RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

(A global point of view)

BY

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ABSTRACT

This work deals with some of the fundamental aspects of retarded functional differential equations (RFDE's) on a differentiable manifold. We start off by giving a solution of the Cauchy initial value problem for a RFDE on a manifold X. Conditions for the existence of global solutions are given.

Using a Riemannian structure on the manifold X, a RFDE may be pulled back into a vector field on the state space of paths on X. This demonstrates a relationship between vector fields and RFDE's by giving a natural embedding of the RFDE's on X as a submodule of the module of vector fields on the state space. For a given RFDE it is shown that a global solution may level out asymptotically to an equilibrium path.

Each differentiable RFDE on a Riemannian manifold linearizes in a natural way, thus generating a semi-flow on the tangent bundle to the state space. Sufficient conditions are given to smooth out the orbits and to obtain the stable bundle theorem for the semi-flow

There are examples of RFDE's on a Riemannian manifold. These include the vector fields, the differential delay equations, the delayed Cartan development and equations of Levin-Nohel type. The retarded heat equation on a compact manifold provides an example of a partial RFDE on a function space.

We conclude by making suggestions for further research.

INTRODUCTION

In this thesis we attempt to lay the foundations of global retarted functional differential equations (RFDE's) on a differentiable manifold. As this is as yet a largely unknown area it seems best that one should start by a description of the general framework in which we operate. Manifolds shall in general be infinite-dimensional and modelled on real Banach (or Hilbert) spaces, unless otherwise indicated. (Eells, [12] ; Lang [32]).

Let X be a manifold, J the negative closed interval $[-r,0]$ for $r > 0$ - called the <u>interval of retardation</u> - and $\mathcal{P}(J,X)$ a manifold of paths $J \to X$ lying within the manifold $\mathcal{C}^0(J,X)$ of continuous paths on X. A (global) <u>time-dependent RFDE</u> F on X is a map $F: [o,K) \times \mathcal{P}(J,X) \to TX$ s.t. $K > 0$ and for each $t \in [o,K)$, $\theta \in \mathcal{P}(J,X)$, the vector $F(t,\theta)$ belongs to the tangent space $T_{\theta(o)}X$ at $\theta(o) \in X$. The autonomous RFDE $F: \mathcal{P}(J,X) \to TX$ is defined in the obvious manner. Solutions of F are sought as paths $\alpha \in \mathcal{P}([-r,\varepsilon),X)$ for some $0 < \varepsilon \leqslant K$ s.t.

$$\begin{aligned} \alpha'(t) &= F(t,\alpha_t) \qquad t \in [o,\varepsilon) \\ \alpha_o &= \theta \in \mathcal{P}(J,X) \end{aligned} \tag{1}$$

where $\alpha_t \in \mathcal{P}(J,X)$, $t \in [o,\varepsilon)$, is defined by $\alpha_t(s) = \alpha(t+s)$ $\forall s \in J$. The initial value problem (1) is the Cauchy problem for RFDE's.

With the exception of a paper by Oliva in 1969 ([38]) on the case $\mathcal{P} = \mathcal{C}^o$, it appears from a study of the existing literature that the problems of RFDE's are not sufficiently well-treated within the above setting. On the other hand the continuous flat case: $\mathcal{P} = \mathcal{C}^o$, $X = R^n$ is a beaten track which has been the subject of vigorous research during the last few decades; this case will therefore not be emphasized in the present thesis, but we shall be preoccupied most of the time with situations in which the ground space X - and hence the state space $\mathcal{P}(J,X)$ - are non-linear. Flat cases in which X is an infinite dimensional linear space, e.g. a function space or a space of sections of a vector bundle, are also interesting because they constitute a natural setting for retarded partial functional differential equations (See Example 4 of Chapter 4, and also [18]).

The RFDE (1) and its autonomous version present us with four major questions:

(i) The classical Cauchy problem of finding unique local and global (i.e. full) solutions of F for a given initial path $\theta \in \mathcal{P}(J,X)$;

(ii) Are there any relationships between antonomous RFDE's and vector fields? What does the critical set $C(F) \equiv \{\theta \in \mathcal{P}(J,X): F(\theta) = 0\}$ of a RFDE look like, and how does its topology relate to that of the ground manifold X?

(iii) Can the autonomous RFDE

$$\alpha'(t) = F(\alpha_t) \qquad t \geq 0 \qquad (2)$$

be linearized in a satisfactory manner, and what are the implications of this linearization upon the behaviour of solutions particularly wrt growth and stability?

(iv) Does the differential equation (2) embrace any examples which are interesting from the global analytic point of view described above?

As a whole this thesis is a contribution to the subject of global RFDE's because it endeavours to attack the hitherto open questions (i) to (iv) by developing some new techniques or by otherwise adopting well-known geometric ideas and applying them in order to answer the above questions.

So X is endowed with a Riemannian structure and for the state space $\mathcal{P}(J,X)$ to be also Riemannian we find it convenient to choose the Sobolev paths $\mathcal{L}^2_1(J,X)$ i.e. $\mathcal{P} = \mathcal{L}^2_1$ (See Chapter 1, §1). This choise is advantagous over that of the continuous paths $\mathcal{C}^0(J,X)$ which is a manifold modelled on a non-Hilbertable Banach space. Moreover, the $\mathcal{L}^2_1$ paths are sufficiently differentiable for parallel transport to be smoothly defined over the whole of the state space $\mathcal{L}^2_1(J,X)$ (Theorem 2.2). Thus in all our considerations, and also for the sake of unification, we shall confine ourselves to the Sobolev ($\mathcal{L}^2_1$) case rather than the continuous ($\mathcal{C}^0$) one, while the latter is only referred to in passing remarks and suggestions.

The thesis falls into five chapters. Each of the first four chapters is primarily intended to shed some light on one of the above mentioned major topics (i), (ii), (iii) and (iv).

Chapter I uses a new localization technique (Lemma 1.1) to solve the Cauchy initial value problem for a RFDE F on a Banach manifold X which admits a linear connection. Our main contributions here are the local existence and uniqueness theorem (Theorem 1.1), together with Theorem (1.5) and its corollary which give sufficient growth conditions on F to guarantee full solutions defined for all future times.

