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$f-$LACUNARY STATISTICAL CONVERGENCE AND STRONG $f-$LACUNARY SUMMABILITY OF ORDER α OF DOUBLE SEQUENCES

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Abstract. The main objective of this article is to introduce the concepts of $f-$lacunary statistical convergence of order α and strong $f-$lacunary summability of order α of double sequences and give some inclusion relations between these concepts.

Keywords: $f-$lacunary statistical convergence; strong $f-$lacunary summability; sequence spaces.

1. Introduction

In 1951, Steinhaus [41] and Fast [19] introduced the concept of statistical convergence while later in 1959, Schoenberg [40] reintroduced it independently. Bhardwaj and Dhawan [4], Caserta et al. [5], Connor [6], Çakallı [11], Çınar et al. [12], Çolak [13], Et et al. ([15],[17]), Fridy [21], Işık [27], Salat [39], Di Maio and Kočinac [14], Mursaleen et al. ([31],[30],[32]), Belen and Mohiuddine [3] and many authors investigated the arguments related to this notion.

A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f-$density of a subset $E\subset\mathbb{N}$ for any unbounded modulus f by

$$d^{f}(E)=\lim_{n\to\infty}\frac{f(|\{k\leq n:k\in E\}|)}{f(n)},\ \textit{if the limit exists}$$

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and defined $f-$statistical convergence for any unbounded modulus f by

$$d^f\left(\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\}\right)=0$$

i.e.

$$\lim_{n\to\infty}\frac{1}{f(n)}f\left(|\{k\leq n:|x_k-\ell|\geq\varepsilon\}|\right)=0,$$

and we write it as $S^f-\lim x_k=\ell$ or $x_k\to\ell\left(S^f\right)$. Every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence does not need to be $f-$statistically convergent for every unbounded modulus f.

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_r)$ of non-negative integers such that $k_0=0$ and $h_r=(k_r-k_{r-1})\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1=k_1$ for convenience.

In [22], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (x_k) of real numbers is called lacunary statistically convergent to a real number ℓ, if

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:|x_k-\ell|\geq\varepsilon\}|=0$$

for every positive real number ε.

Lacunary sequence spaces were studied in ([7],[8],[9],[10],[18],[20],[22],[23],[25],[26],[28],[36],[43]).

A double sequence $x=(x_{j,k})_{j,k=0}^{\infty}$ has Pringsheim limit ℓ provided that given for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{j,k}-\ell|<\varepsilon$ whenever $j,k>N$. In this case, we write $P-\lim x=\ell$ (see Pringsheim [38]).

Let $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K(m,n)=\{(j,k):j\leq m,k\leq n\}$. The double natural density of K is defined by

$$\delta_2(K)=P-\lim_{m,n}\frac{1}{mn}|K(m,n)|\,,\textit{ if the limit exists.}$$

A double sequence $x=(x_{jk})_{j,k\in\mathbb{N}}$ is said to be statistically convergent to a number ℓ if for every $\varepsilon>0$ the set $\{(j,k):j\leq m,k\leq n:|x_{jk}-\ell|\geq\varepsilon\}$ has double natural density zero (see Mursaleen and Edely [31]).

In [35], Patterson and Savaş introduced the concept of double lacunary sequence in the sense that double sequence $\theta''=\{(k_r,l_s)\}$ is called double lacunary sequence, if there exists two increasing sequences of integers such that

$$k_0=0,h_r=k_r-k_{r-1}\to\infty\textit{ as }r\to\infty$$

and

$$l_0=0,\overline{h}_s=l_s-l_{s-1}\to\infty\textit{ as }s\to\infty.$$

where $k_{r,s}=k_r l_s$, $h_{r,s}=h_r\overline{h}_s$ and the following intervals are determined by θ'', $I_r=\{(k):k_{r-1}<k\leq k_r\}$, $I_s=\{(l):l_{s-1}<l\leq l_s\}$, $I_{r,s}=\{(k,l):k_{r-1}<k\leq k_r \text{ and } l_{s-1}<l\leq l_s\}$, $q_r=\frac{k_r}{k_{r-1}}$, $\overline{q}_s=\frac{l_s}{l_{s-1}}$ and $q_{r,s}=q_r\overline{q}_s$.

