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Research article

On the Harnack inequality for non-divergence parabolic equations†

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Abstract: In this paper we propose an elementary proof of the Harnack inequality for linear parabolic equations in non-divergence form.

Keywords: parabolic partial differential equations; Green function; Harnack inequality

To Italo with friendship and admiration.

1. Introduction

The literature about parabolic equations is very large, and it is extremely hard to provide a satisfactory description of all the results. Very nice books such as [3, 5, 8] try and collect the most significant contributions to this wide field. If one restricts the attention to the field of fully nonlinear parabolic equations, a quite extensive and recent account is given in [1].

In this paper we propose a proof of the Harnack inequality for linear parabolic equations in non-divergence form, originated long time ago from several discussions with Eugene Fabes. For simplicity we consider non-negative solutions of the equation

$$Lu = D_t u - Tr\left(A\,(x,t)\,D^2 u\right) = f \tag{1.1}$$

in a space-time cylinder, where f is smooth, and the matrix $A = (a_{ij})$ is smooth and uniformly elliptic, i.e.,

$$\lambda\,|\xi|^2 \le a_{ij}\,(x,t)\,\xi_i\xi_j \le |\xi|^2\,,$$

with $\lambda \in (0,1)$. In the following, whenever we talk of class of L, we mean an operator as in (1.1), where the matrix A has the same lower and upper eigenvalues.

Precisely, let $K_R(x_0)$ be the $R-$cube in $\mathbb{R}^n$, centered at x_0, and $Q_R(x_0,t_0) = K_R(x_0)\times(t_0-R^2,t_0)$. We have

Theorem 1.1. *Let u be a non-negative solution of $Lu = f$ in $Q_{2R}(x_0,t_0)$. Then there exists a constant $c = c(\lambda,n)$ such that*

$$\sup_{Q_R\left(x_0,t_0-2R^2\right)} u \le c\left\{\inf_{Q_R(x_0,t_0)} u + R^{n/(n+1)}\|f\|_{L^{n+1}(Q_{2R}(x_0,t_0))}\right\}. \tag{1.2}$$

The proof we present follows the usual strategy: Local L^∞ estimates for subsolutions, weak Harnack inequalities for supersolutions and has some resemblance with the proof in [8]. The main differences are the use of the Green function and a more, we believe, systematic and elementary use of the growth lemmas in Section 3. There is no problem in extending the proof to operators with bounded drift and zero order terms.

The Green function for the operator L can be introduced due the following theorem of Krylov ([2]).

Theorem 1.2. *Let u be the solution of the following problem in $Q_{r,T} = K_r(0)\times(0,T)$:*

$$\begin{cases} Lu = f & \text{in } Q_{r,T} \\ u = 0 & \text{on } \partial_p Q_{r,T}. \end{cases}$$

Then

$$\|u\|_{L^\infty(Q_{r,T})} \le c(\lambda,n)\, r^{n/(n+1)}\|f\|_{L^{n+1}(Q_{r,T})}. \tag{1.3}$$

From (1.3) one gets the representation formula

$$u(x,t) = \int_{Q_{r,t}} G_{r,T}(x,t;\xi,s)\, f(\xi,s)\, d\xi ds$$

$(x,t)\in Q_{r,t}$, $t\le T$, where $G_{r,T}$ is the Green function for the cylinder $Q_{r,T}$, and the estimate

$$\sup_{(x,t)\in K_r(0)\times(0,T)}\left(\int_{Q_{r,t}} [G_{r,T}(x,t;\xi,s)]^{(n+1)/n} d\xi ds\right)^{n/(n+1)} \le c(\lambda,n)\, r^{n/(n+1)}. \tag{1.4}$$

As a consequence of the Krylov estimate (1.3), an easy check shows that it is enough to prove (1.2) with $f = 0$. In turn, this inequality follows by the combination of the local L^∞ estimate (2.1) and the weak Harnak inequality (3.1).

