is group–like and therefore satisfies the coproduct identity (4.1), which for the symmetric category $\mathsf{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_i}$ reduces precisely to the condition $\Delta(\widetilde{s}_i)=\widetilde{s}_i\otimes\widetilde{s}_i$. □

12.11. **Universal braided Coxeter structures on Kac–Moody algebras.** Let $\mathsf{DY}^{\hbar,\mathsf{int}}_{\overline{\mathfrak{b}}_{B,-}}$ be the category of integrable deformation Drinfeld–Yetter $\overline{\mathfrak{b}}_{B,-}$–modules. Recall that $\widehat{\mathcal{U}}^n_B$ and $\widehat{\mathcal{U}}^n_{B,0}$ denote the algebras of endomorphisms of the forgetful functors $\mathsf{f}_B^{\boxtimes n}:(\mathsf{DY}^{\hbar}_{\overline{\mathfrak{b}}_{B,-}})^n\to\mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$ and $\mathsf{f}_{B,0}^{\boxtimes n}:(\mathsf{DY}^{\hbar}_{\overline{\mathfrak{b}}_{B,-}})^n\to\mathsf{DY}^{\hbar}_{\overline{\mathfrak{h}}}$, respectively. For any $X\in\widehat{\mathcal{U}}^n_B$, we denote by $\mathsf{p}(X)$ the induced endomorphism of the forgetful functor $(\mathsf{DY}^{\hbar,\mathsf{int}}_{\overline{\mathfrak{b}}_{B,-}})^n\to\mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$.

Definition. A braided Coxeter structure of type $(D,\underline{m})$ with diagrammatic categories $\{\mathsf{DY}^{\hbar,\mathsf{int}}_{\overline{\mathfrak{b}}_{B,-}}\}_{B\subseteq D}$ is *universal* if the underlying braided pre–Coxeter structure is (cf. 9.8), and its local monodromies have the form

$$S_i=\widetilde{s}_i\cdot\mathsf{p}(\underline{S}_i)$$

where $\underline{S}_i\in\widehat{\mathcal{U}}^1_{\{i\},0}$, $\underline{S}_i=1\mod\hbar$, and $\widetilde{s}_i=\exp(e_i)\exp(-f_i)\exp(e_i)$.

Remark. Since $\mathsf{DY}^{\hbar}_{\overline{\mathfrak{b}}_i}\simeq\mathrm{Rep}\,U\overline{\mathfrak{g}}_i^{(2)}[\![\hbar]\!]$ with $\mathfrak{g}_i=\mathfrak{sl}_2^{\alpha_i}$, we have $\widehat{\mathcal{U}}^n_{\{i\}}=(U\overline{\mathfrak{g}}_i^{(2)})^{\otimes n}[\![\hbar]\!]$. In particular, $\mathsf{p}(\underline{S}_i)$ is an element in $(U\overline{\mathfrak{g}}_i)^{\overline{\mathfrak{h}}_i}[\![\hbar]\!]$.

13. Quantum Kac–Moody algebras

We show in this section that integrable, category $\mathcal{O}_\infty$ representations of a quantised extended Kac–Moody algebra $U_\hbar\overline{\mathfrak{g}}$ give rise to a braided Coxeter category, with local monodromies given by Lusztig's quantum Weyl group operators. Using the fact that $U_\hbar\overline{\mathfrak{g}}$ is isomorphic to the Etingof–Kazhdan quantisation of $\overline{\mathfrak{g}}$ [17], together with the results of Section 10, we then transport this structure to integrable, category $\mathcal{O}_\infty$ representations of $\overline{\mathfrak{g}}$.

13.1. **The extended Drinfeld–Jimbo quantum group.** Throughout this section, $\mathsf{A}=\{a_{ij}\}_{i,j\in\mathbf{I}}$ denotes a fixed, symmetrisable generalised Cartan matrix, *i.e.,* $a_{ii}=2$, $a_{ij}\in\mathbb{Z}_{\leqslant 0}$ if $i\neq j$, and there is a non–singular diagonal matrix D such that $\mathsf{B}=\mathsf{DA}$ is symmetric (in particular, $a_{ij}=0$ if and only if $a_{ji}=0$). The matrix D is determined uniquely by requiring that $d_i\in\mathbb{Z}_+$ and $\mathsf{gcd}\{d_i\}=1$.

Let $\overline{\mathfrak{g}}=\overline{\mathfrak{g}}(\mathsf{A})$ be the corresponding extended Kac–Moody algebra with the standard diagrammatic Lie bialgebra structure described in 12.7, and set $q_i=\exp(\hbar/2\cdot d_i)$, $i\in\mathbf{I}$. The following is a straightforward generalisation to extended Kac–Moody algebras of the Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{g}$ [11, Example 6.2],[22].

Definition. The Drinfeld–Jimbo quantum group of $\overline{\mathfrak{g}}$ is the unital associative algebra $U_\hbar\overline{\mathfrak{g}}$ over $\mathsf{k}[\![\hbar]\!]$ topologically generated by $\overline{\mathfrak{h}}$ and the elements $\{E_i,F_i\}_{i\in\mathbf{I}}$, with relations

$$[h,h']=0 \qquad [h,E_i]=\alpha_i(h)E_i \qquad [h,F_i]=-\alpha_i(h)F_i$$
$$[E_i,F_i]=\frac{q_i^{h_i}-q_i^{-h_i}}{q_i-q_i^{-1}}$$

for any $h,h'\in\overline{\mathfrak{h}}$, $i\in\mathbf{I}$, where $h_i=\alpha_i^\vee$, and

$$\sum_{m=0}^{1-a_{ij}}(-1)^m X_i^{(1-a_{ij}-m)}X_jX_i^{(m)}=0$$

for $X=E,F$, $i\neq j\in\mathbf{I}$, where $X_i^{(r)}=X_i^r/[r]_{q_i}!$

$U_\hbar\overline{\mathfrak{g}}$ is a Hopf algebra, with counit $\varepsilon(h)=\varepsilon(E_i)=\varepsilon(F_i)=0$, coproduct

$$\Delta(h)=h\otimes 1+1\otimes h \qquad \Delta(E_i)=E_i\otimes q_i^{h_i}+1\otimes E_i \qquad \Delta(F_i)=F_i\otimes 1+q_i^{-h_i}\otimes F_i$$

and antipode $S(h)=-h$, $S(E_i)=-E_iq_i^{-h_i}$, and $S(F_i)=-q_i^{h_i}F_i$, for any $h\in\overline{\mathfrak{h}}$ and $i\in\mathbf{I}$.

