

Lindley distribution with its sub-models

Ramajeyam Tharshan^{1 2} Pushpakanthie Wijekoon³

ABSTRACT

In recent years, modifications of the classical Lindley distribution have been considered by many authors. In this paper, we introduce a new generalization of the Lindley distribution based on a mixture of exponential and gamma distributions with different mixing proportions and compare its performance with its sub-models. The new distribution accommodates the classical Lindley, Quasi Lindley, Two-parameter Lindley, Shanker, Lindley distribution with location parameter, and Three-parameter Lindley distributions as special cases. Various structural properties of the new distribution are discussed and the size-biased and the length-biased are derived. A simulation study is conducted to examine the mean square error for the parameters by means of the method of maximum likelihood. Finally, simulation studies and some real-world data sets are used to illustrate its flexibility in terms of its location, scale and shape parameters.

Key words: Lindley distribution, mixture distributions, size-biased distributions, maximum likelihood estimation.

1. Introduction

In the modeling of the lifetime data, especially biomedical science, engineering, actuarial science, several continuous distributions bounded to 0 and ∞ have been developed, which may have one or more parameter(s). Examples of such distributions are exponential, gamma, Lindley, log-normal, Weibull and their modifications. These distributions may have various abilities to cover the tail-heaviness of a data set. The tail-heaviness of a data set may be measured by the excess kurtosis (EK) and EK is defined as $\tau - 3$, where τ is the kurtosis of the data set. The $EK > 0$ is called a fatter tail (Leptokurtic) and $EK < 0$ is called a thinner tail (Platykurtic) distributions. Among the distributions mentioned-above, Lindley distribution (LD) which was developed by Lindley (1958), and its modifications are more flexible than the above-mentioned distributions, especially when considering less complexity of their mathematical forms, shapes, and failure rate criteria.

¹Postgraduate Institute of Science, University of Peradeniya, Peradeniya, Sri Lanka.
E-mail: tharshan10684@gmail.com. ORCID: <https://orcid.org/0000-0002-6112-2517>.

²Department of Mathematics and Statistics, University of Jaffna, Sri Lanka

³Department of Statistics and Computer Science, University of Peradeniya, Peradeniya, Sri Lanka.
E-mail: pushpaw@pdn.ac.lk. ORCID: <https://orcid.org/0000-0003-4242-1017>.

The LD is a one-parameter exponential family lifetime distribution defined over the interval $[0, \infty)$ having the density function:

$$f_Y(y) = \frac{\theta^2}{1+\theta} (1+y)e^{-\theta y}; y > 0, \theta > 0, \quad (1)$$

where θ is the shape parameter, and y is the respective random variable. The density function of this distribution can be verified that a two-component mixture of two different continuous distributions namely exponential (θ) and gamma ($2, \theta$) distributions with the mixing proportion, $p = \frac{\theta}{1+\theta}$. Ghitany (2008) has done a comprehensive study on the mathematical and statistical properties of the LD and showed that the LD is more flexible and provides a better fit than the exponential distribution for lifetime data.

Even though the LD is used for modeling of the lifetime data, researchers are more keen on its modified forms in terms of increasing the flexibility of LD's shapes and failure rate criteria in recent years. Therefore, many researchers have proposed several modified forms of the LD as an alternative to LD in the past few years. Proposed new distributions are developed in terms of introducing new parameter(s) to the existing distributions. The new parameter(s) might be introduced from the latent variable distribution or mixing components that may be exponential and gamma or gamma and gamma. In this line of new proposed distributions, we may make references to a considerable number of existing distributions that are actual mixing components of LD with an exponential(θ) and a gamma($2, \theta$) distributions mixture but different mixing proportions. The existing distributions are listed below.

Shanker et. al. (2013a) obtained Quasi Lindley distribution (QLD), and discussed its various statistical properties. The distribution is a two-parameter family distribution with density function:

$$f_Y(y) = \frac{\theta(\alpha + y\theta)}{1 + \alpha} e^{-\theta y}; y > 0, \theta > 0, \alpha > -1, \quad (2)$$

where α and θ are shape, and scale parameters, respectively. The mixing proportion, $p = \frac{\alpha}{\alpha + 1}$. Note that the LD is a special case of the QLD when $\alpha = \theta$.

Shanker et. al. (2013b) introduced the two-parameter Lindley distribution (TwPLD) and discussed its statistical properties. Its density function is given by:

$$f_Y(y) = \frac{\theta^2(1 + \alpha y)}{\theta + \alpha} e^{-\theta y}; y > 0, \theta > 0, \alpha > -\theta, \quad (3)$$

where θ and α are shape parameters. The mixing proportion, $p = \frac{\theta}{\theta + \alpha}$. Note that the LD is a special case of TwPLD when $\alpha = 1$.

Shanker (2015) introduced a one-parameter family distribution, namely Shanker distribu-

tion (SD) with the probability density function:

$$f_Y(y) = \frac{\theta^2}{\theta^2 + 1}(\theta + y)e^{-\theta y}; y > 0, \theta > 0, \quad (4)$$

where θ is the shape parameter. The mixing proportion, $p = \frac{\theta^2}{\theta^2 + 1}$.

To increase more flexibility in this line of development, Abdol-Monsef (2016) introduced a new three-parameter family generalized Lindley distribution (TPLwLD) by adding the location parameter for the exponential and gamma components. In this paper, a clear clarification is given that the location parameter is an important parameter in a statistical model to estimate the starting point of the distribution. The density function of TPLwLD is given by:

$$f_Y(y) = \frac{\theta^2}{\theta + \alpha}(1 + \alpha(y - \beta))e^{-\theta(y - \beta)}; y > \beta > 0, 1 + \alpha y > 0, \theta > 0, \alpha + \theta > 0, \quad (5)$$

where θ and α are shape parameters and β is a location parameter. Equation (5) presents two-component mixture of an exponential (θ, β) and gamma $(2, \theta, \beta)$ distributions with the mixing proportion, $p = \frac{\theta}{\theta + \alpha}$. Here the location parameter is added from the mixing components when comparing with TwPLD. Note that LD is a special case of the TPLwLD when $\alpha = 1, \beta = 0$.

Shanker et. al. (2017) obtained the Three-parameter Lindley distribution (ThPLD) with the following density function:

$$f_Y(y) = \frac{\theta^2}{\theta\alpha + \beta}(\alpha + \beta y)e^{-\theta y}; y > 0, \theta > 0, \beta > 0, \theta\alpha + \beta > 0, \quad (6)$$

where θ, α and β are shape parameters. The mixing proportion, $p = \frac{\theta\alpha}{\theta\alpha + \beta}$. Note that the LD is a special case of ThPLD when $\alpha = 1, \beta = 1$.

It is clear that when introducing a new such types of LDs, the researchers incorporate with three types of parameters, namely shape parameters from the latent variable distribution, scale and location parameters from the mixing components. Table 1 summarizes the application of the three types of parameters of the above-mentioned distributions.

The aim of this paper is to introduce a new generalized LD that accommodates all the distributions given in Table 1, and study the importance of the location parameter in the model and different mixing proportions in the development process of the new Lindly family distributions. Further, the new distribution is based on the two-component mixture of exponential and gamma distributions with different mixing proportions and it will be called as the five-parameter generalized Lindley distribution (FPGLD). A simulation study will

be done to study the performance of the maximum likelihood estimators of FPGLD. Further, a comparison study will be done with its sub-models by using simulated data sets, and real-world applications. The characteristics of the data sets will be differentiated by their skewness, Excess kurtosis (EK), and Fano factor values.

Organization of this paper is as follows: in section 2 we introduce the FPGLD and its sub-models. Its statistical properties and reliability properties are presented in section 3 and section 4, respectively. Further, section 5 covers the size-biased form of the FPGLD. The parameter estimation is discussed in section 6. Finally, a simulation study is conducted to examine the performance of the maximum likelihood estimators for FPGLD, and simulated data sets and real-world data sets are used for the comparison study with its sub-models.

Table 1. Application of three types of parameters

Distribution	Authors	Parameters		
		shape	scale	location
LD(θ)	Lindley (1958)	θ	-	-
TwPLD(θ, α)	Shanker et.al.(2013a)	θ, α	-	-
QLD(θ, δ)	Shanker et.al.(2013b)	α	θ	-
SD(θ)	Shanker (2015)	θ	-	-
TPLwLD(θ, α, β)	Monsef (2016)	θ, α	-	β
ThPLD(θ, α, β)	Shanker et.al.(2017)	θ, α, β	-	-

2. Five parameter generalized Lindley distribution

In this section, we introduce the five-parameter generalized Lindley distribution (FPGLD) with its sub-models.

The probability density function (pdf) of the FPGLD with parameters $\theta, \beta, \alpha, \delta$ and η is defined by;

$$f_Y(y) = \frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha + \eta\theta(y - \beta) \right) e^{-\theta(y - \beta)}, \quad (7)$$

where $y > \beta \geq 0, \theta > 0, \delta\alpha > -\eta, \delta\alpha > -\eta\theta(y - \beta)$, and the range of the parameters are based on the log-likelihood function. The proposed distribution is a two-component mixture of exponential distribution with parameters θ and β , and gamma distribution with parameters $2, \theta$ and β with mixing proportion, $p = \frac{\delta\alpha}{\delta\alpha + \eta}$, where δ, α, η are shape parameters, and θ and β are scale and location parameters, respectively. Note that the FPGLD has the same mixing components of TPLwLD but different mixing proportion.

The probability density function of the FPGLD has some desirable properties:

$$(i) f(\beta) = \frac{\theta\delta\alpha}{\delta\alpha + \eta} \quad (ii) \lim_{y \rightarrow \infty} f(y) = 0$$

The first derivative of equation (7) is derived as:

$$f'(y) = \frac{\theta^2 e^{-\theta(y-\beta)}}{\delta\alpha + \eta} \left(-(\delta\alpha + \eta\theta(y-\beta)) + \eta \right).$$

Then, $f'(y) = 0$ gives $y_0 = \frac{\eta(1 + \theta\beta) - \delta\alpha}{\eta\theta}$, when $\eta > \frac{\delta\alpha}{1 + \theta\beta}$.

Therefore, the mode of the FPGLD is given by:

$$mode(y) = \begin{cases} \frac{\eta(1 + \theta\beta) - \delta\alpha}{\eta\theta} & \text{if } \eta > \frac{\delta\alpha}{1 + \theta\beta} \text{ and } \eta > 0. \\ \beta & \text{otherwise} \end{cases}$$

Graphs in Figure 1 have drawn by fixing four parameters and changing the fifth parameter. Figure 1 presents the possible shapes of the pdf of the FPGLD at different parameter values.

