The Aamodt and Kjus problem

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Abstract

We conjecture that an interesting special case of the known *NP*-complete problem, Multiprocessor Scheduling (MPS), is well inside *P*. We present a number of results that supports the conjecture. We also give some empirical results that strengthen our beliefs concerning the conjecture.

1 Introduction

The TV show "Hvem kan slå Aamodt og Kjus?", from now on referred to as A&K, has an interesting way of awarding points to the teams. The show consists of two teams and a series of eleven rounds, $t_1, t_2, \ldots, t_{11}$, where the winner of round t_i gets i points. Since there will be awarded at most $1+2+\ldots+11 = 66$ points, the first team that reaches more than 33 points, wins. However, a problem arises if the score is $33-33$ after the 11th round. This is bad because the show must have a winner. The easy fix is to drop the last round and just make use of 10, but we will not settle with that. We have notified the producers, giving them an infinite number of alternatives that guarantee a winner. If we generalize and let there be m teams and r rounds, and look for ties between all teams, we get a new and interesting complexity problem, A&K, as presented below.

We assume that the reader is familiar with basic concepts of complexity theory, e.g. *NP*-completeness. The standard definitions of the complexity classes *P* and *NP* are found in [1]. Furthermore, we assume that the reader has elementary knowledge about number theory, e.g. modular arithmetic. See [2]. We use $\triangle_n$ to denote the n'th triangle number, i.e. $\triangle_n = 0+1+\cdots+n = \frac{n(n+1)}{2}$. The notation $x \mid y$ reads "x divides y".

The Multiprocessor Scheduling Problem

In the Multiprocessor Scheduling problem (MPS) we are given a multiset A of natural numbers representing tasks, a number of processors $m \in \mathbb{N}$, and a deadline $D \in \mathbb{N}$. The problem is deciding whether or not it is possible to partition A into m subsets such that no subset sum is greater than D. MPS was shown to be *NP*-complete in [3]. We present MPS mainly to be able to relate A&K to a known complexity problem. There exist other *NP*-complete problems that would have served the same purpose, e.g. Bin Packing, Multi-Way Partition.

We are now ready to formally present the A&K-problem.

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2 The A&K-problem

Given a number of rounds $r \in \mathbb{N}$, and a number of teams $m \in \mathbb{N}^+$, is it possible to partition the set $S_r = \{0, 1, 2, \ldots, r\}$ into m subsets, such that each subset sums to the same number, namely $\frac{\triangle_r}{m}$?

An instance of the A&K-problem is a tuple $\langle r, m\rangle$. An instance is a *yes-instance* if it is possible to partition S_r into m subsets with equal sums, and a *no-instance* if not. We will occasionally, (not always), omit the $0 \in S_r$, as this affects the presentability of our partitions.

A&K is a special case of MPS, with $A = S_r = \{0, 1, 2, \ldots, r\}$ and $D = \frac{\sum_{a \in A} a}{m}$, forcing all subset sums to equal $\frac{\triangle_r}{m}$. There cannot exist a polynomial time reduction from A&K to MPS, because merely printing the set S_r requires exponential time with respect to $|r| = O(log(r))$. This should not lead us to believe that A&K is a hard problem; it is just a result of compact representation. As a matter of fact, we are now ready to present the main conjecture of this paper, which states that A&K is really easy.

Conjecture 1. *An instance of the A&K-problem, $\langle r, m\rangle$, is a yes-instance if and only if the following two criteria hold*

$$m \mid \triangle_r \tag{C_1}$$

$$\frac{\triangle_r}{m} \geq r\,. \tag{C_2}$$

Should this conjecture hold, then A&K $\in P$, since checking satisfiability of C_1 and C_2 can be done in polynomial time.

The conjecture is a two-way implication, saying whether a certain instance is a yes- or a no-instance. In order to prove the conjecture, we need to show that an instance is a yes-instance if the two criteria are satisfied, and a no-instance if at least one of them is not.

Conjecture proof — the trivial direction

The first criterion, C_1, is necessary because the set S_r only contains natural numbers. If $m \nmid \triangle_r$, then $\frac{\triangle_r}{m}$ is not a natural number, and no subset of S_r can add up to it. The second criterion, C_2, ensures that the element $r \in S_r$ has a subset that it fits into without overflowing it. Consider the opposite case, namely, $\frac{\triangle_r}{m} < r$. In this case, all subsets are supposed to sum to $\frac{\triangle_r}{m}$, but no matter where you place r, that subset will already sum up to more than it is supposed to.

