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Exact asymptotics of component-wise extrema of two-dimensional Brownian motion

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Abstract: We derive the exact asymptotics of

$$\mathbb{P}\left\{\sup_{t\geq 0}\Big(X_1(t)-\mu_1 t\Big)>u,\ \sup_{s\geq 0}\Big(X_2(s)-\mu_2 s\Big)>u\right\},\quad u\to\infty,$$

where $(X_1(t),X_2(s))_{t,s\geq 0}$ is a correlated two-dimensional Brownian motion with correlation $\rho\in[-1,1]$ and $\mu_1,\mu_2>0$. It appears that the play between ρ and μ_1,μ_2 leads to several types of asymptotics. Although the exponent in the asymptotics as a function of ρ is continuous, one can observe different types of prefactor functions depending on the range of ρ, which constitute a phase-type transition phenomena.

Key Words: Two-dimensional Brownian motion; exact asymptotics; component-wise extrema; quadratic programming problem; generalised Pickands-Piterbarg constants.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Distributional properties of component-wise extrema of stochastic processes attract growing interest in recent literature. On one side, it is a natural object of interest in the extreme value theory of random fields. On the other side, strong motivation to investigate component-wise extrema stems for example from multivariate stochastic models applied to modern multidimensional risk theory, financial mathematics or advanced communication networks, to name some of the applied-probability areas.

We consider a standard correlated Brownian motion $(X_1(t),X_2(t))_{t\geq 0}$ with constant correlation $\rho\in[-1,1]$, and let $(X_1(t),X_2(s))_{t,s\geq 0}$ be its two parameter extension, where

$$\mathbb{E}\left\{X_1(t)X_2(s)\right\}=\rho\min(t,s).$$

The aim of this paper is to find exact asymptotics of

$$P(u):=\mathbb{P}\left\{Q_1>u,Q_2>u\right\},\qquad u\to\infty, \tag{1}$$

where $Q_j=\sup_{t\geq 0}(X_j(t)-\mu_j t)$ with $\mu_j>0$, $j=1,2$.

Due to its importance in, e.g., quantitative finance or ruin theory, the component-wise maxima

$$(Q_1(T),Q_2(T))=\left(\sup_{t\in[0,T]}(X_1(t)-\mu_1 t),\ \sup_{s\in[0,T]}(X_2(s)-\mu_2 s)\right)$$

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have been studied extensively; see, e.g., [4, 12, 15, 19, 23, 24]. In particular, some formulas for the joint distribution of $(Q_1(T),\ Q_2(T))$ are known. Unfortunately, they are in the form of infinite-sums of integrals of some special functions, which makes them of limited use in drawing out asymptotic properties of $P(u)$ as $u \to \infty$.

Interestingly, in [15] it was worked out a formula for joint survival function of $(Q_1(\mathcal{E}_p), Q_2(\mathcal{E}_p))$, where $\mathcal{E}_p$ is an independent exponential random variable with parameter $p > 0$. Vector $(Q_1(\mathcal{E}_p), Q_2(\mathcal{E}_p))$ as well as (Q_1, Q_2) have bivariate exponential distribution (BVE) in the sense of the terminology of Kou and Zhong [15], that is: (i) it has exponential marginals and (ii) it is absolute continuous with respect to two-dimensional Lebesgue measure. The later property for (Q_1, Q_2) follows from Theorem 7.1 in [2] combined with the fact that $\mathbb{P}\{Q_j = 0\} = 0$, see also related Lemma 4.4 in [7]. We remark that requirement (ii) implies that (Q_1, Q_2) does not belong to the classical examples of *Marshall-Olkin*-type BVE; see [18]. Since there are no results in the literature on qualitative properties of our BVE distribution, as a by-product of the results of this contribution, we analyze the dependence structure of Q_1 and Q_2 in an asymptotical sense of Resnick [22]; see Remarks 2.2 (b) and Remarks 2.4 (b) for more details. We refer also to a related work of Rogers and Shepp [23] who considered correlation structure of $(Q_1(T), Q_2(T))$ for two Brownian motions without drift.

A need to consider the joint survival function for (Q_1, Q_2) appeared also in Lieshout and Mandjes [16] who considered two parallel queues sharing the same Brownian input (which is the case of $\rho = 1$) and also a Brownian tandem queue. We refer to [17] for further discussions on Gaussian-related queueing models and to [3, 6] for the analysis of a related *simultaneous ruin* problem for the correlated Brownian motion model.

It is worth noting that in recent papers [26, 13], the component-wise maxima in discrete models defined by

$$(\max_{1\le i\le n} X_i^1, \ldots, \max_{1\le i\le n} X_i^d),$$

with $(X_i^1, \ldots, X_i^d)$ $(i = 1, 2, \ldots)$ independent and identically distributed Gaussian random vectors, were discussed.

The first step in understanding the asymptotics of (1) is to find its logarithmic asymptotics. This was done recently in [8], in an insurance context, where $P(u)$ was interpreted as the probability of component-wise ruin. More precisely, by an application of Theorem 1 in [9]

$$\frac{\ln P(u)}{u} \sim -\frac{g(\boldsymbol{t}_0)}{2}, \quad u \to \infty, \tag{2}$$

where (with $\boldsymbol{t} = (t, s)^\top$ a column vector and $^\top$ denoting the transpose sign)

$$g(\boldsymbol{t}_0) = \inf_{\boldsymbol{t}>\boldsymbol{0}} \inf_{\substack{x\ge 1+\mu_1 t\\ y\ge 1+\mu_2 s}} (x, y)\ \Sigma_{ts}^{-1}\ (x, y)^\top \tag{3}$$

and Σ_{ts}^{-1} is the inverse matrix of $\Sigma_{ts} = \begin{pmatrix} t & \rho\ (t\wedge s) \\ \rho\ (t\wedge s) & s \end{pmatrix}$, with $t \wedge s = \min(t, s)$. The main contribution of [8] includes the detailed analysis of the two-layer minimisation problem involved in $g(\boldsymbol{t}_0)$, which results in an explicit logarithmic asymptotics of $P(u)$; see also Proposition 3.1 below.

In order to get the exact asymptotics of $P(u)$ as $u \to \infty$, we employ a modification of the *double-sum* technique, accommodated to the analysis of multivariate extremes investigated in this contribution; see Theorems 2.1 and 2.3, which constitute the main results of this paper. It appears that the play between ρ and μ_1, μ_2 leads to several types of asymptotics. Although in [8] it was noticed, that the exponent in the asymptotics as a function of ρ, called therein an adjustment coefficient, is continuous, one can observe different types of prefactor functions depending on the range of ρ. This *phase*-type phenomena has no intuitive explanations.

In the rest of the paper we assume that $\rho \in (-1,1)$ and without loss of generality suppose that $\mu_1 \le \mu_2$. Note that for $\rho = 1$,

$$P(u) = \mathbb{P}\left\{\sup_{s\ge 0}(X_2(s) - \mu_2 s) > u\right\} = e^{-2\mu_2 u}, \quad \forall u > 0$$

and, for $\rho = 0$,

$$P(u) = \mathbb{P}\left\{\sup_{t\ge 0}(X_1(t) - \mu_1 t) > u\right\}\mathbb{P}\left\{\sup_{s\ge 0}(X_2(s) - \mu_2 s) > u\right\} = e^{-2(\mu_1+\mu_2)u}, \quad \forall u > 0. \tag{4}$$

To work out the case $\rho = -1$, one can use a result from [25], to show that

$$P(u) \sim e^{-(2\mu_2+6\mu_1)u}(2I_{\{\mu_1=\mu_2\}} + I_{\{\mu_1<\mu_2\}}), \quad u \to \infty, \tag{5}$$

where $I_{\{\cdot\}}$ is the indicator function.

The rest of this paper is organised as follows. In Section 2, we present the exact asymptotics of $P(u)$, given in Theorems 2.1, 2.3. Section 3 recalls the explicit expressions for $g(\boldsymbol{t}_0)$ and $\boldsymbol{t}_0$ derived in [8]. The main lines of proofs are displayed in Section 4 and Section 5, respectively, followed by the Appendix consisting of technical calculations.

We conclude this section by showing some notation and conventions used in this work. All vectors here are 2-dimensional column vectors written in bold letters. For instance $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top$. Operations with vectors are meant component-wise, so $\lambda \boldsymbol{x} = \boldsymbol{x}\lambda = (\lambda x_1, \lambda x_2)^\top$ for any $\lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^2$. For any set $D \subseteq [0,\infty)^2$, any $\lambda > 0$ and any $(a_1, a_2) \in [0,\infty)^2$ denote

$$\lambda D = \{(\lambda t, \lambda s) : (t,s) \in D\}, \quad (a_1,a_2) + D = \{(a_1 + t, a_2 + s) : (t,s) \in D\}.$$

Next, let us briefly mention the following standard notation for two given positive functions $f(\cdot)$ and $h(\cdot)$. We write $f(x) = h(x)(1+o(1))$ or simply $f(x) \sim h(x)$, if $\lim_{x\to a} f(x)/h(x) = 1$ ($a \in \mathbb{R}\cup\{\infty\}$). Further, write $f(x) = o(h(x))$ if $\lim_{x\to a} f(x)/h(x) = 0$, and write $f(x) \lesssim h(x)$ if $\lim_{x\to a} f(x)/h(x) \le 1$.

2. Main results

In this section we present the exact asymptotics of $P(u)$, for which we need some additional notation. First, define

$$\hat\rho_1 = \frac{\mu_1 + \mu_2 - \sqrt{(\mu_1+\mu_2)^2 - 4\mu_1(\mu_2-\mu_1)}}{4\mu_1} \in [0, \frac{1}{2}), \qquad \hat\rho_2 = \frac{\mu_1+\mu_2}{2\mu_2}. \tag{6}$$

These are key points, based on which we consider different scenarios of ρ. Next, let

$$\Sigma_* = \begin{pmatrix} t^* & \rho s^* \\ \rho s^* & s^* \end{pmatrix}, \qquad \boldsymbol{b}_* = (1+\mu_1 t^*, 1+\mu_2 s^*)^\top, \tag{7}$$

with

$$t^* = t^*(\rho) = s^* = s^*(\rho) := \sqrt{\frac{2(1-\rho)}{\mu_1^2 + \mu_2^2 - 2\rho\mu_1\mu_2}}. \tag{8}$$

Moreover, denote, for any fixed $T, S > 0$,

$$\triangle_{T,S} = \{(t,s) : t \in [0,T], s \in [t, t+S]\} \cup \{(t,s) : s \in [0,T], t \in [s, s+S]\}, \tag{9}$$

and define

$$\mathcal{H}(T,S) := \int_{\mathbb{R}^2} e^{\boldsymbol{x}^\top \Sigma_*^{-1} \boldsymbol{b}_*} \mathbb{P}\left\{ \exists_{(t,s)\in\triangle_{T,S}} \begin{array}{c} X_1(t) - \mu_1 t > x_1 \\ X_2(s) - \mu_2 s > x_2 \end{array} \right\} dx_1 dx_2 \in (0,\infty),$$

where the finiteness can be proved by following a standard argument in proving the finiteness of Pickands and Piterbarg type constants; see, e.g., [20] (or Lemma 4.2 in [3]). Interestingly, a new Pickands-Piterbarg constant

$$\widetilde{\mathcal{H}} := \lim_{S\to\infty} \lim_{T\to\infty} \frac{1}{T} \mathcal{H}(T,S) \in (0,\infty)$$

appears in the scenario $\hat\rho_1 < \rho < \hat\rho_2$; the existence, finiteness and positiveness of this constant are proved in Theorem 2.1 below.

We split the statement of the main results on the exact asymptotics into two scenarios: $\mu_1 < \mu_2$ and $\mu_1 = \mu_2$ respectively.

Theorem 2.1. *Suppose that $\mu_1 < \mu_2$. We have, as $u \to \infty$,*

$$P(u) \sim \begin{cases} e^{-2(\mu_2 + (1-2\rho)\mu_1)u}, & \text{if } -1 < \rho < \hat\rho_1; \\ \frac{1}{2}\, e^{-2(\mu_2 + (1-2\hat\rho_1)\mu_1)u}, & \text{if } \rho = \hat\rho_1; \\ \frac{\widetilde{\mathcal{H}}\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}}\, u^{-1/2} e^{-\frac{\mu_1+\mu_2+2/t^*}{1+\rho} u}, & \text{if } \hat\rho_1 < \rho < \hat\rho_2; \\ e^{-2\mu_2 u}, & \text{if } \hat\rho_2 < \rho < 1, \end{cases}$$

where

$$0 < \frac{t^* \boldsymbol{\mu}^\top \Sigma_*^{-1} \boldsymbol{b}_*}{16 \prod_{i=1}^2 (\Sigma_*^{-1} \boldsymbol{b}_*)_i} < \widetilde{\mathcal{H}} < \infty.$$

Remarks 2.2. *(a). It turns out that the special scenario $\rho = \hat\rho_2$ is of different nature than the scenarios analyzed in Theorem 2.1. Note that in this case we have $b_1 = b_2 = 0$ in Lemma A.1, which implies that around its optimizing point $(t^*, s^*) = (1/\mu_2, 1/\mu_2)$ function $g(t,s)$ defined in Section 3 takes different form than for other scenarios. This makes its analysis go out of the approach that works for the other scenarios. In Section 4.4, following the same lines of reasoning as given in the proof of case $\hat\rho_2 < \rho < 1$ in Theorem 2.1, we find the following bounds for the case of $\rho = \hat\rho_2$*

$$\frac{1}{2}\, e^{-2\mu_2 u} \ \lesssim\ P(u) \ \lesssim\ e^{-2\mu_2 u}, \text{ as } u \to \infty. \tag{10}$$

(b). It follows from Theorem 2.1 and (5) that for any $-1 \le \rho < \hat\rho_2$

$$\mathbb{P}\{Q_1(\infty) > u | Q_2(\infty) > u\} = \frac{\mathbb{P}\{Q_1(\infty) > u, Q_2(\infty) > u\}}{\mathbb{P}\{Q_2(\infty) > u\}} \to 0, \quad u \to \infty.$$

According to the terminology from [22], this means that $Q_1(\infty)$ is asymptotically independent of $Q_2(\infty)$. Similarly, one can see that $Q_2(\infty)$ is also asymptotically independent of $Q_1(\infty)$ (note that the notion of asymptotically independence is not symmetric). Furthermore, for $\hat\rho_2 \le \rho \le 1$, we have that $Q_2(\infty)$ is asymptotically independent of $Q_1(\infty)$, but $Q_1(\infty)$ is asymptotically dependent of (equivalent to) $Q_2(\infty)$.

Next we give the result for the case where $\mu := \mu_1 = \mu_2$. In this case, we have $t^* = s^* \equiv 1/\mu$ and

$$\widetilde{\mathcal{H}} := \lim_{S\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_{\mathbb{R}^2} e^{\frac{2\mu}{1+\rho}(x_1+x_2)} \mathbb{P}\left\{ \exists_{(t,s)\in\triangle_{T,S}} \begin{array}{c} X_1(t) - \mu t > x_1 \\ X_2(s) - \mu s > x_2 \end{array} \right\} dx_1 dx_2.$$

Theorem 2.3. *Suppose that $\mu_1 = \mu_2$. We have, as $u \to \infty$,*

$$P(u) \sim \begin{cases} 2\ e^{-4(1-\rho)\mu u}, & \text{if } -1 < \rho < 0; \\ e^{-4\mu u}, & \text{if } \rho = 0; \\ \frac{\widetilde{\mathcal{H}}}{2\sqrt{\pi\mu(1-\rho)}}\ u^{-1/2} e^{-\frac{4\mu}{1+\rho}u}, & \text{if } 0 < \rho < 1, \end{cases}$$

where $(1+\rho)/16 < \widetilde{\mathcal{H}} < \infty$.

Remarks 2.4. *(a). Note that comparing scenario $-1 < \rho < 0$ of Theorem 2.3 with $-1 < \rho < \hat{\rho}_1$ of Theorem 2.1, there is an additional 2 appearing in the asymptotics. The reason for this is that there are two equally important minimizers of $g(t,s), (t,s) \in (0,\infty)^2$ in the case of $\mu_1 = \mu_2$.*
(b). For any $-1 < \rho < 1$, we have that $Q_1(\infty)$ and $Q_2(\infty)$ are mutually asymptotically independent.

