FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 2 (2020), 541–547
https://doi.org/10.22190/FUMI2002541J

ON THE SIGNED MATCHINGS OF GRAPHS

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Abstract. For a graph G and any $v \in V(G)$, $E_G(v)$ is the set of all edges incident with v. A function $f : E(G) \to \{-1, 1\}$ is called a *signed matching* of G if $\sum_{e \in E_G(v)} f(e) \leq 1$ for every $v \in V(G)$. The weight of a signed matching f, is defined by $w(f) = \sum_{e \in E(G))} f(e)$. The signed matching number of G, denoted by $\beta_1'(G)$, is the maximum $w(f)$ where the maximum is taken over all signed matchings over G. In this paper, we have obtained the signed matching number of some families of graphs and studied the signed matching number of subdivision and the edge deletion of edges of a graph.
Keywords: signed matching; signed matching number; bipartite graphs.

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ and size $|E|$ of G is denoted by $n(G)$ and $m(G)$, respectively.
Let $G = (V, E)$ be a graph. For $u \in V$, $E_G(v)$=$\{uv \in E | u \in V\}$ are called the *edge-neighborhood* of v in G. For simplicity $E_G(v)$ is denoted by E(v). The *degree* of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |E(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbour is called a *support vertex*. A graph, G, is called r-regular graph if $\deg_G(v) = r$ for every $v \in V(G)$. For a nonempty subset $X \subseteq E$ the *edge induced subgraph* of G, induced by X, denote by $\langle X \rangle$, is a subgraph with edge set X and a vertex v belong to $\langle X \rangle$ if v is incident with at least one edge in X. A *k-partite graph* is a graph which its vertex set can be partitioned into k sets $V_1, V_2, \cdots, V_k$ such that every edge of the graph has an end point in V_i and an end point in V_j for some $1 \leq i \neq j \leq k$. A *complete k-partite graph* is a k-partite graph that every vertex of each partite set is adjacent to all vertices of the other partite sets. We denote the complete k-partite graph

Received December 31, 2019; accepted January 26, 2020
2010 *Mathematics Subject Classification.* Primary 05C69
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by $K_{n_1,n_2,\cdots,n_k}$, where $|V_i| = n_i$ for $1 \leq i \leq k$. In the case $k = 2$, the k-partite and complete k-partite graph are called *bipartite* and *complete bipartite* graphs. We denote by P_n, C_n, K_n and $\overline{K_n}$, the path, the cycle, complete graph and the empty graph of order n, respectively. A *double star* $DS_{a,b}$ is a graph containing exactly two non-leaf vertices which one is adjacent to a leaves and the other is adjacent to b leaves. These two non-leaf of double star are called *centers* of double star. For a graph $G = (V, E)$ and $e = uv \in E$, a *subdivision* of G respect to e, denote by $S(G)$, is a graph obtained from G by deleting the edge e and add new vertex x and new edges xu and xv. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint vertex sets. A graph $G = (V, E)$ is the *join graph* of G_1 and G_2, if $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. If G is the join graph of G_1 and G_2, we shall write $G = G_1 + G_2$. The graphs $W_n = C_n + K_1, F_n = P_n + K_1$, and $Fr_n = nK_2 + K_1$ are called *wheel, fan* and *friendship* graphs, respectively. For all graph-theoretic terminology not defined here, the reader is referred to [2].

Let $f : E(G) \rightarrow \{-1, 1\}$ be a function. For every vertex v, we define $f_G(v) = \sum_{e\in E_G(v)} f(e)$. A function $f : E(G) \rightarrow \{-1, 1\}$ is called a *signed matching* of G if $f_G(v) \leq 1$ for every $v \in V(G)$. The weight of a signed matching f is defined by $w(f) = f(E(G)) = \sum_{e\in E(G)} f(e)$. The *signed matching number* of G is $\beta_1'(G)$ =max $w(f)$, where the maximum is taken over all signed matchings. It seems natural to define $\beta_1'(\overline{K_n}) = 0$ for all totally disconnected graphs $\overline{K_n}$. A signed matching f on G, with $w(f) = \beta_1'(G)$ is called a β_1'- signed matching.