The double number sequence x is $S_{\theta''}-$convergent to ℓ provided that for every $\varepsilon>0$,

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:|x_{k,l}-\ell|\geq\varepsilon\}\right|=0.$$

In this case, we write $S_{\theta''}-\lim x_{k,l}=\ell$ or $x_{k,l}\to\ell\,(S_{\theta''})$ (see [35]).

The notion of a modulus was given by Nakano [33]. Maddox [29] used a modulus function to construct some sequence spaces. Afterwards, different sequence spaces defined by modulus have been studied by Altın and Et [2], Et et al. [16], Işık [27], Gaur and Mursaleen [24], Nuray and Savaş [34], Pehlivan and Fisher [37], Şengül [42] and many others.

2. Main Results

In this section, we will introduce the concepts of $f-$lacunary statistical convergence of order α and strong $f-$lacunary summability of order α of double sequences, where f is an unbounded modulus and also give some results related to these concepts.

Definition 2.1. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence and α be a real number such that $0<\alpha\leq 1$. We say that the double sequence $x=(x_{k,l})$ is $f-$lacunary statistically convergent of order α, if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{[f(h_{r,s})]^{\alpha}}f\left(\left|\{(k,l)\in I_{r,s}:|x_{k,l}-\ell|\geq\varepsilon\}\right|\right)=0.$$

This space will be denoted by $S_{\theta''}^{f,\alpha}$. In this case, we write $S_{\theta''}^{f,\alpha}-\lim x_{k,l}=\ell$ or $x_{k,l}\to\ell\left(S_{\theta''}^{f,\alpha}\right)$. In the special case $\theta''=\{(2^r,2^s)\}$, we shall write $S''^{f,\alpha}$ instead of $S_{\theta''}^{f,\alpha}$.

Definition 2.2. Let f be a modulus function, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and α be a positive real number. We say that the double sequence $x=(x_{k,l})$ is strongly $w^{\alpha}\left[\theta'',f,p\right]-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{[h_{r,s}]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left[f(|x_{k,l}-\ell|)\right]^{p_k}=0.$$

In this case we write $w^{\alpha}\left[\theta'', f, p\right]-\lim x_{k,l}=\ell$. The set of all strongly $w^{\alpha}\left[\theta'', f, p\right]-$summable sequences will be denoted by $w^{\alpha}\left[\theta'', f, p\right]$. If we take $p_k=1$ for all $k\in\mathbb{N}$, we write $w^{\alpha}\left[\theta'', f\right]$ instead of $w^{\alpha}\left[\theta'', f, p\right]$.

Definition 2.3. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and α be a positive real number. We say that the double sequence $x=(x_{k,l})$ is strongly $w_{\theta''}^{f,\alpha}(p)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k}=0.$$

In the present case, we write $w_{\theta''}^{f,\alpha}(p)-\lim x_{k,l}=\ell$. The set of all strongly $w_{\theta''}^{f,\alpha}(p)-$summable sequences will be denoted by $w_{\theta''}^{f,\alpha}(p)$. In case of $p_k=p$ for all $k\in\mathbb{N}$ we write $w_{\theta''}^{f,\alpha}[p]$ instead of $w_{\theta''}^{f,\alpha}(p)$.

Definition 2.4. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and α be a positive real number. We say that the double sequence $x=(x_{k,l})$ is strongly $w_{\theta'',f}^{\alpha}(p)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left|x_{k,l}-\ell\right|^{p_k}=0.$$

In the present case, we write $w_{\theta'',f}^{\alpha}(p)-\lim x_{k,l}=\ell$. The set of all strongly $w_{\theta'',f}^{\alpha}(p)-$summable sequences will be denoted by $w_{\theta'',f}^{\alpha}(p)$. In case of $p_k=p$ for all $k\in\mathbb{N}$ we write $w_{\theta'',f}^{\alpha}[p]$ instead of $w_{\theta'',f}^{\alpha}(p)$.

The proof of each of the following results is fairly straightforward, so we choose to state these results without proof, where we shall assume that the sequence $p=(p_k)$ is bounded and $0<h=\inf_k p_k\le p_k\le\sup_k p_k=H<\infty$.

