From Linear to Additive Cellular Automata

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Abstract

This paper proves the decidability of several important properties of additive cellular automata over finite abelian groups. First of all, we prove that equicontinuity and sensitivity to initial conditions are decidable for a nontrivial subclass of additive cellular automata, namely, the linear cellular automata over $\mathbb{K}^n$, where $\mathbb{K}$ is the ring $\mathbb{Z}/m\mathbb{Z}$. The proof of this last result has required to prove a general result on the powers of matrices over a commutative ring which is of interest in its own.

Then, we extend the decidability result concerning sensitivity and equicontinuity to the whole class of additive cellular automata over a finite abelian group and for such a class we also prove the decidability of topological transitivity and all the properties (as, for instance, ergodicity) that are equivalent to it. Finally, a decidable characterization of injectivity and surjectivity for additive cellular automata over a finite abelian group is provided in terms of injectivity and surjectivity of an associated linear cellular automata over $\mathbb{K}^n$.

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1 Introduction

Cellular automata (CA) are widely known formal models for studying and simulating complex systems (for recent results, an up-to date bibliography on CA, and simulations of complex systems, see for instance [18, 1, 8, 9, 22, 5]). They are used in many disciplines ranging from physics to biology, stepping through sociology, ecology and many others. In computer science they are used for designing security schemes, random number generation, image processing, etc. This extensive use is essentially due to three main ingredients: the huge variety of distinct dynamical behaviors; the emergence of complex behaviors from local interactions; the ease of implementation (even at a hardware level). In practical applications one needs to know if the CA used for modelling a system has or not some specific property and this can be an issue. Indeed, Jarkko Kari proved a strong result stating (roughly speaking) that all non-trivial dynamical behaviors are undecidable [27]. From this seminal result, a long sequence followed (see [2, 21, 25], just to cite some of them).

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The undecidability issue can be tackled by adding more constraints on the model. These constraints may consist of conservation laws over the evolutions [24, 20, 23, 3, 34] or superposition principles induced by imposing a rich algebraic structure over the CA alphabet [26, 31, 30, 28, 7, 6, 19] (in both cases the literature is huge and only a very small excerpt is cited here).

In this paper we follow the latter trend: the alphabet of the CA is a finite abelian group G and its global update map is an additive function, i.e., an endomorphism of $G^{\mathbb{Z}}$. This pretty broad requirement provides a class of CA generalizing those with linear local rule defined by $n \times n$ matrices (see the previous citations for $n = 1$ and [28, 4] for $n > 1$).

Even if the superposition principle still allows us to prove deep and interesting results on the asymptotic behavior of linear CA over $(\mathbb{Z}/m\mathbb{Z})^n$ (for some integers $m, n > 1$), their dynamics is definitely more interesting and expressive than that of linear CA over $\mathbb{Z}/m\mathbb{Z}$ (the classical linear CA setting) and exhibits much more complex features.

In [13, 10], we proved that ergodicity coincides with topological transitivity (and many other properties) for additive CA over finite abelian groups and in [12] we proved the decidability of those properties for the restricted case of linear CA over $(\mathbb{Z}/m\mathbb{Z})^n$.

The present paper adds the following important results to the panorama of the existing ones for additive CA over finite abelian groups:

- a lifting of the decidability of topological transitivity, ergodicity, and all the related properties from linear CA to the general case of additive CA over finite abelian groups;
- a decidable characterization of sensitivity to initial conditions for linear CA over $(\mathbb{Z}/m\mathbb{Z})^n$ which is then lifted to additive CA over finite abelian groups;
- a dichotomy property of sensitivity to initial conditions vs. equicontinuity;
- a characterization of surjectivity and injectivity properties extending the known results for linear CA given in [28, 4].

The above results are important features of the dynamics of additive CA over finite abelian groups which are involved in the most complex CA behaviors. Two main tools were used in the proofs:

- an embedding of an additive CA over a finite abelian group into a linear CA over a commutative ring;
- a deep result about commutative algebras defined over a commutative ring which is of interest in its own;

The paper is structured as follows. The next section introduces all the necessary background and formal definitions. Section 3 recalls the known results about linear CA over $(\mathbb{Z}/m\mathbb{Z})^n$ and proves the new ones, including the non trivial algebra result about powers of matrix over commutative rings. Section 4 explains the embedding allowing to lift results from linear CA over $(\mathbb{Z}/m\mathbb{Z})^n$ to generic additive CA over abelian groups. It also contains all the main results. In the last section we draw our conclusion and provide some perspectives.

2 Background on DTDS and Cellular Automata

We begin by reviewing some general notions about discrete time dynamical systems and cellular automata.

A *discrete time dynamical system* (DTDS) is a pair $(\mathcal{X}, \mathcal{F})$, where $\mathcal{X}$ is any set equipped with a distance function d (i.e., $(\mathcal{X}, d)$ is a metric space) and $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a map that is continuous on $\mathcal{X}$ according to the topology induced by d.

Let $(\mathcal{X}, \mathcal{F})$ be a DTDS. We say that it is *surjective*, resp., *injective*, if $\mathcal{F}$ is *surjective*, resp., *injective*. The DTDS $(\mathcal{X}, \mathcal{F})$ is *sensitive to the initial conditions* (or simply *sensitive*) if there exists $\varepsilon > 0$ such that for any $x \in \mathcal{X}$ and any $\delta > 0$ there is an element $y \in \mathcal{X}$ such that $0 < d(y, x) < \delta$ and $d(\mathcal{F}^k(y), \mathcal{F}^k(x)) > \varepsilon$ for some $k \in \mathbb{N}$. The system $(\mathcal{X}, \mathcal{F})$ is said to be *equicontinuous* if $\forall\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathcal{X}$, $d(x, y) < \delta$ implies that $\forall k \in \mathbb{N}$, $d(\mathcal{F}^k(x), \mathcal{F}^k(y)) < \varepsilon$. As dynamical properties, sensitivity and equicontinuity represent the main features of unstable and stable dynamical systems, respectively. The former is the well-known basic component and essence of the chaotic behavior of discrete time dynamical systems, while the latter is a strong form of stability.