In Chapter 2 we discuss the general relationships between RFDE's and vector fields on the state space $\mathcal{L}^2_1(J,X)$ with an eye towards the topological structure of the critical set C(F) of an antonomous RFDE F. The Chapter starts off with a new theorem (viz. Theorem 2.1) saying that solutions of the RFDE (2) may reach equilibrium by converging asymptotically to a constant critical path, a behaviour which is analogous to that of trajectories of vector fields. We then go on to introduce the main idea which is to show that a smooth RFDE F pulls back by the Riemannian structure into a smooth vector field ξ^F on the state space $\mathcal{L}_1^2(J,X)$ (Theorem 2.2). As a consequence of this construction the differentiable RFDE's on X are embedded as a sub-module of the module of vector fields on $\mathcal{L}_1^2(J,X)$ over the ring of differentiable functions on $\mathcal{L}_1^2(J,X)$ (Corollaries 2.2.1,2.2.2). The vector field ξ^F is again used to define a class of gradient RFDE's (§2.4) for which the Morse inequalities hold (Theorem 2.4). There were two main stumbling blocks in the course of the development here: the high degree of degeneracy of C(F), and a workable definition of the Morse index of a critical path in C(F). The first difficulty is overcome by taking a viewpoint of Bott ([4]) which amounts to counting components of C(F) rather than the individual critical paths; the second difficulty is resolved by proving Theorem (2.3) to get an explicit formula for the Hessians of F and ξ^F at a critical path. Almost all the results in this Chapter are new except perhaps for Proposition (2.3) which is well-known ([37]) and Proposition (2.5) which was first proved by Bott in the compact case ([4]); our proof of this last proposition is however carried out independently of Bott's and we believe that it can be made to work even when X is infinite dimenstional.

The fundamental question (iii) of linearization is treated in Chapter 3. Here the vector field ξ^F of Chapter 2 is differentiated covariantly along the path space $\mathcal{L}_1^2(J,X)$. It then turns out that this linearization defines a linear semi-flow $\{T_t\}_{t\geq 0}$ on the tangent bundle

$T\mathcal{L}_1^2(J,X)$ (Theorem 3.3). Along the fibres of $T\mathcal{L}_1^2(J,X)$ the methods of strongly continuous linear semi-groups of operators apply giving the stable bundle theorem (Theorem 3.6). These semi-groups methods were applied by Shimanov and Hale to the continuous linear case with $X = R^n$, $\mathcal{P} = \mathcal{C}^0$ ([43],[21]), and our proof of the stable bundle theorem follows Hale closely. Since the linearization consists essentially in differentiating the differential equation (2) covariantly wrt t, this entails some technicalities in establishing smoothness properties of the semi-flow wrt time - mainly because of the Sobolev topology. As a by-product we obtain a general theorem on the smoothness of orbits of the non-linear RFDE F (Theorem 3.1), together with an estimate on the growth of time derivatives of orbits of solutions of F (Corollary 3.3.1). Throughout this chapter two main tool results are frequently used: the well-known Sobolev embedding theorem (Theorem 3.2) and a geometric "bridge" lemma (Lemma 3.2) which is probably new and in any case we provide an independent proof valid when X is finite dimensional. Another new result is Corollary (3.4.1) which gives a criterion for the orbit of a full solution to contain a geodesic segment in X.

The relationship between vector fields and RFDE's is again emphasized in Chapter 4 by way of examples. Vector fields on the ground manifold X are used to construct RFDE's. Among the RFDE's thus obtained are the ODE's (i.e. the non-retarded ones), the differential delay equations (DDE's), the delayed development, and equations of Levin-Nohel type. Theorem (4.1) says that in the gradient case equations of Levin-Nonel type on a Riemanian manifold may not admit non-trivial periodic solutions. Our final contribution in this direction is an example on the retarded heat equation (RHE) as a special case of retarded parabolic partial differential equations.

This is actually shown to be a discontinuous - but closed - RFDE on the linear Fréchet space of smooth functions on a compact manifold. Because of the linearity and symmetry of the situation, and despite the discontinuity of the equation and the infinite dimensionality of the ground space, the RHE still displays very similar dynamical properties to those of the continuous finite dimensional case of Chapter 3. One basic difference however is that the RHE can in general be solved in the forward direction only along a closed Frechet subspace of the state space; if the equation is hyperbolic (See §5 Chapter 4), then backward solutions do exist on the complementary subspace. The delayed heat equation (DHE) is also of interest because then solutions exist on the whole of the state space.

Chapter 5 is the last chapter, and it sketches - in terms of conjectures - new horizons for further development and generalizations of the ideas and results of the previous chapters. Some of these conjectures are almost certainties and we believe that they may become theorems as soon as the loose ends are successfully tied up. The rest of the conjectures - especially those concerned with the continuous case $\mathcal{P} = \mathcal{C}^0$ - are still in a wild state at present, but there are reasons to expect that they can be tamed in the future by extrapolating on the ideas of Chapters 2, 3 of this thesis.

CHAPTER I

The Cauchy Problem

We give a solution of the classical initial value problem of Cauchy for a retarded functional differential equation on a Banach manifold. To that end we shall require the following:

1. Preliminaries:

X is a C^P $(p \geqslant 4)$ metrizable manifold without boundary and modelled on a real Banach space E. Let $\pi_0 : TX \to X$ denote the tangent bundle of X, and assume throughout that X admits a C^{P-2} connection (Eliasson [17], Nomizu [37]). Let $0 < K \leqslant \infty$ and $r \geqslant 0$. Set $J = [-r,0]$, the interval of retardation, and denote by $\mathcal{L}^2_1(J,X)$ the collection of all C^0 paths $\theta : J \to X$ s.t. for each $s \in J$, $\exists$ a chart (U,ϕ) at $\theta(s)$ in X with $\phi \circ \theta$ absolutely continuous, $(\phi \circ \theta)'$ defined a.e. and

$$\int_{\theta^{-1}(U)} |(\phi \circ \theta)'(s)|^2_E \, ds < \infty .$$

where $| \ |_E$ denotes the norm in E. Using a construction of Eells ([13]) or otherwise applying a theorem of Eliasson ([17] Theorem 5.1), we see that $\mathcal{L}^2_1(J,X)$ is a C^{p-3} Banach manifold. Furthermore, define the map ρ_0: $[o,K) \times \mathcal{L}^2_1(J,X) \to X$ to be the evaluation at 0. i.e.

$$\rho_0(t,\theta) = \theta(0) \quad \forall \ \theta \in \mathcal{L}^2_1(J,X), \forall t \in [o,K).$$

Then ρ_0 is C^{p-3} because its local representation is the restriction of the evaluation at 0 in the flat model space. Observe that the tangent bundle $T\mathcal{L}^2_1(J,X)$ is naturally identified with $\mathcal{L}^2_1(J,TX)$ (Eliasson [17], Theorem 5.2).