The following result is well–known for $U_\hbar\mathfrak{g}$ (cf. [11, Sec. 13] and [8, Sec. 8.3]). It readily extends to $U_\hbar\overline{\mathfrak{g}}$ through the isomorphism of Hopf algebras $U_\hbar\overline{\mathfrak{g}}\simeq U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{c}$, where $U_\hbar\mathfrak{c}=S\mathfrak{c}[\![\hbar]\!]$, which quantises the decomposition $\overline{\mathfrak{g}}\simeq\mathfrak{g}\oplus\mathfrak{c}$ (cf. 12.6).

Proposition. [11, 8]

1. *The Hopf algebra $U_\hbar\overline{\mathfrak{g}}$ is a quantisation of the Lie bialgebra $\overline{\mathfrak{g}}$.*
2. *Let $U_\hbar\overline{\mathfrak{b}}_\mp\subset U_\hbar\overline{\mathfrak{g}}$ be the Hopf subalgebra topologically generated by $\mathfrak{h}$ and $\{F_i\}_{i\in\mathbf{I}}$ (resp. $\mathfrak{h}$ and $\{E_i\}_{i\in\mathbf{I}}$). Then, $U_\hbar\overline{\mathfrak{b}}_\mp$ is a quantisation of the Lie bialgebra $\overline{\mathfrak{b}}_\mp$, and there is a unique non–degenerate Hopf pairing $\langle\cdot,\cdot\rangle_{\mathcal{D}}:U_\hbar\overline{\mathfrak{b}}_-\otimes U_\hbar\overline{\mathfrak{b}}_+\to\mathsf{k}(\!(\hbar)\!)$, defined on the generators by*
$$\langle 1,1\rangle_{\mathcal{D}}=1 \qquad \langle h,h'\rangle_{\mathcal{D}}=\frac{1}{\hbar}\langle h,h'\rangle \qquad \langle F_i,E_j\rangle_{\mathcal{D}}=\frac{\delta_{ij}}{q-q^{-1}}$$
and zero otherwise.
3. *The Hopf pairing $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ induces an isomorphism of finitely $\mathbb{N}$–graded QUEs $U_\hbar\overline{\mathfrak{b}}_-\simeq(U_\hbar\overline{\mathfrak{b}}_+)^\star$, where the latter is the restricted QUE dual (cf. 6.3). This gives rise to an isomorphism of QUE $U_\hbar\overline{\mathfrak{g}}\simeq(DU_\hbar\overline{\mathfrak{b}}_-)^{\mathrm{res}}/(\mathfrak{h}\simeq\mathfrak{h}^*)$. In particular, $U_\hbar\overline{\mathfrak{g}}$ is a quasitriangular Hopf algebra, with R–matrix*
$$\overline{R}=q^{\sum_i u_i\otimes u^i}\cdot\sum_p X_p\otimes X^p, \tag{13.1}$$
where $\{u_i\},\{u^i\}\subset\overline{\mathfrak{h}}$ are dual bases with respect to $\langle\cdot,\cdot\rangle$, and $\{X_p\}\subset U_\hbar\overline{\mathfrak{n}}_-$, $\{X^p\}\subset U_\hbar\overline{\mathfrak{n}}_+$ are dual bases with respect to $\langle\cdot,\cdot\rangle_{\mathcal{D}}$.

13.2. **Diagrammatic structures on $U_\hbar\overline{\mathfrak{g}}$.** The quantum group $U_\hbar\overline{\mathfrak{g}}$ is canonically endowed with the structure of diagrammatic Hopf algebra, with subalgebras $U_\hbar\overline{\mathfrak{g}}_B=\langle\alpha_i^\vee,\lambda_i^\vee,E_i,F_i\rangle_{i\in B}$, $B\subseteq D$.

As in the classical case (cf. 12.7), the diagrammatic structure of $U_\hbar\overline{\mathfrak{g}}$ induces a split diagrammatic one on $U_\hbar\overline{\mathfrak{b}}_\pm$. Namely, for any $B\subseteq D$, let $U_\hbar\overline{\mathfrak{b}}_{\pm,B}=U_\hbar\overline{\mathfrak{b}}_\pm\cap U_\hbar\overline{\mathfrak{g}}_B$ be the Hopf sualgebras topologically generated by $\{\alpha_i^\vee,\lambda_i^\vee,e_i\}_{i\in B}$ and $\{\alpha_i^\vee,\lambda_i^\vee,f_i\}_{i\in B}$ respectively. For $B'\subseteq B$, let $i_{\pm,BB',\hbar}:U_\hbar\overline{\mathfrak{b}}_{\pm,B'}\to U_\hbar\overline{\mathfrak{b}}_{\pm,B}$ be the standard embedding, and regard $p_{\pm,B'B,\hbar}=i^t_{\mp,BB',\hbar}$ as a map $U_\hbar\overline{\mathfrak{b}}_{\pm,B}\to U_\hbar\overline{\mathfrak{b}}_{\pm,B'}$ via the identifications $U_\hbar\overline{\mathfrak{b}}^\star_{\mp,C}\cong U_\hbar\overline{\mathfrak{b}}_{\pm,C}$ given by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{D}}$. The map $(DU_\hbar\overline{\mathfrak{b}}_-)^{\mathrm{res}} \to U_\hbar\overline{\mathfrak{g}}$ from Proposition 13.1 (3) is then a morphism of diagrammatic Hopf algebras.

13.3. **Coxeter structures on quantum groups.** Let W be the Weyl group of $\overline{\mathfrak{g}}$, $\{s_i\}_{i\in\mathbf{I}}$ its generators, and set $\underline{m} = (m_{ij})$, where m_{ij} is the order of s_is_j in W. Thus, for any $B \subseteq D$, the generalised braid group $\mathcal{B}_B^{\underline{m}}$ is the Tits braid group of the standard parabolic subgroup of W generated by $\{s_i\}_{i\in B}$.

