



A METHOD TO DETERMINE ALGEBRAICALLY INTEGRAL CAYLEY DIGRAPHS ON FINITE ABELIAN GROUP

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ABSTRACT. Researchers in the past have studied eigenvalues of Cayley digraphs or graphs. We are interested in characterizing Cayley digraphs on a finite commutative group G whose eigenvalues are algebraic integers in a given number field K . We succeed in finding a method to do so. The number of such Cayley digraphs are computed.

1. INTRODUCTION AND MAIN RESULTS

In this paper, the finite digraphs without loops and multiple edges are considered. Let G be a group and let S be a subset of G that does not contain the identity. Then the Cayley digraph $D(G, S)$ is the digraph with vertex set G and edge set $E(D(G, S)) = \{gh \mid hg^{-1} \in S\}$ (see [7]). If S is inverse-closed, then the definition reduces to that of a Cayley graph. When G is cyclic, $D(G, S)$ is called a circulant digraph. For the prerequisites for Cayley digraphs, we refer to [4, 7].

A digraph is called algebraically integral over a number field K if its spectrum consists of some algebraic integers of K . Since 1974, Harary and Schwenk (see [8]) began to research integral graphs, which in fact are algebraically integral over the rational number field \mathbf{Q} . For some literature studying integral Cayley graphs and the spectrums of Cayley digraphs, see e.g., [1, 2, 3, 5, 12, 14].

We call $D(G, S)$ Gaussian integral if it is algebraically integral over the Gaussian field $\mathbf{Q}(i)$. Several authors have studied digraphs with Gaussian spectra (see [6, 9, 15]). In our previous results, algebraically integral circulant digraphs were determined completely (see [11]).

In this paper, we obtain the necessary and sufficient conditions for Cayley digraphs on finite abelian groups to be algebraically integral. Also, the number of such algebraically integral Cayley digraphs are calculated.

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Let n be the order of G . By the theorem of finite abelian groups, G is isomorphic to a direct sum of some cyclic groups. In the following, we assume that $G = C_1 \oplus C_2 \oplus \cdots \oplus C_m$, where $C_i = \mathbf{Z}/n_i\mathbf{Z}$ (n_i not necessary being a power of a prime number) for some integer $n_i, 1 \leq i \leq m$. Here \mathbf{Z} is the set of integers. We assume without loss of generality the fixed number field K is contained in the n th cyclotomic field $\mathbf{Q}(\zeta_n)$. The main results are stated as follows:

- (Theorem 2.1) The Cayley digraph $D(G, S)$ is algebraically integral over K if and only if S is a union of some orbits Hg_i 's.
- (Proposition 3.2) For $G = (\mathbf{Z}/n_1\mathbf{Z}) \oplus (\mathbf{Z}/n_2\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_m\mathbf{Z})$, there are at most $2^{r(G,K)}$ Cayley digraphs algebraically integral over K .

Remark. Based on Theorem 2.1, we can get a method to determine integral and Gaussian integral Cayley digraphs on a finite abelian group completely.

2. PROOF OF MAIN THEOREM

Let $H = \text{Gal}(\mathbf{Q}(\zeta_n)/K)$ be the Galois group of $\mathbf{Q}(\zeta_n)$ over K . Since $H \subseteq \text{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q}) \cong (\mathbf{Z}/n\mathbf{Z})^*$, for each $\sigma \in H$, there is an element $a \in (\mathbf{Z}/n\mathbf{Z})^*$ such that $\sigma(\zeta_n) = \zeta_n^a$. Notice that $n = n_1 n_2 \cdots n_m$. For each n_i , we define a group operation π_i of H on $\mathbf{Z}/n_i\mathbf{Z}$ by $\pi_i(\sigma, x) = \sigma(x) = ax \pmod{n_i}$ for $x \in \mathbf{Z}/n_i\mathbf{Z}$. Then we have a operation of H on G by $\sigma(x_1, x_2, \dots, x_m) = (\pi_1(\sigma, x_1), \pi_2(\sigma, x_2), \dots, \pi_m(\sigma, x_m))$ for $(x_1, x_2, \dots, x_m) \in G$. Using the orbit decomposition formula, $G \setminus \{0\}$ is the disjoint union of the distinct orbits, and we can write $G \setminus \{0\} = \bigsqcup_{i \in I} Hg_i$, where I is some index set, and the g_i are elements of distinct orbits.