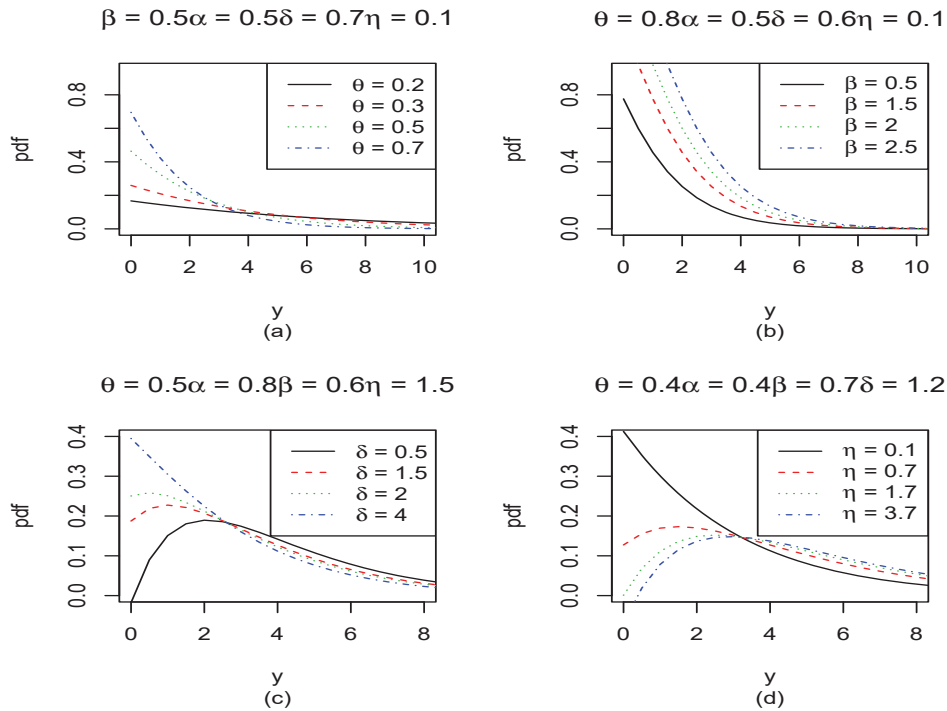


Figure 1: The probability density of FPGLD at different parameter values

(a) β, α, δ and η are fixed, and θ values are changed, (b) θ, α, δ and η are fixed, and β values are changed, (c) θ, β, α and η are fixed, and δ values are changed, (d) θ, β, α and δ are fixed, and η values are changed.

The corresponding cumulative distribution function of the FPGLD is given by:

$$F(y) = 1 - \left(1 + \frac{\eta\theta(y-\beta)}{\delta\alpha + \eta} \right) e^{-\theta(y-\beta)}, \quad (8)$$

where, $y > \beta \geq 0, \theta > 0, \delta\alpha > -\eta, \delta\alpha > -\eta\theta(y-\beta)$.

Sub-models of the FPGLD

The Five-parameter generalized Lindley distribution is nested with six existing Lindley family distributions when setting different particular numerical values of subsets of parameters, namely Lindley distribution (Lindley, 1958), Two-parameter Lindley distribution (Shanker et.al.,2013b), Quasi Lindley distribution (Shanker et.al.,2013a), Shanker distribution (Shanker, 2015), Lindley distribution with location parameter (Monsef, 2016), and Three-parameter Lindley distribution (Shanker et.al.,2017). Table 2 summarizes these modified Lindley distributions as sub-models of the FPGLD. From the knowledge of parameters in the sub-models of the FPGLD, the performance of the newly introduced shape parameters, δ and α in FPGLD in a data set could be studied comparing with TPLwLD, and the performance of the location parameter in a data set could be studied comparing TPLwLD and TwPLD.

Table 2. Sub-models of the FPGLD

Distribution	Parameters				References	
	Shape		Scale	Location		
FPGLD($\theta, \beta, \alpha, \delta, \eta$)	δ	α	η	θ	β	in this paper
LD(θ)	θ	1	1	θ	0	Lindley (1958)
TwPLD(θ, η)	θ	1	η	θ	0	Shanker et. al.(2013)
QLD (θ, δ)	δ	1	1	θ	0	Shanker et. al.(2013)
SD(θ)	θ	θ	1	θ	0	Shanker (2015)
TPLwLD(θ, η, β)	θ	1	η	θ	β	Monsef (2016)
ThPLD(θ, α, η)	θ	α	η	θ	0	Shanker et.al.(2017)

3. Statistical properties

In this section, we provide basic statistical properties of the FPGLD such as r^{th} moment about the origin, central moments, moment generating function, and characteristic function.

3.1. Moments and related measures

The statistical properties of the central tendency, dispersion, skewness, and kurtosis can be studied through the moments. The following theorem gives the r^{th} moment about the origin.

Theorem 1. The r^{th} moment about the origin of the FPGLD is given by:

$$\mu'_r = \frac{e^{\theta\beta}}{(\delta\alpha + \eta)\theta^r} \left(r\Gamma(r, \theta\beta)(\delta\alpha - \eta\beta\theta + \eta(r+1)) + \delta\alpha(\theta\beta)^r e^{-\theta\beta} + \eta(r+1)(\theta\beta)^r e^{-\theta\beta} \right). \tag{9}$$

Proof.

$$\begin{aligned} \mu'_r &= \int_{\beta}^{\infty} y^r \frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha + \eta\theta(y - \beta) \right) e^{-\theta(y-\beta)} dy \\ &= \frac{\theta e^{\theta\beta}}{\delta\alpha + \eta} \left(\delta\alpha \int_{\beta}^{\infty} y^r e^{-\theta y} dy + \eta\theta \int_{\beta}^{\infty} y^{r+1} e^{-\theta y} dy - \eta\theta\beta \int_{\beta}^{\infty} y^r e^{-\theta y} dy \right) \\ &= \frac{\theta e^{\theta\beta}}{\delta\alpha + \eta} \left(\frac{\delta\alpha}{\theta^{r+1}} \Gamma(r+1, \theta\beta) + \frac{\eta}{\theta^{r+1}} \Gamma(r+2, \theta\beta) - \frac{\eta\beta}{\theta^r} \Gamma(r+1, \theta\beta) \right) \\ &= \frac{e^{\theta\beta}}{(\delta\alpha + \eta)\theta^r} \left(r\Gamma(r, \theta\beta)(\delta\alpha - \eta\beta\theta + \eta(r+1)) + \delta\alpha(\theta\beta)^r e^{-\theta\beta} + \eta(r+1)(\theta\beta)^r e^{-\theta\beta} \right). \end{aligned}$$

Substituting $r = 1, 2, 3$ and 4 in equation (9), the first four moments about the origin are derived as:

$$\begin{aligned} \mu'_1 &= \frac{1}{(\delta\alpha + \eta)\theta} \left(\delta\alpha(1 + \theta\beta) + \eta(2 + \theta\beta) \right) = \mu, \\ \mu'_2 &= \frac{1}{(\delta\alpha + \eta)\theta^2} \left(\delta\alpha(2 + \theta\beta(2 + \theta\beta)) + \eta(6 + \theta\beta(4 + \theta\beta)) \right), \\ \mu'_3 &= \frac{1}{(\delta\alpha + \eta)\theta^3} \left(\delta\alpha \left(6 + \theta\beta(6 + \theta\beta(3 + \theta\beta)) \right) + \eta \left(24 + \theta\beta(18 + \theta\beta(6 + \theta\beta)) \right) \right), \end{aligned}$$

and

$$\mu'_4 = \frac{1}{(\delta\alpha + \eta)\theta^4} \left(\delta\alpha \left(24 + \theta\beta(24 + \theta\beta(12 + \theta\beta(4 + \theta\beta))) \right) + \eta \left(120 + \theta\beta(96 + \theta\beta(36 + \theta\beta(8 + \theta\beta))) \right) \right).$$

Then, the r^{th} -order moments about the mean can be obtained by using the relationship between moments about the mean and moments about the origin, i.e)

$$\mu_r = E \left[(Y - \mu)^r \right] = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \mu'_i \mu^{r-i}.$$

Therefore, some r^{th} -order moments about the mean are:

$$\begin{aligned} \mu_2 &= -\mu^2 + \mu'_2 = \frac{\delta\alpha(\delta\alpha + 4\eta) + 2\eta^2}{(\delta\alpha + \eta)^2\theta^2} = \sigma^2, \\ \mu_3 &= 2\mu^3 - 3\mu'_2\mu + \mu'_3 = \frac{2 \left(\delta\alpha \left((\delta\alpha)^2 + 6\eta^2 + 6\eta(\delta\alpha) \right) + 2\eta^3 \right)}{(\delta\alpha + \eta)^3\theta^3}, \text{ and} \\ \mu_4 &= -3\mu^4 + 6\mu'_2\mu^2 - 4\mu'_3\mu + \mu'_4 \\ &= \frac{3 \left(\delta\alpha \left(3(\delta\alpha)^3 + 24\eta(\delta\alpha)^2 + 44\eta^2(\delta\alpha) + 32\eta^3 \right) + 8\eta^4 \right)}{(\delta\alpha + \eta)^4\theta^4}. \end{aligned}$$

Now, the coefficient of variation ($c.v$), measures of skewness (γ_1), measures of kurtosis (γ_2), and the Index of dispersion/Fano factor (γ_3) of the FPGLD can be derived as:

$$\begin{aligned} c.v &= \frac{(\mu_2)^{1/2}}{\mu'_1} = \frac{\sqrt{\delta\alpha(\delta\alpha + 4\eta) + 2\eta^2}}{\delta\alpha(1 + \theta\beta) + \eta(2 + \theta\beta)}, \\ \gamma_1 &= \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{2 \left(\delta\alpha \left((\delta\alpha)^2 + 6\eta^2 + 6\eta(\delta\alpha) \right) + 2\eta^3 \right)}{(\delta\alpha(\delta\alpha + 4\eta) + 2\eta^2)^{3/2}}, \\ \gamma_2 &= \frac{\mu_4}{(\mu_2)^2} = \frac{3 \left(\delta\alpha \left(3(\delta\alpha)^3 + 24\eta(\delta\alpha)^2 + 44\eta^2(\delta\alpha) + 32\eta^3 \right) + 8\eta^4 \right)}{(\delta\alpha(\delta\alpha + 4\eta) + 2\eta^2)^2}, \text{ and} \\ \gamma_3 &= \frac{\mu_2}{\mu'_1} = \frac{\delta\alpha(\delta\alpha + 4\eta) + 2\eta^2}{(\delta\alpha + \eta)\theta(\delta\alpha(1 + \theta\beta) + \eta(2 + \theta\beta))}. \end{aligned}$$