Note that this direction holds for all instances of MPS with $D = \frac{\sum_{a \in A} a}{m}$, by the same arguments as above. That is, the longest task must not exceed the deadline, and the sum of all tasks must be divisible by the number of processors. With this observation, we leave the trivial direction.

On proving the non-trivial direction

The rest of this paper is devoted to the proof of the non-trivial direction of the conjecture. We need to show that there exists a satisfying partition for all instances where both C_1 and C_2 hold. We will call these instances *conjectured yes-instances*. Giving a satisfying partition for a conjectured yes-instance will be referred to as *proving* an instance.

We describe the general plan for proving the statement for different values of m, by giving an example proof. The main idea is that we fix m, and prove the conjectured yes-instances by induction on r.

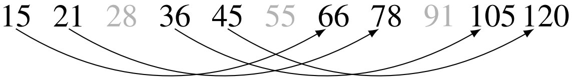
Example proof. Here we will prove the conjecture when $m = 3$. The proof where $m = 2$ is slightly simpler and less instructive and is left to the reader. We need to prove that the cases $\langle r, 3\rangle$, where $3 \mid \triangle_r$ and $\frac{\triangle_r}{3} \geq r$, actually are yes-instances.

Straightforward arithmetic reveals that $\frac{\triangle_r}{3} \geq r \Leftrightarrow r \geq 5$. Thus C_2 holds for all $r \geq 5$. So when does 3 divide $\triangle_r$? Well, this is the case if and only if one of the following holds

- $r \equiv 2 \pmod 3$
- $r \equiv 0 \pmod 3$.

Observe that C_1 holds at a regular basis. It does so for all m, as we will prove later. The following figure illustrates how we will use induction on r to complete the proof. The dotted line marks the point from where C_2 is satisfied. The black cells on the bottom line are the conjectured yes-instances. We will show that all these conjectured yes-instances are in fact yes-instances via induction on r. There are four base cases, namely, $\langle 5,3\rangle$,$\langle 6,3\rangle$,$\langle 8,3\rangle$ and $\langle 9,3\rangle$ (the definition of a base case can be found in Section 4). Here is how the partitions could look in the base cases:

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\triangle_r$	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120



- when $r = 5$ we have the partition $\{\{5,0\},\{1,4\},\{2,3\}\}$
- when $r = 6$ we have the partition $\{\{6,1\},\{2,5\},\{3,4\}\}$
- when $r = 8$ we have the partition $\{\{8,1,3\},\{7,0,5\},\{6,2,4\}\}$
- when $r = 9$ we have the partition $\{\{9,2,4\},\{8,1,6\},\{7,3,5\}\}$

Now we are left with the induction step. We will show that if we could partition S_r for some r, we can also do it for $r + 6$. Assume that we have found a correct partition $\{p_0, p_1, p_2\}$ for some r. We know that each p_i sums to $\frac{\triangle_r}{3}$. Then we extend this partition to be correct for $r + 6$ like this:

$$\begin{aligned}\{p_0 \cup \{r+1, r+6\},\\ p_1 \cup \{r+2, r+5\},\\ p_2 \cup \{r+3, r+4\}\}.\end{aligned}$$

This is a correct extension of our partition from r to $r + 6$. All new elements, $r, r+1, r+2, \ldots, r+6$, are used exactly one time each, and each subset now sums to $2r + 7$ more than they used to. Now we have covered all conjectured yes-instances, thus the conjecture holds for $m = 3$. □

3 Induction step and investigation of the two criteria

The Example proof consisted of four minor steps. Completing those four steps, for all m, is sufficient to prove the conjecture.

1. Show that C_2 is satisfied when r is greater than some limit
2. Show when C_1 is satisfied, and prove that this happens at a regular basis
3. Make actual partitions for all base cases
4. Show how the induction step can be carried out

In this section, we will completely cover steps 4, 2 and 1, (in that order), for all m.

The induction step

The construction we present in this subsection will be used as the induction step for all m.

