

**Fibonacci Lacunary Statistical Convergence In Intuitionistic Fuzzy Normed Linear Spaces**Ömer Kişi<sup>1</sup>, Ibrahim Tuzcuoğlu<sup>2</sup>**Abstract**

We investigate the concept of Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces. We also introduce here a new concept, that is, Fibonacci lacunary statistical completeness and show that every intuitionistic fuzzy normed linear space is Fibonacci lacunary statistically complete.

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## 1. INTRODUCTION AND BACKGROUND

The concept of statistical convergence for real number sequences was first originated by Fast [1]. Later, it was further investigated from sequence point of view and linked with summability theory by Fridy [2] and Šalát [3]. Fridy and Orhan [4] initiated the concept of lacunary statistical convergence.

The concept of 2-normed spaces was introduced and studied by S. Gähler [5]. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. [6] of USA in 1969 entitled 2-Banach spaces. In the same year Gähler [7] published another paper on this theme. A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler and S.C. Gupta of 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by A.H. Siddiqi [8].

Fuzziness has revolutionized many areas such as mathematics, science, engineering, medicine. This concept was given by Zadeh [9]. In fact the fuzzy set theory has become an area of active research for the last 40 years. The notion of fuzzyness are using by many persons for Cybernetics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting.

The fuzzy set theory has been used widely in many engineering applications, such as, in bifurcation of non-linear dynamical systems, in the control of chaos, in the non-linear operator, in population dynamics. The fuzzyness of all the subjects of mathematical sciences has been investigated. It attracted many workers on sequence spaces and summability theory to introduce different types of sequence spaces and study their different properties.

The notion of a fuzzy norm on a linear space was first originated by Katsaras [10]. Felbin [11] gave an alternative idea of a fuzzy norm whose concerned metric is of Kaleva and Seikkala [12] type.

Intuitionistic fuzzy sets was examined by Atanassov [13] is appropriate for such a situation. The notion of intuitionistic fuzzy metric space has been introduced by Park [14]. Furthermore, the concept of intuitionistic fuzzy normed space is given by Saadati and Park [15]. A lot of improvement has been made in the area of intuitionistic fuzzy normed space after the studies of ([16], [17], [18], [19], [22], [20], [21], [22]).

Fibonacci sequence was initiated in the book Liber Abaci of Fibonacci which was written in 1202. However, the sequence is based on older history. The sequence had been described earlier as Virahanka numbers in Indian mathematics [23]. In Liber Abaci, the sequence starts with 1, nowadays the sequence begins either with  $f_0 = 0$  or with  $f_1 = 1$ .

The numbers in the bottom row are called Fibonacci numbers, and the number sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

is the Fibonacci sequence [24].

The Fibonacci numbers are a sequence of numbers  $(f_n)$  for  $n = 1, 2, \dots$  defined by the linear recurrence equation  $f_n = f_{n+1} - f_{n-2}$ ,  $n \geq 2$ . From this definition, it means that the first two numbers in Fibonacci sequence are either 1 and 1 (or 0 and 0) depending on the chosen starting point of the sequence and all subsequent numbers is the sum of the previous two.

Some properties of Fibonacci numbers are given by

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha, \text{ (Golden ratio)}$$

$$\sum_{k=0}^n f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}),$$

$$\sum_k \frac{1}{f_k} \text{ converges,}$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1}, \quad n \geq 1. \text{ (Cassini formula)}$$

The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Başarır [25]. Afterward, Kara [26] defined the Fibonacci difference matrix  $\widehat{F}$  by using the Fibonacci sequence  $(f_n)$  for  $n \in \{1, 2, 3, \dots\}$  and introduced the new sequence spaces related to the matrix domain of  $\widehat{F}$ .

Following [25] and [26], high quality papers have been produced on the Fibonacci matrix by many mathematicians ([27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]).

Kirişçi and Karaisa [39] defined Fibonacci type statistical convergence and investigated some fundamental properties. Afterward, Kirişçi [40] examined Fibonacci statistical convergence on intuitionistic fuzzy normed spaces.

