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Bargaining under Liquidity Constraints:
Unified Strategic Foundations of the Nash and Kalai Solutions[*]

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Abstract

We provide unified strategic foundations for the Nash (1950) and Kalai (1977) solutions in the context of negotiations under liquidity constraints. We propose an N-round game where in each round a seller and a buyer with limited payment capacity negotiate a bundle of divisible goods, where bundle sizes can vary across rounds, according to Rubinstein's (1982) alternating-offer game. The game implements the Nash solution if $N = 1$ and the Kalai solution if $N = +\infty$ and bundle sizes are infinitesimal. If N is set by one player ex ante, the buyer chooses $N = 1$ while the seller chooses $N = +\infty$. We endogenize liquidity constraints and show they binds for all $N < +\infty$, even when there is no cost in holding liquidity.

JEL Classification: D83
Keywords: bargaining with an agenda, Nash program, bargaining solution.

1 Introduction

We propose an extensive-form game that generalizes Rubinstein (1982) and provides strategic foundations to both the generalized Nash (1950) solution and the proportional solution of Kalai (1977) for environments with pairwise meetings and liquidity constraints commonly used in search-theoretic models. For instance, our game can describe a negotiation between a worker and a firm over the terms of an employment contract, hours of work and compensation, subject to the cash constraint of the firm. Another example is when two traders in an over-the-counter (OTC) market negotiate the size of a security trade financed with a collateralized loan.[1]

Two axiomatic solutions have been routinely used to tackle these bargaining problems: the generalized Nash solution and the Kalai (proportional) solution.[2] The Nash solution has the considerable advantage of having well-established strategic foundations (see, e.g., Osborne and Rubinstein, 1990). It has several disadvantages: the buyer's surplus from trade is non-monotone in his payment capacity, which makes the assumption that the buyer cannot under-report his asset holdings critical; it does not guarantee the concavity of the players' surpluses, which reduces its tractability in general equilibrium settings. In contrast, the proportional solution is highly tractable and it has the natural prediction that traders' surpluses increase as gains from trade expand (e.g., Aruoba et al., 2007). However, the proportional solution is not scale invariant and it does not have solid strategic foundations, such as an extensive-form game with alternating offers. This lack of strategic foundations is problematic for a literature that thrives on rigorous micro-foundations.

In this paper, we propose an extensive-form game indexed by a single parameter and show that it can rationalize both the Nash and the Kalai solutions for two particular values of that parameter. Such unified strategic foundations allow us to identify the underlying agenda of the negotiation as the fundamental difference between the two solutions. Formally, we describe a bargaining game between a buyer with a payment capacity (e.g., liquid wealth, borrowing limit) $z \in \mathbb{R}_+$ and a producer. The consumption good, y, is perfectly divisible and the gains from trade are maximized for some $y = y^*$. The game is composed of $N \in \mathbb{N}$ rounds. In each round, the amount of goods up for negotiation is y^*/N, and the players bargain according to a Rubinstein (1982) alternating-offer game with exogenous risk of termination (or, equivalently, a stochastic horizon). We show that for all N this game admits a (essentially) unique subgame-perfect equilibrium (SPE)

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[1] Examples of these different applications include: Lehmann (2012) or Gu et al. (2019) for a labor search model where wages are negotiated subject to the cash constraint of the firm; Lagos and Zhang (2019) for a model of an OTC market where traders' purchases are constrained by their cash holdings; Dugast et al. (2019) where OTC traders are constrained by a market trading capacity; Rocheteau et al. (2018) for a corporate finance model where loan contracts are negotiated bilaterally between a bank and an entrepreneur; and all the New Monetarist literature surveyed in Rocheteau and Nosal (2017) and Lagos et al. (2017).

[2] The Kalai solution should not be confused with the Kalai-Smorodinsky (1975) solution that is scale invariant but not strongly monotone.

and the terminal allocation can be solved in closed form. When the risk of termination goes to zero in each round, the game implements the generalized Nash solution when $N = 1$, all the output is sold at once, and the Kalai solution when $N = +\infty$, the output is sold one infinitesimal bundle at a time. The limiting case $N = +\infty$ also implements the axiomatic ordinal solution O'Neill et al. (2004) for bargaining problems with an agenda. If we allow one of the two players to set the agenda of the negotiation, N, then the buyer chooses $N = 1$ (Nash) whereas the producer chooses $N = +\infty$ (Kalai). The producer prefers to sell his output gradually to delay the time at which the liquidity constraint of the buyer binds, thereby avoiding distorting surplus sharing in a way that is favorable to the buyer.

We generalize our bargaining game by allowing output limits to vary across N rounds, in which case the agenda is an arbitrary sequence of output levels, $\{\Delta \bar{y}_n\}_{n=1}^N$, that are negotiated sequentially. We show that the insights from the game with constant limits are robust to this extension. In particular, the final allocation is obtained from a recursion that involves solving a generalized Nash bargaining problem in each iteration. We give a full characterization of the set of agendas that implement the Kalai solution for a given payment capacity. The only agendas that generate the Kalai outcome uniformly for all payment capacities are obtained at the limit when N becomes large and the quantities negotiated in each round, $\Delta \bar{y}_n$, become arbitrarily small.

We endogenize the buyer's payment capacity by assuming the buyer can borrow z at some interest rate before the negotiation starts. Even if the interest rate is zero, and hence liquidity is costless, the buyer's choice of z is such that the liquidity constraint binds for all $N < +\infty$ and output is less than its first-best level, y^*, which generalizes a key result of Lagos and Wright (2005) obtained for $N = 1$. Moreover, the choice of z is not monotone in N. It is only at the limit, when $N = +\infty$, that a zero interest rate implements y^*.

Finally, we endogenize both the agenda of the negotiation and the buyer's payment capacity. If bundle sizes are constant across rounds, and if the agent setting the agenda is chosen at random before the negotiation starts, then the optimal payment capacity of the buyer decreases with the probability that she sets the agenda. If the agenda is set unilaterally by the seller who can specify different bundle sizes across rounds, any subgame perfect equilibrium implements the proportional solution. Moreover, if the buyer's payment capacity is not observed at the time the seller chooses the agenda, then the only agenda that is part of an equilibrium is gradual.

Related literature

Extensive-form bargaining games with alternating offers a la Rubinstein (1982) have been applied to the description of decentralized markets by Rubinstein and Wolinsky (1985). A risk of breakdown between

consecutive offers was introduced by Binmore et al. (1986). A survey of this literature can be found in Osborne and Rubinstein (1990). The first application to models with liquidity constraints came from Shi (1995) and Trejos and Wright (1995) in economies where the buyer's payment capacity is indivisible. Versions with divisible payments have been developed by Shi (1997) and Lagos and Wright (2005) and are surveyed in Lagos et al. (2017).

There are strategic foundations for the proportional solution, e.g., Bossert and Tan (1995) and Dutta (2012), but they are based on the Nash demand game and they assume interpersonal utility comparisons through penalties for incompatible demands or revoking costs.[3] In contrast to those approaches, we propose an extensive-form game with alternating offers, in the spirit of Rubinstein (1982), that generates a unique SPE allocation invariant to rescaling of players' utilities. Our approach allows us to provide unified foundations for both the Nash and the Kalai solutions where the scaling factor of the proportional solution is determined endogenously.

Fershtman (1990) describes a two-round game where agents negotiate sequentially the split of two pies of different sizes. The negotiation in each round is conducted according to the Rubinstein game, like in our setting. The agenda consists in the order according to which the two pies are negotiated. In our model, the agenda is identified with the number of rounds or, equivalently, the bundle size to be negotiated in each round, and the total gains from trade are endogenous due to a liquidity constraint. Similar to Fershtman (1990), we also find that players can disagree on the agenda with implications for efficiency.[4]

Our game is also related to the Stole and Zwiebel (1996) game in the literature on intra-firm wage bargaining where a firm bargains sequentially with N workers. See Brugemann et al. (2018) for a recent re-examination of this game. In our model the buyer negotiates repeatedly with the same seller who supplies a divisible commodity. Hence, contracts are two-dimensional and specify a price and a quantity. If the agents fail to reach an agreement in one round, they move to the next round, but the agreements of earlier rounds are preserved. In the Stole-Zwiebel game, all previous agreements are erased. In Smith (1999), the firm treats each worker as the marginal worker. It is also the case in our model that in each round agents are effectively negotiating over the marginal surplus but the disagreements points of both players are endogenous and determined recursively.

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[3]Bossert and Tan (1995) adopts a multi-stage arbitration game where incompatible demands are penalized for the player who asks a higher surplus. Dutta (2012) introduces another stage after the Nash demand game where players face revoking costs, and obtained the proportional solution when such cost approaches infinity.

[4]Fershtman (1990) shows that the players disagree on the agenda (the ordering of the pies) if they value the pies differently, i.e., each player would like to negotiate first the pie that she values the least and her opponent values the most.

2 Basic setting

We describe a bargaining game whereby two players negotiate the sale of a divisible commodity in exchange for a payment subject to a liquidity constraint. This game is consistent with a wide class of search-theoretic models where buyers and sellers are matched pairwise at random and negotiate the terms of a mutually beneficial trade subject to some payment constraints.[5]

2.1 Allocations and preferences

The two players are called *buyer* and *seller*. The quantity of the divisible commodity produced by the seller for the buyer is denoted $y \in \mathbb{R}^+$. The payment made by the buyer is denoted $p \in \mathbb{R}_+$. It is subject to a liquidity constraint, $p \leq z$, where z is the payment capacity. Preferences over outcomes of the negotiation, $(y, p) \in \mathbb{R}_+ \times [0, z]$, are represented by the following quasi-linear utility functions:

$$u^b = u(y) - p, \tag{1}$$

$$u^s = -v(y) + p, \tag{2}$$

where the superscripts b and s stand for buyer and seller. As is standard in search-theoretic models, payoffs are linear in the payment.[6] The function $u(y)$ is the buyer's utility from his consumption y. The quantity $v(y)$ is the seller's disutility from producing y. We assume $u'(y) > 0$, $u''(y) < 0$, $u'(0) = +\infty$, $u(0) = v(0) = v'(0) = 0$, $v'(y) > 0$, $v''(y) > 0$, and $u'(y^*) = v'(y^*)$ for some $y^* > 0$. The no-trade allocation, $(y, p) = (0, 0)$, yields $u^b = u^s = 0$.

2.2 Two axiomatic solutions

The generalized Nash solution A common approach to the above bargaining problem is to impose the generalized Nash solution, i.e., $(y, p) \in \arg\max(u^b)^\theta (u^s)^{1-\theta}$, where $\theta \in [0, 1]$ is the buyer's bargaining power, subject to the feasibility constraint, $p \leq z$. The solution (y, p) is determined by $p = p_\theta^{GN}(y) = \min\left\{z, p_\theta^{GN}(y^*)\right\}$, where

$$p_\theta^{GN}(y) \equiv [1 - \Theta(y)]u(y) + \Theta(y)v(y), \tag{3}$$

and where the share of the surplus accruing to the buyer is

$$\Theta(y) = \frac{\theta u'(y)}{\theta u'(y) + (1 - \theta)v'(y)}. \tag{4}$$

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[5] The commodity can be a consumption good (e.g., Lagos and Wright, 2005), hours of work (e.g., Pissarides, 2000), or an over-the-counter security (e.g., Duffie et al., 2005), and the payment can be interpreted as money, collateralized loan, or unsecured debt.

[6] In a general equilibrium setting, the payoffs of the buyer and seller would be given by $u(y) + W^b(\omega^b - p)$ and $-v(y) + W^s(\omega^s + p)$ where W^b and W^s are continuation value functions and ω^b and ω^s represent wealth levels. Under quasi-linear preferences, the value functions are linear in wealth, hence the payoffs in (1)-(2).

Note that $\Theta(y)$ is decreasing in y with $\Theta(y^*) = \theta$. Agents trade y^* if it is feasible, $z \geq p_\theta^{GN}(y^*)$, or the buyer exhausts his payment capacity to consume as much as possible, $p_\theta^{GN}(y) = z$.

The Kalai solution The proportional solution of Kalai (1977) selects the allocation (y, p) that maximizes $u^b = u(y) - p$, subject to $u^b = \theta u^s / (1 - \theta)$ and $p \leq z$ for some $\theta \in [0, 1]$ interpreted as the buyer's bargaining share. The solution (y, p) is determined by $p = p_\theta^K(y) = \min\{z, p_\theta^K(y^*)\}$, where

$$p_\theta^K(y) \equiv (1 - \theta)u(y) + \theta \upsilon(y), \tag{5}$$

The Kalai solution coincides with Nash when the liquidity constraint does not bind, but it differs from it when it binds. The Kalai solution is strongly monotone, i.e., $u^b = u(y) - p_\theta^K(y) = \theta\left[u(y) - \upsilon(y)\right]$ increases with y (for $y \leq y^*$) and hence with z. This result does not hold under Nash bargaining where $u^b = \Theta(y)\left[u(y) - \upsilon(y)\right]$ decreases with y when y is close to y^* (e.g., Aruoba et al., 2007). A second advantage of the Kalai solution is its tractability: it preserves the concavity of the surpluses with respect to z and it generates simple closed-form expressions.[7] Relative to Nash, however, the Kalai solution is not scale invariant and lacks solid strategic foundations.

2.3 The Rubinstein game with sliced bundles

We propose an extensive-form game that implements the Nash and Kalai solutions as two polar cases. The game is composed of $N \geq 1$ rounds. In each round, $n \in \{1, ..., N\}$, there are (potentially) infinitely many stages during which the two players bargain over the sale of at most $\Delta \bar{y}_n$ units of consumption good following an alternating-offer protocol as in Rubinstein (1982).[8] With no loss of generality, we assume that $\sum_{n=1}^N \Delta \bar{y}_n = y^*$, and hence over the N rounds allocations with $y = y^*$ are feasible. The quantity limits at each round, $\{\Delta \bar{y}_n : n = 1, .., N\}$, can be interpreted as resulting from various technological constraints: sellers produce bundles of goods sequentially, and each bundle is negotiated immediately after being produced. Alternatively, production is continuous over time but the consumer and the seller meet infrequently but periodically to negotiate the terms of trade. Another view is that the rules of the extensive-form game that determine the allocations in pairwise meetings are not manipulable by the players and are a primitive of the economy (which is the standard assumption in the literature on decentralized markets).

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[7] One example that showcases the tractability provided by the proportional solution is Lester et al. (2012) on costly information acquisition. Another example is Geromichalos et al. (2016) on term structure in OTC markets.

[8] The infinite number of stages in each round can be interpreted as an approximation for a large but finite number of stages. Indeed, it is well known that the equilibrium of the finite horizon version of the Rubinstein game converges to the equilibrium of the infinite horizon game as the horizon of the negotiation becomes large. For a similar formalization, Aumann and Hart (2003) has an extensive-form game which contains infinitely many stages before another phase of action to occur in the context of a cheap talk game.

The round-game is illustrated in Figure 1. In the initial stage of round-n, the buyer makes an offer $(\Delta y_n, \Delta p_n)$ where $\Delta y_n \leq \Delta \bar{y}_n$ is the amount of goods he is willing to purchase in exchange of some payment $\Delta p_n \leq z - p_{n-1}$ where $p_{n-1} = \sum_{j=1}^{n-1} \Delta p_j$ is the sum of payments that have been agreed upon over the first $n-1$ rounds and hence $z - p_{n-1}$ is the remaining payment capacity of the buyer in round n. The seller either accepts the offer or rejects it. If the offer is accepted, $(\Delta y_n, \Delta p_n)$ is implemented (Δy_n is produced and given to the buyer in exchange for Δp_n) and the negotiation moves to round $n + 1$. If the offer is rejected, then there are two possibilities. With probability $1 - \xi^s$, round n is terminated and the players move to round $n+1$ without having reached an agreement in round n, i.e., $(\Delta y_n, \Delta p_n) = (0,0)$. With probability ξ^s, the round-n negotiation continues and the seller becomes the proposer in the following stage. If the seller's offer is rejected, the negotiation moves to the next stage with probability ξ^b, in which case the buyer is the proposer. While the game in Figure 1 assumes that the buyer makes the first offer in all rounds, we also consider the version of the game where the seller is the first proposer in all rounds.
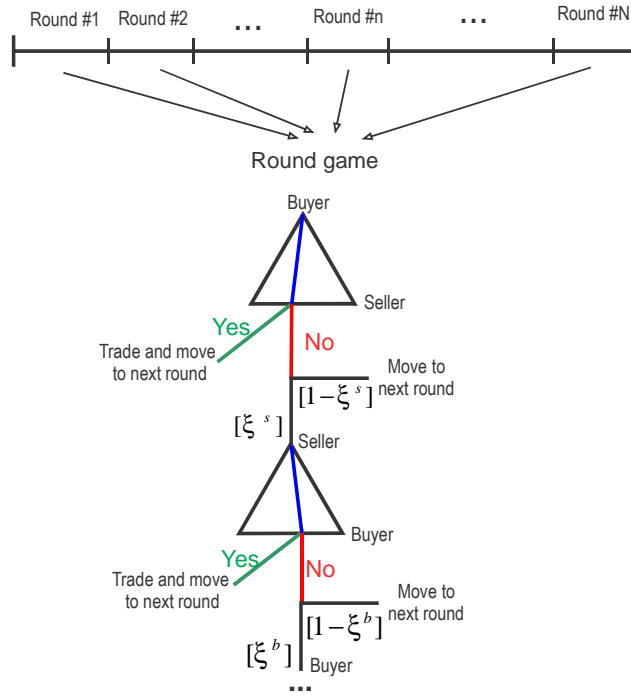
Figure 1: Game tree

The utility levels associated with the interim allocation, $(y_n, p_n) = \left(\sum_{j=1}^{n} \Delta y_j, \sum_{j=1}^{n} \Delta p_j \right)$, are denoted $u_n^b = u(y_n) - p_n$ and $u_n^s = -v(y_n) + p_n$. Interim allocations are subject to two feasibility conditions: $y_n \leq \sum_{j=1}^{n} \Delta \bar{y}_j$ and $p_n \leq z$.

We define a family of Pareto frontiers, $H(u^b, u^s, \bar{y}; z) = 0$, indexed by $\bar{y}$ and z. Each frontier is the

solution to $u^b = \max_{y \leq \bar{y}, p \leq z} \{u(y) - p\}$ subject to $-v(y) + p \geq u^s$. Hence,

$$H(u^b, u^s, \bar{y}; z) = \begin{cases} u(\bar{y}) - v(\bar{y}) - u^b - u^s \text{ if } u^s \leq z - v(\bar{y}), \\ z - v[u^{-1}(u^b + z)] - u^s \text{ otherwise.} \end{cases} \quad (6)$$

We illustrate the Pareto frontiers for a given z but different values for $\bar{y}$ in Figure 2. As long as $\bar{y}$ is sufficiently small relative to z, the Pareto frontier is linear. If $u(\bar{y}) > z$, then the payment constraint binds for u^s sufficiently large, in which case the Pareto frontier is strictly concave over some range.
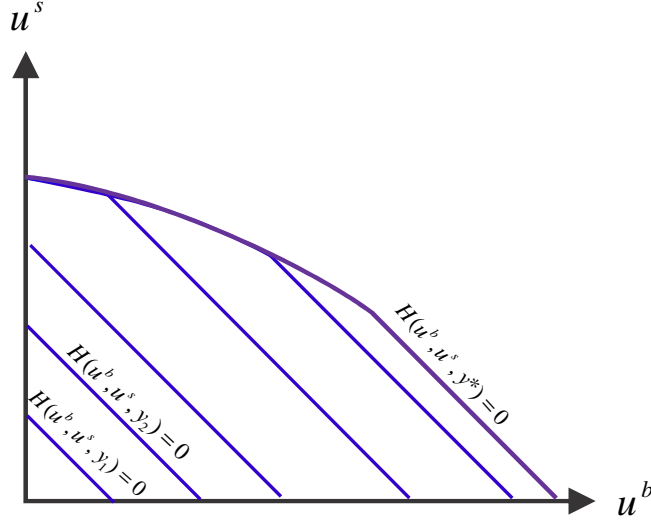
Figure 2: Pareto frontiers for $\bar{y} \in \{y_1, y_2, y^*\}$ and for given z

3 The game with equal-sized bundles

In this section, we impose that all bundles have the same size, i.e., $\Delta\bar{y}_n = y^*/N$ for all n. We will consider the general case where $\Delta\bar{y}_n$ varies across rounds in Section 5. In order to build some intuition, we first solve the two-round game, $N = 2$. We then turn to the general N-round game for arbitrary probabilities of termination within each round and we specialize our analysis later to the case where these probabilities vanish, which eliminates the first-mover advantage.

3.1 The two-round game

In each of the two rounds, the players negotiate over $\Delta\bar{y} = y^*/2$ units of consumption goods and the buyer is the first proposer. We solve the game by backward induction. The subgame starting in the last round is a game with $N = 1$ and quantity limit $\Delta\bar{y}$. Suppose that the initial allocation is $(0,0)$, which would happen if no agreement was reached in the first round. Equilibrium offers are denoted by $(y_1^b, p_1^b, y_1^s, p_1^s)$, where (y_1^χ, p_1^χ) is the offer made by player $\chi \in \{b, s\}$ and the subscript refers to the number of rounds left.