Definition (1.1) (Oliva[38])

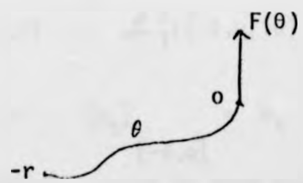
Let F: $[o,K) \times \mathcal{L}^2_1(J,X) \longrightarrow TX$ be a map covering ρ_0, viz. one s.t. the diagram

$$\begin{tikzcd}
 & TX \arrow[d, "\pi_0"] \\
{[0,K)\times\mathcal{L}^2_1(J,X)} \arrow[ur, "F"] \arrow[r, "\rho_0"'] & X
\end{tikzcd}$$

commutes. Then the 4-tuple $(F, [0,K), J, X)$ is called a <u>time-dependent retarded functional differential equation</u> (RFDE) <u>on X with retardation time J</u>. An <u>antonomous RFDE</u> (F,J,X) is defined in the obvious way:

$$\begin{tikzcd}
 & TX \arrow[d, "\pi_0"] \\
\mathcal{L}^2_1(J,X) \arrow[ur, "F"] \arrow[r, "\rho_0"'] & X
\end{tikzcd}$$

i.e. F assigns to each path θ a vector $F(\theta) \in T_{\theta(0)}X$ at its end-point.



<u>Definition (1.2)</u>.

Let $0 < \varepsilon \leq K$ and $\alpha \in \mathcal{L}^2_1([0,\varepsilon),X)$. Define the <u>canonical lift</u> $\alpha' \in \mathcal{L}^2([0,\varepsilon),TX)$ of α via the commutative diagram

$$\begin{tikzcd}
{[0,\varepsilon)\times R} \arrow[dr, shift left, "P_1"] & T[0,\varepsilon) \arrow[l, "\cong"', "\Psi"] \arrow[r, "T\alpha"] \arrow[d, "\pi'"] & TX \arrow[d, "\pi_0"] \\
 & {[0,\varepsilon)} \arrow[ul, shift left, "\iota"] \arrow[ur, dashed, "\alpha'"] \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

where $\bar{\psi}$ is the trivialization of $T[0,\varepsilon)$, P_1 is the projection onto the first factor and ι the canonical section defined by

$$\iota(t) = (t,1) \quad \forall\, t \in [0,\varepsilon)$$

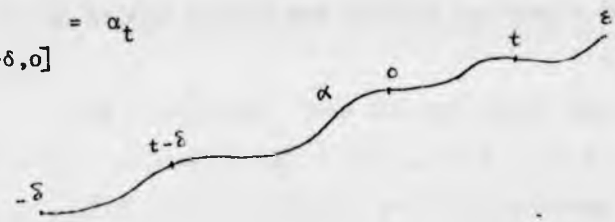
<u>Definition (1.3)</u>:

Let $U \subseteq X$ be open, $0 < \delta \leq r$, $0 < \varepsilon \leq K$, and $\alpha : [-\delta,\varepsilon) \to U$ a (continuous) map. For each $t \in [0,\varepsilon)$ define the map $[\alpha_t]_{[-\delta,0]} : [-\delta,0] \to U$ by

$$[\alpha_t]_{[-\delta,0]}(s) = \alpha(t+s) \qquad \forall\ s \in [-\delta,0]$$

If no ambiguity arises as regards the interval $[-\delta,0]$ we may write

$$[\alpha_t]_{[-\delta,0]} = \alpha_t$$



We therefore have the "<u>memory map</u>"

$$m : [0,\varepsilon) \times \mathcal{L}^2_1([-\delta,\varepsilon),U) \longrightarrow \mathcal{L}^2_1([-\delta,0],U)$$

$$(t,\alpha) \longmapsto [\alpha_t]_{[-\delta,0]} = \alpha_t$$

with "<u>past history</u>" $[-\delta,0]$. Thus at each time t, m picks up the slice of α on $[t-\delta,t]$ and shifts it to the left by t.

The RFDE $(F, [0,K),J,X)$ is said to have a <u>local solution with initial path</u> $\theta \in \mathcal{L}^2_1(J,X)$ if $\exists\ 0 < \varepsilon \leq K$ and $\alpha \in \mathcal{L}^2_1([-r,\varepsilon),X)$ s.t. $\alpha|[0,\varepsilon)$ is C' and

$$\alpha'(t) = F(t,[\alpha_t]_{[-r,0]}) \qquad \forall\, t \in [0,\varepsilon)$$

$$[\alpha_0]_{[-r,0]} = \theta$$

It will turn out that the smoothness properties of the memory map m are essential to the study of the general behaviour of solutions of RFDE's, and will be discussed in greater detail later on.

§2. Local Existence and Uniqueness:

The main objective of this section is to establish existence and uniqueness of a local solution for the RFDE $(F, [o,K),J,X)$ with given initial path $\theta \in \mathcal{L}^2_1(J,X)$. This is achieved by imposing sufficient and reasonable smoothness conditions on the manifold X and the RFDE F. The key step in that direction is to localize F via a "localizing map" whose existence is guaranteed, in a canonical manner, by the following lemma. In this, $(F, [o,K),J,X)$ satisfy the standing hypotheses of §1.