We recall some useful definitions and results. Recently, Kişi and Tuzcuoğlu [41] investigated Fibonacci lacunary statistical convergence on intuitionistic fuzzy  $n$ -normed spaces.

**Definition 1.** ([42]) A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -norm if  $*$  satisfies the following conditions:

- (i)  $*$  is commutative and associative,
- (ii)  $*$  is continuous,
- (iii)  $a * 1 = a$ , for all  $a \in [0, 1]$ ,
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  and  $a, b, c, d \in [0, 1]$ .

**Definition 2.** ([42]) A binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -conorm if  $\diamond$  satisfies the following conditions:

- (i)  $\diamond$  is commutative and associative,
- (ii)  $\diamond$  is continuous,
- (iii)  $a \diamond 0 = a$  for all  $a \in [0, 1]$ ,
- (iv)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$  and  $a, b, c, d \in [0, 1]$ .

Using the continuous  $t$ -norm and  $t$ -conorm, Saadati and Park [15] have introduced the concept of intuitionistic fuzzy normed space as follows:

**Definition 3.** ([15]) The five-tuple  $(\mathcal{X}, \phi, \omega, *, \diamond)$  is said to be intuitionistic fuzzy normed linear space or in short IFNS if  $\mathcal{X}$  is a vector space,  $*$  is a continuous  $t$ -norm,  $\diamond$  is a continuous  $t$ -conorm and  $\phi, \omega$  are fuzzy sets on  $\mathcal{X} \times (0, \infty)$  satisfying the following conditions: For every  $x, y \in \mathcal{X}$  and  $s, t > 0$ :

- (i)  $\phi(x, t) + \omega(x, t) \leq 1$ ,
- (ii)  $\phi(x, t) > 0$ ,
- (iii)  $\phi(x, t) = 1$  if and only if  $x = 0$ ,
- (iv)  $\phi(cx, t) = \phi\left(x, \frac{t}{|c|}\right)$  if  $c \neq 0$ ,
- (v)  $\phi(x, t) * \phi(y, s) \leq \phi(x + y, t + s)$ ,
- (vi)  $\phi(x, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous in  $t$ ;
- (vii)  $\lim_{t \rightarrow \infty} \phi(x, t) = 1$  and  $\lim_{t \rightarrow 0} \phi(x, t) = 0$ ,
- (viii)  $\omega(x, t) < 1$ ,
- (ix)  $\omega(x, t) = 0$  if and only if  $x = 0$ ,
- (x)  $\omega(cx, t) = \omega\left(x, \frac{t}{|c|}\right)$  if  $c \neq 0$ ,
- (xi)  $\omega(x, t) \diamond \omega(y, s) \geq \omega(x + y, t + s)$ ,
- (xii)  $\omega(x, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous in  $t$ ;
- (xiii)  $\lim_{t \rightarrow \infty} \omega(x, t) = 0$  and  $\lim_{t \rightarrow 0} \omega(x, t) = 1$ .

**Definition 4.** ([15]) Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space. A sequence  $x = (x_k)$  is said to convergent to  $\xi \in \mathcal{X}$  with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$  if for every  $\varepsilon > 0$  and  $t > 0$ , there exists a positive integer  $n_0$  such that  $\phi(x_k - \xi, t) > 1 - \varepsilon$  and  $\omega(x_k - \xi, t) < \varepsilon$  for all  $k \geq n_0$ . In this case, we write  $(\phi, \omega) - \lim x = \xi$  or  $x_k \xrightarrow{(\phi, \omega)} \xi$  as  $k \rightarrow \infty$ .

The element  $\xi$  is called the limit of the sequence  $\{x_{n_k}\}$  with respect to the intuitionistic fuzzy  $n$ -norm  $(\phi, \omega)$  and is denoted as  $(\phi, \omega) - \lim x_{n_k} = \xi$ .

**Definition 5.** ([15]) A sequence  $x = (x_k)$  is said to be Cauchy with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$  if for every  $\varepsilon > 0$  and  $t > 0$ , there exists a positive integer  $m_0$  such that  $\phi(x_k - x_l, t) > 1 - \varepsilon$  and  $\omega(x_k - x_l, t) < \varepsilon$  for all  $k, l \geq m_0$ .