The DTDS $(\mathcal{X}, \mathcal{F})$ is *topologically transitive* (or, simply, *transitive*) if for all nonempty open subsets U and V of $\mathcal{X}$ there exists a natural number h such that $\mathcal{F}^h(U) \cap V \neq \emptyset$, while it is said to be *topologically mixing* if for all nonempty open subsets U and V of $\mathcal{X}$ there exists a natural number h_0 such that the previous intersection condition holds for every $h \geq h_0$. Clearly, topological mixing is a stronger condition than transitivity. Moreover, $(\mathcal{X}, \mathcal{F})$ is *topologically weakly mixing* if the DTDS $(\mathcal{X} \times \mathcal{X}, \mathcal{F} \times \mathcal{F})$ is topologically transitive, while it is *totally transitive* if $(\mathcal{X}, \mathcal{F}^h)$ is topologically transitive for all $h \in \mathbb{N}$.

Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a probability space and let $(\mathcal{X}, \mathcal{F})$ be a DTDS where $\mathcal{F}$ is a measurable map which preserves μ, i.e., $\mu(E) = \mu(\mathcal{F}^{-1}(E))$ for every $E \in \mathcal{M}$. The DTDS $(\mathcal{X}, \mathcal{F})$, or, the map $\mathcal{F}$, is *ergodic* with respect to μ if for every $E \in \mathcal{M}$

$$\big(E = \mathcal{F}^{-1}(E)\big) \Rightarrow \mu(E)(1 - \mu(E)) = 0$$

It is well known that $\mathcal{F}$ is ergodic iff for any pair of sets $A, B \in \mathcal{M}$ it holds that

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=0}^{h-1} \mu(\mathcal{F}^{-i}(A) \cap B) = \mu(A)\mu(B)$$

The DTDS $(\mathcal{X}, \mathcal{F})$ is *(ergodic) mixing*, if for any pair of sets $A, B \in \mathcal{M}$ it holds that

$$\lim_{h\to\infty} \mu(\mathcal{F}^{-h}(A) \cap B) = \mu(A)\mu(B) \ ,$$

while it is *(ergodic) weak mixing*, if for any pair of sets $A, B \in \mathcal{M}$ it holds that

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=0}^{h-1} |\mu(\mathcal{F}^{-i}(A) \cap B) - \mu(A)\mu(B)| = 0$$

We now recall some general notions about cellular automata.

Let S be a finite set. A configuration over S is a map from $\mathbb{Z}$ to S. We consider the following *space of configurations* $S^{\mathbb{Z}} = \{\boldsymbol{c} |\ \boldsymbol{c}\colon \mathbb{Z} \to S\}$. Each element $\boldsymbol{c} \in S^{\mathbb{Z}}$ can be visualized as an infinite one-dimensional cell lattice in which each cell $i \in \mathbb{Z}$ contains the element $\boldsymbol{c}_i \in S$.

Let $r \in \mathbb{N}$ and $\delta\colon S^{2r+1} \to S$ be any map. We say that r is the radius of δ.

▶ **Definition 1** (Cellular Automaton). *A one-dimensional CA based on a radius r local rule δ is a pair $(S^{\mathbb{Z}}, F)$, where $F\colon S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the* global transition map *defined as follows:*

$$\forall \boldsymbol{c} \in S^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(\boldsymbol{c})_i = \delta\left(\boldsymbol{c}_{i-r}, \ldots, \boldsymbol{c}_{i+r}\right). \tag{1}$$

We stress that the local rule δ completely determines the global rule F of a CA.

In order to study the dynamical properties of one-dimensional CA, we introduce a distance over the space of the configurations. Namely, $S^{\mathbb{Z}}$ is equipped with the Tychonoff distance d defined as follows

$$\forall \boldsymbol{c}, \boldsymbol{c}' \in S^{\mathbb{Z}}, \quad d(\boldsymbol{c}, \boldsymbol{c}') = \begin{cases} 0, & \text{if} \quad \boldsymbol{c} = \boldsymbol{c}', \\ 2^{-\min\{i \in \mathbb{N} \,:\, \boldsymbol{c}_i \neq \boldsymbol{c}'_i \text{ or } \boldsymbol{c}_{-i} \neq \boldsymbol{c}'_{-i}\}} & \text{otherwise} \ . \end{cases}$$

It is easy to verify that metric topology induced by d coincides with the product topology induced by the discrete topology on $S^{\mathbb{Z}}$. With this topology, $S^{\mathbb{Z}}$ is a compact and totally disconnected space and the global transition map F of any CA $(S^{\mathbb{Z}}, F)$ turns out to be (uniformly) continuous. Therefore, any CA itself is also a discrete time dynamical system. Moreover, any map $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the global transition rule of a CA if and only if F is (uniformly) continuous and $F \circ \sigma = \sigma \circ F$, where $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the *shift map* defined as $\forall \boldsymbol{c} \in S^{\mathbb{Z}}$, $\forall i \in \mathbb{Z}$, $\sigma(\boldsymbol{c})_i = \boldsymbol{c}_{i+1}$. From now, when no misunderstanding is possible, we identify a CA with its global rule. Moreover, whenever an ergodic property is considered for CA, μ is the well-known Haar measure over the collection $\mathcal{M}$ of measurable subsets of $S^{\mathbb{Z}}$, i.e., the one defined as the product measure induced by the uniform probability distribution over S.

2.1 Additive and Linear Cellular Automata

Let us introduce the background of additive CA. The alphabet S will be a finite abelian group G, with group operation $+$, neutral element 0, and inverse operation $-$. In this way, the configuration space $G^{\mathbb{Z}}$ turns out to be a finite abelian group, too, where the group operation of $G^{\mathbb{Z}}$ is the componentwise extension of $+$ to $G^{\mathbb{Z}}$. With an abuse of notation, we denote by the same symbols $+$, 0, and $-$ the group operation, the neutral element, and the inverse operation, respectively, both of G and $G^{\mathbb{Z}}$. Observe that $+$ and $-$ are continuous functions in the topology induced by the metric d. A configuration $\boldsymbol{c} \in G^{\mathbb{Z}}$ is said to be *finite* if the number of positions $i \in \mathbb{Z}$ with $\boldsymbol{c}_i \neq 0$ is finite.