The concept of signed matching is defined by Wang [4], and further studied in, for example [3, 5, 6]. In [4], it is shown that a maximum signed matching can be found in strongly polynomial time. In addition, the exact value of $\beta_1'(G)$ for paths, cycles, complete graphs and complete bipartite graphs were found [4].

In this paper, we have studied the signed matchings of subdivision and edge deletion of a graph. Also, we have studied the signed matchings of join of graphs.

2. Main Results

In this section, we first stated some of the results which would be useful in the remaining part of the paper. The following proposition provides a relation between $|E(G)|$ and $\beta_1'(G)$.

Proposition 2.1. *For any graph* $G = (V(G), E(G))$*, we have* $\beta_1'(G) \equiv |E(G)| \ (mod\, 2)$.

Proof. Let f be a β_1'-signed matching on G. Suppose that P and M are the numbers of positive and negative edges respect to f, respectively. Hence

$$P + M = |E(G)|, P - M = \beta_1'(G).$$

Therefore, $\beta_1'(G) - |E(G)| = -2M$ and we conclude that $\beta_1'(G) \equiv |E(G)| \ (mod\, 2)$. □

In [4], $\beta_1'(G)$ for Eulerian graphs is given as follows.

Theorem 2.1. *[4] Let G be a Eulerian graph of order n and size m. Then*

$$\beta_1'(G) = \frac{1}{2}((-1)^m - 1).$$

Corollary 2.1. *[4]Let n be a natural number. Then*

$$\beta_1'(C_n) = \begin{cases} -1, & \text{if } n \neq 2k, \\ 0, & \text{if } n = 2k. \end{cases}$$

For non-Eulerian graph, the following theorem was given in [4]. Here we give an alternative proof for this theorem.

Theorem 2.2. *Let G be a graph of order n with $2k (k \geq 1)$ odd vertices. Then*

$$0 \leq \beta_1'(G) \leq k.$$

Proof. Let $f : E(G) \longrightarrow \{1, -1\}$ be a β_1'-signed matching of G. Hence $f_G(v) \leq 0$ for any even vertex v and $f_G(v) \leq 1$ for any odd vertex v. Therefore

$$2\beta_1'(G) = 2\sum_{e\in E} f(e) = \sum_{v\in V} f_G(v)) \leq 2k,$$

and hence $\beta_1'(G) \leq k$.
For the lower bound, note that, the edges of G can be partitioned to subsets $E_1, E_2, \cdots E_k$, such that for each i, the induced subgraph $\langle E_i \rangle$ is a trail connected odd vertices and at most one of these trails has odd length (see Theorem 5.3 of [2]). If we label the edges of each E_i alternately by 1 and -1, we can find a signed matching with positive weight. Hence $\beta_1'(G) \geq 0$. □

As a straight result of Theorems 2.1 and 2.2, we have the following corollary.

Corollary 2.2. *Let G be a graph. Hence $\beta_1'(G) = -1$ if and only if G is a Eulerian graph of odd size.*

Theorem 2.3. *[4] Let m and n be two natural numbers. Then*

$$\beta_1'(K_{m,n}) = \begin{cases} 0 & \text{if } mn \equiv 0 \ (mod\ 2), \\ min\{m, n\} & \text{if } mn \equiv 1 \ (mod\ 2). \end{cases}$$

Theorem 2.4. *Let m, n, p be positive integers. Then*

$$\beta_1'(K_{m,n,p}) = \begin{cases} 0 & \text{if } m \equiv n \equiv p \equiv 0 \ (mod\ 2), \\ -1 & \text{if } m \equiv n \equiv p \equiv 1 \ (mod\ 2), \\ 0 & \text{if } m \equiv n \equiv 0 \ (mod\ 2), p \equiv 1 \ (mod\ 2), \\ min\{m, n\} & \text{if } m \equiv n \equiv 1 \ (mod\ 2), p \equiv 0 (mod\ 2) \end{cases}$$