The horizontal symmetry, and dispersion can be measured by γ_1 , and γ_3 , respectively. Figures 4 and 5 (Appendix) show various patterns of the kurtosis and the skewness functions of FPGLD at different parameter values, respectively. From these figures, it is clear that the kurtosis value is increasing when δ is increasing and decreasing when η is increasing for $\delta \leq 1$. Among the different formats of α ; $\alpha = 1$, $\alpha = \delta$, $\alpha = \delta^2$, and $\alpha = \delta^3$, the maximum flexibility is obtained when $\alpha = 1$, i-e) $\delta\alpha = \delta$, in terms of having higher kurtosis value for $\delta \leq 1$. Further, the skewness value is increasing with δ and decreasing with η for $\delta \leq 1$. Figures 6 and 7 (Appendix) represent different shapes of the Fano factor function of FPGLD at different parameter values. Figure 6 (a), (b), (c), and (d) have drawn by fixing θ , β , and η and changing δ and α . Note that all shapes are anti-U shaped and the higher Fano factor values are obtained mostly when $\alpha = 1$. Figure 7 (a), and (b) have drawn by fixing α , δ and

η and changing θ , and β , respectively. All graphs show a monotonic decreasing pattern, and the Fano factor value is increasing when β or θ value is decreasing. When comparing Figures 6 and 7, it is clear that the effect on the Fano factor function of changing δ is totally different than the effect of changing θ .

3.2. Moment generating and characteristic function

The moment generating function is useful to determine the distribution of a random variable. The following theorem provides the moment generating function of the FPGLD.

Theorem 2. The moment generating function say $M_Y(t)$ of the FPGLD is given as follows:

$$M_Y(t) = \frac{\theta e^{\beta t}}{(\delta\alpha + \eta)(t - \theta)^2} \left(-\delta\alpha(t - \theta) + \eta\theta \right). \tag{10}$$

Proof.

$$\begin{aligned} M_Y(t) &= E(e^{tY}) \\ &= \int_{\beta}^{\infty} e^{ty} \frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha + \eta\theta(y - \beta) \right) e^{-\theta(y-\beta)} dy \\ &= \frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha \int_{\beta}^{\infty} e^{ty-\theta(y-\beta)} dy + \eta\theta \int_{\beta}^{\infty} ye^{ty-\theta(y-\beta)} dy - \eta\theta\beta \int_{\beta}^{\infty} ye^{ty-\theta(y-\beta)} dy \right) \end{aligned}$$

The integrals of the above equation will be taken separately as follows:

$$\begin{aligned} \delta\alpha \int_{\beta}^{\infty} e^{ty-\theta(y-\beta)} dy &= \frac{e^{\theta\beta} \delta\alpha}{t - \theta} \left(-e^{\beta(t-\theta)} \right) = \frac{\delta\alpha e^{\beta t}}{(\theta - t)} \\ \eta\theta \int_{\beta}^{\infty} ye^{ty-\theta(y-\beta)} dy &= e^{\theta\beta} \eta\theta \int_{\beta(\theta-t)}^{\infty} \frac{z}{\theta - t} e^{-z} \frac{dz}{\theta - t} \quad ; \quad z = y(\theta - t) \\ &= \frac{\eta\theta e^{\theta\beta}}{(\theta - t)^2} \Gamma(2, \beta(\theta - t)) = \frac{\eta\theta e^{\beta\theta}}{(\theta - t)^2} \left(1 + \beta(\theta - t) \right) \end{aligned}$$

Therefore,

$$\begin{aligned} M_Y(t) &= \frac{\theta}{\delta\alpha + \eta} \left(\frac{-\delta\alpha e^{\beta t}}{(t - \theta)} + \frac{\eta\theta e^{\beta t}}{(\theta - t)^2} (1 + \beta(\theta - t)) + \eta\theta\beta e^{\beta t} (t - \theta) \right) \\ &= \frac{\theta e^{\beta t}}{(\delta\alpha + \eta)(t - \theta)^2} \left(\delta\alpha(\theta - t) + \eta\theta \right). \end{aligned}$$

Similarly, the characteristic function say, $\psi(t)$ of the FPGLD can be derived as follows:

$$\psi_Y(t) = E(e^{itY}) = \frac{\theta e^{\beta it}}{(\delta\alpha + \eta)(\theta - it)^2} \left(\delta\alpha(\theta - it) + \eta\theta \right). \quad (11)$$

3.3. Quantile function

The quantile function of FPGLD can be found by solving $F(y) = u, 0 < u < 1$. It is useful for the quantile estimations and for simulation studies. So, the u^{th} quantile function of FPGLD is derived as:

$$\begin{aligned} F(y) &= 1 - \left(1 + \frac{\eta\theta(y-\beta)}{\delta\alpha + \eta} \right) e^{-\theta(y-\beta)} = u \\ \Rightarrow \left(\delta\alpha + \eta + \eta\theta(y-\beta) \right) e^{-\theta(y-\beta)} &= (1-u)(\delta\alpha + \eta). \end{aligned}$$

This equation can be rewritten as:

$$-\left(\frac{\delta\alpha}{\eta} + 1 + \theta(y-\beta) \right) e^{-\theta(y-\beta) - \frac{\delta\alpha}{\eta} - 1} = \frac{(u-1)(\delta\alpha + \eta)}{\eta} e^{-\frac{\delta\alpha}{\eta} - 1}.$$

Clearly $-\left(\frac{\delta\alpha}{\eta} + 1 + \theta(y-\beta) \right)$ is the negative branch of Lambert function, and one writes it symbolically as W_{-1} . Therefore, the quantile function of the FPGLD can be written in terms of the negative branch of the Lambert function as:

$$-\left(\frac{\delta\alpha}{\eta} + 1 + \theta(y-\beta) \right) = W_{-1} \left(\frac{(u-1)(\delta\alpha + \eta)}{\eta} e^{-\frac{\delta\alpha}{\eta} - 1} \right).$$

Hence,

$$y = \beta - \frac{\delta\alpha + \eta}{\eta\theta} - \frac{1}{\theta} W_{-1} \left(\frac{(u-1)(\delta\alpha + \eta)}{\eta} e^{-\frac{\delta\alpha}{\eta} - 1} \right); y > \beta, 0 < u < 1. \quad (12)$$

Then, the first three quartiles of the FPGLD can be derived by substituting $u = 0.25, 0.5$ and 0.75 in equation (12) and given by:

$$\begin{aligned} Q_1 &= \beta - \frac{\delta\alpha + \eta}{\eta\theta} - \frac{1}{\theta} W_{-1} \left(\frac{(-0.75)(\delta\alpha + \eta)}{\eta} e^{-\frac{\delta\alpha}{\eta} - 1} \right), \\ Q_2 &= \beta - \frac{\delta\alpha + \eta}{\eta\theta} - \frac{1}{\theta} W_{-1} \left(\frac{(-0.5)(\delta\alpha + \eta)}{\eta} e^{-\frac{\delta\alpha}{\eta} - 1} \right), \text{ and} \end{aligned}$$

$$Q_3 = \beta - \frac{\delta\alpha + \eta}{\eta\theta} - \frac{1}{\theta} W_{-1} \left(\frac{(-0.25)(\delta\alpha + \eta)}{\eta} e^{-\frac{\delta\alpha}{\eta} - 1} \right).$$

4. Reliability properties

In this section, we study some important reliability properties of FPGLD, namely the survival function/reliability function $S(y)$, hazard rate function/failure rate function $h(y)$, reversed hazard rate function $r(y)$, cumulative hazard rate function $H(y)$, mean residual life function $m(y)$, Lorenz curve $L(F(y))$, and Benferroni curve $B(F(y))$.

4.1. Hazard rate and mean residual life function

1. The survival function of equation (7) is defined as:

$$S(y) = 1 - F(y) = \left(1 + \frac{\eta\theta(y - \beta)}{\delta\alpha + \eta} \right) e^{-\theta(y - \beta)}; y > \beta. \tag{13}$$

It is clear that, $S(\beta) = 1$ and $\lim_{y \rightarrow \infty} S(y) = 0$.

2. The hazard rate function(hrf) of the FPGLD is defined as:

$$h(y) = \lim_{\Delta y \rightarrow 0} \frac{P(y < Y < y + \Delta y | Y > y)}{\Delta y} = \frac{f(y)}{S(y)} = \frac{\theta \left(\delta\alpha + \eta\theta(y - \beta) \right)}{\delta\alpha + \eta + \eta\theta(y - \beta)}; y > \beta. \tag{14}$$

Further, it can be seen that, $h(\beta) = \frac{\theta\delta\alpha}{\delta\alpha + \eta} = f(\beta)$ and $\lim_{y \rightarrow \infty} h(y) = \theta$.

Figure 2 illustrates the hazard rate function of FPGLD at different parameter values. It is approximately same hazard rate shape of the TPLwLD.

