A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications

Int. J. Nonlinear Anal. Appl. 5 (2014) No. 1, 89-97
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications

Sever S. Dragomir[a]

[a] School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.

Dedicated to the Memory of Charalambos J. Papaioannou

(Communicated by Th. M. Rassias)

Abstract

A companion of Ostrowski's inequality for functions of bounded variation and applications are given.

Keywords: Ostrowski's Inequality, Trapezoid Rule, Midpoint Rule.
2000 MSC: 26D15, 41A55.

1. Introduction

In [11], the author has proved the following inequality of Ostrowski type [24] for functions of bounded variation.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Denote by $\bigvee_a^b (f)$ its total variation on $[a, b]$. Then, for any $x \in [a, b]$, one has the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f). \tag{1.1}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

The above inequality (1.1) has as a remarkable particular case, the *mid-point inequality*, namely

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{1}{2} \bigvee_a^b (f).$$

Here $\frac{1}{2}$ is a best constant as well.

The corresponding version for the generalized trapezoid inequality was obtained in [4].

*Corresponding author
Email address: sever.dragomir@vu.edu.au (Sever S. Dragomir)

Received: January 2013 *Revised:* January 2014

Theorem 1.2. *With the assumptions in Theorem 1.1, one has the inequality*

$$\left|\frac{(x-a) f(a)+(b-x) f(b)}{b-a}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \bigvee_{a}^{b}(f) \qquad (1.2)$$

for any $x \in [a,b]$.

Here the constant $\frac{1}{2}$ is also best possible.

The *trapezoid inequality*

$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq \frac{1}{2} \bigvee_{a}^{b}(f)$

is the best inequality one can derive from (1.2). Here the constant $\frac{1}{2}$ is also sharp.

Recently, Guessab and Schmeisser [23], in the effort of incorporating together the mid-point and trapezoid inequality, have proved amongst others, the following companion of Ostrowski's inequality.

Theorem 1.3. *Assume that the function $f:[a,b]\to\mathbb{R}$ is of $H-r-$Hölder type with $r\in(0,1]$, i.e.,*

$$|f(t)-f(s)| \leq H|t-s|^{r} \text{ for any } t, s \in[a, b]. \qquad (1.3)$$

Then, for each $x\in\left[a,\frac{a+b}{2}\right]$, one has the inequality

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq\left[\frac{2^{r+1}(x-a)^{r+1}+(a+b-2 x)^{r+1}}{2^{r}(r+1)(b-a)}\right] H. \qquad (1.4)$$

This inequality is sharp for each admissible x. Equality is obtained if and only if $f=\pm Hf_*+c$, with $c\in\mathbb{R}$ and

$$f_{*}(t)=\begin{cases}(x-t)^{r}, & \text{for } a \leq t \leq x \\ (t-x)^{r}, & \text{for } x \leq t \leq \frac{1}{2}(a+b) \\ f_{*}(a+b-t), & \text{for } \frac{1}{2}(a+b) \leq t \leq b.\end{cases} \qquad (1.5)$$

Remark 1.4. *For $r=1$, i.e., f is Lipschitzian with the constant $L>0$, and since*

$$\frac{4(x-a)^{2}+(a+b-2 x)^{2}}{4(b-a)}=\left[\frac{1}{8}+2\left(\frac{x-\frac{3 a+b}{4}}{b-a}\right)^{2}\right](b-a)$$

then, by (1.4), we get the following companion of Ostrowski's inequality for Lipschitzian functions

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq\left[\frac{1}{8}+2\left(\frac{x-\frac{3 a+b}{4}}{b-a}\right)^{2}\right](b-a) L, \qquad (1.6)$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

The constant $\frac{1}{8}$ is best possible in (1.6) in the sense that it cannot be replaced by a smaller constant.

