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A Stage-Structure Rosenzweig-MacArthur Model with Effect of Prey Refuge

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Abstract

We proposed and analyzed the stage-structure Rosenzweig-MacArthur model incorporating a prey refuge. It is assumed that the prey is a stage-structure population consisting of two compartments known as immature prey and mature prey. The model incorporates the functional response Holling type-II. In this work, we investigate all the biologically feasible equilibrium points, and it is shown that the system has three equilibrium points. Sufficient conditions for the local stability of the non-negative equilibrium point of the model are also derived. All points are conditionally locally asymptotically stable. By constructing the Jacobian matrix and determining its eigenvalues, we analyzed the local stability of the trivial and non-predator points. Specially for the local stability of the coexistence point is analyzed by using the Routh-Hurwitz criterion. In addition, we investigated the effect of immature prey refuge. Our mathematical analysis exhibits that the immature prey refuge have played a crucial role in the behavioral system. When the effect of immature prey refuge (constant m) increases, it is can stabilize the non-predator point, where all the species can not exist together. And conversely, if contant m decreases, it is can stabilize the coexistence point then all the species can exist together. The work is completed with the numerical simulations to confirmed the analytical results.

Keywords: *Rosenzweig-MacArthur Model; Stage-Structure; Refuge; Routh-Hurwitz Criterion*

1. Introduction

Population dynamics on the predator-prey interactions are an interesting things in the mathematical biology. This problem can be studied through the system of differential equations, and it has been widely studied by many scholars. The refuge and stage-structure are two facets that affects the predator-prey interactions in the nature. In the recent decade, many researchers have considered the effect of prey refuge on their model. For examples, Tao et al. [1] investigated the impact of refuge on a predator-prey model with harvesting. The analytical and numerical results showed that dynamical of the model depend on the constant of prey refuge. In [2], Chen et al. analyzed the global stability of the positive equilibrium point on a Lotka-Volterra model with considered a constant refuge. Next, the influence of the infected prey refuge on a Leslie-Gower model is discussed by Sharma and Samanta [3]. The analysis results showed that there is a great influence of this infected prey refuge on each population. Increasing the amount of infected prey refuge can decrease susceptible prey density as well as the predator density. In [4], Yue developed a Leslie-Gower model incorporating Holling tipe-II schemes and refuge of prey, and obtained that increasing the amount of refuge can ensure the coexistence of the two species more easily, since the existence of alternate prey can prevent the predator from extinction and increasing the amount of refuge could protect more prey from predation and become permanent. Next in [5], Moustofa et al. considered a constant refuge of prey on a fractional Rosenzweig-MacArthur model.

In the other hand, stage-structure use is important in the predator-prey model. Generally, there are many species whose individual members have a life history that takes them through two stage i.e. immature and mature [6–9]. In recent years, a combination of the stage-structure and refuge in the prey has attracted the attention of the many researchers. In [7], Devi analyzed the role of the prey refuge on a predator-prey model with stage-structute and ratio-dependent functional response. Their assumed that predator only attacks and eats mature prey as well as two types of refuges used by prey population. As a result, predator populations will become extinct if the prey refuge is increases. Using the same functional response, Khajanchi and Banerjee [8] studied how the refuges used by mature prey population influences the dynamic behavior. Their finds that the constant prey refuge becomes sufficiently large, implying very less access to the preys by predators and resulting in decrease of predator density. In [9], Wei and Fu analyzed the global stability of a stage-strucutre predator-prey model incorporating refuge in the mature prey. The analysis results showed that the prey refuge will enhance the density of the prey species, and it will decrease the density of predator species. A mathematical model dealing with two species of predator-prey system with the refuge capability is analyzed by Naji and Majeed [10]. The analysis model exhibits the refuge factor plays a vital role in the stabilizing of the system and can prevent extinction in the prey population.

Motivated from the various studies mentioned above, the goal of this paper is to study the effect of immature prey refuge on the Rosenzweig-MacArthur model with stage-structure. We note that in [6], the stage-structure Rosenzweig-MacArthur model without the effect of refuge has been done. The organization of this paper is as follows. In the next section, the methods in our work is given. In section 3, our entire work is discussed. In section 4, a short discussion is given to conclude this work.