3. Analysis of the two-layer minimization problem

In this section, for completeness and for reference we recall some notation and the result on the two-layer minimization problem (3) derived in [8]. Recall that

$$g(t,s) := \inf_{\substack{x\ge 1+\mu_1 t \\ y \ge 1+\mu_2 s}} (x,y)\ \Sigma_{ts}^{-1}\ (x,y)^\top, \qquad t,s>0.$$

Function $g(t,s)$ has a natural interpretation as $g^{-1}(t,s)$ plays the same role as variance of one-dimensional centered normal random variable, in the sense that according to [10]

$$\ln \mathbb{P}\left\{X_1(t) > (1+\mu_1 t)\sqrt{u}, X_2(s) > (1+\mu_2 s)\sqrt{u}\right\} \sim -\frac{g(t,s)}{2} u, \qquad u\to\infty;$$

see also (15) below. Properties of $g(t,s)$ play crucial role for the asymptotics of $P(u)$ as $u\to\infty$. In particular, as mentioned above we know that

$$\ln P(u) \sim -\frac{g(\boldsymbol{t}_0)}{2} u,$$

where $g(\boldsymbol{t}_0) = \inf_{(t,s)\in(0,\infty)^2} g(t,s)$. We refer to [9] for more detailed and general discussions on the logarithmic asymptotics of supremas of multidimensional Gaussian processes and fields. We refer also to [3, 7] for analogs of generalized variance function in the context of extremes of vector-valued Gaussian processes.
For the exact asymptotics of $P(u)$ as $u\to\infty$, the local behaviour of $g(t,s)$ around point $\boldsymbol{t}_0$ has to be analyzed. For this we define for $t,s>0$ the following functions:

$$g_1(t) = \frac{(1+\mu_1 t)^2}{t}, \qquad g_2(s) = \frac{(1+\mu_2 s)^2}{s},$$
$$g_3(t,s) = (1+\mu_1 t, 1+\mu_2 s)\ \Sigma_{ts}^{-1}\ (1+\mu_1 t, 1+\mu_2 s)^\top.$$

Since $t\wedge s$ appears in the above formula, we shall consider a partition of the quadrant $(0,\infty)^2$, namely

$$(0,\infty)^2 = A\cup L\cup B, \quad A=\{s<t\},\ L=\{s=t\},\ B=\{s>t\}. \tag{11}$$

For convenience we denote $\overline{A} = \{s \le t\} = A \cup L$ and $\overline{B} = \{s \ge t\} = B \cup L$. Hereafter, all sets are defined on $(0,\infty)^2$, so $(t,s) \in (0,\infty)^2$ will be omitted.

Note that $g_3(t,s)$ can be represented in the following two different forms:

$$g_3(t,s) = \begin{cases} g_A(t,s) := \frac{(1+\mu_1 t)^2 s - 2\rho s(1+\mu_1 t)(1+\mu_2 s)+(1+\mu_2 s)^2 t}{ts-\rho^2 s^2}, & \text{if } (t,s) \in \overline{A} \\ g_B(t,s) := \frac{(1+\mu_1 t)^2 s - 2\rho t(1+\mu_1 t)(1+\mu_2 s)+(1+\mu_2 s)^2 t}{ts-\rho^2 t^2}, & \text{if } (t,s) \in \overline{B} \end{cases} \tag{12}$$

$$= \begin{cases} \frac{(1+\mu_2 s)^2}{s} + \frac{((1+\mu_1 t)-\rho(1+\mu_2 s))^2}{t-\rho^2 s}, & \text{if } (t,s) \in \overline{A} \\ \frac{(1+\mu_1 t)^2}{t} + \frac{((1+\mu_2 s)-\rho(1+\mu_1 t))^2}{s-\rho^2 t}, & \text{if } (t,s) \in \overline{B}. \end{cases} \tag{13}$$

Denote further

$$g_L(s) := g_A(s,s) = g_B(s,s) = \frac{(1+\mu_1 s)^2 + (1+\mu_2 s)^2 - 2\rho(1+\mu_1 s)(1+\mu_2 s)}{(1-\rho^2)s}, \quad s > 0. \tag{14}$$

The following result gives a full analysis of the two-layer minimization problem (3), which is crucial for our derivation of the exact asymptotics of $P(u)$. We refer to [8] for its detailed proof.

Proposition 3.1. (i). *Suppose that* $-1 < \rho < 0$.

For $\mu_1 < \mu_2$ *we have*

$$g(\boldsymbol{t}_0) = g_A(t_A, s_A) = 4(\mu_2 + (1-2\rho)\mu_1),$$

where, $(t_A, s_A) = (t_A(\rho), s_A(\rho)) := \left(\frac{1-2\rho}{\mu_1}, \frac{1}{\mu_2 - 2\mu_1\rho}\right) \in A$ *is the unique minimizer of* $g(t,s), (t,s) \in (0,\infty)^2$.

For $\mu_1 = \mu_2 =: \mu$ *we have*

$$g(\boldsymbol{t}_0) = g_A(t_A, s_A) = g_B(t_B, s_B) = 8(1-\rho)\mu,$$

where $(t_A, s_A) = \left(\frac{1-2\rho}{\mu}, \frac{1}{(1-2\rho)\mu}\right), (t_B, s_B) := \left(\frac{1}{(1-2\rho)\mu}, \frac{1-2\rho}{\mu}\right) \in B$ *are the only two minimizers of* $g(t,s), (t,s) \in (0,\infty)^2$.

(ii). *Suppose that* $0 \le \rho < \hat{\rho}_1$. *We have*

$$g(\boldsymbol{t}_0) = g_A(t_A, s_A) = 4(\mu_2 + (1-2\rho)\mu_1),$$

where (t_A, s_A) *is the unique minimizer of* $g(t,s), (t,s) \in (0,\infty)^2$.

(iii). *Suppose that* $\rho = \hat{\rho}_1$. *We have*

$$g(\boldsymbol{t}_0) = g_A(t_A, s_A) = 4(\mu_2 + (1-2\rho)\mu_1),$$

where $(t_A, s_A) = (t_A(\hat{\rho}_1), s_A(\hat{\rho}_1)) = (t^*(\hat{\rho}_1), s^*(\hat{\rho}_1)) \in L$, *is the unique minimizer of* $g(t,s), (t,s) \in (0,\infty)^2$, *with* $(t^*, s^*) = (t^*(\hat{\rho}_1), s^*(\hat{\rho}_1))$ *defined in* (8).

(iv). *Suppose that* $\hat{\rho}_1 < \rho < \hat{\rho}_2$. *We have*

$$g(\boldsymbol{t}_0) = g_A(t^*, s^*) = g_L(t^*) = \frac{2}{1+\rho}(\mu_1 + \mu_2 + 2/t^*),$$

where $(t^*, s^*) = (t^*(\rho), s^*(\rho)) \in L$ *is the unique minimizer of* $g(t,s), (t,s) \in (0,\infty)^2$.

(v). *Suppose that* $\rho = \hat\rho_2$. *We have* $t^*(\hat\rho_2) = s^*(\hat\rho_2) = 1/\mu_2$, *and*

$$g(\boldsymbol{t}_0) = g_A(1/\mu_2, 1/\mu_2) = g_L(1/\mu_2) = g_2(1/\mu_2) = 4\mu_2,$$

where the minimum of $g(t,s), (t,s) \in (0,\infty)^2$ *is attained at* $(1/\mu_2, 1/\mu_2)$, *with* $g_3(1/\mu_2, 1/\mu_2) = g_2(1/\mu_2)$, *and* $1/\mu_2$ *is the unique minimizer of* $g_2(s), s \in (0,\infty)$.

(vi). *Suppose that* $\hat\rho_2 < \rho < 1$. *We have*

$$g(\boldsymbol{t}_0) = \inf_{(t,s)\in D_2} g_2(s) = g_2(1/\mu_2) = 4\mu_2,$$

where the minimum of $g(t,s), (t,s) \in (0,\infty)^2$ *is attained when* $g(t,s) = g_2(s)$.

4. Proof of Theorem 2.1

Note that, by a change of variables and the self-similarity of Brownian motion,

$$\begin{aligned} P(u) &= \mathbb{P}\left\{\exists_{t,s>0}\ (X_1(ut) > (1+\mu_1 t)u,\ X_2(us) > (1+\mu_2 s)u)\right\} \\ &= \mathbb{P}\left\{\exists_{(t,s)\in(0,\infty)^2}\ (X_1(t) > (1+\mu_1 t)\sqrt{u},\ X_2(s) > (1+\mu_2 s)\sqrt{u})\right\}, \end{aligned} \tag{15}$$

and recall the notation for the optimizer points (t_A, s_A) as introduced in Proposition 3.1.
The proof of Theorem 2.1 will be presented in the order of cases (i) $-1 < \rho < \hat\rho_1$, (ii) $\hat\rho_1 < \rho < \hat\rho_2$, (iii) $\rho = \hat\rho_1$, (iv) $\hat\rho_2 \le \rho < 1$ in the following subsections.
For each of these cases (particularly for cases (i)-(iii)), we employ a modification of the *double-sum technique*. The idea here is first to split the region $(0,\infty)^2$ into several subregions; see sections *Splitting on subregions* below. Then we can show that the main contributor to the exact asymptotics of $P(u)$ is the maxima on a small, appropriately chosen region which includes the optimizer point (t_A, s_A) or (t^*, s^*), since the contributions of the maxima on other regions are negligible; see sections *Upper bounds and estimates*. The derivation of the asymptotics for the contributing region follows by an application of the double-sum method, where we use that asymptotically the probability of interest behaves as a sum of tail probabilities of maxima over sets of even smaller size, which in the literature on extremes of Gaussian processes is referred to as the *Pickands' size*.

4.1. (i) Scenario $-1 < \rho < \hat\rho_1$.

4.1.1. *Splitting on subregions.* We first split the region $(0,\infty)^2$ into the following two parts:

$$U_1 := [t_A - \theta_0, t_A + \theta_0] \times [s_A - \theta_0, s_A + \theta_0] \subset A. \quad U_2 := (0,\infty)^2 \setminus U_1,$$

where $\theta_0 > 0$ is some small constant which can be identified later on. It follows from (15) that

$$\begin{aligned} & P_0(u) := \mathbb{P}\left\{\exists_{(t,s)\in U_1}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ & \le P(u) \le \mathbb{P}\left\{\exists_{(t,s)\in U_1}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ & \qquad +\mathbb{P}\left\{\exists_{(t,s)\in U_2}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ & =: P_0(u) + r_0(u) \end{aligned} \tag{16}$$

Furthermore, we have, for all large u,

$$p(u) \le P_0(u) \le p(u) + r_1(u), \tag{17}$$

where

$$p(u) := \mathbb{P}\left\{\exists_{(t,s)\in\triangle_u^{(1)}\times\triangle_u^{(2)}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},$$
$$r_1(u) := \mathbb{P}\left\{\exists_{(t,s)\in U_1\setminus\triangle_u^{(1)}\times\triangle_u^{(2)}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},$$

with

$$\triangle_u^{(1)} = \left[t_A - \frac{\ln(u)}{\sqrt{u}}, t_A + \frac{\ln(u)}{\sqrt{u}}\right],\ \triangle_u^{(2)} = \left[s_A - \frac{\ln(u)}{\sqrt{u}}, s_A + \frac{\ln(u)}{\sqrt{u}}\right].$$

Next, we further split the rectangle $\triangle_u^{(1)} \times \triangle_u^{(2)}$ into smaller rectangles. To this end, we denote, for any fixed $T, S > 0$

$$\triangle_{j;u}^{(1)} = \triangle_{j;u}^{(1)}(T) = [t_A + jTu^{-1}, t_A + (j+1)Tu^{-1}], \quad -N_u^{(1)} \le j \le N_u^{(1)},$$
$$\triangle_{l;u}^{(2)} = \triangle_{l;u}^{(2)}(S) = [s_A + lSu^{-1}, s_A + (l+1)Su^{-1}], \quad -N_u^{(2)} \le l \le N_u^{(2)},$$

where $N_u^{(1)} = \lfloor T^{-1}\ln(u)\sqrt{u}\rfloor$, $N_u^{(2)} = \lfloor S^{-1}\ln(u)\sqrt{u}\rfloor$ (we denote by $\lfloor a\rfloor$ the smallest integer that is larger than a). Define

$$p_{j,l;u} = \mathbb{P}\left\{\exists_{(t,s)\in\triangle_{j;u}^{(1)}\times\triangle_{l;u}^{(2)}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}$$

and

$$p_{j,l_1,l_2;u} = \mathbb{P}\left\{\exists_{t\in\triangle_{j;u}^{(1)}} X_1(t) > \sqrt{u}(1+\mu_1 t), \exists_{s\in\triangle_{l_1;u}^{(2)}} X_2(s) > \sqrt{u}(1+\mu_2 s), \exists_{s\in\triangle_{l_2;u}^{(2)}} X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}$$
$$\overline{p}_{j_1,j_2,l;u} = \mathbb{P}\left\{\exists_{t\in\triangle_{j_1;u}^{(1)}} X_1(t) > \sqrt{u}(1+\mu_1 t), \exists_{t\in\triangle_{j_2;u}^{(1)}} X_1(t) > \sqrt{u}(1+\mu_1 t), \exists_{s\in\triangle_{l;u}^{(2)}} X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}.$$

We have from the generalized Bonferroni's inequality (see Lemma A.2 in Appendix A)

$$p_1(u) \ge p(u) \ge p_2(u) - \Pi_1(u) - \Pi_2(u), \tag{18}$$

where

$$p_1(u) = \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} \sum_{l=-N_u^{(2)}}^{N_u^{(2)}} p_{j,l;u}, \quad p_2(u) = \sum_{j=-N_u^{(1)}+1}^{N_u^{(1)}-1} \sum_{l=-N_u^{(2)}+1}^{N_u^{(2)}-1} p_{j,l;u},$$
$$\Pi_1(u) = \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} \sum_{-N_u^{(2)}\le l_1<l_2\le N_u^{(2)}} p_{j,l_1,l_2;u}, \quad \Pi_2(u) = \sum_{l=-N_u^{(2)}}^{N_u^{(2)}} \sum_{-N_u^{(1)}\le j_1<j_2\le N_u^{(1)}} \overline{p}_{j_1,j_2,l;u}.$$

4.1.2. *Upper bounds and estimates.* In what follows, we shall derive upper bounds for $r_0(u), r_1(u)$ in Lemma 4.1, the exact asymptotics of $p_1(u), p_2(u)$ in Lemma 4.2 and asymptotic behaviour for $\Pi_1(u), \Pi_2(u)$ in Lemma 4.3. The proofs of the lemmas are displayed in Appendix A. Recall that we assume $-1 < \rho < \hat{\rho}_1$.

Lemma 4.1. *For any chosen small $\theta_0 > 0$, we have, for all large u,*

$$r_0(u) \le e^{-\frac{(\sqrt{u}-C_0)^2}{2}\widehat{g}}, \quad \textit{with } \widehat{g} = \inf_{(t,s)\in U_2} g(t,s) > g_A(t_A, s_A),$$
$$r_1(u) \le C_1 u^{3/2} e^{-\frac{u}{2} g_A(t_A,s_A) - K_1(\ln(u))^2}$$

hold for some constants $C_0, C_1, K_1 > 0$ not depending on u.

Below we discuss the asymptotics of $p_1(u), p_2(u)$. Define

$$\mathcal{H}(\mu; T) := \int_{\mathbb{R}} e^{2\mu x_1} \mathbb{P}\left\{\exists_{t\in[0,T]}\ B_1(t) - \mu t > x_1\right\} dx_1.$$

Lemma 4.2. *We have, as $u \to \infty$,*

$$p_1(u) \sim p_2(u) \sim \frac{\mathcal{H}(\mu_1; T)\mathcal{H}(\mu_2 - 2\mu_1\rho; S)}{TS} \frac{1}{\mu_1(\mu_2 - 2\mu_1\rho)} e^{-\frac{g_A(t_A,s_A)}{2}u}.$$

The last lemma is concerned with the asymptotic behaviour of $\Pi_1(u), \Pi_2(u)$.