By substituting $x=\frac{3a+b}{4}$ into the above inequality, we obtain the following trapezoid type inequality, which is the best in the class,

$$\left|\frac{f\left(\frac{3 a+b}{4}\right)+f\left(\frac{a+3 b}{4}\right)}{2}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq \frac{1}{8}(b-a) L. \qquad (1.7)$$

The constant $\frac{1}{8}$ here is also best possible in the above sense.

For a monograph devoted to Ostrowski type inequalities, see [18].
For research papers on Ostrowski's inequality see [1]-[17], [19]-[21] and [22].
The main aim of this paper is to provide a sharp bound for the difference

$$\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt,$$

where f is assumed to be of bounded variation. Some applications for cumulative distribution function and quadrature rules are also given.

2. Some Integral Inequalities

The following identity holds.

Lemma 2.1. *Assume that the function $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then we have the equality*

$$\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.1}$$
$$=\frac{1}{b-a}\left[\int_a^x (t-a)\,df(t)+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\ +\int_{a+b-x}^b (t-b)\,df(t)\right]$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proof . Obviously, all the Riemann-Stieltjes integrals from the right hand side of (2.1) exist because the functions $(\cdot-a)$, $\left(\cdot-\frac{a+b}{2}\right)$ and $(\cdot-b)$ are continuous on these intervals and f is of bounded variation.

Using the integration by parts formula for Riemann-Stieltjes integrals, we have, for any $x\in\left[a,\frac{a+b}{2}\right]$, that

$$\int_a^x (t-a)\,df(t)=f(x)(x-a)-\int_a^x f(t)\,dt,$$

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\ =f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x} f(t)\,dt$$

and

$$\int_{a+b-x}^b (t-b)\,df(t)=(x-a)\,f(a+b-x)-\int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities we deduce (2.1). □

Remark 2.2. *A version of this identity for piecewise continuously differentiable functions has been obtained in [23, Lemma 3.2].*

The following companion of Ostrowski's inequality holds.

Theorem 2.3. *Assume that the function $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.2}$$

$$\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right]$$

$$\leq \begin{cases} \left[14+\left|x-\frac{3a+b}{4}b-a\right|\right]_a^b(f) \\ \left[2(x-ab-a)^\alpha+\left(\frac{a+b}{2}-xb-a\right)^\alpha\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_a^x(f)\right]^\beta+\left[\bigvee_x^{a+b-x}(f)\right]^\beta+\left[\bigvee_{a+b-x}^b(f)\right]^\beta\right]^{\frac{1}{\beta}}, \quad if \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \left[x-a+\frac{b-a}{2}b-a\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\} \end{cases} \textit{for any } x\in\left[a,\frac{a+b}{2}\right],$$

where $\bigvee_c^d(f)$ *denotes the total variation of* f *on* $[c,d]$. *The constant* $\frac{1}{4}$ *is best possible in the first branch of the second inequality in (2.2).*

Proof . We use the fact that for a continuous function $p:[c,d]\to\mathbb{R}$ and a function $v:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_c^d p(t)\,dv(t)\right|\leq \sup_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.3}$$

Taking the modulus in (2.1) we have

$$\left|\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{b-a}\left[\left|\int_a^x(t-a)\,df(t)\right|+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\right|\right.$$
$$\left.+\left|\int_{a+b-x}^b(t-b)\,df(t)\right|\right]$$
$$\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right]=:M(x)$$

and the first inequality in (2.2) is obtained.

Now, observe that

$$\mathrm{M}(x)\leq \frac{1}{b-a}\max\left\{x-a,\frac{a+b}{2}-x\right\}\left[\bigvee_a^x(f)+\bigvee_x^{a+b-x}(f)+\bigvee_{a+b-x}^b(f)\right]$$
$$=\frac{1}{b-a}\left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f)$$

and the first branch in the second inequality in (2.2) is proved.