Lemma (1.1):

Let $\theta \in \mathcal{L}^2_1(J,X)$. Then for each chart (U,ϕ) at $\theta(o)$ in X $\exists\ o < \delta \leq r$ s.t. $\theta([-\delta,o]) \subset U$, and if $o < \varepsilon \leq \delta$, $\exists$ a map $C : [o,\varepsilon) \times \mathcal{L}^2_1([-\delta,o],U) \to \mathcal{L}^2_1(J,X)$ with the following properties.

i) the diagram

$$\begin{tikzcd} & X & \\ {[o,\varepsilon)\times\mathcal{L}^2_1([-\delta,o],U)} \arrow[ur, "\rho_o"] \arrow[rr, "C"'] & & \mathcal{L}^2_1(J,X) \arrow[ul, "\rho_o"'] \end{tikzcd}$$

commutes, where ρ_o is evaluation at o.

ii) if $\beta \in \mathcal{L}^2_1([-\delta,\varepsilon),U)$ is s.t. $\beta|[-\delta,o] = \theta|[-\delta,o]$ and $\alpha \in \mathcal{L}^2_1([-r,\varepsilon),X)$ is defined by

$$\alpha(t) = \begin{cases} \theta(t) & t \in J \\ \beta(t) & t \in [o,\varepsilon) \end{cases},$$

then for each $t \in [o,\varepsilon)$

$$[\alpha_t]_{[-r,o]} = C(t, [\beta_t]_{[-\delta,o]})$$

iii) Define the sets

$$Y_\theta^U \equiv \{(t,\gamma) : t \in [o,\varepsilon),\ \gamma \in \mathcal{L}_1^2([-\delta,o],U)\ ,\ \ \theta(t-\delta) = \gamma(-\delta)\}$$

$$\subset [o,\varepsilon) \times \mathcal{L}_1^2([-\delta,o],U).$$

$$Y_\theta^U(t) \equiv \{\ \gamma : \gamma \in \mathcal{L}_1^2([-\delta,o],U)\ ,\ \ \theta(t-\delta) = \gamma(-\delta)\}\ ,\ \text{for each}$$

$t \in [o,\varepsilon)$. Then Y_θ^U is closed in $[o,\varepsilon) \times \mathcal{L}_1^2([-\delta,o],U)$; and, for each $t \in [o,\varepsilon)$, $Y_\theta^U(t)$ is a closed C^{p-3} submanifold of $\mathcal{L}_1^2([-\delta,o],U)$, where we take $\mathcal{L}_1^2([-\delta,o],U)$ to be naturally embedded as an open C^{p-3} submanifold of $\mathcal{L}_1^2(J,X)$. Moreover, $C|Y_\theta^U$ is continuous and each $C(t,.)|Y_\theta^U(t)$, $t\in [o,\varepsilon)$, is of class C^{p-3}.

<u>Proof</u>.

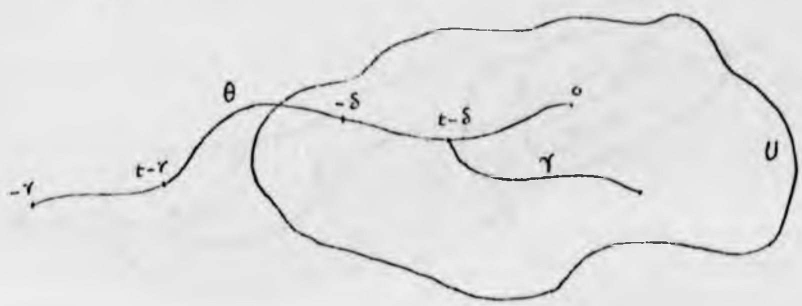
By continuity of θ at o, for each chart (U,ϕ) at $\theta(o) \ \exists\ 0 < \delta \leq r$ s.t. $\theta\{[-\delta,o]\} \subset U$. For $o < \varepsilon \leq \delta$ define C as follows: if $(t,\gamma) \in Y_\theta^U$, write

$$C(t,\gamma)(s) = \begin{cases} \theta(s+t) & s \in [-r,-\delta] \\ \gamma(s) & s \in [-\delta,o] \end{cases} ,$$

if $(t,\gamma) \notin Y_\theta^U$, take

$$C(t,\gamma)(s) = \gamma(o) \qquad \forall\ s \in J$$

i.e. on Y_θ^U C looks like



It is easily seen that C makes the diagram in (i) commutative.

To check (ii), let $\beta \in \mathcal{L}^2_1([-\delta,\varepsilon),U)$ and $\alpha \in \mathcal{L}^2_1([-r,\varepsilon),X)$ be as given. Then, for each $t \in [o,\varepsilon)$, $(t,[\beta_t]_{[-\delta,o]}) \in Y^U_\theta$ and so by definition

$$C(t,[\beta_t]_{[-\delta,o]})(s) = \begin{cases} \theta(s+t) & s \in [-r,-\delta] \\ \beta(s+t) & s \in [-\delta,o] \end{cases}$$

$$= \alpha(s+t) \qquad s \in J$$

$$= [\alpha_t]_{[-r,o]}(s) \qquad s \in J$$

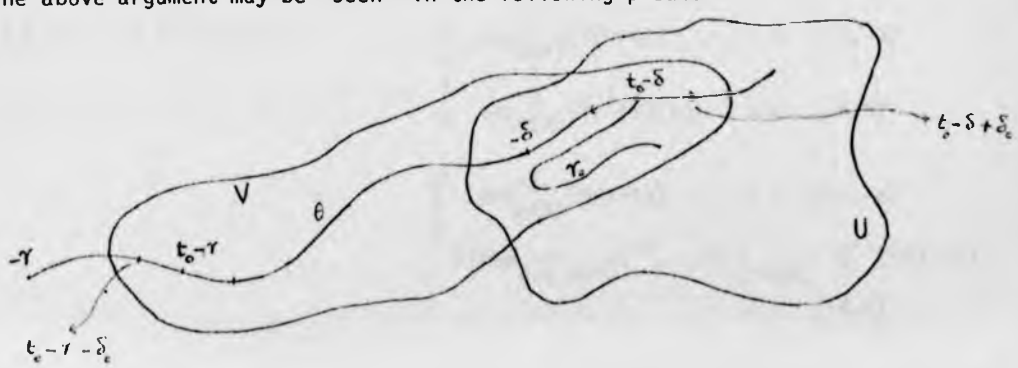
which is the required property.

For the continuity of $C|Y^U_\theta$, it is sufficient to show that for each $(t_o,\gamma_o) \in Y^U_\theta$, $t_o > o$, and each open set V in X s.t. $C(t_o,\gamma_o) \in \mathcal{L}^2_1(J,V)$, $C^{-1}\{\mathcal{L}^2_1(J,V)\}$is a neighbourhood of (t_o,γ_o) in Y^U_θ. In this context, the continuity of θ allows us to choose $\delta_o > o$ s.t.$(t_o-\delta-\delta_o,t_o-\delta+\delta_o) \subset [-\delta,o]$, $(t_o-r-\delta_o,t_o-r+\delta_o) \subset [-r,o]$, $\theta\{(t_o-\delta-\delta_o,t_o-\delta+\delta_o)\} \subseteq V$ and $\theta\{(t_o-r-\delta_o,t_o-r+\delta_o)\} \subseteq V$. But $\gamma_o \in \mathcal{L}^2_1([-\delta,o],V)$ so we define

$$G = \{(t_o-\delta_o,t_o+\delta_o) \times \mathcal{L}^2_1([-\delta,o],V)\} \cap Y^U_\theta$$

then G is an open neighbourhood of (t_o,γ_o) in Y^U_θ. Also by the definition of C and the choice of δ_o, it follows easily that $C(t,\gamma) \in \mathcal{L}^2_1(J,V)$ $\forall (t,\gamma) \in G$.