3. The reversed hazard function of FPGLD is defined as:

$$r(y) = \lim_{\Delta y \rightarrow 0} \frac{P(y < Y < y + \Delta y | Y < y)}{\Delta y} = \frac{\theta \left(\delta\alpha + \eta\theta(y - \beta) \right) e^{-\theta(y - \beta)}}{\delta\alpha + \eta - \left(1 + \eta\theta(y - \beta) \right) e^{-\theta(y - \beta)}}; y > \beta. \tag{15}$$

4. The cumulative hazard rate function of FPGLD is defined as:

$$H(y) = \int_{\beta}^y h(t) dt = -\log[S(y)] = -\log \left(1 + \frac{\eta\theta(y - \beta)}{\delta\alpha + \eta} \right) e^{-\theta(y - \beta)}. \tag{16}$$

5. The following theorem gives the mean residual life function of FPGLD.

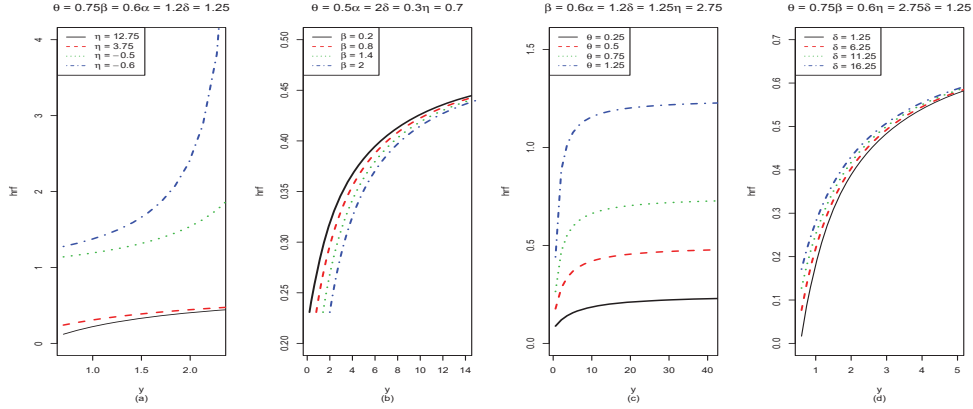


Figure 2: The hazard rate function of FPGLD at different parameter values

(a) θ, β, α , and δ are fixed, and η values are changed, (b) θ, α, δ and η are fixed, and β values are changed, (c) β, α, δ and η are fixed, and θ values are changed, (d) θ, β, η , and δ are fixed, and δ values are changed.

Theorem 3. The mean residual life function of FPGLD is given by:

$$m(y) = \frac{\delta\alpha + \eta(2 + \theta(y - \beta))}{\theta(\delta\alpha + \eta + \eta\theta(y - \beta))}. \tag{17}$$

Proof.

$m(y) = \frac{1}{1 - F(y)} \int_y^\infty tf(t)dt - y$, consider the integrals separately as follows:

$$\begin{aligned} \int_y^\infty tf(t)dt &= \int_y^\infty t \frac{\theta}{\theta\alpha + \eta} (\delta\alpha + \eta\theta(y - \beta)) e^{-\theta(y-\beta)} dt \\ &= \frac{\theta e^{\theta\beta}}{\delta\alpha + \eta} \left(\delta\alpha \int_y^\infty te^{-\theta t} dt + \eta\theta \int_y^\infty t^2 e^{-\theta t} dt - \beta\eta\theta \int_y^\infty te^{-\theta t} dt \right) \\ &= \frac{\theta e^{\theta\beta}}{\delta\alpha + \eta} \left(\delta\alpha \left(\frac{\delta\alpha}{\theta^2} \Gamma(2, \theta y) + \frac{\eta\theta}{\theta^3} \Gamma(3, \theta y) - \frac{\beta\eta\theta}{\theta^2} \Gamma(2, \theta y) \right) \right) \\ &= \frac{(1 + \theta y)(\delta\alpha + 2\eta - \beta\eta\theta) + \eta(\theta y)^2}{\theta(\delta\alpha + \eta)} e^{-\theta(y-\beta)}. \end{aligned}$$

Therefore,

$$m(y) = \frac{(1 + \theta y)(\delta\alpha + 2\eta - \beta\eta\theta) + \eta(\theta y)^2}{\theta(\delta\alpha + \eta + \eta\theta(y - \beta))} - y = \frac{\delta\alpha + \eta(2 + \theta(y - \beta))}{\theta(\delta\alpha + \eta + \eta\theta(y - \beta))}.$$

Then, equation (17) satisfies the following properties:

$$m(y) \geq 0, m(\beta) = \frac{\delta\alpha + 2\eta}{\theta(\delta\alpha + \eta)}, \text{ and } \lim_{y \rightarrow \infty} m(y) = \frac{1}{\theta}.$$

4.2. Lorenz and Bonferroni curves

The concept of the Lorenz and Bonferroni curves were formulated by Bonferroni to measure the income inequalities. They are widely used in economics, reliability, demography, medicine, and insurance. The following theorem gives the function of the Lorenz curve of FPGLD.

Theorem 4. The Lorenz curve is defined for FPGLD as:

$$L(F(y)) = 1 - \frac{\int_y^\infty xf(x)dx}{\mu} = 1 - \frac{e^{-\theta(y-\beta)} \left[(1+y\theta) \left(\delta\alpha + \eta(2 + \theta(y-\beta)) \right) \right]}{\alpha\delta(1 + \theta\beta) + \eta(2 + \theta\beta)}. \quad (18)$$

Proof.

$$L(F(y)) = 1 - \frac{\int_y^\infty xf(x)dx}{\mu}.$$

Note that

$$\begin{aligned} & \int_y^\infty xf(x)dx \\ &= \int_y^\infty x \frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha + \eta\theta(y-\beta) \right) e^{-\theta(y-\beta)} dx \\ &= \frac{\theta}{\delta\alpha + \eta} \left[e^{\theta\beta} \delta\alpha \int_y^\infty xe^{-\theta y} dx + e^{\theta\beta} \eta\theta \int_y^\infty x^2 e^{-\theta y} dy - \eta\theta\beta e^{\theta\beta} \int_y^\infty xe^{-\theta y} dy \right] \\ &= \frac{\theta}{\delta\alpha + \eta} \left[\frac{e^{\theta\beta} \delta\alpha}{\theta^2} \Gamma(2, y\theta) + \frac{e^{\theta\beta} \eta\theta}{\theta^3} \Gamma(3, y\theta) - \frac{e^{\theta\beta} \eta\theta\beta}{\theta^2} \Gamma(2, y\theta) \right] \\ &= \frac{e^{-\theta(y-\beta)}}{(\delta\alpha + \eta)\theta} \left[(1+y\theta) \left(\delta\alpha + \eta(2 + \theta(y-\beta)) \right) \right]. \end{aligned}$$

Therefore,
$$L(F(y)) = 1 - \frac{e^{-\theta(y-\beta)} \left[(1+y\theta) \left(\delta\alpha + \eta(2 + \theta(y-\beta)) \right) \right]}{\alpha\delta(1 + \theta\beta) + \eta(2 + \theta\beta)}.$$

Then, the function of Bonferroni curve for the FPGLD is defined as:

$$\begin{aligned}
 B(F(y)) &= \frac{L(F(y))}{F(y)} \\
 &= \frac{(\delta\alpha + \eta) \left[\delta\alpha(1 + \theta\beta) + \eta(2 + \theta\beta) - e^{-\theta(y-\beta)} \left((1 + y\theta) \left(\delta\alpha + \eta(2 + \theta(y-\beta)) \right) \right) \right]}{\left[(\delta\alpha + \eta) - \left((\delta\alpha + \eta)\eta\theta(y-\beta) \right) e^{-\theta(y-\beta)} \right] \left[\alpha\delta(1 + \theta\beta) + \eta(2 + \theta\beta) \right]}. \quad (19)
 \end{aligned}$$

4.3. Renyi entropy

The Renyi entropy (Renyi, 1961) is a basic uncertainty measure of a distribution say $H_R(\gamma)$ and an extension of Shannon entropy (Shannon et al., 1949). This entropy is widely used in ecology and quantum information. The following theorem gives the Renyi entropy of FPGLD.

Theorem 5. The Renyi entropy of the FPGLD is given by:

$$\begin{aligned}
 H_R(\gamma) &= \frac{1}{1-\gamma} \log \int_{\beta}^{\infty} (f(y))^{\gamma} dy \\
 &= \frac{1}{1-\gamma} \log \left(\frac{\theta^{\gamma-1} (\delta\alpha)^{\gamma}}{(\delta\alpha + \eta)^{\gamma}} \sum_{k=0}^{\gamma} \binom{\gamma}{k} \left(\frac{\eta}{\delta\alpha\gamma} \right)^k k\Gamma(k) \right); \gamma \geq 0, \gamma \neq 1. \quad (20)
 \end{aligned}$$

Proof.

$$\begin{aligned}
 H_R(\gamma) &= \frac{1}{1-\gamma} \log \int_{\beta}^{\infty} (f(y))^{\gamma} dy \\
 &= \frac{1}{1-\gamma} \log \int_{\beta}^{\infty} \left(\frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha + \eta\theta(y-\beta) \right) e^{-\theta(y-\beta)} \right)^{\gamma} dy \\
 &= \frac{1}{1-\gamma} \log \int_{\beta}^{\infty} \frac{\theta^{\gamma} (\delta\alpha)^{\gamma}}{(\delta\alpha + \eta)^{\gamma}} \left(1 + \frac{\eta\theta(y-\beta)}{\delta\alpha} \right)^{\gamma} e^{-\gamma\theta(y-\beta)} dy \\
 &= \frac{1}{1-\gamma} \log \int_{\beta}^{\infty} \frac{\theta^{\gamma} (\delta\alpha)^{\gamma}}{(\delta\alpha + \eta)^{\gamma}} \sum_{k=0}^{\gamma} \binom{\gamma}{k} \left(\frac{\eta\theta(y-\beta)}{\delta\alpha} \right)^k e^{-\gamma\theta(y-\beta)} dy \\
 &= \frac{1}{1-\gamma} \log \left(\frac{\theta^{\gamma} (\delta\alpha)^{\gamma}}{(\delta\alpha + \eta)^{\gamma}} \sum_{k=0}^{\gamma} \binom{\gamma}{k} \left(\frac{\eta\theta}{\delta\alpha} \right)^k \int_{\beta}^{\infty} (y-\beta)^k e^{-\gamma\theta(y-\beta)} dy \right) \\
 &= \frac{1}{1-\gamma} \log \left(\frac{\theta^{\gamma-1} (\delta\alpha)^{\gamma}}{(\delta\alpha + \eta)^{\gamma}} \sum_{k=0}^{\gamma} \binom{\gamma}{k} \left(\frac{\eta}{\delta\alpha\gamma} \right)^k k\Gamma(k) \right).
 \end{aligned}$$

5. The size-biased of FPGLD

The application of the size-biased distributions known as weighted distributions has been significantly used in forestry and wood product studies (Gove, 2003a) incorporating sampling probabilities that are proportional to weighted function $w(y)$. The size-biased distributions is defined as:

$$f_w(y) = \frac{w(y)f(y)}{E(w(y))}, \tag{21}$$

where, $w(y) = y^\gamma$ is a non-negative weighted function of order γ . Then, equation (21) can be rewritten as $f_Y^\gamma(y) = \frac{y^\gamma f(y)}{E(y^\gamma)}$, where $Y_r \sim f_Y^\gamma(y)$ is the size-biased random variable. The following theorem gives the density function for the size-biased FPGLD.