Assume that we are proving the conjecture for some particular m, and that we have made a correct partition, $\{p_0, p_1, \ldots, p_{m-1}\}$, for some r. Then we can always extend this partition by $2m$, to hold for $r+2m$ in the following fashion

$$\begin{aligned} \{p_0 &\cup \{r+1, r+2m\} \\ p_1 &\cup \{r+2, r+(2m-1)\} \\ &\vdots \\ p_{m-1} &\cup \{r+m, r+(m+1)\}\}. \end{aligned}$$

Now each subset has increased by $2r+2m+1$ and all numbers between r and $r+2m$ has been used exactly once. Extending a partition by $2m$ like this will occasionally be referred to as *jumping*.

The first criterion, C_1

This first theorem states that C_1 holds at a regular basis. The pattern in which C_1 holds repeats every $2m$ steps as r grows larger. The theorem also states that when the fixed m is odd, the pattern repeats every m steps.

Theorem 1. *For any $r \in \mathbb{N}$ and $m \in \mathbb{N}^+$ we have*

$$m \mid \triangle_r \quad \Leftrightarrow \quad m \mid \triangle_{r+2m} \, .$$

Furthermore, if m is an odd number, the following holds

$$m \mid \triangle_r \quad \Leftrightarrow \quad m \mid \triangle_{r+m} \, .$$

Proof. Assume $m \mid \triangle_r$. We have

$$\triangle_{r+2m} = \triangle_r + (r+1) + (r+2) + \cdots + (r+2m) = \triangle_r + \triangle_{2m} + 2rm \, . \quad (1)$$

Since

$$\triangle_{2m} = \frac{2m(2m+1)}{2} = m(2m+1) \, ,$$

all summands that make up $\triangle_{r+2m}$ are divisible by m.

Now assume $m \mid \triangle_{r+2m}$. By (1) , $m \mid \triangle_r + \triangle_{2m} + 2rm$. Since $m \mid \triangle_{2m}$ and $m \mid 2rm$ we have that $m \mid \triangle_r$.

Furthermore, we have that

$$\triangle_{r+m} = \triangle_r + (r+1) + (r+2) + \cdots + (r+m) = \triangle_r + \triangle_m + rm \, .$$

Now, the equivalence $m \mid \triangle_r \;\Leftrightarrow\; m \mid \triangle_{r+m}$ holds iff $m \mid \frac{m(m+1)}{2}$ which is exactly when m is odd. □

The following theorem states exactly when C_1 holds for different values of r and m.

Theorem 2. *Assume m is of the form $P_0^{k_0}P_1^{k_1}...P_n^{k_n}$, where P_i is the i'th prime, (in particular $P_0 = 2$), P_n is the greatest prime that divides m and $k_i \in \mathbb{N}$.*[1] *We have that $m \mid \triangle_r$ if and only if one of the following holds for all $i \in \{1,\dots,n\}$*

- $r \equiv -1 \pmod{P_i^{k_i}}$
- $r \equiv 0 \pmod{P_i^{k_i}}$

and one of the following holds

- $r \equiv -1 \pmod{2^{k_0+1}}$
- $r \equiv 0 \pmod{2^{k_0+1}}$.

Before we prove the theorem we introduce a few lemmas.

Lemma 1. *For all $r,n \in \mathbb{N}$, we have*

$$\triangle_{(n+1)r} = \triangle_{nr} + nr^2 + \triangle_r \ .$$

Proof. We have

$$\triangle_{(n+1)r} = \triangle_{nr} + (nr+1) + (nr+2) + ... + (nr+r) =$$

$$\triangle_{nr} + \underbrace{nr + nr + ... + nr}_{r} + \underbrace{1+2+3+...+r}_{r} = \triangle_{nr} + nr^2 + \triangle_r \ .$$

□

Lemma 2. *Let $P^k \in \mathbb{N}$, where P is a prime strictly greater than 2, and $k \in \mathbb{N}$. We have $P^k \mid \triangle_r$ if and only if one of the following holds*

- $r \equiv -1 \pmod{P^k}$
- $r \equiv 0 \pmod{P^k}$.

Proof. We first prove the if-part of the lemma. Assume that either $r \equiv -1 \pmod{P^k}$ or $r \equiv 0 \pmod{P^k}$ holds. Then r is of the form $aP^k - 1$ or aP^k, where $a \in \mathbb{N}^+$. We show both cases by induction over a, starting with the base cases (where $a = 1$).

Case (A) $r = P^k - 1$**.** Here $\triangle_r = \triangle_{P^k-1} = \frac{(P^k-1)P^k}{2}$ which is divisible by P^k.