Let $\mathsf{DY}^{\mathrm{adm}}_{U_\hbar\overline{\mathfrak{b}}_{B,-}}$ be the braided monoidal category of admissible Drinfeld–Yetter $U_\hbar\overline{\mathfrak{b}}_{B,-}$–modules. As in 12.8, denote by $\mathsf{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_{B,-}}$ the full subcategory of $\overline{\mathfrak{h}}_B$–diagonalisable, integrable Drinfeld–Yetter $U_\hbar\overline{\mathfrak{b}}_{B,-}$–modules $\mathcal{V}$ such that the action and coaction of $\overline{\mathfrak{h}}$ on $\mathcal{V}$ coincide under $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{h}}}$, that is satisfy

$$\rho_{\mathcal{V}} = \langle\cdot,\cdot\rangle_{\overline{\mathfrak{h}}}\otimes \mathrm{id}_{\mathcal{V}}\circ \mathrm{id}_{\overline{\mathfrak{h}}}\otimes\rho^*_{\mathcal{V}}$$

so as to give rise to integrable modules over $U_\hbar\overline{\mathfrak{g}}_B$.

Proposition. *There is a canonical* (a,Υ)*–strict braided Coxeter category* $\mathbb{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_-}$ *of type* $(D,\underline{m})$*, with*

- *diagrammatic categories* $\mathsf{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_{B,-}}$, $B\subseteq D$
- *standard restriction functors* $\mathsf{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_{B,-}} \to \mathsf{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_{B',-}}$ *determined by the split diagrammatic structure of* $U_\hbar\overline{\mathfrak{b}}_-$
- *local monodromies given by Lusztig's quantum Weyl group operators* $S_i^\hbar$

Proof. The (a,Υ)–strict braided pre–Coxeter structure on $\mathbb{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_-}$ is defined in 6.7. For the Coxeter structure, we proceed as in 12.10. Denote by $\mathcal{M}^{\mathrm{int}}_{U_\hbar\overline{\mathfrak{g}}}$ the category of integrable $U_\hbar\overline{\mathfrak{g}}$–modules. Following [28], the quantum Weyl group operator of $U_\hbar\overline{\mathfrak{g}}$ corresponding to $i\in\mathbf{I}$ is the element $S_i^\hbar \in \mathsf{End}\left(\mathcal{M}^{\mathrm{int}}_{U_\hbar\overline{\mathfrak{g}}}\to \mathrm{Vect}_{\mathsf{K}}\right)$ acting on $\mathcal{V}\in\mathcal{M}^{\mathrm{int}}_{U_\hbar\overline{\mathfrak{g}}}$ as

$$S_i^\hbar(v) = \sum_{\substack{a,b,c\in\mathbb{Z}\\ a-b+c=-\lambda(\alpha_i^\vee)}} (-1)^b q_i^{\frac{h_i^2}{4}+b-ac} E_i^{(a)}F_i^{(b)}E_i^{(c)}v$$

where $v\in\mathcal{V}_\lambda$ for $\lambda\in\mathfrak{h}^*$. The quantum Weyl group operators $S_i^\hbar$ satisfy the braid relations (3.9), together with the coproduct identity

$$\Delta(S_i^\hbar) = R_i^{21}\cdot(S_i^\hbar\otimes S_i^\hbar)$$

Each $S_i^\hbar$, acts on any $\mathcal{V}_i\in\mathsf{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_{\{i\},-}}$ and they complete the (a,Υ)–strict braided Coxeter structure on $\mathbb{DY}^{\mathrm{adm,int}}_{U_\hbar\overline{\mathfrak{b}}_-}$. □

13.4. **Etingof–Kazhdan quantisation.** Let $\mathcal{Q}(\overline{\mathfrak{g}})$ (resp. $\mathcal{Q}(\overline{\mathfrak{b}}_\pm)$) be the Etingof–Kazhdan quantisation of the extended Kac–Moody algebra $\overline{\mathfrak{g}}$ (resp. the Borel subalgebras $\overline{\mathfrak{b}}_\pm\subset\overline{\mathfrak{g}}$).

Proposition.

(1) $\mathcal{Q}(\overline{\mathfrak{g}})$ *is a diagrammatic QUE, with subalgebras* $\mathcal{Q}(\overline{\mathfrak{g}}_B)$, $B\subseteq D$.
(2) $\mathcal{Q}(\overline{\mathfrak{b}}_\pm)$ *is a split diagrammatic QUE, with subalgebras* $\mathcal{Q}(\overline{\mathfrak{b}}_{B,\pm})$.
(3) *The quantised embeddings* $\mathcal{Q}(\overline{\mathfrak{b}}_{B,-})\to\mathcal{Q}(\overline{\mathfrak{g}}_B)$, $B\subseteq D$*, give rise to a morphism of diagrammatic QUEs* $\mathcal{Q}(\overline{\mathfrak{b}}_-)\to\mathcal{Q}(\overline{\mathfrak{g}})$.

(4) *The following data defines an* (a,Υ)*–strict braided pre–Coxeter category* $\mathbb{DY}^{\mathrm{adm,int}}_{\mathcal{Q}(\overline{\mathfrak{b}}_-)}$
- *For any* $B\subseteq D$, $\mathbb{DY}^{\mathrm{adm,int}}_{\mathcal{Q}(\overline{\mathfrak{b}}_-),B}$ *is the braided monoidal category* $\mathsf{DY}^{\mathrm{adm,int}}_{\mathcal{Q}(\mathfrak{b}_{B,-})}$
- *For any* $B'\subseteq B$, *the functor* $F_{B'B}:\mathbb{DY}^{\mathrm{adm,int}}_{\mathcal{Q}(\overline{\mathfrak{b}}_-),B}\to\mathbb{DY}^{\mathrm{adm,int}}_{\mathcal{Q}(\overline{\mathfrak{b}}_-),B'}$ *is the restriction functor* $\mathrm{Res}_{\mathcal{Q}(\overline{\mathfrak{b}}_{B',-}),\mathcal{Q}(\overline{\mathfrak{b}}_{B,-})}$.