2. Methods

The behavioral of the system (1) is analyzed by applying several stages as follows.

1. Modifying the predator-prey model in [6] by adding a new parameter of the immature prey refuge.
2. Solving the nullcline equations of the system (1) to determine the non-negative equilibrium points of the model.
3. Analyzing the local stability of the equilibrium points through the eigenvalues of a Jacobian matrix of each point, except for the coexistence point were analyzed using Routh-Hurwitz criterion.
4. Analyzing the implement of prey refuge by assessing the derivative alone the coexistence point.
5. Portraying the numerical solutions of the system to confirm the analysis results by using the fourth-order Runge–Kutta method.

3. Results and Discussion

3.1. Mathematical Model

In this article, we incorporate a parameter of the prey refuge into the stage-structure Rosenzweig-MacArthur model. The dynamics of this model can be represented mathematically with the following set of differential equations:

$$\begin{aligned}\frac{dx_1}{dt} &= rx_2\left(1-\frac{x_1}{k}\right)-\alpha x_1-\frac{\beta\left(1-m\right)x_1x_3}{\left(1-m\right)x_1+n_1}\\ \frac{dx_2}{dt} &= \alpha x_1-\delta_1 x_2\\ \frac{dx_3}{dt} &= \frac{\varphi\beta\left(1-m\right)x_1x_3}{\left(1-m\right)x_1+n_1}-\delta_2 x_3\end{aligned} \tag{1}$$

where x_1, x_2 and x_3 are respectively the densities of immature and mature prey as well as predator population at time t. In this study, the system (1) is supplemented with the initial conditions i.e., $x_1(0) > 0$, $x_2(0) > 0$ and $x_3(0) > 0$. The following assumptions are made in formulating model:

1. For the immature prey: the growth is assumed to be logistically with constant intrinsic rate $r > 0$ and $k > 0$ is the carrying capacity of the environment. $\beta > 0$ is maximum value which per capita reduction rate of immature prey can attain. $n_1 > 0$ measure the extent to which enviroment provides protection to immature prey. $m \geq 0$ is a refuge protection of the immature prey.
2. For the mature prey: $\alpha > 0$ denotes the surviving rate of immaturity to reach maturity. The per capita death rate of the mature prey is $\delta_1 > 0$.
3. For the predator: the predator not attacks and eats the mature prey. A conversion rate of the consumed prey into the predator births is $\varphi > 0$. $\delta_2 > 0$ is the per capita death rate of the predator.

3.2. Existence and Stability of Equilibrum Points

In this section, we discuss the existence and stability of equilibrium points. In order to obtain the equilibrium points of system (1), we consider the immature prey nullcline, mature prey nullcline and predator nullcline of this system, which are given by:

$$
\begin{aligned}
rx_2\left(1-\frac{x_1}{k}\right)-\alpha x_1-\frac{\beta\left(1-m\right)x_1x_3}{\left(1-m\right)x_1+n_1}&=0\\
\alpha x_1-\delta_1 x_2&=0\\
\frac{\varphi\beta\left(1-m\right)x_1x_3}{\left(1-m\right)x_1+n_1}-\delta_2 x_3&=0
\end{aligned}
\tag{2}
$$

Next, from system (2), we get the following three non-negative equilibrium points, namely,

1. A trivial equilibrium point $E_0=(0,0,0)$, which shown extinct of all species in the ecosystem.
2. A non-predator equilibrium point $E_1=(x_{11},x_{21},0)$. Notice that equilibrium point E_1 will exist if
$$r>\delta_1 \tag{3}$$
where $x_{11}=\frac{k(r-\delta_1)}{r}$ and $x_{21}=\frac{\alpha}{\delta_1}x_{11}$.
3. A coexistence equilibrium point $E_*=(x_{1*},x_{2*},x_{3*})$. Notice that equilibrium point E_* will exist if
$$\beta\varphi>\delta_2 \text{ and } m<1 \tag{4}$$
where $x_{1*}=\frac{\delta_2 n_1}{(1-m)(\beta\varphi-\delta_2)}$, $x_{2*}=\frac{\alpha}{\delta_1}x_{1*}$ and $x_{3*}=\frac{\alpha\varphi x_{1*}}{\delta_2}\left(\frac{(r-\delta_1)}{\delta_1}+\frac{rx_{2*}}{\alpha k}\right)$.