**Definition 6.** ([40]) Take an IFNS  $(\mathcal{X}, \phi, \omega, *, \diamond)$ . A sequence  $(x_k)$  is said to be Fibonacci statistical convergence with respect to IFN  $(\phi, \omega)$ , if there is a number  $\xi \in \mathcal{X}$  such that for every  $\varepsilon > 0$  and  $t > 0$ , the set

$$K_\varepsilon(\widehat{F}) := \left\{ k \leq n : \phi(\widehat{F}x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \omega(\widehat{F}x_k - \xi, t) \geq \varepsilon \right\}$$

has natural density zero, i.e.,  $d(K_\varepsilon(\widehat{F})) = 0$ . That is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \phi(\widehat{F}x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \omega(\widehat{F}x_k - \xi, t) \geq \varepsilon \right\} \right| = 0.$$

In this case, we write  $d(\widehat{F})_{IFN} - \lim x_k = \xi$  or  $x_k \rightarrow \xi(S(\widehat{F})_{IFN})$ .

## 2. MAIN RESULTS

**Definition 7.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. Then, a sequence  $x = (x_k)$  is said to be Fibonacci lacunary statistically convergent to  $\xi \in \mathcal{X}$  with regards to the intuitionistic fuzzy norm  $(\phi, \omega)$ , if for every  $\varepsilon > 0$  and  $p > 0$ ,

$$(2.1) \quad \delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) \leq 1 - \varepsilon \text{ or } \omega(\widehat{F}x_k - \xi, p) \geq \varepsilon \right\} \right) = 0,$$

or equivalently

$$(2.2) \quad \delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) > 1 - \varepsilon \text{ and } \omega(\widehat{F}x_k - \xi, p) < \varepsilon \right\} \right) = 1.$$

In this case, we write  $S_\theta^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$  or  $x_k \xrightarrow{(\phi, \omega)} \xi(S_\theta(\widehat{F}))$ , where  $\xi$  is said to be  $S_\theta^{(\phi, \omega)}(\widehat{F}) - \lim x$  and we denote the set of all Fibonacci  $S_\theta$ -convergent sequences with regards to intuitionistic fuzzy norm  $(\phi, \omega)$  by  $S_\theta^{(\phi, \omega)}(\widehat{F})$ .

By using (2.1) and (2.2), we easily get the following lemma.

**Lemma 1.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. For every  $\varepsilon > 0$  and  $p > 0$ , the following statements are equivalent:

- (a)  $S_\theta^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$ ;
- (b)  $\delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) \leq 1 - \varepsilon \right\} \right) = \delta_\theta \left( \left\{ k \in \mathbb{N} : \omega(\widehat{F}x_k - \xi, p) \geq \varepsilon \right\} \right) = 0$ ;
- (c)  $\delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) > 1 - \varepsilon \text{ and } \omega(\widehat{F}x_k - \xi, p) < \varepsilon \right\} \right) = 1$ ;
- (d)  $\delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) > 1 - \varepsilon \right\} \right) = \delta_\theta \left( \left\{ k \in \mathbb{N} : \omega(\widehat{F}x_k - \xi, p) < \varepsilon \right\} \right) = 1$ ;
- (e)  $S_\theta(\widehat{F}) - \lim \phi(\widehat{F}x_k - \xi, p) = 1$  and  $S_\theta(\widehat{F}) - \lim \omega(\widehat{F}x_k - \xi, p) = 0$ .

**Theorem 1.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. If a sequence  $x = (x_k)$  is Fibonacci lacunary statistically convergent with regards to the intuitionistic fuzzy norm  $(\phi, \omega)$ , then  $S_\theta^{(\phi, \omega)}(\widehat{F})$ -limit is unique.

