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## HYPERBOLIC TYPE SOLUTIONS FOR THE COUPLE BOITI-LEON-PEMPINELLI SYSTEM

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**Abstract.** In this paper, the  $(1/G')$ -expansion method is proposed to construct hyperbolic type solutions of the nonlinear evolution equations. To assess the applicability and effectiveness of the method, two cases of the coupled Boiti-Leon-Pempinelli (CBLP) system have been investigated in this study. It is shown that, with the help of symbolic computation, the  $(1/G')$ -expansion method provides a powerful and straightforward mathematical tool for solving nonlinear partial differential equations.

**Keywords:** nonlinear evolution equations; partial differential equations; symbolic computation.

### 1. Introduction

Nonlinear evolution equations usually used to describe the nonlinear phenomena of waves in plasma physics, ocean engineering, quantum mechanics, fluid dynamics, solid state physics, hydrodynamics and many other branches of sciences and engineering. These types of equations have been used to describe the liquid flow containing gas bubbles, the propagation of waves, fluid flow in elastic oceans, rivers, tubes, lakes as well as a gravity waves in a smaller domain and Spatio-temporal rescaling of the nonlinear wave motion.

There are several approaches for finding solutions of nonlinear partial differential equations which have been developed and employed successfully. Some of these are a new sub equation method [1], homotopy analysis method [2, 3], homotopy-Pade method [4], homotopy perturbation method [5, 6],  $(G'/G)$ -expansion method [7, 8], modified variational iteration algorithm-I [9, 10, 11], sub equation method [12], Variational iteration method with an auxiliary parameter [13, 14, 15, 16], sumudu transform approach [17],  $(1/G')$ -expansion method [18, 19], variational iteration method [20, 21], auto-Bäcklund transformation method [22], Clarkson–Kruskal direct method [23], Bernoulli sub-equation function technique [24], decomposition

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method [25, 26, 27, 28], modified variational iteration algorithm-II [29, 30, 31], first integral method [32], homogeneous balance method [33], modified Kudryashov technique [34], residual power series approach [35], collocation method [36], extended rational SGEEM [37], sine-Gordon expansion method [38, 39] and many more [40, 41, 42, 43].

Consider the following coupled Boiti-Leon-Pempinelli System [44]

$$(1.1) \quad \begin{aligned} u_{ty} &= (u^2 - u_x)_{xy} + 2v_{xxx}, \\ v_t &= v_{xx} + 2uv_x. \end{aligned}$$

There have been numerous studies about the analytical treatment of CBLP System. In some of the studies, new traveling wave solutions of CBLP System have been attained utilizing the generalized  $(G'/G)$ -expansion method [44], while the analytic solutions of CBLP System have been obtained in [45].

In current work, we will construct the exact solutions of the CBLP System employing  $(1/G')$ -expansion method.

The remaining portion of this paper is as follows: In section 2,  $(1/G')$ -expansion method is elaborated, in section 3,  $(1/G')$ -expansion method's applications are discussed and utilized to obtain hyperbolic type solutions of the CBLP System, applicability and reliability of the proposed techniques are shown through 3D, contour and 2D graphics. The conclusion is discussed in the last section.

## 2. Description of the Method

Consider a general form of the following nonlinear PDE,

$$(2.1) \quad \sigma \left( u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \dots \right) = 0.$$

Here, let  $u = u(\xi) = u(x, y, t)$ ,  $\xi = x + y - ct$ ,  $c \neq 0$ , where  $c$  is a constant and the speed of the wave. We can convert it into the following nODE for  $u(\xi)$

$$(2.2) \quad \tau(u, -cu', u', u', u'', \dots) = 0.$$

The solution of Eq. (2.2) is assumed to have the form

$$(2.3) \quad u(\xi) = a_0 + \sum_{i=1}^n a_i \left( \frac{1}{G'} \right)^i,$$

whereas  $a_i$ ,  $i = 0, 1, \dots, n$  are nonzero constants,  $G = G(\xi)$  provides the following second order IODE

$$(2.4) \quad G'' + \lambda G' + \mu = 0,$$

where  $\mu$  and  $\lambda$  are constants to be determined after,

$$(2.5) \quad \frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + B \cosh[\xi\lambda] - B \sinh[\xi\lambda]},$$

where  $B$  is integral constant. If the desired derivatives of the Eq. (2.3) are calculated and substituting in the Eq. (2.2), a polynomial with the argument  $(1/G')$  is attained. An algebraic equation system is created by equalizing the coefficients of this polynomial to zero. The equation are solved using a package program and put into place in the default Eq. (2.2) solution function. Lastly, the solutions of Eq. (1.1) are found.