Lemma 4.3. *It holds that*

$$\limsup_{S\to\infty}\limsup_{T\to\infty}\lim_{u\to\infty} \frac{\Pi_1(u)}{\exp(-g_A(t_A,s_A)u/2)} = \limsup_{S\to\infty}\limsup_{T\to\infty}\lim_{u\to\infty} \frac{\Pi_2(u)}{\exp(-g_A(t_A,s_A)u/2)} = 0.$$

4.1.3. *Asymptotics of $P(u)$.* By Lemmas 4.1, 4.2, 4.3 applied to (16) - (18) we obtain that

$$P(u) \sim \lim_{S\to\infty}\lim_{T\to\infty} \frac{\mathcal{H}(\mu_1; T)\mathcal{H}(\mu_2 - 2\mu_1\rho; S)}{TS} \frac{1}{\mu_1(\mu_2 - 2\mu_1\rho)} e^{-\frac{g_A(t_A,s_A)}{2}u} = e^{-\frac{g_A(t_A,s_A)}{2}u},$$

where we used that, for any $\mu > 0$

$$\mathcal{H}(\mu) := \lim_{T\to\infty} \frac{1}{T}\mathcal{H}(\mu; T) = \mu, \tag{19}$$

see, e.g., [3]. Hence, using that $g_A(t_A, s_A) = 4(\mu_2 + (1 - 2\rho)\mu_1)$ (see (i)-(ii) of Proposition 3.1) we conclude the proof for scenario $-1 < \rho < \hat{\rho}_1$ in Theorem 2.1. □

4.2. **(ii) Scenario $\hat{\rho}_1 < \rho < \hat{\rho}_2$.**

4.2.1. *Splitting on subregions.* We split the region $(0, \infty)^2$ into five pieces as shown in Figure 1 (left). Namely, with some small $\theta_0 > 0$ and u large, let

$$D_0 = \{(t,s) : t^* - \ln(u)/\sqrt{u} \le t \le t^* + \ln(u)/\sqrt{u},\ 0 \le s - t \le \ln(u)^2/u\} \cup$$
$$\{(t,s) : s^* - \ln(u)/\sqrt{u} \le s \le s^* + \ln(u)/\sqrt{u},\ 0 \le t - s \le \ln(u)^2/u\},$$
$$D_2 = \{(t,s) : t^* + \ln(u)/\sqrt{u} \le t \le t^* + \theta_0,\ s^* + \ln(u)/\sqrt{u} \le s \le s^* + \theta_0,\},$$
$$D_3 = \{(t,s) : s^* - \ln(u)/\sqrt{u} \le s \le s^* + \ln(u)/\sqrt{u},\ s + \ln(u)^2/u \le t \le t^* + \theta_0,\},$$
$$D_4 = \{(t,s) : t^* - \ln(u)/\sqrt{u} \le t \le t^* + \ln(u)/\sqrt{u},\ t + \ln(u)^2/u \le s \le s^* + \theta_0,\},$$
$$D_1 = [t^* - \theta_0, t^* + \theta_0] \times [s^* - \theta_0, s^* + \theta_0] \setminus (D_0 \cup D_2 \cup D_3 \cup D_4),$$

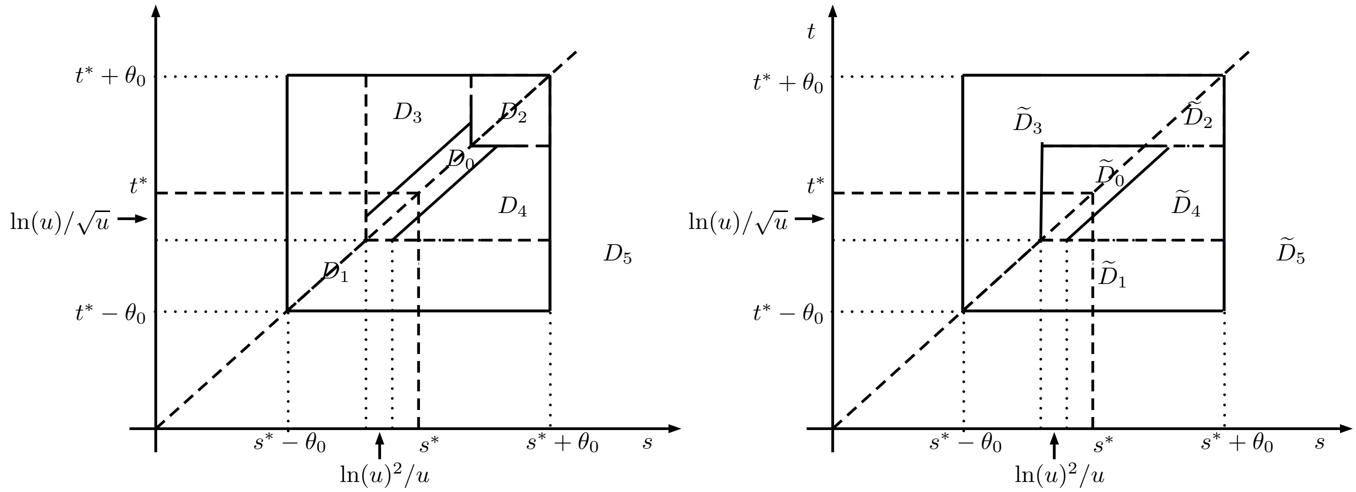

FIGURE 1. Partition of $(0,\infty)^2$: Left for $\hat{\rho}_1 < \rho < \hat{\rho}_2$; right for $\rho = \hat{\rho}_1$

$$D_5 = (0,\infty)^2 \setminus [t^*-\theta_0, t^*+\theta_0] \times [s^*-\theta_0, s^*+\theta_0].$$

Clearly, we have the following bounds

$$p(u) \le P(u) \le p(u) + r_1(u) + r_2(u) + r_3(u), \tag{20}$$

where

$$p(u) := \mathbb{P}\left\{\exists_{(t,s)\in D_0}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},$$

$$r_1(u) := \mathbb{P}\left\{\exists_{(t,s)\in D_5}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},$$

$$r_2(u) := \mathbb{P}\left\{\exists_{(t,s)\in D_1\cup D_2}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},$$

$$r_3(u) := \mathbb{P}\left\{\exists_{(t,s)\in D_3\cup D_4}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}.$$

Next, we consider a further partition of D_0. Recall $\triangle_{T,S}$ given in (9). Denote, for any $T, S > 0$ and $u > 0$,

$$\triangle^{(1)}_{j;u} = \triangle^{(1)}_{j;u}(T) = [t^* + jTu^{-1}, t^* + (j+1)Tu^{-1}], \quad -N^{(1)}_u \le j \le N^{(1)}_u,$$

$$\triangle^{(2)}_{l;u} = \triangle^{(2)}_{l;u}(S) = [lSu^{-1}, (l+1)Su^{-1}], \quad 1 \le l \le N^{(2)}_u,$$

where $N^{(1)}_u = \lfloor T^{-1}\ln(u)\sqrt{u}\rfloor$, $N^{(2)}_u = \lfloor S^{-1}\ln(u)^2\rfloor$. Define further

$$\begin{aligned}
p_{j;u} &:= \mathbb{P}\left\{\exists_{(t,s)\in(t^*+\frac{jT}{u}, s^*+\frac{jT}{u})+u^{-1}\triangle_{T,S}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},\\
p_{j,l;u} &:= \mathbb{P}\left\{\exists_{t\in\triangle^{(1)}_{j;u}, s-t\in\triangle^{(2)}_{l;u}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},\\
\overline{p}_{j,l;u} &:= \mathbb{P}\left\{\exists_{s\in\triangle^{(1)}_{j;u}, t-s\in\triangle^{(2)}_{l;u}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\},
\end{aligned}$$

and

$$q_{j_1,j_2;u} = \mathbb{P}\left\{\begin{array}{l}\exists_{(t,s)\in(t^*+\frac{j_1T}{u}, s^*+\frac{j_1T}{u})+u^{-1}\triangle_{T,S}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\\ \exists_{(t,s)\in(t^*+\frac{j_2T}{u}, s^*+\frac{j_2T}{u})+u^{-1}\triangle_{T,S}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\end{array}\right\}.$$

Thus, it follows from the Bonferroni's inequality that

$$\overline{\Pi}_1(u) + \Pi_1(u) + p_1(u) \ge p(u) \ge p_2(u) - \Pi_{21}(u) - \Pi_{22}(u), \tag{21}$$

where

$$p_1(u) = \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} p_{j;u}, \quad p_2(u) = \sum_{j=-N_u^{(1)}+1}^{N_u^{(1)}-1} p_{j;u}, \quad \overline{\Pi}_1(u) = \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} \sum_{1\le l\le N_u^{(2)}} \overline{p}_{j,l;u},$$

$$\Pi_1(u) = \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} \sum_{1\le l\le N_u^{(2)}} p_{j,l;u}, \quad \Pi_{21}(u) := \sum_{j_1=-N_u^{(1)}}^{N_u^{(1)}} \sum_{j_2>j_1+1} q_{j_1,j_2;u}, \quad \Pi_{22}(u) := \sum_{j_1=-N_u^{(1)}}^{N_u^{(1)}} q_{j_1,j_1+1;u}.$$

4.2.2. *Upper bounds and estimates.* In what follows, we shall derive upper bounds for $r_i(u), i = 1, 2, 3$ in Lemma 4.4, the exact asymptotics of $p_1(u), p_2(u)$ in Lemma 4.5 and asymptotic behaviour for $\overline{\Pi}_1(u), \Pi_1(u), \Pi_{21}(u), \Pi_{22}(u)$ in Lemma 4.7. The proofs of the lemmas are displayed in Appendix A.

Lemma 4.4. *For any chosen small $\theta_0 > 0$, we have, for all large u,*

$$r_1(u) \le e^{-\frac{(\sqrt{u}-C_1)^2}{2}\widehat{g}}, \quad \widehat{g} = \inf_{(t,s)\in D_5} g(t,s) > g_L(t^*),$$

$$r_2(u) \le C_2 u^{3/2} e^{-\frac{u}{2} g_L(t^*) - K_2(\ln(u))^2},$$

$$r_3(u) \le C_3 u^{3/2} e^{-\frac{u}{2} g_L(t^*) - K_3(\ln(u))^2}$$

hold for some constants $C_1, C_2, C_3, K_2, K_3 > 0$ not depending on u.

Lemma 4.5. *For any $T, S > 0$, we have, as $u \to \infty$,*

$$p_1(u) \sim p_2(u) \sim \frac{\mathcal{H}(T,S)}{T} \frac{\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}}\ u^{-1/2}\ e^{-\frac{u g_L(t^*)}{2}}.$$

Below, we show, for any fixed $S > 0$, the sub-additivity property of $\mathcal{H}(T,S)$ as a function of $T > 0$.

Lemma 4.6. *Let $S > 0$ be fixed, we have for any $T_1, T_2 > 0$*

$$\mathcal{H}(T_1 + T_2, S) \le \mathcal{H}(T_1, S) + \mathcal{H}(T_2, S)$$

and further,

$$0 < \frac{t^* \boldsymbol{\mu}^\top \Sigma_*^{-1} \boldsymbol{b}_*}{16 \prod_{i=1}^2 (\Sigma_*^{-1} \boldsymbol{b}_*)_i} < \lim_{T\to\infty} \frac{1}{T}\mathcal{H}(T,S) = \inf_{T>0} \frac{1}{T}\mathcal{H}(T,S) < \infty.$$

The last lemma gives some asymptotic results for $\overline{\Pi}_1(u), \Pi_1(u), \Pi_{21}(u), \Pi_{22}(u)$.

Lemma 4.7. *For any $T > S > 1$,*

$$\lim_{u\to\infty} \frac{\max(\Pi_1(u), \overline{\Pi}_1(u))}{u^{-1/2}\exp(-g_L(t^*)u/2)} \le C_0 \lfloor S \rfloor \sum_{l\ge 1} e^{-K_0 l S},$$

$$\lim_{u\to\infty} \frac{\Pi_{21}(u)}{u^{-1/2}\exp(-g_L(t^*)u/2)} \le C_1(S) \lfloor T \rfloor \sum_{l\ge 1} e^{-K_1 l T},$$

and

$$\lim_{u\to\infty} \frac{\Pi_{22}(u)}{u^{-1/2}\exp(-g_L(t^*)u/2)} = \frac{\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}} \left(\frac{2\mathcal{H}(T,S)}{T} - \frac{\mathcal{H}(2T,S)}{T} \right),$$

where $C_0, K_0, K_1 > 0$ are three constants which do not dependent on T, S, u, and $C_1(S)$ does not dependent on T, u.

4.2.3. *Asymptotics of $P(u)$.* Combining (20)-(21) and the results in Lemmas 4.4, 4.5 and 4.7, yields that, for any large T_1, T_2, S_1, S_2 such that $S_i < T_i, i = 1, 2$,

$$\begin{aligned} &\frac{\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}} \frac{\mathcal{H}(T_1,S_1)}{T_1} + 2C_0\lfloor S_1\rfloor \sum_{l\ge 1} e^{-K_0 l S_1} \\ &\ge \limsup_{u\to\infty} \frac{P(u)}{u^{-1/2}\exp(-g_L(t^*)u/2)} \ge \liminf_{u\to\infty} \frac{P(u)}{u^{-1/2}\exp(-g_L(t^*)u/2)} \\ &\ge \frac{\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}} \frac{\mathcal{H}(T_2,S_2)}{T_2} - C_1(S_2)\lfloor T_2\rfloor \sum_{k\ge 1} e^{-K_1 k T_2} - \frac{\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}} \left(\frac{2\mathcal{H}(T_2,S_2)}{T_2} - \frac{\mathcal{H}(2T_2,S_2)}{T_2} \right). \end{aligned}$$

Letting first $T_2 \to \infty$ and then $S_2 \to \infty$, we have from the above formula, (19) and Lemma 4.6 that

$$\lim_{S\to\infty} \lim_{T\to\infty} \frac{\mathcal{H}(T,S)}{T} \in (0,\infty).$$

The proof for scenario $\hat\rho_1 < \rho < \hat\rho_2$ in Theorem 2.1 follows then by letting $T_1 \to \infty$ and then $S_1 \to \infty$, and (iv) of Proposition 3.1. □

4.3. **(iii) Scenario $\rho = \hat\rho_1$.** Since the idea of the proof of this case is similar to that of scenarios (i) and (ii), we present only main steps. We split the region $(0,\infty)^2$ into five pieces as shown in Figure 1 (right). Namely, with some small $\theta_0 > 0$ and u large, let

$$\begin{aligned} \widetilde{D}_0 &= \{(t,s) : t^* - \ln(u)/\sqrt{u} \le t \le t^* + \ln(u)/\sqrt{u},\ 0 \le s-t \le \ln(u)^2/u\} \cup \\ &\qquad \{(t,s) : s^* - \ln(u)/\sqrt{u} \le s \le s^* + \ln(u)/\sqrt{u},\ s < t \le t^* + \ln(u)/\sqrt{u}\} =: \widetilde{D}_{0B} \cup \widetilde{D}_{0A}, \\ \widetilde{D}_3 &= \{(t,s) : s^* - \theta_0 \le s \le s^* + \theta_0,\ s < t \le t^* + \theta_0\} \setminus \widetilde{D}_{0A}, \\ \widetilde{D}_1 &= D_1 \cap \overline{B}, \quad \widetilde{D}_2 = D_2 \cap \overline{B}, \quad \widetilde{D}_4 = D_4, \quad \widetilde{D}_5 = D_5. \end{aligned}$$

Clearly, we have the following bounds

$$p(u) \le P(u) \le p(u) + \tilde r_0(u) + \tilde r_1(u) + \tilde r_2(u), \tag{22}$$

where

$$\begin{aligned} p(u) &:= \mathbb{P}\left\{\exists_{(t,s)\in\widetilde{D}_{0A}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}, \\ \tilde r_0(u) &:= \mathbb{P}\left\{\exists_{(t,s)\in\widetilde{D}_{0B}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}, \\ \tilde r_1(u) &:= \mathbb{P}\left\{\exists_{(t,s)\in\widetilde{D}_1\cup\widetilde{D}_2\cup\widetilde{D}_4\cup\widetilde{D}_5}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}, \\ \tilde r_2(u) &:= \mathbb{P}\left\{\exists_{(t,s)\in\widetilde{D}_3}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}. \end{aligned}$$

