SMOOTH CENTRALLY SYMMETRIC POLYTOPES IN DIMENSION 3 ARE IDP

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ABSTRACT. In 1997 Oda conjectured that every smooth lattice polytope has the integer decomposition property. We prove Oda's conjecture for centrally symmetric 3-dimensional polytopes, by showing they are covered by lattice parallelepipeds and unimodular simplices.

1. Introduction

A **lattice polytope** in $\mathbb{R}^d$ is the convex hull of finitely many points in the integer lattice $\mathbb{Z}^d$. All polytopes in this paper will be assumed to be lattice polytopes. They appear naturally in a variety of different fields, such as combinatorics, commutative algebra, toric geometry and optimization, where their geometric and arithmetic behavior has been intensively studied in recent decades. In [5], Oda posed the following fundamental problem:

Problem 1.1. *Given two lattice polytopes $P, Q \subseteq \mathbb{R}^d$, when can every lattice point p in the Minkowski sum $P + Q := \{x + y : x \in P, y \in Q\}$ be written as the sum of two lattice points $p_1 \in P$ and $p_2 \in Q$, i.e., $p = p_1 + p_2$?*

In general, for arbitrary lattice polytopes, not every lattice point in $P + Q$ is the sum of a lattice point in P and a lattice point in Q, not even in the special case $P = Q$. For example, let P be the convex hull of $(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 2, 1)$ and consider $2P$. Then $2P$ contains the lattice point $(1, 1, 1)$ but this cannot be written as the sum of any two lattice points in P. Of particular interest in this context are so-called *IDP polytopes* – a lattice polytope has the **Integer Decomposition Property** (or is **IDP** for short) if for every integer $n \geq 1$ and every lattice point $p \in nP \cap \mathbb{Z}^d$ there are lattice points $p_1, \ldots, p_n \in P \cap \mathbb{Z}^d$ such that $p = p_1 + \cdots + p_n$. IDP polytopes are of great interest when studying the arithmetic behavior of dilated polytopes (*Ehrhart theory*) as well as in commutative algebra and toric geometry. The following basic fact will play a crucial role in this note:

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Proposition 1.2 (See, e.g., [1]). *Unimodular simplices, parallelepipeds, and zonotopes are IDP.*

A natural notion in toric geometry is that of a smooth polytope: a lattice polytope P is **smooth** if it is simple and if its primitive edge directions at every vertex form a basis of the lattice $(\operatorname{aff} P) \cap \mathbb{Z}^d$. In particular, every face of a smooth lattice polytope is itself smooth.

Due to its relation with projective normality of projective toric varieties, the following specialization of Problem 1.1 was also asked by Oda [5]. It has since become known as *Oda's Conjecture*.

Problem 1.3 (Oda's Conjecture). *Is every smooth lattice polytope IDP?*

The purpose of this note is to prove the following case of Oda's conjecture.

Theorem 1.4. *Every centrally symmetric* 3*-dimensional smooth polytope is IDP.*

We have organized the paper as follows. In Section 2 we recall some basic facts about smooth lattice polytopes which we will apply in the proof of Theorem 1.4. In Section 3 we provide a proof of Theorem 1.4. We have structured the crucial steps of the proof into subsequent subsections. Finally in Section 4 we conclude the paper with some open questions which might help to settle Problem 1.3 for the 3-dimensional case.

2. Preliminaries

The following lemma is an immediate consequence of having IDP.

Lemma 2.1 ([2, p. 65]). *Let $P, P_1, \dots, P_m \subseteq \mathbb{R}^d$ be lattice polytopes such that $P = P_1 \cup \cdots \cup P_m$. If $P_1, \dots, P_m$ are IDP, then so is P.*

From the definition of a smooth lattice polytope, the following fact straightforwardly follows.