### 3. Solutions of CBLP System

The traveling wave transmutation  $\xi = x + y - ct$ , allows us to convert Eq. (1.1) into an ODE for  $u = u(\xi)$

$$(3.1) \quad -cu'' = (u^2 - u')'' + 2v,$$

$$(3.2) \quad -cv' = v'' + 2uv',$$

here by integrating twice the Eq. (3.1), we attain

$$(3.3) \quad v' = \frac{1}{2}u' - \frac{1}{2}cu - \frac{1}{2}u^2.$$

According to  $\xi$  in Eq. (3.3) and considering zero constants for integration, we attain

$$(3.4) \quad v = \frac{1}{2}u - \frac{1}{2} \int (cu + u^2) d\xi.$$

Replacing Eq. (3.3) into the Eq. (3.2),

$$(3.5) \quad u'' - 2u^3 - 3cu^2 - c^2u = 0.$$

In Eq. (3.5), we get balancing term  $n = 1$  and in Eq.(2.3), the following situation is obtained:

$$(3.6) \quad u(\xi) = a_0 + a_1 \left( \frac{1}{G'} \right), \quad a_1 \neq 0.$$

Replacing Eq. (3.6) into Eq. (3.5) and the coefficients of the algebraic Eq. (1.1) are equal to zero, can find the following algebraic equation systems

$$(3.7) \quad \begin{aligned} Const & : -c^2a_0 - 3ca_0^2 - 2a_0^3 = 0, \\ \left( \frac{1}{G'[\xi]} \right)^1 & : -c^2a_1 + \lambda^2a_1 - 6ca_0a_1 - 6a_0^2a_1 = 0, \\ \left( \frac{1}{G'[\xi]} \right)^2 & : 3\lambda\mu a_1 - 3ca_1^2 - 6a_0a_1^2 = 0, \\ \left( \frac{1}{G'[\xi]} \right)^3 & : 2\mu^2a_1 - 2a_1^3 = 0. \end{aligned}$$

**Case1:**

$$(3.8) \quad a_0 = 0, \quad a_1 = -\mu, \quad c = -\lambda,$$

replacing Eq.(3.8) into the Eq.(3.6) and the following hyperbolic type solutions is obtained for Eq. (1.1):

$$(3.9) \quad u_1(x, y, t) = -\frac{\mu}{-\frac{\mu}{\lambda} + B \cosh[\lambda(t\lambda + x + y)] - B \sinh[\lambda(t\lambda + x + y)]},$$

$$(3.10) \quad v_1(x, y, t) = \frac{1}{4\mu}(\lambda\mu \left( \lambda(t\lambda + x + y) + 2 \log \left[ \begin{array}{l} (-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right] \\ + (B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right] \end{array} \right] \right) - \lambda\mu(\lambda(t\lambda + x + y) + 2 \log \left[ \begin{array}{l} (-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right] \\ + (B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right] \end{array} \right] - \frac{4B\lambda\mu \sinh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right]}{(A\lambda - \mu) \left( \begin{array}{l} (-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right] \\ + (B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(t\lambda + x + y) \right] \end{array} \right)}) - \frac{\mu}{2(-\frac{\mu}{\lambda} + B \cosh[\lambda(t\lambda + x + y)] - B \sinh[\lambda(t\lambda + x + y)])}.$$

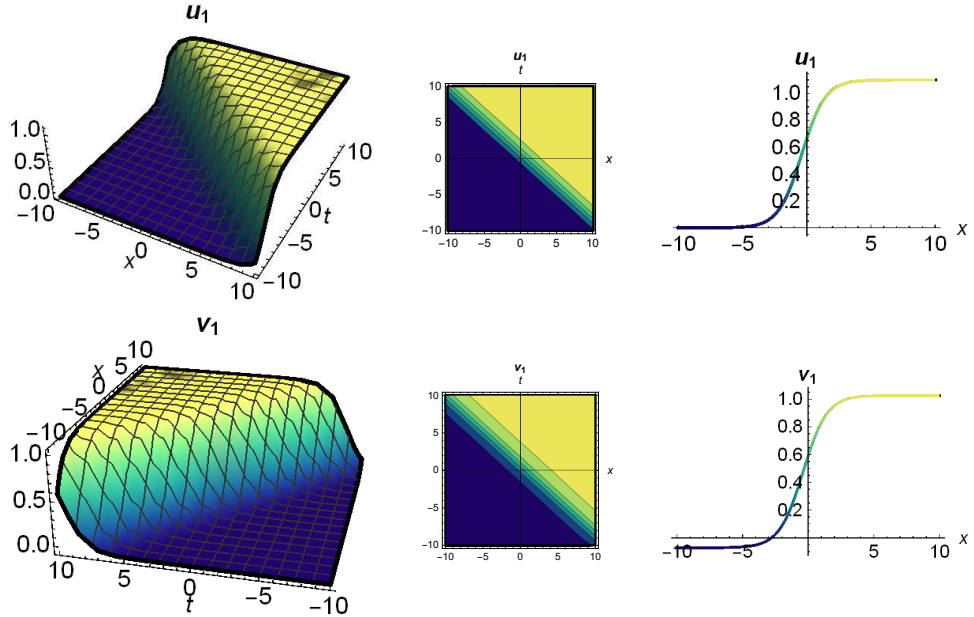


FIG. 3.1: 3D, contour and 2D graphs respectively for  $B = 0.6$ ,  $\mu = -0.1$ ,  $y = 1$ ,  $\lambda = 1.1$  values of Eqs. (3.9) and (3.10).