If the liquidity constraint, $p \le z$, does not bind on and off the equilibrium path, then the equilibrium outcome specifies $y_1^b = y_1^s = \Delta \bar{y}$ since otherwise there would be unexploited gains from trade, and $p = p_1^b$ where (p_1^b, p_1^s) solves

$$-v(\Delta \bar{y}) + p_1^b = \xi^s \left[-v(\Delta \bar{y}) + p_1^s \right] \tag{7}$$

$$u(\Delta \bar{y}) - p_1^s = \xi^b \left[u(\Delta \bar{y}) - p_1^b \right]. \tag{8}$$

From (7) the buyer offers $(\Delta \bar{y}, p_1^b)$ that makes the seller indifferent between accepting and rejecting. From (8), if she rejects and if the negotiation is not terminated, the seller offers $(\Delta \bar{y}, p_1^s)$ in the next stage, so as to make the buyer indifferent between accepting or rejecting. The solution to (7)-(8) is $p_1^b = p^b(\Delta \bar{y})$ and $p_1^s = p^s(\Delta \bar{y})$, where

$$p^b(y) \equiv \frac{\xi^s \left(1 - \xi^b \right) u(y) + (1 - \xi^s) v(y)}{1 - \xi^b \xi^s} \tag{9}$$

$$p^s(y) \equiv \frac{\left(1 - \xi^b \right) u(y) + \xi^b (1 - \xi^s) v(y)}{1 - \xi^b \xi^s}. \tag{10}$$

From (9) the buyer's payment when it is his turn to make an offer coincides with $p_\theta^K(\Delta \bar{y})$ for $\theta = (1 - \xi^s)/(1 - \xi^b \xi^s)$. If the seller makes the offer, the buyer's share is reduced to $\xi^b (1 - \xi^s) / (1 - \xi^b \xi^s)$. The liquidity constraint does not bind if $p^s(\Delta \bar{y}) \le z$.

The same logic goes through if the buyer's liquidity constraint binds when either the buyer or the seller makes an offer: the equilibrium offers maximize the offering party's utility subject to the constraint that the responding party is willing to accept the offer. Formally, the buyer's and seller's offers solve:

$$(y_1^b, p_1^b) \in \arg\max_{y,p} \{ u(y) - p \} \tag{11}$$

$$\text{s.t.} \quad -v(y) + p \ge \xi^s \left[-v(y_1^s) + p_1^s \right] \tag{12}$$

$$p \in [0, z], \quad y \le \bar{\Delta} y, \tag{13}$$

and

$$(y_1^s, p_1^s) \in \arg\max_{y,p} \{ -v(y) + p \} \tag{14}$$

$$\text{s.t.} \quad u(y) - p \ge \xi^b \left[u(y_1^b) - p_1^b \right] \tag{15}$$

$$p \in [0, z], \quad y \le \Delta \bar{y}. \tag{16}$$

Following the logic in Rubinstein (1982), these two problems can be solved jointly to deliver a unique $(y_1^b, p_1^b, y_1^s, p_1^s)$. The outcome is illustrated in the left panel of Figure 3, where o^χ is the offer made by player $\chi \in \{b, s\}$ in the utility space. As the figure indicates, both offers are located on the Pareto frontier,

$H(u^b, u^s, \Delta\bar{y}; z) = 0$, and are determined as follows. The buyer's offer generates a utility level to the seller that is equal to a fraction ξ^s of what the seller's offer generates, i.e., $u^s(o^b) = \xi^s u^s(o^s)$. Similarly, $u^b(o^s) = \xi^b u^b(o^b)$.
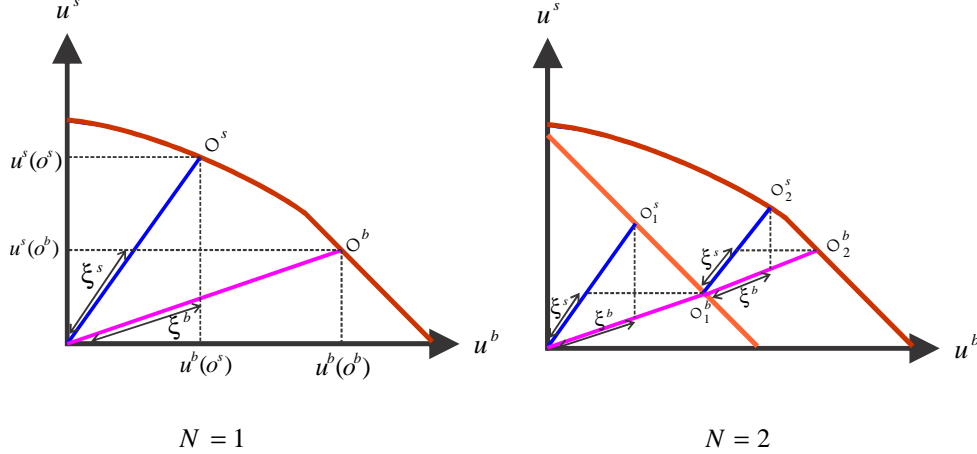

$$N = 1 \qquad\qquad\qquad N = 2$$

Figure 3: Left: one-round negotiation; Right: two-round negotiation.

We now move backward in the game tree to the beginning of round 1. If the players fail to reach an agreement in round 1, then they move to round 2 and the equilibrium offers solve (11)-(16), with the final allocation given by the first offer made by the buyer, (y_1^b, p_1^b). Graphically, the disagreement offer is represented by o_1^b in the right panel of Figure 3. It is located on the lower Pareto frontier corresponding to the output limit $\Delta\bar{y}$. Since the players' beliefs regarding round 2 are consistent with equilibrium play, in round-1 they are effectively negotiating over all feasible allocations achievable from the two rounds combined, i.e., $(y_2, p_2) \in [0, 2\Delta\bar{y}] \times [0, z]$, with the understanding that a failed negotiation leads to the allocation (y_1^b, p_1^b). Applying the same logic as before to this grand negotiation, the first-round offers generate final allocations that solve:

$$(y_2^b, p_2^b) \in \arg\max_{y,p} \{u(y) - p\} \tag{17}$$

$$\text{s.t.} \quad -v(y) + p \geq (1 - \xi^s)[-v(y_1^b) + p_1^b] + \xi^s[-v(y_2^s) + p_2^s] \tag{18}$$

$$p \in [0, z], \quad y \in [0, y^*], \tag{19}$$

and

$$(y_2^s, p_2^s) \in \arg\max_{y,p} \{-v(y) + p\} \tag{20}$$

$$\text{s.t.} \quad u(y) - p \geq (1 - \xi^b)[u(y_1^b) - p_1^b] + \xi^b[u(y_2^b) - p_2^b] \tag{21}$$

$$p \in [0, z], \quad y \in [0, y^*]. \tag{22}$$

10

The problems (17)-(22) are analogous to (11)-(16), except that (18) and (21) include the payoffs from the allocation, (y_1^b, p_1^b), in case the negotiation breaks down.

Graphically, the final allocation is represented by o_2^b in the right panel of Figure 3. The associated utility levels are located on the upper frontier corresponding to the cumulative output limit, $2\Delta\bar{y} = y^*$. The seller is indifferent between accepting o_2^b or receiving a lottery that assigns the disagreement allocation, o_1^b, with probability $1 - \xi^s$ and the seller's offer, o_2^s, with probability ξ^s, that is, $u^s(o_2^b) - u^s(o_1^b) = \xi^s[u^s(o_2^s) - u^s(o_1^b)]$ and $u^b(o_2^s) - u^b(o_1^b) = \xi^b[u^b(o_2^b) - u^b(o_1^b)]$.

Finally, while (17)-(22) computes the final allocation, (y_2^b, p_2^b), we need to specify the equilibrium intermediate allocation at the end of round-1, denoted by $(\hat{y}_1, \hat{p}_1)$, and to check its existence and feasibility. Subgame perfection requires that, in the subgame at the beginning of round-2 with initial allocation $(\hat{y}_1, \hat{p}_1)$, the equilibrium outcome coincides with the final allocation, (y_2^b, p_2^b). The equilibrium offers in that subgame solve (17)-(22) with $(y_1^b, p_1^b) = (\hat{y}_1, \hat{p}_1)$ and feasibility conditions

$$p \in [\hat{p}_1, z], \quad y - \hat{y}_1 \in [0, \bar{\Delta} y]. \tag{23}$$

If $(y_2^b, p_2^b, y_2^s, p_2^s)$ satisfies these feasibility constraints, then it immediately follows that $(\hat{y}_1, \hat{p}_1) = (y_1^b, p_1^b)$, that is, starting with (y_1^b, p_1^b) as the intermediate equilibrium allocation at the end of round 1, (y_2^b, p_2^b) would be the equilibrium round-2 allocation. To see that these feasibility conditions hold, consider two cases. First, suppose that $p_1^b < z$. Then, $y_1^b = \bar{\Delta} y$ and hence (y_1^b, p_1^b) is not located on the upper Pareto frontier, i.e., there are unexploited gains from trade. Since (y_2^b, p_2^b) and (y_2^s, p_2^s) belong to the upper Pareto frontier, it follows that $p_2^\chi > p_1^b$ and $y_2^\chi > y_1^b$, i.e., the feasibility constraints hold. Second, suppose that $p_1^b = z$, and hence $y_1^b \leq \bar{\Delta} y$ and (y_1^b, p_1^b) is on the upper Pareto frontier. In that case, $(y_2^b, p_2^b) = (y_2^s, p_2^s) = (y_1^b, p_1^b)$ and the feasibility conditions are also satisfied.

Our analysis shows that there are two interpretations for (y_1^b, p_1^b). First, (y_1^b, p_1^b) is the final allocation from the last round if the players fail to reach agreement in the first round. Second, (y_1^b, p_1^b) is the equilibrium intermediate allocation at the end of round 1.

3.2 Equilibrium: the recursive solution

Now we extend the above logic to a general N-round game, and the following proposition characterizes the final allocation for an arbitrary number of rounds and arbitrary initial allocation for any subgame.

Proposition 1 (SPE of the Rubinstein game with sliced bundles.) *Consider a subgame of the Rubinstein game with sliced bundles where the identity of the first proposer in each round is $\chi \in \{b, s\}$. The subgame starts in round $n \in \{1, ..., N\}$ with an interim allocation $(y_{n-1}, p_{n-1}) \in [0, (n-1)y^*/N] \times [0, z)$.*

There is a sequence, $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=n}^N$, defined uniquely by $(y_{n-1}^b, p_{n-1}^b) = (y_{n-1}, p_{n-1}) = (y_{n-1}^s, p_{n-1}^s)$, and the following recursion:

$$(y_j^b, p_j^b) \quad \in \quad \arg\max_{y,p} \{u(y) - p\} \tag{24}$$

$$s.t. \quad -v(y) + p \quad \geq \quad (1 - \xi^s)\left[-v(y_{j-1}^\chi) + p_{j-1}^\chi\right] + \xi^s\left[-v(y_j^s) + p_j^s\right] \tag{25}$$

$$p \quad \in \quad \left[p_{j-1}^\chi, z\right], \quad y - y_{j-1}^\chi \leq \frac{y^*}{N}, \tag{26}$$

and

$$(y_j^s, p_j^s) \quad \in \quad \arg\max_{y,p} \{-v(y) + p\} \tag{27}$$

$$s.t. \quad u(y) - p \quad \geq \quad (1 - \xi^b)\left[u(y_{j-1}^\chi) - p_{j-1}^\chi\right] + \xi^b\left[u(y_j^b) - p_j^b\right] \tag{28}$$

$$p \quad \in \quad \left[p_{j-1}^\chi, z\right], \quad y - y_{j-1}^\chi \leq \frac{y^*}{N}. \tag{29}$$

The final allocation of any SPE is $(y, p) = (y_N^\chi, p_N^\chi)$ where $\chi = b$ if the buyer is the first proposer in each round and $\chi = s$ if the seller is the first proposer in each round. The whole game corresponds to $n = 1$ and $(y_0^b, p_0^b) = (0, 0) = (y_0^s, p_0^s)$.

Proposition 1 states that the SPE of the Rubinstein game with sliced bundles is computed as the sequence of subgame perfect equilibria of one-round Rubinstein games. The termination payoffs in the j-th iteration correspond to the payoffs in the $(j-1)$-th iteration, $u_{j-1}^b = u(y_{j-1}^\chi) - p_{j-1}^\chi$ and $u_{j-1}^s = -v(y_{j-1}^\chi) + p_{j-1}^\chi$, where $\chi \in \{b, s\}$ is the identity of the first proposer. As in the two-round game, the two interpretations of (y_1^b, p_1^b) also apply to the sequence $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=n}^N$, and we express the feasibility conditions, (26) and (29), in terms of incrementals, similar to (23). To describe the two interpretations more precisely here, consider the game with the buyer as the first proposer, $\chi = b$.

According to the first interpretation, (y_j^b, p_j^b) is the equilibrium final allocation in the (sub)game composed of the *last* j rounds with initial allocation $(y_{N-j}, p_{N-j}) = (0, 0)$, i.e., no agreement has been reached in the first $N - j$ rounds. As in the two-round game, (y_1^b, p_1^b) is the equilibrium allocation of a game composed of a single round (the N-th round) with initial allocation $(0, 0)$, and (y_2^b, p_2^b) is the final allocation in a game composed of two rounds only (the subgame starting in round $N-1$) with initial allocation $(0, 0)$. By pursuing the same logic, we can conclude that (y_N^b, p_N^b) is the final allocation of the *whole* game.

Alternatively, the terms of the sequence $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=1}^N$ are also intermediate allocations along the equilibrium path leading to the final allocation. More precisely, as the buyer makes the first offer in each round, (y_n^b, p_n^b) is the equilibrium intermediate allocation achieved at the end of the n-th round. Indeed, if (y_N^b, p_N^b) is the final allocation of the entire game, then subgame perfection requires that it is the final

allocation in the subgame starting in round $(n+1)$ with allocation (y_n^b, p_n^b). To verify this claim, consider the sequence, $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=1}^N$, obtained from (24)-(29) with $(y_0^b, p_0^b) = (0,0)$. Then, (y_N^b, p_N^b) is the final allocation at the end of the entire game. However, if we run the program with initial allocation (y_n^b, p_n^b), then we obtain the truncated sequence, $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=n+1}^N$, which confirms that (y_N^b, p_N^b) is also the final allocation of the subgame that start at round $(n+1)$ with intermediate allocation (y_n^b, p_n^b).[9]

In our proof we extend this logic to a general subgame with arbitrary intermediate payoffs, and the key part is to establish that, in the j-th iteration, $u_{j-1}^b = u(y_{j-1}^\chi) - p_{j-1}^\chi$ and $u_{j-1}^s = -v(y_{j-1}^\chi) + p_{j-1}^\chi$, are the relevant payoffs in case the negotiation terminates. The general result is obtained by backward induction. To illustrate the logic of our proof, consider the subgame that starts at round $N-1$ with some interim allocation, (y_{N-2}, p_{N-2}), reached along the equilibrium path, and assume that the buyer makes the first offer in each round. If no agreement is reached in this round, then the game moves to the last round, N. In that subgame, which begins with interim allocation (y_{N-2}, p_{N-2}), the construction and the uniqueness of the SPE follows the standard argument of Rubinstein (1982), and the equilibrium offers, denoted by (y_{N-1}^b, p_{N-1}^b) and (y_{N-1}^s, p_{N-1}^s), are computed according to (24)-(29) with $j = N-1$. Hence, the players understand that, if the negotiation breaks down in round $N-1$, the final allocation is given by (y_{N-1}^b, p_{N-1}^b). Finally, consider the original subgame at round $N-1$. The players' final payoffs will lie within the Pareto frontier $H(u^b, u^s, y_{N-2} + 2y^*/N; z) = 0$, and hence the players are effectively negotiating over outcomes reachable within that Pareto frontier with the termination payoffs corresponding to (y_{N-1}^b, p_{N-1}^b). By the standard logic of a Rubinstein game, its solution, $(y_N^b, p_N^b, y_N^s, p_N^s)$, is computed according to (24)-(29) with (y_{N-1}^b, p_{N-1}^b) as the disagreement point.

A corollary of Proposition 1 is that if $z \geq p^s(y^*)$, then agents trade y^* and the payment is identical to the one of the one-round game, i.e., the solution coincides with the proportional solution with endogenous bargaining shares determined by termination probabilities. Indeed, in that case, the liquidity constraint never binds along the sequence $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=1}^N$ starting with $(y_0, p_0) = (0,0)$. In contrast, if $z < p^s(y^*)$ does not hold, then the liquidity constraint will be binding somewhere along that sequence. If the seller makes the first offer in all rounds, then there exists $\hat{N} \leq N$ such that $p_n^s = p^s(\bar{y}_n) \leq z$ and $p_n^b \leq z$ are slack for all $n < \hat{N}$. In round $\hat{N}$, the game ends with the final allocation $(y_{\hat{N}}^s, z)$.

3.3 Two remarks

We give two remarks, the first regarding the uniqueness and the second regarding robustness. First, while Proposition 1 states that the final allocations are uniquely determined, the SPE is not unique if there is a

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[9]This second interpretation requires that bundle sizes are constant across rounds. Otherwise, the intermediate payoffs need to be computed with a different, but similar, recursion. See Section 5 for details.

rank $J < N$ in the sequence $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=n-1}^N$ such that for all $j \geq J$, $(y_j^b, p_j^b, y_j^s, p_j^s) = (y_N^b, p_N^b, y_N^s, p_N^s)$.
In that case there is indeterminacy regarding the timing of the interim agreements.

Second, we comment on the assumption that the rules of the game specify that, if the negotiation over a bundle fails in one round, the players commit not to renegotiate the unsold bundle when the N rounds are over. We view this assumption as similar to the commitment of the players to obey the rules of the game in any negotiation. For instance, in a take-it-or-leave-it bargaining game, the players are committed to walk away if the first and only offer is rejected. In an alternating offer game, the players are committed to take turns to make offers. In the context of our game, the commitment not to renegotiate is not critical as we could assume that the number of rounds is larger than the number of bundles that are up for negotiation, $\bar{N} > N$, in which case, if the negotiation fails in one round, agents would have the option to renegotiate an unsold bundle later. This case is discussed at the end of our proof of Proposition 1. Indeed, the argument for the first interpretation of the sequence, $\left\{(y_j^b, p_j^b, y_j^s, p_j^s)\right\}_{j=n}^N$, is not affected by the change from N to $\bar{N}$ before y^* is reached, and if it is reached, it remains constant. Hence, the final outcome is not affected either.

3.4 The chained Nash solution

The player making the first offer in each round has an advantage that allows him to capture a larger share of the surplus. In order to eliminate this advantage, we now assume that $\left(\xi^b, \xi^s\right) = \left(e^{-(1-\theta)\varepsilon}, e^{-\theta\varepsilon}\right)$ and we let ε go to 0. So, the probabilities of termination of the current round following rejected offers, $1 - \xi^b$ and $1 - \xi^s$, vanish while their ratio converges to $(1-\theta)/\theta$.

Proposition 2 (The chained Nash solution.) *Suppose* $\left(\xi^b, \xi^s\right) = \left(e^{-(1-\theta)\varepsilon}, e^{-\theta\varepsilon}\right)$ *for some* $\theta \in [0, 1]$ *and consider the limit as* $\varepsilon \to 0$. *The final allocation of the Rubinstein game with sliced bundles is the last term of the sequence* $\{(y_n, p_n)\}_{n=1}^N$ *where* $(y_0, p_0) = (0, 0)$ *and*

$$(y_n, p_n) \in \arg\max_{y,p} \left[u(y) - u(y_{n-1}) - (p - p_{n-1})\right]^\theta \left[-\upsilon(y) + \upsilon(y_{n-1}) + (p - p_{n-1})\right]^{1-\theta} \tag{30}$$

$$\text{s.t.} \quad y - y_{n-1} \leq \frac{y^*}{N} \text{ and } p \leq z.$$

The solution to (30) is such that:

$$\begin{cases} p_n = z = g(y_n, \bar{y}_{n-1}) & z \leq g(\bar{y}_n, \bar{y}_{n-1}) \\ p_n = z \text{ and } y_n = \bar{y}_n & \text{if} \quad g(\bar{y}_n, \bar{y}_{n-1}) < z < p_\theta^K(\bar{y}_n) \\ p_n = p_\theta^K(\bar{y}_n) \text{ and } y_n = \bar{y}_n & z \geq p_\theta^K(\bar{y}_n), \end{cases} \tag{31}$$

where

$$g(y, \bar{y}) \equiv [1 - \Theta(y)]u(y) + \Theta(y)\upsilon(y) + [\Theta(y) - \theta][u(\bar{y}) - \upsilon(\bar{y})]. \tag{32}$$

As is well known (see, e.g., Osborne and Rubinstein (1990), Proposition 310.3), if $N = 1$, then the limit of the Rubinstein game as the risk of breakdown vanishes coincides with the generalized Nash solution with buyer's bargaining power equal to θ. We extend this result to the case where there are multiple rounds of the Rubinstein game, $N > 1$. In each round, the interim allocation, (y_n, p_n), is given by the generalized Nash solution where the disagreement point corresponds to the interim payoffs from the previous round, $u(y_{n-1}) - p_{n-1}$ and $-v(y_{n-1}) + p_{n-1}$. We illustrate this solution in Figure 4 where the bargaining set in each round is represented by a purple area and the solution is given by the tangency point between the red curve representing the generalized Nash product and the bargaining set. In rounds where the liquidity constraint, $p_n \leq z$, does not bind, $y = \bar{y}_n$ and $p = p_\theta^K(\bar{y}_n)$, i.e., the solution coincides with the proportional solution. In later rounds when $p_n \leq z$ binds, the Nash solution differs from the proportional solution and assigns a larger share of the surplus to the buyer than θ.
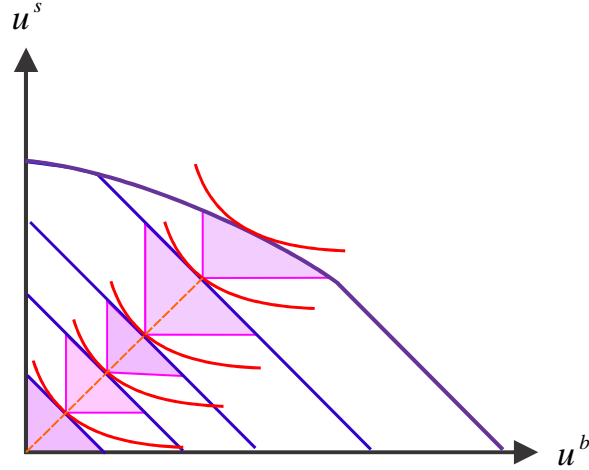

Figure 4: Chained Nash solution

The function $g(y_n, \bar{y}_{n-1})$ in (32) specifies the total payment for y_n units of output given $\bar{y}_{n-1}$ has been secured in previous rounds and provided that the feasibility constraint, $y_n \leq \bar{y}_n$, is slack. It is increasing in both y_n (for all $y_n < y^*$) and $\bar{y}_{n-1}$. It is related to p^{GN} and p^K as follows. The function $g(y, 0) \equiv p_\theta^{GN}(y)$ is the payment in the first round, which coincides with the payment function of the generalized Nash solution in (3) while $g(\bar{y}, \bar{y}) = p_\theta^K(\bar{y})$ is the payment function of the proportional solution, (5). Note also that $g(y^*, \bar{y}) = p_\theta^K(y^*) = p_\theta^{GN}(y^*)$ for any $\bar{y} \leq y^*$.