Theorem 2.1. *The space $w_{\theta''}^{f,\alpha}(p)$ is paranormed by*

$$g(x)=\sup_{r,s}\left\{\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|x_{k,l}\right|\right)\right]^{p_k}\right\}^{\frac{1}{M}}$$

where, $M=\max(1,H)$.

Proposition 2.1. ([37]) *Let f be a modulus and $0<\delta<1$. Then for each $\|u\|\ge\delta$, we have $f(\|u\|)\le 2f(1)\delta^{-1}\|u\|$.*

Theorem 2.2. *Let f be an unbounded modulus, α be a real number such that $0<\alpha\leq 1$ and $p>1$. If $\lim_{u\to\infty}\inf\frac{f(u)}{u}>0$, then $w_{\theta''}^{f,\alpha}[p]=w_{\theta'',f}^{\alpha}[p]$.*

Proof. Let $p>1$ be a positive real number and $x\in w_{\theta''}^{f,\alpha}[p]$. If $\lim_{u\to\infty}\inf\frac{f(u)}{u}>0$ then there exists a number $c>0$ such that $f(u)>cu$ for $u>0$. Clearly

$$\begin{aligned}\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}[f(|x_{k,l}-\ell|)]^{p} &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}[c\,|x_{k,l}-\ell|]^{p}\\ &= \frac{c^{p}}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-\ell|^{p},\end{aligned}$$

and therefore $w_{\theta''}^{f,\alpha}[p]\subset w_{\theta'',f}^{\alpha}[p]$.

Now let $x\in w_{\theta'',f}^{\alpha}[p]$. Then we have

$$\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-\ell|^{p}\to 0 \text{ as } r,s\to\infty.$$

Let $0<\delta<1$. We can write

$$\begin{aligned}\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-\ell|^{p} &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{\substack{(k,l)\in I_{r,s}\\ |x_{k,l}-\ell|\geq\delta}}|x_{k,l}-\ell|^{p}\\ &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{\substack{(k,l)\in I_{r,s}\\ |x_{k,l}-\ell|\geq\delta}}\left[\frac{f(|x_{k,l}-\ell|)}{2f(1)\,\delta^{-1}}\right]^{p}\\ &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\frac{\delta^{p}}{2^{p}f(1)^{p}}\sum_{(k,l)\in I_{r,s}}[f(|x_{k,l}-\ell|)]^{p}\end{aligned}$$

by Proposition 2.1. Therefore $x\in w_{\theta''}^{f,\alpha}[p]$.

If $\lim_{u\to\infty}\inf\frac{f(u)}{u}=0$, the equality $w_{\theta''}^{f,\alpha}[p]=w_{\theta'',f}^{\alpha}[p]$ cannot be hold as shown in the following example:

Let $f(x)=2\sqrt{x}$ and define a double sequence $x=(x_{k,l})$ by

$$x_{k,l}=\begin{cases}\sqrt[3]{h_{r,s}}, & \text{if } k=k_r \text{ and } l=l_s\\ 0, & \text{otherwise}\end{cases}\qquad r,s=1,2,....$$

For $\ell=0$, $\alpha=\frac{3}{4}$ and $p=\frac{6}{5}$, we have

$$\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}[f(|x_{k,l}|)]^{p}=\frac{\left(2\,[h_{r,s}]^{\frac{1}{6}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r,s}}\right)^{\frac{3}{4}}}=\frac{\left(2\left(h_r\overline{h_s}\right)^{\frac{1}{6}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_r\overline{h_s}}\right)^{\frac{3}{4}}}\to 0 \text{ as } r,s\to\infty$$

hence $x \in w_{\theta''}^{f,\alpha}[p]$, but

$$\frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} |x_{k,l}|^{p} = \frac{\left(\sqrt[3]{h_{r,s}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r,s}}\right)^{\frac{3}{4}}} \to \infty \ as \ r,s \to \infty$$

and so $x \notin w_{\theta'',f}^{\alpha}[p]$. □

Maddox [29] showed that the existence of an unbounded modulus f for which there is a positive constant c such that $f(xy) \geq cf(x) f(y)$, for all $x \geq 0$, $y \geq 0$.