*Proof.* Assume that there exist two distinct elements  $\xi_1, \xi_2 \in X$  such that  $S_\theta^{(\phi, \omega)}(\widehat{F}) - \lim x_k = \xi_1$  and  $S_\theta^{(\phi, \omega)}(\widehat{F}) - \lim x_k = \xi_2$ . Given  $\varepsilon > 0$ , choose  $\gamma > 0$  such that  $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$  and  $\gamma \diamond \gamma < \varepsilon$ . Hence, for any  $p > 0$ , define the following sets as:

$$\mathcal{K}_{\phi,1}(\gamma, p) = \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi_1, \frac{p}{2}) \leq 1 - \gamma \right\},$$

$$\mathcal{K}_{\phi,2}(\gamma, p) = \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi_2, \frac{p}{2}) \leq 1 - \gamma \right\},$$

$$\mathcal{K}_{\omega,1}(\gamma, p) = \left\{ k \in \mathbb{N} : \omega(\widehat{F}x_k - \xi_1, \frac{p}{2}) \geq \gamma \right\},$$

$$\mathcal{K}_{\omega,2}(\gamma, p) = \left\{ k \in \mathbb{N} : \omega(\widehat{F}x_k - \xi_2, \frac{p}{2}) \geq \gamma \right\}.$$

Since  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x_k = \xi_1$ , we have by Lemma 1

$$\delta_{\theta}(\mathcal{K}_{\phi,1}(\gamma, p)) = \delta_{\theta}(\mathcal{K}_{\omega,1}(\gamma, p)) = 0 \text{ for all } p > 0.$$

Futhermore, using  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x_k = \xi_2$ , we get

$$\delta_{\theta}(\mathcal{K}_{\phi,2}(\gamma, p)) = \delta_{\theta}(\mathcal{K}_{\omega,2}(\gamma, p)) = 0 \text{ for all } p > 0.$$

Now let  $\mathcal{K}_{\phi, \omega}(\gamma, p) = (\mathcal{K}_{\phi,1}(\gamma, p) \cup \mathcal{K}_{\phi,2}(\gamma, p)) \cap (\mathcal{K}_{\omega,1}(\gamma, p) \cup \mathcal{K}_{\omega,2}(\gamma, p))$ .

Then, observe that  $\delta_{\theta}(\mathcal{K}_{\phi, \omega}(\gamma, p)) = 0$  which implies  $\delta_{\theta}(\mathbb{N} \setminus \mathcal{K}_{\phi, \omega}(\gamma, p)) = 1$ . If  $k \in \mathbb{N} \setminus \mathcal{K}_{\phi, \omega}(\gamma, p)$ , then we have two possible cases.

Case (i)  $k \in \mathbb{N} \setminus (\mathcal{K}_{\phi,1}(\gamma, p) \cup \mathcal{K}_{\phi,2}(\gamma, p))$  and case (ii)  $k \in \mathbb{N} \setminus (\mathcal{K}_{\omega,1}(\gamma, p) \cup \mathcal{K}_{\omega,2}(\gamma, p))$ .

We first consider that  $k \in \mathbb{N} \setminus (\mathcal{K}_{\phi,1}(\gamma, p) \cup \mathcal{K}_{\phi,2}(\gamma, p))$ . Then, we have

$$\phi(\xi_1 - \xi_2, p) \geq \phi\left(\widehat{F}x_k - \xi_1, \frac{p}{2}\right) * \phi\left(\widehat{F}x_k - \xi_2, \frac{p}{2}\right) > (1 - \gamma) * (1 - \gamma)$$

Since  $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$ , it follows that  $\phi(\xi_1 - \xi_2, p) > 1 - \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, we get  $\phi(\xi_1 - \xi_2, p) = 1$  for all  $p > 0$ , which yields  $\xi_1 = \xi_2$ .