Similar arguments as used in scenarios (i), (ii) give that

$$\lim_{u\to\infty} \frac{p(u)}{\exp\left(-g_A(t_A,s_A)u/2\right)} = \frac{\mu_1^{3/2}(\mu_2-2\mu_1\rho)^2}{2\pi\sqrt{\mu_2-2(\mu_1+\mu_2)\rho+3\mu_1\rho^2}} \int_{\mathbb{R}}\int_{x_2}^{\infty} e^{\frac{-(a_1x_1^2-2a_2x_1x_2+a_3x_2^2)}{4}}\, dx_1 dx_2, \tag{23}$$

and

$$\lim_{u\to\infty} \frac{\tilde r_0(u)}{u^{-1/2}\exp\left(-g_A(t_A,s_A)u/2\right)} \le \frac{\widetilde{\mathcal{H}}\sqrt{t^*}}{2\sqrt{\pi(1-\rho)}}, \tag{24}$$

and the asymptotically negligibility of $\tilde{r}_1(u), \tilde{r}_2(u)$. Note that in proving the bound for $\tilde{r}_2(u)$, in addition to (33) as in the proof of Lemma 4.1, we also need the fact that (for $t > s$)

$$\begin{aligned} g_A(t_A+t, s_A+s) \quad &\geq \quad g_A(t_A, s_A) + \frac{a_1}{2}(1-\varepsilon)\Bigg(\left(t + \frac{\mu_2 - 2\mu_1\rho}{\mu_1}\rho s\right)^2 \\ &\quad + \left(\frac{(\mu_2 - 2\mu_1\rho)^3(\mu_2 - 2(\mu_1+\mu_2)\rho + 3\mu_1\rho^2)}{\mu_1^3} \right) s^2 \Bigg). \end{aligned}$$

Consequently, the claim follows by formulas (22)-(24) and the asymptotically negligibility of $\tilde{r}_1(u), \tilde{r}_2(u)$. This completes the proof of scenario $\rho = \hat{\rho}_1$ in Theorem 2.1. □

4.4. **(iv) Scenario $\hat{\rho}_2 \leq \rho < 1$.** First note that

$$e^{-2\mu_2 u} = \mathbb{P}\left\{\sup_{s\geq 0}(X_2(s) - \mu_2 s) > u\right\} \geq P(u) \geq \mathbb{P}\left\{\exists_{t\geq 0}\ X_1(t) - \mu_1 t > u,\ X_2(t) - \mu_2 t > u\right\} =: \pi(u).$$

Furthermore, the exact asymptotics for $\pi(u)$ has been discussed in Corollary 4.3 in [14] (where we take $r = 0$). Thus, we have, for $\rho = \hat{\rho}_2$,

$$\pi(u) \sim \frac{1}{2}e^{-2\mu_2 u}, \quad u \to \infty,$$

and for $\hat{\rho}_2 < \rho < 1$,

$$\pi(u) \sim e^{-2\mu_2 u}, \quad u \to \infty.$$

Therefore, the claims in scenario $\hat{\rho}_2 < \rho < 1$ of Theorem 2.1 and $\hat{\rho}_2 = \rho$ in (a) of Remark 2.2 follow. □

5. Proof of Theorem 2.3

For $\mu_1 = \mu_2 = \mu$, we have that $\hat{\rho}_1 = 0, \hat{\rho}_2 = 1$. The case $\rho = 0$ follows from (4). Thus the interesting scenarios include (i) $-1 < \rho < 0$ and (ii) $0 < \rho < 1$. The claim for (ii) $0 < \rho < 1$ follows directly from (iii) in Theorem 2.1, with $t^* = 1/\mu$. Next, we shall focus on the proof for (i) $-1 < \rho < 0$. The proof goes with the same arguments as in the proof of scenario (i) in Theorem 2.1, but now there are two minimizers of the function $g(t,s), (t,s) \in (0,\infty)^2$, namely, $(t_A, s_A) = (t_0, s_0) \in A, (t_B, s_B) = (s_0, t_0) \in B$, with $t_0 = \frac{1-2\rho}{\mu}, s_0 = \frac{1}{(1-2\rho)\mu}$.
We first split the region $(0,\infty)^2$ into three parts. Namely, with some small $\theta_0 > 0$, let

$$U_{11} = [t_0 - \theta_0, t_0 + \theta_0] \times [s_0 - \theta_0, s_0 + \theta_0] \subset A,$$

$$U_{12} = [s_0 - \theta_0, s_0 + \theta_0] \times [t_0 - \theta_0, t_0 + \theta_0] \subset B,$$

$$U_2 = (0,\infty)^2 \setminus (U_{11} \cup U_{12}).$$

As in the proof of scenario (i) of Theorem 2.1, the main contribution of the asymptotics comes from $U_{11} \cup U_{12}$. Note further that

$$\begin{aligned} &\mathbb{P}\left\{\exists_{(t,s)\in(U_{11}\cup U_{12})}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ &\quad = \quad \mathbb{P}\left\{\exists_{(t,s)\in U_{11}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ &\qquad + \mathbb{P}\left\{\exists_{(t,s)\in U_{12}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \end{aligned}$$

$$
-\mathbb{P}\left\{\begin{array}{l} \exists_{(t_1,s_1)\in U_{11}}\ X_1(t_1) > \sqrt{u}(1+\mu_1 t_1), X_2(s_1) > \sqrt{u}(1+\mu_2 s_1) \\ \exists_{(t_2,s_2)\in U_{12}}\ X_1(t_2) > \sqrt{u}(1+\mu_1 t_2), X_2(s_2) > \sqrt{u}(1+\mu_2 s_2) \end{array}\right\}
$$
$$
=: \quad P_{\theta_0,1}(u) + P_{\theta_0,2}(u) - P_{\theta_0,0}(u).
$$

By symmetric property of the model we know that $P_{\theta_0,1}(u) = P_{\theta_0,2}(u)$. Next, we show in Lemma 5.1 that $P_{\theta_0,0}(u)$ is asymptotically negligible compared with $P_{\theta_0,1}(u)$. The proof of it is displayed in Appendix A.

Lemma 5.1. *For any chosen small $\theta_0 > 0$, we have for all large u*

$$
P_{\theta_0,0}(u) \le e^{-\frac{(\sqrt{u}-C_0)^2 g_A(t_A,s_A)}{2\sigma_0^2}}
$$

holds for some constant $C_0 > 0, \sigma_0^2 \in (0,1)$ which do not depend on u.

The rest of the proof is the same as those in the proof of scenario (i) in Theorem 2.1, and thus omitted. This completes the proof. □

Appendix A. Proofs of Lemmas 4.1-5.1

In this section we give proofs of Lemmas 4.1-5.1 that are the building blocks of the proofs of Theorems 2.1 and 2.3. We begin with the analysis of the local behaviour of function $g(t,s),(t,s) \in (0,\infty)$ at its minimizer in scenarios (i)–(iv) of Proposition 3.1, respectivelly.

Lemma A.1. *Assume that $\mu_1 < \mu_2$. We have*

(i). *If $-1 < \rho < \hat{\rho}_1$, then as $(t,s) \to (0,0)$,*

$$
g(t_A + t, s_A + s) = g_A(t_A, s_A) + \frac{a_1}{2} t^2(1+o(1)) - a_2 ts(1+o(1)) + \frac{a_3}{2} s^2(1+o(1)),
$$

where, with $h(\rho) := \mu_2 - 2(\mu_1+\mu_2)\rho + 3\mu_1\rho^2 > 0$,

$$
a_1 := \frac{2\mu_1^3(\mu_2 - 2\mu_1\rho)}{h(\rho)} > 0, \quad a_2 := \frac{-2\rho\mu_1^2(\mu_2 - 2\mu_1\rho)^2}{h(\rho)}, \quad a_3 := \frac{2(\mu_2 - 2\mu_1\rho)^4(1-2\rho)}{h(\rho)} > 0.
$$

(ii). *If $\hat{\rho}_1 < \rho < \hat{\rho}_2$, then*

– *(ii.1), as $(t,s) \to (0,0)$, with $s < t$ (i.e., $(t^*+t, s^*+s) \in A$),*

$$
g(t^*+t, s^*+s) = g_L(t^*) + b_1(t-s)(1+o(1)) + \frac{c_1}{2} s^2(1+o(1)),
$$

where

$$
b_1 := \frac{(\rho - 1 - 2\rho^2) + 2\rho(\mu_2 - \mu_1\rho)s^* + (1+\rho)\mu_1^2 s^{*2}}{(1-\rho)(1+\rho)^2 s^{*2}} > 0,
$$
$$
c_1 := \frac{2}{s^{*3}}\left(1 + \frac{\rho^2(\rho(1-\rho) - (\mu_2 - \mu_1\rho)s^*)^2}{(1-\rho^2)^3}\right) > 0.
$$

– *(ii.2), as $(t,s) \to (0,0)$, with $s > t$ (i.e., $(t^*+t, s^*+s) \in B$),*

$$
g(t^*+t, s^*+s) = g_L(t^*) + b_2(s-t)(1+o(1)) + \frac{c_2}{2} t^2(1+o(1)),
$$

where

$$
b_2 := \frac{(\rho - 1 - 2\rho^2) + 2\rho(\mu_1 - \mu_2\rho)t^* + (1+\rho)\mu_2^2 t^{*2}}{(1-\rho)(1+\rho)^2 t^{*2}} > 0,
$$

$$c_2 \quad := \quad \frac{2}{t^{*3}}\left(1+\frac{\rho^2(\rho(1-\rho)-(\mu_1-\mu_2\rho)t^*)^2}{(1-\rho^2)^3}\right)>0.$$

– *(ii.3), as $(t,s)\to(0,0)$, with $s=t$ (i.e., $(t^*+t,s^*+s)\in L$),*

$$g(t^*+t,s^*+t)=g_L(t^*)+\frac{b_0}{2}t^2(1+o(1)),$$

where $b_0:=\frac{4}{(1+\rho)t^{*3}}$.

(iii). *If $\rho=\hat\rho_1$ (in this case $t_A=s_A=t^*=s^*$), then*

– *(iii.1), as $(t,s)\to(0,0)$, with $s<t$,*

$$g(t_A+t,s_A+s)=g_A(t_A,s_A)+\frac{a_1}{2}t^2(1+o(1))-a_2ts(1+o(1))+\frac{a_3}{2}s^2(1+o(1)),$$

– *(iii.2), as $(t,s)\to(0,0)$, with $s>t$,*

$$g(t^*+t,s^*+s)=g_L(t^*)+b_2(s-t)(1+o(1))+\frac{c_2}{2}t^2(1+o(1)).$$

– *(iii.3), as $(t,s)\to(0,0)$, with $s=t$,*

$$g(t^*+t,s^*+t)=g_L(t^*)+\frac{b_0}{2}t^2(1+o(1)).$$

The proof of Lemma A.1 is tedious but only involves basic calculations using Taylor expansion, and thus it is omitted.

Next we present below a generalized version of the Bonferroni's inequality. The proof can be found in, e.g., [11].

Lemma A.2. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $A_1,\cdots,A_n$ and $B_1,\cdots,B_m$ be $n+m$ events in $\mathcal{F}$ with $n,m\ge 2$. Then*

$$\sum_{k=1}^n\sum_{l=1}^m\mathbb{P}\left\{A_k\cap B_l\right\}\ge\mathbb{P}\left\{\bigcup_{\substack{k=1,\dots,n\\ l=1,\dots,m}}(A_k\cap B_l)\right\}\ge\sum_{k=1}^n\sum_{l=1}^m\mathbb{P}\left\{A_k\cap B_l\right\}$$
$$-\sum_{k=1}^n\sum_{1\le l_1<l_2\le m}\mathbb{P}\left\{A_k\cap B_{l_1}\cap B_{l_2}\right\}-\sum_{l=1}^m\sum_{1\le k_1<k_2\le n}\mathbb{P}\left\{A_{k_1}\cap A_{k_2}\cap B_l\right\}.$$

A.1. **Proof of Lemma 4.1.** Let $T_0>0$ be a fixed large constant (will be determined later). It is easily seen that

$$r_0(u)\quad\le\quad\mathbb{P}\left\{\exists_{(t,s)\in[0,T_0]^2\setminus U_1}\ X_1(t)>\sqrt{u}(1+\mu_1t),X_2(s)>\sqrt{u}(1+\mu_2s)\right\}$$
$$+\mathbb{P}\left\{\exists_{t\ge T_0}\ X_1(t)>\sqrt{u}(1+\mu_1t)\right\}+\mathbb{P}\left\{\exists_{s\ge T_0}\ X_2(s)>\sqrt{u}(1+\mu_2s)\right\}.$$

Next we consider upper bounds for each term on the right-hand side. According to Lemma 5 of [8], for any fixed t,s, there exists a unique index set

$$I(t,s)\subseteq\{1,2\}$$

such that

$$g(t,s)=(1+\mu_1t,1+\mu_2s)_{I(t,s)}\ (\Sigma_{ts})^{-1}_{I(t,s),I(t,s)}\ (1+\mu_1t,1+\mu_2s)^\top_{I(t,s)}, \tag{25}$$

and

$$(\Sigma_{ts})^{-1}_{I(t,s),I(t,s)}\ (1+\mu_1t,1+\mu_2s)^\top_{I(t,s)}>\mathbf{0}_{I(t,s)}. \tag{26}$$

In the above, we use notation that if $I \subset \{1,2\}$, then for a vector $\boldsymbol{a} \in \mathbb{R}^2$ we denote by $\boldsymbol{a}_I = (a_i, i \in I)$ a sub-block vector of $\boldsymbol{a}$. Similarly, if further $J \subset \{1,2\}$, for a matrix $M = (m_{ij})_{i,j\in\{1,2\}} \in \mathbb{R}^{2\times 2}$ we denote by $M_{I,J} = (m_{ij})_{i\in I, j\in J}$ the sub-block matrix of M determined by I and J. Furthermore, we write $M_{II}^{-1} = (M_{II})^{-1}$ for the inverse matrix of M_{II} whenever it exists.

Thus,

$$
\begin{aligned}
&\mathbb{P}\left\{\exists_{(t,s)\in[0,T_0]^2\setminus U_1}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}\\
&\le \mathbb{P}\left\{\exists_{(t,s)\in[0,T_0]^2\setminus U_1}\ (1+\mu_1 t, 1+\mu_2 s)_{I(t,s)}\ (\Sigma_{ts})^{-1}_{I(t,s),I(t,s)}\ (X_1(t), X_2(s))^\top_{I(t,s)} > \sqrt{u} g(t,s)\right\}\\
&= \mathbb{P}\left\{\exists_{(t,s)\in[0,T_0]^2\setminus U_1}\ \frac{Z(t,s)}{g(t,s)} > \sqrt{u}\right\},
\end{aligned} \tag{27}
$$

where

$$
Z(t,s) := (1+\mu_1 t, 1+\mu_2 s)_{I(t,s)}\ (\Sigma_{ts})^{-1}_{I(t,s),I(t,s)}\ (X_1(t), X_2(s))^\top_{I(t,s)}. \tag{28}
$$

Note that

$$
\mathrm{Var}\left(\frac{Z(t,s)}{g(t,s)}\right) = \frac{1}{g(t,s)}. \tag{29}
$$

In order to apply the Borell-TIS inequality, we first show that

$$
\limsup_{(t,s)\to(t^{(b)},s^{(b)})} \frac{|Z(t,s)|}{g(t,s)} < \infty, \qquad \text{almost surely}
$$

holds for any $(t^{(b)}, s^{(b)})$ on the boundary $\{(t,s) : t \ge 0, s = 0\} \cup \{(t,s) : t = 0, s \ge 0\}$.