▶ **Definition 2** (Additive Cellular Automata). *An* additive CA *over a abelian finite group G is a CA $(G^{\mathbb{Z}}, F)$ where the global transition map $F : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ is an endomorphism of $G^{\mathbb{Z}}$.*

The *sum of two additive CA* F_1 and F_2 over G is naturally defined as the map on $G^{\mathbb{Z}}$ denoted by $F_1 + F_2$ and such that

$$\forall \boldsymbol{c} \in G^{\mathbb{Z}}, \quad (F_1 + F_2)(\boldsymbol{c}) = F_1(\boldsymbol{c}) + F_2(\boldsymbol{c})$$

Clearly, $F_1 + F_2$ is an additive CA over G.

We now recall the notion of linear CA, an important subclass of additive CA. We stress that, whenever the term *linear* is involved, the alphabet S is $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{Z}/m\mathbb{Z}$ for some positive integer m. Both $\mathbb{K}^n$ and $(\mathbb{K}^n)^{\mathbb{Z}}$ become $\mathbb{K}$-modules in the obvious (i.e., entrywise) way.

A local rule $\delta \colon (\mathbb{K}^n)^{2r+1} \to \mathbb{K}^n$ of radius r is said to be linear if it is defined by $2r+1$ matrices $A_{-r}, \ldots, A_0, \ldots, A_r \in \mathbb{K}^{n \times n}$ as follows:

$$\forall (x_{-r}, \ldots, x_0, \ldots, x_r) \in (\mathbb{K}^n)^{2r+1}, \quad \delta(x_{-r}, \ldots, x_0, \ldots, x_r) = \sum_{i=-r}^{r} A_i \cdot x_i \ .$$

▶ **Definition 3** (Linear Cellular Automata (LCA)). *A* linear CA (LCA) *over $\mathbb{K}^n$ is a CA based on a linear local rule.*

Let $\mathbb{K}^n[X, X^{-1}]$ and $\mathbb{K}^n[[X, X^{-1}]]$ denote the set of *Laurent polynomials* and the set of *Laurent series*, respectively, with coefficients in $\mathbb{K}^n$. Before proceeding, let us recall that such formalisms have been successfully used to study the dynamical behaviour of LCA in the case $n = 1$ [26, 31]. Indeed, global rules and configurations are represented by Laurent polynomials and Laurent series, respectively, and the application of a global rule turns into a polynomial-series multiplication. In the more general case of LCA over $\mathbb{K}^n$, a configuration $\boldsymbol{c} \in (\mathbb{K}^n)^{\mathbb{Z}}$ can be associated with the Laurent series

$$\boldsymbol{P_c}(X) = \sum_{i\in\mathbb{Z}} \boldsymbol{c}_i X^i = \begin{bmatrix} c^1(X) \\ \vdots \\ c^n(X) \end{bmatrix} = \begin{bmatrix} \sum_{i\in\mathbb{Z}} c_i^1 X^i \\ \vdots \\ \sum_{i\in\mathbb{Z}} c_i^n X^i \end{bmatrix} \in \left(\mathbb{K}[[X, X^{-1}]]\right)^n \cong \mathbb{K}^n[[X, X^{-1}]] \ .$$

Then, if F is the global rule of a LCA defined by $A_{-r}, \dots, A_0, \dots, A_r$, one finds

$$\boldsymbol{P}_{F(\boldsymbol{c})}(X) = A \cdot \boldsymbol{P_c}(X)$$

where

$$A = \sum_{i=-r}^{r} A_i X^{-i} \in \mathbb{K}[X, X^{-1}]^{n\times n}$$

is the *the matrix associated with the LCA F*. In this way, for any integer $k > 0$ the matrix associated with F^k is A^k, and then $\boldsymbol{P}_{F^k(\boldsymbol{c})}(X) = A^k \cdot \boldsymbol{P_c}(X)$.

A matrix $A \in \mathbb{K}[X, X^{-1}]^{n\times n}$ is in *Frobenius normal form* if

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \mathbb{a}_0 & \mathbb{a}_1 & \mathbb{a}_2 & \dots & \mathbb{a}_{n-2} & \mathbb{a}_{n-1} \end{bmatrix} \tag{2}$$

where each $\mathbb{a}_i \in \mathbb{K}[X, X^{-1}]$. Recall that the coefficients of the characteristic polynomial of A are just the elements $\mathbb{a}_i$ of the n-th row of A (up to sign).

▶ **Definition 4** (Frobenius LCA). *A LCA $((\mathbb{K}^n)^{\mathbb{Z}}, F)$ is said to be a* Frobenius LCA *if the matrix $A \in \mathbb{K}[X, X^{-1}]^{n\times n}$ associated with F is in Frobenius normal form.*

3 Decidability Results about Linear CA

We now deal with sensitivity and equicontinuity for LCA over $\mathbb{K}^n$. First of all, we remind that a dichotomy between sensitivity and equicontinuity holds for LCA. Moreover, these properties are characterized by the behavior of the powers of the matrix associated with a LCA.

▶ **Proposition 5** ([14]). *Let $\left((\mathbb{K}^n)^{\mathbb{Z}}, F\right)$ be a LCA over $\mathbb{K}^n$ and let A be the matrix associated with F. The following statements are equivalent:*
1. *F is sensitive to the initial conditions;*
2. *F is not equicontinuous;*
3. *$\left|\{A^1, A^2, A^3, \dots\}\right| = \infty$.*

An immediate consequence of Proposition 5 is that any decidable characterization of sensitivity to the initial conditions in terms of the matrices defining LCA over $\mathbb{K}^n$ would also provide a characterization of equicontinuity. In the sequel, we are going to show that such a characterization actually exists. First of all, we remind that a decidable characterization of sensitivity and equicontinuity was provided for the class of Frobenius LCA in [14]. In particular, the following result holds.