Lemma 2.2. *Let $P \subseteq \mathbb{R}^d$ be a smooth d-dimensional lattice polytope. Let v be a vertex of P and let $p_1, \dots, p_d$ denote the primitive ray generators on the edges on v. Then the parallelepiped spanned by $p_1, \dots, p_d$ from v does not contain any lattice points aside from its vertices.*

The following two lemmas are known to the experts – we include them for the sake of completeness. We start by introducing some notation.

Definition 2.3. Let P be a polytope and a a linear function. For a real number c, let P_c be the hyperplane cut of P:

$$P_c := \{x \in P \mid a(x) = c\}.$$

We call c *special* if P_c contains a vertex of P. For fixed P and a the set of special c's is finite.

Recall that a fan Σ is said to *coarsen* another fan Σ' if any $\sigma' \in \Sigma'$ is contained in some cone $\sigma \in \Sigma$. We refer to [2, Section 1] for details and references on fans.

In the following lemma, we assume the notation as in Definition 2.3.

Lemma 2.4. *For $c_1 < c_2$ the normal fans of P_{c_1} and P_{c_2} coincide if the interval $[c_1, c_2]$ does not contain special values. If c_2 is the only special value in this interval, then the normal fan of P_{c_2} coarsens that of P_{c_1} (see Figure 1).*

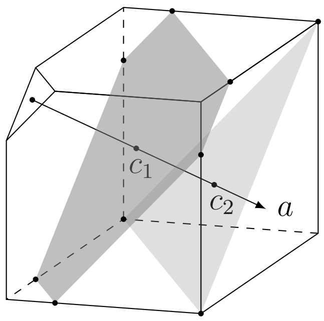

FIGURE 1. Illustration of Lemma 2.4.

Proof. This is a consequence of [6, Lemma 2.2.2], where we regard the hyperplane cuts P_c as fibers of a projection defined by a, from the polytope P to the line. See also [7, Lemmas 2.4.12 and 13]. □

Lemma 2.5. *Let $P \subseteq \mathbb{R}^d$ be a smooth d-dimensional lattice polytope, F a facet of P and $a\colon \mathbb{R}^d \to \mathbb{R}$ the primitive linear functional defining F, i.e., $a(\mathbb{Z}^d) = \mathbb{Z}$, $F = \{x \in P \mid a(x) = c\}$ for some $c \in \mathbb{Z}$ and $a(x) \geq c$ for all $x \in P$. Then $F' := P_{c+1}$ is a lattice polytope whose normal fan coarsens that of F.*

Proof. As P is simple all but one of the edge directions from each vertex of F lie in F. Further the smoothness condition implies that there is a lattice point on any edge adjacent to a vertex in F but not contained in F at lattice distance 1 from the affine hull of F. Hence F' is the convex hull of primitive ray generators of edges adjacent to the vertices in F, but not belonging to F.

The statement about the normal fan is a general fact about simple polytopes. Let $P' \supset P$ be a (not necessarily lattice) polytope with the same normal fan as P constructed as follows: The supporting hyperplanes of P' coincide with those of P, apart from the hyperplane supporting F, which is shifted parallelly by $1 \gg \epsilon > 0$ in the outer direction. As P is simple there are no vertices of P' in P'_c (recall that a vertex is contained in at least d facets). The values in $[c, c+1)$ are nonspecial for P', as a is primitive. Further, for $l \in [c, c+1]$ we have $P'_l = P_l$. By Lemma 2.4, $P'_{c+1} = P_{c+1}$ may only have a fan that coarsens that of $F = P'_c$. □

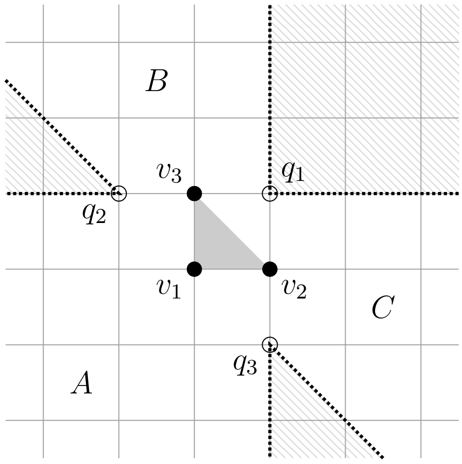

FIGURE 2. Illustration of the proof of Lemma 3.1.