**Case 2:**

$$(3.11) \quad a_0 = -\lambda, \quad a_1 = -\mu, \quad c = \lambda,$$

replacing values Eq. (3.11) into Eq. (3.6) and the following hyperbolic type solutions are obtained for Eq. (1.1):

$$(3.12) \quad u_2(x, y, t) = -\lambda - \frac{\mu}{-\frac{\mu}{\lambda} + B \cosh[\lambda(x - t\lambda + y)] - B \sinh[\lambda(x - t\lambda + y)]},$$

$$\begin{aligned}
 v_2(x, y, t) = & \frac{1}{2}(\lambda^2(-x - y + t\lambda) \\
 & + \frac{1}{\mu}\lambda \left( \lambda(x - t\lambda + y)\mu + \mu \left( \lambda(x - t\lambda + y) + 2 \log \left[ \frac{(-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(x + y - t\lambda) \right] + (B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(x + y - t\lambda) \right]}{(B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]} \right] \right) \right) \\
 & + \frac{1}{2\mu}(-\lambda\mu \left( \lambda(x - t\lambda + y) + 2 \log \left[ \frac{(-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]}{(B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]} \right] \right) \\
 & - \lambda\mu(\lambda(x - t\lambda + y) + 2 \log \left[ \frac{(-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]}{(B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]} \right] \\
 & - \frac{4B\lambda\mu \sinh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]}{(B\lambda - \mu) \left( \frac{(-B\lambda + \mu) \cosh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]}{(B\lambda + \mu) \sinh \left[ \frac{1}{2}\lambda(x - t\lambda + y) \right]} \right) \\
 & + \frac{1}{2} \left( -\lambda - \frac{\mu}{-\frac{\mu}{\lambda} + B \cosh[\lambda(x - t\lambda + y)] - B \sinh[\lambda(x - t\lambda + y)]} \right).
 \end{aligned}
 \tag{3.13}$$

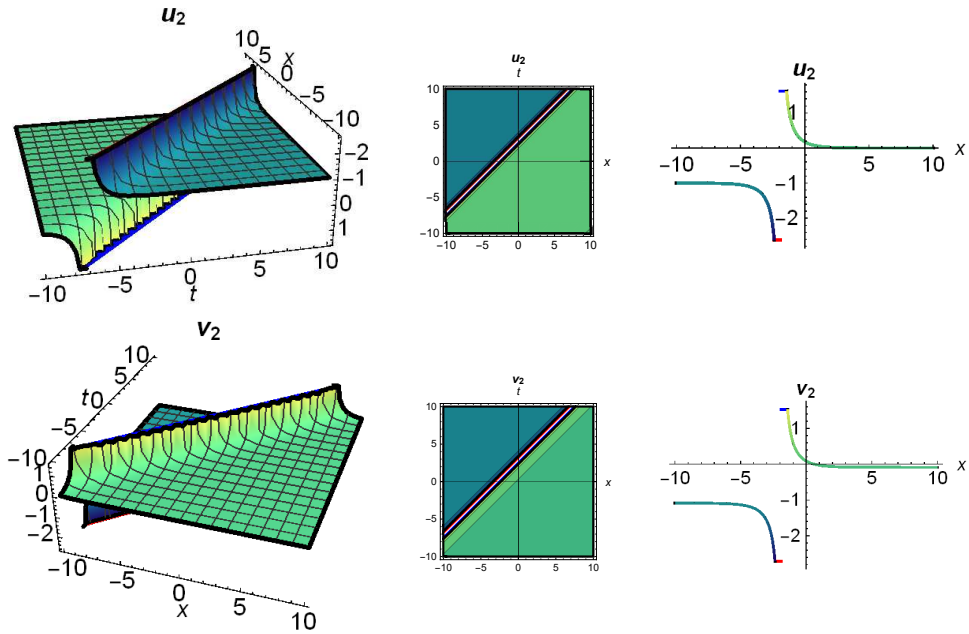


FIG. 3.2: 3D, contour and 2D graphs respectively for  $B = 0.6$ ,  $\mu = -0.1$ ,  $y = 1$ ,  $\lambda = 1.1$  values of Eqs. (3.12) and (3.13).

#### 4. Conclusion

In this work, we have achieved hyperbolic type exact solutions of the CBLP System with the help of  $(1/G')$ -expansion method. Computer technology utilized in the construction of 3D, 2D and contour graphics of the obtained solutions. The CBLP System, which plays an important role in mathematical physics, has been investigated analytically for the effectiveness and reliability of the proposed method.

Furthermore, the applied method is an effective, powerful method and can be used to establish new exact solutions of many other nonlinear partial differential equations arising in applied sciences and engineering.

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