(5) *The braided pre–Coxeter category* $\mathbb{DY}^{\mathrm{adm,int}}_{\mathcal{Q}(\overline{\mathfrak{b}}_-)}$ *is a deformation of* $\mathbb{DY}^{\mathrm{int}}_{\overline{\mathfrak{b}}_-}$.

Proof. (1) and (2) follow from the compatibility of the quantisation functor with the diagrammatic and split diagrammatic structures of $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{b}}_\pm$, respectively (Corollary 6.8). (3) follows from the functoriality of $\mathcal{Q}$, and the canonical morphism of diagrammatic Lie bialgebras $\overline{\mathfrak{b}}_-\to\overline{\mathfrak{g}}$ (Proposition 12.7 (4)). (4) is given by Corollary 6.9. (5) is clear. □

13.5. **Quantum double construction of $\mathcal{Q}(\overline{\mathfrak{g}})$.** By [13], the Etingof–Kazhdan quantisation functor $\mathcal{Q}$ is compatible with taking duals and doubles. This is used in [17] to show that $\mathcal{Q}(\mathfrak{g})$ is a quotient of the quantum double of $\mathcal{Q}(\mathfrak{b}_-)$, and that it is isomorphic to the quantum group $U_\hbar\mathfrak{g}$. The argument is easily adapted to the extended Kac–Moody algebra $\overline{\mathfrak{g}}$, since the latter is a central extension of the former $\mathfrak{g}$ (cf. 12.6). Specifically, by Proposition 11.7, $\overline{\mathfrak{g}}$ is isomorphic to the quotient of the Drinfeld double of $\overline{\mathfrak{b}}_-$ by the ideal generated by the identification of $\phi:\overline{\mathfrak{h}}\to\overline{\mathfrak{h}}^*$, *i.e.*, $\overline{\mathfrak{g}}\simeq(\mathfrak{D}\overline{\mathfrak{b}}_-)^{\mathrm{res}}/(\overline{\mathfrak{h}}\simeq\overline{\mathfrak{h}}^*)$. Since $\mathcal{Q}$ is compatible with doubling operations, there is an isomorphism $\mathcal{Q}((\mathfrak{D}\overline{\mathfrak{b}}_-)^{\mathrm{res}})\simeq(D\mathcal{Q}(\overline{\mathfrak{b}}_-))^{\mathrm{res}}$, which is the identity on $\overline{\mathfrak{h}}\oplus\overline{\mathfrak{h}}^*$. This yields an isomorphism of Hopf algebras

$$(D\mathcal{Q}(\overline{\mathfrak{b}}_-))^{\mathrm{res}}\Big/\overline{\mathfrak{h}}\simeq\overline{\mathfrak{h}}^*\simeq\mathcal{Q}(\overline{\mathfrak{g}}).$$

which shows, in particular, that $\mathcal{Q}(\overline{\mathfrak{g}})$ is quasitriangular. Finally, one proves the following

Theorem. [17]

(1) *There is a (non–canonical) isomorphism of QUEs* $\varphi^{\overline{\mathfrak{b}}_-}:U_\hbar\overline{\mathfrak{b}}_-\to\mathcal{Q}(\overline{\mathfrak{b}}_-)$, *which is the identity on* $\overline{\mathfrak{h}}$.

(2) *By the quantum double construction of* $\mathcal{Q}(\overline{\mathfrak{g}})$ *and* $U_\hbar\overline{\mathfrak{g}}$ *(cf. Proposition 13.1 (3)),* $\varphi^{\overline{\mathfrak{b}}_-}$ *induces an isomorphism of quasitriangular QUEs* $\varphi^{\overline{\mathfrak{g}}}:U_\hbar\overline{\mathfrak{g}}\to\mathcal{Q}(\overline{\mathfrak{g}})$.

13.6. **Diagrammatic isomorphism between $\mathcal{Q}(\overline{\mathfrak{g}})$ and $U_\hbar\overline{\mathfrak{g}}$.** We now show that the isomorphism between $\mathcal{Q}(\overline{\mathfrak{g}})$ and $U_\hbar\overline{\mathfrak{g}}$ can be chosen so as to preserve the diagrammatic structures.

Proposition.

(1) *There is an isomorphism of split diagrammatic QUEs* $\psi^{\overline{\mathfrak{b}}_-}:U_\hbar\overline{\mathfrak{b}}_-\to\mathcal{Q}(\overline{\mathfrak{b}}_-)$, *which is the identity on* $\overline{\mathfrak{h}}$.

(2) *By the quantum double construction,* $\psi^{\overline{\mathfrak{b}}_-}$ *induces an isomorphism of diagrammatic QUEs* $\psi^{\overline{\mathfrak{g}}}:U_\hbar\overline{\mathfrak{g}}\to\mathcal{Q}(\overline{\mathfrak{g}})$.

PROOF. (1) For any $j \in D$, use Theorem 13.5 (1) to choose an isomorphism of Hopf algebras $\psi_j^{\overline{\mathfrak{b}}_-} \coloneqq \varphi^{\overline{\mathfrak{b}}_{\{j\},-}} : U_\hbar \overline{\mathfrak{b}}_{\{j\},-} \to \mathcal{Q}(\overline{\mathfrak{b}}_{\{j\},-})$. Then, for any $B \subseteq D$, we get an isomorphism of Hopf algebras $\psi_B^{\overline{\mathfrak{b}}_-} : U_\hbar \overline{\mathfrak{b}}_B \to \mathcal{Q}(\overline{\mathfrak{b}}_B)$ by

$$\psi_B^{\overline{\mathfrak{b}}_-}(F_j) \coloneqq \mathcal{Q}(i_j^{\overline{\mathfrak{b}}_-}) \circ \psi_j^{\overline{\mathfrak{b}}_-}(F_j)$$

where $j \in B$. The collection $\psi^{\overline{\mathfrak{b}}_-} = \{\psi_B^{\overline{\mathfrak{b}}_-}\}_{B \subseteq D}$ gives an isomorphism of split diagrammatic Hopf algebras. (2) is clear. $\square$

13.7. **An equivalence of braided Coxeter categories.** We now prove the main result of this paper. We show that the Coxeter structure on integrable Drinfeld–Yetter modules for $U_\hbar \overline{\mathfrak{b}}$, which accounts for the quantum Weyl group operators of $U_\hbar \overline{\mathfrak{g}}$, can be transferred to a Coxeter structure on integrable Drinfeld–Yetter modules for $\overline{\mathfrak{b}}$, with standard restriction functors.