▶ **Theorem 6** (Theorem 31 in [14]). *Sensitivity and equicontinuity are decidable for Frobenius LCA over* $\mathbb{K}^n$.

In order to prove that equicontinuity and sensitivity are decidable for the whole class of LCA over $\mathbb{K}^n$, we need to prove the following result whose proof is strongly far from trivial and, for a lack of space, is omitted (the proof can be found in [11]).

Notation. Let $\mathbb{K}$ be a commutative ring. Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix over $\mathbb{K}$. We denote by χ_A the *characteristic polynomial* of A which is as usual defined as the polynomial $\det(tI_n - A) \in \mathbb{K}[t]$, where I_n stands for the $n \times n$ identity matrix and $tI_n - A$ is considered as an $n \times n$-matrix over the polynomial ring $\mathbb{K}[t]$.

▶ **Theorem 7.** *Let* $\mathbb{K}$ *be a finite commutative ring. Let* $\mathbb{L}$ *be a commutative* $\mathbb{K}$*-algebra. Let* $n \in \mathbb{N}$. *Let* A *and* B *be two* $n \times n$*-matrices over* $\mathbb{L}$ *such that* $\chi_A = \chi_B$. *Then, the set* $\{A^0, A^1, A^2, \dots\}$ *is finite if and only if the set* $\{B^0, B^1, B^2, \dots\}$ *is finite.*

We are now able to prove the following

▶ **Theorem 8.** *Sensitivity and equicontinuity are decidable for LCA over* $\mathbb{K}^n$.

Proof. Let $\left((\mathbb{K}^n)^{\mathbb{Z}}, G\right)$ be any LCA over $\mathbb{K}^n$ and let A be the matrix associated with G. Consider the Frobenius LCA $\left((\mathbb{K}^n)^{\mathbb{Z}}, F\right)$ such that $\chi_A = \chi_B$, where B is the matrix (in Frobenius normal form) associated with F. By Theorem 7 and Proposition 5 the former LCA is equicontinuous if and only if the latter is. Theorem 6 concludes the proof. ◀

For a sake of completeness, we recall that injectivity and surjectivity are decidable for LCA over $\mathbb{K}^n$. This result follows from a characterization of these properties in terms of the determinant of the matrix associated with a LCA and from the fact that injectivity and surjectivity are decidable for LCA over $\mathbb{K}$ (for the latter, see [26]).

▶ **Theorem 9** ([4, 28]). *Injectivity and surjectivity are decidable for LCA over* $\mathbb{K}^n$. *In particular, a LCA over* $\mathbb{K}^n$ *is injective (resp., surjective) if and only if the determinant of the matrix associated with it is the Laurent series associated with an injective (resp., surjective) LCA over* $\mathbb{K}$.

The decidability of topologically transitivity, ergodicity, and other mixing and ergodic properties for LCA over $\mathbb{K}^n$ has been recently proved in [13]. In particular, authors showed the equivalence of all the mixing and ergodic properties for additive CA over a finite abelian group and the decidability for LCA over $\mathbb{K}^n$ (see also [10]).

▶ **Theorem 10** ([13, 10]). *Let* F *be any additive CA over a finite abelian group. The following statements are equivalent: (1)* F *is topologically transitive; (2)* F *is ergodic; (3)* F *is surjective and for every* $k \in \mathbb{N}$ *it holds that* $F^k - I$ *is surjective; (4)* F *is topologically mixing; (5)* F *is weak topologically transitive; (6)* F *is totally transitive; (7)* F *is weakly ergodic mixing; (8)* F *is ergodic mixing. Moreover, all the previously mentioned properties are decidable for LCA over* $\mathbb{K}^n$.

4 From Linear to Additive CA

In this section we are going to prove that sensitivity, equicontinuity, topological transitivity, and all the properties equivalent to the latter are decidable also for additive CA over a finite abelian group. For each of them we will reach the decidability result by extending the analogous one obtained for LCA to the wide class of additive CA over a finite abelian group. In a similar way, we provide a decidable characterization of injectivity and surjectivity for additive CA over a finite abelian group.

We recall that the local rule $\delta : G^{2r+1} \to G$ of an additive CA of radius r over a finite abelian group G can be written as

$$\forall (x_{-r}, \dots, x_r) \in G^{2r+1}, \qquad \delta(x_{-r}, \dots, x_r) = \sum_{i=-r}^{r} \delta_i(x_i) \tag{3}$$

where the functions δ_i are endomorphisms of G.

The fundamental theorem of finite abelian groups states that every finite abelian group G is isomorphic to $\bigoplus_{i=1}^{h} \mathbb{Z}/k_i\mathbb{Z}$ where the numbers $k_1, \dots, k_h$ are powers of (not necessarily distinct) primes and $\oplus$ is the direct sum operation. Hence, the global rule F of an additive CA over G splits into the direct sum of a suitable number h' of additive CA over subgroups $G_1, \dots, G_{h'}$ with $h' \le h$ and such that $\gcd(|G_i|, |G_j|) = 1$ for each pair of distinct $i, j \in \{1, \dots, h'\}$. Each of them can be studied separately and then the analysis of the dynamical behavior of F can be carried out by combining together the results obtained for each component.

In order to make things clearer, consider the following example. If F is an additive CA over $G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$ then F splits into the direct sum of 3 additive CA F_1, F_2, and F_3 over $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/25\mathbb{Z}$, respectively. Therefore, F will be sensitive to initial conditions iff at least one F_i is sensitive to the initial conditions, while F will be topological transitive iff every F_i is topological transitive.

The above considerations lead us to three distinct scenarios:

1) $G \cong \mathbb{Z}/p^k\mathbb{Z}$. Then, G is cyclic and we can define each δ_i simply assigning the value of δ_i applied to the unique generator of G. Moreover, every pair δ_i, δ_j commutes, i.e., $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, and this makes it possible a detailed analysis of the global behavior of F. Indeed, additive cellular automata over $\mathbb{Z}/p^k\mathbb{Z}$ are nothing but LCA over $\mathbb{Z}/p^k\mathbb{Z}$ and almost all dynamical properties, including sensitivity to the initial conditions, equicontinutity, injectivity, surjectivity, topological transitivity and so on are well understood and characterized (see [31]).