2. Local L^∞ estimates for subsolutions

Theorem 2.1. *Let u be a non-negative subsolution in $Q_{2r}(x_0,t_0)$. Then, for all $p > 0$,*

$$\|u\|_{L^\infty(Q_{r/2}(x_0,t_0))} \le c(p,\lambda,n)\left(\frac{1}{|Q_r(x_0,t_0)|}\int_{Q_{2r}(x_0,t_0)} u^p dxdt\right)^{1/p}. \tag{2.1}$$

Proof. The function

$$v(x,t) = u\left(x_0 + rx, t_0 - 4r^2 + r^2 t\right)$$

satisfies an equation $L_r v = 0$ with L_r in the same class of L, in the cylinder $Q_2(0,4)$. Let $Q_s = Q_s(0,4)$. We want to show that

$$\|u\|_{L^\infty(Q_{1/2})} \leq c(p,\lambda,n)\left(\int_{Q_1} u^p dxdt\right)^{1/p}. \tag{2.2}$$

It is well-known that it is enough to prove that

$$\|u\|_{L^\infty(Q_r)} \leq \frac{c(p,\lambda,n)}{(\rho - r)^2}\|u\|_{L^{2(n+1)}(Q_\rho)}. \tag{2.3}$$

for $1/2 \leq r \leq \rho \leq 1$. For completeness, we show that (2.3) implies (2.2).

From the Young inequality

$$ab \leq \frac{(\eta a)^\theta}{\theta} + \frac{(b/\eta)^\gamma}{\gamma}, \quad a, b \geq 0, \quad \eta > 0, \quad \frac{1}{\theta} + \frac{1}{\gamma} = 1$$

with $\eta = (\varepsilon\theta)^{1/\theta}$, and

$$\theta = \frac{2(n+1)}{2(n+1) - p}, \quad \gamma = \frac{2(n+1)}{p}, \quad p < 2(n+1),$$

since $\gamma\theta^{\gamma/\theta} > 1$, we get

$$ab \leq \varepsilon a^\theta + \varepsilon^{-\gamma/\theta} b^\gamma.$$

Choosing

$$a = (\sup_{Q_\rho} u)^{(2(n+1)-p)/2(n+1)}, \qquad b = \frac{c(p,\lambda,n)}{(\rho - r)^2}\left(\int_{Q_\rho} u^p dxdt\right)^{1/2(n+1)}$$

and using (2.3) we get

$$\sup_{Q_r} u \leq \varepsilon \sup_{Q_\rho} u + \frac{c(\varepsilon,p,\lambda,n)}{(\rho - r)^{4(n+1)/p}}\left(\int_{Q_\rho} u^p dxdt\right)^{1/p}.$$

This inequality is of the form

$$f(r) \leq \varepsilon f(\rho) + H(\rho - r)^{-\alpha}, \tag{2.4}$$

where

$$f(\rho) = \sup_{Q_\rho} u, \quad \alpha = 4(n+1)/p, \quad \text{and} \quad H = c(\varepsilon,p,\lambda,n)\left(\int_{Q_\rho} u^p dxdt\right)^{1/p}.$$

Let $r_0 = r$, $r_{j+1} = r_j + (1-\tau)(\rho - r)\tau^j$, $j \geq 0$, with $\varepsilon < \tau^\alpha < 1$. Then

$$\begin{aligned} f(r) &\leq \varepsilon f(r_1) + H(1-\tau)^{-\alpha}(\rho - r)^{-\alpha} \\ &\leq \varepsilon^2 f(r_2) + H(1-\tau)^{-\alpha}(\rho - r)^{-\alpha}(\varepsilon\tau^{-\alpha} + 1)\ldots. \\ &\leq \varepsilon^k f(r_k) + H(1-\tau)^{-\alpha}(\rho - r)^{-\alpha}\sum_{l=0}^{k-1}(\varepsilon\tau^{-\alpha})^l. \end{aligned}$$

Letting $k \to \infty$, we get (2.2).