3. Proof of the Main Result

3.1. Covering of Lattice Polygons.

Lemma 3.1. *Let $F \subseteq \mathbb{R}^2$ be a smooth lattice polygon. Every unimodular simplex $\Delta \subsetneq F$ can be extended to a lattice unit square in F.*

Proof. After a unimodular transformation, we may assume that Δ is the standard simplex, i.e., the central triangle in Figure 2. Assume to the contrary that Δ cannot be extended to a unit square. This means that the three points q_1, q_2 and q_3 in Figure 2 are not contained in F. By convexity, it follows that F does not contain any lattice point in the three shaded regions. On the other hand, we assumed that $\Delta \neq F$, so F has to contain at least one further lattice point besides v_1, v_2 and v_3. Without loss of generality, we may assume that there is another lattice point in the region A. Further, by symmetry, we may even assume that there is a lattice point in A that is strictly to the left (and possibly below) of v_1 with respect to Figure 2.

This implies that all further lattice points in region B have to lie on the vertical line through v_3, as otherwise q_2 would lie in F. Let v be the point furthest up on this line, where $v = v_3$ is possible. This is a vertex of F, and we consider the parallelepiped spanned by the two primitive ray generators on the edges on it. One of the edges goes down and leftwards into region A, but misses v_1. The other one goes down and rightwards into region C, possibly hitting v_2. Hence, v_1 lies in the interior of the parallelepiped, contradicting Lemma 2.2. □

3.2. Pushing Facets.

Lemma 3.2. *Let $P \subseteq \mathbb{R}^3$ be a 3-dimensional, smooth lattice polytope with a facet F that is a unimodular triangle. Then (up to translation) the section of P defined in Lemma 2.5 coincides with rF for some integer $r \geq 0$.*

If P has interior lattice points, (in particular, if $P = -P$) then $r \geq 2$.

Proof. The normal fan of F has no proper coarsenings. Hence, by Lemma 2.5, F and F' are similar and since F is a unimodular triangle and F' is a lattice polytope, $F' = rF$ for some integer $r \geq 0$. We note that if $r = 0$ or $r = 1$ then P does not contain interior lattice points. □

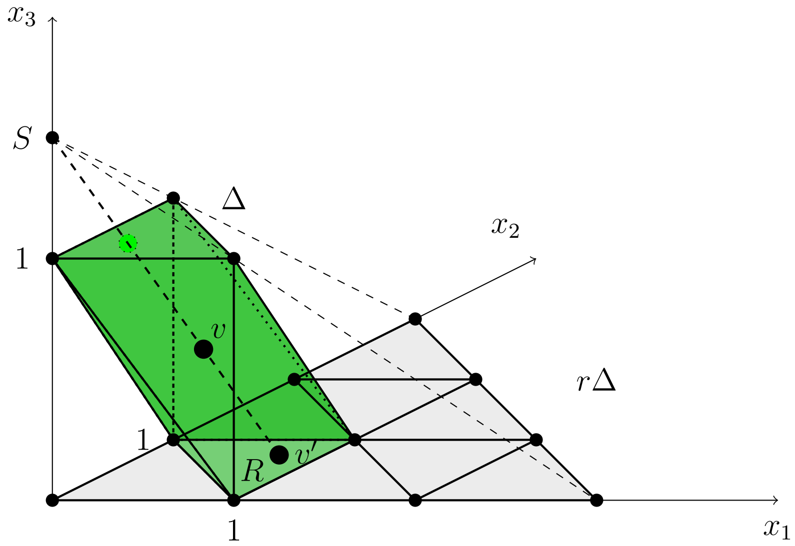

FIGURE 3. Illustration of the proof of Lemma 3.3.