Case (B) $r = P^k$**.** Here $\triangle_r = \triangle_{P^k} = \frac{P^k(P^k+1)}{2}$ which is divisible by P^k.

We will only continue the induction for case (B), and later show that the statement holds in case (A).

Assume by the induction hypothesis that $P^k \mid \triangle_{aP^k}$. We now need to prove that $P^k \mid \triangle_{(a+1)P^k}$.

We have $\triangle_{(a+1)P^k} = \triangle_{aP^k} + aP^{k^2} + \triangle_{P^k}$ by Lemma 1. All summands are divisible by P^k. The first, by the induction hypothesis, and the third by the base case. To conclude the if-part of the proof, observe that the statement also holds in case (A) because $\triangle_{(a+1)P^k-1} = \triangle_{(a+1)P^k} - (a+1)P^k$ and everything is divisible by P^k.

For the other direction, assume $P^k \mid \triangle_r$. Then $P^k \mid \frac{r(r+1)}{2}$, which implies that $P^k \mid r(r+1)$. Since P^k only has one factor, namely P, and r and $r+1$ shares no factors, exactly one of the following holds:

[1]All natural numbers can be written in this form, and by the Fundamental Theorem of Arithmetic the factorization is unique.

- $P^k \mid r$
- $P^k \mid r+1$.

Now, if $P^k \mid r$, then the lemma holds because $r \equiv 0 \pmod{P^k}$. If $P^k \mid r+1$, then $r+1 = qP^k$ for some $q \in \mathbb{N}$. Thus $r = qP^k - 1$, and the lemma holds, because $r \equiv -1 \pmod{P^k}$. $\square$

Lemma 3. *Let $k \in \mathbb{N}$. We have $2^k \mid \triangle_r$ if and only if one of the following holds*

- $r \equiv -1 \pmod{2^{k+1}}$
- $r \equiv 0 \pmod{2^{k+1}}$.

Proof. This proof is very similar to the proof of Lemma 2. The only real difference is in the only-if direction after we assume $2^k \mid \triangle_r$. Now $2^k \mid \frac{r(r+1)}{2}$, which implies that $2^{k+1} \mid r(r+1)$. We leave the rest to the reader. $\square$

Proof of Theorem 2. Assume m is of the form $P_0^{k_0} P_1^{k_1} \ldots P_n^{k_n}$, where P_i is the i'th prime and P_n is the greatest prime that divides m and $k_i \in \mathbb{N}$. Furthermore, assume that $m \mid \triangle_r$. We see that $m \mid \triangle_r$ if and only if $P_i^{k_i} \mid \triangle_r$ for all $i \leq n$. This is equivalent to the fact that $2^{k_0} \mid \triangle_r$ *and* $P_i^{k_i} \mid \triangle_r$ for all $i \in \{1, \ldots, n\}$. By Lemma 2 we know that $P_i^{k_i} \mid \triangle_r$ if and only if one of the following holds for all $i \in \{1, \ldots, n\}$

- $r \equiv -1 \pmod{P_i^{k_i}}$
- $r \equiv 0 \pmod{P_i^{k_i}}$.

Finally, by Lemma 3 we get that $2^{k_0} \mid \triangle_r$ if and only if one of the following holds

- $r \equiv -1 \pmod{2^{k_0+1}}$
- $r \equiv 0 \pmod{2^{k_0+1}}$.

The assumption has been shown equivalent to the conclusion, and the theorem has been proven. $\square$

Please note that if $k_0 = 0$, i.e. m is an odd number, the second requirement of Theorem 2 is always fulfilled and can be dropped.

The second criterion, C_2

C_2 was presented in a way that made it easy to see why we needed it. Now we state it in a more useful way, with respect to proving the conjecture, just like we did with C_1 above. Stating it like this also grants some useful insight for free, as discussed below the proof.