Theorem 6. The density function for sized-biased FPGLD is given by:

$$f_Y^\gamma(y) = y^\gamma \theta^{\gamma+1} \left(\frac{\delta\alpha + \eta\theta(y - \beta)}{A} \right) e^{-\theta y}; y > \beta, \gamma > 0, \tag{22}$$

where, $A = \gamma\Gamma(\gamma, \theta\beta) \left(\delta\alpha + \eta(\gamma + 1 - \theta\beta) \right) + e^{-\theta\beta} (\theta\beta)^\gamma \left(\delta\alpha + \eta(\gamma + 1) \right)$.

Proof.

$$f_Y^\gamma(y) = \frac{y^\gamma f(y)}{E(y^\gamma)}.$$

Note that

$$\begin{aligned} E(y^\gamma) &= \int_{\beta}^{\infty} y^\gamma f(y) dy \\ &= \int_{\beta}^{\infty} y^\gamma \frac{\theta}{\delta\alpha + \eta} \left(\delta\alpha + \eta\theta(y - \beta) \right) e^{-\theta(y-\beta)} dy \\ &= \frac{\theta e^{\theta\beta}}{\delta\alpha + \eta} \left(\frac{\delta\alpha}{\theta^{\gamma+1}} \Gamma(\gamma + 1, \theta\beta) + \frac{\eta\theta}{\theta^{\gamma+2}} \Gamma(\gamma + 2, \theta\beta) - \frac{\eta\theta\beta}{\theta^{\gamma+1}} \Gamma(\gamma + 1, \theta\beta) \right) \\ &= \frac{\theta e^{\theta\beta}}{(\delta\alpha + \eta)\theta^{\gamma+1}} \left(\Gamma(\gamma, \theta\beta) \left(\delta\alpha\gamma + \eta\gamma(\gamma + 1) - \eta\theta\beta\gamma \right) + \delta\alpha(\theta\beta)^\gamma e^{-\theta\beta} + \right. \\ &\qquad \qquad \qquad \left. \eta(\gamma + 1)(\theta\beta)^\gamma e^{-\theta\beta} \right) \\ &= \frac{\theta e^{\theta\beta}}{(\delta\alpha + \eta)\theta^\gamma} \left(\gamma\Gamma(\gamma, \theta\beta) \left(\delta\alpha + \eta(\gamma + 1 - \theta\beta) \right) + \right. \\ &\qquad \qquad \qquad \left. e^{-\theta\beta} (\theta\beta)^\gamma \left(\delta\alpha + \eta(\gamma + 1) \right) \right). \end{aligned}$$

Therefore,

$$f_Y^\gamma(y) = \frac{\frac{y^\gamma \theta}{\delta \alpha + \eta} \left(\delta \alpha + \eta \theta (y - \beta) \right) e^{-\theta(y-\beta)}}{\frac{\theta e^{\theta \beta}}{(\delta \alpha + \eta) \theta^\gamma} \left(\gamma \Gamma(\gamma, \theta \beta) \left(\delta \alpha + \eta (\gamma + 1 - \theta \beta) \right) + e^{-\theta \beta} (\theta \beta)^\gamma \left(\delta \alpha + \eta (\gamma + 1) \right) \right)}$$

$$= y^\gamma \theta^{\gamma+1} \left(\frac{\delta \alpha + \eta \theta (y - \beta)}{\gamma \Gamma(\gamma, \theta \beta) (\delta \alpha + \eta (\gamma + 1 - \theta \beta)) + e^{-\theta \beta} (\theta \beta)^\gamma (\delta \alpha + \eta (\gamma + 1))} \right) e^{-\theta y}.$$

The length biased probability density function can be derived from size-biased pdf of FPGLD by substituting $\gamma = 1$. The length-biased probability density function is given by:

$$f_Y^1(y) = y \theta^2 \left(\frac{\delta \alpha + \eta \theta (y - \beta)}{\delta \alpha + \eta (2 - \theta \beta) + \theta \beta (\delta \alpha + 2 \eta)} \right) e^{-\theta(y-\beta)}; y > \beta, \gamma > 0. \quad (23)$$

6. Parameter estimation and inference

In this section, the parameter estimation and inference are given. In the parameter estimation of FPGLD, the method of moment estimators (MME) and maximum likelihood estimators (MLE) methods are introduced.

6.1. Method of moment estimation

The method of moment estimators can be derived by equating the raw-moments, say μ'_r ,

to the sample moments, say $\frac{\sum_{i=1}^n y_i^r}{n}$, $r = 1, 2, 3, 4, 5$

Then, we need to solve the following system of non-linear equations.

$$n \left(\delta \alpha (1 + \theta \beta) + \eta (2 + \theta \beta) \right) - \theta (\delta \alpha + \eta) \sum_{i=1}^n y_i = 0$$

$$n \left(\delta \alpha (2 + \theta \beta (2 + \theta \beta)) + \eta (6 + \theta \beta (4 + \theta \beta)) \right) - \theta^2 (\delta \alpha + \eta) \sum_{i=1}^n y_i^2 = 0$$

$$n \left(\delta \alpha \left(6 + \theta \beta (6 + \theta \beta (3 + \theta \beta)) \right) + \eta \left(24 + \theta \beta (18 + \theta \beta (6 + \theta \beta)) \right) \right) - \theta^3 (\delta \alpha + \eta) \sum_{i=1}^n y_i^3 = 0$$

$$n \left(\delta \alpha \left(24 + \theta \beta (24 + \theta \beta (12 + \theta \beta (4 + \theta \beta))) \right) + \eta \left(120 + \theta \beta (96 + \right.$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial \beta^2} &= \sum_{i=1}^n \frac{(\eta \theta)^2}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \alpha^2} &= \sum_{i=1}^n \frac{-\delta^2}{(\delta \alpha + \eta \theta (y_i - \beta))^2} + \frac{n \delta^2}{(\delta \alpha + \eta)^2}, \\
\frac{\partial^2 l}{\partial \delta^2} &= \sum_{i=1}^n \frac{-\alpha^2}{(\delta \alpha + \eta \theta (y_i - \beta))^2} + \frac{n \alpha^2}{(\delta \alpha + \eta)^2}, \\
\frac{\partial^2 l}{\partial \eta^2} &= \sum_{i=1}^n \frac{-\theta^2 (y_i - \beta)^2}{(\delta \alpha + \eta \theta (y_i - \beta))^2} + \frac{n}{(\delta \alpha + \eta)^2}, \\
\frac{\partial^2 l}{\partial \theta \partial \beta} &= \sum_{i=1}^n \frac{-\eta \delta \alpha}{(\delta \alpha + \eta \theta (y_i - \beta))^2} + n, \\
\frac{\partial^2 l}{\partial \theta \partial \alpha} &= \sum_{i=1}^n \frac{-\eta \delta (y_i - \beta)}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \theta \partial \delta} &= \sum_{i=1}^n \frac{-\eta \alpha (y_i - \beta)}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \theta \partial \eta} &= \sum_{i=1}^n \frac{\delta \alpha (y_i - \beta)}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \beta \partial \alpha} &= \sum_{i=1}^n \frac{\eta \theta \delta}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \beta \partial \delta} &= \sum_{i=1}^n \frac{\eta \theta \alpha}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \beta \partial \eta} &= \sum_{i=1}^n \frac{-\eta \delta \theta}{(\delta \alpha + \eta \theta (y_i - \beta))^2}, \\
\frac{\partial^2 l}{\partial \alpha \partial \delta} &= \sum_{i=1}^n \frac{\eta \theta (y_i - \beta)}{(\delta \alpha + \eta \theta (y_i - \beta))^2} - \frac{n \eta}{(\delta \alpha + \eta)^2}, \\
\frac{\partial^2 l}{\partial \alpha \partial \eta} &= \sum_{i=1}^n \frac{-\delta \theta (y_i - \beta)}{(\delta \alpha + \eta \theta (y_i - \beta))^2} + \frac{n \delta}{(\delta \alpha + \eta)^2}, \text{ and} \\
\frac{\partial^2 l}{\partial \delta \partial \eta} &= \sum_{i=1}^n \frac{-\alpha \theta (y_i - \beta)}{(\delta \alpha + \eta \theta (y_i - \beta))^2} + \frac{n \alpha}{(\delta \alpha + \eta)^2}.
\end{aligned}$$

Let $\hat{p} = (\hat{\theta}, \hat{\beta}, \hat{\alpha}, \hat{\delta}, \hat{\eta})$ be MLE of p . By the asymptotic theory the estimators are asymptotically normal 5-variate with mean $(\theta, \beta, \alpha, \delta, \eta)$, and observed information matrix is given

by:

$$I(y) = \begin{pmatrix} -\frac{\partial^2 l}{\partial \theta^2} & -\frac{\partial^2 l}{\partial \theta \partial \beta} & -\frac{\partial^2 l}{\partial \theta \partial \alpha} & -\frac{\partial^2 l}{\partial \theta \partial \delta} & -\frac{\partial^2 l}{\partial \theta \partial \eta} \\ -\frac{\partial^2 l}{\partial \beta \partial \theta} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \alpha} & -\frac{\partial^2 l}{\partial \beta \partial \delta} & -\frac{\partial^2 l}{\partial \beta \partial \eta} \\ -\frac{\partial^2 l}{\partial \alpha \partial \theta} & -\frac{\partial^2 l}{\partial \alpha \partial \beta} & -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \delta} & -\frac{\partial^2 l}{\partial \alpha \partial \eta} \\ -\frac{\partial^2 l}{\partial \delta \partial \theta} & -\frac{\partial^2 l}{\partial \delta \partial \beta} & -\frac{\partial^2 l}{\partial \delta \partial \alpha} & -\frac{\partial^2 l}{\partial \delta^2} & -\frac{\partial^2 l}{\partial \delta \partial \eta} \\ -\frac{\partial^2 l}{\partial \eta \partial \theta} & -\frac{\partial^2 l}{\partial \eta \partial \beta} & -\frac{\partial^2 l}{\partial \eta \partial \alpha} & -\frac{\partial^2 l}{\partial \eta \partial \delta} & -\frac{\partial^2 l}{\partial \eta^2} \end{pmatrix}$$

at $\theta = \hat{\theta}, \beta = \hat{\beta}, \alpha = \hat{\alpha}, \delta = \hat{\delta}, \eta = \hat{\eta}$. By the asymptotic theory, the estimates are approximately multivariate normal. Therefore, the $(1 - \alpha)100\%$ confidence interval for the parameters $\theta, \beta, \alpha, \delta, \eta$ are given by:

$$\begin{aligned} \hat{\theta} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\theta})}, & \quad \hat{\beta} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\beta})}, & \quad \hat{\alpha} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\alpha})}, \\ \hat{\delta} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\delta})}, & \quad \hat{\eta} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\eta})} \end{aligned}$$

wherein, the $\text{var}(\hat{\theta}), \text{var}(\hat{\beta}), \text{var}(\hat{\alpha}), \text{var}(\hat{\delta})$, and $\text{var}(\hat{\eta})$ are the variance of $\hat{\theta}, \hat{\beta}, \hat{\alpha}, \hat{\delta}$, and $\hat{\eta}$, respectively, and can be derived by diagonal elements of $I^{-1}(y)$ and $z_{\alpha/2}$ is the critical value at α level of significance.