We obtain a necessary and sufficient condition for the Cayley digraph $D(G, S)$ to be algebraically integral over K as in the following theorem.

Theorem 2.1. *The Cayley digraph $D(G, S)$ is algebraically integral over K if and only if S is a union of some orbits Hg_i 's.*

In the following, we give a proof of Theorem 2.1. Let \widehat{G} denote the group of multiplicative homomorphisms from G to \mathbf{C}^* (see [13]), where \mathbf{C} is the complex field. Then we have an isomorphism $G \cong \widehat{G}$. Also we set $G = \{h_u | h_u = (h_{u1}, h_{u2}, \dots, h_{um}), 1 \leq u \leq n\}$. For later use, we suppose $h_n = (0, 0, \dots, 0)$. For each $h_u \in G$, denote by $\chi_u \in \widehat{G}$ the homomorphism satisfying $\chi_u(h_v) = \prod_{i=1}^m \zeta_{n_i}^{h_{ui}h_{vi}}, h_v \in G$. In fact, χ_u is also called a character of G . The spectrum of the Cayley digraph $D(G, S)$ is given by [2], $\text{spec}(D(G, S)) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where $\lambda_u = \sum_{h_v \in S} \chi_u(h_v)$.

Proposition 2.2. *If S is a union of some orbits Hg_i 's, then $D(G, S)$ is algebraically integral over K .*

Proof. For every element $\sigma \in H$ and each orbit Hg_i , we have $\sigma Hg_i = \{\sigma(hg_i) | h \in H\} = Hg_i$. So $\sigma S = S$, since S is a union of some orbits Hg_i 's.

Thus

$$\begin{aligned}\sigma(\lambda_u) &= \sum_{h_v \in S} \sigma(\chi_u(h_v)) = \sum_{h_v \in S} \sigma\left(\prod_{i=1}^m \zeta_{n_i}^{h_{ui}h_{vi}}\right) = \sum_{h_v \in S} \prod_{i=1}^m \sigma(\zeta_{n_i}^{h_{ui}h_{vi}}) \\ &= \sum_{h_v \in S} \prod_{i=1}^m \zeta_{n_i}^{h_{ui}\sigma(h_{vi})} = \sum_{h_v \in S} \chi_u(\sigma(h_v)) = \lambda_u.\end{aligned}$$

By the Galois theory (see [10]) for finite Galois extensions, we get $\lambda_u \in K$. Notice that λ_u are algebraic integers, hence $D(G, S)$ is algebraically integral over K . \square

In the other hand, we have the following proposition.

Proposition 2.3. *If $D(G, S)$ is algebraically integral over K , then S is a union of some orbits Hg_i 's .*

Before the proof, we require some preparation. Let Γ be an $(n-1)$ -order square matrix. Namely, we suppose $\Gamma = (\gamma_{uv})$, where γ_{uv} is the entry of the u th row and v th column, with $\gamma_{uv} = \chi_u(h_v)$. For the matrix Γ , we have the following lemma.

Lemma 2.4. *The matrix Γ is nonsingular.*

Proof. Let χ be a nontrivial character of finite abelian group G , we know that $\sum_{g \in G} \chi(g) = 0$. Then the sum of all entries in each row of Γ is -1 . Therefore, to prove this lemma, it suffices to show the $n \times n$ block matrix

$$\Gamma' = \begin{pmatrix} \Gamma & 1 \\ 1 & 1 \end{pmatrix}$$

is invertible.

Let F be the space of complex valued class functions on G . All the elements of \widehat{G} form an orthonormal basis of F (see [13]). Suppose there exist n complex numbers $k_u (1 \leq u \leq n)$ such that $\sum_{u=1}^n k_u R_u = (0, 0, \dots, 0)$, where R_u is the u th row of the matrix Γ' . Then we have the class function $\sum_{u=1}^n k_u \chi_u = 0$. So $k_u = 0$ for all $u, 1 \leq u \leq n$, which shows that the row vectors of Γ' are linearly independent. Hence Γ' is invertible. \square

Let τ be an $(n-1)$ -dimension column vector,

$$\tau = (v_1, v_2, \dots, v_{n-1})^T,$$

with $v_i = 1$ for $h_i \in S$ and 0, otherwise. It is easy to see that

$$\Gamma\tau = (\lambda_1, \lambda_2, \dots, \lambda_{n-1})^T.$$

Let τ_i be the $(n-1)$ -dimension column vector for the orbit Hg_i just as τ for S . We denote by W the vector space $\{\omega \in K^{n-1} \mid \Gamma\omega \in K^{n-1}\}$ and by $V \subset K^{n-1}$ the vector space spanned by the vectors $\{\tau_i \mid i \in I\}$. We obtain the following lemma for W and V .