Thus, the non-predator equilibrium point E_1 is exists if condition (3) is applies. It is occur if the growth rate of immature prey greater than the death rate of mature prey. Next, if the death rate of predator is less than its birth rate and the parameter value m is less than 1, then condition (4) holds and the coexistence point E_* is exists.

Now to study the local stability of these equilibrium points, the Jacobian matrix from system (1) is determined as

$$
J=\begin{pmatrix}
-\frac{rx_2}{k}-\alpha-\frac{n_1\beta(1-m)x_3}{((1-m)x_1+n_1)^2} & r\left(1-\frac{x_1}{k}\right) & -\frac{\beta(1-m)x_1}{(1-m)x_1+n_1}\\
\alpha & -\delta_1 & 0\\
\frac{n_1\varphi\beta(1-m)x_3}{((1-m)x_1+n_1)^2} & 0 & \frac{\varphi\beta(1-m)x_1}{(1-m)x_1+n_1}-\delta_2
\end{pmatrix}
\tag{5}
$$

By observing the eigenvalues of the Jacobian matrix (5) at each equilibrium point, we have the following stability properties.

Theorem 1. *The trivial equilibrium point E_0 is locally asymptotically stable, if condition (6) is satisfied:*

$$\alpha<\delta_1 \tag{6}$$

proof. The Jacobian matrix (5) evaluated at the equilibrium point E_0 is given by

$$
J_{(0,0,0)}=\begin{pmatrix}
-\alpha & r & 0\\
\alpha & -\delta_1 & 0\\
0 & 0 & -\delta_2
\end{pmatrix}
\tag{7}
$$

From Jacobian matrix (7) the eigenvalues are obtained: $\lambda_1=-\delta_2$ and $\lambda_{2,3}=-\frac{b_1\pm\sqrt{D}}{2}$, where

$$
\begin{aligned}
D&={b_1}^2-4c_1,\\
b_1&=\delta_1-\alpha,\\
c_1&=-\alpha\left(r+\delta_1\right).
\end{aligned}
$$

If $\alpha<\delta_1$ then equilibrium point E_0 is locally asymptotically stable. It means that if the death rate of mature prey greather then the convertion rate of the immature prey into mature prey, then the trivial equilibrium point E_0 is locally asymptotically stable. ■

Theorem 2. *The non-predator equilibrium point E_1 is locally asymptotically stable, if condition (8) is satisfied:*

$$
\begin{aligned}
&(i) \quad z_1<\delta_1,\\
&(ii) \quad z_4<0.
\end{aligned}
\tag{8}
$$

proof. The Jacobian matrix (5) evaluated at the equilibrium point E_1 is given by

$$J_{(x_{11},x_{21},0)} = \begin{pmatrix} z_1 & z_2 & z_3 \\ \alpha & -\delta_1 & 0 \\ 0 & 0 & z_4 \end{pmatrix} \tag{9}$$

where

$$\begin{aligned} z_1 &= -\frac{rx_{21}}{K} - \alpha, \\ z_2 &= r\left(1 - \frac{x_{11}}{K}\right), \\ z_3 &= -\frac{\beta(1-m)x_{11}}{(1-m)x_{11}+n_1}, \\ z_4 &= \frac{\varphi\beta(1-m)x_{11}}{(1-m)x_{11}+n_1} - \delta_2. \end{aligned}$$

From Jacobian matrix (9), the eigenvalues are obtained: $\lambda_1 = z_4$ and $\lambda_{2,3} = -\frac{b_2 \pm \sqrt{D}}{2}$. If $z_1 < \delta_1$ and $z_4 < 0$ then the non-predator equilibrium point E_1 is locally asymptotically stable, where

$$\begin{aligned} D &= {b_2}^2 - 4c_2, \\ b_2 &= \delta_1 - z_1, \\ c_2 &= \alpha z_2 - \delta_1 z_1. \end{aligned}$$