Theorem. *Let $\Phi \in \widehat{\mathfrak{U}}^3_{\mathsf{LBA}}$ be a factorisable associator.*

(1) *There is an equivalence of braided pre–Coxeter categories*

$$\mathbb{H}_{\overline{\mathfrak{b}}_-} : \mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-} \longrightarrow \mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_-}$$

where

(a) $\mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_-}$ *is the (a, Υ)–strict structure defined in 13.3, with*
- *diagrammatic categories* $\mathsf{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_{B,-}}$
- *standard monoidal restriction functors*
$$\mathrm{Res}_{U_\hbar \overline{\mathfrak{b}}_{B',-}, U_\hbar \overline{\mathfrak{b}}_{B,-}} : \mathsf{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_{B,-}} \to \mathsf{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_{B',-}}$$

(b) $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ *is a–strict and universal (see 9.8), with*
- *diagrammatic categories* $\mathsf{DY}^{\Phi_B,\mathrm{int}}_{\overline{\mathfrak{b}}_{B,-}}$
- *restriction functors of the form* $(\mathrm{Res}_{\overline{\mathfrak{b}}_{B',-}\overline{\mathfrak{b}}_{B,-}}, J_{\mathcal{F}}) : \mathsf{DY}^{\Phi_B,\mathrm{int}}_{\overline{\mathfrak{b}}_{B,-}} \to \mathsf{DY}^{\Phi_B,\mathrm{int}}_{\overline{\mathfrak{b}}_{B',-}}$, *for some monoidal structure $J_{\mathcal{F}}$*

(c) *the equivalence $\mathbb{H}_{\overline{\mathfrak{b}}_-}$ is given the composition*

$$\mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-} \to \mathbb{DY}^{\Phi,\Upsilon\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-} \to \mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{\mathcal{Q}(\overline{\mathfrak{b}}_-)} \to \mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_-} \tag{13.2}$$

where the first equivalence is given by Corollary 10.2, the second one by the transfer Theorem 10.10, and the third one by the isomorphism of diagrammatic QUEs $\mathcal{Q}(\overline{\mathfrak{b}}_-) \simeq U_\hbar \overline{\mathfrak{b}}_-$ (Prop. 13.6).

(2) *There is a unique braided Coxeter structure on $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-}$, which extends the pre–Coxeter structure, and is such that*

$$\mathbb{H}_{\overline{\mathfrak{b}}_-} : \mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-} \to \mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_-}$$

is an equivalence of Coxeter categories with respect to the Coxeter structure on $\mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar \overline{\mathfrak{b}}_-}$ arising from the quantum Weyl group operators of $U_\hbar \overline{\mathfrak{g}}$ (cf. 13.3). Moreover, the braided Coxeter structure on $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ is universal in the sense of Definition 12.11.

(3) *The representations of the generalised braid groups $\mathcal{B}_B^m$, $B \subseteq D$, arising from the quantum Weyl group operators of $U_\hbar\overline{\mathfrak{g}}$ and the Coxeter category $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}$ are equivalent.*

PROOF. (1) Let $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str}}$ be the universal a–strict braided pre–Coxeter category associated to $\overline{\mathfrak{b}}_-$ by Corollary 10.2. By Theorem 10.10, there is an equivalence of braided pre–Coxeter categories $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str}} \to \mathbb{DY}_{\mathcal{Q}(\overline{\mathfrak{b}}_-)}^{\mathrm{adm}}$. The isomorphism of split diagrammatic Hopf algebras $\mathcal{Q}(\overline{\mathfrak{b}}_-) \simeq U_\hbar\overline{\mathfrak{b}}_-$ constructed in Proposition 13.6, then allows to extend it to an equivalence

$$\widetilde{\mathbb{H}}_{\overline{\mathfrak{b}}_-} : \mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str}} \to \mathbb{DY}_{U_\hbar\overline{\mathfrak{b}}_-}^{\mathrm{adm}}$$

Let $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}$ be the braided pre–Coxeter subcategory of $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str}}$, with underlying diagrammatic categories $\mathsf{DY}_{\overline{\mathfrak{b}}_{B,-}}^{\Phi_B,\mathrm{int}}$, $B \subseteq D$. Since the Etingof–Kazhdan functors preserve integrable modules [1, Prop. 6.5], the restriction of $\widetilde{\mathbb{H}}_{\overline{\mathfrak{b}}_-}$ give rise to an equivalence of braided pre–Coxeter categories $\mathbb{H}_{\overline{\mathfrak{b}}_-} : \mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}} \to \mathbb{DY}_{U_\hbar\overline{\mathfrak{b}}_-}^{\mathrm{adm},\mathrm{int}}$.

(2) By 13.3, the quantum Weyl group operators of $U_\hbar\overline{\mathfrak{g}}$ define a Coxeter structure on $\mathbb{DY}_{U_\hbar\overline{\mathfrak{b}}_-}^{\mathrm{adm},\mathrm{int}}$. The requirement that $\mathbb{H}_{\overline{\mathfrak{b}}_-}$ be an equivalence of braided Coxeter categories therefore uniquely determines a Coxeter structure on $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}$. Namely, let Ψ_i denote the pullback on the algebra of endomorphisms of the forgetful functor along

$$H_i : \mathsf{DY}_{\overline{\mathfrak{b}}_{\{i\},-}}^{\Phi_{\{i\}},\mathrm{int}} \to \mathsf{DY}_{\mathcal{Q}(\overline{\mathfrak{b}}_{\{i\},-})}^{\mathrm{adm},\mathrm{int}} \to \mathsf{DY}_{U_\hbar\overline{\mathfrak{b}}_{\{i\},-}}^{\mathrm{adm},\mathrm{int}}$$

Then, the operators $\Psi_i(S_i^\hbar)$ extend the braided pre–Coxeter structure of $\mathbb{DY}_{\overline{\mathfrak{b}}_-}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}$ to a braided Coxeter structure. It is then clear by 13.3 that $\Psi_i(S_i^\hbar)$ satisfies the conditions of Definition 12.11, and therefore that this structure is universal.