From (31), there are three cases to distinguish regarding the determination of the interim allocations in round n. If $z < g(\bar{y}_n, \bar{y}_{n-1})$ then the liquidity constraint binds, $p_n = z$, and the quantity traded is less than the size of the bundle that is up for negotiation, $y_n - \bar{y}_{n-1} < y^*/N$. If $z \in (g(\bar{y}_n, \bar{y}_{n-1}), p_\theta^K(\bar{y}_n))$, then the buyer's payment capacity is sufficiently large to finance the purchase of the whole bundle but it is not large

enough to give the seller a share $(1 - \theta)$ of the surplus. In that case, both $p \leq z$ and $y_n \leq \bar{y}_n$ bind. In that regime, as z increases, output is constant at $\bar{y}_n$ but the payment increases one-for-one with z.[10] Finally, if $z \geq p_\theta^K(\bar{y}_n)$ then the liquidity constraint is slack, $y_n = \bar{y}_n$, and the payment coincides with the proportional solution.
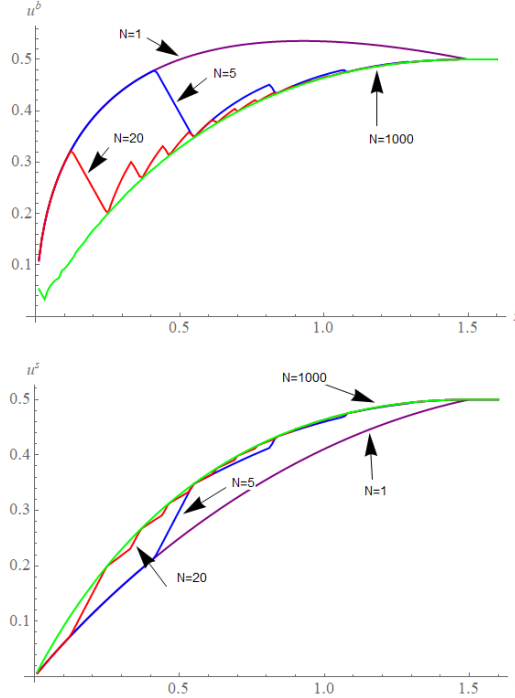

Figure 5: Buyer's and seller's surpluses. $u(y) = 2\sqrt{y}$, $v(y) = y$, $\theta = 0.5$

From (31) the buyer's surplus as a function of his payment capacity is:

$$
\begin{aligned}
u_N^b(z) &= \sum_{n=1}^N \left[\mathbb{I}_{\{p_\theta^K(\bar{y}_{n-1}) \leq z \leq g(\bar{y}_n, \bar{y}_{n-1})\}} \left\{ u\left[y_n(z)\right] - z \right\} + \mathbb{I}_{\{g(\bar{y}_n, \bar{y}_{n-1}) < z < p_\theta^K(\bar{y}_n)\}} \left[u\left(\bar{y}_n\right) - z \right] \right] \\
&\quad + \mathbb{I}_{\{z \geq p_\theta^K(y^*)\}} \theta \left[u(y^*) - v(y^*) \right].
\end{aligned}
\tag{33}
$$

The first term corresponds to the first case in (31) and is equal to $u(y_n) - g(y_n, \bar{y}_{n-1})$, where $y_n(z)$ is the solution to $z = g(y_n, \bar{y}_{n-1})$. The second term is the surplus when all feasibility constraints bind, corresponding to the second case in (31). Note that the second term does not exist when $n = N$. The last term is the surplus when the liquidity constraint is slack.

In Figure 5, we represent the buyer's and seller's surpluses for different values of N. The top panel shows that the buyer's surplus is non-monotone in z. This non-monotonicity has two origins. First, in the absence of output limits, the Nash solution generates non-monotone payoffs as the bargaining set expands — a well-known property of models with liquidity constraints (e.g., Aruoba et al., 2007). This result is

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[10] The allocations generated by our game for all $N \geq 2$ do not satisfy the axiom of monotonicity of Gu and Wright (2016), which underscores the importance of providing strategic foundations to the trading mechanisms in monetary economies.

illustrated by the purple curve for the case $N = 1$. Intuitively, the buyer's surplus is the product of the buyer's share, $\Theta(y)$, and the total surplus of the match, $u(y) - v(y)$. As z increases, and hence y increases toward y^*, $\Theta(y)$ decreases while $u(y) - v(y)$ increases. Close to y^*, the first effect dominates.

The second origin of the non-monotonicity of the buyer's surplus comes from the output limits, $y_n \leq \bar{y}_n$. In all rounds before the last, the output limit starts to bind when z reaches the threshold $g(\bar{y}_n, \bar{y}_{n-1})$, in which case the buyer's surplus exhibits a kink at a local maximum. As z increases above $g(\bar{y}_n, \bar{y}_{n-1})$, the output remains unchanged at $y_n = \bar{y}_n$ while the payment by the buyer keeps increasing until it reaches $p_\theta^K(\bar{y}_n)$ at which point the buyer's share in the total surplus is exactly θ. Hence, the buyer spends more to consume the same amount and his surplus decreases. If z increases above $p_\theta^K(\bar{y}_n)$, then the buyer has extra payment capacity, $z - p_\theta^K(\bar{y}_n)$, to spend in round $n+1$ and hence his surplus increases with z. Both origins for the non-monotonicity are based on the fact that the buyer can capture a share of the match surplus that is greater than his natural share represented by θ when his liquidity constraint is binding.

4 Foundations for Kalai and Nash bargaining

We now turn to the main contribution of the paper and provide unified strategic foundations for the Nash and Kalai solutions. We begin with the Kalai solution and show that as N goes to infinity, i.e., the size of the bundle negotiated in each round becomes infinitesimal, the final payoffs and allocations converge to the ones induced by the proportional solution. This result holds irrespective of the identity of the player making the first offer in each round and whether or not the risk of termination within each round is significantly different from zero or not. In contrast, the Nash solution is obtained when $N = 1$ and the risk of termination vanishes.

4.1 Main result

Proposition 3 (Strategic foundations for Kalai bargaining: Infinitesimal slicing.) *Consider the limit of the Rubinstein game with sliced bundles as N tends to infinity. Suppose that $\left(\xi^b, \xi^s\right) = \left(e^{-(1-\theta)\varepsilon}, e^{-\theta\varepsilon}\right)$ for some $\theta \in [0, 1]$.*

1. *For any $\varepsilon > 0$ and initial proposer $\chi \in \{b, s\}$, the final allocation, (y_N^χ, p_N^χ), converges to (y, p) that solves*

$$p = p^\chi(y) = \min\{z, p^\chi(y^*)\}. \tag{34}$$

2. *At the limit $\varepsilon \to 0$, the final allocation, (y_N^χ, p_N^χ), converges to (y, p) that solves*

$$p = p_\theta^K(y) = \min\{z, p_\theta^K(y^*)\} \tag{35}$$

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regardless of the identity of initial proposer $\chi \in \{b, s\}$. The sequence of buyer's surplus functions given by (33), $\{u_N^b : \mathbb{R}_+ \to \mathbb{R}\}_{N=1}^{+\infty}$, converges uniformly to the surplus under Kalai bargaining defined as:

$$u_\infty^b(z) = \theta \{u[y(z)] - v[y(z)]\} \ \ with \ y(z) = \left(p_\theta^K\right)^{-1}(z) \ for \ z \leq p_\theta^K(y^*) \ and \ y(z) = y^* \ otherwise.$$

According to the first part of Proposition 3, the limit of the SPE of the game as N approaches infinity coincides with the proportional solution with buyer's share equal to $(1 - \xi^s) / (1 - \xi^b \xi^s)$ when the buyer is the first proposer in all rounds, while that limit coincides with the proportional solution with buyer's share reduced to $\xi^b (1 - \xi^s) / (1 - \xi^b \xi^s)$ when the seller is the first proposer. Thus, the first proposer still has the first-mover advantage even as N approaches infinity.

The second part of Proposition 3 describes the limit as $N \to \infty$ when the round game has no first-mover advantage and it shows that the SPE allocation converges to the proportional solution with buyer's bargaining share equal to θ and a scaling of the utility functions such that both players assign the same value to the payment good. Interestingly, the bargaining power of the Nash solution and the bargaining share of the proportional solution have the same microfoundations based on the relative probabilities that a round of negotiation ends after an offer has been rejected. Moreover, from the second part of Proposition 3, we have the following Corollary:

Corollary 1 *(Unified foundations for Kalai and Nash.)* *At the limit $\varepsilon \to 0$, the SPE of the Rubinstein game with sliced bundles coincides with:*

1. *the generalized Nash solution with buyer's bargaining power θ if $N = 1$;*

2. *the Kalai's proportional solution with buyer's share θ if $N = +\infty$.*

Corollary 1 states that the Rubinstein game with sliced bundles provides unified strategic foundations to two of the most commonly used bargaining solutions in the search-theoretic literature by changing a single parameter, N. Thus, the key difference between the generalized Nash solution and the proportional solution is the agenda of the negotiation in terms of the sliced output into bundles to be negotiated sequentially. If the output is sold in a single round, then the outcome corresponds to the generalized Nash solution. If output is sliced into infinitesimal bundles, then the outcome is the proportional solution.

4.2 Relation to bargaining games with an agenda

Proposition 3 showed that the convergence to a proportional solution as N goes to infinity is robust to some details of the game, e.g., who is making the first offer and whether or not the risk of termination vanishes.[11]

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[11] In Rocheteau et al. (2019) we provide yet another game with alternating ultimatum offers that also generates the proportional solution as N goes to infinity. The round-game corresponds to a two-stage take-it-or-leave-it-offer game: in the first stage

We can establish the robustness of this result more generally by turning to an axiomatic approach. O'Neill et al. (2004) formalize bargaining games with an agenda where the agenda in our context is the collection of Pareto frontiers, $\langle H(u^b, u^s, y; z) = 0, y \in [0, y^*] \rangle$. The ordinal solution of O'Neill et al. (2004) imposes five axioms (Pareto optimality, scale invariance, symmetry, directional continuity, and time-consistency) and it is shown that the solution is given by the following differential equations (See Rocheteau et al. 2019 for details):

$$u^{\chi'}(y) = -\frac{1}{2} \frac{\partial H(u^b, u^s, y; z)/\partial y}{\partial H(u^b, u^s, y; z)/\partial u^\chi}, \quad \chi \in \{b, s\}, \tag{36}$$

where $[u^b(y), u^s(y)]$ is the path of the solution for $y \in [0, y^*]$. An increase in y by one unit expands the bargaining set by $\partial H/\partial y$. The maximum utility gain that the buyer could enjoy from this expansion is $-(\partial H/\partial y)/(\partial H/\partial u^b)$. According to (36), the buyer enjoys half of this gain. From (6), as long as the liquidity constraint does not bind (if it binds, the game ends), $\partial H(u^b, u^s, y; z)/\partial y = u'(y) - v'(y)$ and $\partial H(u^b, u^s, y; z)/\partial u^b = -1$. Hence, $u^{b'}(y) = [u'(y) - v'(y)]/2$. Integrating from $y = 0$, $u^b = [u(y) - v(y)]/2$, which corresponds to the egalitarian solution.[12] While scale invariance was imposed as an axiom, the solution exhibits ordinality endogenously: the solution is covariant with respect to any order-preserving transformation.

4.3 Setting the agenda

Our model can be extended to endogenize the choice of the bargaining solution in an internally consistent way. To do so, we add an initial stage before the negotiation takes place where one of the players is chosen at random to unilaterally set the number of rounds of the negotiation, N.

Proposition 4 (Endogenous bargaining solution.) *Suppose $\left(\xi^b, \xi^s\right) = \left(e^{-(1-\theta)\varepsilon}, e^{-\theta\varepsilon}\right)$ for some $\theta \in [0, 1]$ and consider the limit as $\varepsilon \to 0$.*

  1. *If the buyer sets the agenda, then $N = 1$ and the output, y_θ^b, solves $p_\theta^{GN}(y_\theta^b) = \min\{z, p_\theta^{GN}(y^*)\}$.*

  2. *If the seller sets the agenda, then $N = +\infty$ and the output, y_θ^s, solves $p_\theta^K(y_\theta^s) = \min\{z, p_\theta^K(y^*)\}$.*

Moreover, if $z < p_\theta^K(y^*)$, then $y_\theta^s < y_\theta^b$.

If the buyer sets the agenda of the negotiation, then he chooses to negotiate all the output at once in a single round and the outcome coincides with the generalized Nash solution. If the seller sets the agenda, then he sells one infinitesimal bundle at a time. One can build some intuition for this result by comparing (3) and (5). If the negotiation takes place in a single round, the buyer's share in the match surplus is $\Theta(y) > \theta$

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an offer is made; in the second stage the offer is accepted or rejected. The identity of the proposer alternates across rounds.

[12]One can relax the symmetry axiom in which case the solution of O'Neill et al. (2004) to the bargaining problem with agenda $\langle H(u^b, u^s, y; z) = 0, y \in [0, y^*] \rangle$ corresponds to the proportional solution for some arbitrary bargaining share.

for all $y < y^*$. The buyer extracts a larger share of the surplus because the players recognize that when the liquidity constraint binds any transfer of output generates a gain for the buyer that is larger than the seller's cost, $u'(y) > v'(y)$. In contrast, by negotiating small bundles of output, the seller can avoid binding liquidity constraints in almost all rounds, which allows the surplus to be shared according to θ.

5   Bargaining with time-varying quantity limits

We now extend our analysis to allow for heterogeneous quantity limits across rounds, and derive conditions under which the proportional solution is implemented. We consider the version of the game where the risk of breakdown between offers and counter-offers converges to zero. Hence, the game is equivalent to multiple rounds of generalized Nash bargaining with endogenous disagreement points.

5.1   Two-round game

Consider the following two-round game with the buyer payment capacity equal to z. In the first round, the quantity limit is $\Delta\bar{y}_1$ and in the second it is $\Delta\bar{y}_2$, with $\Delta\bar{y}_1 + \Delta\bar{y}_2 = y^*$. The game is solved by backward induction.

We start at the beginning of the second round with the agreement from the first round being $(y_1, p_1) \in [0, \Delta\bar{y}_1] \times [0, z]$. Since the risk of breakdown is almost zero, the outcome, (y_2, p_2), is given by the generalized Nash solution with disagreement payoffs equal to $u_1^b = u(y_1) - p_1$ and $u_1^s = p_1 - v(y_1)$. Formally,

$$(y_2, p_2) \in \arg\max_{y,p} \left[u(y) - p - u_1^b\right]^\theta \left[-v(y) + p - u_1^s\right]^{1-\theta} \quad \text{s.t.} \quad y - y_1 \leq \Delta\bar{y}_2 \text{ and } p \leq z. \tag{37}$$

We move backward to round 1. First we compute the disagreement allocation for round-1, denoted by $(\hat{y}_1, \hat{p}_1)$. This allocation is the outcome from round-2 negotiation in case round-1 reaches no trade. Thus, $(\hat{y}_1, \hat{p}_1)$ solves (37) with $(y_1, p_1) = (0, 0)$, i.e.,

$$(\hat{y}_1, \hat{p}_1) \in \arg\max_{y,p} \left[u(y) - p\right]^\theta \left[-v(y) + p\right]^{1-\theta} \quad \text{s.t.} \quad y \leq \Delta\bar{y}_2 \text{ and } p \leq z. \tag{38}$$

The corresponding payoffs are denoted by $(\hat{u}_1^b, \hat{u}_1^s) \in \mathcal{P}(\Delta\bar{y}_2)$, where $\mathcal{P}(y) \equiv \{(u^b, u^s) : H(u^b, u^s, y; z) = 0\}$ is the Pareto frontier when the output limit is y. The first-order conditions are:

$$\frac{\hat{u}_1^s}{\hat{u}_1^b} \geq \frac{1 - \Theta(\hat{y}_1)}{\Theta(\hat{y}_1)}, \quad "=" \text{ if } \hat{y}_1 < \Delta\bar{y}_2$$

$$\frac{\hat{u}_1^s}{\hat{u}_1^b} \leq \frac{1 - \theta}{\theta}, \quad\quad\quad "=" \text{ if } \hat{p}_1 < z.$$

If the constraint, $\hat{p}_1 \leq z$, does not bind, then $\hat{y}_1 = \Delta\bar{y}_2$ and $\hat{u}_1^s / \hat{u}_1^b = (1 - \theta)/\theta$. If $\hat{p}_1 \leq z$ binds, then either $\hat{y}_1 = \Delta\bar{y}_2$, in which case the solution is at the kink of the frontier $\mathcal{P}(\Delta\bar{y}_2)$, or $\hat{y}_1 < \Delta\bar{y}_2$.

Since agents in the first round can perfectly foresee the final outcome following any round-1 allocation, (y_1, p_1), they effecitvely negotiate the final allocation, (y_2, p_2), that is determined as follows:

$$(y_2, p_2) \in \arg \max_{(y,p) \in O} [u(y) - p - \hat{u}_1]^\theta [-v(y) + p - \hat{u}_1^s]^{1-\theta} \quad \text{s.t.} \quad y \leq y^* \text{ and } p \leq z, \tag{39}$$

where O is the set of final allocations that are part of an equilibrium. We solve the game taking $O = [0, y^*] \times [0, z]$. Later on we will verify that this is without loss of generality by constructing round-1 allocations that are feasible and lead to the final allocation, (y_2, p_2).

The solution to (39) can take two forms. First, if $\hat{p}_1 \leq z$ binds, then $(u_2^b, u_2^s) = (\hat{u}_1^b, \hat{u}_1^s)$ and a round-1 equilibrium allocation is $(y_1, p_1) = (0, 0)$, which suggests that the first round of the negotiation is irrelevant. This case is represented in the top panel of Figure 6. The second possibility is that $\hat{p}_1 \leq z$ does not bind, in which case $\hat{y}_1 = \Delta \bar{y}_2$ and $(\hat{u}_1^b, \hat{u}_1^s) \in \mathcal{P}(\Delta \bar{y}_2)$ is determined by

$$\frac{\hat{u}_1^s}{\hat{u}_1^b} = \frac{1 - \theta}{\theta}. \tag{40}$$

In the bottom panels of Figure 6, $(\hat{u}_1^b, \hat{u}_1^s)$ is located at the intersection of the yellow frontier, $\mathcal{P}(\Delta \bar{y}_2)$, and the orange dashed line with slope $(1 - \theta)/\theta$. The solution (u_2^b, u_2^s) to (39) has two subcases. In the first subcase, $p_2 \leq z$ does not bind, and hence

$$\frac{u_2^s - \hat{u}_1^s}{u_2^b - \hat{u}_1^b} = \frac{1 - \theta}{\theta}. \tag{41}$$

This solution is depicted in the bottom left panel in Figure 6 at the intersection of the orange dashed line with slope $(1-\theta)/\theta$ and the upper Pareto frontier, $\mathcal{P}(\Delta \bar{y}_1 + \Delta \bar{y}_2) = \mathcal{P}(y^*)$. The final outcome then coincides with the proportional solution that implements y^*. Otherwise, the liquidity constraint binds, and

$$\frac{u_2^s - \hat{u}_1^s}{u_2^b - \hat{u}_1^b} = \frac{1 - \Theta(y_2)}{\Theta(y_2)}, \tag{42}$$

with $\Theta(y)$ given by (4). This case is depicted in the bottom right panel in Figure 6. Using that $\Theta(y_2) > \theta$ when $y_2 < y^*$, it follows that (u_2^b, u_2^s) is located to the right of the red dashed line.