7. Applications

In this section, we perform a simulation study to examine the behavior of FPGLD's parameter estimates by MLE method, performance of location parameter β , and performance of scale parameter θ when it is incorporated in the mixing proportion. Further, the real-world applications are used to study the performance of the FPGLD with TPLwLD, LD, TwPLD, QLD, SD, and ThPLD. The estimates of the parameters for each distribution has been derived by the MLE method.

7.1. Simulation study

7.1.1 Performance of maximum likelihood method

Here, we discuss the simulation study for the unknown parameter estimations of FPGLD by maximum likelihood method for different sample sizes. The combination of parameter values are set to $\theta = 0.5, \delta = 0.1, \alpha = 0.2, \eta = 0.4, \beta = 1.5$. Then, the steps of the simulation study are given below:

1. Generate 1000 samples for each of the sample size, $n = 20, n = 50, n = 80$ and $n = 100$ using equation (12).
2. Calculate the average MSE for the parameters of FPGLD using the equation

$$\text{MSE}(p) = \frac{\sum_{i=1}^{1000} (\hat{p}_i - p)^2}{1000}, \text{ where } p = (\theta, \delta, \alpha, \eta, \beta), \text{ represents the parameter set.}$$

Table 3 summarizes the average mean square error(MSE) values of FPGLD at different sample sizes. According to Table 3, the average MSE values for parameters $\theta, \beta, \alpha, \delta$ and η decreases when sample size increases. Further, it is notable that decreasing rates of average MSE for the parameters θ and β are higher than decreasing rates of average MSE for the parameters $\delta, \alpha,$ and η . This indicates that the parameters of the mixing components, θ and β are highly sensitive than the parameters $\delta, \alpha,$ and η that are introduced from the latent variable distribution in the unknown parameter estimations for this model.

Table 3. Simulation results for the average MSE values

Parameter	MSE			
	$n = 20$	$n = 50$	$n = 80$	$n = 100$
$\theta = 0.5$	0.014061	0.004528	0.004479	0.002029
$\delta = 0.1$	0.009918	0.009916	0.009871	0.009870
$\alpha = 0.2$	0.039809	0.039541	0.039524	0.039456
$\eta = 0.4$	0.158916	0.158795	0.156714	0.156594
$\beta = 1.5$	0.135841	0.040434	0.025789	0.019406

7.1.2 Performance of the FPGLD when the location parameter $\beta = 0$

In this subsection, the performance of the FPGLD is examined by a simulation study when the location parameter $\beta = 0$. It was done by comparing FPGLD $(\theta, \beta, \alpha, \delta, \eta)$ and FPGLD $(\theta, \beta = 0, \alpha, \delta, \eta)$ for selected values of skewness, EK, and Fano factor. The study is designed as follows:

1. Generate random samples of size, $n = 150$ from FPGLD $(\theta, \beta, \alpha, \delta, \eta)$ with various skewness (SK), Excees kurtosis (EK), and Fano factor (FF) values by setting the parameter values.
2. Fit the FPGLD $(\theta, \beta, \alpha, \delta, \eta)$ and FPGLD $(\theta, \beta = 0, \alpha, \delta, \eta)$ to the generated data sets.
3. Calculate the differences of negative log-likelihood $(-2\log L)$ values for every generated data sets as:

$$\left(-2\log L(\text{FPGLD}(\theta, \beta = 0, \alpha, \delta, \eta)) \right) - \left(-2\log L(\text{FPGLD}(\theta, \beta, \alpha, \delta, \eta)) \right)$$

The table 6 (Appendix) summarizes the differences of $-2\log L$ values between FPGLD $(\theta, \beta = 0, \alpha, \delta, \eta)$ and FPGLD $(\theta, \beta, \alpha, \delta, \eta)$. We may notice that $-2\log L$ difference is decreasing when skewness, EK, and Fano factor values are increasing. Hence, this simulation study reveals that the inclusion of the location parameter in this distribution resists the flexibility to cover the higher skewness, EK, and Fano factor values.

7.1.3 Performance of scale parameter θ when that is incorporated in the mixing proportion

Here, we compare the LD and QLD using a simulation study since they just differ in their defined mixing proportion. i-e) while the LD's mixing proportion is defined incorporating the scale parameter of the mixing component θ , the QLD's mixing proportion is not incorporated with θ . The similar steps that have designed in section 7.1.2 are followed and $-2\log L$ differences are calculated as: $(-2\log L(\text{QLD}(\theta, \alpha)) - (-2\log L(\text{LD}(\theta)))$. Table 7 summarizes the differences of $-2\log L$ values between QLD and LD. We may notice that $-2\log L$ difference is decreasing when skewness, EK, and Fano factor values are increasing. The results indicates that the incorporation of the scale parameter in the mixing proportion in LD resists the flexibility to cover the higher skewness, EK, and Fano factor values.

7.2. Real-world applications

The performance of the FPGLD with respect to the sub-models is now considered by using real-world applications. The negative log-likelihood ($-2\log L$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Kolmogorov-Smirnov Statistics (K-S Statistics) are utilized to compare the performance of distributions. The estimates of the parameters for each distribution has been derived by the MLE method. The following four real-world data sets have been fitted to the distributions for the goodness of fit of distributions.

Data set 1: This data set is the relief times (in minutes) of the 20 patients receiving an analgesic and reported by Gross and Clark (1975).

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

Data set 2: The data set reported by Bjerkedal (1960) that represents the survival times(in days) of 72 guinea pigs infected with virulent tubercle bacilli is given below:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

Data set 3: The data set was given by Fuller et. al. (1994) that represents the strength data of glass of the aircraft window is given below:

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Data set 4: The data set was used by Lawless (1982) and the data were recorded in tests on the endurance of deep groove ball bearings. The corresponding random variable is the number of million revolutions before failure for each of the 23 ball bearings in the life tests.

17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

Some of the important statistical measures for the data set 1 to 4 are summarized in Table 4:

Table 4. Statistical measures for data set 1 to 4

Data	Sample size	Minimum value	Mean	Median	Skewness	EK	Fano factor
Data 1	20	1.100	1.900	1.700	1.720	2.924	0.261
Data 2	72	12.000	99.819	70.000	1.796	2.614	65.920
Data 3	31	18.830	30.811	29.900	0.405	-0.713	1.708
Data 4	23	17.880	72.230	67.800	0.941	0.488	19.448

Figure 3 (Appendix) shows the density plots that compare the fitted densities of each model with the empirical histogram of the real-world data sets. We can observe that the fitted densities for the FPGLD and TPLwLD show a closer fit with the empirical distributions for real-data sets 1, 3 and 4, and both fitted densities are approximately the same. Further, QLD shows a closer fit with the empirical distribution for the data set 2. Table 8 (Appendix) shows the values of $-2\log L$, AIC, BIC and K-S statistics and critical values of the K-S statistics. According to Table 8, we may note that AIC and BIC values increase when the number of parameters of the distributions increases. Therefore, we use $-2\log L$ values and K-S statistics for the comparison of all models.

Based on the minimum $-2\log L$, and the significant results by K-S statistics, FPGLD and TPLwLD provide a better fit than all other sub-models for the data sets 1, 3, and 4. Data set 1, 3, and 4 have considerably smaller skewness and EK values or smaller Fano factor value. There is no difference between the log-likelihood values of FPGLD and TPLwLD for these data sets. This indicates that $\delta\alpha = \theta$ and the performance of both distributions are the same. Further, when we compare TPLwLD and TwPLD, the likelihood ratio (LR) test statistics for the hypothesis testing $H_0 : \beta = 0$ versus $H_a : \beta \neq 0$ for data 1, 3 and 4 are 22.686, 53.413, and 16.663, respectively, and all are greater than $\chi_{1,0.05}^2 = 3.841$. These results indicate the importance of the location parameter in such type of lifetime data analysis than introducing new shape parameters from the latent variable distribution to give different weights.

On the other hand, it is notable that in most of the real-data applications, the performance of TwPLD, QLD, and ThPLD are the same except for data set 2, where the QLD shows the minimum $-2\log L$ significant result by K-S statistic. The data set 2 has considerably higher skewness, EK and Fano factor values. To show the effect of the higher skewness, EK , and Fano factor values, data set 5 (Appendix) was also used to fit the distributions, and Table 5 summarizes the results of the goodness of fit test. These results indicate that QLD performs well than other distributions for the data sets with considerably higher skewness, EK and Fano factor values. A possible reason may be that it has the flexibility with the format $\delta\alpha = \delta \neq \theta$ and exclusion of the location parameter. Therefore, when developing a best-fitted distribution for the data sets that have higher skewness, EK , and Fano factor values, it is recommended to use proper mixing weights and mixing components without a location parameter.

We hope these findings could be helpful for the researchers when they develop a new Lindley family distribution.

Table 5. $-2\log L$, and K-S statistics of the NGAD and LwLD for different data sets with various EK values

Data	Distribution	Sample size	Skewness	EK	Fano factor	$-2\log L$	AIC
Data 5	FPGLD	60	2.437	7.018	2.547	252.416	262.416
	TPLwLD					252.416	258.416
	TwPLD					266.401	270.401
	QLD					250.920	254.920
	ThPLD					266.401	272.401
	LD					259.171	261.171
	SD					257.096	259.096

8. Conclusions

In this paper, we have introduced a new five-parameter generalized Lindley distribution(FPGLD) based on exponential and gamma mixtures with different mixing proportions and done a comparison study with its sub-models. The FPGLD generalizes the Lindley distribution with location parameter(TPLwLD), Quasi Linley distribution (QLD), Two-parameter Lindley distribution (TwPLD), Three-parameter Lindley distribution (ThPLD), Shanker distribution (SD), and classical Lindley distribution (LD). Hence, using FPGLD a researcher can compare the other existing lifetime distributions without considering its sub-models separately. The statistical properties and estimates of parameters are obtained for the FPGLD and compared it with its sub-models.