■

Theorem 3. *The coexistence equilibrium point E_* is locally asymptotically stable if conditions (10) is satisfied:*

$$\begin{aligned} (i)\ \ & \phi_i > 0, \text{ where } i = 1,2,3, \\ (ii)\ \ & \phi_1\phi_2 - \phi_3 > 0. \end{aligned} \tag{10}$$

proof. The Jacobian matrix (5) evaluated at the equilibrium point E_* is given by

$$J_{(x_{1*},x_{2*},x_{3*})} = \begin{pmatrix} z_{1*} & z_{2*} & z_{3*} \\ \alpha & -\delta_1 & 0 \\ z_{4*} & 0 & z_{5*} \end{pmatrix} \tag{11}$$

where

$$\begin{aligned} z_{1*} &= -\frac{rx_{2*}}{K} - \alpha - \frac{n_1\beta(1-m)x_{3*}}{((1-m)x_{1*}+n_1)^2}, \\ z_{2*} &= r\left(1 - \frac{x_{1*}}{k}\right), \\ z_{3*} &= -\frac{\beta(1-m)x_{1*}}{(1-m)x_{1*}+n_1}, \\ z_{4*} &= \frac{n_1\varphi\beta(1-m)x_{3*}}{((1-m)x_{1*}+n_1)^2}, \\ z_{5*} &= \frac{\varphi\beta(1-m)x_{1*}}{(1-m)x_{1*}+n_1} - \delta_2. \end{aligned}$$

■

The characteristics equation of the Jacobian matrix (11) is written by:

$$\lambda^3 + \phi_1\lambda^2 + \phi_2\lambda + \phi_3 = 0 \tag{12}$$

where

$$\begin{aligned} \phi_1 &= \delta_1 - (z_{1*} + z_{5*}), \\ \phi_2 &= \alpha z_{2*} + z_{1*}z_{5*} - (z_{3*}z_{4*} + \delta_1(z_{1*} + z_{5*})), \\ \phi_3 &= \delta_1 z_{1*}z_{5*} - (\alpha z_{2*}z_{5*} + \delta_1 z_{3*}z_{4*}). \end{aligned}$$

The stability of equilibrium point E_* is studied using the Routh-Hurwitz criterion [11]. Thus, it can be shown that the coexistence point E_* is locally asymptotically stable if $\phi_i > 0,\ i = 1,2,3$ and

$$\begin{aligned}\phi_* &= \phi_1\phi_2 - \phi_3 > 0\\ &= (z_{1*} + z_{5*})\left[\delta_1\left(z_{1*} + z_{5*}\right) + z_{3*}z_{4*} - \left(\delta_1^2 + z_{1*}z_{5*}\right)\right] + \alpha z_{2*}\left(\delta_1 - z_{1*}\right) > 0.\end{aligned}$$

3.3. The influence of the Prey Refuge

In this section, we investigate the role of immature prey refuge by assessing the derivative along the non-negative coexistence equilibrium point E^* with respect to the parameter m. Since both x_{1*} and x_{2*} are the continuous functions of parameter m, we have

$$\frac{dx_{1*}}{dm} = \frac{\delta_2 n_1}{(1-m)^2\left(\beta\varphi - \delta_2\right)} > 0 \tag{13}$$

$$\frac{dx_{2*}}{dm} = \frac{\alpha}{\delta_1}\left(\frac{\delta_2 n_1}{(1-m)^2\left(\beta\varphi - \delta_2\right)}\right) > 0 \tag{14}$$

The (13) and (14) inequalities shows that increasing the amount of the immature prey refuge can increase the densities of both immature prey and mature prey. Next, since x_{3*} is a continuous function of parameter m, we have

$$\frac{dx_{3*}}{dm} = -\left(\frac{\left(r - \delta_1\right)\varphi n_1\alpha}{\delta_1(1-m)^2\left(\beta\varphi - \delta_2\right)} + \frac{2x_{3*}}{m-1}\right) < 0 \tag{15}$$

From the inequality (15) obtained $\frac{dx_{3*}}{dm} < 0$. Thus, increment amount of the constant immature prey refuge (m) can decrease the predator density. For this case, the predator population will be extinction.