Proof. If $m \equiv n \equiv p$ $(mod\ 2)$, then each vertex of $K_{m,n,p}$ has even degree and hence $K_{m,n,p}$ is an Eulerian graph. Therefore, the first and the second parts of theorem are obtained by Theorem 2.1. Now suppose that $V_1 = \{v_1, v_2, \ldots, v_m\}, V_2 = \{u_1, u_2, \ldots, u_n\}$ and $V_3 = \{w_1, w_2, \ldots w_p\}$ are three parts of $K_{m,n,p}$ of sizes m, n and p, respectively. Let $f : E(G) \longrightarrow \{1, -1\}$ be a signed matching of $K_{m,n,p}$. At first consider the case $m \equiv n \equiv 0$ $(mod\ 2)$ and $p \equiv 1$ $(mod\ 2)$. Hence every vertex of V_3 has even degree. Therefore $f_{K_{m,n,p}}(v) \leq 0$ for any $v \in V_3$. On the other hand $K_{m,n,p} \cong K_{m+n,p} \cup K_{m,n}$ and hence

$$w(f) = \sum_{v \in V_3} f_{K_{m,n,p}}(v) + \sum_{v \in V_2} f_{K_{m,n}}(v).$$

Note that For any $v \in V_2$, the degree of v in $K_{m,n}$ is even and hence $f_{K_{m,n}}(v) \leq 0$. Therefore $w(f) \leq 0$. Hence $\beta_1'(K_{m,n,p}) \leq 0$. Now consider the function $g : E(K_{m,n,p}) \longrightarrow \{1, -1\}$ as follows:

$$g(u_i v_j) = (-1)^{i+j},\ g(u_i w_j) = (-1)^{i+j},\ g(w_i v_j) = (-1)^{i+j}.$$

It is not difficult to see that g is a signed matching and $w(g) = 0$. Therefore,, in this case $\beta_1'(K_{m,n,p}) = 0$.
Now suppose that $m \equiv n \equiv 1$ $(mod\ 2)$ and $p \equiv 0$ $(mod\ 2)$. Again, every vertex of V_3 has even degree. Therefore $f_{K_{m,n,p}}(v) \leq 0$ for any $v \in V_3$. By the same argument as above we have

$$w(f) = \sum_{v \in V_3} f_{K_{m,n,p}}(v) + f(E(K_{m,n})) \leq f(E(K_{m,n})).$$

But $f(E(K_{m,n})) \leq min\{m, n\}$ by Theorem 2.3. Hence $\beta_1'(K_{m,n,p}) \leq min\{m, n\}$ By the same argument as above $\beta_1'(K_{m,n,p}) = min\{m, n\}$. □

Theorem 2.5. *Suppose that a and b are two integers. Then*

$$\beta_1'(DS_{a,b}) = \begin{cases} 3 & \text{if } a \equiv b \equiv 0\ (mod\ 2) \\ 1 & \text{if } a \equiv b \equiv 1\ (mod\ 2) \\ 2 & \text{if } a \equiv 1\ (mod\ 2), b \equiv 0\ (mod\ 2) \end{cases}$$

Proof. Let u and v be centers of double star $DS_{a,b}$ of degrees $a+1$ and $b+1$. Suppose that $f : E(DS_{a,b}) \longrightarrow \{1, -1\}$ is a signed matching set. Hence $w(f) = f_{DS_{a,b}}(u) + f_{DS_{a,b}}(v) - f(uv)$.
If $a \equiv b \equiv 1$ $(mod\ 2)$, then $\deg(u)$ and $\deg(v)$ are even. Therefore, it follows that $f_{DS_{a,b}}(u), f_{DS_{a,b}}(v) \leq 0$. We conclude $w(f) \leq -f(uv) \leq 1$. Hence $\beta_1'(DS_{a,b}) \leq 1$. Now consider $g : E(DS_{a,b}) \longrightarrow \{1, -1\}$ such that $g(e) = 1$ for $\frac{a+1}{2}$ edges of $E_{DS_{a,b}}(v) \setminus \{uv\}$ and $\frac{b+1}{2}$ edges of $E_{DS_{a,b}}(u) \setminus \{uv\}$ and $g(e) = -1$ for the remaining edges of $E_{DS_{a,b}}(v) \cup E_{DS_{a,b}}(u)$. Clearly g is a signed matching and $w(g) = 1$. Hence $\beta_1'(DS_{a,b}) \geq 1$ and we conclude $\beta_1'(DS_{a,b}) = 1$.