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REFERENCES

BJERKEDAL, T., (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli, *American Journal of Hygiene*, 72(1), pp. 130–148.

FULLER, E. R., FREIMAN, S. W., QUINN, J. B., QUINN, G., CARTER, W., (1994). Fracture mechanics approach to the design of glass aircraft windows - A case study, *Proceedings of SPIE - The International Society for Optical Engineering*, pp. 419–430.

GHITANY, M. E., ATIEH, B., NADARAJAH, S., (2008). Lindley distribution and its Applications, *Mathematics and Computers in Simulation*, 78(4), pp. 493–506.

GOVE, J. H., (2003a). Estimation and applications of size-biased distributions in forestry, *Modeling forest systems*. Edited by A. Amaro, D. Reed, and P. Soares. CABI Publishing, Wallingford, UK., pp. 201–212.

- GROSS, A. J., CLARK, V. A., (1975). Survival Distributions, *Reliability Applications in the Biometrical Sciences*, John Wiley, New York, USA.
- HIBATULLAH, R., WIDYANINGSIH, Y., ABDULLAH, S., (2018). Marshall - Olkin extended power Lindley distribution with application, *J. Ris. and Ap. Mat.*, 2(2), pp. 84 – 92.
- LAWLESS, J. F., (1982). Statistical models and methods for lifetime data, John Wiley and Sons, New York, USA.
- LINDLEY, D. V., (1958). Fiducial distributions and Bayes' theorem, *Journal of the Royal Statistical Society, Series B*, 20(1), pp. 102–107.
- LINDLEY, D. V., (1965). Introduction to Probability and Statistics from Bayesian viewpoint, part II, Inference, *Cambridge university press, New York*.
- MONSEF, M. M. E. A., (2016). A new Lindley distribution with location parameter, *Communications in Statistics-Theory and Methods*, 45(17), pp. 5204–5219.
- NICHOLS, M. D., PADGETT, W. J., (2006). A bootstrap control chart for Weibull percentiles, *Quality and Reliability Engineering International*, 22(2), pp. 141–151.
- RENYI, A. (1961). On measures of entropy and information, In Fourth Berkeley Symposium on Mathematical Statistics and Probability, pp. 547–561.
- SHANKER, R., MISHR, A., (2013a). A Quasi Lindley Distribution, *African Journal of Mathematics and Computer Science Research*, 6(4), pp. 64 –71.
- SHANKER, R., MISHR, A., (2013b). A two-parameter Lindley distribution, *Statistics in Transition new Series*, 14(1), pp. 45–56.
- SHANKER, R., SHARMA, S., SHANKER, R., (2013). A Two-Parameter Lindley Distribution for Modeling Waiting and Survival Times Data, *Applied Mathematics*, 4, pp. 363–368.
- SHANKER, R., (2015). Shanker Distribution and Its Applications, *International Journal of Statistics and Applications*, 5(6), pp. 338–348.
- SHANKER, R., SHUKLA, K. K., SHANKER, R., TEKIE, A. L., (2017). A Three-parameter Lindley Distribution, *American Journal of Mathematics and Statistics*, 7(1), pp. 15–26.
- SHANNON, C., WEAVER, W., (1949). *The mathematical theory of communication*, Chicago: University of Illinois Press.

APPENDIX

Table 6. Differences of $-2\log L$ values between FPGLD $(\theta, \beta = 0, \alpha, \delta, \eta)$ and FPGLD $(\theta, \beta, \alpha, \delta, \eta)$

$\downarrow SK(EK)$	FF \rightarrow						
	6.70	7.70	12.70	15.80	20.50	28.30	34.60
0.918 (0.496)	37.442	32.105	19.087	15.009	11.112	7.656	6.147
0.920 (0.498)	34.758	29.892	16.788	12.822	9.151	5.896	4.435
0.928 (0.509)	31.223	26.807	13.746	9.840	6.392	3.392	2.239
0.969 (0.571)	28.037	23.353	10.201	6.563	3.370	1.160	0.421
1.044 (0.709)	27.116	22.081	8.693	5.069	2.164	0.368	0.014
1.102 (0.837)	27.108	22.031	8.321	4.656	1.822	0.189	0.002
1.208 (1.107)	27.082	22.003	8.242	4.540	1.634	0.089	0.001

Table 7. Differences of $-2\log L$ values between QLD (θ, α) and LD (θ)

$\downarrow SK(EK)$	FF \rightarrow						
	6.70	7.70	12.70	15.80	20.50	28.30	34.60
0.918 (0.496)	71.804	70.212	64.911	62.578	59.815	56.668	54.969
0.920 (0.498)	70.585	69.019	63.197	60.659	57.701	54.250	52.264
0.928 (0.509)	68.506	66.906	60.250	57.248	53.771	49.493	47.189
0.969 (0.571)	65.186	63.199	54.971	51.261	46.605	41.265	38.026
1.044 (0.709)	62.391	60.046	50.563	46.135	40.695	34.049	29.988
1.102 (0.837)	60.962	58.641	48.365	43.576	37.731	30.405	27.083
1.208 (1.107)	57.306	52.694	41.795	36.615	30.199	22.009	20.084

Data set 5 (Hibatullah.et.al.,(2018): average wind speed per month.

1.04525, 2.78426, 2.54918, 6.90446, 2.46577, 2.83905, 2.09819, 0.47927, 1.41378, 4.77888, 2.28740, 4.79976, 1.32359, 1.71967, 3.52471, 0.38095, 10.9028, 1.38314, 1.89628, 1.03046, 2.44529, 13.1893, 2.16495, 3.78884, 2.20266, 0.71543, 16.4941, 3.14792, 7.72747, 2.84926, 2.68460, 5.45061, 1.32353, 1.48582, 5.10102, 3.00342, 1.77735, 4.88295, 0.80280, 5.02584, 1.50003, 2.01266, 1.74341, 3.11761, 0.80668, 2.65187, 4.64156, 1.65586, 6.95507, 5.83996, 3.33749, 1.27453, 2.29751, 3.26983, 2.65993, 4.53323, 5.73434, 2.09596, 1.52554, 2.71060.

Table 8. MLE, AIC, BIC, K-S statistics and its critical value of the fitted distributions

Data	Model	MLE	$-2logL$	AIC	BIC	K-S Statistic	Critical value
Data 1	FPGLD	$\hat{\theta} = 2.284, \hat{\beta} = 1.024, \hat{\alpha} = 2.655 * 10^{-10}, \hat{\delta} = 7.554 * 10^{-6}, \hat{\eta} = 2.319 * 10^{-5}$	30.988	40.988	45.966	0.121	
	TPLwLD	$\hat{\theta} = 2.284, \hat{\beta} = 1.024, \hat{\alpha} = 4.982 * 10^3, \hat{\delta} = 0.526, \hat{\alpha} = -5.583 * 10^{-6}$	30.988	36.988	39.975	0.121	0.304
	TwPLD	$\hat{\theta} = 0.527, \hat{\alpha} = 1.322 * 10^3$	65.674	69.674	71.666	0.390	
	QLD	$\hat{\theta} = 0.053, \hat{\beta} = 1.025, \hat{\alpha} = 8.175 * 10^3$	65.674	71.674	74.661	0.390	
	ThPLD	$\hat{\theta} = 0.816$	60.499	62.499	63.495	0.341	
Data 2	LD	$\hat{\theta} = 0.795$	59.788	61.788	62.784	0.309	
	FPGLD	$\hat{\theta} = 0.021, \hat{\beta} = 6.886, \hat{\alpha} = 7.174 * 10^{-12}, \hat{\delta} = 1.635 * 10^{-7}, \hat{\eta} = 4.475 * 10^{-8}$	786.511	796.511	807.894	0.145	
	TPLwLD	$\hat{\theta} = 0.022, \hat{\beta} = 6.891, \hat{\alpha} = 225.796, \hat{\delta} = 0.010, \hat{\alpha} = 2.407 * 10^{-15}$	806.844	810.884	815.438	0.198	0.160
	TwPLD	$\hat{\theta} = 0.023, \hat{\alpha} = -0.241$	783.145	787.145	791.698	0.152	
	QLD	$\hat{\theta} = 0.010, \hat{\beta} = 2.356 * 10^{-14}, \hat{\alpha} = 145.12$	806.844	812.884	819.714	0.198	
Data 3	ThPLD	$\hat{\theta} = 0.020$	789.039	791.039	793.316	0.133	
	LD	$\hat{\theta} = 0.020$	788.570	790.570	792.847	0.133	
	FPGLD	$\hat{\theta} = 0.159, \hat{\beta} = 1.821, \hat{\alpha} = 6.031 * 10^{-4}, \hat{\delta} = 1.519 * 10^{-12}, \hat{\eta} = 7.327 * 10^{-5}$	209.115	219.115	226.285	0.106	
	TPLwLD	$\hat{\theta} = 0.159, \hat{\beta} = 18.211, \hat{\alpha} = 566.720, \hat{\delta} = 0.032, \hat{\alpha} = 1.552 * 10^{-18}$	209.115	215.115	219.418	0.106	0.244
	TwPLD	$\hat{\theta} = 0.032, \hat{\alpha} = 2.274 * 10^3$	274.528	278.528	281.396	0.426	
Data 4	QLD	$\hat{\theta} = 0.003, \hat{\beta} = 1.352 * 10^{-13}, \hat{\beta} = 1.352 * 10^{-13}, \hat{\alpha} = 13.603$	274.528	280.529	284.831	0.426	
	ThPLD	$\hat{\theta} = 0.063$	253.988	255.988	257.422	0.333	
	LD	$\hat{\theta} = 0.065$	252.343	254.353	255.787	0.326	
	FPGLD	$\hat{\theta} = 0.035, \hat{\beta} = 14.277, \hat{\alpha} = 2.969 * 10^{-12}, \hat{\delta} = 1.320 * 10^{-6}, \hat{\eta} = 6.005 * 10^{-7}$	226.210	236.210	241.887	0.090	
	TPLwLD	$\hat{\theta} = 0.035, \hat{\beta} = 14.281, \hat{\alpha} = 239.790, \hat{\delta} = 0.014, \hat{\alpha} = 1.635 * 10^{-14}$	226.210	232.210	235.616	0.090	0.284
Data 4	TwPLD	$\hat{\theta} = 0.014, \hat{\alpha} = 3.815 * 10^4$	242.873	246.874	249.141	0.263	
	QLD	$\hat{\theta} = 0.014, \hat{\beta} = 15.372, \hat{\alpha} = -2.652 * 10^{-15}$	242.873	248.873	252.300	0.263	
	ThPLD	$\hat{\theta} = 0.027$	231.471	233.471	234.607	0.149	
	LD	$\hat{\theta} = 0.028$	231.061	233.061	234.196	0.145	
	SD						