The above argument may be "seen" in the following picuture



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$$C(t, [\beta_t]_{[-\delta,o]})(s) = \begin{cases} \theta(s+t) & s \in [-r,-\delta] \\ \beta(s+t) & s \in [-\delta,o] \end{cases}$$

$$= \alpha(s+t) \qquad s \in J$$

$$= [\alpha_t]_{[-r,o]}(s) \qquad s \in J$$

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For the continuity of $C|Y^U_\theta$, it is sufficient to show that for each $(t_o,\gamma_o) \in Y^U_\theta$, $t_o > o$, and each open set V in X s.t. $C(t_o,\gamma_o) \in \mathcal{L}^2_1(J,V)$, $C^{-1}\{\mathcal{L}^2_1(J,V)\}$is a neighbourhood of (t_o,γ_o) in Y^U_θ. In this context, the continuity of θ allows us to choose $\delta_o > o$ s.t.$(t_o-\delta-\delta_o, t_o-\delta+\delta_o) \subset [-\delta,o]$, $(t_o-r-\delta_o, t_o-r+\delta_o) \subset [-r,o]$, $\theta\{(t_o-\delta-\delta_o, t_o-\delta+\delta_o)\} \subseteq V$ and $\theta\{(t_o-r-\delta_o, t_o-r+\delta_o)\} \subseteq V$. But $\gamma_o \in \mathcal{L}^2_1([-\delta,o],V)$ so we define

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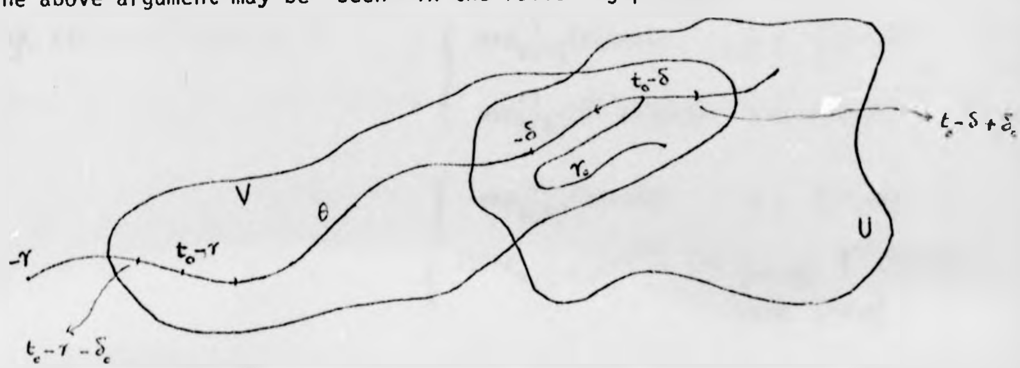
then G is an open neighbourhood of (t_o,γ_o) in Y^U_θ. Also by the definition of C and the choice of δ_o, it follows easily that $C(t,\gamma) \in \mathcal{L}^2_1(J,V)$ $\forall (t,\gamma) \in G$.

The above argument may be "seen" in the following picuture



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Thus ker $\tilde{\rho}_{-\delta}$ splits and $Y_\theta^U(t)$ is a closed C^{p-3} submanifold of $\mathcal{L}_1^2([-\delta,o],U)$, of codimension = dimension of E.

We finally show that for each $t \in [o,\varepsilon)$, $C(t,.)|Y_\theta^U(t)$ is C^{p-3}. Let $\gamma_o \in Y_\theta^U(t)$. Choose a C^{p-2} connection on X. This induces a C^{p-2} exponential map $\exp:\mathfrak{D} \subset TX \rightarrow X$ where $\mathfrak{D}$ is an open neighbourhood of the zero section $(TX)_o$ in TX. Since $C(t,\gamma_o)$ is continuous and J is compact, we can choose $\psi \in \mathcal{C}^p(J,X)$ and a tubular neighbourhood $\mathcal{U} \subset J \times X$ of graph (ψ) through the C^{p-2} diffeomorphism $(\Pi_\psi, \exp)$: $\psi^*(\mathfrak{D}) \rightarrow \mathcal{U} \subset J \times X$ where $\Pi_\psi : \psi^*(\mathfrak{D}) \rightarrow J$ is the pull-back of the disc bundle $\Pi_o|\mathfrak{D}:\mathfrak{D} \longrightarrow X$ over ψ (Lang [32] Chapter III §1) and graph $(C(t,\gamma_o)) \subset \mathcal{U}$. Call this diffeomorphism Exp_ψ. Define a natural chart $(\mathcal{V},\mathcal{Y})$ centred at ψ and containing $C(t,\gamma_o)$ by

$$\mathcal{V} = \{\eta \in \mathcal{L}_1^2(J,X) : \text{graph } (\eta) \subset \mathcal{U}\}$$

$$\mathcal{Y} : \mathcal{V} \rightarrow \Gamma_1^2(\psi^*(\mathfrak{D})) \subset \Gamma_1^2(\psi^*(TX)) \quad ,$$

$$\mathcal{Y}(\eta) = (\mathrm{Exp}_\psi) \circ (\mathrm{id}_J, \eta) \quad ,$$

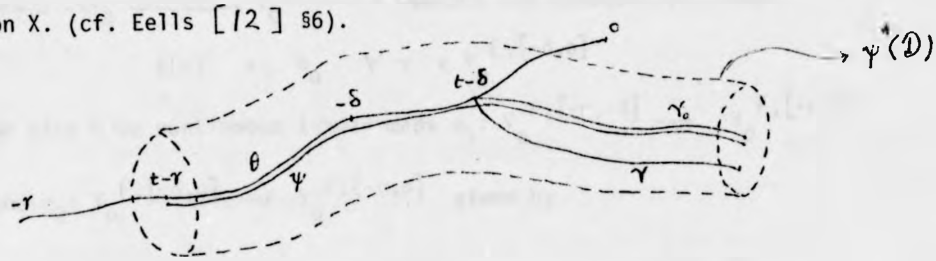
where $\Gamma_1^2(\psi^*(TX))$ is the Banachable space of all $\mathcal{L}_1^2$ sections of the bundle $\psi^*(TX) \rightarrow J$. As $C(t,.)$ is continuous, $\exists$ an open set $\mathcal{N}$ in $\mathcal{L}_1^2([-\delta,o],\phi(U))$ s.t. $\bar{\phi}(\gamma_o) \in \mathcal{N}$ and $\forall \tilde{\gamma} \in \mathcal{N}$, graph $C(t,\tilde{\gamma}) \subset \mathcal{U}$. Because $\bar{\phi}$ is a diffeomorphism, it is sufficient to prove that the composition