Lemma 3.3. *Let $\Delta \subseteq \mathbb{R}^2$ be a unimodular triangle and $r \geq 1$ an integer. Then the Cayley polytope of Δ and its r-th dilate, i.e., $Q = \operatorname{conv}((\Delta, 1), (r\Delta, 0)) \subseteq \mathbb{R}^3$, can be covered by unimodular simplices. In particular, it is IDP.*

Proof. The following straightforward argument shows that Q can be covered by lattice polytopes isomorphic to either $\operatorname{conv}((\Delta, 1), (\Delta, 0))$ or $\operatorname{conv}((\Delta, 1), (-\Delta, 0))$ as illustrated by Figure 3:

The statement is clear when $r = 1$. Let $r \geq 2$. Every dilate $r\Delta$ can be triangulated by translates of Δ and $-\Delta$. Let v be a point in Q and let S be the *center of similarity* of Δ and $r\Delta$, i.e., the center of the scaling transformation which in our case is $S = (0, 0, r/(r-1)) \in \mathbb{R}^3$. Let v' be the intersection of the straight line connecting S and v with the hyperplane $\{x_3 = 0\}$ and let R be a triangle in the triangulation containing v'. Then v is contained in $\operatorname{conv}((\Delta, 1), (R, 0))$.

The polytopes $\operatorname{conv}((\Delta, 1), (\Delta, 0))$ and $\operatorname{conv}((\Delta, 1), (-\Delta, 0))$ in turn are easily seen to have a unimodular triangulation since every 3-dimensional lattice simplex contained in $\operatorname{conv}((\Delta, 1), (\Delta, 0))$ and $\operatorname{conv}((\Delta, 1), (-\Delta, 0))$ is unimodular. One can say much more on triangulations of such polytopes e.g. by the Cayley trick [4,8]. □

3.3. **Conclusion.**

Proof of Theorem 1.4. By Lemma 2.1, it suffices to cover P by parallelepipeds and unimodular simplices. Let $v \in P$ be distinct from 0. Let v' be the intersection of the half ray $\mathbb{R}_{\geq 0}v$ with a facet F of P.

(1) If F is not a unimodular simplex, then by Lemma 3.1 there exists a unit square D such that $v' \in D \subseteq F$. Hence $v \in \operatorname{conv}(D, -D)$, which is a parallelepiped since it is unimodularly equivalent to the parallelepiped spanned by $(1,0,0), (0,1,0), (2a+1, 2b+1, 2\ell)$, where ℓ is the lattice distance of D from the origin and a, b are two integers.

(2) If F is a unimodular simplex let F' be as in Lemma 3.2. If $v \in \operatorname{conv}(F, F')$ we are done by Lemma 3.3. Otherwise, let $\tilde{v}$ be the intersection of the half ray $\mathbb{R}_{\geq 0}\, v$ with F'. We proceed as in point (1) replacing v' by $\tilde{v}$. □

Example 3.4. Let $C_d = [-1,1]^d \subset \mathbb{R}^d$ and consider its n-th dilate nC_d. Then nC_d is a centrally symmetric smooth polytope. By *chiseling off* antipodal vertices of nC_d at distance 1, there appear two unimodular facets and the smoothness is preserved. (See, e.g., [3] for details on chiselings.) Successive chiselings give us various examples of centrally symmetric smooth polytopes containing unimodular facets.

4. Summary

We have proved that any centrally symmetric 3-dimensional smooth polytope P is covered by parallelepipeds and unimodular simplices. It would be desirable to strengthen the statement to show that P admits a unimodular covering. This would follow from a positive answer to one of the following questions.

Question 4.1. *Do 3-dimensional parallelepipeds admit a unimodular covering? Do centrally symmetric parallelepipeds of the form* $\operatorname{conv}(D, -D)$ *where D is a unit square admit a unimodular covering?*

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