In fact, if the above does not hold for some boundary point $(t^{(b)}, s^{(b)})$, then for any $M > 0$ there exist a sequence $\{(t_k, s_k)\}_{k=1}^\infty$ and some measurable set E such that $(t_k, s_k) \to (t^{(b)}, s^{(b)})$, $\mathbb{P}\{E\} > 0$ and

$$
\frac{|Z(t_k,s_k)|}{g(t_k,s_k)} \ge M \quad \text{on } E
$$

for all large enough k. Then we have

$$
\mathrm{Var}\left(\frac{Z(t_k,s_k)}{g(t_k,s_k)}\right) \ge M^2 \mathbb{P}\{E\} > 0. \tag{30}
$$

On the other hand, by Lemma 6 of [8] we have $g(t,s) = g_3(t,s)$ for all $(t,s) \in \{(t,s) : t \ge 0, s = 0\} \cup \{(t,s) : t = 0, s \ge 0\}$, and thus by (29) and (13) we have $\lim_{k\to\infty} \mathrm{Var}\left(\frac{Z(t_k,s_k)}{g(t_k,s_k)}\right) = 0$. This is a contradiction with (30). Therefore, $\frac{Z(t,s)}{g(t,s)}, (t,s) \in [0,T_0]^2 \setminus U_1$ is almost surely bounded. Consequently, by the Borell-TIS inequality (see, e.g., [1]) we have, for any fixed small constant $\theta_0 > 0$

$$
\mathbb{P}\left\{\exists_{(t,s)\in[0,T_0]^2\setminus U_1}\ \frac{Z(t,s)}{g(t,s)} > \sqrt{u}\right\} \le e^{-\frac{(\sqrt{u}-C_0)^2}{2}\widehat{g}}
$$

holds for all u such that

$$
\sqrt{u} > C_0 := \mathbb{E}\left\{\sup_{(t,s)\in[0,T_0]^2\setminus U_1} \frac{Z(t,s)}{g(t,s)}\right\}.
$$

Moreover, since X_i is the standard Brownian motion,

$$
\lim_{t\to\infty} \frac{X_i(t)}{1+\mu_i t} = 0 \qquad \text{almost surely,}
$$

showing that the random process $\frac{X_i(t)}{1+\mu_i t}, t \ge T_0$ has almost surely bounded sample paths on $[T_0, \infty)$. Again by the Borell-TIS inequality

$$\mathbb{P}\left\{\exists_{t\ge T_0}\ X_i(t) > \sqrt{u}(1+\mu_i t)\right\} \le e^{-\frac{(\sqrt{u}-C_i)^2}{2}\frac{(1+\mu_i T_0)^2}{T_0}}$$

holds for all $\sqrt{u} > C_i := \mathbb{E}\left\{\sup_{t\in[T_0,\infty)} \frac{X_1(t)}{1+\mu_i t}\right\}$. Since for all large enough T_0 it holds that $\frac{(1+\mu_i T_0)^2}{T_0} > \widehat{g}$, the claim for $r_0(u)$ is established.

Below we consider $r_1(u)$. Since $(t_A, s_A) \in A$, we have from Proposition 3.1 that for any chosen small θ_0

$$g(t,s) = g_A(t,s), \quad (t,s) \in U_1 \subset A,$$

and further (cf. (28))

$$Z(t,s) = (1+\mu_1 t, 1+\mu_2 s)\ \Sigma_{ts}^{-1}\ (X_1(t), X_2(s))^\top =: h_1(t,s)X_1(t) + h_2(t,s)X_2(s), \quad (t,s) \in U_1,$$

with

$$h_1(t,s) = \frac{(1+\mu_1 t)s - \rho s(1+\mu_2 s))}{ts - \rho^2 s^2}, \quad h_2(t,s) = \frac{(1+\mu_2 s)t - \rho s(1+\mu_1 t))}{ts - \rho^2 s^2}.$$

Thus, similarly to (27) we conclude that

$$r_1(u) \le \mathbb{P}\left\{\exists_{(t,s)\in U_1 \setminus \triangle_u^{(1)} \times \triangle_u^{(2)}}\ \frac{Z(t,s)}{g_A(t,s)} > \sqrt{u}\right\}. \tag{31}$$

Since $h_1(t,s), h_2(t,s), g_A(t,s), (t,s) \in U_1$ are all smooth functions and

$$\mathbb{E}\left\{(X_i(t_1) - X_i(t_2))^2\right\} = |t_1 - t_2|, \quad i = 1,2$$

one can check that, for all $(t_1,s_1),(t_2,s_2) \in U_1$,

$$\mathbb{E}\left\{\left(\frac{Z(t_1,s_1)}{g_A(t_1,s_1)} - \frac{Z(t_2,s_2)}{g_A(t_2,s_2)}\right)^2\right\} \le \text{Const}\cdot(|t_1-t_2| + |s_1-s_2|).$$

Therefore, an application of the Piterbarg's inequality in [5][Lemma 5.1] (see also [20][Theorem 8.1] or [21][Theorem 3]) yields that

$$r_1(u) \le \mathbb{P}\left\{\exists_{(t,s)\in U_1 \setminus \triangle_u^{(1)} \times \triangle_u^{(2)}}\ \frac{Z(t,s)}{g_A(t,s)} > \sqrt{u}\right\} \le C_3 u^{3/2} e^{-\frac{u}{2}\widetilde{g}_u}, \tag{32}$$

where $C_3 > 0$ is some constant which does not depend on u and

$$\widetilde{g}_u := \inf_{(t,s)\in U_1 \setminus \triangle_u^{(1)} \times \triangle_u^{(2)}} g_A(t,s).$$

Moreover, we have from (i) of Lemma A.1 that for all $(t_A + t, s_A + s) \in U_1$

$$\begin{aligned} g_A(t_A+t, s_A+s) \quad \ge \quad & g_A(t_A,s_A) + \frac{a_1}{2}(1-\varepsilon)\Bigg((1-\rho^2)t^2 + \left(\rho t + \frac{\mu_2 - 2\mu_1\rho}{\mu_1}s\right)^2 \\ & + \left(\frac{\mu_2-2\mu_1\rho}{\mu_1}\right)^2\left(\frac{\mu_2-2\mu_1\rho}{\mu_1}(1-2\rho) - 1\right)s^2\Bigg) \end{aligned} \tag{33}$$

holds with some small $\varepsilon > 0$, where for all $-1 < \rho < \hat{\rho}_1$ (see also the proof of (b).(i) in Lemma 9 of [8] for $\rho > 0$)

$$\frac{\mu_2 - 2\mu_1\rho}{\mu_1}(1-2\rho) - 1 > 0.$$

Thus

$$\widetilde{g}_u \geq g_A(t_A, s_A) + \frac{a_1}{2}(1-\varepsilon)\min\left((1-\rho^2), \left(\frac{\mu_2 - 2\mu_1\rho}{\mu_1}\right)^2\left(\frac{\mu_2 - 2\mu_1\rho}{\mu_1}(1-2\rho)-1\right)\right)\frac{(\ln(u))^2}{u}.$$

Inserting the above to (32) completes the proof. □

A.2. **Proof of Lemma 4.2.** We first analyze the summand $p_{j,l;u}$. We set

$$\boldsymbol{b}_{j,l;u} = (a_{j;u}, b_{l;u})^\top, \quad a_{j;u} = 1 + \mu_1(t_A + \frac{jT}{u}), \quad b_{l;u} = 1 + \mu_2(s_A + \frac{lS}{u}). \tag{34}$$

It follows that

$$\begin{aligned} p_{j,l;u} &= \mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s\in[0,S]}} \begin{array}{c} X_1(t_A + \frac{jT}{u} + \frac{t}{u}) > a_{j;u}\sqrt{u} + \frac{\mu_1}{\sqrt{u}}t \\ X_2(s_A + \frac{lS}{u} + \frac{s}{u}) > b_{l;u}\sqrt{u} + \frac{\mu_2}{\sqrt{u}}s \end{array}\right\} \\ &= \mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s\in[0,S]}} \begin{array}{c} X_1(t_A + \frac{jT}{u}) + X_1(t_A + \frac{jT}{u} + \frac{t}{u}) - X_1(t_A + \frac{jT}{u}) > a_{j;u}\sqrt{u} + \frac{\mu_1}{\sqrt{u}}t \\ X_2(s_A + \frac{lS}{u}) + X_2(s_A + \frac{lS}{u} + \frac{s}{u}) - X_2(s_A + \frac{lS}{u}) > b_{l;u}\sqrt{u} + \frac{\mu_2}{\sqrt{u}}s \end{array}\right\}. \end{aligned} \tag{35}$$

Since $(t_A + \frac{jT}{u}, s_A + \frac{lS}{u}) \in A$ for all large u, the covariance matrix of $\boldsymbol{Z}_{j,l;u} := (X_1(t_A + \frac{jT}{u}), X_2(s_A + \frac{lS}{u}))^\top$ is given by

$$\Sigma_{j,l;u} = \begin{pmatrix} t_A + \frac{jT}{u} & \rho\ (s_A + \frac{lS}{u}) \\ \rho\ (s_A + \frac{lS}{u}) & s_A + \frac{lS}{u} \end{pmatrix}.$$

Thus, the density function of $\boldsymbol{Z}_{j,l;u}$ is given by

$$\phi_{\Sigma_{j,l;u}}(\boldsymbol{w}) = \frac{1}{\sqrt{(2\pi)^2\,|\Sigma_{j,l;u}|}}\exp\left(-\frac{1}{2}\boldsymbol{w}^\top(\Sigma_{j,l;u})^{-1}\boldsymbol{w}\right), \quad \boldsymbol{w} = (w_1, w_2)^\top.$$

By conditioning on the value of $\boldsymbol{Z}_{j,l;u}$ we rewrite (35) as

$$p_{j,l;u} = \int_{\mathbb{R}^2}\phi_{\Sigma_{j,l;u}}(\boldsymbol{w})\mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s\in[0,S]}} \begin{array}{c} X_1(t_A + \frac{jT}{u} + \frac{t}{u}) - X_1(t_A + \frac{jT}{u}) > a_{j;u}\sqrt{u} + \frac{\mu_1}{\sqrt{u}}t - w_1 \\ X_2(s_A + \frac{lS}{u} + \frac{s}{u}) - X_2(s_A + \frac{lS}{u}) > b_{l;u}\sqrt{u} + \frac{\mu_2}{\sqrt{u}}s - w_2 \end{array}\middle|\boldsymbol{Z}_{j,l;u} = \boldsymbol{w}\right\}d\boldsymbol{w},$$

Using change of variables $\boldsymbol{w} = \sqrt{u}\boldsymbol{b}_{j,l;u} - \boldsymbol{x}/\sqrt{u}$ we further obtain

$$p_{j,l;u} = u^{-1}\int_{\mathbb{R}^2}\phi_{\Sigma_{j,l;u}}(\sqrt{u}\boldsymbol{b}_{j,l;u} - \boldsymbol{x}/\sqrt{u})P_{j,l;u}(\boldsymbol{x})\,d\boldsymbol{x},$$

where

$$P_{j,l;u}(\boldsymbol{x}) := \mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s\in[0,S]}} \begin{array}{c} X_1(t_A + \frac{jT}{u} + \frac{t}{u}) - X_1(t_A + \frac{jT}{u}) > \frac{\mu_1}{\sqrt{u}}t + \frac{x_1}{\sqrt{u}} \\ X_2(s_A + \frac{lS}{u} + \frac{s}{u}) - X_2(s_A + \frac{lS}{u}) > \frac{\mu_2}{\sqrt{u}}s + \frac{x_2}{\sqrt{u}} \end{array}\middle|\boldsymbol{Z}_{j,l;u} = \sqrt{u}\boldsymbol{b}_{j,l;u} - \frac{\boldsymbol{x}}{\sqrt{u}}\right\}.$$

Now, we analyse $P_{j,l;u}(\boldsymbol{x})$. Due to the fact that $(t_A, s_A) \in A$, we have for all $t \in [0,T], s \in [0,S]$, and large enough u

$$t_A + \frac{jT}{u} + \frac{t}{u} \geq t_A + \frac{jT}{u} > s_A + \frac{lS}{u} + \frac{s}{u} \geq s_A + \frac{lS}{u}.$$

Thus, by the properties of Brownian motion

$$\begin{aligned} P_{j,l;u}(\boldsymbol{x}) &= \mathbb{P}\left\{\exists_{t\in[0,T]}\ X_1(t) - \mu_1 t > x_1\right\} \\ &\quad \times\mathbb{P}\left\{\exists_{s\in[0,S]}\ X_2(s_A + \frac{lS}{u} + \frac{s}{u}) - X_2(s_A + \frac{lS}{u}) > \frac{\mu_2}{\sqrt{u}}s + \frac{x_2}{\sqrt{u}}\middle|\boldsymbol{Z}_{j,l;u} = \sqrt{u}\boldsymbol{b}_{j,l;u} - \frac{\boldsymbol{x}}{\sqrt{u}}\right\}, \end{aligned}$$

Next we have

$$\phi_{\Sigma_{j,l;u}}(\sqrt{u}\boldsymbol{b}_{j,l;u}-\boldsymbol{x}/\sqrt{u})=\frac{1}{\sqrt{(2\pi)^2\,|\Sigma_{j,l;u}|}}\exp\left(-\frac{1}{2}(\sqrt{u}\boldsymbol{b}_{j,l;u}-\boldsymbol{x}/\sqrt{u})^\top(\Sigma_{j,l;u})^{-1}(\sqrt{u}\boldsymbol{b}_{j,l;u}-\boldsymbol{x}/\sqrt{u})\right),$$

where the exponent can be rewritten as

$$\begin{aligned}&(\sqrt{u}\boldsymbol{b}_{j,l;u}-\boldsymbol{x}/\sqrt{u})^\top(\Sigma_{j,l;u})^{-1}(\sqrt{u}\boldsymbol{b}_{j,l;u}-\boldsymbol{x}/\sqrt{u})\\&=u(\boldsymbol{b}_{j,l;u})^\top\Sigma_{j,l;u}^{-1}\boldsymbol{b}_{j,l;u}-2\boldsymbol{x}^\top\Sigma_{j,l;u}^{-1}\boldsymbol{b}_{j,l;u}+\frac{1}{u}\boldsymbol{x}^\top\Sigma_{j,l;u}^{-1}\boldsymbol{x}\\&=ug_A(t_A+\frac{jT}{u},s_A+\frac{lS}{u})-2\boldsymbol{x}^\top\Sigma_{j,l;u}^{-1}\boldsymbol{b}_{j,l;u}+\frac{1}{u}\boldsymbol{x}^\top\Sigma_{j,l;u}^{-1}\boldsymbol{x}.\end{aligned}$$

Define

$$f_{j,l;u}(\boldsymbol{x}):=\exp\left(\boldsymbol{x}^\top\Sigma_{j,l;u}^{-1}\boldsymbol{b}_{j,l;u}-\frac{1}{2u}\boldsymbol{x}^\top\Sigma_{j,l;u}^{-1}\boldsymbol{x}\right),\qquad \boldsymbol{x}\in\mathbb{R}^2.$$

Thus, it follows that

$$p_1(u)=\frac{u^{-1}}{2\pi}\sum_{j=-N_u^{(1)}}^{N_u^{(1)}}\sum_{l=-N_u^{(2)}}^{N_u^{(2)}}\frac{1}{\sqrt{|\Sigma_{j,l;u}|}}\exp\left(-\frac{1}{2}ug_A(t_A+\frac{jT}{u},s_A+\frac{lS}{u})\right)\int_{\mathbb{R}^2}f_{j,l;u}(\boldsymbol{x})P_{j,l;u}(\boldsymbol{x})\,d\boldsymbol{x}.$$

Further, we obtain from (i) of Lemma A.1 that, for all large enough u,

$$g_A(t_A+\frac{jT}{u},s_A+\frac{lS}{u})\sim g_A(t_A,s_A)+\frac{1}{2}\left(a_1\left(\frac{jT}{u}\right)^2-2a_2\left(\frac{jT}{u}\right)\left(\frac{lS}{u}\right)+a_3\left(\frac{lS}{u}\right)^2\right)$$

holds uniformly for $-N_u^{(1)}\le j\le N_u^{(1)},-N_u^{(2)}\le l\le N_u^{(2)}$.

Consequently, by Lemma A.3 below we obtain

$$\lim_{u\to\infty}\frac{p_1(u)}{\exp\left(-g_A(t_A,s_A)u/2\right)}=\frac{1}{2\pi\sqrt{|\Sigma_0|}}\frac{\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_2-2\mu_1\rho;S)}{TS}\int_{\mathbb{R}^2}e^{\frac{-(a_1x_1^2-2a_2x_1x_2+a_3x_2^2)}{4}}\,d\boldsymbol{x},$$

which gives the result for $p_1(u)$. The claim for $p_2(u)$ follows with the same arguments. □

Lemma A.3. *For any* $T,S>0$

$$\lim_{u\to\infty}\int_{\mathbb{R}^2}f_{j,l;u}(\boldsymbol{x})P_{j,l;u}(\boldsymbol{x})\,d\boldsymbol{x}=\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_2-2\mu_1\rho;S)$$

holds uniformly for $-N_u^{(1)}\le j\le N_u^{(1)},-N_u^{(2)}\le l\le N_u^{(2)}$.