To show (2.3), take a cutoff $\varphi \in C_0^\infty\left(Q_\rho\right)$, $\varphi = 1$ on Q_r, $0 \le \varphi \le 1$. We have

$$\left|D^\alpha D_t^l \varphi\right| \le \frac{c(\alpha, l)}{(\rho - r)^{|\alpha| + l}}.$$

If $G_2(x, t, y, s)$ is the Green function for the cylinder Q_2, we can write

$$u(x,t)\,\varphi(x,\ t) = \int_0^t \int_{K_2} G_2(x, t; y, s)\, L(u\varphi)(y, s)\, dyds.$$

Since $Lu \le 0$, we have

$$L(u\varphi) = \varphi Lu\ - 2A\nabla u \cdot \nabla\varphi + uL\varphi \le -2A\nabla u \cdot \nabla\varphi + uL\varphi.$$

Then, using (1.4) and choosing $\psi \in C_0^\infty\left(Q_\rho \backslash \overline{Q_r}\right)$, $\psi = 1$ on the support of φ,

$$\begin{aligned} u(x,t)\,\varphi(x,\ t) \quad \le \quad & \frac{c}{(\rho - r)}\left(\int_0^t \int_{K_2} G_2(x, t; y, s)\, |\nabla u(y, s)|^2\, \psi^2(y, s)\, dyds\right)^{1/2} \\ & + \frac{c}{(\rho - r)^2}\left(\int_{Q_\rho} u^{2(n+1)} dyds\right)^{1/2(n+1)}. \end{aligned} \tag{2.5}$$

Let $Eu = \sum_{i,j=1}^n a_{ij}(x,t)\, D_{ij}^2 u$. We have

$$\begin{aligned} \int_0^t \int_{K_2} G_2(x,t;y,s)\,|\nabla u|^2\,\psi^2 dyds \le & \lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\,\psi^2 A\nabla u \cdot \nabla u\ dyds \\ = & \lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\left[\tfrac{1}{2}E(u^2) - uEu\right]\psi^2 dyds \\ = & \tfrac{1}{2}\lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\left[\left(E(u^2\psi^2) - D_s\left(u^2\psi^2\right)\right)\right] dyds \\ & + \lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\left[\tfrac{1}{2}D_s\left(u^2\psi^2\right) - uEu\ \psi^2 - \tfrac{1}{2}u^2 E(\psi^2)\right] dyds \\ & - 4\lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\,(A\nabla u \cdot \nabla\psi) u\psi dyds \\ = & I + II + III. \end{aligned}$$

Let $(x,t) \in Q_r$. Then I vanishes, since equals $-u^2(x,t)\,\psi^2(x,t) = 0$.

For III we find

$$III \le c\varepsilon\lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\,|\nabla u|^2\,\psi^2 dyds + \frac{c}{\varepsilon}\lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\,|\nabla\psi|^2\, u^2 dyds.$$

Using (1.4)

$$III \le c\varepsilon\lambda^{-1}\int_0^t \int_{K_2} G_2(x,t;y,s)\,|\nabla u|^2\,\psi^2 dyds + \frac{c}{\varepsilon}\lambda^{-1}\frac{1}{(\rho - r)^2}\left(\int_{Q_\rho} u^{2(n+1)} dyds\right)^{1/(n+1)}.$$

For II, since $Lu \leq 0$, observe that

$$\tfrac{1}{2}D_s\left(u^2\psi^2\right) - uEu\,\psi^2 = u\psi^2\left(D_s u - Eu\right) + \psi u^2 D_s\psi \leq \psi u^2 D_s\psi$$

so that

$$II \leq c\lambda^{-1}\frac{1}{(\rho-r)^2}\left(\int_{Q_\rho} u^{2(n+1)}dyds\right)^{1/(n+1)}.$$

Chosing ε sufficiently small, we conclude. □

3. Weak Harnack inequality for supersolutions

In this section we prove the weak Harnack inequality for non-negative supersolutions.