On the other hand, if  $k \in \mathbb{N} \setminus (\mathcal{K}_{\omega,1}(\gamma, p) \cup \mathcal{K}_{\omega,2}(\gamma, p))$ , then we may write

$$\omega(\xi_1 - \xi_2, p) < \omega\left(\widehat{F}x_k - \xi_1, \frac{p}{2}\right) \diamond \omega\left(\widehat{F}x_k - \xi_2, \frac{p}{2}\right) < \gamma \diamond \gamma.$$

Now using the fact that  $\gamma \diamond \gamma < \varepsilon$ , we see that  $\omega(\xi_1 - \xi_2, p) < \varepsilon$ . Since arbitrary  $\varepsilon > 0$ , we get  $\omega(\xi_1 - \xi_2, p) = 0$  for all  $p > 0$ . This occurs that  $\xi_1 = \xi_2$ . So, we conclude that  $S_{\theta}^{(\phi, \omega)}(\widehat{F})$ -limit is unique.  $\square$

**Theorem 2.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. If  $(\phi, \omega) - \lim x = \xi$  then  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$ .

*Proof.* Let  $(\phi, \omega) - \lim x = \xi$ . Then, for every  $\varepsilon > 0$  and  $p > 0$ , then there is a number  $k_0 \in \mathbb{N}$  such that

$$\phi(x_k - \xi, p) > 1 - \varepsilon \text{ and } \omega(x_k - \xi, p) < \varepsilon$$

for all  $k \geq k_0$ . Hence, the set

$$\{k \in \mathbb{N} : \phi(x_k - \xi, p) \leq 1 - \varepsilon \text{ or } \omega(x_k - \xi, p) \geq \varepsilon\}$$

has finite number of terms. Since every finite subset of  $\mathbb{N}$  has density zero and hence

$$\delta_{\theta}\left(\left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \leq 1 - \varepsilon \text{ or } \omega\left(\widehat{F}x_k - \xi, p\right) \geq \varepsilon\right\}\right) = 0,$$

that is,  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$ .  $\square$

**Theorem 3.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. Then, for any lacunary sequence  $\theta$ ,  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$  iff there exists a subset  $K = \{j_1 < j_2 < \dots\} \subseteq \mathbb{N}$  such that  $\delta_{\theta}(K) = 1$  and  $(\phi, \omega) - \lim_{n \rightarrow \infty} x_{j_n} = \xi$ .

*Proof.* Necessity. Assume that  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$ . Let for any  $p > 0$  and  $s = 1, 2, \dots$

$$T_{(\phi, \omega)}(s, p) = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) > 1 - \frac{1}{s} \text{ and } \omega\left(\widehat{F}x_k - \xi, p\right) < \frac{1}{s}\right\}$$

and

$$R_{(\phi, \omega)}(s, p) = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \leq 1 - \frac{1}{s} \text{ or } \omega\left(\widehat{F}x_k - \xi, p\right) \geq \frac{1}{s}\right\}.$$

Then  $\delta_{\theta}(R_{(\phi, \omega)}(s, p)) = 0$  since  $S_{\theta}^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$ . Also

$$(2.3) \quad T_{(\phi, \omega)}(s, p) \supset T_{(\phi, \omega)}(s + 1, p)$$

and

$$(2.4) \quad \delta_{\theta}(T_{(\phi, \omega)}(s, p)) = 1$$

for  $p > 0$  and  $s = 1, 2, \dots$

Now we have to show that for  $k \in T_{(\phi, \omega)}(s, p)$ ,  $x_k \xrightarrow{(\phi, \omega)} \xi$ . Suppose that for some  $k \in T_{(\phi, \omega)}(s, p)$ ,  $x_k \not\xrightarrow{(\phi, \omega)} \xi$ . Therefore, there is  $\sigma > 0$  and a positive integer  $k_0$  such that

$$\phi\left(\widehat{F}x_k - \xi, p\right) \leq 1 - \sigma \text{ or } \omega\left(\widehat{F}x_k - \xi, p\right) \geq \sigma$$

for all  $k \geq k_0$ . Let

$$\phi(\widehat{F}x_k - \xi, p) > 1 - \sigma \text{ and } \omega(\widehat{F}x_k - \xi, p) < \sigma$$

for all  $k \geq k_0$ . Then, we get

$$\delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) > 1 - \sigma \text{ and } \omega(\widehat{F}x_k - \xi, p) < \sigma \right\} \right) = 0.$$

Since  $\sigma > \frac{1}{s}$ , we have  $\delta_\theta(T_{(\phi, \omega)}(s, p)) = 0$ , which contradicts (2.4). Therefore  $x_k \xrightarrow{(\phi, \omega)} \xi$ .