We omit the tedious proof of Lemma A.3 since its idea is standard, i.e., it is based on finding a uniform integrable bound for the integrand and then using the dominated convergence theorem.

A.3. **Proof of Lemma 4.3.** Let us begin with $\Pi_1(u)$. It follows that

$$\begin{aligned}\Pi_1(u)&=\sum_{j=-N_u^{(1)}}^{N_u^{(1)}}\sum_{-N_u^{(2)}\le l_1<l_2\le N_u^{(2)}}p_{j,l_1,l_2;u}\\&=\sum_{j=-N_u^{(1)}}^{N_u^{(1)}}\sum_{l=-N_u^{(2)}}^{N_u^{(2)}}p_{j,l,l+1;u}+\sum_{j=-N_u^{(1)}}^{N_u^{(1)}}\sum_{l_1=-N_u^{(2)}}^{N_u^{(2)}}\sum_{l_2=l_1+2}^{N_u^{(2)}}p_{j,l_1,l_2;u}=:\Pi_{11}(u)+\Pi_{12}(u).\end{aligned}$$

In order to deal with $\Pi_{11}(u)$ we note that

$$p_{j,l,l+1;u}=p_{j,l;u}+p_{j,l+1;u}-\widetilde{p}_{j,l;u},$$

where

$$\widetilde{p}_{j,l;u}=\mathbb{P}\left\{\exists_{(t,s)\in\triangle^{(1)}_{j;u}\times(\triangle^{(2)}_{l;u}\cup\triangle^{(2)}_{l+1;u})}\ X_1(t)>\sqrt{u}(1+\mu_1 t),X_2(s)>\sqrt{u}(1+\mu_2 s)\right\}.$$

Then we have

$$\Pi_{11}(u)=\sum_{j=-N_u^{(1)}}^{N_u^{(1)}}\sum_{l=-N_u^{(2)}}^{N_u^{(2)}}(p_{j,l;u}+p_{j,l+1;u}-\widetilde{p}_{j,l;u}).$$

Using the same arguments as in the proof of Lemma 4.2 we obtain

$$\lim_{u\to\infty}\frac{\Pi_{11}(u)}{e^{-g_A(t_A,s_A)u/2}}=\frac{1}{\mu_1(\mu_2-2\mu_1\rho)}\left(\frac{2\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_2-2\mu_1\rho;S)}{TS}-\frac{\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_2-2\mu_1\rho;2S)}{TS}\right),$$

which gives that

$$\limsup_{S\to\infty}\limsup_{T\to\infty}\lim_{u\to\infty}\frac{\Pi_{11}(u)}{e^{-g_A(t_A,s_A)u/2}}=0.$$

Next we consider $\Pi_{12}(u)$ which is more involved. We have (recall (34) for $a_{j;u},b_{l;u}$)

$$\begin{aligned}
p_{j,l_1,l_2;u} &= \mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s_1\in[0,S]\\ s_2\in[0,S]}}\begin{array}{c} X_1(t_A+\frac{jT}{u}+\frac{t}{u})>a_{j;u}\sqrt{u}+\frac{\mu_1}{\sqrt{u}}t\\ X_2(s_A+\frac{l_1S}{u}+\frac{s_1}{u})>b_{l_1;u}\sqrt{u}+\frac{\mu_2}{\sqrt{u}}s_1\\ X_2(s_A+\frac{l_2S}{u}+\frac{s_2}{u})>b_{l_2;u}\sqrt{u}+\frac{\mu_2}{\sqrt{u}}s_2\end{array}\right\}\\
&\le \mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s_1\in[0,S]\\ s_2\in[0,S]}}\begin{array}{c} X_1(t_A+\frac{jT}{u}+\frac{t}{u})>a_{j;u}\sqrt{u}+\frac{\mu_1}{\sqrt{u}}t\\ \frac{1}{2}\left(X_2(s_A+\frac{l_1S}{u}+\frac{s_1}{u})+X_2(s_A+\frac{l_2S}{u}+\frac{s_2}{u})\right)>b_{l_1,l_2;u}\sqrt{u}+\frac{\mu_2}{2\sqrt{u}}(s_1+s_2)\end{array}\right\}=:P_{j,l_1,l_2;u}, \qquad (36)
\end{aligned}$$

with

$$b_{l_1,l_2;u}=1+\mu_2\left(s_A+\frac{l_1S}{u}+\frac{(l_2-l_1)S}{2u}\right).$$

For notational simplicity, we shall denote

$$\widetilde{t_A}=t_A+\frac{jT}{u},\quad \widetilde{s_A}=s_A+\frac{l_1S}{u},\quad \overline{s_A}=\widetilde{s_A}+\frac{(l_2-l_1)S}{2u},\quad \widehat{s_A}=\widetilde{s_A}+\frac{(l_2-l_1)S}{4u}.$$

Again by conditioning on the event

$$E_{j,l_1,l_2;u}(x_1,x_2):=\left\{X_1(\widetilde{t_A})=a_{j;u}\sqrt{u}-\frac{x_1}{\sqrt{u}},\quad \frac{1}{2}\left(X_2(\widetilde{s_A})+X_2(s_A+\frac{l_2S}{u})\right)=b_{l_1,l_2;u}\sqrt{u}-\frac{x_2}{\sqrt{u}}\right\},$$

we have

$$P_{j,l_1,l_2;u}=u^{-1}\int_{\mathbb{R}^2}\phi_{\Sigma_{j,l_1,l_2;u}}(\sqrt{u}\boldsymbol{b}_{j,l_1,l_2;u}-\boldsymbol{x}/\sqrt{u})F(j,l_1,l_2;u,\boldsymbol{x})\,d\boldsymbol{x},$$

where

$$\Sigma_{j,l_1,l_2;u}=\begin{pmatrix}\widetilde{t_A} & \rho\,\overline{s_A}\\ \rho\,\overline{s_A} & \widehat{s_A}\end{pmatrix},\qquad \boldsymbol{b}_{j,l_1,l_2;u}=(a_{j;u},b_{l_1,l_2;u})^\top$$

and

$$F(j,l_1,l_2;u,\boldsymbol{x}):=\mathbb{P}\left\{\exists_{t\in[0,T]}\ X_1(t)-\mu_1t>x_1\right\}\ \mathbb{P}\left\{\exists_{\substack{s_1\in[0,S]\\ s_2\in[0,S]}}Y_{j,l_1,l_2;u}(s_1,s_2)>x_2\ \middle|\ E_{j,l_1,l_2;u}(x_1,x_2)\right\},$$

where

$$Y_{j,l_1,l_2;u}(s_1,s_2) = \frac{\sqrt{u}}{2}\left(X_2(\widetilde{s_A}+\frac{s_1}{u}) - X_2(\widetilde{s_A}) + X_2(s_A+\frac{l_2S}{u}+\frac{s_2}{u}) - X_2(s_A+\frac{l_2S}{u})\right) - \frac{\mu_2}{2}(s_1+s_2).$$

Similarly as in the proof of Lemma 4.2, we obtain

$$\phi_{\Sigma_{j,l_1,l_2;u}}(\sqrt{u}\boldsymbol{b}_{j,l_1,l_2;u} - \boldsymbol{x}/\sqrt{u}) = \frac{1}{\sqrt{(2\pi)^2\,|\Sigma_{j,l_1,l_2;u}|}} \exp\left(-\frac{1}{2}u\ (\boldsymbol{b}_{j,l_1,l_2;u})^\top \Sigma^{-1}_{j,l_1,l_2;u}\boldsymbol{b}_{j,l_1,l_2;u}\right)\ f_{j,l_1,l_2;u}(\boldsymbol{x}),$$

where

$$f_{j,l_1,l_2;u}(\boldsymbol{x}) := \exp\left(\boldsymbol{x}^\top\Sigma^{-1}_{j,l_1,l_2;u}\boldsymbol{b}_{j,l_1,l_2;u} - \frac{1}{2u}\boldsymbol{x}^\top\Sigma^{-1}_{j,l_1,l_2;u}\boldsymbol{x}\right).$$

Next, some elementary calculations give that

$$(\boldsymbol{b}_{j,l_1,l_2;u})^\top\Sigma^{-1}_{j,l_1,l_2;u}\boldsymbol{b}_{j,l_1,l_2;u} = g_A(t_A+\frac{jT}{u}, s_A+\frac{l_1S}{u}+\frac{(l_2-l_1)S}{2u}) + \frac{\widetilde{t_A}g_A(\widetilde{t_A},\overline{s_A}) - a^2_{j;u}}{4(\widetilde{t_A}\widehat{s_A}-\rho^2\overline{s_A}^2)}\ \frac{(l_2-l_1)S}{u}.$$

Further, note that

$$g_A(t_A+\frac{jT}{u}, s_A+\frac{l_1S}{u}+\frac{(l_2-l_1)S}{2u}) = g_A(t_A+\frac{jT}{u}, s_A+\frac{l_1S}{u}) + \frac{\partial g_A(t,s)}{\partial s}\ |_{(\widetilde{t_A},\widetilde{s_A}+\theta_{l_1,l_2;u}\frac{(l_2-l_1)S}{2u})}\ \frac{(l_2-l_1)S}{2u}$$

holds for some $\theta_{l_1,l_2;u}\in(0,1)$ and

$$\frac{\partial g_A(t,s)}{\partial t}\ |_{(\widetilde{t_A},\widetilde{s_A}+\theta_{l_1,l_2;u}\frac{(l_2-l_1)S}{2u})}\ \to\ 0,\qquad u\to\infty$$

holds uniformly for j,l_1,l_2 (hereafter when we write j,l_1,l_2 we mean $-N_u^{(1)}\le j\le N_u^{(1)}, -N_u^{(2)}\le l_1,l_2\le N_u^{(2)}$)).
Consequently

$$\exp\left(-\frac{1}{2}u\ (\boldsymbol{b}_{j,l_1,l_2;u})^\top\Sigma^{-1}_{j,l_1,l_2;u}\boldsymbol{b}_{j,l_1,l_2;u}\right) \sim \exp\left(-\frac{1}{2}u\ g_A(t_A+\frac{jT}{u}, s_A+\frac{l_1S}{u})\right) e^{-Q_0(l_2-l_1)S} \tag{37}$$

holds uniformly for j,l_1,l_2 as $u\to\infty$, where (by (b).(i) of Lemma 9 of [8] or Lemma A.1.(i) with $a_1>0$)

$$Q_0 = \frac{t_Ag_A(t_A,s_A) - (1+\mu_1t_A)^2}{8(t_As_A-\rho^2s_A^2)} > 0.$$

Next, we consider the uniform, in j,l_1,l_2, limit of the following:

$$\mathbb{P}\left\{\exists_{\substack{s_1\in[0,S]\\ s_2\in[0,S]}} Y_{j,l_1,l_2;u}(s_1,s_2) > x_2\ \middle|\ E_{j,l_1,l_2;u}(x_1,x_2)\right\},\quad u\to\infty$$

For the conditional mean we can derive that

$$\begin{aligned}\mathbb{E}\left\{Y_{j,l_1,l_2;u}(s_1,s_2)|E_{j,l_1,l_2;u}(x_1,x_2)\right\} &= -\frac{\mu_2}{2}(s_1+s_2)\\ &\quad + \left(\frac{\rho(s_1+s_2)}{2\sqrt{u}}, \frac{s_1}{4\sqrt{u}}\right)\ \Sigma^{-1}_{j,l_1,l_2;u}(\boldsymbol{b}_{j,l_1,l_2;u} - \boldsymbol{x}/\sqrt{u}),\end{aligned}$$

which further gives that

$$\begin{aligned}\mathbb{E}\left\{Y_{j,l_1,l_2;u}(s_1,s_2)|E_{j,l_1,l_2;u}(x_1,x_2)\right\} &= -\frac{\mu_2}{2}(s_1+s_2) + \frac{2\rho a_{j;u}\widehat{s_A} - \rho a_{j;u}\overline{s_A} - 2\rho^2 b_{j,l_1,l_2;u}\overline{s_A} + b_{j,l_1,l_2;u}\widetilde{t_A}}{4(\widetilde{t_A}\widehat{s_A}-\rho^2\overline{s_A}^2)}s_1\\ &+ \frac{\rho a_{j;u}\widehat{s_A} - \rho^2 b_{j,l_1,l_2;u}\overline{s_A}}{2(\widetilde{t_A}\widehat{s_A}-\rho^2\overline{s_A}^2)}s_2 + \frac{\rho\overline{s_A}s_1 - 2\rho\widehat{s_A}(s_1+s_2)}{4(\widetilde{t_A}\widehat{s_A}-\rho^2\overline{s_A}^2)}\ \frac{x_1}{u} + \frac{2\rho^2\overline{s_A}(s_1+s_2) - \widetilde{t_A}s_1}{4(\widetilde{t_A}\widehat{s_A}-\rho^2\overline{s_A}^2)}\ \frac{x_2}{u}\\ &\to -\frac{1}{2}\left(\mu_2 - 2\mu_1\rho\right)s_2,\qquad u\to\infty.\end{aligned}$$

For the conditional variance of the increments we have

$$\mathrm{Var}\left\{Y_{j,l_1,l_2;u}(s_1,s_2)-Y_{j,l_1,l_2;u}(s_1',s_2')|E_{j,l_1,l_2;u}(x_1,x_2)\right\}=\frac{|s_1-s_1'|+|s_2-s_2'|}{4}$$
$$+\left(\frac{\rho(s_1-s_1'+s_2-s_2')}{2\sqrt{u}},\frac{s_1-s_1'}{4\sqrt{u}}\right)\Sigma^{-1}_{j,l_1,l_2;u}\left(\frac{\rho(s_1-s_1'+s_2-s_2')}{2\sqrt{u}},\frac{s_1-s_1'}{4\sqrt{u}}\right)^\top$$
$$\to\frac{|s_1-s_1'|+|s_2-s_2'|}{4},\quad u\to\infty.$$

Therefore, similarly as in Lemma A.3 we can show that as $u\to\infty$

$$\mathbb{P}\left\{\exists_{\substack{s_1\in[0,S]\\ s_2\in[0,S]}} Y_{j,l_1,l_2;u}(s_1,s_2)>x_2\ \middle|E_{j,l_1,l_2;u}(x_1,x_2)\right\}\to\mathbb{P}\left\{\exists_{\substack{s_1\in[0,S]\\ s_2\in[0,S]}}\frac{1}{2}(B_1(s_1)+B_2(s_2))-\frac{1}{2}\left(\mu_2-2\mu_1\rho\right)s_2>x_2\right\}.$$

Consequently, the dominated convergence theorem gives

$$\begin{aligned}&\int_{\mathbb{R}^2}f_{j,l_1,l_2;u}(\boldsymbol{x})F(j,l_1,l_2;u,\boldsymbol{x})\,d\boldsymbol{x}\\ &\to\int_{\mathbb{R}}e^{2\mu_1x_1}\mathbb{P}\left\{\exists_{t\in[0,T]}\ X_1(t)-\mu_1t>x_1\right\}dx_1\\ &\quad\times\int_{\mathbb{R}}e^{2(\mu_2-2\mu_1\rho)x_2}\mathbb{P}\left\{\exists_{\substack{s_1\in[0,S]\\ s_2\in[0,S]}}\frac{1}{2}(B_1(s_1)+B_2(s_2))-\frac{1}{2}\left(\mu_2-2\mu_1\rho\right)s_2>x_2\right\}dx_2 \qquad (38)\\ &=:\mathcal{H}(\mu_1;T)\ \mathcal{H}(\mu_1,\mu_2;S)\end{aligned}$$

holds uniformly for j,l_1,l_2, as $u\to\infty$.