Theorem 2.3. *Let f be an unbounded modulus and α be a positive real number. If $\lim_{u\to\infty} \frac{[f(u)]^{\alpha}}{u^{\alpha}} > 0$, then $w^{\alpha}\left[\theta'', f\right] \subset S_{\theta''}^{f,\alpha}$.*

Proof. Let $x \in w^{\alpha}[\theta'', f]$ and $\lim_{u\to\infty} \frac{f(u)^{\alpha}}{u^{\alpha}} > 0$. For $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{[h_{r,s}]^{\alpha}} \sum_{(k,l)\in I_{r,s}} f(|x_{k,l}-\ell|) &\geq \frac{1}{[h_{r,s}]^{\alpha}} f\left(\sum_{(k,l)\in I_{r,s}} |x_{k,l}-\ell|\right) \geq \frac{1}{[h_{r,s}]^{\alpha}} f\left(\sum_{\substack{(k,l)\in I_{r,s} \\ |x_{k,l}-\ell|\geq\varepsilon}} |x_{k,l}-\ell|\right) \\
&\geq \frac{1}{[h_{r,s}]^{\alpha}} f\left(|\{(k,l) \in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\}|\,\varepsilon\right) \\
&\geq \frac{c}{[h_{r,s}]^{\alpha}} f\left(|\{(k,l) \in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\}|\right) f(\varepsilon) \\
&= \frac{c}{[h_{r,s}]^{\alpha}} \frac{f\left(|\{(k,l) \in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\}|\right)}{[f(h_{r,s})]^{\alpha}} [f(h_{r,s})]^{\alpha} f(\varepsilon).
\end{aligned}$$

Therefore, $w^{\alpha}\left[\theta'', f\right] - \lim x_{k,l} = \ell$ implies $S_{\theta''}^{f,\alpha} - \lim x_{k,l} = \ell$. □

Theorem 2.4. *Let α_1, α_2 be two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$, f be an unbounded modulus function and let $\theta'' = \{(k_r, l_s)\}$ be a double lacunary sequence, then we have $w_{\theta''}^{f,\alpha_1}(p) \subset S_{\theta''}^{f,\alpha_2}$.*

Proof. Let $x \in w_{\theta''}^{f,\alpha_1}(p)$ and $\varepsilon > 0$ be given and $\sum_1$, $\sum_2$ denote the sums over $(k,l) \in I_{r,s}$, $|x_{k,l}-\ell| \geq \varepsilon$ and $(k,l) \in I_{r,s}$, $|x_{k,l}-\ell| < \varepsilon$ respectively. Since

$f\left(h_{r,s}\right)^{\alpha_1} \leq f\left(h_{r,s}\right)^{\alpha_2}$ for each r and s, we may write

$$\begin{aligned}
& \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_1}} \sum_{(k,l)\in I_{r,s}} \left[f\left(|x_{k,l}-\ell|\right)\right]^{p_k} \\
= & \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_1}} \left[\sum\nolimits_1 \left[f\left(|x_{k,l}-\ell|\right)\right]^{p_k} + \sum\nolimits_2 \left[f\left(|x_{k,l}-\ell|\right)\right]^{p_k}\right] \\
\geq & \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[\sum\nolimits_1 \left[f\left(|x_{k,l}-\ell|\right)\right]^{p_k} + \sum\nolimits_2 \left[f\left(|x_{k,l}-\ell|\right)\right]^{p_k}\right] \\
\geq & \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[\sum\nolimits_1 \left[f\left(\varepsilon\right)\right]^{p_k}\right] \\
\geq & \frac{1}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[f\left(\sum\nolimits_1 \left[\varepsilon\right]^{p_k}\right)\right] \\
\geq & \frac{1}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[f\left(\sum\nolimits_1 \min(\left[\varepsilon\right]^{h}, \left[\varepsilon\right]^{H})\right)\right] \\
\geq & \frac{1}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} f\left(\left|\left\{(k,l)\in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\right\}\right| \left[\min(\left[\varepsilon\right]^{h}, \left[\varepsilon\right]^{H})\right]\right) \\
\geq & \frac{c}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} f\left(\left|\left\{(k,l)\in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\right\}\right|\right) f\left(\left[\min(\left[\varepsilon\right]^{h}, \left[\varepsilon\right]^{H})\right]\right).
\end{aligned}$$

Hence $x \in S_{\theta''}^{f,\alpha_2}$. □

Theorem 2.5. *Let* $\theta'' = \{(k_r, l_s)\}$ *be a double lacunary sequence and* α *be a fixed real number such that* $0 < \alpha \leq 1$. *If* $\liminf_r q_r > 1$, $\liminf_s q_s > 1$ *and* $\lim_{u\to\infty} \frac{[f(u)]^\alpha}{u^\alpha} > 0$, *then* $S''^{f,\alpha} \subset S_{\theta''}^{f,\alpha}$.