Theorem 3.1. *Let u be a non-negative supersolution in $Q_2(0,4)$. Then there exists $p_0 = p_0(\lambda, n)$, $p_0 > 0$, such that*

$$\left(\int_{Q_1(0,1)} u^{p_0}dyds\right)^{1/p_0} \leq c(\lambda,n)\inf_{Q_{1/2}(0,2)} u. \tag{3.1}$$

Proof. We may assume that $\inf_{Q_{1/2}(0,2)} u = 1$. Let

$$\Gamma_z = \{(x,t) \in Q_1(0,1):\ u(x,t) > z\}.$$

For any $p_0 > 0$ we have

$$\int_{Q_1(0,1)} u^{p_0}dyds = p_0\int_0^\infty z^{p_0-1}|\Gamma_z|\,dz \leq c(p_0) + \int_1^\infty z^{p_0-1}|\Gamma_z|\,dz.$$

As we will show in Section 5, we have

$$\Gamma_z \subset \Gamma_{z1}\cup\Gamma_{z2}\cup\Gamma_{z3}, \tag{3.2}$$

where

$$\begin{aligned}
\Gamma_{z1} &= \left\{(x,t)\in Q_1(0,1):\ |\Gamma_z| \leq \frac{c}{z}\right\},\\
\Gamma_{z2} &= \left\{(x,t)\in Q_1(0,1):\ |\Gamma_z| \leq \frac{c}{z^{1/M}}\right\},\\
\Gamma_{z3} &= \left\{(x,t)\in Q_1(0,1):\ |\Gamma_z| \leq \frac{1}{\rho}\left|\Gamma_{\gamma z}\right|\right\},
\end{aligned}$$

with c, M, ρ, γ depending only on λ and n, and $\rho > 1, 0 < \gamma < 1$. Then

$$\begin{aligned}
\int_{Q_1(0,1)} u^{p_0}dyds \leq& c(p_0) + p_0\int_1^\infty z^{p_0-1}|\Gamma_z\cap\Gamma_{z1}|\,dz\\
&+ p_0\int_1^\infty z^{p_0-1}|\Gamma_z\cap\Gamma_{z2}|\,dz + p_0\int_1^\infty z^{p_0-1}|\Gamma_z\cap\Gamma_{z3}|\,dz\\
\leq& c(p_0) + c_1(p_0) + c_2(p_0,M) + \frac{1}{\rho\gamma^{p_0}}\int_{Q_1(0,1)} u^{p_0}dyds.
\end{aligned}$$

Choosing $p_0 \ll 1$ such that $\rho\gamma^{p_0} > 1$, we get

$$\int_{Q_1(0,1)} u^{p_0} dyds \leq c(\lambda, n).$$

□

The next sections will be devoted to the proof of (3.2).

4. Two main Lemmas

In view of the proof of (3.2), we need two fundamental lemmas. The first one states that if u is a non-negative supersolution in $Q_{2r} = Q_{2r}(x_0, t_0)$ and it is greater than z on a sizable portion of Q_r, then $\inf u > cz$ on a full smaller cylinder. Therefore, a measure-theoretical information is converted in a pointwise information.