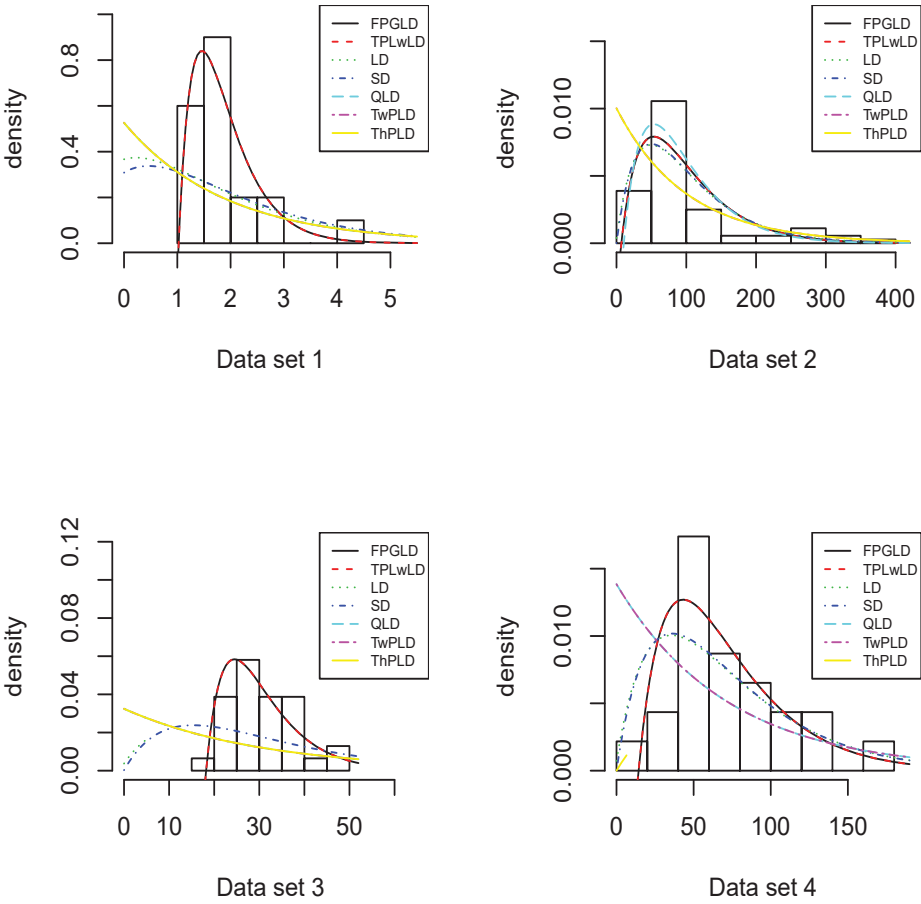


Figure 3: Empirical histograms with fitted densities of distributions

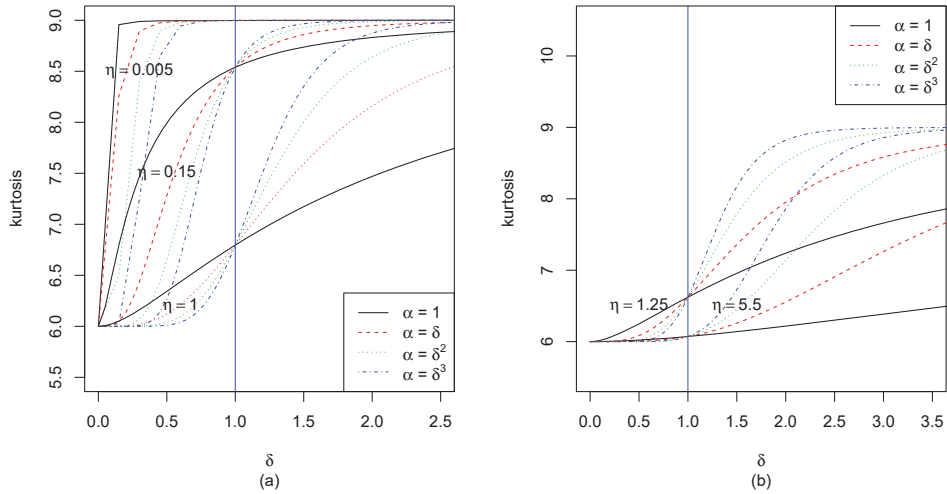


Figure 4: The kurtosis values of FPGLD at different parameter values of δ , α and η

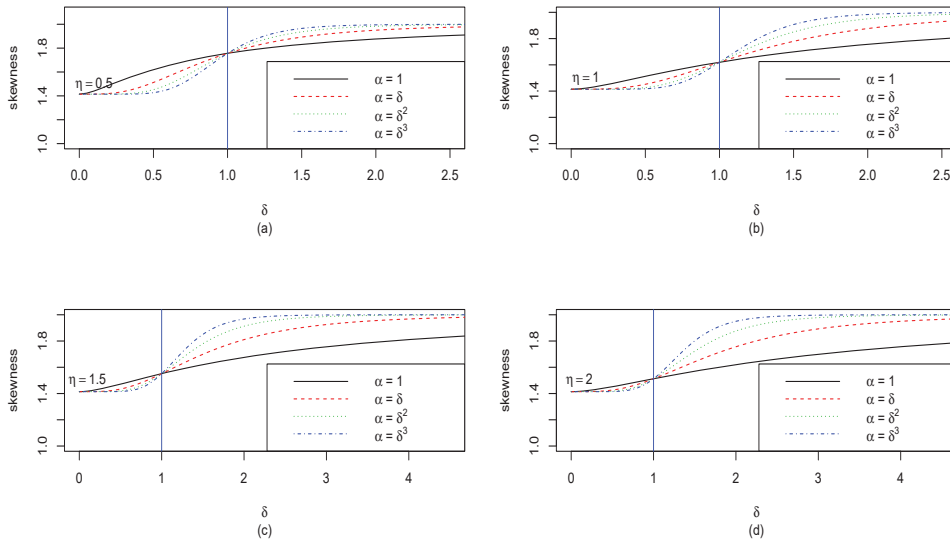


Figure 5: The skewness values of FPGLD at different parameter values of δ , α and η

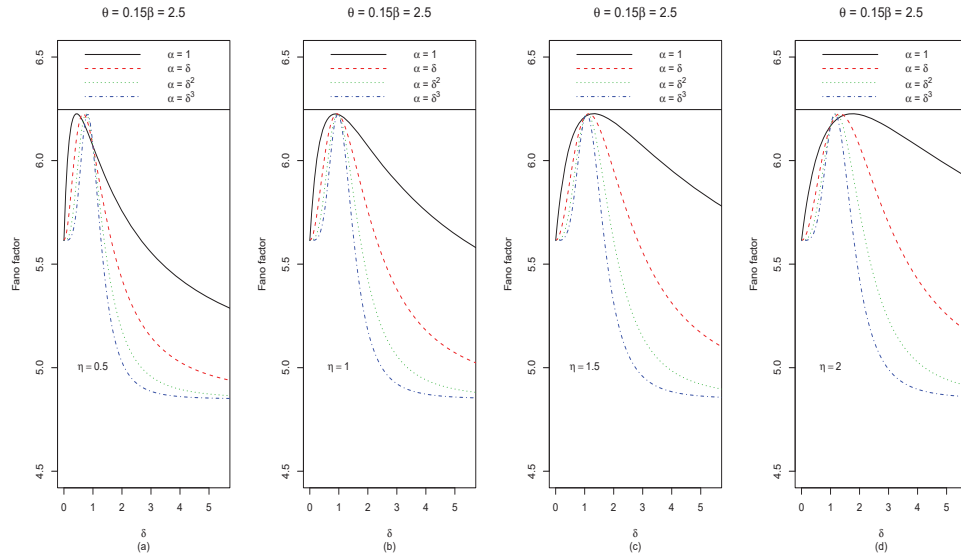


Figure 6: The Fano factor values of FPGLD at different parameter values of δ, α and η

(a) to (d): θ, β and η are fixed, and δ and α values are changed

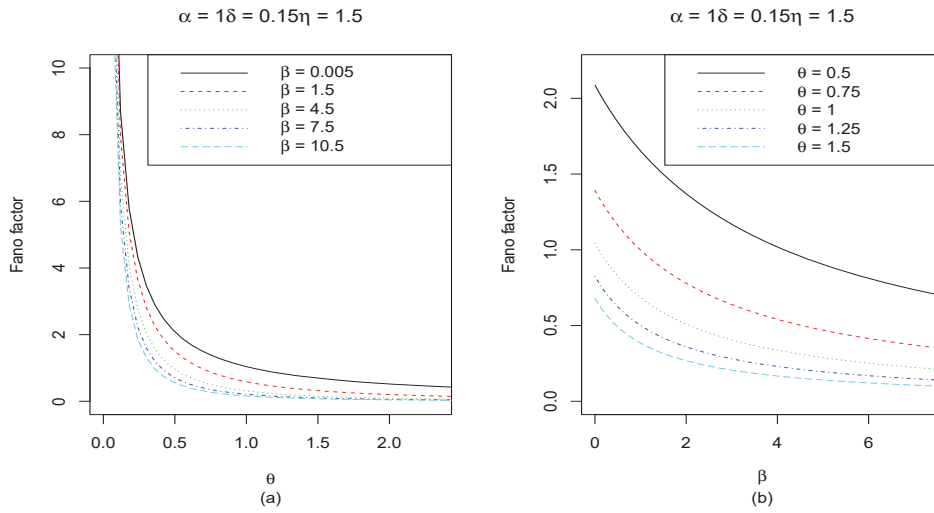


Figure 7: The Fano factor values of FPGLD at different parameter values of β and θ

(a) and (b): δ, α and η are fixed, and θ and β values are changed