Sufficiency. Suppose that there exists a subset  $K = \{j_1 < j_2 < \dots\} \subseteq \mathbb{N}$  such that  $\delta_\theta(K) = 1$  and  $(\phi, \omega) - \lim_{n \rightarrow \infty} x_{j_n} = \xi$ , i.e. there exists  $N \in \mathbb{N}$  such that for every  $\sigma > 0$  and  $t > 0$

$$\phi(\widehat{F}x_k - \xi, p) > 1 - \sigma \text{ and } \omega(\widehat{F}x_k - \xi, p) < \sigma.$$

Now

$$R_{(\phi, \omega)}(\sigma, p) := \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) \leq 1 - \sigma \text{ or } \omega(\widehat{F}x_k - \xi, p) \geq \sigma \right\} \\ \subseteq \mathbb{N} - \{j_{N+1} < j_{N+2} < \dots\}.$$

Therefore  $\delta_\theta(R_{(\phi, \omega)}(\sigma, p)) \leq 1 - 1 = 0$ . Hence  $S_\theta^{(\phi, \omega)}(\widehat{F}) - \lim x = \xi$ . This completes the proof of the theorem.  $\square$

Now, we define Fibonacci lacunary statistically Cauchy sequences with respect to an intuitionistic fuzzy normed space and introduce a new concept of Fibonacci lacunary statistical completeness.

**Definition 8.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. Then, a sequence  $x = (x_k)$  is said to be Fibonacci lacunary statistically Cauchy (or  $\widehat{F}S_\theta$ -Cauchy) with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$  if for every  $\varepsilon > 0$  and  $p > 0$ , there exists  $N = N(\varepsilon)$  such that

$$\delta_\theta \left( \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \widehat{F}x_N, p) \leq 1 - \varepsilon \text{ or } \omega(\widehat{F}x_k - \widehat{F}x_N, p) \geq \varepsilon \right\} \right) = 0.$$

**Theorem 4.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed linear space and  $\theta$  be a lacunary sequence. If a sequence  $x = (x_k)$  is Fibonacci lacunary statistically convergent, then it is Fibonacci lacunary statistically Cauchy with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ .

*Proof.* Let  $x_k \rightarrow \xi(S_\theta^{(\phi, \omega)}(\widehat{F}))$ . For given  $\varepsilon > 0$ , choose  $\gamma > 0$  such that  $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$  and  $\gamma \diamond \gamma < \varepsilon$ . Hence, for any  $p > 0$ , we have

$$(2.5) \quad \delta_\theta(A(\varepsilon, p)) = \delta_\theta \left( \left\{ k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) \leq 1 - \varepsilon \text{ or } \omega\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) \geq \varepsilon \right\} \right) = 0$$

which implies that

$$\delta_\theta(A^c(\varepsilon, p)) = \delta_\theta \left( \left\{ k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) > 1 - \varepsilon \text{ and } \omega\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) < \varepsilon \right\} \right) = 1.$$

Let  $m \in A^c(\varepsilon, p)$ . Then

$$\phi(\widehat{F}x_m - \xi, p) > 1 - \varepsilon \text{ and } \omega(\widehat{F}x_m - \xi, p) < \varepsilon.$$

Now, let

$$B(\varepsilon, p) = \left\{ k \in \mathbb{N} : \phi(\widehat{F}x_k - \widehat{F}x_m, p) \leq 1 - \gamma \text{ or } \omega(\widehat{F}x_k - \widehat{F}x_m, p) \geq \gamma \right\}.$$