Using Hölder's discrete inequality we have (for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$) that

$$\mathrm{M}(x)\leq \frac{1}{b-a}\left[(x-a)^\alpha+\left(\frac{a+b}{2}-x\right)^\alpha+(x-a)^\alpha\right]^{\frac{1}{\alpha}}$$
$$\times\left[\left[\bigvee_a^x(f)\right]^\beta+\left[\bigvee_x^{a+b-x}(f)\right]^\beta+\left[\bigvee_{a+b-x}^b(f)\right]^\beta\right]^{\frac{1}{\beta}}$$

giving the second branch in the second inequality.

Finally, we have

$$\mathrm{M}(x)\leq \frac{1}{b-a}\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}$$
$$\times\left[(x-a)+\left(\frac{a+b}{2}-x\right)+(x-a)\right],$$

which is equivalent with the last inequality in (2.2).

The sharpness of the constant $\frac{1}{4}$ in the first branch of the second inequality in (2.2) will be proved in a particular case later. □

Corollary 2.4. *With the assumptions in Theorem 2.3, one has the trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{2}\bigvee_a^b(f). \tag{2.4}$$

The constant $\frac{1}{2}$ is best possible in (2.4).

Proof . Follows from the first inequality in (2.2) on choosing $x = a$. For the sharpness of the constant, assume that (2.4) holds with a constant $A > 0$, i.e.,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le A\bigvee_a^b(f). \tag{2.5}$$

If we choose $f : [a,b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & if \quad x = a, \\ 0 & if \quad x \in (a,b), \\ 1 & if \quad x = b, \end{cases}$$

then f is of bounded variation on $[a,b]$ and

$$\frac{f(a)+f(b)}{2} = 1, \quad \int_a^b f(t)\,dt = 0, \quad and \quad \bigvee_a^b(f) = 2,$$

giving in (2.5) $1 \le 2A$, thus $A \ge \frac{1}{2}$ and the corollary is proved. □

Remark 2.5. *The inequality (2.4) was first proved in a different manner in [8].*

Corollary 2.6. *With the assumptions in Theorem 2.3, one has the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{2}\bigvee_a^b(f). \tag{2.6}$$

The constant $\frac{1}{2}$ is best possible in (2.6).

Proof . Follows from the first inequality in (2.2) on choosing $x = \frac{a+b}{2}$. For the sharpness of the constant, assume that (2.6) holds with a constant $B > 0$, i.e.,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le B\bigvee_a^b(f). \tag{2.7}$$

If we choose $f : [a,b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & if \quad x \in \left[a, \frac{a+b}{2}\right), \\ 1 & if \quad x = \frac{a+b}{2}, \\ 0 & if \quad x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

then f is of bounded variation on $[a,b]$, and $\mathrm{f}\left(\frac{a+b}{2}\right) = 1$, $\int_a^b f(t)\,dt = 0$, *and* $\bigvee_a^b(f) = 2$, giving in (2.7), $1 \le 2B$, thus $B \ge \frac{1}{2}$. □

Remark 2.7. *The inequality (2.6) was firstly proved in a different manner in [9].*

The best inequality we may get from Theorem 2.3 on using the bound provided by the first branch in the second inequality in (2.2) is incorporated in the following corollary.

Corollary 2.8. *With the assumptions in Theorem 2.3, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}\bigvee_a^b(f). \tag{2.8}$$

The constant $\frac{1}{4}$ is best possible.

Proof . Follows by Theorem 2.3 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (2.8) holds with a constant $C>0$, i.e.,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le C\bigvee_a^b(f). \tag{2.9}$$

Consider the function $f:[a,b]\to\mathbb{R}$, given by

$$f(x)=\begin{cases}1 & if \quad x\in\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\},\\[2ex] 0 & if \quad x\in[a,b]\setminus\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\}.\end{cases}$$

Then f is of bounded variation on $[a,b]$,

$$\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}=1,\qquad\int_a^b f(t)\,dt=0$$

and

$$\bigvee_a^b(f)=4,$$

giving in (2.9) $4C\ge 1$, thus $C\ge\frac{1}{4}$.