Theorem 3. *For all $r, m \in \mathbb{N}^+$ we have*

$$\frac{\triangle_r}{m} \geq r \quad \Leftrightarrow \quad r \geq 2m - 1.$$

Proof. We have

$$\begin{aligned} \frac{\triangle_r}{m} \geq r \quad &\Leftrightarrow \quad \frac{\frac{r(r+1)}{2}}{m} \geq r \\ &\Leftrightarrow \quad r(r+1) \geq 2mr \\ &\Leftrightarrow \quad r \geq 2m - 1 \, . \end{aligned}$$

$\square$

Now we can show that the smallest conjectured yes-instance for any m, is always $\langle 2m-1, m\rangle$. This is because when $r \geq 2m-1$, C_2 is satisfied. Furthermore, C_1 holds because, $m \mid \triangle_{2m-1}$, since $\triangle_{2m-1} = \frac{(2m-1)2m}{2} = (2m-1)m$. Moreover $\langle 2m, m\rangle$ is also a conjectured yes-instance, since $\triangle_{2m} = \triangle_{2m-1} + 2m$, which is divisible by m, satisfying C_1. The second criterion, C_2, is also satisfied since $2m-1 < 2m$.

4 The base cases

Remember that we always fix the value of m in our proofs, just like we did in the example proof where m was equal to 3. The construction we presented in the subsection concerning the induction step goes hand in hand with Theorem 1. The induction step lets us jump $2m$ steps forwards, so if we jump from a yes-instance, i.e. a proven conjectured yes-instance, we must hit a new conjectured yes-instance. This is because the new partition is correct and the conjecture holds in the trivial direction. Theorem 1 states that C_1 holds at a regular basis, creating a pattern that repeats within $2m$ steps. We know from the discussion after Theorem 3 that $\langle 2m-1, m\rangle$ is the smallest conjectured yes-instance, so if we prove that instance, we have proven every $2m$'th instance from that point on, by induction. In particular, we will have proven the instance $\langle 4m-1, m\rangle$. Thus, proving every conjectured yes-instance between $\langle 2m-1, m\rangle$ and $\langle 4m-1, m\rangle$ is sufficient in order to prove the conjecture for that particular m. This is because there can not exist any conjectured yes-instances before $\langle 2m-1, m\rangle$, and an arbitrary conjectured yes-instance after $\langle 4m-1, m\rangle$, let us call it $\langle R+4m-1, m\rangle$ can be reached by jumping from $\langle R+2m-1, m\rangle$. These first instances are called base cases.

Definition 1 (Base case). *An instance of the A&K-problem, $\langle r, m\rangle$, is a base case for m if the following two holds*

- $2m-1 \leq r \leq 4m-2$
- $\langle r, m\rangle$ *is a conjectured yes-instance*

With this definition and the discussion above, we have shown that by extending the base cases we hit all conjectured yes-instances and we hit conjectured yes-instances exclusively. As a result of this, all that remains is to find a correct partition (or at least prove that such a partition exists), for all base cases.

When m only has one factor

At this point, we are able to prove a special case of the conjecture.

Theorem 4. *Let $\langle r, m\rangle$ be an instance of A&K where m is of the form P^k, where P is a prime number and $k \in \mathbb{N}$. Then $\langle r, m\rangle$ is a yes-instance if and only if the following two criteria holds*

$$m \mid \triangle_r$$

$$\frac{\triangle_r}{m} \geq r\,.$$

Proof. By the considerations above, it is sufficient to give a correct partition for all base cases. We split the proof into two parts.

First, consider the case where $m = 2^k$. The only base cases are $\langle 2^{k+1}-1, 2^k\rangle$ and $\langle 2^{k+1}, 2^k\rangle$, since they are the only conjectured yes-instances where $2m-1 \leq r \leq 4m-2$. For these two base cases we have the partitions:

Case $r = 2^{k+1} - 1$: $\{\{r,0\},\{r-1,1\},\{r-2,2\},\dots\{2^k,2^k-1\}\}$

Case $r = 2^{k+1}$: $\{\{r,1\},\{r-1,2\},\{r-2,3\},\dots\{2^k+1,2^k\}\}$

As discussed after the proof of Theorem 3, $\langle 2m-1,m\rangle$ and $\langle 2m,m\rangle$ are always conjectured yes-instances. It is not hard to see that the partitions above work for any m. This shows that the instances $\langle 2m-1,m\rangle$ and $\langle 2m,m\rangle$ in fact are yes-instances, for all m.[2]

Now, consider the case where $m = P^k$ and P is a prime strictly greater than 2. Since m now is odd, we know from Theorem 1 that the pattern in which the conjectured yes-instances appear, repeats every m steps. Thus if $\langle 2m-1,m\rangle$ and $\langle 2m,m\rangle$ are base cases then there must be base cases at $\langle 3m-1,m\rangle$ and $\langle 3m,m\rangle$ too. Our base cases are $\langle 2P^k-1,P^k\rangle$,$\langle 2P^k,P^k\rangle$, $\langle 3P^k-1,P^k\rangle$ and $\langle 3P^k,P^k\rangle$. This can also be directly read from Theorem 2. The cases $\langle 2P^k-1,P^k\rangle$ and $\langle 2P^k,P^k\rangle$ were proven above.