If $a \equiv b \equiv 0$ $(mod\ 2)$, then $\deg(u)$ and $\deg(v)$ are odd. Therefore, it follows $f_{DS_{a,b}}(v)), f_{DS_{a,b}}(u) \leq 1$. We conclude that $w(f) \leq 2 - f(uv) \leq 3$. Hence

$\beta_1'(DS_{a,b}) \leq 3$. Now consider $g : E(DS_{a,b}) \longrightarrow \{1,-1\}$ such that $g(e) = 1$ for $\frac{a+2}{2}$ edges of $E_{DS_{a,b}}(v) \setminus \{uv\}$ and $\frac{b+2}{2}$ edges of $E_{DS_{a,b}}(u) \setminus \{uv\}$ and $g(e) = -1$ for the remaining edges of $E(v) \cup E(u)$. Again g is a signed matching with $w(g) = 3$ and we conclude that $\beta_1'(DS_{a,b}) = 3$.
For the last case, suppose that $a \equiv 0\ (mod\ 2)$ and $b \equiv 1 (mod\ 2)$. By the same argument as above, we conclude that $\beta_1'(S_{a,b}) = 2$. □

Theorem 2.6. *Let n be an integer. Then*

$$\beta_1'(W_n) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor & \text{if } n \equiv 0,3\ (mod\ 4), \\ \lfloor \frac{n+1}{2} \rfloor - 1 & \text{if } n \equiv 1,2\ (mod\ 4). \end{cases}$$

Proof. Suppose that $E(W_n) = \{v_iv_{i+1}, uv_i : 1 \leq i \leq n\}$, where indices computing in module n. Note that the vertex u has degree equal to n, and other vertices have degree 3. If $n \equiv 0\ (mod\ 4)$, then W_n has n vertices of odd degree. Hence $\beta(W_n) \leq \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$ by Theorem 2.2. Now define $f : E(W_n) \longrightarrow \{1,-1\}$ by

$$f(uv_{4i+1}) = f(uv_{4i+2}) = f(v_{4i+2}v_{4i+3}) = f(v_{4i+3}v_{4i+4}) = f(v_{4i+4}v_{4i+5}) = 1$$

for $0 \leq i \leq \frac{n}{4} - 1$ and $f(e) = -1$ for other edges of W_n. Clearly f is a signed matching with $w(f) = \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$. So $\beta_1'(W_n) \geq \lfloor \frac{n+1}{2} \rfloor$. Hence $\beta_1'(W_n) = \lfloor \frac{n+1}{2} \rfloor$. The case $n \equiv 3\ (mod\ 4)$ is obtained by a similar argument as the above.
Now suppose that $n \equiv 2\ (mod\ 4)$. Hence $\beta_1'(W_n) \leq \frac{n}{2}$ by Theorem 2.2. But $\beta_1'(W_n) \neq \frac{n}{2}$ by Proposition 2.1 and therefore $\beta_1'(W_n) \leq \frac{n}{2} - 1$. Now define $f : E(W_n) \longrightarrow \{1,-1\}$ by

$$f(uv_{n-1}) = f(v_1v_n) = 1,$$

$$f(uv_{4i+1}) = f(uv_{4i+2}) = f(v_{4i+2}v_{4i+3}) = f(v_{4i+3}v_{4i+4}) = f(v_{4i+4}v_{4i+5}) = 1$$

for $0 \leq i \leq \frac{n-6}{4}$ and $f(e) = -1$ for other edges of W_n. Clearly f is a signed matching with $w(f) = \frac{n}{2} - 1 = \lfloor \frac{n+1}{2} \rfloor - 1$. So $\beta_1'(W_n) \geq \lfloor \frac{n+1}{2} \rfloor - 1$. □

Theorem 2.7. *Let n be an integer. Then*

$$\beta_1'(F_n) = \begin{cases} \lfloor \frac{n-1}{2} \rfloor - 1 & \text{if } n \equiv 0,3\ (mod\ 4), \\ \lfloor \frac{n-1}{2} \rfloor & \text{if } n \equiv 1,2\ (mod\ 4). \end{cases}$$