3.4. Numerical Simulation

Our analytical findings are justified in this section by performing numerical simulations of the system (1). We numerically simulate the model for the following set of parameter values: $r = 0,5; k = 0,9; \alpha = 0,3; \beta = 0,2; \delta_1 = 0,51;\ \delta_2 = 0,5;\ \varphi = 0,5; n_1 = 0,5; m = 0,0051$. It can be observed that the condition (6) holds, and we find $\lambda_1 = -0,004; \lambda_2 = -0,806$ and $\lambda_3 = -0,500$. And implies that the point E_0 is locally asymptotically stable. This means that all population will go extinct. This behavior is confirmed by our numerical simulation as depicted in Figure 1. If we decrease the value of δ_1, i.e. using $\delta_1 = 0,1$ and consistently using the same parameter then condition (3) holds. The non-predator point $E_1 = (x_{11}, x_{21}, 0)$ is exist and locally asymptotically stable, where $x_{11} = 0,720$ and $x_{21} = 2,160$. The entirety of the eigenvalues of the Jacobian matrix (9) are negative, where $\lambda_1 = -1,521; \lambda_2 = -0,079$ and $\lambda_3 = -0,441$. This means that the immature and mature prey will survive in the system, while predator will go extinct. This situation is shown in Figure 2. If we decrease the value of δ_2 i.e. using $\delta_2 = 0,05$ then conditon (4) holds. The coexistence point $E_* = (x_{1*}, x_{2*}, x_{3*})$ is exists, where $x_{1*} = 0,503;\ x_{2*} = 1,508$ and $x_{3*} = 1,821$. The coefficients of a polynomial (12) are $\phi_1 = 1,4188; \phi_2 = 0,0747; \phi_3 = 0,0009$ and $\phi_* = \phi_1\phi_2 - \phi_3 = 0,1051$ so condition (10) in Theorem 3 holds. Thus coexistence point E_* is locally asymptotically stable. This means that immature and mature prey as well as predator will survive in the system. This situation is clearly shown by our numerical result in Figure 3.

To demonstrate the importance of immature prey refuge, we changed parameter value of m. If we further increase the parameter value of m such that $m = 0,5$, then the non-predator point E_1 is exist and locally asymptotically stable. The entirety of the eigenvalues of a Jacobian matrix (9) are negative, where $\lambda_1 = -1,521; \lambda_2 = -0,079$ and $\lambda_3 = -0,008$. Figure 4a shown the situation for this case. Next, Figure 4b shown the behavior of system (1) for $m = 0,9$. The point E_1 is consistently exist and locally asymptotically stable, where $\lambda_1 = -1,521; \lambda_2 = -0,079$ and $\lambda_3 = -0,037$. In both $m = 0,5$ and $m = 0,9$ condition (15) holds. This means that if the immature prey refuge increase then prey population will reach at its highest population while predator to extinction. This illustrates that the existence of immature prey refuge plays an important role on the system.

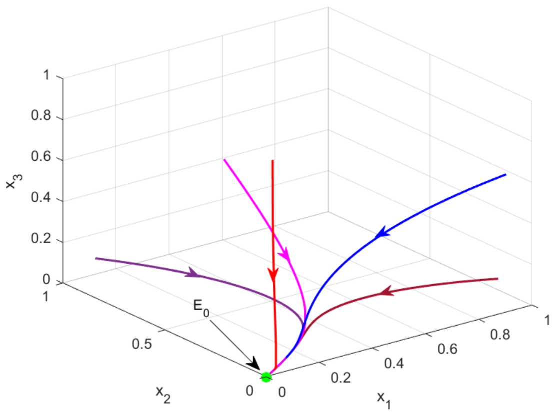


Figure 1. Phase portrait of the system (1) with $r = 0,5; k = 0,9; \alpha = 0,3; \beta = 0,2; \delta_1 = 0,51;\ \delta_2 = 0,5;\ \varphi = 0,5; n_1 = 0,5; m = 0,0051$.