For the remaining two cases, $\langle 3P^k-1,P^k\rangle$ and $\langle 3P^k,P^k\rangle$ we present an algorithm for filling in the subsets. See figure 1. Below we prove that the algorithm not only works for the instances $\langle 3P^k-1,P^k\rangle$ and $\langle 3P^k,P^k\rangle$, but for any instance $\langle 3m-1,m\rangle$ or $\langle 3m,m\rangle$, where m is an odd number.

To better understand how the algorithm works, we imagine that we place the subsets underneath each other with p_0 at the top and p_{m-1} at the bottom. We only prove that the algorithm works for the base case $\langle 3P^k-1,P^k\rangle$. The other case is similar. Since $r = 3m-1$ we know that we started out (the algorithm) with the set $S_{3m-1} = \{0,1,2,\dots,3m-1\}$. In step 1, we distributed the m biggest elements of S_r, such that the set $p_i = \{r-i\}$. In step 2, we distributed the m smallest elements such that each p_i contains exactly two elements. Figure 2a illustrates how the sets look, after step 2, in the base case $\langle 26,9\rangle$. Since we have used the m biggest and the m smallest elements, we are left with the set $\{m,m+1,\dots,r-m\} = \{m,m+1,\dots,2m-1\}$, which has cardinality m. Since we also have m unfinished subsets, we know that we can fit exactly one element in each subset. We want to show that you can fit all remaining elements into the subsets (one in each), so that all subsets sum to $\frac{\triangle_r}{m}$.

The subsets from p_0 down to $p_{\frac{m-1}{2}}$ are from now on called *part A*. The remaining subsets, those from $p_{\frac{m+1}{2}}$ to p_{m-1} we call *part B*. Note that part A contains one more subset than part B, as m is an odd number.

It is clear that after step 2, $p_{\frac{m-1}{2}}$ has the lowest subset sum in part A. Both two elements of each subset in part A, are one bigger than those of the subset below it, and one smaller than those of the subset above. This means that starting at $p_{\frac{m-1}{2}}$ going upwards, the subset sums increase by two at every step until you reach p_0, which has the highest subset sum in part A. The same holds for part B, with p_{m-1} having the smallest subset sum and $p_{\frac{m+1}{2}}$ having the biggest.

Now we show that the subset sum of $p_{\frac{m-1}{2}}$ is one lower than the subset sum of p_{m-1}. Furthermore, we show that $p_{\frac{m-1}{2}}$ needs exactly the biggest remaining element, $2m-1$, to sum to $\frac{\triangle_r}{m}$. Since the parts A and B have the properties explained above, we can then zig-zag between the parts filling in all the remaining elements. See figure 2b. The subset $p_{\frac{m-1}{2}} = \{r-\frac{m-1}{2},0\}$, and the subset sum is $(3m-1)-\frac{m-1}{2}+0 = \frac{5m-1}{2}$. And $p_{m-1} = \{r-(m-1),\frac{m+1}{2}\}$, and the subset sum is $(3m-1)-(m-1)+\frac{m+1}{2} = \frac{5m+1}{2}$. We subtract the subset sum of $p_{\frac{m-1}{2}}$ from $\frac{\triangle_r}{m}$ to check what (element) the subset lacks, and

[2] Thus, the instance < 11,2 > is a yes-instance, which means it is a poor choice of an instance to base your TV show on, if you want to avoid draws.