2) $G \cong (\mathbb{Z}/p^k\mathbb{Z})^n$. In this case, G is not cyclic anymore and has n generators. We can define each δ_i assigning the value of δ_i for each generator of G. This gives rise to the class of linear CA over $(\mathbb{Z}/p^k\mathbb{Z})^n$. Now, δ_i and δ_j do not commute in general and this makes the analysis of the dynamical behavior much harder. Nevertheless, in Section 3 we have proved that sensitivity and equicontinuity are decidable by exploiting Theorem 7. As pointed out in [14], we also recall that linear CA over $(\mathbb{Z}/p^k\mathbb{Z})^n$ allow the investigation of some classes of non-uniform CA over $\mathbb{Z}/p^k\mathbb{Z}$ ((for these latter see [15, 16, 17]).

3) $G \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{k_i}\mathbb{Z}$. In this case ($\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ in the example), G is again not cyclic and F turns out to be a subsystem of a suitable LCA. Then, the analysis of the dynamical behavior of F is even more complex than in **2)**. We do not even know easy checkable characterizations of basic properties like surjectivity or injectivity so far. We will provide them in the sequel as we stated at the beginning of this section.

Therefore, without loss of generality, in the sequel we can assume that $G = \mathbb{Z}/p^{k_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{k_n}\mathbb{Z}$ with $k_1 \ge k_2 \ge \dots \ge k_n$ in order to reach our goal.

For any $i \in \{1, \ldots, n\}$ let us denote by $\boldsymbol{e}^{(i)} \in G^{\mathbb{Z}}$ the bi-infinite configuration such that $\boldsymbol{e}^{(i)}_0 = e_i$ and $\boldsymbol{e}^{(i)}_j = 0$ for every integer $j \neq 0$.

▶ **Definition 11.** *Let $(G^{\mathbb{Z}}, F)$ be an additive CA over G. We say that $\boldsymbol{e}^{(i)} \in G^{\mathbb{Z}}$* spreads under F *if for every $\ell \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $F^k(\boldsymbol{e}^{(i)})_j \neq 0$ for some integer j with $|j| > \ell$.*

▶ Remark 12. Whenever we consider $\boldsymbol{P}_{\boldsymbol{e}^{(i)}}(X) \in G[X, X^{-1}]$, we will say that $\boldsymbol{P}_{\boldsymbol{e}^{(i)}}(X)$ spreads under F if for every $\ell \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\boldsymbol{P}_{F^k(\boldsymbol{e}^{(i)})}(X)$ has at least one component with a non null monomial of degree which is greater than ℓ in absolute value. Clearly, $\boldsymbol{P}_{\boldsymbol{e}^{(i)}}(X)$ spreads under F if and only if $\boldsymbol{e}^{(i)}$ spreads under F.

Let $\hat{G} = (\mathbb{Z}/p^{k_1}\mathbb{Z})^n$. Define the map $\psi : G \to \hat{G}$ as follows

$$\forall h \in G, \quad \forall i = 1, \ldots, n, \quad \psi(h)^i = h^i \, p^{k_1 - k_i} \ ,$$

where, for a sake of clarity, we stress that h^i denotes the i-th component of h, while $p^{k_1-k_i}$ is just the $(k_1 - k_i)$-th power of p.

▶ **Definition 13.** *We define the function $\Psi : G^{\mathbb{Z}} \to \hat{G}^{\mathbb{Z}}$ as the componentwise extension of ψ, i.e.,*

$$\forall \boldsymbol{c} \in G^{\mathbb{Z}}, \quad \forall j \in \mathbb{Z}, \quad \Psi(\boldsymbol{c})_j = \psi(\boldsymbol{c}_j) \ .$$

It is easy to check that Ψ is continuous and injective. Since every configuration $\boldsymbol{c} \in G^{\mathbb{Z}}$ (or $\hat{G}^{\mathbb{Z}}$) is associated with the Laurent series $\boldsymbol{P}_{\boldsymbol{c}}(X) \in G[[X, X^{-1}]]$ (or $\hat{G}[[X, X^{-1}]]$), with an abuse of notation we will sometimes consider Ψ as map from $G[[X, X^{-1}]]$ to $\hat{G}[[X, X^{-1}]]$ with the obvious meaning.

For any additive CA over G, we are now going to define a LCA over $(\mathbb{Z}/p^{k_1}\mathbb{Z})^n$ associated to it. With a further abuse of notation, in the sequel we will write p^{-m} with $m \in \mathbb{N}$ even if this quantity might not exist in $\mathbb{Z}/p^k\mathbb{Z}$. However, we will use it only when it multiplies $p^{m'}$ for some integer $m' > m$. In such a way $p^{m'-m}$ is well-defined in $\mathbb{Z}/p^k\mathbb{Z}$ and we will note it as product $p^{-m} \cdot p^{m'}$.

▶ **Definition 14.** *Let $(G^{\mathbb{Z}}, F)$ be any additive CA and let $\delta : G^{2r+1} \to G$ be its local rule defined, according to* (3)*, by $2r+1$ endomorphisms $\delta_{-r}, \ldots, \delta_r$ of G . For each $z \in \{-r, \ldots, r\}$, we define the matrix $A_z = (a^{(z)}_{i,j})_{1 \leq i \leq n,\, 1 \leq j \leq n} \in (\mathbb{Z}/p^{k_1}\mathbb{Z})^{n \times n}$ as*

$$\forall i, j \in \{1, \ldots, n\}, \qquad a^{(z)}_{i,j} = p^{k_j - k_i} \cdot \delta_z(e_j)^i$$

The LCA *associated with the additive* CA $(G^{\mathbb{Z}}, F)$ *is $(\hat{G}^{\mathbb{Z}}, L)$, where L is defined by $A_{-r}, \ldots, A_r$ or, equivalently, by $A = \sum_{z=-r}^{r} A_z X^{-z} \in \hat{G}[X, X^{-1}]^{n \times n}$.*

▶ Remark 15. Since every δ_z is an endomorphism of G, by construction A turns out to be well-defined.