(3) By construction, the action of the generalised braid groups $\mathcal{B}_B^m$ on $\mathcal{V} \in \mathsf{DY}_{U_\hbar\overline{\mathfrak{b}}_{B,-}}^{\mathrm{adm},\mathrm{int}}$ arising from the Coxeter category $\mathbb{DY}_{U_\hbar\overline{\mathfrak{b}}_-}^{\mathrm{adm},\mathrm{int}}$ coincides with the action of the quantum Weyl group operators of $U_\hbar\overline{\mathfrak{g}}_B$ (cf. 13.3). The result then follows from (2) and Proposition 3.11 (2). $\square$

13.8. **Coxeter structures and category $\mathcal{O}_\infty$.** Fix $B \subseteq D$. Recall that a $U_\hbar\overline{\mathfrak{g}}_B$–module $\mathcal{V}$ is in category $\mathcal{O}_{U_\hbar\overline{\mathfrak{g}}_B}^{\mathrm{int}}$ if it is topologically free over $\mathsf{k}[\![\hbar]\!]$, integrable, and satisfies the conditions $(\mathcal{O}1) - (\mathcal{O}3)$ of 12.8. Let $\mathcal{O}_{\infty,U_\hbar\overline{\mathfrak{g}}_B}^{\mathrm{int}}$ be the category of $U_\hbar\overline{\mathfrak{g}}_B$–modules satisfying conditions $(\mathcal{O}1)$ and $(\mathcal{O}3)$, but not necessarily the finite–dimensionality of weight spaces. The realisation of $U_\hbar\overline{\mathfrak{g}}_B$ as a quotient of the quantum double of $U_\hbar\overline{\mathfrak{b}}_{B,-}$ (13.1) gives rise to a full embedding $\mathcal{O}_{\infty,U_\hbar\overline{\mathfrak{g}}_B}^{\mathrm{int}} \subset \mathsf{DY}_{U_\hbar\overline{\mathfrak{b}}_{B,-}}^{\mathrm{adm},\mathrm{int}}$.

Since the Etingof–Kazhdan functor $\mathsf{DY}_{\overline{\mathfrak{b}}_{B,-}}^{\Phi_B} \to \mathsf{DY}_{\mathcal{Q}(\overline{\mathfrak{b}}_{B,-})}^{\mathrm{adm}}$ is the identity on $\overline{\mathfrak{h}}_B$–modules, the equivalence (13.2) preserves the categories $\mathcal{O}_{\infty,\overline{\mathfrak{g}}_B}^{\Phi_B,\mathrm{int}} \subset \mathsf{DY}_{\overline{\mathfrak{b}}_{B,-}}^{\Phi_B,\mathrm{int}}$ and $\mathcal{O}_{\infty,U_\hbar\overline{\mathfrak{g}}_B}^{\mathrm{int}} \subset \mathsf{DY}_{U_\hbar\overline{\mathfrak{b}}_{B,-}}^{\mathrm{adm},\mathrm{int}}$. This yields the following

Theorem. *Let $\Phi \in \widehat{\mathfrak{U}}_{\mathsf{LBA}}^3$ be factorisable associator. Then, there is an equivalence of braided Coxeter categories*

$$\mathbb{H}_{\overline{\mathfrak{g}}} : \mathbb{O}_{\infty,\overline{\mathfrak{g}}}^{\Phi,\mathrm{int}} \to \mathbb{O}_{\infty,U_\hbar\overline{\mathfrak{g}}}^{\mathrm{int}}$$

where $\mathbb{O}^{\Phi,\mathrm{int}}_{\infty,\overline{\mathfrak{g}}}$ (resp. $\mathbb{O}^{\mathrm{int}}_{\infty,U_\hbar\overline{\mathfrak{g}}}$) is the braided Coxeter category obtained from $\mathbb{DY}^{\Phi,\mathsf{a}\text{-str},\mathrm{int}}_{\overline{\mathfrak{b}}_-}$ (resp. $\mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar\overline{\mathfrak{b}}_-}$) by restriction to integrable, category $\mathcal{O}_\infty$ representations.

13.9. **Coxeter structures, Levi subalgebras and category $\mathcal{O}$.** As mentioned in 12.4 and 12.9, the reason behind the introduction of extended Kac–Moody algebras and of category $\mathcal{O}_\infty$ is the construction of a diagrammatic structure endowed with well–defined restriction functors.

There is, however, a weaker notion of diagrammatic structure which leads to an analogue of Theorem 13.8 expressed solely in terms of standard Kac–Moody algebras and category $\mathcal{O}$ representations. Indeed, the facts that minimal realisations of Kac–Moody algebras do not give rise to a diagrammatic structure (Prop. 12.3), and that category $\mathcal{O}$ representations are not stable under restriction, due to the requirement on the finite–dimensionality of weight spaces, can both be overcome by considering instead the Levi subalgebras $\mathfrak{l}_B = \langle e_i, f_i, \mathfrak{h}\rangle_{i\in B}$ of a given Kac–Moody algebra $\mathfrak{g}$.

The collection $\{\mathfrak{l}_B\}$ does not, however, define a diagrammatic structure on $\mathfrak{g}$, since it does not satisfy the orthogonality condition $[\mathfrak{l}_B, \mathfrak{l}_{B'}] = 0$ for $B \perp B'$. As mentioned in 3.13, this condition is convenient in the construction of PROPic structures, but not required by the axioms of a Coxeter category. It is in fact possible to adapt the definition of universal pre–Coxeter structure, and consequently Sections 7–9, by removing the orthogonal factorisation axiom in Definition 9.4. In this new setting, Proposition 9.10 does not hold, *i.e.,* a non–orthogonal structure cannot be a–strictified in general. With this exception, all other results from Section 10 can be adapted, and applied to the case of the Levi subalgebras $\mathfrak{l}_B$.