The intermediate payoffs in round 1, (u_1^b, u_1^s), are determined such that the solution to (37) coincides with (u_2^b, u_2^s). In the case where $p_2 \leq z$ does not bind, $(u_1^b, u_1^s) \in \mathcal{P}(\Delta \bar{y}_1)$ and $u_1^s/u_1^b = (1 - \theta)/\theta$. See top panel of Figure 7. If $p_2 \leq z$ binds, we distinguish two possibilities. First, suppose that $y_2 \leq \Delta \bar{y}_1$, that is, the final allocation is feasible given the round-1 quantity limit. In this case, we can take $(u_1^b, u_1^s) = (u_2^b, u_2^s)$. Otherwise, $y_2 > \Delta \bar{y}_1$ and we take $y_1 = \Delta \bar{y}_1$. Since $p_2 \leq z$ binds, this implies that

$$\frac{u_2^s - u_1^s}{u_2^b - u_b^1} = \frac{1 - \Theta(y_2)}{\Theta(y_2)} = \frac{u_2^s - \hat{u}_1^s}{u_2^b - \hat{u}_1^b},$$
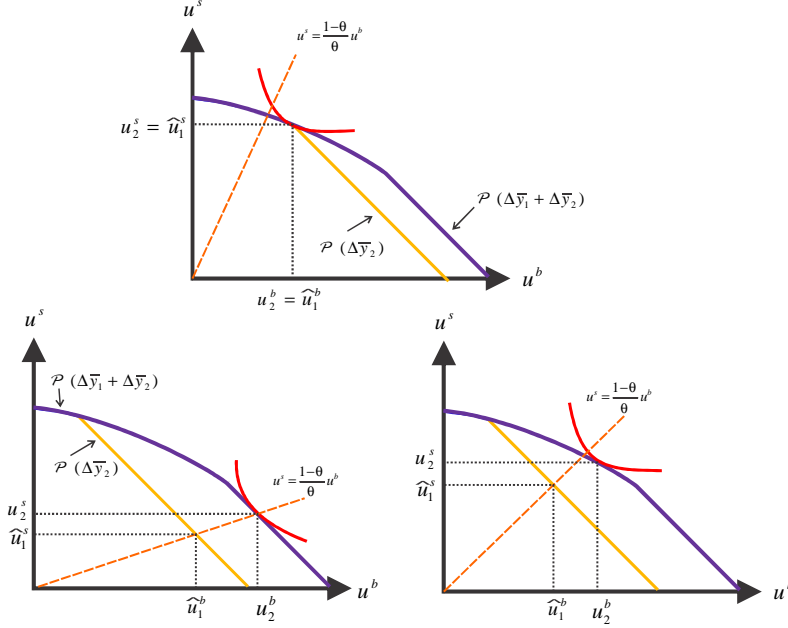
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Figure 6: Two-round game with heterogeneous output limits. Top panel: $\hat{p}_1 \leq z$ binds. Bottom left panel: $\hat{p}_1 \leq z$ and $p_2 \leq z$ are slack. Bottom right panel: $\hat{p}_1 \leq z$ is slack but $p_2 \leq z$ binds.

which uniquely pins down $(u_1^b, u_1^s) \in \mathcal{P}(\Delta \bar{y}_1)$. The solution can be seen geometrically in Figure 7 where (u_1^b, u_1^s) is located at the intersection of the green frontier, $\mathcal{P}(\Delta \bar{y}_1)$, and the red dashed line with slope $[1 - \Theta(y_2)]/\Theta(y_2)$. In the bottom left panel, we assume $\Delta \bar{y}_1 < \Delta \bar{y}_2$ and hence the green frontier, $\mathcal{P}(\Delta \bar{y}_1)$, is below the yellow frontier, $\mathcal{P}(\Delta \bar{y}_2)$; in the bottom right panel, $\Delta \bar{y}_1 > \Delta \bar{y}_2$ and hence $\mathcal{P}(\Delta \bar{y}_1)$ is above $\mathcal{P}(\Delta \bar{y}_2)$. We can now check that $p_1 \in (0, z)$. It is clear from Figure 7 that $u_1^s < u_2^s$, $y_1 < y_2$, but $p_2 = z$. Hence, $p_1 < z$. When $y_1 = \Delta \bar{y}_1 > \Delta \bar{y}_2$, the right panel shows that $u_1^s > \hat{u}_1^s \geq 0$ and hence $p_1 > 0$; when $y_1 = \Delta \bar{y}_1 < \Delta \bar{y}_2$, the left panel shows that $u_1^s > 0$ and hence $p_1 > 0$ as well.

We give two remarks before we move to the general N-round game. First, when $\Delta \bar{y}_1 = \Delta \bar{y}_2$, the green and yellow frontiers, $\mathcal{P}(\Delta \bar{y}_1)$ and $\mathcal{P}(\Delta \bar{y}_2)$, coincide in Figure 7, and hence, $(u_1^b, u_1^s) = (\hat{u}_1^b, \hat{u}_1^s)$. Thus, the disagreement payoffs in round-1 are also the intermediate payoffs at the end of round-1. Second, when $\Delta \bar{y}_1 \neq \Delta \bar{y}_2$, the disagreement payoffs, $(\hat{u}_1^b, \hat{u}_1^s)$, are computed using $\Delta \bar{y}_2$. As a result, $\Delta \bar{y}_2$ can affect the final payoffs even if round-2 is not active in equilibrium. For example, when $\Delta \bar{y}_1 > y_2 > \Delta \bar{y}_2$, we have $(u_1^b, u_1^s) = (u_2^b, u_2^s)$, that is, the final allocation is achieved in round-1, but (u_2^b, u_2^s) is determined by $\Delta \bar{y}_2$ alone according to (42).

5.2 N-round game

We consider now a N-round game with time-varying quantity limits, $\{\Delta \bar{y}_n\}_{n=1}^N$, where $\sum_{n=1}^N \Delta \bar{y}_n = y^*$. The following proposition characterizes the SPE final payoffs, which generalizes Proposition 2 for the case
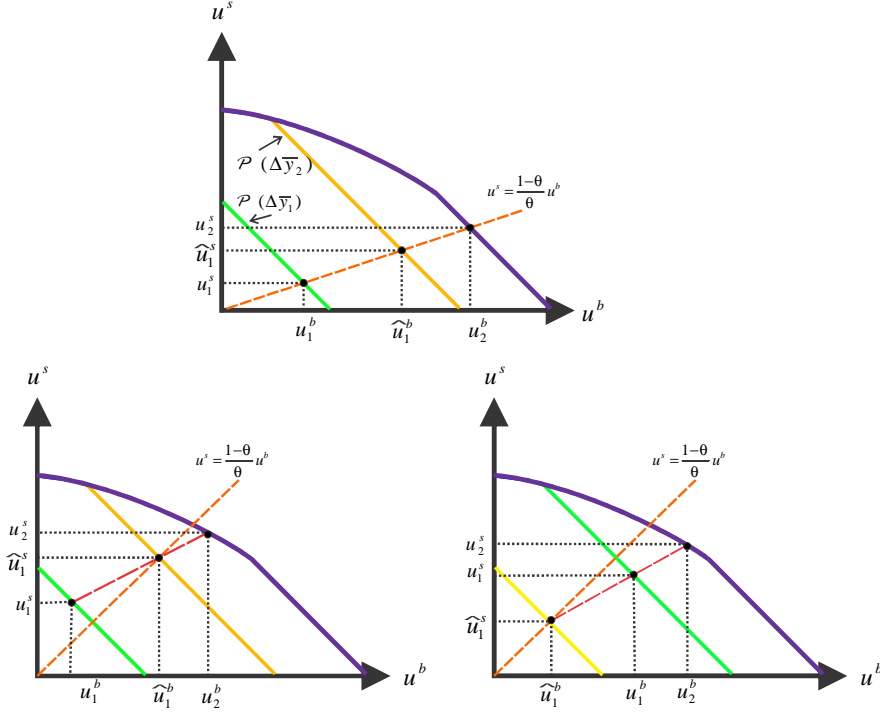
Figure 7: Two-round game with heterogenous output limits: determination of first-round payoffs.

of time-varying quantities.

Proposition 5 *(The chained Nash solution with time-varying quantity limits.)* *Consider the N-round game with quantity limit at round-n equal to $\Delta \bar{y}_n$, $n = 1, 2, ..., N$. The final allocation of any SPE is the last term of the sequence $\{(\hat{y}_n, \hat{p}_n)\}_{n=1}^{N}$ where $(\hat{y}_0, \hat{p}_0) = (0, 0)$ and*

$$(\hat{y}_n, \hat{p}_n) \in \arg \max_{y,p} \left[u(y) - u(\hat{y}_{n-1}) - (p - \hat{p}_{n-1}) \right]^{\theta} \left[-\upsilon(y) + \upsilon(\hat{y}_{n-1}) + (p - \hat{p}_{n-1}) \right]^{1-\theta} \tag{43}$$

$$s.t. \quad y - \hat{y}_{n-1} \leq \Delta \bar{y}_{N-n+1} \ and \ p \leq z.$$

The proof is by induction (see Appendix). As in Proposition 1, we interpret $(\hat{y}_n, \hat{p}_n)$ as the final allocation in the subgame composed of the last n rounds with output limits $\{\Delta \bar{y}_{N-n+j}\}_{j=1}^{n}$ assuming the initial allocation is $(0, 0)$, i.e., no agreement has been reached in the first $N - n$ rounds. To understand this interpretation, note that if $(\hat{y}_{n-1}, \hat{p}_{n-1})$ is the final allocation from the last $(n-1)$ rounds, then $(\hat{y}_{n-1}, \hat{p}_{n-1})$ is also the disagreement allocation of the last n-round game that begins at round-$(N-n+1)$. Hence, $(\hat{y}_n, \hat{p}_n)$ defined by (43) is the final allocation of a game with the last n rounds. Finally, when $n = 1$ the disagreement point is no trade and hence $(\hat{y}_0, \hat{p}_0) = (0, 0)$.

So, as in the case with constant output limits studied earlier, the final outcome can be determined by applying the generalized Nash solution with endogenous disagreement points N consecutive times. In

contrast to the case with constant limits, the terms of the sequence $\{(\hat{y}_n, \hat{p}_n)\}_{n=1}^{N-1}$ are not the allocations in the intermediate rounds. Also in contrast with the constant-limit case, (43) makes the backward-induction logic of the construction more transparent since $(\hat{y}_1, \hat{p}_1)$ is obtained from the quantity limit $\Delta \bar{y}_N$, $(\hat{y}_2, \hat{p}_2)$ is obtained from $\Delta \bar{y}_N + \Delta \bar{y}_{N-1}$, and so on.

Assuming that the constraint $p \leq z$ does not bind before the last round,

$$\frac{\hat{u}_n^s - \hat{u}_{n-1}^s}{\hat{u}_n^b - \hat{u}_{n-1}^b} = \frac{1-\theta}{\theta} \quad \text{for all } n < N,$$

$$\frac{u_N^s - \hat{u}_{N-1}^s}{u_N^b - \hat{u}_{N-1}^b} = \frac{1 - \Theta(y_N)}{\Theta(y_N)}.$$

For all $n < N$, $(\hat{u}_n^b, \hat{u}_n^s)$ lies at the intersection of $\mathcal{P}\left(\sum_{j=1}^n \Delta \bar{y}_{N-n+j}\right)$ and the line going through the origin with slope $(1-\theta)/\theta$. The final outcome is at the intersection of the Pareto frontier, $\mathcal{P}(y^*)$, and the line going through $(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s)$ with slope $[1 - \Theta(y_N)]/\Theta(y_N)$. So the outcome is identical to the one of the $N = 2$ game where the output limit in the first round is $\Delta \bar{y}_1$ and the output limit in the second round is $\sum_{j=2}^N \Delta \bar{y}_j$. In particular, if the constraint $p \leq z$ does not bind in the last round of iteration in (43), then the final allocation coincides with the proportional solution implementing y^*. This happens if and only if $z \geq p_\theta^K(y^*)$, regardless of the output limits. Finally, if the constraint $p \leq z$ binds before the last iteration in (43), the final allocation can be computed using the function g analogous to (31).

5.3 Implementation of the proportional solution

We now ask under which conditions the game with time-varying output limits implements the proportional solution. As mentioned, when $z \geq p_\theta^K(y^*)$, any N-round game with output limits $\{\Delta \bar{y}_n\}_{n=1}^N$ satisfying $\sum_{n=1}^N \Delta \bar{y}_n = y^*$ can implement the proportional solution with output level y^*. In this case, the solution coincides with the generalized Nash solution as the liquidity constraint is slack. So we focus on the case where $z < p_\theta^K(y^*)$, and denote $y_\theta^K(z) < y^*$ the ouput level under the proportional solution, i.e., the solution to $z = p_\theta^K(y_\theta^K)$.

Proposition 6 *(Implementation of the proportional solution with time-varying output limits.)*
Suppose that $z < p_\theta^K(y^*)$. A game characterized by $\left\langle z, \{\Delta \bar{y}_n\}_{n=1}^N \right\rangle$ implements the proportional solution if and only if there exists a round $N_\theta^K(z) \leq N$ such that

$$\sum_{n=1}^{N_\theta^K} \Delta \bar{y}_{N-(n-1)} = y_\theta^K(z). \tag{44}$$

If $N = 2$, the proportional solution is implemented if and only if $\Delta \bar{y}_2 = y_\theta^K$.

Proposition 6 characterizes the set of all finite sequences of output limits, $\{\Delta \bar{y}_n\}_{n=1}^N$, that implement the proportional solution. For $\{\Delta \bar{y}_n\}_{n=1}^N$ to implement y_θ^K, there must be an intermediate round, denoted N_θ^K, such that the sum of the output limits in the last N_θ^K rounds is exactly equal to y_θ^K. In particular, y_θ^K can be obtained from a sequence with $N = 2$ terms, $\Delta \bar{y}_1 = y^* - y_\theta^K$, and $\Delta \bar{y}_2 = y_\theta^K$.[13]

The 'if' part of Proposition 6 provides a condition to implement the proportional solution that is only valid for a given z as $y_\theta^K = \left(p_\theta^K\right)^{-1}(z)$. The 'only if' part implies that a given $\{\Delta \bar{y}_n\}_{n=1}^N$ implements y_θ^K only if $z = p_\theta^K \left(\sum_{j=1}^n \Delta \bar{y}_{N-j+1}\right)$ for $n = 1, ..., N$. As a result, for a given agenda, there are only finitely many payment capacities under which the proportional solution is implementable. So one cannot use a game with a finite agenda to rationalize the use of the proportional solution in models where z is endogenous. In order to implement the proportional solution for all z, we need to look beyond finite agendas and consider limits of finite sequence of output limits as N becomes large.

In order to define those limits, we introduce an alternative representation of $\{\Delta \bar{y}_n\}_{n=1}^N$ as a non-decreasing step function, $\mathbf{y} : [0,1] \to [0, y^*]$, such that $\mathbf{y}(0) = 0$, and

$$\mathbf{y}(t) = \sum_{n=1}^N \Delta \bar{y}_n \mathbb{I}_{\{t > (n-1)/N\}}. \tag{45}$$

In this alternative representation, t is the virtual time of the negotiation and $\mathbf{y}(t)$ is the cumulative output considered for negotiation up to t. We expand this set to include any function, $\bar{\mathbf{y}}$, that can be expressed as the limit of finite agendas of the form given by (45) according to the sup-norm, and we use $\mathcal{A}$ to denote the expanded set of all agendas. We define the outcome from negotiation under a limiting agenda, $\bar{\mathbf{y}} \in \mathcal{A}$, as the limit of a sequence of equilibrium outcomes obtained from a sequence of finite agendas that converge to $\bar{\mathbf{y}}$.[14] We say that a limiting agenda, $\bar{\mathbf{y}}$, is continuous if it is a continuous function from $[0,1]$ to $[0, y^*]$. The following proposition characterizes the range of allocations achievable from agendas in $\mathcal{A}$, and the set of agendas that implement the proportional solution for all payment capacities. For any $z < p_\theta^{GN}(y^*) = p_\theta^K(y^*)$, we use $y_\theta^{GN}(z)$ be denote the solution to $z = p_\theta^{GN}(y)$.

Proposition 7 (Uniform implementation of the proportional solution.) *If $z \geq p_\theta^K(y^*)$, then any $\bar{\mathbf{y}} \in \mathcal{A}$ implements the proportional solution, $p = p_\theta^K(y^*)$ and $y = y^*$. If $z < p_\theta^K(y^*)$, then for all $\bar{\mathbf{y}} \in \mathcal{A}$, $p = z$ and $y \in \left[y_\theta^K(z), y_\theta^{GN}(z)\right]$. Moreover, $\bar{\mathbf{y}} \in \mathcal{A}$ implements $y_\theta^K(z)$ for all $z \geq 0$ if and only if it is continuous.*

Proposition 7 shows that the proportional solution is uniformly implementable across z by any continuous agenda in $\mathcal{A}$. Any such agenda can be discretized into finite agendas of N rounds and is the limit of those

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[13] We thank an anonymous referee for pointing out this result.
[14] See the Appendix for a precise definition of these limits and the proof of their existence and uniqueness.

finite agendas as N goes to infinity, where the output limits converge to zero uniformly.[15] This result extends part 2 of Proposition 3 to the case of heterogeneous output limits. Conversely, Proposition 7 also shows that, for any discontinuous agenda, the final allocation differs from the proportional solution for a positive measure of z's. This result suggests that, in order to provide a strategic foundations to the proportional solution based on extensive-form games with alternating offers, gradualism is necessary.

6 Bargaining with endogenous liquidity constraints

In general equilibrium, the payment capacity of buyers (e.g., their holdings of money or liquid assets, their borrowing capacity...) is endogenous and depends critically on the trading mechanism. In this section, we explore the implications of agenda setting in the Rubinstein game with sliced bundles for the choice of z, allocations, and social efficiency. To do so, we endogenize the buyer's payment capacity by assuming that the buyer chooses z at some cost ιz before being matched. We think of $\iota \geq 0$ as the opportunity cost of holding assets in a liquid form or an interest rate on a loan. For instance, in a model where z represents real money balances (e.g., Lagos and Wright, 2005), ι is the nominal interest rate on illiquid bonds adjusted by the probability of entering a pairwise meeting. Throughout the section we assume that the risk of breakdown in each round of the negotiation converges to zero.

6.1 Endogenous liquidity constraints under an exogenous agenda

For now we consider the game with homogeneous output limits, y^*/N, and we vary N from one to infinity to trace out the role of the agenda for the choice of z. An optimal payment capacity is:

$$z_N^* \in Z_N^* \equiv \arg\max_{z \geq 0} \left\{ -\iota z + u_N^b(z) \right\}, \tag{46}$$

where Z_N^* is the set of all maximizers and the objective between brackets is the buyer's surplus from an N-round negotiation, $u_N^b(z)$, net of the cost of obtaining z units of payment, ιz. For all $N \geq 1$ and all $\iota \geq 0$, (46) admits a solution in $\left[0, p_\theta^K(y^*)\right]$.

Suppose first that buyers have "deep pockets" in that they can borrow at no cost, $\iota = 0$. The next proposition generalizes a key and paradoxical result from Lagos and Wright (2005) for $N = 1$ (generalized Nash bargaining) according to which the buyer does not find it optimal to carry enough liquidity to purchase y^*, i.e., $z_1^* < p_\theta^K(y^*)$ for all $\theta < 1$.

Proposition 8 (Bargaining-induced liquidity constraint.) *Suppose $\iota = 0$. For all $N \geq 1$ and all $\theta \in (0, 1)$, $z_N^* < p_\theta^K(y^*)$ for all $z_N^* \in Z_N^*$. As $N \to \infty$, any selection from Z_N^* converges to $p_\theta^K(y^*)$.*

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[15] In the proof we show that any continuous and increasing function $\mathbf{y} : [0, 1] \to [0, y^*]$ with $\mathbf{y}(0) = 0$ and $\mathbf{y}(1) = y^*$ belong to the set $\mathcal{A}$.

Proposition 8 shows that it is optimal for the buyer to invest in a payment capacity lower than what is necessary to maximize the match surplus, $p_\theta^K(y^*)$, because by tightening his liquidity constraint the buyer raises his share of the surplus in the last round of the negotiation, $\Theta(y)$.[16] It is only at the limit, when slicing is infinitesimal, that the buyer invests in a payment capacity that is large enough to finance y^*, i.e., the strategic effect of a binding liquidity constraint vanishes.
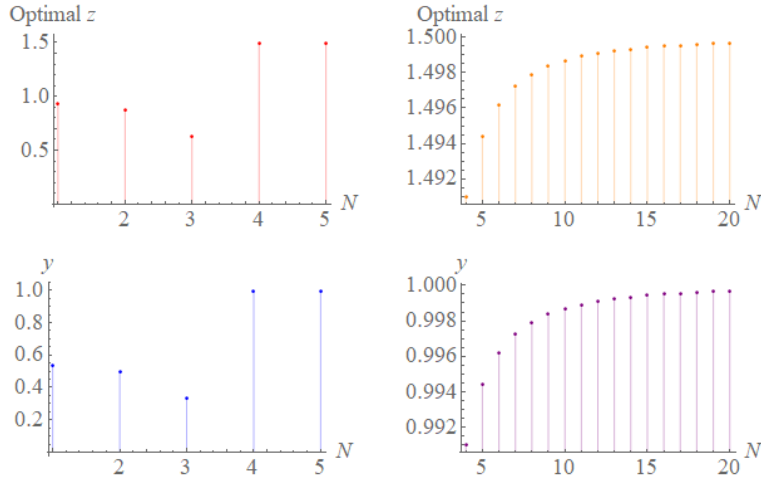

Figure 8: Optimal payment capacity and output when $\iota = 0$

We illustrate this result in the top panels of Figure 8 where we plot the solution to (46) as a function of N.[17] The relation between z_N^* and N is non-monotone as illustrated by the top left panel of Figure 8. In order to explain this non-monotonicity, we plot $u_N^b(z)$ in Figure 9 for $N \in \{1, 2, 3, 4\}$. The maximum surplus of the buyer when $N \in \{2, 3\}$ is obtained at $z = g(\bar{y}_1, 0)$, when the buyer brings just enough liquidity to purchase one bundle. As the bundle size, $\bar{y}_1 = y^*/N$, decreases with N, so does z_N^*. As N increases above 4, $u_N^b(z)$ evaluated at $g(\bar{y}_1, 0)$ no longer reaches a maximum. In that case it is optimal for the buyer to hold enough liquidity to purchase multiple bundles and z_N^* jumps upward closer to $p_{0.5}^K(y^*)$.

The bottom panels plot the equilibrium final y, denoted by $y(z_N^*)$, for different values of N. Under generalized Nash bargaining ($N = 1$), $y(z_1^*)$ is slightly less than 60 percent of the first best. In the language of monetary theory, the Friedman rule ($\iota = 0$) fails to implement the first best when prices are determined according to the generalized Nash solution (Lagos and Wright, 2005). Under proportional bargaining ($N = +\infty$), $y(z_\infty^*) = 1$, which corresponds to the first best. In-between these two polar cases, output varies in a non-monotone fashion with N. The lowest value for y is obtained when $N = 3$, in which case output is about

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[16] For a related result in the context of an over-the-counter market where agents on both sides of the market choose how much asset to bring in a match and bargain according to the generalized Nash solution, see Lebeau (2019).