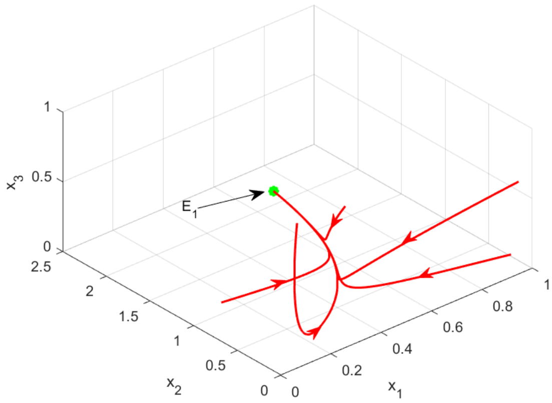


Figure 2. Phase portrait of the system (1) with $\delta_1 = 0,1$.

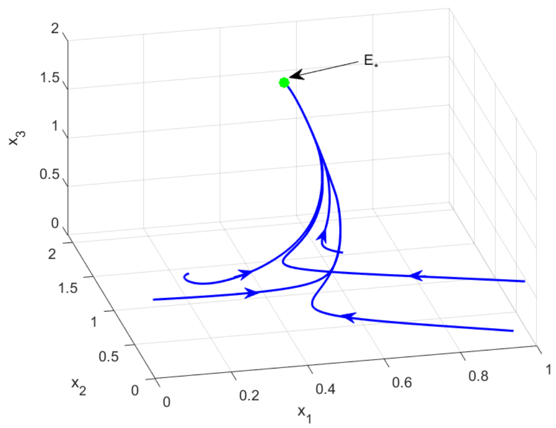
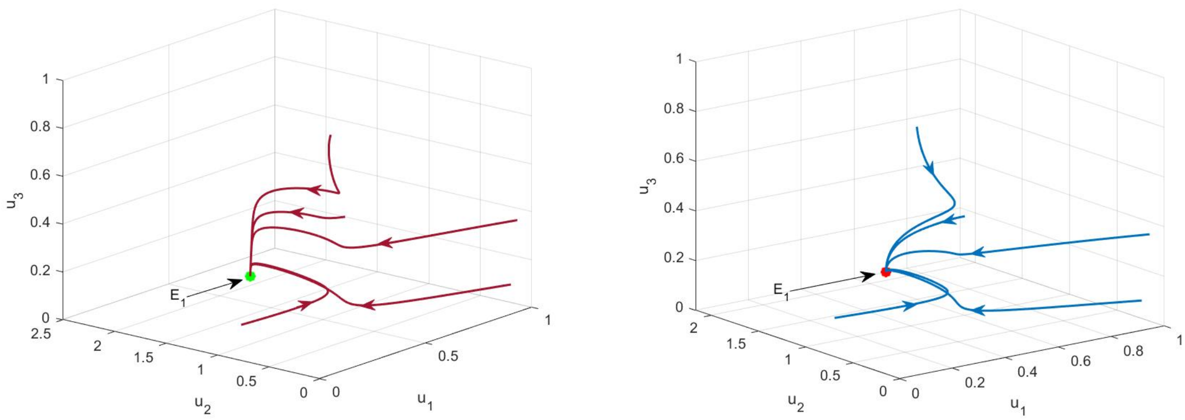


Figure 3. Phase portrait of the system (1) with $\delta_2 = 0,05$.

4. Conclusion

In this paper, a model that describes the Rosenzweig-MacArthur system having a refuge and stage-structure properties in the prey population has been studied analytically and numerically. From the analysis of the system (1), we obtain three equilibrium points namely the trivial point E_0, the non-predator point E_1 and the coexistence point E_*. E_0, E_1 and E_* are stable under certain conditions. Increasing the parameter value of the immature prey refuge may stabilize equilibrium point E_1. It is can prevent extinction on the population of the immature prey



(a) $m = 0.5$ **(b)** $m = 0.9$

Figure 4. Phase portrait of the system (1) with different values of m.

and mature prey. This is caused by very low access to the immature prey by the predator. Thus, the immature prey refuge has a significant impact on the existence of all species in the system.

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