As observed in 13.7 and 13.8, for any $B \subseteq D$, the Etingof–Kazhdan equivalence $\mathsf{DY}^{\Phi_B}_{\mathfrak{b}_{B,-}} \to \mathsf{DY}^{\mathrm{adm}}_{\mathcal{Q}(\mathfrak{b}_{B,-})}$ preserves integrable modules, and is the identity on $\mathfrak{h}$–modules. It therefore restricts to an equivalence of braided monoidal categories

$$H_{\mathfrak{l}_B} : \mathcal{O}^{\Phi_B,\mathrm{int}}_{\mathfrak{l}_B} \to \mathcal{O}^{\mathrm{int}}_{U_\hbar\mathfrak{l}_B}$$

Together with the fact that the universal constructions described in Section 10 are easily seen to yield twists, associators and joins which are invariant under $\mathfrak{h}$, this yields the following analogue of Theorem 13.8.

Theorem. *Let $\Phi \in \widehat{\mathfrak{U}}^3_{\mathsf{LBA}}$ be an associator. Then, there is an equivalence of braided Coxeter categories*

$$\mathbb{H}_{\mathfrak{g}} : \mathbb{O}^{\Phi,\mathrm{int}}_{\mathfrak{g}} \to \mathbb{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g}}$$

where $\mathbb{O}^{\Phi,\mathrm{int}}_{\mathfrak{g}}$ (resp. $\mathbb{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g}}$) is the braided Coxeter category obtained from $\mathbb{DY}^{\Phi,\Upsilon\text{-str},\mathrm{int}}_{\mathfrak{b}_-}$ (resp. $\mathbb{DY}^{\mathrm{adm},\mathrm{int}}_{U_\hbar\mathfrak{b}_-}$) by restriction to the categories $\mathcal{O}^{\Phi,\mathrm{int}}_{\mathfrak{l}_B}$ (resp. $\mathcal{O}^{\mathrm{int}}_{U_\hbar\mathfrak{l}_B}$).[36]

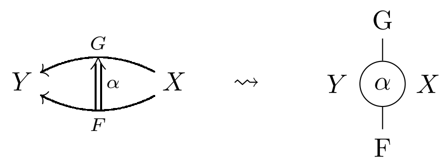
Appendix A. Graphical calculus for Coxeter objects

We describe below the axioms of Coxeter objects in a 2–category $\mathfrak{X}$ in terms of graphical calculus.

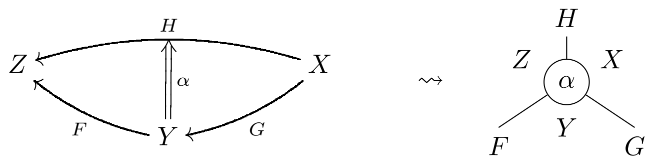
---

[36] Note that in order to have an action of the quantum Weyl group operators $S_i^\hbar$, which do not commute with the action of $\mathfrak{h}$, the diagrammatic categories $(\mathbb{O}^{\Phi,\mathrm{int}}_{\mathfrak{g}})_\emptyset$ and $(\mathbb{O}^{\mathrm{int}}_{U_\hbar\mathfrak{g}})_\emptyset$ have to be taken to be $\mathrm{Vect}_{\mathsf{k}[\![\hbar]\!]}$, rather than category $\mathcal{O}$ for $\mathfrak{l}_\emptyset = \mathfrak{h}$. Note also that the example in Proposition 12.3 shows that the minimal realisation of Kac–Moody algebras does not lead to a diagrammatic structure, even if the orthogonality requirement is omitted. It is therefore not possible in general to formulate an analogue of Theorem 13.9 involving minimal realisations, rather than Levi subalgebras.

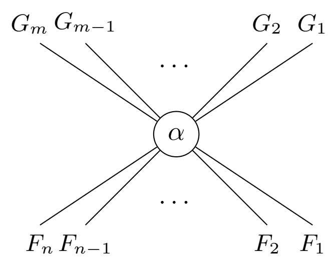
A.1. **Graphical notation.** In the following, we use the graphical notation of string diagrams to describe relations between 2–morphisms in a 2–category $\mathfrak{X}$ (*e.g.*, [23, 26]). We represent objects, 1–morphisms, and 2–morphisms with two dimensional, one dimensional and zero dimensional cells, respectively. Let $X, Y \in \mathfrak{X}$, $F, G \in \mathfrak{X}^{(1)}(X,Y)$ and $\alpha \in \mathfrak{X}^{(2)}(F,G)$. Then, we represent α as



where the diagram on the right–hand side is read from bottom to top, and from right to left. Similarly, a 2–morphism $\alpha : F\circ G \to H$ will be represented as follows:



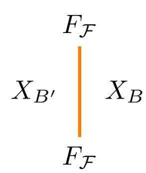
and more generally we represent $\alpha : F_n \circ \cdots \circ F_1 \Rightarrow G_m \circ \cdots \circ G_1$ as



When no confusion is possible, we omit the labels and identify the 1–morphisms with the color of the string, and the 2–morphism with the underlying diagram.

A.2. **Coxeter objects (cf. 3.10).** A Coxeter object in a 2–category $\mathfrak{X}$ is the datum of

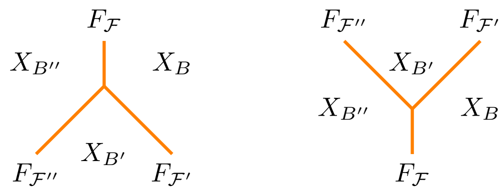
- for any $B \subseteq D$, an object X_B
- for any $\mathcal{F} \in \mathsf{Mns}(B,B')$, a 1–morphism $F_{\mathcal{F}} : X_B \to X_{B'}$ which we represent as the identity 2–morphisms $\mathrm{id}_{F_{\mathcal{F}}}$



- for any $\mathcal{F}' \in \mathsf{Mns}(B,B'), \mathcal{F}'' \in \mathsf{Mns}(B',B'')$ and $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, a 2–morphism

$$F_{\mathcal{F}''} \circ F_{\mathcal{F}'} \underset{(\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''})^{-1}}{\overset{\mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}}{\rightleftarrows}} F_{\mathcal{F}}$$