[17] The examples are obtained for the following functional forms and parameter value: $u(y) = 2\sqrt{y}$, $v(y) = y$, and $\theta = 1/2$. The payment capacity necessary to purchase $y^* = 1$ is $p_{0.5}^K(y^*) = 1.5$.

one third of the first best. Hence, the parameter N, which distinguishes Nash from Kalai bargaining, plays a critical role to assess the size of the inefficiencies in economies with bargaining under liquidity constraints.
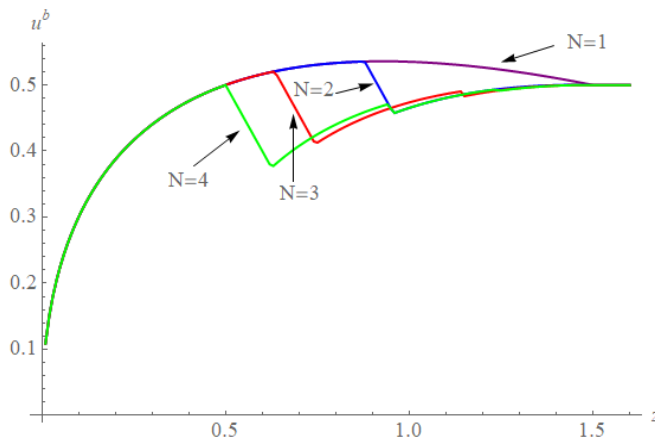

Figure 9: Buyer's surplus for $N \in \{1, 2, 3, 4\}$

We now illustrate the effect of increasing the cost of the payment capacity, ι. In the left panel of Figure 10, we plot z_N^* as a function of N when $\iota = 0.1$. Compared against Figure 8, it shows that when N is large, z_N^* falls from about 1.5 to about 1.2. The fall in z^* is much more dramatic for lower values of N. Suppose, for instance, that $N = 9$, as illustrated in the right panel of Figure 10. If $\iota = 0$, z_N^* is close to 1.5 but as ι rises to 10% z_N^* drops to about 0.25. This result shows that N is of paramount importance to quantify the effects of binding liquidity constraints, e.g., due to inflation (see Lagos and Wright, 2005, for such an exercise), and the Nash solution ($N = 1$) does not provide a lower bound for the size of this effect.

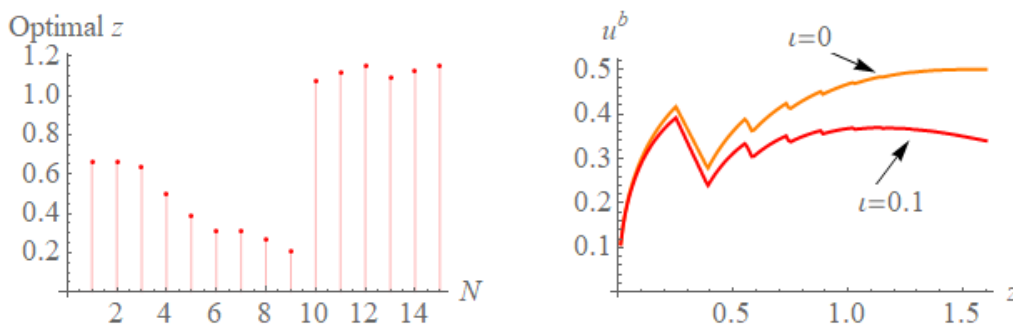Figure 10: Left: Optimal payment capacity when $\iota = 0.1$; Right: Buyer's net surplus for $\iota \in \{0, 0.1\}$

6.2 Liquidity constraint with an endogenous agenda

If the agenda of the negotiation is endogenous, it might depend on the cost of liquidity, ι, thereby providing an additional channel through which ι affects allocations and welfare. For now, we assume that agendas have

constant output limits across rounds so that the choice of the agenda is reduced to a choice of N. Later, we consider the case of a general agenda composed of time-varying output limits.

The timing of the game is as follows. First, before being matched with a seller, the buyer chooses his payment capacity, z^*. Second, once the match is formed, the buyer sets N with probability λ; otherwise, the seller is the one to choose the agenda. Third, agents bargain according to the agenda. From Proposition 4, the optimal payment capacity of the buyer then solves

$$z^* \in \arg\max_{z \geq 0} \left\{ \lambda u_1^b(z) + (1-\lambda) u_\infty^b(z) \right\}. \tag{47}$$

From (47) the buyer chooses his payment capacity to maximize his expected surplus where, from Proposition 4, with probability λ the buyer chooses $N = 1$, and, with complement probability $1 - \lambda$, the seller sets $N = +\infty$. Note that this result remains the same regardless of whether the seller observes the choice of z or not when it is his turn to choose N. For the next proposition, we assume that preferences are such that $u_1^b(z)$ is strictly concave over $\left(0, p^K(y^*)\right)$. (For such conditions, see, e.g., Lagos and Wright, 2005.)

Proposition 9 (Endogenous liquidity constraint and agenda setting.) *Suppose $\iota = 0$, $\lambda \in (0,1)$, and $\theta \in (0,1)$. The optimal payment capacity of the buyer, z^*, decreases with the buyer's probability to set the agenda, λ.*

If $\lambda = 0$, then $z^* = p^K(y^*)$ and ex ante welfare, $\mathbb{E}\left[u(y) - v(y)\right]$, is maximum. As λ increases above 0, the payment capacity falls below $p^K(y^*)$ and ex ante welfare is reduced.

Now we turn to endogenous agendas with time-varying output limits, and we focus on the case where the seller is the one who sets the agenda ($\lambda = 0$). The seller can choose any agenda of the form, $\{\Delta \bar{y}_n\}_{n=1}^N$, including limits of sequences of finite agendas as described in Section 6.3. We consider two versions of the game regarding the observability of the buyer's payment capacity at the time when the seller sets the agenda.

Proposition 10 (Endogenous liquidity constraint with time-varying output limits.) *Consider two versions of the game: in the first, the seller observes the buyer's choice of z before choosing an agenda $\mathbf{y} \in \mathcal{A}$; in the second, the seller chooses an agenda $\mathbf{y} \in \mathcal{A}$ without observing z, but z is revealed when they start negotiating. In either version, a SPE exists, and in any SPE, the buyer chooses $z \in \arg\max \left\{ -\iota z + u\left[y_\theta^K(z)\right] - p_\theta^K(z) \right\}$. Moreover, any continuous $\mathbf{y} \in \mathcal{A}$ is an equilibrium agenda in either version.*

1. *In the first version, any finite agenda $\{\Delta \bar{y}_n\}_{n=1}^N$ that satisfies (44) is an equilibrium agenda.*

2. *In the second version, no finite agenda is an equilibrium agenda.*

When output limits are parts of the design of the agenda, an agenda with $N = 2$ rounds exists that implements the proportional solution, as shown in Proposition 6. However, Proposition 10 shows that such equilibria are not robust when observability of z is dropped. The intuition for this result goes as follows. The best response of the seller to a payment capacity z is an agenda that implements the proportional solution. Suppose that the agenda is finite. There exists a profitable deviation by the buyer that consists in lowering her payment capacity by a small amount so that her liquidity constraints binds in the last round, thereby raising her surplus. In contrast, continuous agendas implement the proportional solution and achieve the highest surplus for the seller for all $z \geq 0$, and hence are weakly dominant strategies. This result holds true even if there is unobservable heterogeneity in buyer's preference or cost of liquidity, ι.

7   Conclusion

We constructed a game according to which a buyer who is subject to a liquidity constraint and a seller bargain over bundles of potentially different sizes sequentially according to a Rubinstein alternating-offer game. The game formalizes the problem of the determination of the terms of trade, both quantities and prices, in bilateral matches commonly found in the search-theoretic literature. We characterized the SPE analytically and show the same game admits the generalized Nash solution as the outcome when $N = 1$, i.e, when the output is negotiated all at once, and the proportional solution when $N = +\infty$ and when the output is negotiated one infinitesimal bundle at a time. If the agenda of the negotiation, i.e., the number of rounds and the output limit at each round, is chosen prior to the bargaining, the buyer sets $N = 1$, which implements the generalized Nash solution, while the seller adopts $N = +\infty$ and infinitesimal bundles in all rounds, which implements the proportional solution, thereby endogenizing the choice of the bargaining solution internally. We also endogenized the buyer's payment capacity and showed the liquidity constraint binds for all $N < +\infty$ even when liquidity is costless. These results show the importance of going deeper into the strategic foundations of bargaining games under liquidity constraints commonly used in the search literature, and open new ways to reconsider bargaining problems with divisible goods and assets.

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Appendix: Proofs of Propositions
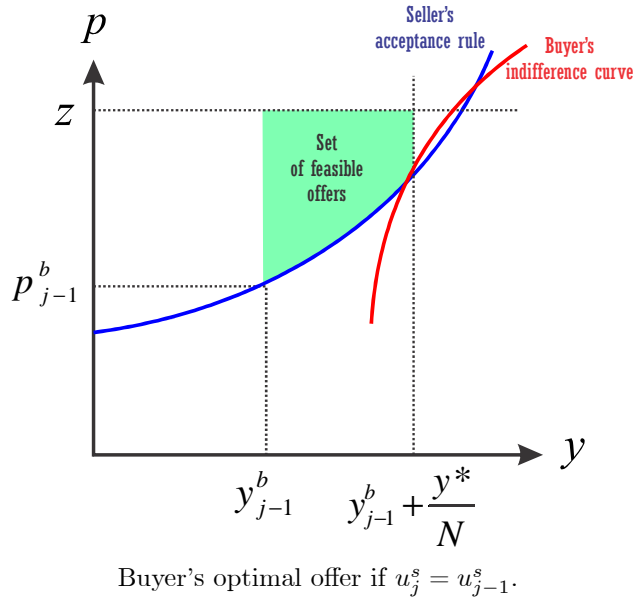
Proof of Proposition 1

Throughout the proof we assume that the buyer is making the first offer in each round, $\chi = b$. The case where the seller is the first proposer is analogous.

<u>Part 1.</u> We first show the existence of a unique joint solution to (24)-(26) and (27)-(29) for given (y^b_{j-1}, p^b_{j-1}). Suppose that $p^b_{j-1} < z$. Let $u^b_j \equiv u(y^b_j) - p^b_j$ and $u^s_j \equiv -v(y^s_j) + p^s_j$ denote the values to the maximization problems. We also show that the solution satisfies $y^b_j > y^b_{j-1}$, $y^s_j > y^b_{j-1}$, $p^b_j > p^b_{j-1}$ and $p^s_j > p^b_{j-1}$.

We first characterize the solution to (24)-(26) when u^s_j is taken as given. If $u^s_j = u^s_{j-1} \equiv -v(y^b_{j-1}) + p^b_{j-1}$ then the set $\{(y,p)$ satisfies (25)-(26)$\}$ is nonempty and convex. See green area in figure below. From the strict concavity of the objective, a solution exists and is unique and it is such that $u^b_j = \bar{u}^b_j$, the solution to $H(\bar{u}^b_j, u^s_{j-1}, y^b_{j-1} + y^*/N, z) = 0$, where H defined by (6) describes the Pareto frontier. With the change of variable $x \equiv p - p^b_{j-1}$ and $\omega(y) \equiv v(y) - v(y^b_{j-1})$, the problem reduces to

$$\bar{u}^b_j = \max_y \left\{ u(y) - \omega(y) - p^b_{j-1} \right\} \quad \text{s.t.} \quad y \leq \min\left\{ \omega^{-1}\left(z - p^b_{j-1}\right), y^b_{j-1} + \frac{y^*}{N} \right\}.$$

Using that $u'(y^b_{j-1}) > \omega'(y^b_{j-1})$ since $y^b_{j-1} < y^*$ and the fact that the upper bound on y is strictly higher than y^b_{j-1} (because $p^b_{j-1} < z$), the optimal $y > y^b_{j-1}$ and hence $\bar{u}^b_j > u(y^b_{j-1}) - p^b_{j-1}$. Graphically, (y^b_{j-1}, p^b_{j-1}) is located to the left of the buyer's indifference curve in the figure below.

Buyer's optimal offer if $u^s_j = u^s_{j-1}$.

If $u^s_j = \bar{u}^s_j$ solves $H(u^b_{j-1}, \bar{u}^s_j, y^b_{j-1} + y^*/N, z) = 0$, then $u^b_j > u^b_{j-1}$ as long as $\xi^s < 1$, where u^b_j solves the buyer's problem with $u^s_j = \bar{u}^s_j$. To see this, note that the allocation $(\bar{y}^s_j, \bar{p}^s_j)$ corresponding to $\bar{u}^s_j \equiv -v(\bar{y}^s_j) + \bar{p}^s_j$

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is such that $u(\bar{y}_j^s) - \bar{p}_j^s = u(y_{j-1}^b) - p_{j-1}^b$. This implies that the seller's payoff is

$$-v(\bar{y}_j^s) + \bar{p}_j^s = u(\bar{y}_j^s) - v(\bar{y}_j^s) - \left[u(y_{j-1}^b) - p_{j-1}^b\right].$$

From (25) at equality, the buyer's surplus is

$$u_j^b - u_{j-1}^b = (1 - \xi^s)\left\{u(\bar{y}_j^s) - v(\bar{y}_j^s) - \left[u(y_{j-1}^b) - v(y_{j-1}^b)\right]\right\} > 0,$$

where we have used that any allocation on the Pareto frontier corresponding to $y_{j-1}^b + y^*/N$ is such that $y > y_{j-1}^b$. Graphically, $\bar{u}_j^s$ corresponds to the upper, dashed acceptance rule of the seller in the figure below. It is such that the dashed buyer's indifference curve goes through (y_{j-1}^b, p_{j-1}^b). The actual acceptance rule of the seller when it is the buyer's turn to make an offer is between this upper curve and the lower dotted curve going through (y_{j-1}^b, p_{j-1}^b). The buyer's optimal offer is located to the right of (y_{j-1}^b, p_{j-1}^b), which illustrates the fact that $u_j^b > u_{j-1}^b$.

Buyer's optimal offer if $u_j^s = \bar{u}_j^s$.
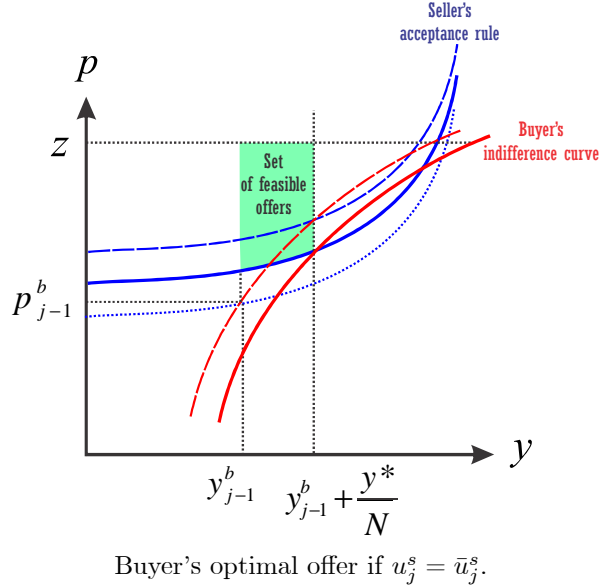
For the clarity of notations, we use $u_j^b(u^s)$ to denote the solution to (24)-(26) with $u_j^s = u^s$. As mentioned, $u_j^b(u^s)$ is well defined since the solution is unique, and, by the Theorem of Maximum, the optimal buyer offer, (y^b, p^b), is continuous in u^s. Moreover, by monotonicity, the relevant range of $u_j^b(u^s)$ is $\left[u_{j-1}^b, \bar{u}_j^b\right]$. Finally, the set of u^s for which $p^b = z$ is compact, and note that $u_j^b(u^s)$ is differentiable when $p^b < z$. The derivitives are given by $\partial u_j^b/\partial u^s = -\xi^s$ if $p \leq z$ does not bind and $\partial u_j^b/\partial u^s = -\left[u'(y^b)/v'(y^b)\right]\xi^s$ whenever differetiable. Note that although there are potentially kinks when $p^b = z$, it will not affect our argument.

Using the same reasoning for the seller's problem, (27)-(29), if $u_j^b = u_{j-1}^b$ then $u_j^s = \bar{u}_j^s > u_{j-1}^s$ and if

$u_j^b = \bar{u}_j^b$ then $u_j^s > u_{j-1}^s$. Moreover, with the same arguments as before, $\partial u_j^s / \partial u^b = -\xi^b$ if $p \le z$ does not bind and $\partial u_j^s / \partial u^b = -\left[v'(y^s)/u'(y^s) \right] \xi^b$ otherwise.

Now we claim that the slope of the solution to the seller's problem, $u_j^s(u^b)$ (represented in blue in the figure below), is steeper in absolute value than the slope of the solution to the buyer's problem, $u_j^b(u^s)$ (represented in red) within the space $(u^s, u^b) \in \left[u_{j-1}^s, \bar{u}_j^s \right] \times \left[u_{j-1}^b, \bar{u}_j^b \right]$. Consider two points, (u_j^s, u_j^b) on the blue curve (that is, u_j^s solves the seller problem with optimal offer (y^s, p^s)) and $(u_j^s, u_j^{b\prime})$ on the red curve (that is, u_j^b solves the seller problem with optimal offer (y^b, p^b)). If the constraint $p \le z$ is not binding at both, then the product of derivitives, $\partial u_j^s / \partial u^b \times \partial u_j^b / \partial u^s$ evaluated at the two corresponding points, is $\xi^b \xi^s < 1$. If the constraint $p \le z$ are binding for both the buyer's problem and the seller's, then it is easy to verify that $y^b > y^s$ and $\partial u_j^s / \partial u^b \times \partial u_j^b / \partial u^s = \left[v'(y^s)/u'(y^s) \right] \left[u'(y^b)/v'(y^b) \right] \xi^s \xi^b < 1$. If the constraint $p \le z$ is binding for the buyer's problem but not the seller's, then $\partial u_j^s / \partial u^b \times \partial u_j^b / \partial u^s = \left[u'(y^b)/v'(y^b) \right] \xi^s \xi^b < 1$. Finally, one can verify that it cannot be the case that the constraint is binding at the seller's problem but not the buyer's for such two points.

Hence, there exists a unique solution in $\left[u_{j-1}^s, \bar{u}_j^s \right] \times \left[u_{j-1}^b, \bar{u}_j^b \right]$. By monotonicity, any solution will also line within this range and hence it is unique globally. The solution is such that: $y_j^b = y_{j-1}^b + y^*/N$ if $p_j^b < z$ and $p_j^b = z$ otherwise; $y_j^s = y_{j-1}^b + y^*/N$ if $p_j^s < z$ and $p_j^s = z$ otherwise. It is also easy to check with a proof by contradiction that $p_j^b \le p_j^s$.
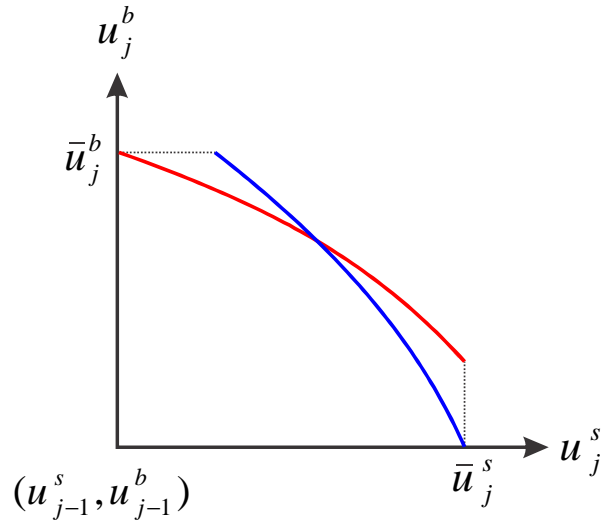
Unique solution to the buyer's and seller's problems

Part 2. Having established that the problems (24)-(26) and (27)-(29) admit a unique solution, we now show that these solutions can be used to construct a SPE of our game and uniqueness of the SPE payoffs.

Let $V_n^b(o)$ and $V_n^s(o)$ denote the values of a buyer and a seller in a subgame starting at the beginning of the nth round with the interim allocation $o = (y_{n-1}, p_{n-1}) \in [0, (n-1)y^*/N] \times [0, z)$. Let $O_n(o) \subseteq [y_{n-1}, y_{n-1} + (N-n+1)y^*/N] \times [p_{n-1}, z]$ the set of all terminal allocations that can be achieved in a SPE in that subgame. By backward induction, we can effectively assume that offers in round n are elements of $O_n(o)$. Then, the players' values in the subgame are $V_n^b(o) = u(y^b) - p^b$ and $V_n^s(o) = -v(y^b) + p^b$ where the buyer's and seller's offers, (p^b, y^b) and (p^s, y^s), solve:

$$(y^b, p^b) \quad \in \quad \arg \max_{(y,p) \in O_n(o)} \{u(y) - p\} \tag{48}$$

$$\text{s.t.} \quad -v(y) + p \quad \geq \quad (1-\xi^s)V_{n+1}^s(o) + \xi^s[-v(y^s) + p^s] \tag{49}$$

and

$$(y^s, p^s) \quad \in \quad \arg \max_{(y,p) \in O_n(o)} \{-v(y) + p\} \tag{50}$$

$$\text{s.t.} \quad u(y) - p \quad \geq \quad (1-\xi^b)V_{n+1}^b(o) + \xi^b[u(y^b) - p^b]. \tag{51}$$

According to (48)-(49) the buyer's offer maximizes his surplus subject to the constraint that it has to be accepted by the seller. The reservation payoff to the seller, on the right side of (49), is the weighted average of the seller's surplus if it is his turn to make the offer and his surplus if the negotiation in round n breaks down where the weights are given by the probabilities of the two events. In the event of a breakdown, the negotiation enters round $n+1$ with interim allocation o, hence the seller's payoff is $V_{n+1}^s(o)$. The determination of the seller's offer according to (50)-(51) has a similar interpretation. Thus, we only need to show that the value functions can be computed according to (24)-(26) and (27)-(29), that is, the final allocation of a subgame starting in round n with initial allocation $o = (y_{n-1}, p_{n-1})$ is (y_N^b, p_N^b) as given by the last term of the sequence $\{(y_j^b, p_j^b, y_j^s, p_j^s)\}_{j=n}^N$ in the statement of the proposition. Once this is proved, by the one-stage-deviation principle, it follows that these strategies form a SPE.