$\mathcal{Y} \circ C(t,.) \circ \bar{\phi}^{-1} : \mathcal{N} \cap Y_\theta^{\phi(U)}(t) \rightarrow \Gamma_1^2(\psi^*(\mathfrak{D}))$ is of class C^{p-3}. Now this is given for $\tilde{\gamma} \in \mathcal{N} \cap Y_\theta^{\phi(U)}(t)$ by

$$(\mathcal{Y} \circ C(t,.) \circ \bar{\phi}^{-1})(\tilde{\gamma})(s) = \begin{cases} \exp_{\psi(s)}^{-1}(\theta(s+t)) & s \in [-r,-\delta] \\ \exp_{\psi(s)}^{-1}(\bar{\phi}^{-1}(\tilde{\gamma})(s)) & s \in [-\delta,o] \end{cases}$$

$$= \begin{cases} \exp_{\psi(s)}^{-1}(\theta(s+t)) & s \in [-r,-\delta] \\ \{(\mathrm{Exp}_\psi|[-\delta,o])^{-1} \circ (\mathrm{id}_{[-\delta,o]}, \bar{\phi}^{-1}(\tilde{\gamma}))\}(s) & s \in [-\delta,o] \end{cases}$$

In view of this observation, the differentiability of $C(t,.)$ is then an immediate consequence of the next lemma (Lemma 1.2) and the fact that the differential structure on $\mathcal{L}_1^2(J,X)$ is independent of the connection on X. (cf. Eells [12] §6).



Lemma (1.2):

Let E, F be Banach spaces. Let $\theta_o \in \mathcal{L}_1^2([-r,-\delta],F)$ and $v \in E$. Define the hyperplane

$$\gamma_v^{E,[-\delta,o]} = \{\gamma : \gamma \in \mathcal{L}_1^2([-\delta,o],E), \quad \gamma(-\delta) = v\}$$

Give $\gamma_v^{E,[-\delta,o]}$ and $\gamma_{\theta_o(-\delta)}^{F,[-\delta,o]}$ the sup metrics, and suppose

$d: \gamma_v^{E,[-\delta,o]} \longrightarrow \gamma_{\theta_o(-\delta)}^{F,[-\delta,o]}$ is a Lipschitz ($C^k, k \geq 1$) map.

Define the map $\bar{d} : \gamma_v^{E,[-\delta,o]} \to \mathcal{L}_1^2(J,F)$

by

$$\bar{d}(\gamma)(s) = \begin{cases} \theta_o(s) & s \in [-r,-\delta] \\ d(\gamma)(s) & s \in [-\delta,o] \end{cases}$$

Then $\bar{d}$ is Lipschitz ($C^k, k \geq 1$, resp.).

Proof:

Without loss of generality, take $v = 0$ and $\theta_o(-\delta) = 0$. Let $j : Y_o^{E,[-\delta,o]} \longrightarrow Y_o^{F,[-r,-\delta]}$ denote the constant (C^∞) map

$$j(\gamma) = \theta_o \quad \forall\ \gamma \in Y_o^{E,[-\delta,o]}$$

We also have continuous linear maps $e_1: Y_o^{F,[-r,-\delta]} \longrightarrow Y_o^{F,[-r,o]}$ and $e_2: Y_o^{F,[-\delta,o]} \longrightarrow Y_o^{F,[-r,o]}$ given by

$$e_1(\gamma)(s) = \begin{cases} \gamma(s) & s \in [-r,-\delta] \\ 0 & s \in [-\delta,o] \end{cases}$$

and

$$e_2(\gamma)(s) = \begin{cases} 0 & s \in [-r,-\delta] \\ \gamma(s) & s \in [-\delta,o] \end{cases}$$

Hence $\bar{d} = e_1 \circ j + e_2 \circ d$ is Lipschitz (C^k), being a composition of such maps. Q.E.D.

Remark:

The above lemma (1.2) still holds if E and F were replaced by C^p vector bundles over compact intervals (with some Finslers on them: Eells [12], Eliasson [17] §4, Abraham-Smale [2] Chapter 1 §5), and $\mathcal{L}_1^2([-r,-\delta],F)$, $\mathcal{L}_1^2([-\delta,o],E)$, etc. , by the corresponding Banach spaces of $\mathcal{L}_1^2$ sections of E and F, with d a mapping between the appropriate spaces of sections.

Proof:

Without loss of generality, take $v = 0$ and $\theta_o(-\delta) = 0$. Let $j : Y_o^{E,[-\delta,o]} \longrightarrow Y_o^{F,[-r,-\delta]}$ denote the constant (C^∞) map

$$j(\gamma) = \theta_o \quad \forall \gamma \in Y_o^{E,[-\delta,o]}$$

We also have continuous linear maps $e_1 : Y_o^{F,[-r,-\delta]} \longrightarrow Y_o^{F,[-r,o]}$ and $e_2 : Y_o^{F,[-\delta,o]} \longrightarrow Y_o^{F,[-r,o]}$ given by

$$e_1(\gamma)(s) = \begin{cases} \gamma(s) & s \in [-r,-\delta] \\ 0 & s \in [-\delta,o] \end{cases}$$

and

$$e_2(\gamma)(s) = \begin{cases} 0 & s \in [-r,-\delta] \\ \gamma(s) & s \in [-\delta,o] \end{cases}$$

Hence $\bar{d} = e_1 \circ j + e_2 \circ d$ is Lipschitz (C^k), being a composition of such maps. Q.E.D.