Next we derive a useful upper bound for $\mathcal{H}(\mu_1,\mu_2;S)$, $S>0$:

$$\mathcal{H}(\mu_1,\mu_2;S)\le(\lfloor S\rfloor)^2e^{Q_0S}\mathcal{H}(\mu_1,\mu_2;1)<\infty. \qquad (39)$$

In order to prove (39), by taking $j=l_1=0,l_2=1$ we arrive at

$$\begin{aligned}P_{0,0,1;u}&=\mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s_1\in[0,S]\\ s_2\in[0,S]}}\begin{array}{c}X_1(t_A+\frac{t}{u})>a_{0;u}\sqrt{u}+\frac{\mu_1}{\sqrt{u}}t\\ \frac{1}{2}(X_2(s_A+\frac{s_1}{u})+X_2(s_A+\frac{S}{u}+\frac{s_2}{u}))>b_{0,1;u}\sqrt{u}+\frac{\mu_2}{2\sqrt{u}}(s_1+s_2)\end{array}\right\}\\ &\sim\frac{u^{-1}}{\sqrt{(2\pi)^2\left|\Sigma_{0,0,0;u}\right|}}\exp\left(-\frac{1}{2}u\ g_A(t_A,s_A)\right)e^{-Q_0S}\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_1,\mu_2;S). \qquad (40)\end{aligned}$$

Define, for any integers $0\le m,n\le\lfloor S\rfloor$,

$$q_{m,n;u}:=\mathbb{P}\left\{\exists_{\substack{t\in[0,T]\\ s_1\in[0,1]\\ s_2\in[0,1]}}\begin{array}{c}X_1(t_A+\frac{t}{u})>a_{0;u}\sqrt{u}+\frac{\mu_1}{\sqrt{u}}t\\ \frac{1}{2}(X_2(s_A+\frac{m}{u}+\frac{s_1}{u})+X_2(s_A+\frac{S+n}{u}+\frac{s_2}{u}))>\widetilde{b}_{m,n;u}\sqrt{u}+\frac{\mu_2}{2\sqrt{u}}(s_1+s_2)\end{array}\right\}$$

with

$$\widetilde{b}_{m,n;u}=1+\mu_2\left(s_0+\frac{m}{u}+\frac{S+n-m}{2u}\right).$$

Using the same arguments as in the derivation of (40) one can show that

$$q_{m,n;u}\sim\frac{u^{-1}}{\sqrt{(2\pi)^2\left|\Sigma_{0,0,0;u}\right|}}\exp\left(-\frac{1}{2}u\ g_A(t_A,s_A)\right)e^{-Q_0(S+n-m)}\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_1,\mu_2;1). \qquad (41)$$

Comparing (40) and (41) we derive

$$\begin{aligned}\mathcal{H}(\mu_1,\mu_2;S) &\le \sum_{m=0}^{\lfloor S\rfloor-1}\sum_{n=0}^{\lfloor S\rfloor-1} e^{-Q_0(n-m)}\mathcal{H}(\mu_1,\mu_2;1)\\ &\le (\lfloor S\rfloor)^2 e^{Q_0 S}\mathcal{H}(\mu_1,\mu_2;1).\end{aligned}$$

The finiteness of $\mathcal{H}(\mu_1,\mu_2;1)$ can be proved by using the Borell-TIS inequality. This justifies bound (39).

Now, we are ready to analyse the triple sum $\Pi_{12}(u)$. We have

$$\begin{aligned}\Pi_{12}(u) &= \sum_{j=-N_u^{(1)}}^{N_u^{(1)}}\sum_{l_1=-N_u^{(2)}}^{N_u^{(2)}}\sum_{l_2=l_1+2}^{N_u^{(2)}} \frac{u^{-1}}{\sqrt{(2\pi)^2\,|\Sigma_{j,l_1,l_2;u}|}}\\ &\quad\times\exp\left(-\frac{1}{2}u\,(\boldsymbol{b}_{j,l_1,l_2;u})^\top\Sigma_{j,l_1,l_2;u}^{-1}\boldsymbol{b}_{j,l_1,l_2;u}\right)\int_{\mathbb{R}^2} f_{j,l_1;u}(\boldsymbol{x})F(j,l_1,l_2;u,\boldsymbol{x})\,d\boldsymbol{x}.\end{aligned}$$

Therefore, we can derive from (37)-(38) and (39) that

$$\lim_{u\to\infty}\frac{\Pi_{12}(u)}{\exp(-ug_A(t_A,s_A)/2)} \le \text{Const}\sum_{k=1}^{\infty} e^{-kQ_0S}\frac{\mathcal{H}(\mu_1;T)\mathcal{H}(\mu_1,\mu_2;1)(\lfloor S\rfloor)^2}{TS}.$$

Consequently, the above implies that

$$\limsup_{S\to\infty}\limsup_{T\to\infty}\lim_{u\to\infty}\frac{\Pi_{12}(u)}{\exp(-ug_A(t_A,s_A)/2)}=0.$$

Thus, the claim for $\Pi_1(u)$ is established. Using similar arguments, one can further show that the claim for $\Pi_2(u)$ holds. □

A.4. **Proof of Lemma 4.4.** The claim for $r_1(u)$ follows from the same arguments as that for $r_0(u)$ of Lemma 4.1. Next, as in the proof of Lemma 4.1, using the Piterbarg's inequality we can show that

$$r_2(u)\le C_2u^{3/2}e^{-\frac{u}{2}\widetilde{g}_u},$$

where $C_2>0$ is some constant which does not depend on u, and thus the claim for $r_2(u)$ follows since

$$\begin{aligned}\widetilde{g}_u &= \inf_{(t,s)\in D_1\cup D_2} g(t,s) = \inf_{s\in[s_0-\theta_0,s_0-\ln(u)/\sqrt{u}]\cup[s_0+\ln(u)/\sqrt{u},s_0+\theta_0]} g_L(s)\\ &\ge g_L(s^*)+\frac{b_0}{2}(1-\varepsilon)\frac{(\ln(u))^2}{u},\end{aligned}$$

where the last inequality follows by (ii.3) of Lemma A.1. Finally, the claim for $r_3(u)$ can be proved similarly, by using Piterbarg's inequality and (ii.1)-(ii.2) of Lemma A.1. □

A.5. **Proof of Lemma 4.5.** We first analyse the summand $p_{j;u}$. Let

$$\boldsymbol{b}_{j;u}=(a_{j;u},b_{j,u})^\top,\quad a_{j;u}=1+\mu_1(t^*+\frac{jT}{u}),\quad b_{j;u}=1+\mu_2(s^*+\frac{jT}{u}).$$

Then

$$p_{j;u}=\mathbb{P}\left\{\exists_{(t,s)\in\triangle_{T,S}}\begin{array}{c}X_1(t^*+\frac{jT}{u}+\frac{t}{u})>a_{j;u}\sqrt{u}+\frac{\mu_1}{\sqrt{u}}t\\ X_2(s^*+\frac{jT}{u}+\frac{s}{u})>b_{j;u}\sqrt{u}+\frac{\mu_2}{\sqrt{u}}s\end{array}\right\}.$$

Define $\boldsymbol{Z}_{j;u}:=(X_1(t^*+\frac{jT}{u}),X_2(s^*+\frac{jT}{u}))^\top$, whose density function is given by

$$\phi_{\Sigma_{j;u}}(\boldsymbol{w})=\frac{1}{\sqrt{(2\pi)^2\,|\Sigma_{j;u}|}}\exp\left(-\frac{1}{2}\boldsymbol{w}^\top(\Sigma_{j;u})^{-1}\boldsymbol{w}\right),\quad \boldsymbol{w}=(w_1,w_2)^\top,$$

with the covariance matrix given by

$$\Sigma_{j;u} = \begin{pmatrix} t^* + \frac{jT}{u} & \rho\ (t^* + \frac{jT}{u}) \\ \rho\ (t^* + \frac{jT}{u}) & s^* + \frac{jT}{u} \end{pmatrix}.$$

By conditioning on the value of $\boldsymbol{Z}_{j;u}$ and using change of variables $\boldsymbol{w} = \sqrt{u}\boldsymbol{b}_{j;u} - \boldsymbol{x}/\sqrt{u}$, we further obtain

$$p_{j;u} = u^{-1} \int_{\mathbb{R}^2} \phi_{\Sigma_{j;u}}(\sqrt{u}\boldsymbol{b}_{j;u} - \boldsymbol{x}/\sqrt{u}) \mathbb{P} \left\{ \exists_{(t,s)\in\triangle_{T,S}} \begin{array}{c} X_1(t) - \mu_1 t > x_1 \\ X_2(s) - \mu_2 s > x_2 \end{array} \right\} d\boldsymbol{x}.$$

Consequently, similar arguments as in the proof of Lemma 4.2 yield

$$\begin{aligned} p_1(u) \quad &\sim \quad p_2(u) \sim \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} p_{j;u} \sim \frac{\mathcal{H}(T,S)u^{-1}}{\sqrt{(2\pi t^*)^2(1-\rho^2)}} \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} e^{-\frac{u}{2} g_L(t^* + \frac{jT}{u})} \\ &\sim \quad \frac{\mathcal{H}(T,S)u^{-1/2}}{T\sqrt{(2\pi t^*)^2(1-\rho^2)}} e^{-\frac{u}{2} g_L(t^*)} \int_{\mathbb{R}} e^{-\frac{b_0}{4}x^2} dx. \end{aligned}$$

This completes the proof. □

A.6. **Proof of Lemma 4.6.** First note that

$$\begin{aligned} &\mathbb{P}\left\{\exists_{(t,s)\in(t^*,s^*)+u^{-1}\triangle_{T_1+T_2,S}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ &\le \mathbb{P}\left\{\exists_{(t,s)\in(t^*,s^*)+u^{-1}\triangle_{T_1,S}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\} \\ &\quad + \mathbb{P}\left\{\exists_{(t,s)\in(t^*+\frac{T_1}{u}, s^*+\frac{T_1}{u})+u^{-1}\triangle_{T_2,S}}\ X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s)\right\}. \end{aligned}$$

Using the same arguments as the proof of Lemma 4.5, we conclude the sub-additivity of $\mathcal{H}(T,S), T > 0$. Thus

$$\lim_{T\to\infty} \frac{1}{T}\mathcal{H}(T,S) = \inf_{T>0} \frac{1}{T}\mathcal{H}(T,S) < \infty$$

follows directly from Fekete's lemma. Moreover, since by definition

$$\mathcal{H}(T,S) \ge \int_{\mathbb{R}^2} e^{\boldsymbol{x}^\top \Sigma_*^{-1} \boldsymbol{b}_*} \mathbb{P}\left\{ \exists_{t\in[0,T]} \begin{array}{c} X_1(t) - \mu_1 t > x_1 \\ X_2(t) - \mu_2 t > x_2 \end{array} \right\} dx_1 dx_2,$$

the positive lower bound follows from Lemma 4.7 in [3]. This completes the proof. □

A.7. **Proof of Lemma 4.7.** We begin with the analysis of $\Pi_1(u)$. We first look at $p_{j,l;u}$. Denote

$$\begin{aligned} &\boldsymbol{b}_u = \boldsymbol{b}_{j,l,m,n;u} := (a_{j,m;u}, b_{j,l,m,n;u})^\top, \\ &a_{j,m;u} = 1 + \mu_1(t^* + \frac{jT+m}{u}), \quad b_{j,l,m,n;u} = 1 + \mu_2(t^* + \frac{jT+m}{u} + \frac{lS+n}{u}). \end{aligned}$$

It is derived that

$$\begin{aligned} p_{j,l;u} \quad &\le \quad \sum_{m=0}^{\lfloor T\rfloor -1} \sum_{n=0}^{\lfloor S\rfloor -1} \mathbb{P}\left\{ \exists_{\substack{t\in[t^*+\frac{jT}{u}+\frac{m}{u}, t^*+\frac{jT}{u}+\frac{m+1}{u}] \\ s-t\in[\frac{lS}{u}+\frac{n}{u}, \frac{lS}{u}+\frac{n+1}{u}]}} X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s) \right\} \\ &= \quad \sum_{m=0}^{\lfloor T\rfloor -1} \sum_{n=0}^{\lfloor S\rfloor -1} \mathbb{P}\left\{ \exists_{\substack{t\in[0,1] \\ s\in[0,1]}} \begin{array}{c} X_1(t^* + \frac{jT+m}{u} + \frac{t}{u}) > a_{j,m;u}\sqrt{u} + \frac{\mu_1}{\sqrt{u}} t \\ X_2(t^* + \frac{jT+m}{u} + \frac{lS+n}{u} + \frac{t+s}{u}) > b_{j,l,m,n;u}\sqrt{u} + \frac{\mu_2}{\sqrt{u}}(t+s) \end{array} \right\} \end{aligned}$$

$$=: \sum_{m=0}^{\lfloor T\rfloor-1}\sum_{n=0}^{\lfloor S\rfloor-1} p_{j,l,m,n;u}.$$

Next, we look at $p_{j,l,m,n;u}$. We define

$$\begin{aligned}
\boldsymbol{Z}_u &:= \left(X_1(t^*+\frac{jT+m}{u}),\ X_2(s^*+\frac{jT+m}{u}+\frac{lS+n}{u})\right)^\top,\\
Y_{1;u}(t) &:= \left(X_1(t^*+\frac{jT+m}{u}+\frac{t}{u})-X_1(t^*+\frac{jT+m}{u})\right)\sqrt{u}-\mu_1 t,\\
Y_{2;u}(t,s) &:= \left(X_2(t^*+\frac{jT+m}{u}+\frac{lS+n}{u}+\frac{t+s}{u})-X_2(t^*+\frac{jT+m}{u}+\frac{lS+n}{u})\right)\sqrt{u}-\mu_2(t+s).
\end{aligned}$$

Consider the conditional process

$$\boldsymbol{W}_u(t,s) := (Y_{1;u}(t), Y_{2;u}(t,s))^\top \mid \boldsymbol{Z}_u = \sqrt{u}\boldsymbol{b}_u - \frac{\boldsymbol{x}}{\sqrt{u}}.$$

We have that $(Y_{1;u}(t), Y_{2;u}(t,s), \boldsymbol{Z}_u^\top)$ is a normally distributed random vector, with mean

$$\widehat{\boldsymbol{\mu}}(t,s) := (-\mu_1 t, -\mu_2(t+s), 0, 0)^\top$$

and covariance matrix given by (suppose $S>1$)

$$\widehat{\Sigma}_u(t,s) := \begin{pmatrix} t & 0 & 0 & \rho\frac{t}{\sqrt{u}}\\ 0 & t+s & 0 & 0\\ 0 & 0 & t^*+\frac{jT+m}{u} & \rho\ (t^*+\frac{jT+m}{u})\\ \rho\frac{t}{\sqrt{u}} & 0 & \rho\ (t^*+\frac{jT+m}{u}) & t^*+\frac{jT+m}{u}+\frac{lS+n}{u}\end{pmatrix}.$$

Thus, for the mean

$$\begin{aligned}
\mathbb{E}\left\{\boldsymbol{W}_u(t,s)\right\} &= (-\mu_1 t, -\mu_2(t+s)) + \begin{pmatrix} 0 & \rho\frac{t}{\sqrt{u}}\\ 0 & 0\end{pmatrix}\begin{pmatrix} t^*+\frac{jT+m}{u} & \rho\ (t^*+\frac{jT+m}{u})\\ \rho\ (t^*+\frac{jT+m}{u}) & t^*+\frac{jT+m}{u}+\frac{lS+n}{u}\end{pmatrix}^{-1}\left(\sqrt{u}\boldsymbol{b}_u-\frac{\boldsymbol{x}}{\sqrt{u}}\right)\\
&= (-\mu_1 t, -\mu_2(t+s)) + \left(\frac{\rho(b_{j,l,m,n;u}-\rho a_{j,m;u})t-\rho t\frac{x_2-\rho x_1}{u}}{(t^*+\frac{jT+m}{u}+\frac{lS+n}{u})-\rho^2(t^*+\frac{jT+m}{u})}, 0\right)\\
&\to (-\nu(\rho)\ t, -\mu_2(t+s))^\top,\quad \nu(\rho) := \frac{\rho^2-\rho+(\mu_1-\mu_2\rho)t^*}{t^*(1-\rho^2)},
\end{aligned}$$

as $u\to\infty$, where the convergence is uniform for $-N_u^{(1)}\le j\le N_u^{(1)}, 1\le l\le N_u^{(2)}$. Similarly, we can derive that, for any $t_1,t_2\in[0,T], s_1,s_2\in[-S,S]$,

$$\mathrm{Cov}(\boldsymbol{W}_u(t_1,s_1)-\boldsymbol{W}_u(t_2,s_2)) \to \mathrm{Cov}((B_1(t_1)-B_1(t_2), B_2(t_1+s_1)-B_2(t_2+s_2))^\top)$$

as $u\to\infty$, uniformly for $-N_u^{(1)}\le j\le N_u^{(1)}, 1\le l\le N_u^{(2)}$. Consequently, we have, as $u\to\infty$,

$$\begin{aligned}
&\mathbb{P}\left\{\exists_{\substack{t\in[0,1]\\ s\in[0,1]}} \begin{matrix} Y_{1;u}(t)>x_1\\ Y_{2;u}(t,s)>x_2\end{matrix}\ \middle|\ \boldsymbol{Z}_u=\sqrt{u}\boldsymbol{b}_u-\frac{\boldsymbol{x}}{\sqrt{u}}\right\}\\
&\to \mathbb{P}\left\{\exists_{\substack{t\in[0,1]\\ s\in[0,1]}} \begin{matrix} B_1(t)-\nu(\rho)t>x_1\\ B_2(t+s)-\mu_2(t+s)>x_2\end{matrix}\right\}.
\end{aligned}$$