Proof. Suppose first that $\liminf_r q_r > 1$ and $\liminf_s q_s > 1$; then there exists $a, b > 0$ such that $q_r \geq 1 + a$ and $q_s \geq 1 + b$ for sufficiently large r and s, which implies that

$$\frac{h_r}{k_r} \geq \frac{a}{1+a} \Longrightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{a}{1+a}\right)^\alpha$$

and

$$\frac{\overline{h}_s}{l_s} \geq \frac{b}{1+b} \Longrightarrow \left(\frac{\overline{h}_s}{l_s}\right)^\alpha \geq \left(\frac{b}{1+b}\right)^\alpha.$$

If $S''^{f,\alpha} - \lim x_{k,l} = \ell$, then for every $\varepsilon > 0$ and for sufficiently large r and s, we have

$$\begin{aligned}
& \frac{1}{[f(k_r l_s)]^{\alpha}} f(|\{k \leq k_r, l \leq l_s : |x_{k,l} - \ell| \geq \varepsilon\}|) \\
\geq\ & \frac{1}{[f(k_r l_s)]^{\alpha}} f(|\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}|) \\
=\ & \frac{[f(h_{r,s})]^{\alpha}}{[f(k_r l_s)]^{\alpha}} \frac{1}{[f(h_{r,s})]^{\alpha}} f(|\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}|) \\
=\ & \frac{[f(h_{r,s})]^{\alpha}}{[h_{r,s}]^{\alpha}} \frac{k_r^{\alpha}}{[f(k_r l_s)]^{\alpha}} \frac{[h_{r,s}]^{\alpha}}{k_r^{\alpha}} \frac{f(|\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}|)}{[f(h_{r,s})]^{\alpha}} \\
=\ & \frac{[f(h_{r,s})]^{\alpha}}{[h_{r,s}]^{\alpha}} \frac{k_r^{\alpha} l_s^{\alpha}}{[f(k_r l_s)]^{\alpha}} \frac{h_r^{\alpha} \overline{h}_s^{\alpha}}{k_r^{\alpha} l_s^{\alpha}} \frac{f(|\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}|)}{[f(h_{r,s})]^{\alpha}} \\
\geq\ & \frac{[f(h_{r,s})]^{\alpha}}{[h_{r,s}]^{\alpha}} \frac{(k_r l_s)^{\alpha}}{[f(k_r l_s)]^{\alpha}} \left(\frac{a}{1+a}\right)^{\alpha} \left(\frac{b}{1+b}\right)^{\alpha} \frac{f(|\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}|)}{[f(h_{r,s})]^{\alpha}}.
\end{aligned}$$

This proves the sufficiency. □

Theorem 2.6. *Let f be an unbounded modulus, $\theta = (k_r)$ and $\theta' = (l_s)$ be two lacunary sequences, $\theta'' = \{(k_r, l_s)\}$ be a double lacunary sequence and $0 < \alpha \leq 1$. If $S_{f,\theta}^{\alpha} - \lim x_k = \ell$ and $S_{f,\theta'}^{\alpha} - \lim x_l = \ell$, then $S_{f,\theta''}^{\alpha} - \lim x_{k,l} = \ell$.*

Proof. Suppose $S_{f,\theta}^{\alpha} - \lim x_k = \ell$ and $S_{f,\theta'}^{\alpha} - \lim x_l = \ell$. Then for $\varepsilon > 0$ we can write

$$\lim_r \frac{1}{[f(h_r)]^{\alpha}} |\{k \in I_r : |x_k - \ell| \geq \varepsilon\}| = 0$$

and

$$\lim_s \frac{1}{\left[f\left(\overline{h}_s\right)\right]^{\alpha}} |\{l \in I_s : |x_l - \ell| \geq \varepsilon\}| = 0.$$