Figure 1: Algorithm 1

0. Create m empty subsets named $p_0, p_1, \ldots, p_{m-1}$.

1. Distribute the m biggest numbers in the subsets, in the following fashion

    for $i \in \{0, 1, \ldots, m-1\}$ **do**
        $p_i = p_i \cup \{r - i\}$
    end for

2. Fill in the m smallest elements in the subsets, in the following fashion

    for $i \in \{0, 1, \ldots, m-1\}$ **do**
        $k = \frac{m-1}{2} - i \pmod{m}$ // $k \in \mathbb{N}$ since m is odd
        if $r = 3m - 1$ **then**
            $p_k = p_k \cup \{i\}$
        else // $r = 3m$
            $p_k = p_k \cup \{i + 1\}$
        end if
    end for

3. Fill in the rest of the numbers in the correct subset

    for $i \in \{0, 1, \ldots, m-1\}$ **do**
        if i is even **then**
            $k = \frac{m-1}{2} - \frac{i}{2}$
        else // i is odd
            $k = m - \frac{i+1}{2}$
        end if
        $p_k = p_k \cup \{2m - 1 - i\}$
    end for

Figure 2: Different stages of Algorithm 1 with the input $\langle 26, 9 \rangle$

(a) After step 2

$$
\begin{aligned}
\{p_0 &= \{26, 4\}, \\
p_1 &= \{25, 3\}, \\
p_2 &= \{24, 2\}, \\
p_3 &= \{23, 1\}, \\
p_4 &= \{22, 0\}, \\
p_5 &= \{21, 8\}, \\
p_6 &= \{20, 7\}, \\
p_7 &= \{19, 6\}, \\
p_8 &= \{18, 5\}\}
\end{aligned}
$$

(b) After step 3

$$
\begin{aligned}
\{p_0 &= \{26, 4, 9\}, \\
p_1 &= \{25, 3, 11\}, \\
p_2 &= \{24, 2, 13\}, \\
p_3 &= \{23, 1, 15\}, \\
p_4 &= \{22, 0, 17\}, \\
p_5 &= \{21, 8, 10\}, \\
p_6 &= \{20, 7, 12\}, \\
p_7 &= \{19, 6, 14\}, \\
p_8 &= \{18, 5, 16\}\}
\end{aligned}
$$

get $\frac{\triangle_r}{m} - \frac{5m-1}{2} = 2m-1$. Now since $\frac{5m-1}{2}$ is one less than $\frac{5m+1}{2}$ we have shown that Algorithm 1 always produces the desired partition. Thus we have proven Theorem 4. □

As discussed above, we have that $\langle 3m-1, m\rangle$ and $\langle 3m, m\rangle$ are base cases whenever m is odd. Now since Algorithm 1 only requires m to be odd to work, we have actually proven the base cases $\langle 3m-1, m\rangle$ and $\langle 3m, m\rangle$ for any odd m.

5 Reducing an instance

Here we present an alternative approach to prove the base cases. Instead of building the partitions bottom up, we reduce the problem of proving a base case $\langle r, m\rangle$, to the problem of proving the instance $\langle r', m'\rangle$, where $r' < r$ and $m' < m$. The next lemma is needed to show that the Greedy algorithm, presented in figure 3, works.

Lemma 4. *If $\langle r, m\rangle$ is a base case, then $\frac{\triangle_r}{m} < 2r$.*

Proof. Since $\langle r, m\rangle$ is a base case it follows that $r < 4m-1$. We leave the actual calculations to the reader, and just claim that

$$r < 4m-1 \Leftrightarrow \frac{\triangle_r}{m} < 2r\ .$$

□

Assume that we want to make a partition for the base case $\langle r, m\rangle$, and let us assume that $\frac{\triangle_r}{m}$ is an odd number. Then we can reduce the instance $\langle r, m\rangle$, to a smaller instance, by the Greedy algorithm, seen in figure 3.

Figure 3: Greedy algorithm

```
e_1 = r
e_2 = △_r/m − r    // △_r/m − r is smaller than r, by Lemma 4, since r + (△_r/m − r) = △_r/m
i = 0
while e_1 − e_2 ≥ 1 do   //the elements will be adjacent at some point since △_r/m is odd
    p_i = {e_1, e_2}
    e_1 = e_1 − 1
    e_2 = e_2 + 1
    i = i + 1
end while
```

The algorithm first fills in r and $\frac{\triangle_r}{m} - r$ in the first subset. As long as the two numbers that have just been added are not adjacent, the algorithm decreases the biggest by one and increases the smallest by one, and places them in the next subset. The algorithm terminates when $e_1 < e_2$.

We have now made $\frac{r+1-(\frac{\triangle_r}{m}-r)}{2}$ subsets, each summing to $\frac{\triangle_r}{m}$. This can be understood by observing that the algorithm will terminate when $e_1 < e_2$. This will be halfway between the difference of r and $\frac{\triangle_r}{m} - r$.