The proof is by induction from $n = N, N-1$, and so on, and the induction base is $n = N$. In round N, the subgame has a single round and $o = (y_{N-1}, p_{N-1})$. In this case, $O_N(o) = [y_{N-1}, y_{N-1} + y^*/N] \times [p_{N-1}, z]$ and $V_{N+1}^b(o) = 0 = V_{N+1}^b(o)$. We have shown the existence and uniqueness of a solution to (48)-(51) in Part 1, and it is given by $(y_N^b, p_N^b, y_N^s, p_N^s)$ according to the program (24)-(26) and (27)-(29). Moreover, it also shows that $V_N^b(o)$ and $V_N^s(o)$ are uniquely determined, which follows from uniqueness of the one-round game given in Rubinstein (1982) (see also Osborne and Rubinstein (1994) for how to extend the argument to the model with risk of breakdown).

Suppose, by induction, that the statement holds for the subgame starting in round $n+1$. Then, $V_{n+1}^b(o) = u(y_{N-1}^b) - p_{N-1}^b$ and $V_{n+1}^s(o) = -v(y_{N-1}^b) - p_{N-1}^b$. We now move to the beginning of round n, and consider

two cases. First, if $p_{N-1}^b = z$, then the buyer can guarantee a payoff of $V_{n+1}^b(o)$ while the seller can guarantee $V_{n+1}^s(o)$. Since there is no feasible allocation that can make both parties better off, it is optimal for both to propose and accept no trade in this round. This shows that it must be the case that $V_n^b(o) = V_{n+1}^b(o)$ and $V_n^s(o) = V_{n+1}^s(o)$, and this coincides with the solution to the program (24)-(26) and (27)-(29).

Second, suppose that $p_{N-1}^b < z$. Suppose, for a moment, that the set of terminal allocations starting from o is the set of all feasible allocations. This implies that the solutions to (48)-(49) and (50)-(51) coincide with (p_N^b, y_N^b) and (p_N^s, y_N^s). To see this, note that the maximization problems are identical to (24)-(26) and (27)-(29) when the set of terminal allocations is

$$O_n(o) = \left\{ (y,p) \in \left[y_{N-1}^b, y_{N-1}^b + \frac{y^*}{N} \right] \times \left[p_{N-1}^b, z \right] \right\}.$$

We showed in Part 1 that the constraints $y \geq y_{N-1}^b$ and $p \geq p_{N-1}^b$ do not bind for both the buyer and the seller's problems. Moreover, $y_{N-1}^b = y_{n-1} + (N-n)y^*/N$ if $p_{N-1}^b \leq z$ does not bind. So, even if we allow agents to negotiate over allocations in the largest set of feasible allocations,

$$O_n'(o) = \left[y_{n-1}, y_{n-1} + \frac{(N-n+1)y^*}{N} \right] \times \left[p_{n-1}, z \right].$$

the outcome is still (y_N^b, p_N^b) and (y_N^s, p_N^s). Since we know that this solution is an outcome of the SPE by the induction hypothesis, and since $O_n'(o)$ is a superset of all achievable allocations from the SPE, it follows that the solution is the unique final equilibrium allocation. Again, uniqueness of the SPE within the round game (when $p_{N-1}^b < z$) follows the standard argument. This completes the induction argument. Note also that the solutions to (48)-(49) and (50)-(51) specify equilibrium strategies for all subgames, and the implied equilibrium strategies are such that there is no trade in round n if $V_n^b(o) = V_{n+1}^b(o)$ and $V_n^s(o) = V_{n+1}^s(o)$.

Finally, we add two remarks. First, although we have shown the uniqueness of the SPE final payoffs, there can be multiple SPE that achieve the same payoffs. In particular, if $p_n^b = z$ for some $n < N$ in the sequence $\{(y_n^b, p_n^b, y_n^s, p_n^s)\}_{n=1}^N$, then there is always a SPE where agents propose no trade in the first $N - \widehat{N}$ rounds, where $\widehat{N}$ is the smallest n such that $p_n^b = z$. However, the players can also agree on some trades in earlier rounds that lead to the same outcome. Second, the proof is not affected if we assumed the number of rounds is $\overline{N} > N$ and hence $\overline{N}y^*/N > y^*$. The only place that requires modification is that, besides the case $p_{N-1}^b = z$, we also need to consider $y_{N-1}^b \geq y^*$. But this also leads to the same result that $V_n^b(o) = V_{n+1}^b(o)$ and $V_n^s(o) = V_{n+1}^s(o)$ and all the rest still follows.

Proof of Proposition 2

<u>Part 1.</u> First, we characterize the solution to (30), which can be solved recursively by considering the following problem (where $(\hat{y}, \hat{p})$ corresponds to the disagreement point):

$$\max_{(y,p)}\{[u(y) - p] - [u(\hat{y}) - \hat{p}]\}^{\theta}\{[-v(y) + p] - [-v(\hat{y}) + \hat{p}]\}^{1-\theta},$$

$$\text{s.t. } p \leq z, \ y \leq \bar{y}.$$

$$(52)$$

The FOC (after taking log on the objective function) is given by

$$\frac{\theta u'(y)}{[u(y) - p] - [u(\hat{y}) - \hat{p}]} - \frac{(1 - \theta)v'(y)}{[-v(y) + p] - [-v(\hat{y}) + \hat{p}]} \geq 0, \text{ ” } = \text{ ” if } y < \bar{y}; \quad (53)$$

$$\frac{-\theta}{[u(y) - p] - [u(\hat{y}) - \hat{p}]} + \frac{1 - \theta}{[-v(y) + p] - [-v(\hat{y}) + \hat{p}]} \geq 0, \text{ ” } = \text{ ” if } p < z. \quad (54)$$

The solution depends on two critical values of z. To define the thresholds, first define

$$\bar{p} = [(1 - \theta)u(\bar{y}) + \theta v(\bar{y})] - [(1 - \theta)u(\hat{y}) + \theta v(\hat{y})] + \hat{p},$$

and, for all $0 \leq \hat{y} \leq y \leq y^{*}$, define

$$h(y, \hat{y}, \hat{p}) = \{[1 - \Theta(y)]u(y) + \Theta(y)v(y)]\} - \{[1 - \Theta(y)]u(\hat{y}) + \Theta(y)v(\hat{y})\} + \hat{p}. \quad (55)$$

Note that $\Theta(y)$ is given by (4) and $h(y, \hat{y}, \hat{p}) = g(y, \hat{y})$ with $\hat{p} = p_{\theta}^{K}(\hat{y}) = (1 - \theta)u(\hat{y}) + \theta v(\hat{y})$, where the function g is given by (32). By concavity of u and convexity of v, $h(y, \hat{y}, \hat{p}) < \bar{p}$ as long as $y < y^{*}$.

Then, it is straightforward to verify that, to satisfy (53) and (54),

1. if $z \geq \bar{p}$, the solution is $(y, p) = (\bar{y}, \bar{p})$;

2. if $h(\bar{y}, \hat{y}, \hat{p}) < z \leq \bar{p}$, the solution is $(y, p) = (\bar{y}, z)$;

3. if $\hat{p} < z \leq h(\bar{y}, \hat{y}, \hat{p})$, the solution is $p = z$ and y satisfying $h(y, \hat{y}, \hat{p}) = z$.

Now, we apply these solutions iteratively to solve the sequence (30). To simplify notation we denote $\bar{y}_n \equiv ny^{*}/N$. Let $\hat{N}$ satisfy

$$p_{\theta}^{K}(\bar{y}_{\hat{N}-1}) < z \leq p_{\theta}^{K}(\bar{y}_{\hat{N}}). \quad (56)$$

Then, for all $n < \hat{N}$, case 1 applies and the constraint $p \leq z$ never binds. By induction the payment is given by $p_n = p_{\theta}^{K}(\bar{y}_n)$. When $n = \hat{N}$, the solution is

$$(y_{\hat{N}}, p_{\hat{N}}) = \begin{cases} (\bar{y}_{\hat{N}}, z) & \text{if } z > g(\bar{y}_{\hat{N}}, \bar{y}_{\hat{N}-1}), \\ (y, z) \text{ that solves } g(y, \bar{y}_{\hat{N}-1}) = z & \text{if } z \leq g(\bar{y}_{\hat{N}}, \bar{y}_{\hat{N}-1}), \end{cases} \quad (57)$$

and the sequence remains constant after $\hat{N}$. Note that the solution $(y_{\hat{N}}, p_{\hat{N}})$ is continuous in z.

<u>Part 2.</u> Here we show convergence. We assume that the buyer makes the first offers, and the seller case is analogous. The standard argument (c.f. Osborne and Rubinstein (1994), Proposition 310.3) shows that as the risk of termination vanishes, the SPE payoff in the round-game converges to the Nash solution. We need two modifications. First, the disagreement payoff in our round game is endogenously determined by the outcome from the previous round; second, we allow for asymmetry. To do so, recall that ξ^b and ξ^s are given by

$$\xi^b = e^{-(1-\theta)\varepsilon} \text{ and } \xi^s = e^{-\theta\varepsilon}. \tag{58}$$

Consider the sequence computed according to (24)-(29). To emphasize that the solution $\{(y_n^b, p_n^b, y_n^s, p_n^s)\}_{n=1}^N$ depends on ε through (ξ^b, ξ^s), we denote $o_n^b(\varepsilon) = (y_n^b, p_n^b)$ and $o_n^s(\varepsilon) = (y_n^s, p_n^s)$, and $u^b(o) = u(y) - p$ and $u^s(o) = -v(y) + p$ for $o = (y, p)$.

Now, for each n, let o_n^N denote the solution to (30), and let $\hat{o}_n(\varepsilon)$ denote the solution to (30) but with (y_{n-1}, p_{n-1}) given by $o_{n-1}^b(\varepsilon)$ for $n > 1$. We show by induction that

$$\lim_{\varepsilon \to 0} \hat{o}_n(\varepsilon) = o_n^N = \lim_{\varepsilon \to 0} o_n^b(\varepsilon) = \lim_{\varepsilon \to 0} o_n^s(\varepsilon) \text{ for all } n = 1, .., N. \tag{59}$$

Consider $n = 1$. By (25) and (28) and (58), we have (note that $u^b[(0,0)] = 0 = u^s[(0,0)]$)

$$\{u^b[o_1^b(\varepsilon)]\}^\theta \{u^s[o_1^b(\varepsilon)]\}^{1-\theta} = \{u^b[o_1^s(\varepsilon)]\}^\theta \{u^s[o_1^s(\varepsilon)]\}^{1-\theta}. \tag{60}$$

As in the standard argument, since $(u^b[o_1^b(\varepsilon)], u^s[o_1^b(\varepsilon)])$ and $(u^b[o_1^s(\varepsilon)], u^s[o_1^s(\varepsilon)])$ both lie on the Pareto frontier $H(u^b, u^s, \bar{y}_1; z) = 0$ while o_1^N maximizes $(u^b)^\theta (u^s)^{1-\theta}$ over the same frontier (which, as shown in the main text, is strictly concave), and since $u^b[o_1^s(\varepsilon)] < u^b[o_1^b(\varepsilon)]$, it follows that

$$u^b[o_1^s(\varepsilon)] < u^b(o_1^N) < u^b[o_1^b(\varepsilon)]. \tag{61}$$

Taking ε to zero, (25) and (28) imply that $o_1^s(\varepsilon)$ and $o_1^b(\varepsilon)$ coincide at the limit and (61) implies that the limit is o_1^N.

Now, suppose, by induction, that (59) holds for n. By (25) and (28) for $n + 1$, we have

$$u^b[o_{n+1}^s(\varepsilon)] - u^b[o_n^b(\varepsilon)] = \xi^b\{u^b[o_{n+1}^b(\varepsilon)] - u^b[o_n^b(\varepsilon)]\},$$
$$u^s[o_{n+1}^b(\varepsilon)] - u^s[o_n^b(\varepsilon)] = \xi^s\{u^s[o_{n+1}^s(\varepsilon)] - u^s[o_n^b(\varepsilon)]\},$$

which, together with (58), in turn imply that

$$\{u^b[o_{n+1}^s(\varepsilon)] - u^b[o_n^b(\varepsilon)]\}^\theta \{u^s[o_{n+1}^s(\varepsilon)] - u^s[o_n^b(\varepsilon)]\}^{1-\theta}$$
$$= \{u^b[o_{n+1}^b(\varepsilon)] - u^b[o_n^b(\varepsilon)]\}^\theta \{u^s[o_{n+1}^b(\varepsilon)] - u^s[o_n^b(\varepsilon)]\}^{1-\theta}.$$

Following the same argument as $n = 1$, we have

$$u^b[o^s_{n+1}(\varepsilon)] \leq u^b[\hat{o}_{n+1}(\varepsilon)] \leq u^b[o^b_{n+1}(\varepsilon)]. \tag{62}$$

Note that we may have equality as the constraint $p \leq z$ may be binding in $o^b_n(\varepsilon)$. Since the Nash solution is continuous in the disagreement payoff, it follows that $\lim_{\varepsilon \to 0} \hat{o}_{n+1}(\varepsilon) = o^N_{n+1}$, and since $o^s_{n+1}(\varepsilon)$ and $o^b_{n+1}(\varepsilon)$ coincide at the limit, (62) implies that the limit is o^N_{n+1}.

Proof of Proposition 3

<u>Part 1</u>. Here we prove the case $\chi = b$. The other case is similar. First we characterize the binding payment constraint when the buyer is making the first offers. Again, we denote $\bar{y}_n \equiv ny^*/N$. In this case, when the constraint $p \leq z$ is not binding in either the buyer's or the seller's problem, the payment at round n is given by $\bar{p}^s(\bar{y}_n, \bar{y}_{n-1})$, where the function $\bar{p}^s(y, \bar{y})$ is

$$\bar{p}^s(y, \bar{y}) = \frac{\left(1 - \xi^b\right) u(y) + \xi^b \left(1 - \xi^s\right) v(y)}{1 - \xi^b \xi^s} - \frac{\left(1 - \xi^b\right)\left(1 - \xi^s\right)[u(\bar{y}) - v(\bar{y})]}{1 - \xi^b \xi^s}.$$

Note that $\bar{p}^s(y, y) = p^b(y)$. Then,

1. if $z \geq \bar{p}^s(\bar{y}_m, \bar{y}_{m-1})$ for all $m \leq n$, then $(y^b_n, p^b_n) = [\bar{y}_n, p^b(\bar{y}_n)]$ and $(y^s_n, p^s_n) = [\bar{y}_n, \bar{p}^s(\bar{y}_n, \bar{y}_{n-1})]$;

2. otherwise, either $p^b_n = z$, or $y^b_n = \bar{y}_n$.

The payments $\bar{p}^s(\bar{y}_n, \bar{y}_{n-1})$ may not be increasing with n. However, for N sufficiently large, $\bar{p}^s(\bar{y}_n, \bar{y}_{n-1})$ strictly increases with n. Given these results, we show convergence. We consider two cases: first, $z < p^b(y^*)$; second, $z \geq p^b(y^*)$.

First consider the case where $z < p^b(y^*)$. For N large, there is a unique $\hat{N}(N)$ satisfying

$$\bar{p}^s(\bar{y}_{\hat{N}}, \bar{y}_{\hat{N}-1}) \leq z < \bar{p}^s(\bar{y}_{\hat{N}+1}, \bar{y}_{\hat{N}}). \tag{63}$$

Define $\hat{y}_N \equiv \bar{y}_{\hat{N}(N)}$ for such N's. It then follows that $(y^b_n, p^b_n) = [\bar{y}_n, p^b(\bar{y}_n)]$ for all $n \leq \hat{N}$. We can then rewrite (63) as

$$\bar{p}^s(\hat{y}_N, \hat{y}_N - y^*/N) \leq z < \bar{p}^s(\hat{y}_N + y^*/N, \hat{y}_N).$$

Taking N to infinity, it follows that $\hat{y}_N$ converges to the unique y that solves $p^b(y) = z$. Note that $\bar{p}^s(y, y) = p^b(y)$. This also implies that

$$\lim_{N \to \infty} \bar{p}^s(\hat{y}_N, \hat{y}_N - y^*/N) = z = \lim_{N \to \infty} p^b(\hat{y}_N).$$

The result then follows from the observation that, the final y in the whole game with N rounds, y_N, satisfies

$$\hat{y}_N \le y_N \le \hat{y}_N + v^{-1}[z - p^b(\hat{y}_N)].$$

Hence, both y_N and $\hat{y}_N$ converge to the unique $y < y^*$ that solves $p^b(y) = z$.

Now, consider the case where $z > p^b(y^*)$. Following the same reasoning as above, for N large, $\hat{N}(N) = N$ and $\hat{y}_N = y^*$. The knife-edge case where $z = p^b(y^*)$ follows from continuity in z.

<u>Part 2</u>. From Proposition 2, for given N, the final allocation is the last term of the sequence $\{(y_n, p_n)\}_{n=1}^N$ obtained from (30) and $(y_0, p_0) = (0, 0)$. If $p_\theta^K(y^*) > z$, then there exists $\hat{N}(N) \le N$ such that $p_n = p_\theta^K(\bar{y}_n)$ for all $n < \hat{N}$ and the final allocation is $(y_{\hat{N}}, z)$. By the same logic as in Part 1 and 2, as $N \to +\infty$, $(y_{\hat{N}}, p_{\hat{N}}) \to (y, p)$ solution to $p = p_\theta^K(y) = z$.

We now show the uniform convergence of the buyer's payoff to his payoff under proportional bargaining. Note that for $z \ge p_\theta^K(y^*)$, $u_N^b(z) = \theta[u(y^*) - v(y^*)]$ for all N and hence we only need to show uniform convergence over $z \in [0, p_\theta^K(y^*)]$. For all $z \in [p_\theta^K(\bar{y}_{n-1}), p_\theta^K(\bar{y}_n)]$, the buyer's surplus defined in (33), $u_N^b(z)$, is bounded from above by

$$\Theta(\bar{y}_{n-1})\{[u(\bar{y}_n) - v(\bar{y}_n)] - [u(\bar{y}_{n-1}) - v(\bar{y}_{n-1})]\} + \theta[u(\bar{y}_{n-1}) - v(\bar{y}_{n-1})].$$

Indeed, for $z \in [p_\theta^K(\bar{y}_{n-1}), g(\bar{y}_n, \bar{y}_{n-1})]$, this bound is obtained since $y_n(z)$ is bounded by $\bar{y}_n$ while $\Theta[y_n(z)]$ is bounded by $\Theta(\bar{y}_{n-1})$; for $z \in [g(\bar{y}_n, \bar{y}_{n-1}), p_\theta^K(\bar{y}_n)]$, $u_N^b(z)$ is maximized at $z = g(\bar{y}_n, \bar{y}_{n-1})$ and hence the same bound applies. We now establish that $u_N^b(z)$ converges uniformly to $u_\infty^b(z) = \theta[u(y) - v(y)]$, where $y(z)$ is defined by $p_\theta^K(y) = z$, as $N \to +\infty$. First, note that $u_N^b(z) \ge u_\infty^b(z)$ for all z. Second, for all $z \in [p_\theta^K(\bar{y}_{n-1}), p_\theta^K(\bar{y}_n)]$, $u_\infty^b(z)$ is bounded below by $\theta[u(\bar{y}_{n-1}) - v(\bar{y}_{n-1})]$. It follows that

$$\left|u_N^b(z) - u_\infty^b(z)\right| \le \sum_{n=1}^N \mathbb{I}_{\{p_\theta^K(\bar{y}_{n-1}) \le z \le p_\theta^K(\bar{y}_{n-1})\}} \Theta(\bar{y}_{n-1})\{[u(\bar{y}_n) - v(\bar{y}_n)] - [u(\bar{y}_{n-1}) - v(\bar{y}_{n-1})]\}$$

for all $z \in [0, p_\theta^K(y^*)]$. Using that $\Theta(y)$ is decreasing and $u(y) - v(y)$ is concave,

$$\left|u_N^b(z) - u_\infty^b(z)\right| \le u(\bar{y}_1) - v(\bar{y}_1) = u\left(\frac{y^*}{N}\right) - v\left(\frac{y^*}{N}\right) \text{ for all } z \in [0, p_\theta^K(y^*)].$$

As N tends to ∞, the right side converges to 0. Hence, u_N^b converges to u_∞^b uniformly.

Proof of Corollary 1

The case where $N = 1$ follows from Proposition 2. Note that $g(y, 0) = p_\theta^{GN}(y)$. The case where $N = +\infty$ follows from Proposition 3 (2).

Proof of Proposition 4

<u>Part 1</u>: Buyer's optimal choice. From (31) the payment made by the buyer to purchase $y \in [\bar{y}_{n-1}, \bar{y}_n]$ is bounded below by $g(y, \bar{y}_{n-1})$. By the definition of the function g in (32), $g(y, \bar{y}) > g(y, 0)$ for all $y < y^*$ and for all $\bar{y} > 0$. It follows immediately that it is weakly optimal for the buyer to choose $N = 1$, which corresponds to the payment function $g(y, 0)$.

<u>Part 2</u>: Seller's optimal choice. From (32)

$$g(y, \bar{y}) = p_\theta^K(y) - [\Theta(y) - \theta]\{[u(y) - \upsilon(y)] - [u(\bar{y}) - \upsilon(\bar{y})]\} < p_\theta^K(y) \text{ for all } y \in (\bar{y}, y^*).$$

Hence, from (31) the payment for given y is bounded above by $p_\theta^K(y)$. For any y the payment corresponds to some $p \in \left[p_\theta^K(y - \frac{y^*}{N}), p_\theta^K(y + \frac{y^*}{N})\right]$. Hence, as N tends to infinity, the payment converges to its upper bound, $p_\theta^K(y)$. Therefore, it is optimal for the seller to choose $N = +\infty$.