Lemma 2.5. *W and V are the same vector space.*

Proof. By Proposition 2.2, $\Gamma\tau_i \in K^{n-1}, i \in I$. So $V \subset W$. Let $\omega \in W$ and $\omega = (\omega_1, \omega_2, \dots, \omega_{n-1})^T, \mu = \Gamma\omega = (\mu_1, \mu_2, \dots, \mu_{n-1})^T$. First, we show that $\mu_u = \mu_v$ if h_u, h_v are in the same orbit Hg_i . Because $h_u, h_v \in Hg_i$, there exists an element $\sigma \in H$ such that $\sigma(h_u) = h_v$, i.e., $\sigma(h_{u1}, h_{u2}, \dots, h_{um}) = (h_{v1}, h_{v2}, \dots, h_{vm})$. In fact,

$$\begin{aligned} \mu_u = \sigma(\mu_u) &= \sigma\left(\sum_{k=1}^{n-1} \omega_k \chi_u(h_k)\right) = \sum_{k=1}^{n-1} \omega_k \sigma(\chi_u(h_k)) \\ &= \sum_{k=1}^{n-1} \omega_k \prod_{l=1}^m \sigma(\zeta_{n_l}^{h_{ul}h_{kl}}) = \sum_{k=1}^{n-1} \omega_k \prod_{l=1}^m \zeta_{n_l}^{\sigma(h_{ul})h_{kl}} \\ &= \sum_{k=1}^{n-1} \omega_k \prod_{l=1}^m \zeta_{n_l}^{h_{vl}h_{kl}} = \sum_{k=1}^{n-1} \omega_k \chi_v(h_k) = \mu_v, \end{aligned}$$

which implies that $\Gamma(W) \subset V$. Notice that the matrix Γ is nonsingular by Lemma 2.4 and $V \subset W$. Hence $\dim W = \dim V$, and $W = V$. \square

Now it comes to prove Proposition 2.3.

Proof of Proposition 2.3. Since $D(G, S)$ is algebraically integral over $K, \Gamma\tau \in K^{n-1}$. We have $\tau \in W$. By Lemma 2.5, $\tau \in V$ and $\tau = \sum_{i \in I} c_i \tau_i$ for some coefficients $c_i \in K$. By the construction of τ and the τ_i 's, we conclude that S is the union of the Hg_i 's with $c_i = 1$. The proof is completed. \square

Merging Proposition 2.2 and Proposition 2.3 together, we obtain the result of Theorem 2.1.

3. CALCULATING THE NUMBER OF ALGEBRAICALLY INTEGRAL CAYLEY DIGRAPHS

Let $G_{n_i} (1 \leq i \leq m)$ be the set of all the orbits of $\mathbf{Z}/n_i\mathbf{Z}$ under H . Denote by P_G the collection of Cartesian product $\{P = p_1 \times p_2 \times \dots \times p_m \mid P \neq 0, p_i \in G_{n_i}\}$. For a Cartesian product $P \in P_G$, choose one element $\rho \in P, \rho = (a_1, a_2, \dots, a_m)$. Denote by $\mathbf{Q}(P)$ the cyclotomic field

$$\mathbf{Q}(\zeta_{n_1}^{a_1}, \zeta_{n_2}^{a_2}, \dots, \zeta_{n_m}^{a_m}) = \mathbf{Q}(\zeta_{n_1}^{a_1}) \cdot \mathbf{Q}(\zeta_{n_2}^{a_2}) \cdot \dots \cdot \mathbf{Q}(\zeta_{n_m}^{a_m}),$$

and by $[\mathbf{Q}(P) : \mathbf{Q}(P) \cap K]$ the dimension of $\mathbf{Q}(P)$ as a vector space over $\mathbf{Q}(P) \cap K$. Here $|P|$ represents the cardinal number of P . It is easy to see that $\mathbf{Q}(P)$ is well-defined. By Galois theory (see [10]) in number fields, we have $|P| = \prod_{i=1}^m [\mathbf{Q}(\zeta_{n_i}^{a_i}) : \mathbf{Q}(\zeta_{n_i}^{a_i}) \cap K]$.