Lemma 4.1. *Let $u \geq 0$, $Lu \geq 0$ in Q_{2r}. There exist $\xi \in (0,1)$ and $c > 0$, both depending only on λ and n, such that, if*

$$|\{(x,t) \in Q_r : u(x,t) > z\}| \geq \xi |Q_r|,$$

then

$$\inf_{Q_{r/2}} u > cz.$$

Proof. We may assume $r = 1, z = 1, x_0 = 0, t_0 = 1$. As in the previous Section, we let

$$\Gamma_1 = \{(x,t) \in Q_1 : \; u(x,t) > 1\}.$$

Consider the function

$$w_{\Gamma_1}(x,t) = \int_{\Gamma_1} G_1(x,t;y,s)\, dyds.$$

If we let $Q_1 = Q_1(0,1)$, we have

$$Lw_{\Gamma_1}(x,t) = \mathcal{X}_{\Gamma_1}(x,t) \qquad \text{and} \qquad w_{\Gamma_1}(x,t) = 0 \;\text{ on }\; \partial_p Q_1.$$

From Theorem 1.2

$$\sup_{Q_1} w_{\Gamma_1}(x,t) \leq c_0.$$

Since $u \geq 0$ on $\partial_p Q_1$ and $u > 1$ on Γ_1, the maximum principle gives

$$u(x,t) \geq c_0^{-1} w_{\Gamma_1}(x,t) \quad \text{in} \quad Q_1.$$

Thus, it is enough to show that $w_{\Gamma_1}(x,t) \geq c > 0$ in $Q_{1/2}$. We have:

$$\begin{aligned} w_{\Gamma_1}(x,t) &= w_{Q_1}(x,t) - w_{Q_1\backslash\Gamma_1}(x,t) \\ &\geq w_{Q_1}(x,t) - |Q_1\backslash\Gamma_1|^{1/(n+1)} \left(\int_{Q_1} G_1(x,t;y,s)^{(n+1)/n}\, dyds\right)^{n/(n+1)} \\ &\geq w_{Q_1}(x,t) - c(1-\xi)^{1/(n+1)}. \end{aligned}$$

Take a cutoff $\psi \in C_0^\infty(Q_1)$, $\psi = 1$ on $Q_{1/2}$. Then $v = w_{Q_1} - \psi / \|L\psi\|_\infty$ satisfies

$$Lv = 1 - \frac{L\psi}{\|L\psi\|_\infty} \geq 0 \text{ in } Q_1 \quad \text{and} \quad v = 0 \text{ on } \partial_p Q_1.$$

Thus, $w_{Q_1}(x,t) \geq \psi / \|L\psi\|_\infty$ in Q_1 and, since $\psi = 1$ on $Q_{1/2}$, $\|L\psi\|_\infty = c_1(\lambda, n) > 0$, we conclude that

$$w_{\Gamma_1}(x,t) \geq w_{Q_1}(x,t) - c(1-\xi)^{1/(n+1)} \geq c_1^{-1} - c(1-\xi)^{1/(n+1)} \geq c_2(\lambda,n) > 0,$$

provided $\xi = \xi(\lambda, n)$ is suitably chosen. □

The second lemma is a variant of the so-called *growth lemma*: if u is strictly positive in a small ball at level $t = 0$, this positivity expands parabolically upwards.

The growth lemma was originally introduced by Landis ([6]), first in the context of elliptic equations, and later extended to parabolic ones (for an exposition of these ideas, we refer the interested reader to [7]). The deep significance of the growth lemma was later shown by Krylov and Safonov, in their celebrated proof of the local Hölder continuity of solution to equations like (1.1) ([4]). A similar and slightly easier version was used by Safonov for the corresponding proof in the elliptic case ([9]). Since then, the growth lemma has become a sort of standard tool in the regularity theory of elliptic and parabolic equations in non-divergence form.

Lemma 4.2. *Let $u \geq 0$, $Lu \geq 0$ in $B_{2r}(0) \times \left(0, 4r^2\right)$. Assume that $u(x,0) \geq 1$ for $|x| \leq \varepsilon r$, $0 < \varepsilon \leq 1$. Then, there exist $M = M(\lambda, n)$ and $c = c(\lambda, n)$ such that, for each $\alpha \in (0, 4 - \varepsilon^2)$,*

$$u\left(x, \alpha r^2\right) \geq c\varepsilon^M \quad \textit{for} \quad |x| \leq \frac{\sqrt{\alpha + \varepsilon^2}}{4} r.$$