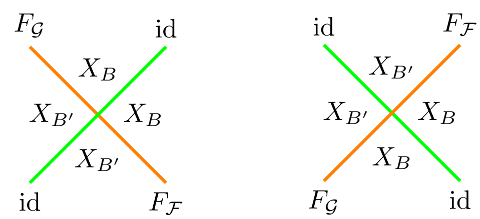
represented as



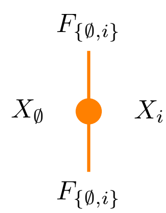
- for any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$ a pair of 2–morphisms

$$F_{\mathcal{F}} \underset{\Upsilon^{-1}_{\mathcal{G}\mathcal{F}}}{\overset{\Upsilon_{\mathcal{G}\mathcal{F}}}{\rightleftarrows}} F_{\mathcal{G}}$$

represented as *fake crossings*

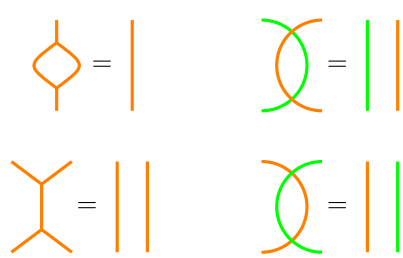


- for any $i \in D$, an invertible 1–morphism $S_i : F_{\{\emptyset,i\}} \to F_{\{\emptyset,i\}}$

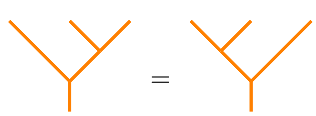


satisfying the following relations. To alleviate the notation, the labels of objects and 1–morphisms are omitted unless necessary.

- **Invertibility.**

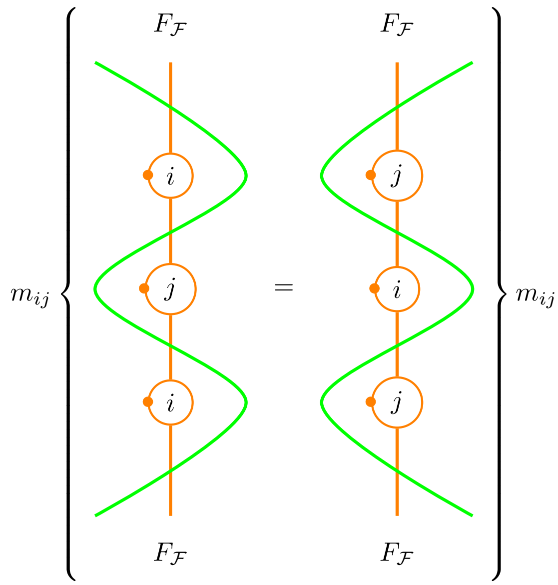


- **Associativity.**

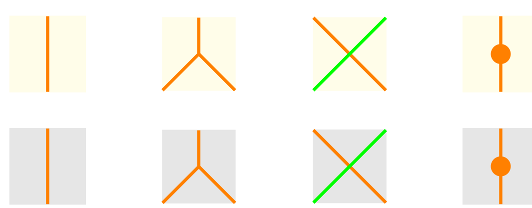


- **Vertical and horizontal factorisation.**

- **Braid relations.** For any $i, j \in B \subseteq D$, $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mns}(B)$ such that $i \neq j$, $m_{ij} < \infty$, $\{i\} \in \mathcal{H}$, $\{j\} \in \mathcal{G}$,

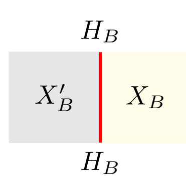


A.3. 1–**Morphisms.** Let X, X' be Coxeter objects in $\mathfrak{X}$. We distinguish between them by assigning a different color to their 2–cells (specifically, yellow for X, gray for X'). We represent their defining data as



Then a 1–morphisms of Coxeter objects $H : X \Rightarrow X'$ is the datum of

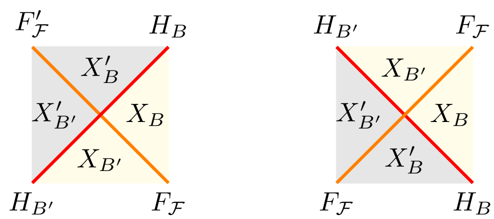
- for any $B \subseteq D$, a 1–morphism $H_B : X_B \to X'_B$



- for any $\mathcal{F} \in \mathsf{Mns}(B, B')$ a pair of 2–morphisms

$$F'_{\mathcal{F}} \circ H_B \underset{\gamma_{\mathcal{F}}^{-1}}{\overset{\gamma_{\mathcal{F}}}{\rightleftarrows}} H_{B'} \circ F_{\mathcal{F}}$$

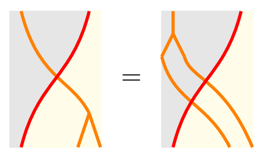
represented as



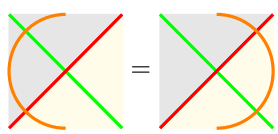
satisfying the following relations

- **Invertibility.**

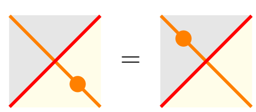
- **Vertical factorization.**



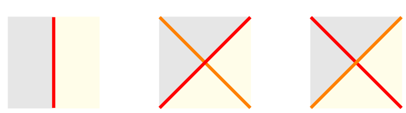
- **Preserving associators.** [37]



- **Preserving local monodromies.**



A.4. 2–**Morphisms.** Let $H, H' : X \to X'$ be two 1–morphisms,



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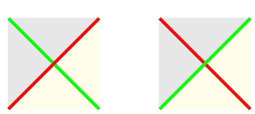
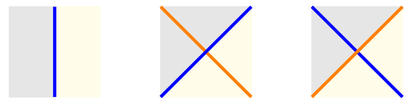
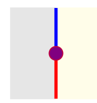
[37]The crossings



represent the identity on H_B.



A 2–morphism $u : H \Rightarrow H'$ is the datum, for any $B \subseteq D$, of an invertible 2–morphism $u_B : H_B \Rightarrow H'_B$



satisfying

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