▶ Remark 16. The following diagram commutes

$$\begin{array}{ccc} G^{\mathbb{Z}} & \xrightarrow{\ F\ } & G^{\mathbb{Z}} \\ {\scriptstyle\Psi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi} \\ \hat{G}^{\mathbb{Z}} & \xrightarrow[L]{} & \hat{G}^{\mathbb{Z}} \end{array} ,$$

i.e., $L \circ \Psi = \Psi \circ F$. Therefore we say that $(\hat{G}^{\mathbb{Z}}, L)$ is the LCA associated with $(G^{\mathbb{Z}}, F)$ *via the embedding* Ψ.

4.1 Sensitivity and Equicontinuity for Additive Cellular Automata

Let us start with the decidability of sensitivity and equicontinuity.

▶ **Lemma 17.** *Let $(G^{\mathbb{Z}}, F)$ be any additive CA. If for some $i \in \{1, \dots, n\}$ the configuration $\boldsymbol{e}^{(i)} \in G^{\mathbb{Z}}$ spreads under F then $(G^{\mathbb{Z}}, F)$ is sensitive to the initial conditions.*

Proof. We prove that F is sensitive with constant $\varepsilon = 1$. Let $\boldsymbol{e}^{(i)} \in G^{\mathbb{Z}}$ be the configuration spreading under F. Choose arbitrarily an integer $\ell \in \mathbb{N}$ and a configuration $\boldsymbol{c} \in G^{\mathbb{Z}}$. Let $t \in \mathbb{N}$ and $j \notin \{-\ell, \dots, \ell\}$ be the integers such that $F^t(\boldsymbol{e}^{(i)})_j \neq 0$. Consider the configuration $\boldsymbol{c}' = \boldsymbol{c} + \sigma^j(\boldsymbol{e}^{(i)})$. Clearly, it holds that $d(\boldsymbol{c}, \boldsymbol{c}') < 2^{-\ell}$ and $F^t(\boldsymbol{c}') = F^t(\boldsymbol{c}) + F^t(\sigma^j(\boldsymbol{e}^{(i)})) = F^t(\boldsymbol{c}) + \sigma^j(F^t(\boldsymbol{e}^{(i)}))$. So, we get $d(F^t(\boldsymbol{c}'), F^t(\boldsymbol{c})) = 1$ and this concludes the proof. ◀

In order to prove the decidability of sensitivity, we need to deal with the following notions about Laurent polynomials.

▶ **Definition 18.** *Given any polynomial $\mathbb{p}(X) \in \mathbb{Z}/p^{k_1}\mathbb{Z}\left[X, X^{-1}\right]$, the* positive *(resp.,* negative*) degree of $\mathbb{p}(X)$, denoted by deg$^+[\mathbb{p}(X)]$ (resp., deg$^-[\mathbb{p}(X)]$) is the maximum (resp., minimum) degree among those of the monomials having both positive (resp., negative) degree and coefficient which is not multiple of p. If there is no monomial satisfying both the required conditions, then deg$^+[\mathbb{p}(X)] = 0$ (resp., deg$^-[\mathbb{p}(X)]$=0).*

▶ **Lemma 19.** *Let $(\hat{G}^{\mathbb{Z}}, L)$ be a LCA and let $A \in \mathbb{Z}/p^{k_1}\mathbb{Z}\left[X, X^{-1}\right]^{n\times n}$ be the matrix associated to it. If $(\hat{G}^{\mathbb{Z}}, L)$ is sensitive then for every integer $m \geq 1$ there exists an integer $k \geq 1$ such that at least one entry of A^k has either positive or negative degree with absolute value which is greater than m.*

Proof. We can write $A = B + p \cdot C$ for some $B, C \in \mathbb{Z}/p^{k_1}\mathbb{Z}\left[X, X^{-1}\right]^{n\times n}$, where the monomials of all entries of B have coefficient which is not multiple of p. Assume that there exists a bound $b \geq 1$ such that for every $k \geq 1$ all entries of A^k have degree less than b in absolute value. Therefore, it holds that $\left|\{A^k, k \geq 1\}\right| < \infty$ and so, by Proposition 5, $(\hat{G}^{\mathbb{Z}}, L)$ is not sensitive. ◀

We are now able to prove the following important result.

▶ **Theorem 20.** *Let $(G^{\mathbb{Z}}, F)$ be any additive CA over G and let $(\hat{G}^{\mathbb{Z}}, L)$ be the LCA associated to it via the embedding Ψ. Then, the CA $(G^{\mathbb{Z}}, F)$ is sensitive to the initial conditions if and only if $(\hat{G}^{\mathbb{Z}}, L)$ is. Moreover, the CA $(G^{\mathbb{Z}}, F)$ is equicontinuous if and only if $(\hat{G}^{\mathbb{Z}}, L)$ is.*

Proof. Let us start with the equivalence between sensitivity of $(G^{\mathbb{Z}}, F)$ and sensitivity of $(\hat{G}^{\mathbb{Z}}, L)$.
$\Longrightarrow$: Assume that $(\hat{G}^{\mathbb{Z}}, L)$ is not sensitive. Then, by Proposition 5, there exist two integers $k \in \mathbb{N}$ and $m > 0$ such that $L^{k+m} = L^k$. Therefore, we get $\Psi \circ F^{k+m} = L^{k+m} \circ \Psi = L^k \circ \Psi = \Psi \circ F^k$. Since Ψ is injective, it holds that $F^{k+m} = F^k$ and so $(G^{\mathbb{Z}}, F)$ is not sensitive.
$\Longleftarrow$: Assume that $(\hat{G}^{\mathbb{Z}}, L)$ is sensitive and for any natural k let $A^k = (a^{(k)}_{i,j})_{1\leq i\leq n,\, 1\leq j\leq n}$ be the k-th power of $A \in \mathbb{Z}/p^{k_1}\mathbb{Z}\left[X, X^{-1}\right]^{n\times n}$, where A is the matrix associated to $(\hat{G}^{\mathbb{Z}}, L)$. We are going to show that at least one configuration among $\boldsymbol{e}^{(1)}, \dots, \boldsymbol{e}^{(n)}$ spreads under F. Choose arbitrarily $\ell \in \mathbb{N}$. By Lemma 19, there exist an integer $m \geq 1$ and one entry (i, j) such that either deg$^-[a^{(m)}_{i,j}] < -\ell$ or deg$^+[a^{(m)}_{i,j}] > \ell$. Without loss of generality suppose that deg$^+[a^{(m)}_{i,j}] > \ell$. The i–th component of $\boldsymbol{P}_{F^m(\boldsymbol{e}^{(j)})}(X)$ is the well defined polynomial $p^{k_i-k_1} \cdot p^{k_1-k_j} \cdot a^{(m)}_{i,j}$. Since deg$^+[a^{(m)}_{i,j}] > \ell$, we can state that $\boldsymbol{e}^{(j)}$ spreads under F. By Lemma 17, it follows that $(G^{\mathbb{Z}}, F)$ is sensitive.