Similar arguments as those in the proof of Lemma 4.2 gives that

$$p_{j,l,m,n;u} \sim \frac{\widehat{\mathcal{H}}(1,1)u^{-1}}{\sqrt{(2\pi)^2\,|\Sigma_*|}} e^{-\frac{u}{2} g_B(t^*+\frac{jT+m}{u}, s^*+\frac{jT+m}{u}+\frac{lS+n}{u})},$$

where (recall notation in (7))

$$\widehat{\mathcal{H}}(1,1) = \int_{\mathbb{R}^2} e^{\boldsymbol{x}^\top \Sigma_*^{-1}\boldsymbol{b}_*} \mathbb{P}\left\{ \exists_{\substack{t\in[0,1]\\ s\in[0,1]}} \begin{array}{c} B_1(t) - \nu(\rho)t > x_1 \\ B_2(t+s) - \mu(t+s) > x_2 \end{array} \right\} dx_1 dx_2 \in (0,\infty).$$

It follows further from (ii.2) of Lemma A.1 that there exists some $\varepsilon > 0$ such that, for all $t < s$ small,

$$g_B(t^*+t, t^*+s) \geq g_L(t^*) + b_2(1-\varepsilon)(s-t) + \frac{c_2(1-\varepsilon)}{2} t^2,$$

thus, for u sufficiently large

$$g_B(t^* + \frac{jT+m}{u}, t^* + \frac{jT+m}{u} + \frac{lS+n}{u}) \geq g_L(t^*) + b_2(1-\varepsilon)\frac{lS}{u} + \frac{c_2(1-\varepsilon)}{2}\left(\frac{\widehat{j}T}{u}\right)^2$$

holds for all $-N_u^{(1)} \leq j \leq N_u^{(1)}, 1 \leq l \leq N_u^{(2)}, 0 \leq m \leq \lfloor T \rfloor - 1, 0 \leq n \leq \lfloor S \rfloor - 1$, where $\widehat{j} = jI_{\{j\geq 0\}} + (j+1)I_{\{j<0\}}$.
This implies that, for u large

$$e^{-\frac{u}{2} g_A(t^*+\frac{jT+m}{u}, s^*+\frac{jT+m}{u}+\frac{lS+n}{u})} \leq e^{-\frac{u}{2} g_L(t^*)} \frac{\sqrt{u}}{T}\left(e^{-\frac{c_2(1-\varepsilon)}{4}\left(\frac{\widehat{j}T}{\sqrt{u}}\right)^2} \frac{T}{\sqrt{u}}\right) e^{-\frac{b_2(1-\varepsilon)}{2} lS}.$$

Based on the above discussions we obtain

$$\lim_{u\to\infty} \frac{\Pi_1(u)}{u^{-1/2}\exp(-g_L(t^*)u/2)} \leq \frac{\widehat{\mathcal{H}}(1,1)}{\sqrt{(2\pi)^2\,|\Sigma_*|}} \frac{\lfloor T \rfloor \lfloor S \rfloor}{T} \sum_{l\geq 1} e^{-\frac{b_2(1-\varepsilon)}{2} lS} \int_{\mathbb{R}} e^{-\frac{c_2(1-\varepsilon)}{4} x^2} dx.$$

Similar bounds can be found for $\overline{\Pi}_1(u)$, and thus the first claim follows.

Next we consider $\Pi_{21}(u)$. For any $j_2 > j_1 + 1$ we have

$$\begin{aligned} q_{j_1,j_2;u} &= \mathbb{P}\left\{ \begin{array}{c} \exists_{(t,s)\in(t^*+\frac{j_1T}{u}, s^*+\frac{j_1T}{u})+u^{-1}\triangle_{T,S}} X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s) \\ \exists_{(t,s)\in(t^*+\frac{j_2T}{u}, s^*+\frac{j_2T}{u})+u^{-1}\triangle_{T,S}} X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s) \end{array} \right\} \\ &\leq u^{-1} \int_{\mathbb{R}^2} \phi_{\Sigma_{j_1,j_2;u}}(\sqrt{u}\boldsymbol{b}_{j_1,j_2;u} - \boldsymbol{x}/\sqrt{u}) \overline{F}(j_1,j_2;u,\boldsymbol{x})\, d\boldsymbol{x} =: Q_{j_1,j_2;u}, \end{aligned}$$

where, with $a_{j;u} = 1 + \mu_1(t^* + \frac{j_1T}{u}), b_{j;u} = 1 + \mu_2(s^* + \frac{j_1T}{u})$,

$$\Sigma_{j_1,j_2;u} = \left(t^* + \frac{j_1T}{u} + \frac{(j_2-j_1)S}{4u}\right)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \boldsymbol{b}_{j_1,j_2;u} = \left(\frac{a_{j_1;u}+a_{j_2;u}}{2}, \frac{b_{j_1;u}+b_{j_2;u}}{2}\right),$$

$$\overline{F}(j_1,j_2;u,\boldsymbol{x}) := \mathbb{P}\left\{ \exists_{\substack{(t,s)\in\triangle_{T,S}\\ (t',s')\in\triangle_{T,S}}} \begin{array}{c} Y_{1;u}(t,t') > x_1 \\ Y_{2;u}(s,s') > x_2 \end{array} \middle| \begin{array}{c} Y_{3;u} = \frac{a_{j_1;u}+a_{j_2;u}}{2}\sqrt{u} - \frac{x_1}{\sqrt{u}} \\ Y_{4;u} = \frac{a_{j_1;u}+a_{j_2;u}}{2}\sqrt{u} - \frac{x_2}{\sqrt{u}} \end{array} \right\},$$

with

$$Y_{1;u}(t,t') = \frac{\sqrt{u}}{2}\left(X_1(t^*+\frac{j_1T}{u}+\frac{t}{u}) - X_1(t^*+\frac{j_1T}{u}) + X_1(t^*+\frac{j_2T}{u}+\frac{t'}{u}) - X_1(t^*+\frac{j_2T}{u})\right) - \frac{\mu_1}{2}(t+t'),$$

$$Y_{2;u}(s,s') = \frac{\sqrt{u}}{2}\left(X_2(s^*+\frac{j_1T}{u}+\frac{s}{u}) - X_2(s^*+\frac{j_1T}{u}) + X_2(s^*+\frac{j_2T}{u}+\frac{s'}{u}) - X_2(s^*+\frac{j_2T}{u})\right) - \frac{\mu_2}{2}(s+s'),$$

$$Y_{3;u} = \frac{1}{2}\left(X_1(t^*+\frac{j_1T}{u}) + X_1(t^*+\frac{j_2T}{u})\right), \quad Y_{4;u} = \frac{1}{2}\left(X_2(s^*+\frac{j_1T}{u}) + X_2(s^*+\frac{j_2T}{u})\right).$$

Next we have that $(Y_{1;u}(t,t'), Y_{2;u}(s,s'), Y_{3;u}, Y_{4;u})$ is a normally distributed random vector, with mean

$$\widehat{\boldsymbol{\mu}}(t,t',s,s') = \Big(-\frac{\mu_1}{2}(t+t'), -\frac{\mu_2}{2}(s+s'), 0, 0\Big)^\top$$

and covariance matrix given by (suppose $T > S$)

$$\widehat{\Sigma}_u(t,s) = \begin{pmatrix} \frac{t+t'}{4} & \frac{\rho(t\wedge s + t'\wedge s')}{4} & \frac{t}{4\sqrt{u}} & \frac{\rho t}{4\sqrt{u}} \\ \frac{\rho(t\wedge s + t'\wedge s')}{4} & \frac{s+s'}{4} & \frac{\rho s}{4\sqrt{u}} & \frac{s}{4\sqrt{u}} \\ \frac{t}{4\sqrt{u}} & \frac{\rho s}{4\sqrt{u}} & t^* + \frac{j_1 T}{u} + \frac{(j_2-j_1)T}{4u} & \rho\left(t^* + \frac{j_1 T}{u} + \frac{(j_2-j_1)T}{4u}\right) \\ \frac{\rho t}{4\sqrt{u}} & \frac{s}{4\sqrt{u}} & \rho\left(t^* + \frac{j_1 T}{u} + \frac{(j_2-j_1)T}{4u}\right) & s^* + \frac{j_1 T}{u} + \frac{(j_2-j_1)T}{4u} \end{pmatrix}.$$

Similarly as before, one can get

$$Q_{j_1,j_2;u} \sim \frac{\widetilde{\mathcal{H}}(T,S)u^{-1}}{\sqrt{(2\pi)^2 |\Sigma_*|}} e^{-\frac{u}{2}\left(1+\frac{\frac{(j_2-j_1)T}{4u}}{t^*+\frac{j_1 T}{u}+\frac{(j_2-j_1)T}{4u}}\right) g_L(t^* + \frac{j_1 T}{u} + \frac{(j_2-j_1)T}{2u})},$$

as $u \to \infty$, where

$$\widetilde{\mathcal{H}}(T,S) := \int_{\mathbb{R}^2} e^{\boldsymbol{x}^\top \Sigma_*^{-1} \boldsymbol{b}_*} \mathbb{P}\left\{ \exists_{\substack{(t,s)\in\triangle_{T,S} \\ (t',s')\in\triangle_{T,S}}} \begin{array}{c} \frac{1}{2}(X_1(t) + \widetilde{X}_1(t')) - \frac{\mu_1 t^* - 1}{4t^*} t - \frac{\mu_1}{2} t' > x_1 \\ \frac{1}{2}(X_2(s) + \widetilde{X}_2(s')) - \frac{\mu_2 s^* - 1}{4s^*} s - \frac{\mu_2}{2} s' > x_2 \end{array} \right\} dx_1 dx_2 \in (0,\infty),$$

with $(\widetilde{X}_1, \widetilde{X}_2)$ an independent copy of (X_1, X_2). Particularly, letting $j_1 = 0, j_2 = 2$ we can show, similarly as in (39), that

$$\widetilde{\mathcal{H}}(T,S) \le \widetilde{\mathcal{H}}(1,S)(\lfloor T \rfloor)^2 e^{\frac{g_L(t^*)}{8t^*}T}.$$

Therefore, as $u \to \infty$,

$$\begin{aligned} \Pi_{21}(u) &\lesssim \frac{\widetilde{\mathcal{H}}(T,S)u^{-1/2}}{T\sqrt{(2\pi)^2|\Sigma_*|}} e^{-\frac{u}{2}g_L(t^*)} \left(\sum_{j_1=-N_u^{(1)}}^{N_u^{(1)}} e^{-\frac{b_0}{4}\left(\frac{j_1 T}{\sqrt{u}}\right)^2} \frac{T}{\sqrt{u}} \right) \left(\sum_{j_2=j_1+2}^{N_u^{(1)}} e^{-\frac{g_L(t^*)}{8t^*}(j_2-j_1)T} \right) \\ &\lesssim \frac{\widetilde{\mathcal{H}}(1,S)\lfloor T \rfloor^2 u^{-1/2}}{T\sqrt{(2\pi)^2|\Sigma_*|}} e^{-\frac{u}{2}g_L(t^*)} \int_{\mathbb{R}} e^{-\frac{b_0}{4}x^2} dx \sum_{k=1}^{\infty} e^{-\frac{g_L(t^*)}{8t^*}kT}. \end{aligned}$$

Finally, we consider $\Pi_{22}(u)$. Note that

$$\Pi_{22}(u) = \sum_{j=-N_u^{(1)}}^{N_u^{(1)}} p_{j;u} + p_{j+1;u} - \widetilde{p}_{j;u},$$

where

$$\widetilde{p}_{j;u} = \mathbb{P}\left\{ \exists_{(t,s)\in(t^*+\frac{jT}{u}, t^*+\frac{jT}{u})+u^{-1}\triangle_{2T,S}} X_1(t) > \sqrt{u}(1+\mu_1 t), X_2(s) > \sqrt{u}(1+\mu_2 s) \right\}.$$

Consequently, the claim for $\Pi_{22}(u)$ follows directly by using Lemma 4.5. $\square$

A.8. **Proof of Lemma 5.1.** Similarly as in (27) we obtain

$$\begin{aligned} P_{\theta_0,0}(u) &\leq \mathbb{P}\left\{\exists_{(t_1,s_1)\in U_{11}}\ Z(t_1,s_1) > \sqrt{u}g(t_1,s_1),\ \ \exists_{(t_2,s_2)\in U_{12}}\ Z(t_2,s_2) > \sqrt{u}g(t_2,s_2)\right\} \\ &\leq \mathbb{P}\left\{\exists_{(t_1,s_1)\in U_{11}}\ \overline{Z}(t_1,s_1) > \sqrt{u}\sqrt{g(t_0,s_0)},\ \ \exists_{(t_2,s_2)\in U_{12}}\ \overline{Z}(t_2,s_2) > \sqrt{u}\sqrt{g(s_0,t_0)}\right\} \\ &\leq \mathbb{P}\left\{\exists_{(t_1,s_1)\in U_{11},(t_2,s_2)\in U_{12}}\ \overline{Z}(t_1,s_1) + \overline{Z}(t_2,s_2) > 2\sqrt{u}\sqrt{g(t_0,s_0)}\right\}, \end{aligned}$$

where, we used the fact that $g(t_0,s_0) = g(s_0,t_0) \leq \inf_{(t,s)\in(U_{11}\cup U_{12})} g(t,s)$, and

$$\overline{Z}(t_i,s_i) := \frac{Z(t_i,s_i)}{\sqrt{\mathrm{Var}(Z(t_i,s_i))}} = \frac{Z(t_i,s_i)}{\sqrt{g(t_i,s_i)}}, \quad i = 1,2.$$

Further note that

$$\begin{aligned} \mathbb{E}\left\{Z(t_0,s_0)Z(s_0,t_0)\right\} &= \mathbb{E}\left\{(2\mu X_1(t_0) + 2(1-2\rho)\mu X_2(s_0))(2(1-2\rho)\mu X_1(s_0) + 2\mu X_2(t_0))\right\} \\ &= 8(1+2\rho)(1-\rho)\mu. \end{aligned}$$

We obtain

$$\begin{aligned} \mathbb{E}\left\{(\overline{Z}(t_0,s_0) + \overline{Z}(s_0,t_0))^2\right\} &= 2 + 2\mathbb{E}\left\{\overline{Z}(t_0,s_0)\overline{Z}(s_0,t_0)\right\} \\ &= 2 + 2\frac{\mathbb{E}\left\{Z(t_0,s_0)Z(s_0,t_0)\right\}}{g(t_0,s_0)} \\ &= 2 + 2(1+2\rho) < 4. \end{aligned}$$

Thus, for sufficiently small $\theta_0 > 0$,

$$\sigma^2 := \sup_{\substack{(t_1,s_1)\in U_{11} \\ (t_2,s_2)\in U_{12}}} \mathbb{E}\left\{(\overline{Z}(t_1,s_1) + \overline{Z}(t_2,s_2))^2\right\} < 4,$$

where we use continuity of the functions involved. Again, using the Borell-TIS inequality we obtain

$$\mathbb{P}\left\{\exists_{(t_1,s_1)\in U_{11},(t_2,s_2)\in U_{12}}\ \overline{Z}(t_1,s_1) + \overline{Z}(t_2,s_2) > 2\sqrt{u}\sqrt{g(t_0,s_0)}\right\} \leq e^{-\frac{(2\sqrt{u}\sqrt{g(t_0,s_0)}-C_0)^2}{2\sigma^2}}$$

holds for all large u such that

$$2\sqrt{u}\sqrt{g(t_0,s_0)} > C_0 := \mathbb{E}\left\{\sup_{\substack{(t_1,s_1)\in U_{11} \\ (t_2,s_2)\in U_{12}}} (\overline{Z}(t_1,s_1) + \overline{Z}(t_2,s_2))\right\}.$$

Thus, the claim follows. ☐

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