We need to show that  $B(\varepsilon, p) \subset A(\varepsilon, p)$ . Let  $k \in B(\varepsilon, p) \setminus A(\varepsilon, p)$ . Then, we get

$$\phi(\widehat{F}x_k - \widehat{F}x_m, p) \leq 1 - \gamma \text{ and } \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) > 1 - \varepsilon,$$

in particular  $\phi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) > 1 - \varepsilon$ . Then

$$1 - \gamma \geq \phi(\widehat{F}x_k - \widehat{F}x_m, p) \geq \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) * \phi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) > (1 - \varepsilon) * (1 - \varepsilon) > 1 - \gamma,$$

which is not possible. On the other hand,

$$\omega(\widehat{F}x_k - \widehat{F}x_m, p) \geq \gamma \text{ and } \omega\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) < \varepsilon,$$

in particular  $\omega\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) < \varepsilon$ . Then

$$\gamma \leq \omega(\widehat{F}x_k - \widehat{F}x_m, p) \leq \omega\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) \diamond \omega\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) < \varepsilon \diamond \varepsilon < \gamma,$$

which is not possible. Hence  $B(\varepsilon, p) \subset A(\varepsilon, p)$ . Hence, by (2.5)  $\delta_\theta(B(\varepsilon, p)) = 0$ . Therefore,  $x$  is Fibonacci lacunary statistically Cauchy with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ .  $\square$

**Definition 9.** An intuitionistic fuzzy normed linear space  $(\mathcal{X}, \phi, \omega, *, \diamond)$  is said to be  $\widehat{FS}_\theta$ -complete if every  $\widehat{FS}_\theta$ -Cauchy sequence with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$  is  $\widehat{FS}_\theta$ -convergent with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ .

**Theorem 5.** Let  $\theta$  be a lacunary sequence. Then every intuitionistic fuzzy normed space  $(\mathcal{X}, \phi, \omega, *, \diamond)$  is  $\widehat{FS}_\theta$ -complete.

*Proof.* Let  $x = (x_k)$   $\widehat{FS}_\theta$ -Cauchy but not  $\widehat{FS}_\theta$ -convergent with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ . For given  $\varepsilon > 0$ , choose  $\gamma > 0$  such that  $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$  and  $\gamma \diamond \gamma < \varepsilon$ . Now

$$\phi\left(\widehat{F}x_k - \widehat{F}x_m, p\right) \geq \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) * \phi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) > (1 - \varepsilon) * (1 - \varepsilon) > 1 - \gamma$$

and

$$\omega\left(\widehat{F}x_k - \widehat{F}x_m, p\right) \leq \omega\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) \diamond \omega\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) < \varepsilon \diamond \varepsilon < \gamma,$$

since  $x$  is not  $\widehat{FS}_\theta^{(\phi, \omega)}$ -convergent. Therefore,  $\delta_\theta(B^c(\varepsilon, p)) = 0$ , where

$$B(\varepsilon, p) = \{k \in \mathbb{N} : \omega_{x_k - x_m} \leq 1 - r\}$$

and so  $\delta_\theta(B(\varepsilon, p)) = 1$ , which is a contradiction, since  $x$  was  $\widehat{FS}_\theta$ -Cauchy with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ . So,  $x$  must be  $\widehat{FS}_\theta$ -convergent with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ . Hence, every intuitionistic fuzzy normed space is  $\widehat{FS}_\theta$ -complete.  $\square$

From Theorem 3, 4, 5, we can state the following:

**Theorem 6.** Let  $(\mathcal{X}, \phi, \omega, *, \diamond)$  be an intuitionistic fuzzy normed space. Then, for any sequence  $x = (x_k)$  the following conditions are equivalent:

- (i)  $x$  is Fibonacci lacunary statistically convergent with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ .
- (ii)  $x$  is Fibonacci lacunary statistically Cauchy with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ .
- (iii) Intuitionistic fuzzy normed space  $(\mathcal{X}, \phi, \omega, *, \diamond)$  is  $\widehat{FS}_\theta$ -complete.
- (iv) There exists an increasing index sequence  $K = (k_n)$  of natural numbers such that  $\delta_\theta(K) = 1$  and the subsequence  $(x_{k_n})$  is a  $\widehat{FS}_\theta$ -Cauchy with respect to the intuitionistic fuzzy norm  $(\phi, \omega)$ .

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