So we have

$$\begin{aligned}
& \frac{1}{[f(h_{r,s})]^{\alpha}} |\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}| \\
\leq\ & \frac{1}{\left[c f(h_r) f\left(\overline{h}_s\right)\right]^{\alpha}} |\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}| \\
\leq\ & \frac{1}{c^{\alpha} [f(h_r)]^{\alpha} \left[f\left(\overline{h}_s\right)\right]^{\alpha}} |\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}| \\
\leq\ & \left[\frac{1}{[f(h_r)]^{\alpha}} |\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|\right] \left[\frac{1}{\left[f\left(\overline{h}_s\right)\right]^{\alpha}} |\{l \in I_s : |x_l - \ell| \geq \varepsilon\}|\right].
\end{aligned}$$

Hence $S_{f,\theta''}^{\alpha} - \lim x_{k,l} = \ell$. □

Theorem 2.7. *Let f be an unbounded modulus. If $\lim p_k > 0$, then $w_{\theta''}^{f,\alpha}(p) - \lim x_{k,l} = \ell$ uniquely.*

Proof. Let $\lim p_k = s > 0$. Assume that $w_{\theta''}^{f,\alpha}(p) - \lim x_{k,l} = \ell_1$ and $w_{\theta''}^{f,\alpha}(p) - \lim x_{k,l} = \ell_2$. Then

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} = 0,$$

and

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} = 0.$$

By definition of f, we have

$$\begin{aligned} & \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|\ell_1 - \ell_2|)]^{p_k} \\ \leq\ & \frac{D}{[f(h_{r,s})]^{\alpha}} \left(\sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} + \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} \right) \\ =\ & \frac{D}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} + \frac{D}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} \end{aligned}$$

where $\sup_k p_k = H$ and $D = \max\left(1, 2^{H-1}\right)$. Hence

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|\ell_1 - \ell_2|)]^{p_k} = 0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $\ell_1 - \ell_2 = 0$. Thus the limit is unique. $\square$

Theorem 2.8. *Let $\theta_1'' = \{(k_r, l_s)\}$ and $\theta_2'' = \{(s_r, t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$ and α_1, α_2 two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If*

$$\lim_{r,s\to\infty} \inf \frac{[f(h_{r,s})]^{\alpha_1}}{[f(\ell_{r,s})]^{\alpha_2}} > 0 \tag{2.1}$$

then $w_{\theta_2''}^{f,\alpha_2}(p) \subset w_{\theta_1''}^{f,\alpha_1}(p)$, where $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r$ and $l_{s-1} < l \leq l_s\}$, $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h}_s$ and $J_{r,s} = \{(s,t) : s_{r-1} < s \leq s_r$ and $t_{s-1} < l \leq t_s\}$, $s_{r,s} = s_r t_s$, $\ell_{r,s} = \ell_r \overline{\ell}_s$.

Proof. Let $x \in w_{\theta_2''}^{f,\alpha_2}(p)$. We can write

$$\begin{aligned}\frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in J_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} &= \frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in J_{r,s}-I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} \\ &\quad + \frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} \\ &\geq \frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} \\ &\geq \frac{[f(h_{r,s})]^{\alpha_1}}{[f(\ell_{r,s})]^{\alpha_2}} \frac{1}{[f(h_{r,s})]^{\alpha_1}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k}.\end{aligned}$$

Thus if $x \in w_{\theta_2''}^{f,\alpha_2}(p)$, then $x \in w_{\theta_1''}^{f,\alpha_1}(p)$. □

From Theorem 2.8. we have the following results.

Corollary 2.1. *Let $\theta_1'' = \{(k_r, l_s)\}$ and $\theta_2'' = \{(s_r, t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$ and α_1, α_2 two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If (2.1) holds then*

(i) $w_{\theta_2''}^{f,\alpha}(p) \subset w_{\theta_1''}^{f,\alpha}(p)$, *if* $\alpha_1 = \alpha_2 = \alpha$,

(ii) $w_{\theta_2''}^{f}(p) \subset w_{\theta_1''}^{f,\alpha_1}(p)$, *if* $\alpha_2 = 1$,

(iii) $w_{\theta_2''}^{f}(p) \subset w_{\theta_1''}^{f}(p)$, *if* $\alpha_1 = \alpha_2 = 1$.

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