Proof. The result follows by a similar argument as the proof of Theorem 2.6. □

Theorem 2.8. *Let n be an integer. Then*

$$\beta_1'(Fr_n) = \begin{cases} 0 & \text{if } n \equiv 0\ (mod\ 2) \\ -1 & \text{if } n \equiv 1\ (mod\ 2) \end{cases}$$

Proof. Since the graph Fr_n is an Eulerian graph, the result follows from Theorem 2.1. □

Theorem 2.9. *Let G be a graph and e be an edge of G. If $S(G)$ is the subdivision of G by edge e, then*

$$\beta_1'(G) - 1 \leq \beta_1'(S(G)) \leq \beta_1'(G) + 1.$$

In addition these bounds are sharp.

Proof. Suppose that $e = uv$ and $S(G) = G \setminus \{e\} \cup \{xu, xv\}$, where x is the new vertex. Let f be a β_1'-signed matching of G. If $f(e) = 1$ (or $f(e) = -1$), then define $g : E(S(G)) \longrightarrow \{1, -1\}$ by $g(xu) = 1$(or $g(xu) = -1$), $g(xv) = -1$ and $g(w) = f(w)$ for other edges of $S(G)$. Clearly g is a signed matching on $S(G)$ and $w(g) = \beta_1'(G) - 1$. Hence $\beta_1'(G) - 1 \leq \beta_1'(S(G))$.
Now suppose that f is a β_1'-signed matching of $S(G)$. Define signed matching g on G by $g(e) = -1$ and $g(w) = f(w)$ for other edges of G. we conclude that $\beta_1'(S(G)) \leq \beta_1'(G) + 1$.
For any positive integer n, we have $S(C_n) = C_{n+1}$. If n is even, then $\beta_1'(C_n) = 0$ and $\beta_1'(C_{n+1}) = -1$ by Corollary 2.1 and the lower bound is occurred. If n is odd, then $\beta_1'(C_n) = -1$ and $\beta_1'(C_{n+1}) = 0$ by Corollary 2.1 and the we obtain the upper bound. □

Theorem 2.10. *Let G be a graph. Then*

$$\beta_1'(G) - 3 \leq \beta_1'(G - e) \leq \beta_1'(G) + 1.$$

In addition these bounds are sharp.

Proof. Suppose that $e = uv$. Let f be a β_1'-signed matching of G. If $f(e) = 1$, then define $g : E(G-e) \longrightarrow \{1, -1\}$ by $g(x) = f(x)$ for any edge x of $G-e$. Clearly g is a signed matching on $G - e$ and $w(g) = \beta_1'(G) - 1$. Hence $\beta_1'(G) - 1 \leq \beta_1'(G - e)$.
If $f(e) = -1$, change the label of two edges e_1 and e_2 (which are adjacent to u and v in $G - e$, respectively) from 1 to -1. Hence we have a signed matching on $G - e$ of weight $\beta_1'(G) - 3$ and hence $\beta_1'(G) - 3 \leq \beta_1'(G - e)$.
Now suppose that f is a β_1'-signed matching of $G - e$. Define signed matching g on G by $g(e) = -1$ and $g(w) = f(w)$ for other edges of G. We conclude that $\beta_1'(G - e) \leq \beta_1'(G) + 1$.
Suppose that n is an even integer. We have $\beta_1'(DS_{n,n}) = 3$ by Theorem2.5. If x, y are centers of double star and $e = xy$, then $DS_{n,n} - e = 2K_{1,n}$ and we have $\beta_1'(DS_{n,n} - e) = 0$ by Theorem 2.3. Hence the lower bound is obtained. If n is even and m is odd, then $\beta_1'(K_{1,n} \cup K_{1,m}) = 1$ by Theorem 2.3. But $K_{1,n} \cup K_{1,m} + e = DS_{m,n}$, where e joint two stars $K_{1,m}$ and $K_{1,n}$. Hence $\beta_1'(DS_{m.n}) = 2$ and upper bound is occurred. □

Acknowledgments

The authors would like to thank the referees for their helpful remarks which have contributed to improve the presentation of the article.

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