As to the equicontinuity equivalence, the above first implication also proves that if $(\hat{G}^{\mathbb{Z}}, L)$ is equicontinuous (i.e., by Proposition 5, it is not sensitive) then $F^{k+m} = F^k$, i.e., by [29], $(G^{\mathbb{Z}}, F)$ is equicontinuous. Conversely, if $(G^{\mathbb{Z}}, F)$ is equicontinuous then it trivially follows that it is not sensitive, i.e., by the above second implication, $(\hat{G}^{\mathbb{Z}}, L)$ is not sensitive, i.e., by Proposition 5, $(\hat{G}^{\mathbb{Z}}, L)$ is equicontinuous. ◀

As immediate consequence of Theorem 20 we can state that the dichotomy between sensitivity and equicontinuity also holds for additive CA.

▶ **Corollary 21.** *Any additive CA over a finite abelian group is sensitive to the initial conditions if and only if it is not equicontinuous.*

The following decidability result follows from Theorem 20 and the decidability of sensitivity for LCA.

▶ **Corollary 22.** *Equicontinuity and sensitivity to the initial conditions are decidable for additive CA over a finite abelian group.*

Proof. Use Theorem 8 and 20. ◀

4.2 Surjectivity and Injectivity for Additive Cellular Automata

We now study injectivity and surjectivity for additive CA.

▶ **Lemma 23.** *Let $(\hat{G}^{\mathbb{Z}}, L)$ be any LCA over $\hat{G}$. If there exists a configuration $\boldsymbol{b} \in \hat{G}^{\mathbb{Z}}$ with $\boldsymbol{b} \neq 0$ and $L(\boldsymbol{b}) = 0$, then there exists a configuration $\boldsymbol{b}' \in \Psi(G^{\mathbb{Z}})$ such that $\boldsymbol{b}' \neq 0$ and $L(\boldsymbol{b}') = 0$. In particular, if $\boldsymbol{b}$ is finite then $\boldsymbol{b}'$ is finite too.*

Proof. Let $\boldsymbol{b} \in \hat{G}^{\mathbb{Z}}$ any configuration with $\boldsymbol{b} \neq 0$ and $L(\boldsymbol{b}) = 0$. Set $\boldsymbol{b}^{(1)} = p \cdot \boldsymbol{b}$. If $\boldsymbol{b}^{(1)} = 0$ then for every $i \in \mathbb{Z}$ each component of $\boldsymbol{b}_i$ has p^{k_1-1} as factor. So, $\boldsymbol{b} \in \Psi(G^{\mathbb{Z}})$ and $\boldsymbol{b}' = \boldsymbol{b}$ is just one possible configuration the thesis requires to exhibit. Otherwise, by repeating the same argument, set $\boldsymbol{b}^{(2)} = p \cdot \boldsymbol{b}^{(1)}$. If $\boldsymbol{b}^{(2)} = 0$ then, for every $i \in \mathbb{Z}$, each component of $\boldsymbol{b}_i^{(1)}$ has p^{k_1-1} as factor and so $\boldsymbol{b}^{(1)} \in \Psi(G^{\mathbb{Z}})$. Since $L(\boldsymbol{b}^{(1)}) = 0$, a configuration we are looking for is $\boldsymbol{b}' = \boldsymbol{b}^{(1)}$. After $k_1 - 1$ iterations, i.e., once we get $\boldsymbol{b}^{(k_1-1)} = p \cdot \boldsymbol{b}^{(k-2)}$ (with $\boldsymbol{b}^{(k-2)} \neq 0$), if $\boldsymbol{b}^{(k_1-1)} = 0$ holds we conclude that $\boldsymbol{b}' = \boldsymbol{b}^{(k_1-2)}$ by using the same argument of the previous steps. Otherwise, by definition, for every $i \in \mathbb{Z}$ each component of $\boldsymbol{b}_i^{(k_1-1)}$ itself certainly contains p^{k_1-1} as factor. Therefore, $\boldsymbol{b}^{(k_1-1)} \in \Psi(G^{\mathbb{Z}})$. Moreover, $L(\boldsymbol{b}^{(k_1-1)}) = 0$. Hence, we can set $\boldsymbol{b}' = \boldsymbol{b}^{(k_1-1)}$ and this concludes the proof. ◀

The following lemma will be useful for studying both surjectivity and other properties.