Remark:

The above lemma (1.2) still holds if E and F were replaced by C^p vector bundles over compact intervals (with some Finslers on them: Eells [12], Eliasson [17] §4, Abraham-Smale [2] Chapter 1 §5), and $\mathcal{L}_1^2([-r,-\delta],F)$, $\mathcal{L}_1^2([-\delta,o],E)$, etc. , by the corresponding Banach spaces of $\mathcal{L}_1^2$ sections of E and F, with d a mapping between the appropriate spaces of sections.

Definition (1.4):

Let $\theta, (U,\phi)$, ε, δ and $C: [0,\varepsilon)\times \mathcal{L}^2_1([-\delta,0],U) \to \mathcal{L}^2_1(J,X)$ be as in Lemma (1.1). Suppose $(F,[0,K),J,X)$ is a RFDE (satisfying the hypotheses of §1). Define the map $\tilde{C}: [0,\varepsilon)\times \mathcal{L}^2_1([-\delta,0],U) \to [0,\varepsilon)\times\mathcal{L}^2_1(J,X)$ by

$$\tilde{C}(t,\gamma) = (t, C(t,\gamma)) \qquad t \in [0,\varepsilon),\ \gamma \in \mathcal{L}^2_1([-\delta,0],U).$$

Call the composition $F \circ \tilde{C}$ a local representation of F at θ, and denote it by F^U_θ. Observe that $(F^U_\theta, [0,\varepsilon), [-\delta,0], U)$ is a RFDE on U. F is said to be locally Lipschitz at θ if $\exists$ a chart (U,ϕ) at $\theta(0)$ in X and a trivialization $\psi: TU = \Pi_0^{-1}(U) \to U\times E$ of TU, so that when $f_0: [0,\varepsilon)\times\mathcal{L}^2_1([-\delta,0],\phi(U)) \to E$ denotes the composite map:

$$\begin{tikzcd}
{[0,\varepsilon)\times \mathcal{L}^2_1([-\delta,0],\phi(U))} \arrow[rr, "\cong", "(id_{[0,\varepsilon)},\, \phi^{-1})"'] \arrow[d, dashed, "f_0"'] & & {[0,\varepsilon)\times \mathcal{L}^2_1([-\delta,0],U)} \arrow[d, "F^U_\theta"] \\
E & U\times E \arrow[l, "p_E"] & TU \arrow[l, "\cong", "\psi"']
\end{tikzcd}$$

then

$$|f_0(t,\gamma_1) - f_0(t,\gamma_2)|_E \leq k \sup_{s\in[-\delta,0]} |\gamma_1(s) - \gamma_2(s)|_E$$

$\forall (t,\gamma_1), (t,\gamma_2) \in \gamma^{\phi(U)}_{\phi(\theta^*)}$, where $\theta^* = \theta|[-\delta,0]$ and $k > 0$ is some constant depending on θ, ϕ, U but independent of $t \in [0,\varepsilon)$. We say F is strongly locally Lipschitz (near θ) if, together with the trivialization $\psi: TU \to U\times E$, $\exists$ a chart $(\mathcal{U}, \mathcal{Y})$ at θ in $\mathcal{L}^2_1(J,X)$ s.t. $p_E \circ \psi \circ F \circ (id, \mathcal{Y}^{-1})$ is Lipschitz wrt the supremum metric on the corresponding target space of $\mathcal{Y}$, in the second variable uniformly wrt the first.

At this point we observe that the effect of the localizing map C is to shorten the "memory" of the system $(F,[o,K),J,X)$ by curtailing the interval of retardation beyond $-\delta$, so that, thinking of the chart U in X as a piece of the flat Banach space E, we reduce the problem to solving the classical RFDE f_o in linear space. We are therefore lead to prove a version of the classical local existence and uniqueness theorem in the flat, which apparently is non-existent in the literature: (cf. Driver [10], Cruz and Hale [7], Hale [21]).

Theorem (1.1):

Let $V \subseteq E$ be open, and $0 < \varepsilon \leqslant \delta$. Let $\theta_o \in \mathcal{L}_1^2([-\delta,o],V)$ and $Y_{\theta_o}^V$ be the cylinder $Y_{\theta_o}^V = \{(t,\gamma): t \in [o,\varepsilon),\ \gamma \in \mathcal{L}_1^2([-\delta,o],V),$ $\theta_o(t-\delta) = \gamma(-\delta)\}$. Suppose that $f: [o,\varepsilon) \times \mathcal{L}_1^2([-\delta,o],V) \to E$ is a map s.t. $f|Y_{\theta_o}^V$ is continuous and is Lipschitz in the second variable uniformly wrt the first, and with $\mathcal{L}_1^2([-\delta,o],V)$ given the supremum metric. Then the RFDE $(f,[o,\varepsilon),[-\delta,o],V)$ has a unique local solution with initial path θ_o.

Proof:

We use a contraction argument.

$Y_{\theta_o}^V$ is dense in the cylinder

$Y_{\theta_o}^V(\mathcal{C}^o) = \{(t,\gamma): t \in [o,\varepsilon),\ \gamma \in \mathcal{C}^o([-\delta,o],V),\ \theta_o(t-\delta) = \gamma(-\delta)\}$,

and because of the uniform Lipschitz condition $f|Y_{\theta_o}^V$ can be extended uniquely to a continuous map $\tilde{f} : Y_{\theta_o}^V(\mathcal{C}^o) \to E$ which is Lipschitz in the 2nd variable uniformly wrt the first i.e. $\exists\ k > 0$ s.t.

$$|\tilde{f}(t,\gamma_1) - \tilde{f}(t,\gamma_2)|_E \leqslant k\ \|\gamma_1 - \gamma_2\| \qquad (1)$$

$$\forall\ (t,\gamma_1),(t,\gamma_2) \in Y_{\theta_o}^V(\mathcal{C}^o)$$

where $\|.\|$ is the supremum norm on the Banach space $\mathcal{C}^0([-\delta,o],E)$. Since $\tilde{f}$ is continuous, it is locally bounded, so $\exists\, M > 0$, $0 < \varepsilon_1 < \varepsilon$ and $\ell_1 > 0$ s.t.