Proof of Proposition 5

We prove the proposition, assuming only that $\sum_{n=1}^{N} \Delta\bar{y}_n \leq y^*$. To compute the final allocation, we show by induction that in a subgame consisting of the last n rounds with output limits $\{\Delta\bar{y}_{N-n+j}\}_{j=1}^{n}$ and with intermediate allocation $(\hat{y}_0, \hat{p}_0) = (0, 0)$, the equilibrium final allocation is given by the term $(\hat{y}_n, \hat{p}_n)$ obtained from (43). The induction base is $n = 1$, which corresponds to the subgame in the last round. In this case, the disagreement point is no trade, and hence $(\hat{y}_1, \hat{p}_1)$ is the generalized Nash solution to the bargaining problem with disagreement payoff $(0, 0)$ and quantity limit $\Delta\bar{y}_N$.

Now, suppose that the claim holds for n, and consider the subgame consisiting of the last $(n + 1)$ rounds with intermediate allocation $(0, 0)$. This subgame starts in round-$(N - n)$, and the disagreement point follows from the induction hypothesis: if the agents move to the next round with intermediate allocation $(0, 0)$, the final allocation would be $(\hat{y}_n, \hat{p}_n)$. The agents negotiate on outcomes in the set

$$\left\{(y, p) : y \in \left[0, \sum_{j=1}^{n+1} \Delta\bar{y}_{N-(n+1)+j}\right], p \in [0, z]\right\}.$$

With disagreement allocation $(\hat{y}_n, \hat{p}_n)$, the generalized Nash solution is then the term $(\hat{y}_{n+1}, \hat{p}_{n+1})$ obtained from (43), and this completes the induction argument. For this argument to hold, however, we need to ensure that the solution is sustainable as a SPE outcome, which requires existence of intermediate offers that lead to the final allocation, an issue we deal with next.

Intermediate payoffs

Following the same logic as above, the final allocation for a subgame that starts in round-$(n + 1)$ with some arbitrary intermediate allocation (y_n, p_n) (and with output limits $\{\Delta\bar{y}_{n+j}\}_{j=1}^{N-n}$) is the last term of the

following sequence with $j = 1, 2, ..., N - n$, and with $(\tilde{y}_n, \tilde{p}_n) = (y_n, p_n)$,

$$(\tilde{y}_{n+j}, \tilde{p}_{n+j}) \in \arg \max_{y,p} \left[u(y) - u(\tilde{y}_{n+j-1}) - (p - \tilde{p}_{n+j-1}) \right]^\theta \left[-\upsilon(y) + \upsilon(\tilde{y}_{n+j-1}) + (p - \tilde{p}_{n+j-1}) \right]^{1-\theta}$$

$$\text{s.t.} \quad y - \tilde{y}_{n+j-1} \leq \Delta \bar{y}_{N+1-j} \text{ and } p \leq z. \tag{64}$$

In terms of final payoffs, if the liquidity constraint $\tilde{p}_N \leq z$ does not bind, the final payoffs, denoted by (u_N^b, u_N^s), satisfy

$$(u_N^b, u_N^s) \in \mathcal{P} \left(y_n + \sum_{j=1}^{N-n} \Delta \bar{y}_{n+j} \right) \text{ and } \frac{u_N^s - u_n^s}{u_N^b - u_n^b} = \frac{1 - \theta}{\theta}.$$

Otherwise, the constraint binds somewhere in the sequence, say at $j = J \leq N - n$. The final payoffs satisfy

$$(u_N^b, u_N^s) \in \mathcal{P} \left(y_n + \sum_{j=1}^{J} \Delta \bar{y}_{N-j+1} \right), \tag{65}$$

and

$$\frac{\tilde{u}_{n+J-1}^s - u_n^s}{\tilde{u}_{n+J-1}^b - u_n^b} = \frac{1 - \theta}{\theta}, \tag{66}$$

$$\frac{u_N^s - \tilde{u}_{n+J-1}^s}{u_N^b - \tilde{u}_{n+J-1}^b} < \frac{1 - \theta}{\theta}, \tag{67}$$

$$\frac{u_N^s - \tilde{u}_{n+J-1}^s}{u_N^b - \tilde{u}_{n+J-1}^b} \geq \frac{1 - \Theta(y_N)}{\Theta(y_N)}, \quad \text{``} = \text{''} \text{ if } y_N < y_n + \sum_{j=1}^{J} \Delta \bar{y}_{N-j+1}. \tag{68}$$

Note that (65) and (66)-(68) uniquely pin down (u_N^b, u_N^s).

We are now in the position to construct the intermediate payoffs. First, consider the case where the liquidity constraint $\hat{p}_N \leq z$ does not bind in (43). In this case, the final payoffs are given by

$$(u_N^b, u_N^s) \in \mathcal{P} \left(\sum_{j=1}^{N} \Delta \bar{y}_j \right) \text{ and } \frac{u_N^s}{u_N^b} = \frac{1 - \theta}{\theta}. \tag{69}$$

The following intermediate payoffs at the end of round-n will lead to the final allocation given by (69):

$$(\tilde{u}_n^b, \tilde{u}_n^s) \in \mathcal{P} \left(\sum_{j=1}^{n} \Delta \bar{y}_j \right) \text{ and } \frac{\tilde{u}_n^s}{\tilde{u}_n^b} = \frac{1 - \theta}{\theta}. \tag{70}$$

Now we turn to the case where the liquidity constraint $\hat{p}_n \leq z$ binds for some n in (43). With no loss of generality, we assume that it binds only at the very end, i.e., at $(y_N, p_N) = (\hat{y}_N, \hat{p}_N)$ when we compute the sequence according to (43). If it binds earlier, say at $(\hat{y}_n, \hat{p}_n)$, then we can use the same argument but assume no trade from rounds 1 to $N - n$. This assumption is with no loss of generality because $\sum_{n=1}^{N} \Delta \bar{y}_n \leq y^*$.

In this case, the sequence $\{(\hat{u}_n^b, \hat{u}_n^s)\}_{n=1}^N$ satisfies

$$\frac{\hat{u}_n^s}{\hat{u}_n^b} = \frac{1-\theta}{\theta} \text{ and } (\hat{u}_n^b, \hat{u}_n^s) \in \mathcal{P}\left(\sum_{j=1}^{n} \Delta\bar{y}_{N-n+j}\right) \text{ for } n < N,$$

$$\frac{u_N^s - \hat{u}_{N-1}^s}{u_N^b - \hat{u}_{N-1}^b} < \frac{1-\theta}{\theta}, \tag{71}$$

$$\frac{u_N^s - \hat{u}_{N-1}^s}{u_N^b - \hat{u}_{N-1}^b} \geq \frac{1-\Theta(y_N)}{\Theta(y_N)}, \text{ `` = '' if } y_N < \sum_{j=1}^{N} \Delta\bar{y}_j. \tag{72}$$

Now, take $\theta' \in (0,1)$ satisying

$$\frac{1-\theta'}{\theta'} = \frac{u_N^s - \hat{u}_{N-1}^s}{u_N^b - \hat{u}_{N-1}^b}, \tag{73}$$

the buyer's share of surplus from the last round. Then, y_N and $p_N = z$ satisfy

$$z = (1-\theta')\left[u(y_N) - u\left(\sum_{j=2}^{N}\Delta\bar{y}_j\right)\right] + \theta'\left[v(y_N) - v\left(\sum_{j=2}^{N}\Delta\bar{y}_j\right)\right] \tag{74}$$

$$+ (1-\theta)u\left(\sum_{j=2}^{N}\Delta\bar{y}_j\right) + \theta v\left(\sum_{j=2}^{N}\Delta\bar{y}_j\right).$$

By (71) and (72), $\theta < \theta' \leq \Theta(y_N)$.

Now we claim that for each $n = 1,...,N-1$, there exist intermediate payoffs (u_n^b, u_n^s) with associated allocation, (y_n, p_n), such that the final allocation of the subgame starting at round-$(n+1)$ with intermediate allocation (y_n, p_n) will be (y_N, p_N) according to (64). We consider two cases.

(a) Suppose that $\sum_{j=1}^{n}\Delta\bar{y}_j \geq y_N$. Then, take $(y_n, p_n) = (y_N, p_N)$.

(b) Suppose that $\sum_{j=1}^{n}\Delta\bar{y}_j < y_N$. Then, take $y_n = \sum_{j=1}^{n}\Delta\bar{y}_j$, and hence the payoffs (u_n^b, u_n^s) will lie on the Pareto frontier $\mathcal{P}\left(\sum_{j=1}^{n}\Delta\bar{y}_j\right)$. For the final payoffs to be (u_N^b, u_N^s) from (u_n^b, u_n^s), we need them to satisfy (65)-(68). We do this in two steps: first we construct J and $(\tilde{u}_{n+J-1}^b, \tilde{u}_{n+J-1}^s)$, and from there we compute (u_n^b, u_n^s).

(b.1) We use (65) to uniquely determine J and hence J satisfies

$$\sum_{j=1}^{n}\Delta\bar{y}_j + \sum_{j=1}^{J-1}\Delta\bar{y}_{N+1-j} < y_N \leq \sum_{j=1}^{n}\Delta\bar{y}_j + \sum_{j=1}^{J}\Delta\bar{y}_{N+1-j}.$$

To determine $(\tilde{u}_{n+J-1}^b, \tilde{u}_{n+J-1}^s)$, take θ' from (73) and let $(\tilde{u}_{n+J-1}^b, \tilde{u}_{n+J-1}^s)$ solve

$$\frac{u_N^s - \tilde{u}_{n+J-1}^s}{u_N^b - \tilde{u}_{n+J-1}^b} = \frac{1-\theta'}{\theta'}, \ (\tilde{u}_{n+J-1}^b, \tilde{u}_{n+J-1}^s) \in \mathcal{P}\left(\sum_{j=1}^{n}\Delta\bar{y}_j + \sum_{j=1}^{J-1}\Delta\bar{y}_{N+1-j}\right). \tag{75}$$

This implies that $\tilde{y}_{n+J-1} = \sum_{j=1}^{n}\Delta\bar{y}_j + \sum_{j=1}^{J-1}\Delta\bar{y}_{N+1-j}$. Since $\theta' \in (\theta, \Theta(y_N)]$, it can be verified that $(\tilde{u}_{n+J-1}^b, \tilde{u}_{n+J-1}^s)$ satisfy the FOCs (67)-(68).

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(b.2) From $(\tilde{u}^b_{n+J-1}, \tilde{u}^s_{n+J-1})$ we can compute (u^b_n, u^s_n) according to (66):

$$\frac{\tilde{u}^s_{n+J-1} - u^s_n}{\tilde{u}^b_{n+J-1} - u^b_n} = \frac{1-\theta}{\theta}, \quad (u^b_n, u^s_n) \in \mathcal{P}\left(\sum_{j=1}^n \Delta\bar{y}\right). \tag{76}$$

Finally, we show that $p_n \in (0, z)$. From (75) and (76) we can compute p_n as follow:

$$\begin{aligned}
p_n &= z - (1-\theta')[u(y_N) - u(\tilde{y}_{n+J-1})] - \theta'[v(y_N) - v(\tilde{y}_{n+J-1})] \\
&\quad - (1-\theta)\left[u(\tilde{y}_{n+J-1}) - u\left(\sum_{j=1}^n \Delta\bar{y}_j\right)\right] - \theta\left[v(\tilde{y}_{n+J-1}) - v\left(\sum_{j=1}^n \Delta\bar{y}_j\right)\right].
\end{aligned} \tag{77}$$

This implies that $p_n < z$, because $y_N > \tilde{y}_{n+J-1} > \sum_{j=1}^n \Delta\bar{y}_j$. Moreover, by (74), after some algebra it can be verified that $p_n > 0$ if and only if

$$(1-\theta)u\left(\sum_{j=1}^n \Delta\bar{y}_j\right) + \theta v\left(\sum_{j=1}^n \Delta\bar{y}_j\right) > (\theta' - \theta)\left[S(\tilde{y}_{n+J-1}) - S\left(\sum_{j=2}^N \Delta\bar{y}_j\right)\right], \tag{78}$$

where $S(y) \equiv u(y) - v(y)$. If $\tilde{y}_{n+J-1} \leq \sum_{j=2}^N \Delta\bar{y}_j$, $S(\tilde{y}_{n+J-1}) - S\left(\sum_{j=2}^N \Delta\bar{y}_j\right) \leq 0$, (78) holds immediately. Now consider the case where $\tilde{y}_{n+\tilde{J}-1} > \sum_{j=2}^N \Delta\bar{y}_j$, which implies that

$$\sum_{i=1}^n \Delta\bar{y}_i > \Delta\bar{y}_1 - (\Delta\bar{y}_{n+1} + \cdots + \Delta\bar{y}_{N-J+1}) = \tilde{y}_{n+J-1} - \sum_{j=2}^N \Delta\bar{y}_j. \tag{79}$$

Now,

$$\begin{aligned}
(1-\theta)u\left(\sum_{j=1}^n \Delta\bar{y}_j\right) + \theta v\left(\sum_{j=1}^n \Delta\bar{y}_j\right) &> (\theta' - \theta)S\left(\sum_{j=1}^n \Delta\bar{y}_j\right) \\
> (\theta' - \theta)S\left(\tilde{y}_{n+J-1} - \sum_{j=2}^N \Delta\bar{y}_j\right) &> (\theta' - \theta)\left[S(\tilde{y}_{n+J-1}) - S\left(\sum_{j=2}^N \Delta\bar{y}_j\right)\right],
\end{aligned}$$

where the first inequality follows from $\theta' > \theta$, the second from (79), and the third from the concavity of S. This proves (78).

Proof of Proposition 6

Let $z < p^K_\theta(y^*)$ be given. Suppose first that there exists a round $N^K_\theta \leq N$ such that $\sum_{n=1}^{N^K_\theta} \Delta\bar{y}_{N-(n-1)} = y^K_\theta(z)$. Then, it is straightforward to verify that the liquidity constraint in (43) never binds from term 1 to term N^K_θ, and

$$\frac{\hat{u}^s_{N^K_\theta}}{\hat{u}^b_{N^K_\theta}} = \frac{1-\theta}{\theta} \quad \text{and} \quad (\hat{u}^b_{N^K_\theta}, \hat{u}^s_{N^K_\theta}) \in \mathcal{P}[y^K_\theta(z)],$$

which is the definition of the proportional solution. Conversely, suppose that no such N^K_θ exists. Define $\widehat{N} = \min\{n \leq N : \hat{p}_n = z\}$, the first round of the sequence $\{(\hat{y}_n, \hat{p}_n)\}_{n=1}^N$ where $\hat{p}_n \leq z$ binds. It follows

that $\hat{u}^s_{\widehat{N}-1}/\hat{u}^b_{\widehat{N}-1} = (1-\theta)/\theta$, $\widehat{y}_{\widehat{N}-1} = \sum_{n=1}^{\widehat{N}-1} \Delta\bar{y}_{N-(n-1)}$, and $\sum_{n=1}^{\widehat{N}} \Delta\bar{y}_{N-(n-1)} \neq y^K_\theta(z)$. Suppose, by contradiction, that the game still implements the proportional solution. Then, it must be the case that $\widehat{y}_{\widehat{N}} = y^K_\theta(z) < \sum_{n=1}^{\widehat{N}} \Delta\bar{y}_{N-(n-1)} \leq y^*$. This then implies that the output limit is not binding, but the liquidity constraint is, and hence the solution $(\hat{u}^b_{\widehat{N}}, \hat{u}^s_{\widehat{N}})$ satisfies $(\hat{u}^s_{\widehat{N}} - \hat{u}^s_{\widehat{N}-1})/(\hat{u}^b_{\widehat{N}} - \hat{u}^b_{\widehat{N}-1}) = [1 - \Theta(\widehat{y}_{\widehat{N}})]/\Theta(\widehat{y}_{\widehat{N}})$ but with $\Theta(\widehat{y}_{\widehat{N}}) > \theta$ since $\widehat{y}_{\widehat{N}} < y^*$, a contradiction.

Outcomes of limiting agendas

Here we define agendas in $\mathcal{A}$ and their outcomes more precisely. First, a function $\bar{\mathbf{y}} : [0,1] \to [0, y^*]$ is in $\mathcal{A}$ if and only if there is a sequence of finite agendas, $\{\mathbf{y}^k\}_{k=1}^\infty$, such that

$$\lim_{k\to\infty} \sup_{t\in[0,1]} |\mathbf{y}^k(t) - \bar{\mathbf{y}}(t)| = 0. \tag{80}$$

Any finite agenda $\mathbf{y}$ is in $\mathcal{A}$ by taking the constant sequence. It also includes any continuous and increasing $\mathbf{y} : [0,1] \to [0, y^*]$ such that $\mathbf{y}(0) = 0$, $\mathbf{y}(1) = y^*$, which is the limit of the sequence, $\{\mathbf{y}^N\}_{N=1}^\infty$, given by

$$\mathbf{y}^N(t) = \mathbf{y}\left(\frac{n}{N}\right) \text{ for } t \in \left(\frac{n-1}{N}, \frac{n}{N}\right], \ n = 1, .., N.$$

Each $\mathbf{y}^N$ is a finite agenda that discretizes $\mathbf{y}$ by N rounds of output limits $\{\Delta\bar{y}_n\}_{n=1}^N$ such that $\Delta\bar{y}_n = \mathbf{y}\left(\frac{n}{N}\right) - \mathbf{y}\left(\frac{n-1}{N}\right)$. Since $\mathbf{y}$ is uniformly continuous, $\mathbf{y}^N$ converges to $\mathbf{y}$ under the sup-norm. One instance is $\bar{\mathbf{y}}(t) = ty^*$, which is the limit of the sequence $\{\mathbf{y}^N\}_{N=1}^\infty$ such that each $\mathbf{y}^N$ represents $\{\Delta\bar{y}_n\}_{n=1}^N$ with $\Delta\bar{y}_n = y^*/N$. For any agenda $\mathbf{y} \in \mathcal{A}$, which is the limit of a sequence of finite agendas, $\{\mathbf{y}^k\}_{k=1}^\infty$, we define the outcome for agenda $\mathbf{y}$ as the limit of the outcomes from those finite agendas. Denote the equilibrium allocation from the finite agenda $\mathbf{y}^k$ and payment capacity z by $[y(\mathbf{y}^k; z), p(\mathbf{y}^k; z)]$. We define

$$[y(\mathbf{y}; z), p(\mathbf{y}; z)] = \lim_{k\to\infty} [y(\mathbf{y}^k; z), p(\mathbf{y}^k; z)] \tag{81}$$

to be the allocation from $(\mathbf{y}, z)$. The following lemma shows that the limit (81) always exists and is independent of the sequence that converges to the limiting agenda.

Lemma 1 *For any $\mathbf{y} \in \mathcal{A}$, $\mathbf{y}$ is left-continuous and the limit (81) exists and is the same for any sequence $\{\mathbf{y}^k\}_{k=1}^\infty$ that converges to $\mathbf{y}$ under the sup-norm:*

(1) If $z \geq p^K_\theta(y^*)$, then $[y(\mathbf{y}; z), p(\mathbf{y}; z)] = [y^*, p^K_\theta(y^*)]$.

(2) Otherwise, $p(\mathbf{y}^k; z) = z$. We have two cases.

(2a) If $\mathbf{y}$ is continuous, then $y(\mathbf{y}; z) = y^K_\theta(z)$.

(2b) If $\mathbf{y}$ is discontinuous, then there exists t such that

$$p^K_\theta[y^* - \mathbf{y}(t)] \geq z > p^K_\theta[y^* - \lim_{t'\downarrow t} \mathbf{y}(t')], \tag{82}$$

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and

$$y(\mathbf{y}; z) \ solves \ z \ = \ g[y, y^* - \lim_{t' \downarrow t} \mathbf{y}(t')] \ if \ z \le g[y^* - \mathbf{y}(t), y^* - \lim_{t' \downarrow t} \mathbf{y}(t')], \tag{83}$$

$$y(\mathbf{y}; z) \ = \ y^* - \mathbf{y}(t) \ otherwise.$$

Proof. Let $\mathbf{y} \in \mathcal{A}$ and let $\{\mathbf{y}^k\}_{k=1}^{\infty}$ converge to $\mathbf{y}$ under the sup-norm.

(1) Note that if $z \ge p_{\theta}^K(y^*)$, then $[y(\mathbf{y}^k; z), p(\mathbf{y}^k; z)] = [y^*, p_{\theta}^K(y^*)]$ for all k, and hence the limit exists and is equal to $[y^*, p_{\theta}^K(y^*)]$.

(2) Since $z < p_{\theta}^K(y^*)$, for any k we have $p(\mathbf{y}^k; z) = z$ and hence the limit. To find the output level so that the liquidity constraint exactly binds, the relevant output level is $y^* - \mathbf{y}(t)$ from Proposition 5. If N_k, the number of rounds in the finite agenda $\mathbf{y}^k$, is bounded by some $M < +\infty$, then $\mathbf{y}$ must be a finite agenda. Indeed, this implies that each $\mathbf{y}^k$ is fully determined by the vector $[\mathbf{y}^k(1/M'), \mathbf{y}^k(2/M'), ..., \mathbf{y}^k(M'/M')]$, where $M' = 1 \times 2 \times ... \times M$, and $\mathbf{y}^k \to \mathbf{y}$ implies that $[\mathbf{y}^k(1/M'), \mathbf{y}^k(2/M'), ..., \mathbf{y}^k(M'/M')]$ converges to $[\mathbf{y}(1/M'), \mathbf{y}(2/M'), ..., \mathbf{y}(M'/M')]$, with $\mathbf{y}$ constant in between any two adjacent points. Thus, $\mathbf{y}$ is a finite agenda.