We have also used each number between r and $\frac{\triangle_r}{m} - r$ exactly once. This means we have reduced the problem to solving the instance

$$\left\langle \frac{\triangle_r}{m} - r - 1,\ m - \left(\frac{r+1-(\frac{\triangle_r}{m}-r)}{2}\right)\right\rangle = \langle r', m'\rangle\ .$$

Now all remaining subsets still need to sum to $\frac{\triangle_r}{m}$, which means that $\frac{\triangle_r}{m} = \frac{\triangle_{r'}}{m'}$. It now follows that $m' \mid \triangle_{r'}$ and $\frac{\triangle_{r'}}{m'} \geq r'$ since $r > r'$. Since both criteria are satisfied, $\langle r', m' \rangle$ is a conjectured yes-instance. The instance $\langle r', m' \rangle$ is not necessarily, (probably never), a base case for m'. However, since we know that all conjectured yes-instances are covered by extending all base cases, there must exist a base case $\langle r'', m' \rangle$, that can be extended to prove $\langle r', m' \rangle$. The problem now is that $\frac{\triangle_{r''}}{m'}$ is not necessarily an odd number, so we can not further reduce this base case by using the Greedy algorithm. Note that the Greedy Algorithm is actually the one used to prove the base cases $\langle 2m-1, m \rangle$ and $\langle 2m, m \rangle$. We present an example that illustrates how the Greedy algorithm is used, and also shows its limitation, that is, when $\frac{\triangle_{r''}}{m'}$ is even.

Example $\langle 2869, 779 \rangle$

The instance $\langle 2869, 779 \rangle$ is a base case for 779 since

- $2m-1 \leq r \leq 4m-2$, since $r = 3m+532$.
- $\langle 2869, 779 \rangle$ is a conjectured yes-instance, since $\frac{\triangle_{2869}}{779} = 5285$.

Furthermore, $\frac{\triangle_{2869}}{779} = 5285$ is an odd number, so we can run the Greedy algorithm on this instance.

$$\begin{aligned} p_0 &= \{2869, 2416\}, \\ p_1 &= \{2868, 2417\}, \\ &\vdots \\ p_{226} &= \{2643, 2642\} \end{aligned}$$

Now we have made 227 subsets each summing to 5285, and we have used all numbers between 2869 and 2416 exactly once. We are left with the case $\langle 2415, 552 \rangle$, which originates, (can be extended), from the base case $\langle 1311, 552 \rangle$. But, $\frac{\triangle_{1311}}{552} = 1558$, which is an even number, and we cannot run the Greedy algorithm to reduce the instance further.

If we had an algorithm for reducing base cases where $\frac{\triangle_r}{m}$ is even, we would be finished with the whole proof of the conjecture.

6 Empirical results

In this section, we have both good and bad news. We present the bad news first.

We have made a Python program that prints out the number of base cases for different m. Based on the outputs of the program, we believe that the number of base cases for a given m is 2^{d+1}, where

$$d = |\{P \,:\, P \text{ is prime}, \; P > 2 \text{ and } P \mid m\}| \;.$$

If this is true, there is no limit to the number of base cases, as m may have an arbitrary number of prime factors. We have confirmed that 2^{d+1} equals the number of base cases, for all m up to 16000.

On the bright side, we have made another program, that brute forces A&K for different values of m. The program looks for satisfying partitions for all base cases where $\frac{\triangle_r}{m}$ is even, for all m with more than one unique prime factor. So far we have proven the conjecture for all m up to 129. This may seem like a modest number, but remember that for now, we have no better algorithms than those used to solve *NP*-complete problems.

7 Conclusions and future work

We have proven the conjecture for all m of the form P^k, where P is any prime. For all other m, we know that we can do induction over r, and we are done with the induction step. We also know exactly where the base cases we need to prove are. What remains is to make partitions for the base cases, or prove that such partitions exist.

We could find a way to construct the partitions for the base cases for any composite m, based on the partitions of the prime factors of m. This was our initial plan, but so far our attempts have been fruitless.

As suggested after the example above, it is sufficient to find an algorithm that reduces a base case where $\frac{\triangle_r}{m}$ is even, to a smaller instance. Then we could reduce any base case until we reach an instance proved earlier.

Remember that we do not actually need to construct the partitions at all. It suffices to prove that they exist. This may be simpler than giving a constructive proof, but it is not inconceivable that a constructive proof is needed in order to prove the conjecture.

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