Proof. We may assume $r = 1$. Let

$$\psi(x,t) = \begin{cases} \left(1 - \dfrac{4|x|^2}{t+\varepsilon^2}\right)^4 \dfrac{\varepsilon^{2q}}{(t+\varepsilon^2)^q} & \text{if} \quad \dfrac{4|x|^2}{t+\varepsilon^2} \leq 1, \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

Claim: *if $q = q(\lambda, n)$ is large, then $L\psi \leq 0$ in $\mathbb{R}^{n+1}_+$*. Indeed,

$$\begin{aligned} L\psi = & \frac{\varepsilon^{2q}}{(t+\varepsilon^2)^{q+1}} \left(1 - \frac{4|x|^2}{t+\varepsilon^2}\right)^2 \left\{ \frac{16|x|^2}{t+\varepsilon^2} \left(1 - \frac{4|x|^2}{t+\varepsilon^2}\right) - q\left(1 - \frac{4|x|^2}{t+\varepsilon^2}\right)^2 \right. \\ & \left. -768 \frac{Ax \cdot x}{t+\varepsilon^2} + 32\left(1 - \frac{4|x|^2}{t+\varepsilon^2}\right)(\mathrm{tr}A) \right\}. \end{aligned}$$

If $1 - \delta \leq \frac{4|x|^2}{t+\varepsilon^2} \leq 1$, with a suitable $\delta = \delta(\lambda, n)$, then $L\psi \leq 0$. If $0 < \frac{4|x|^2}{t+\varepsilon^2} \leq 1 - \delta$, then $L\psi \leq 0$ for q large.

Applying the maximum principle, we find

$$u(x,t) \geq \psi(x,t) \text{ in } Q_1.$$

In particular, for $0 \leq t = \alpha \leq 4 - \varepsilon^2$ and $|x| \leq \sqrt{\alpha + \varepsilon^2}/4$, letting $M = 2q$ yields

$$u(x, \alpha) \geq \left(\frac{3}{4}\right)^4 \frac{1}{4^{M/2}} \varepsilon^M = c(\lambda, n)\, \varepsilon^M,$$

since $M = M(\lambda, n)$, due to its definition in terms of q. □

Remark 4.3. *The extension to a fully nonlinear parabolic equation such as*

$$D_t u - F(D^2 u) = 0,$$

where F falls within the same class of ellipticity as the matrix governing the problem in (1.1)*, should not require too much of an effort. Indeed, as shown in [1, Section 4], the key-point is the proof of the weak Harnack inequality. Since the barrier employed in the proof of Lemma 4.2 is a radial function, the same argument works also in the fully nonlinear case considered above. The difficult step is represented by Lemma 4.1: This should require the construction of a second, proper barrier, as in [1], and then the analogue of the Krylov-Safonov estimates. We refrain to speculate further on this issue here, since it goes beyond the limits of the present manuscript. We plan to address this topic in a separate paper.*

5. Estimates for the level sets of u

In this section we prove (3.2), concluding the proof of the weak Harnack inequality. As it will be clear in the following, Lemma 5.1 is a straightforward consequence of Lemmas 4.1–4.2: whenever one has at disposal these two results, the structure of the equation plays no further role in the proof.

We let Γ_z and ξ as in Lemma 4.1, and introduce the notation

$$Q_s^+(x_0, t_0) = B_s(x_0) \times \left(t_0 + bs^2, t_0 + \frac{s^2}{\eta}\right],$$

where b and η are to be chosen depending only on λ, n. We have

Lemma 5.1. *If $|\Gamma_z| > \xi$ then*

$$|\Gamma_z| \leq \frac{c}{z} \inf_{Q_{1/2}(0,2)} u \tag{5.1}$$

with $c = c(\lambda, n)$.