Let f be the map $\text{Gal}(\mathbf{Q}(P)/\mathbf{Q}(P) \cap K) \rightarrow \prod_{i=1}^m \text{Gal}(\mathbf{Q}(\zeta_{n_i}^{a_i})/\mathbf{Q}(\zeta_{n_i}^{a_i}) \cap K)$ by restriction, namely, $\sigma \mapsto (\sigma|_{\mathbf{Q}(\zeta_{n_1}^{a_1})}, \sigma|_{\mathbf{Q}(\zeta_{n_2}^{a_2})}, \dots, \sigma|_{\mathbf{Q}(\zeta_{n_m}^{a_m})})$. It is easy to see that f is injective. So every orbit contained in the Cartesian product P has $[\mathbf{Q}(P) : \mathbf{Q}(P) \cap K]$ elements, which implies that P can be divided equally into $|P|/[\mathbf{Q}(P) : \mathbf{Q}(P) \cap K]$ orbits under H . Totally, we get the following Lemma.

Lemma 3.1. *Under the operation of H , the Cartesian product P described as above is divided into*

$$\frac{\prod_{i=1}^m [\mathbf{Q}(\zeta_{n_i}^{a_i}) : \mathbf{Q}(\zeta_{n_i}^{a_i}) \cap K]}{[\mathbf{Q}(P) : \mathbf{Q}(P) \cap K]}$$

orbits.

Let

$$r(G, K) = \sum_{P \in P_G} \frac{\prod_{i=1}^m [\mathbf{Q}(\zeta_{n_i}^{a_i}) : \mathbf{Q}(\zeta_{n_i}^{a_i}) \cap K]}{[\mathbf{Q}(P) : \mathbf{Q}(P) \cap K]}.$$

By Lemma 3.1, $r(G, K)$ is the orbit number of group operation of H on $G \setminus \{0\}$. So we obtain the following proposition by Theorem 2.1.

Proposition 3.2. *For $G = (\mathbf{Z}/n_1\mathbf{Z}) \oplus (\mathbf{Z}/n_2\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_m\mathbf{Z})$, there are at most $2^{r(G,K)}$ Cayley digraphs algebraically integral over K .*

REFERENCES

1. R. C. Alperin and B. L. Peterson, *Integral sets and Cayley graphs of finite groups*, Electron. J. Combin. **19** (2012), #P44.
2. K. Babai, *Spectra of Cayley graphs*, J. Comb. Theory Ser. B **27** (1979), 180–189.
3. K. Balinska, D. Cvetkovic, Z. Radosavljevic, S. Simic, and D. Stevanovic, *A survey on integral graphs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **13** (2002), 42–65.
4. N. Biggs, *Algebraic graph theory*, North–Holland, Amsterdam, 1985.
5. W. G. Bridges and R. A. Mena, *Rational G -matrices with rational eigenvalues*, J. Comb. Theory Ser. A **32** (1982), no. 2, 264–280.
6. F. Esser and F. Harary, *Digraphs with real and Gaussian spectra*, Discrete Appl. Math. **2** (1980), 113–124.
7. C. Godsil and G. Royle, *Algebraic graph theory*, Springer–Verlag, New York, 2001.
8. F. Harary and A. Schwenk, *Which graphs have integral spectra*, Graphs Combin. (R. Bari and F. Harary, eds.), Springer–Verlag, Berlin, 1974, pp. 45–51.
9. X. D. Hou, *On the G -matrices with entries and eigenvalues in $Q(i)$* , Graphs Combin. **8** (1992), 53–64.
10. S. Lang, *Algebra*, 3rd ed., Springer–Verlag, New York, 2002.
11. F. Li, *Circulant digraphs integral over number fields*, Discrete Math. **313** (2013), 821–823.
12. J. Sander and T. Sander, *The exact maximal energy of integral circulant graphs with prime power order*, Contrib. Discrete Math. **8** (2013), no. 2, 19–40.
13. J. P. Serre, *Linear representations of finite groups*, Springer–Verlag, New York, 1977.
14. W. So, *Integral circulant graphs*, Discrete Math. **306** (2005), 153–158.
15. Y. Xu and J. X. Meng, *Gaussian integral circulant digraphs*, Discrete Math. **311** (2011), 45–50.

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