(2a) $\mathbf{y}$ is continuous for all $t \in [0, 1]$. We prove $y(\mathbf{y}; z) = y_{\theta}^K(z)$ by two steps. First, we claim that for any output limits, $\{\Delta \bar{y}_n\}_{n=1}^N$, and any $\varepsilon > 0$,

$$\sup_n |\Delta \bar{y}_n| < \varepsilon \implies |y\left(\{\Delta \bar{y}_n\}_{n=1}^N; z\right) - y_{\theta}^K(z)| < \varepsilon \ \text{for all} \ z < p_{\theta}^K(y^*), \tag{84}$$

where $y\left(\{\Delta \bar{y}_n\}_{n=1}^N; z\right)$ is the equilibrium output under output limits, $\{\Delta \bar{y}_n\}_{n=1}^N$, and payment capacity, z. To see this, suppose that $\sup_n |\Delta \bar{y}_n| < \varepsilon$. For each n, let $\bar{y}_n = \sum_{j=1}^n \Delta \bar{y}_{N-(j-1)}$, and let $z \in (p_{\theta}^K(\bar{y}_{n-1}), p_{\theta}^K(\bar{y}_n)]$. Thus, $y_{\theta}^K(z) \in (\bar{y}_{n-1}, \bar{y}_n]$ and $y(\{\Delta \bar{y}_n\}_{n=1}^N, z) \in (\bar{y}_{n-1}, \bar{y}_n]$. This then impies that

$$|y_{\theta}^K(z) - y(\{\Delta \bar{y}_n\}_{n=1}^N; z)| \le \Delta \bar{y}_{N-n+1} < \varepsilon.$$

Since this is true for all $z < p_{\theta}^K(y^*)$, (84) follows.

Second, we claim that for any $\varepsilon > 0$, for k large, $\sup_n \Delta \bar{y}_n^k < \varepsilon$, where $\{\Delta \bar{y}_n^k\}_{n=1}^{N_k}$ is the output limits corresponding to $\mathbf{y}^k$. Given this claim, the result follows immediately from (84). To prove the claim, first note that continuity of $\mathbf{y}$ implies that it cannot be a finite agenda, and hence, by the earlier argument, $N_k \to \infty$. Moreover, since $[0, 1]$ is a compact set, $\mathbf{y}$ is uniformly continuous. Thus, for any $\varepsilon > 0$, for sufficiently small δ, $|\mathbf{y}(t) - \mathbf{y}(t')| < \varepsilon/3$ whenever $|t - t'| < \delta$. Since $\mathbf{y}^k \to \mathbf{y}$ in sup-norm and since $N_k \to \infty$, for k sufficiently large, $1/N_k < \delta$ and $\sup_{t \in [0,1]} |\mathbf{y}^k(t) - \mathbf{y}(t)| < \varepsilon/3$ and hence, for all n,

$$\Delta \bar{y}_n^k \ = \ \mathbf{y}^k\left(\frac{n}{N_k}\right) - \mathbf{y}^k\left(\frac{n-1}{N_k}\right)$$

$$< \ \left|\mathbf{y}^k\left(\frac{n}{N_k}\right) - \mathbf{y}\left(\frac{n}{N_k}\right)\right| + \left|\mathbf{y}\left(\frac{n}{N_k}\right) - \mathbf{y}\left(\frac{n-1}{N_k}\right)\right| + \left|\mathbf{y}^k\left(\frac{n-1}{N_k}\right) - \mathbf{y}\left(\frac{n-1}{N_k}\right)\right| < \varepsilon.$$

This proves that $y(\mathbf{y}; z)$ exists and is equal to $y_\theta^K(z)$.

(2b) Suppose that $\mathbf{y}$ is not continuous. Since $\mathbf{y}^k \to \mathbf{y}$ under the sup-norm for a sequence of finite agendas, and since each $\mathbf{y}^k$ is left-continuous, $\mathbf{y}$ is also left-continuous. The existence of t satisfying (82) then follows from strict monotonicity of $p_\theta^K(y)$ up to y^*.

Now, if $\{N_k\}$ is bounded by M, by the earlier argument, each $\mathbf{y}^k$ can be regarded as output limits consisting of $M' = M!$ rounds (including potentially limits of zeros in some rounds, which does not affect the equilibrium outcome). Since the equilibrium outcome is continuous in $\{\Delta\bar{y}_n\}_{n=1}^{M'}$ for the fixed M', the limit of the outcome sequence must coincide with the outcome of $\mathbf{y}$.

So assume form now on that $\{N_k\}$ is unbounded. By applying subseqeucne if necessary, we consider only the case where $N_k \to \infty$. We consider two cases for z.

(2b.1) Suppose that $z = p_\theta^K[y^* - \mathbf{y}(t)]$ does not hold for any t, and hence

$$p_\theta^K[y^* - \mathbf{y}(t)] > z > p_\theta^K[y^* - \lim_{t' \downarrow t} \mathbf{y}(t')] \text{ for some } t. \tag{85}$$

Denote $\mathbf{y}^+(t) \equiv \lim_{t' \downarrow t} \mathbf{y}(t') > \mathbf{y}(t)$ by (85). For any finite k, let n_k be the unique n such that

$$t \in [n_k/N_k, (n_k + 1)/N_k). \tag{86}$$

Since both inequalities in (85) are strict, let $\epsilon > 0$ be such that the middle term is at least ϵ away from the left and the right. Convergence under sup-norm implies that for k large, $\mathbf{y}^k[(n_k + 1)/N_k] > \mathbf{y}[(n_k + 1)/N_k] - \epsilon > \mathbf{y}^+(t) - \epsilon$, and $\mathbf{y}^k(n_k/N_k) < \mathbf{y}(n_k/N_k) - \epsilon < \mathbf{y}(t) - \epsilon$. This then implies that

$$p_\theta^K\left[y^* - \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right)\right] < z < p_\theta^K\left[y^* - \mathbf{y}^k\left(\frac{n_k}{N_k}\right)\right].$$

As a result,

$$y(\mathbf{y}^k; z) \text{ solves } z = g\left[y, y^* - \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right)\right] \text{ if } z \leq g\left[y^* - \mathbf{y}^k\left(\frac{n_k}{N_k}\right), y^* - \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right)\right],$$

$$y(\mathbf{y}^N; z) = y^* - \mathbf{y}^k\left(\frac{n_k}{N_k}\right) \text{ otherwise.} \tag{87}$$

Compare (87) to (83), we only need to show that

$$\lim_{k \to \infty} \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right) = \mathbf{y}^+(t) \text{ and } \lim_{k \to \infty} \mathbf{y}^k\left(\frac{n_k}{N_k}\right) = \mathbf{y}(t).$$

This follows form (86), which implies that $\frac{n_k + 1}{N_k}$ converges to t from the right and $\frac{n_k}{N_k}$ converges to t from the left, and from the uniform convergence of $\mathbf{y}^k$ to $\mathbf{y}$.

(2b.2) Suppose that $z = p_\theta^K[y^* - \mathbf{y}(t)]$ for some t. We claim that

$$y(\mathbf{y}; z) = y_\theta^K(z) = y^* - \mathbf{y}(t). \tag{88}$$

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Let T be the set of t's such that $z = p_\theta^K[y^* - \mathbf{y}(t)]$. Then $T = [t_1, t_2]$ or $T = (t_1, t_2]$ for some $t_1 \leq t_2$ since $\mathbf{y}$ is left-continuous and increasing. Moreover, $\mathbf{y}(t) > \mathbf{y}(t_2)$ for all $t > t_2$ and $\mathbf{y}(t) < \mathbf{y}(t_1)$ for all $t < t_1$, and $\mathbf{y}(t_1) < \mathbf{y}(t_2)$ if $T = (t_1, t_2]$. For each k, let n_k be the unique n such that

$$p_\theta^K\left[y^* - \mathbf{y}^k\left(\frac{n_k}{N_k}\right)\right] \geq z > p_\theta^K\left[y^* - \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right)\right]. \tag{89}$$

This implies that

$$\mathbf{y}^k\left(\frac{n_k}{N_k}\right) \leq \mathbf{y}(t_2) < \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right). \tag{90}$$

We consider two cases.

First, suppose that $\mathbf{y}$ is continuous for all $t \in [t_1, t_2]$. We claim that

$$\lim_{k \to \infty} \mathbf{y}^k\left(\frac{n_k}{N_k}\right) = \lim_{k \to \infty} \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right) = \mathbf{y}(t_2). \tag{91}$$

This implies that $\lim_{k \to \infty} y(\mathbf{y}^k; z) = y^* - \mathbf{y}(t_2)$ because, by (89), $y^* - \mathbf{y}^k\left(\frac{n_k+1}{N_k}\right) \leq y(\mathbf{y}^k; z) \leq y^* - \mathbf{y}^k\left(\frac{n_k}{N_k}\right)$. Since $\left\{\frac{n_k}{N_k}\right\}$ lies in the compact set $[0, 1]$, we may assume it converges by applying subsequences if necessary. By uniform convergence and monotonicity, (90) implies that its limit $\bar{t}$ must be within $[t_1, t_2]$. (91) then follows from continuity of $\mathbf{y}$ at $\bar{t}$ and uniform convergence of $\mathbf{y}^k$ to $\mathbf{y}$.

Second, suppose that $\mathbf{y}$ is discontinuous at both t_1 and t_2, and hence $\mathbf{y}(t_1) < \mathbf{y}^+(t_1) = \mathbf{y}(t_2) < \mathbf{y}^+(t_2)$. These inequalities, together with uniform convergence, monotonicity, and (90), imply that $\left\{\frac{n_k}{N_k}\right\}$ is bounded above by t_2 except for finitely many elements, and $\left\{\frac{n_k+1}{N_k}\right\}$ is bounded below by t_1 except for finitely many elements. Thus, by applying subsequence if necessary, we may assume that $\lim_{k \to \infty}\left\{\frac{n_k}{N_k}\right\} = \bar{t} = \lim_{k \to \infty}\left\{\frac{n_k+1}{N_k}\right\}$ with $\bar{t} \in [t_1, t_2]$. We consider three subcases. First, $\bar{t} \in (t_1, t_2)$. Using the same argument as in the first case, we can show (91) holds. Seond, $\bar{t} = t_1$. If $\left\{\frac{n_k}{N_k}\right\}$ is bounded below by t_1, then the argument is the same and (91) holds. Otherwise, we may assume that $\left\{\frac{n_k}{N_k}\right\}$ converges to t_1 from the left and $\left\{\frac{n_k+1}{N_k}\right\}$ converges to t_1 from the right. Uniform convergene of $\mathbf{y}^k$ implies that

$$\lim_{k \to \infty} \mathbf{y}^k\left(\frac{n_k}{N_k}\right) = \mathbf{y}(t_1) < \mathbf{y}^+(t_1) = \mathbf{y}(t_2) = \lim_{k \to \infty} \mathbf{y}^k\left(\frac{n_k + 1}{N_k}\right). \tag{92}$$

For each k, by (89), we can then compute $y(\mathbf{y}^k; z)$ in the same way as in (87). Since $z = p_\theta^K[y^* - \mathbf{y}(t_2)] = g[y^* - \mathbf{y}(t_2), y^* - \mathbf{y}(t_2)]$, it follows that for k large, $y(\mathbf{y}^k; z)$ solves $z = g\left[y, y^* - \mathbf{y}^k\left(\frac{n_k+1}{N_k}\right)\right]$. Thus, by continuity of g and (92), $\lim_{k \to \infty} y(\mathbf{y}^k; z) = y^* - \mathbf{y}(t_2)$. The third subcase where $\bar{t} = t_2$ follows an analogous argument. This proves (88). Finally, the case with $\mathbf{y}$ discontinuous at t_1 but not t_2, or the other way around, follows analogous arguments. ∎

Proof of Proposition 7

The result that for any $\mathbf{y} \in \mathcal{A}$ and any $z < p_\theta^K(y^*)$, $p(\mathbf{y}; z) = z$ and $y(\mathbf{y}; z) \geq y_\theta^K(z)$ follows from Lemma 1 (1). Lemma 1 (2) implies that only continuous agendas implement the proportional solution for all z. Indeed, let $\mathbf{y}$ be discontinuous at some $t \in [0, 1]$ such that $\mathbf{y}^+(t) > \mathbf{y}(t)$. Then, (83) implies that $y(\mathbf{y}; z) > y_\theta^K(z)$ for all $z \in (p_\theta^K[y^* - \mathbf{y}^+(t)], p_\theta^K[y^* - \mathbf{y}(t)])$. Finally, the result that $y(\mathbf{y}; z) \leq y_\theta^{GN}(z)$ follows from the fact that $g(y, 0) = p_\theta^{GN}(y)$ and g is strictly increasing in its second argument.

Proof of Proposition 8

We denote $z_N^* \in Z_N^*$ a maximizer of $u_N^b(z)$ where

$$Z_N^* \equiv \arg \max_{z \in [0, p_\theta^K(y^*)]} u_N^b(z).$$

The set Z_N^* is nonempty by the Extreme Value Theorem, because $u_N^b(z)$ is continuous and $[0, p_\theta^K(y^*)]$ is compact. We first establish that for all $N \geq 1$, $z_N^* < p_\theta^K(y^*)$. From (32),

$$g\left(y^*, \frac{(N-1)y^*}{N}\right) \equiv (1 - \theta) u(y^*) + \theta v(y^*) = p_\theta^K(y^*).$$

Hence, $\left[g(y^*, (N-1)y^*/N), p_\theta^K(y^*)\right] = \{p_\theta^K(y^*)\}$. It follows from (33) that in the last round of the negotiation, if $z \in \left[p_\theta^K(\bar{y}_{N-1}), p_\theta^K(y^*)\right]$, then the buyer's surplus is

$$u^b(z) = u[y(z)] - g[y(z), \bar{y}_{N-1}],$$

where $y(z)$ is implicitly defined by $z = g(y, \bar{y}_{N-1})$. From (32),

$$u(y) - g(y, \bar{y}) \equiv \Theta(y) \{[u(y) - v(y)] - [u(\bar{y}) - v(\bar{y})]\} + \theta[u(\bar{y}) - v(\bar{y})]$$

It can be checked that $u'(y) - \partial g(y, \bar{y})/\partial y < 0$ when evaluated at $y = y^*$ for all $\theta < 1$. Indeed, $\Theta'(y^*) < 0$ while $u'(y^*) - v'(y^*) = 0$. So, the optimal z over the interval $\left[p_\theta^K(\bar{y}_{N-1}), p_\theta^K(y^*)\right]$ is $z < p_\theta^K(y^*)$. For all $z \geq p_\theta^K(y^*)$, the buyer's surplus is constant and equal to $\theta[u(y^*) - v(y^*)]$. Hence, $z_N^* < p_\theta^K(y^*)$.

Next we establish that any sequence, $\{z_N^*\}_{N=1}^{+\infty}$, with $z_N^* \in Z_N^*$ for all N, converges to $p_\theta^K(y^*)$. By the Bolzano-Weierstrass Theorem, since the sequence $\{z_N^*\}_{N=1}^{+\infty}$ is bounded, it admits a convergent subsequence. Consider any such convergent subsequence, $\{z_{\tilde{N}}^*\}$, where $z_{\tilde{N}}^* \to z_\infty^*$ denotes the limit. We show that $z_\infty^* = p_\theta^K(y^*)$. To this end, we show that for any $\epsilon > 0$,

$$u_\infty^b(z) \leq u_\infty^b(z_\infty^*) + \epsilon \text{ for all } z \in \left[0, p_\theta^K(y^*)\right], \tag{93}$$

which then implies that u_∞^b obtains maximum at z_∞^*. Since $Z_\infty^* = \{p_\theta^K(y^*)\}$, i.e., there is a unique maximizer under Kalai bargaining and it is $p_\theta^K(y^*)$. This then implies that $z_\infty^* = p_\theta^K(y^*)$. It then follows that all convergent subsequences of $\{z_N^*\}_{N=1}^{+\infty}$ converge to same limit $p_\theta^K(y^*)$, hence $\{z_N^*\}_{N=1}^{+\infty}$ converges to $p_\theta^K(y^*)$.

Now we prove (93). Let $\epsilon > 0$ be given. By the definition of a maximizer,

$$u_{\tilde{N}}^b(z_{\tilde{N}}^*) \geq u_{\tilde{N}}^b(z) \text{ for all } z \in \left[0, p_\theta^K(y^*)\right].$$

By Proposition 3, as $\tilde{N}$ goes to infinity, $u_{\tilde{N}}^b$ converges uniformly to u_∞^b. Thus, there exists $\hat{N}$ such that for all $\tilde{N} \geq \hat{N}$, $u_{\tilde{N}}^b(z_{\tilde{N}}^*) \leq u_\infty^b(z_{\tilde{N}}^*) + \epsilon/2$. Hence,

$$u_\infty^b(z) \leq u_{\tilde{N}}^b(z) \leq u_{\tilde{N}}^b(z_{\tilde{N}}^*) \leq u_\infty^b(z_{\tilde{N}}^*) + \epsilon/2 \text{ for all } z \in \left[0, p_\theta^K(y^*)\right],$$

where we used that $u_\infty^b(z) \leq u_N^b(z)$ for all $N < +\infty$. As $z_{\tilde{N}}^*$ converges to z_∞^* as $\tilde{N}$ goes to infinity, and by the continuity of $u_\infty^b(z)$, there exists $\check{N}$ such that for all $\tilde{N} \geq \check{N}$, $\left|u_\infty^b(z_{\tilde{N}}^*) - u_\infty^b(z_\infty^*)\right| \leq \epsilon/2$. This then implies that, for all $z \in \left[0, p_\theta^K(y^*)\right]$,

$$u_\infty^b(z) \leq u_\infty^b(z_{\tilde{N}}^*) + \epsilon/2 \leq u_\infty^b(z_\infty^*) + \epsilon.$$

This then proves (93).

Proof Proposition 9

Both $u_1^b(z) \equiv u\left[y_\theta^{GN}(z)\right] - z$, where $y_\theta^{GN}(z)$ is defined implicitly by $p_\theta^{GN}(y) = z$, and $u_\infty^b(z) \equiv u\left[y_\theta^K(z)\right] - z$, where $y_\theta^K(z)$ is defined by $p_\theta^K(y) = z$, are differentiable over $[0, p_\theta^K(y^*))$. For all $z \geq p_\theta^K(y^*)$, $\lambda u_1^b(z) + (1 - \lambda)u_\infty^b(z) = \theta\left[u(y^*) - v(y^*)\right]$. The derivative of the buyer's expected surplus, $\lambda u_1^b(z) + (1 - \lambda)u_\infty^b(z)$, when evaluated at $z = p_\theta^K(y^*)^-$ is negative for all $\lambda > 0$ and $\theta < 1$. Hence, $z^* < p_\theta^K(y^*)$. Since we assume that $u_1^b(z)$ is strictly concave (and since $u_\infty^b(z)$ is strictly concave by the strict concavity of u and convexity of v), from (47) the optimal z^* is the unique solution to the first-order condition

$$\lambda u_1^{b\prime}(z^*) + (1 - \lambda)u_\infty^{b\prime}(z^*) = 0.$$

Using that $u_\infty^{b\prime}(z) > 0$ for all $z < p_\theta^K(y^*)$, the optimal solution must satisfy $u_1^{b\prime}(z^*) < 0$. By the implicit function theorem,

$$\frac{\partial z^*}{\partial \lambda} = \frac{u_\infty^{b\prime}(z^*) - u_1^{b\prime}(z^*)}{\lambda u_1^{b\prime\prime}(z^*) + (1 - \lambda)u_\infty^{b\prime\prime}(z^*)} < 0.$$

Proof of Proposition 10

We first show that a SPE exists, where the buyer chooses z that satisfies

$$z \in \arg\max\left\{-\iota z + u\left[y_\theta^K(z)\right] - p_\theta^K(z)\right\}. \tag{94}$$

and the seller chooses a continuous $\mathbf{y}$. Since by Proposition 7, for any z the seller's payoffs are bounded above by the proportional solution which can be achieved by a continuous $\mathbf{y}$, in either version such agenda

is optimal, regardless of the buyer's choice. Given that the seller chooses a continuous $\mathbf{y}$, z is optimal if it solves (94). This proves the existence. Note that this also pins down the equilibrium payoffs for both agents.

Now we consider the version where z is observed by the seller before setting the agenda. Since for given z any agenda that satisfies (44) also implements the proportional solution for z, it is also a best response for the seller and hence an equilibrium agenda.

Finally, suppose that the seller does not observe z when setting the agenda. We claim that any finite agenda, $\{\Delta \bar{y}_n\}_{n=1}^{N}$, cannot be an equilibrium agenda. The proof is by contradiction and suppose that $\{\Delta \bar{y}_n\}_{n=1}^{N}$ is an equilibrium agenda. Moreover, suppose that z is the equilibrium payment capacity chosen by the buyer. We claim that (i) $z < p_\theta^K(y^*)$ and (ii) $\sum_{j=1}^{J} \Delta \bar{y}_{N+1-j} = y_\theta^K(z)$ for some $J < N$. To prove (i), suppose that, by contradiction, $z \geq p_\theta^K(y^*)$. Then the outcome is $[y^*, p_\theta^K(y^*)]$. But then the buyer can deviate to $z' = p_\theta^K(y^*) - \epsilon$ for some small $\epsilon > 0$ and do better, which follows from the fact the buyer's surplus is decreasing when y is close to y^*. This proves (i). Now consider (ii). Since the seller can expect the equilibrium z, his optimal response is to obtain the surplus according to the proportional solution, and this requires $\sum_{j=1}^{J} \Delta \bar{y}_{N+1-j} = y_\theta^K(z)$ for some J. Since equilibrium $z < p_\theta^K(y^*)$ by (i), this implies that $J < N$. Now, if the buyer brings $z - \epsilon'$ for $\epsilon' > 0$ but small, then

$$g \left(\sum_{j=1}^{J} \Delta \bar{y}_{N+1-j}, \sum_{j=1}^{J-1} \Delta \bar{y}_{N+1-j} \right) < z - \epsilon' < p_\theta^K \left(\sum_{j=1}^{J} \Delta \bar{y}_{N+1-j} \right)$$

and hence the final output would still be $y = \sum_{j=1}^{J} \Delta \bar{y}_{N+1-j}$, and this is a profitable deviation for the buyer.