On the other hand, if $|\Gamma_z| \leq \xi$, there exist $c_1 > 0$, $\gamma \in (0, 1)$, $\rho > 1$, $M > 0$, all depending only on λ and n, such that, either

$$|\Gamma_z|^M \leq \frac{c_1}{z} \inf_{Q_{1/2}(0,2)} u, \tag{5.2}$$

or

$$|\Gamma_z| \leq \frac{1}{\rho} \left|\Gamma_{\gamma z}\right|. \tag{5.3}$$

Proof. Let $|\Gamma_z| > \xi$. Then, Lemma 4.1 gives

$$\inf_{Q_{1/2}(0,1)} u \geq cz$$

and Lemma 4.2 with $\varepsilon = 1$ gives

$$\inf_{Q_{1/2}(0,2)} u \geq c_1 z \geq c_1 z \left|\Gamma_z\right|$$

which is (5.1).

Let now $|\Gamma_z| \leq \xi$. We apply the Calderon-Zygmund decomposition to $f = \mathcal{X}_{\Gamma_z}$ to find a sequence of non overlapping cylinders Q_{r_j} contained in $Q_1 = Q_1(0,1)$, satisfying the following conditions:

i) $\left|\Gamma_z \backslash \cup Q_{r_j}\right| = 0$;

ii) $\left|\Gamma_z \cap Q_{r_j}\right| > \xi \left|Q_{r_j}\right|$;

iii) each Q_{r_j} is contained in a predecessor $\tilde{Q}_j \subset Q_1$ such that the $\tilde{Q}_j$ are non overlapping and

$$\left|\Gamma_z \cap \tilde{Q}_j\right| \leq \xi \left|\tilde{Q}_j\right|.$$

Let $D = \bigcup_j \tilde{Q}_j$ and $D^+ = \bigcup_j \tilde{Q}_j^+$. From *ii*) and Lemma 4.1 we infer

$$\inf_{Q_{r_j/2}} u \geq c(\lambda, n)\, z,$$

and from Lemma 4.2 we get

$$\inf_{\tilde{Q}_j^+} u > \gamma z,$$

where $\gamma \in (0,1)$ depends on b and η in the definition of Q_j^+. In turn, b is chosen accordingly to Lemma 4.2 and η will be chosen later. Thus

$$u > \gamma z \ \text{ in } D^+. \tag{5.4}$$

Now we use the following lemma, whose proof we postpone to the end.

Lemma 5.2. *With the same notation as before, we have*

$$\left|D^+\right| \geq \frac{1 - b\eta}{\eta + 1} |D|. \tag{5.5}$$

Let δ be a small positive number and assume first that

$$\left|D^+ \backslash Q_1\right| \leq \delta \left|\Gamma_z\right|.$$

Then, from Lemma 5.2,

$$\left|D^+ \cap Q_1\right| = \left|D^+\right| - \left|D^+ \backslash Q_1\right| \geq \left|D^+\right| - \delta\left|\Gamma_z\right| \geq \frac{1 - b\eta}{\eta + 1} |D| - \delta \left|\Gamma_z\right|. \tag{5.6}$$

Since $|\Gamma_z \backslash D| = 0$,

$$|D| = \sum_j \left|\tilde{Q}_j\right| \geq \frac{1}{\xi} \sum_j \left|\tilde{Q}_j \cap \Gamma_z\right| = \frac{1}{\xi} \left|\Gamma_z\right|.$$

It follows form (5.4) that

$$\left|\Gamma_{\gamma z}\right| \geq \left(\frac{1 - b\eta}{(\eta + 1)\,\xi} - \delta\right) \left|\Gamma_z\right|.$$

Choosing η and δ such that
$$\rho = \frac{1 - b\eta}{(\eta + 1)\xi} - \delta > 1,$$
(5.3) follows.

Now assume that
$$\left|D^{+} \backslash Q_1\right| > \delta |\Gamma_z|. \tag{5.7}$$
We distinguish two cases.