This example can be used to prove the sharpness of the constant $\frac{1}{4}$ in (2.2) as well. □

3. Applications for P.D.F.'s

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,\infty)$ and with the cumulative distribution function $F(x)=\Pr(X\le x)=\int_a^x f(t)\,dt$.

We may state the following theorem.

Theorem 3.1. *With the above assumptions, we have the inequality*

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right| \tag{3.1}$$
$$\le\tfrac{1}{b-a}\left\{\left(2x-\tfrac{3a+b}{4}\right)\left[F(x)-F(a+b-x)\right]+(x-a)\right\}$$
$$\le\tfrac{1}{4}+\left|x-\tfrac{3a+b}{4}b-a\right|,$$

$foranyx\in\left[a,\frac{a+b}{2}\right]$, where $E(X)$ denotes the expectation of X, namely $E(X)=\int_a^b t\,dF(t)$.

Proof . If we apply Theorem 2.3 for F, which is monotonic nondecreasing, we get

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{1}{b-a}\int_a^b F(t)\,dt\right| \tag{3.2}$$
$$\leq \tfrac{1}{b-a}\Big[(x-a)F(x)+\left(\tfrac{a+b}{2}-x\right)(F(a+b-x)-F(x))$$
$$+(x-a)(1-F(a+b-x))\Big]$$
$$\leq \tfrac{1}{4}+\left|x-\tfrac{3a+b}{4}b-a\right|.$$

Since
$E(X)=\int_a^b t\,dF(t)=b-\int_a^b F(t)\,dt,$
thenby(3.2)*weget*(3.1)*andthetheoremisproved.*□

In particular, we have:

Corollary 3.2. *With the above assumptions, we have:*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{4}.$$

4. A Composite Quadrature Formula

Let $I_n : a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$ and $h_i:=x_{i+1}-x_i$ $(i=0,\ldots,n-1)$ and $\nu(I_n):=\max\{h_i|i=0,\ldots,n-1\}$.

Consider the composite quadrature rule

$$Q_n(I_n,f)=\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i. \tag{4.1}$$

The following result holds.

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt=Q_n(I_n,f)+R_n(I_n,f) \tag{4.2}$$

where $Q_n(I_n,f)$ is defined in formula (4.1), and the remainder $R_n(I_n,f)$ satisfies the estimate

$$|R_n(I_n,f)|\leq\frac{1}{4}\nu(I_n)\bigvee_a^b(f). \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible.

Proof . Applying Corollary 2.8 on the interval $[x_i,x_{i+1}]$ we may state that

$$\left|\int_{x_i}^{x_{i+1}}f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i\right|\leq\frac{1}{4}h_i\bigvee_{x_i}^{x_{i+1}}(f), \tag{4.4}$$

for any $i\in\{0,\ldots,n-1\}$.

Summing the inequality (4.4) over i from 0 to $n-1$, and using the generalized triangle inequality we get

$$|R_n(I_n,f)|\leq\frac{1}{4}\sum_{i=0}^{n-1}h_i\bigvee_{x_i}^{x_{i+1}}(f)\leq\frac{1}{4}\nu(I_n)\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)=\frac{1}{4}\nu(I_n)\bigvee_a^b(f),$$

and the proof is completed. □

For the particular case when the division I_n is equidistant, i.e.,

$$I_n : x_i = a + i \cdot \frac{b-a}{n}, \quad i = 0, \dots, n,$$

we may consider the quadrature rule:

$$Q_n(f) := \frac{b-a}{2n} \sum_{i=0}^{n-1} \left\{ f\left[a + \left(\frac{4i+1}{4n}\right)(b-a)\right] + f\left[a + \left(\frac{4i+3}{4n}\right)(b-a)\right] \right\}.$$

The following corollary will be more useful in practice.