▶ **Lemma 24.** *Let $(G^{\mathbb{Z}}, F)$ and $(\hat{G}^{\mathbb{Z}}, L)$ be any additive CA over G and any LCA over $\hat{G}$, respectively, such that $L \circ \Psi = \Psi \circ F$. Then, the CA $(G^{\mathbb{Z}}, F)$ is surjective if and only if $(\hat{G}^{\mathbb{Z}}, L)$ is.*

Proof. $\Longleftarrow$: Assume that F is not surjective. Then, by the Garden of Eden theorem [32, 33], F is not injective on the finite configurations, i.e., there exist two distinct and finite configurations $\boldsymbol{c}', \boldsymbol{c}'' \in G^{\mathbb{Z}}$ with $F(\boldsymbol{c}') = F(\boldsymbol{c}'')$. Therefore, the element $\boldsymbol{c} = \boldsymbol{c}' - \boldsymbol{c}'' \in G^{\mathbb{Z}}$ is a finite configuration such that $\boldsymbol{c} \neq 0$ and $F(\boldsymbol{c}) = 0$. So, we get both $\Psi(\boldsymbol{c}) \neq 0$ and $L(\Psi(\boldsymbol{c})) = \Psi(F(\boldsymbol{c})) = 0$. Since $\Psi(\boldsymbol{c}) \neq 0$, it follows that L is not surjective.
$\Longrightarrow$: Assume that L is not surjective. Then it is not injective on the finite configurations. Thus, there exist a finite configuration $\boldsymbol{b} \neq 0$ with $L(\boldsymbol{b}) = 0$. By Lemma 23, there exists a finite configuration $\boldsymbol{b}' \in \Psi(G^{\mathbb{Z}})$ such that $\boldsymbol{b}' \neq 0$ and $L(\boldsymbol{b}') = 0$. Let $\boldsymbol{c} \in G^{\mathbb{Z}}$ be the finite

configuration such that $\Psi(\boldsymbol{c}) = \boldsymbol{b}'$. Clearly, it holds that $\boldsymbol{c} \neq 0$. We get $\Psi(F(\boldsymbol{c})) = L(\Psi(\boldsymbol{c})) = 0$. Since Ψ is injective, it follows that $F(\boldsymbol{c}) = 0$. Therefore, we conclude that F is not surjective. ◀

Next two theorems state that surjectivity and injectivity behave as sensitivity when looking at an additive CA over G and the associated LCA via the embedding Ψ.

▶ **Theorem 25.** *Let $(G^{\mathbb{Z}}, F)$ be any additive CA over G and let $(\hat{G}^{\mathbb{Z}}, L)$ be the LCA associated with it via the embedding Ψ. Then, the CA $(G^{\mathbb{Z}}, F)$ is surjective if and only if $(\hat{G}^{\mathbb{Z}}, L)$ is.*

Proof. Use Lemma 24. ◀

▶ **Theorem 26.** *Let $(G^{\mathbb{Z}}, F)$ be any additive CA and let $(\hat{G}^{\mathbb{Z}}, L)$ be the LCA associated with it via the embedding Ψ. Then, the CA $(G^{\mathbb{Z}}, F)$ is injective if and only if $(\hat{G}^{\mathbb{Z}}, L)$ is.*

Proof. $\Longleftarrow$: Assume that F is not injective. Then, there exist two distinct configurations $\boldsymbol{c}, \boldsymbol{c}' \in G^{\mathbb{Z}}$ with $F(\boldsymbol{c}) = F(\boldsymbol{c}')$. We get $L(\Psi(\boldsymbol{c})) = \Psi(F(\boldsymbol{c})) = \Psi(F(\boldsymbol{c}')) = L(\Psi(\boldsymbol{c}'))$ and, since Ψ is injective, it follows that L is not injective.
$\Longrightarrow$: Assume that L is not injective. Then, there exists a configuration $\boldsymbol{b} \in \hat{G}^{\mathbb{Z}}$ such that $\boldsymbol{b} \neq 0$ and $L(\boldsymbol{b}) = 0$. By Lemma 23, there exists a configuration $\boldsymbol{b}' \in \Psi(G^{\mathbb{Z}})$ such that $\boldsymbol{b}' \neq 0$ and $L(\boldsymbol{b}') = 0$. Let $\boldsymbol{c} \in G^{\mathbb{Z}}$ be the configuration such that $\Psi(\boldsymbol{c}) = \boldsymbol{b}'$. Clearly, it holds that $\boldsymbol{c} \neq 0$. We get $\Psi(F(\boldsymbol{c})) = L(\Psi(\boldsymbol{c})) = 0$. Since Ψ is injective, it follows that $F(\boldsymbol{c}) = 0$. Since $F(0) = 0$, we conclude that F is not injective. ◀

4.3 Topological transitivity and ergodicity

We start by proving that the embedding Ψ also preserves topological transitivity between an additive CA over G and the associated LCA.

▶ **Theorem 27.** *Let $(G^{\mathbb{Z}}, F)$ be any additive CA over G and let $(\hat{G}^{\mathbb{Z}}, L)$ be the LCA associated with it via the embedding Ψ. Then, the CA $(G^{\mathbb{Z}}, F)$ is topologically transitive if and only if $(\hat{G}^{\mathbb{Z}}, L)$ is.*

Proof. Since $\Psi \circ F = L \circ \Psi$, for every $k \in \mathbb{N}$ it holds that $\Psi \circ (F^k - I) = \Psi \circ F^k - \Psi = L^k \circ \Psi - \Psi = (L^k - I) \circ \Psi$. By Lemma 24 , $F^k - I$ is surjective iff $L^k - I$ is. Theorem 25 and 10 conclude the proof. ◀

As a final result, we get the decidability of many mixing and ergodic properties for additive CA over any finite abelian group, including topological transitivity and ergodicity.

▶ **Corollary 28.** *All the following properties are decidable for additive CA over any finite abelian group: (1) topological transitivity; (2) ergodicity; (3) topological mixing; (4) weak topological transitivity; (5) total transitivity; (6) weak ergodic mixing; (7) ergodic mixing.*

Proof. It is an immediate consequence of Theorem 10 and 27. ◀

5 Conclusions

In this paper we have provided many decidability and characterization results about the dynamical behavior of additive CA over finite abelian groups. These results were obtained using an embedding of linear CA over $(\mathbb{Z}/m\mathbb{Z})^n$ to additive CA over finite abelian groups and a deep algebra result about powers of matrices over commutative rings.

The are at least three main research directions that are worth investigating. First, one might ask which results and characterizations are still true when considering non-abelian groups. Second, it would be very interesting to find characterizations or decidability results about positive expansivity and strong transitivity for the case of additive CA over finite abelian groups. Finally, an important research direction consists in generalizing our results to higher dimensions (see [18] for recent results about D-dimensional CA).

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