Case a) There exists j such that $r_j^2 \geq \frac{\delta}{2b} |\Gamma_z|$. Since
$$\inf_{Q_{r_j/2}} u \geq cz,$$
from Lemma 4.2 with $\varepsilon = r_j/2$, $r = 2$, we get
$$\inf_{Q_{1/2}(0,2)} u \geq c\left(\frac{r_j}{2}\right)^M z \geq c |\Gamma_z|^{M/2} z,$$
which gives (5.2).

Case b) For all j, $r_j^2 \leq \frac{\delta}{2b} |\Gamma_z|$. We show that there exist r_j and $c = c(\delta, \eta)$ such that
$$r_j^2 \geq c |\Gamma_z|. \tag{5.8}$$
It is enough to show that the height of one of the $\tilde{Q}_j^+$ is greater than $\delta |\Gamma_z| /2$. To prove it we show that the bottom of at least one of the $\tilde{Q}_j^+$ is at time level $t < 1 + \delta |\Gamma_z| /2$, and its top at a time level $t > 1 + \delta |\Gamma_z|$.

Indeed, since $r_j^2 \leq \frac{\delta}{2b} |\Gamma_z|$, the bottom of $\tilde{Q}_j^+$ is at a time level $t < 1 + br_j^2 < 1 + \delta |\Gamma_z| /2$. If the top of every $\tilde{Q}_j^+$ were at a time level $\leq 1 + \delta |\Gamma_z|$, we would have
$$D^{+} \subset Q_1 \cup [B_1(0) \times (0, 1 + \delta |\Gamma_z|)],$$
so that
$$\left|D^{+} \backslash Q_1\right| \leq \delta |\Gamma_z|$$
contradicting (5.7).

Acting as in case a) we obtain (5.2). □

We are left with the proof of Lemma 5.2. Let
$$\tilde{Q}_j = K_{\tilde{r}_j}(x_j) \times (t_0 - \tilde{r}_j^2, t_0]$$
and set
$$\hat{Q}_j = \tilde{Q}_j \cup [K_{\tilde{r}_j}(x_j) \times (t_0, t_0 + b\tilde{r}_j^2]], \quad \hat{D} = \cup \hat{Q}_j.$$
Note that
$$\tilde{Q}_j^+ = K_{\tilde{r}_j}(x_j) \times \left(t_0 + b\tilde{r}_j^2, t_0 + \frac{1}{\eta}\tilde{r}_j^2\right] = K_{\tilde{r}_j}(x_j) \times \tilde{I}_j^+$$

and $\left|\tilde{I}_j^+\right| = \dfrac{1-\eta b}{\eta}\tilde{r}_j^2$. Moreover,

$$\hat{Q}_j \cup \tilde{Q}_j^+ = K_{\tilde{r}_j}(x_j) \times \left(t_0 - \tilde{r}_j^2, t_0 + \frac{1}{\eta}\tilde{r}_j^2\right] = K_{\tilde{r}_j}(x_j) \times \hat{I}_j^+$$

and $\left|\hat{I}_j^+\right| = \dfrac{1+\eta}{\eta}\tilde{r}_j^2 = \dfrac{1+\eta}{1-\eta b}\left|\tilde{I}_j^+\right|$.

Let $x \in K_{\tilde{r}_j}(x_j)$. Then

$$\left|\left\{t: (x,t) \in D^+ \cup \hat{D}\right\}\right| \leq \frac{1+\eta}{1-\eta b}\left|\tilde{I}_j^+\right| = \frac{1+\eta}{1-\eta b}\left|\{t: (x,t) \in D^+\}\right|,$$

so that, integrating with respect to x, we get

$$\left|D^+\right| \geq \frac{1-b\eta}{\eta+1}\left|D^+ \cup \hat{D}\right|,$$

which implies (5.5). □

Conflict of interest

The authors declare no conflict of interest.

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