Corollary 4.2. *With the assumption of Theorem 4.1, we have*

$$\int_a^b f(t)\,dt = Q_n(f) + R_n(f), \tag{4.5}$$

where $Q_n(f)$ is defined by (4) and the remainder $R_n(f)$ satisfies the estimate

$$|R_n(f)| \le \frac{1}{4} \cdot \frac{b-a}{n} \bigvee_a^b (f). \tag{4.6}$$

The constant $\frac{1}{4}$ is sharp.

Remark 4.3. *If one is interested in finding the minimal number of points for the equidistant partition I_n so that the theoretical error in (4.6) is smaller that $\varepsilon > 0$, then this number n_ε is given by*

$$n_\varepsilon := \left[\frac{1}{4} \cdot \frac{b-a}{\varepsilon} \bigvee_a^b (f)\right] + 1, \tag{4.7}$$

where $[a]$ denotes the integer part of the positive number a.

References

[1] G. A. Anastassiou, *Univariate Ostrowski inequalities*, revisited. Monatsh. Math., 135 (3) (2002) 175-189.

[2] P. Cerone and S. S. Dragomir, *Midpoint-type rules from an inequalities point of view*, Ed. G. A. Anastassiou, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press, New York, 135-200.

[3] P. Cerone and S. S. Dragomir, *New bounds for the three-point rule involving the Riemann-Stieltjes integrals*, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 53-62 (2002).

[4] P. Cerone, S. S. Dragomir and C. E. M. Pearce, *A generalised trapezoid inequality for functions of bounded variation*, Turkish J. Math., 24 (2) (2000) 147-163.

[5] P. Cerone, S. S. Dragomir and J. Roumeliotis, *Some Ostrowski type inequalities for n-time differentiable mappings and applications*, Demonstratio Mathematica, 32 (2) (1999) 697-712.

[6] S. S. Dragomir, *Ostrowski's inequality for monotonous mappings and applications*, J. KSIAM, 3 (1) (1999) 127-135.

[7] S. S. Dragomir, *The Ostrowski's integral inequality for Lipschitzian mappings and applications*, Comp. Math. Appl., 38 (1999) 33-37.

[8] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull. Austral. Math. Soc., 60 (1) (1999) 495-508.

[9] S. S. Dragomir, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math., 22 (2000) 13-18.

[10] S. S. Dragomir, *On the Ostrowski's inequality for Riemann-Stieltjes integral*, Korean J. Appl. Math., 7 (2000) 477-485.

[11] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. & Appl., 4 (1) (2001) 33-40.

[12] S. S. Dragomir, *On the Ostrowski inequality for Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *where* f *is of Hölder type and* u *is of bounded variation and applications*, J. KSIAM, 5 (1) (2001) 35-45.

[13] S. S. Dragomir, *Ostrowski type inequalities for isotonic linear functionals*, J. Inequal. Pure & Appl. Math., 3 (5) (2002) Art. 68.

[14] S. S. Dragomir, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math., 3 (2) (2002) Article 31, 8 pages.

[15] S. S. Dragomir, *An Ostrowski like inequality for convex functions and applications*, Revista Math. Complutense, 16 (2) (2003) 373-382.

[16] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[17] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, *A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis*, Bull. Math. Soc. Sci. Math. Romanie, 42 (90) (4) (1999) 301-314.

[18] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrechy/Boston/London, 2002.

[19] S. S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in* L_1-*norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. of Math., 28 (1997) 239-244.

[20] S. S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett., 11 (1998) 105-109.

[21] S. S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in* L_p-*norm and applications to some special means and to some numerical quadrature rules*, Indian J. of Math., 40 (3) (1998) 245-304.

[22] A. M. Fink, *Bounds on the deviation of a function from its averages*, Czechoslovak Math. J., 42 (2) (1992) 298-310.

[23] A. Guessab and G. Schmeisser, *Sharp integral inequalities of the Hermite-Hadamard type*, J. Approx. Th., 115 (2002) 260-288.

[24] A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Hel, 10 (1938) 226-227.