$$|\tilde{f}(t,\gamma)| \leq \tfrac{1}{2}M \quad \forall\,(t,\gamma) \in Y^V_{\theta_0}(\mathcal{C}^0) \cap \{[o,\varepsilon_1] \times B(\ell_1)\} \tag{2}$$

where

$$B(\ell_1) = \{\gamma : \gamma \in \mathcal{C}^0([-\delta,o],E),\ \|\gamma - \theta_0\| \leq \ell_1\}$$

Since V is open, $\exists\ \ell_0 > 0$ s.t.

$$\{v : v \in E,\ |v - \theta_0(o)|_E \leq \ell_0\} \subset V \tag{3}$$

Define $\ell > 0$ by

$$\ell = \min(\varepsilon_1 M, \ell_1, \ell_0) \tag{4}$$

Now θ_0 is continuous, so it is uniformly continuous on the compact interval $[-\delta,o]$; hence $\exists\,\delta_0 > o$ s.t.

$$s,s' \varepsilon [-\delta,o],\ |s - s'| < \delta_0 \Longrightarrow |\theta_0(s) - \theta_0(s')|_E < \tfrac{1}{2}\ell$$

Choose ε_0 s.t. $o < \varepsilon_0 < \min(\frac{1}{k}, \delta_0, \varepsilon_1, \frac{\ell}{M})$ and define

$$A(\varepsilon_0,\ell) = \{\beta : \beta \in \mathcal{C}^0([-\delta,\varepsilon_0],E),\ \beta_0 = \theta_0,\ \beta_t \in B(\ell)\ \forall\, t \in [o,\varepsilon_0]\}$$

where β_t stands for $[\beta_t]_{[-\delta,o]}$ (Definition 1.3). Observe that $A(\varepsilon_0,\ell)$ is non-empty, indeed define $\beta^* \varepsilon\ \mathcal{C}^0([-\delta,\varepsilon_0],E)$ by

$$\beta^*(t) = \begin{cases} \theta_0(t) & t \in [-\delta,o] \\ \theta_0(o) & t \in [o,\varepsilon_0] \end{cases}$$

Then by the choice of ε_0 and the uniform continuity of θ_0,

$$|\beta^*_t(s) - \theta_0(s)| = \begin{cases} |\theta_0(t+s) - \theta_0(s)| & s \in [-\delta,-t] \\ |\theta_0(o) - \theta_0(s)| & s \in [-t,o] \end{cases}$$

$$< \tfrac{1}{2}\ell \qquad \forall\, s \in [-\delta,o],\quad \forall\, t \in [o,\varepsilon_0]$$

$\therefore \beta^* \varepsilon A(\varepsilon_0, \ell)$.

$$m : [0,\varepsilon_0] \times \mathcal{C}^0([-\delta,\varepsilon_0],E) \to \mathcal{C}^0([-\delta,0],E)$$

$$m(t,\beta) = \beta_t \qquad t \in [0,\varepsilon_0] \ , \ \beta \in \mathcal{C}^0([-\delta,\varepsilon_0],E)$$

is the memory map with past history $[-\delta,0]$. By continuity and compactness, it follows that the map

$m(.,\beta) : [0,\varepsilon_0] \longrightarrow \mathcal{C}^0([-\delta,0],E)$ is continuous, for each $\beta \in \mathcal{C}^0([-\delta,\varepsilon_0],E)$, and

$$\| m(t,\beta)\| = \| \beta_t \| \leq \| \beta \| \qquad \forall\ t \in [0,\varepsilon_0]. \tag{5}$$

It is therefore easily seen that m is (jointly) continuous and is continuous linear in the 2nd variable. Because $m(t,.)$ is continuous and $B(\ell)$ is closed, it is clear that $A(\varepsilon_0,\ell)$ is a closed subset of the complete metric space $\mathcal{C}^0([-\delta,\varepsilon_0],E)$

Furthermore, $\beta \in A(\varepsilon_0,\ell) \Rightarrow (t,\beta_t) \in \gamma^V_{\theta_0}(\mathcal{C}^0) \cap \{[0,\varepsilon_0] \times B(\ell)\}$ $\forall\ t \in [0,\varepsilon_0]$. To see this, notice that by the choice of ℓ_0 in (3) it is an easy matter to check that for each $\beta \in A(\varepsilon_0,\ell)$ $\beta_t \in \mathcal{C}^0([-\delta,0],V)\ \forall\ t \in [0,\varepsilon_0]$. We can therefore define the map $T: A(\varepsilon_0,\ell) \longrightarrow \mathcal{C}^0([-\delta,\varepsilon_0],E)$ by

$$(T\beta)(t) = \begin{cases} \theta_0(0) + \int_0^t \bar{f}(u,\beta_u)du & t \in [0,\varepsilon_0] \\ \theta_0(t) & t \in [-\delta,0] \end{cases} \tag{6}$$

for each $\beta \in A(\varepsilon_0,\ell)$. The continuity of $\bar{f}$ and m imply that

$$[0,\varepsilon_0] \longrightarrow E$$
$$u \mapsto \bar{f}(u,\beta_u)$$

is also continuous, so that T is well-defined and its fixed point(s) are precisely the solution(s) of the RFDE f on $[0,\varepsilon_0]$. It remains to show that T is a contraction mapping of $A(\varepsilon_0,\ell)$ into itself.

Let $\beta \in A(\varepsilon_o,\ell)$ and $t \in [o,\varepsilon_o]$. If $s \in [-t,o]$, then

$$|(T\beta)_t(s) - \theta_o(s)| \leq |\theta_o(o) - \theta_o(s)| + \int_o^{t+s} |\tilde{f}(u,\beta_u)|\,du$$

$$< \tfrac{1}{2}\ell + \tfrac{1}{2}M(t+s) \leq \tfrac{1}{2}\ell + \tfrac{1}{2}M\varepsilon_o \qquad \text{(by (2))}$$

$$< \ell$$

If $s \in [-\delta,-t]$, $|(T\beta)_t(s) - \theta_o(s)| = |\theta_o(t+s) - \theta_o(s)| < \tfrac{1}{2}\ell$. Thus $T\beta \in A(\varepsilon_o,\ell)$. T is a contraction, because if $\beta^1,\beta^2 \in A(\varepsilon_o,\ell)$ we have $\forall\, t \in [-\delta,\varepsilon_o]$,

$$|(T\beta^1)(t) - (T\beta^2)(t)| \leq \int_o^t |\tilde{f}(u,\beta_u^1) - \tilde{f}(u,\beta_u^2)|\,du$$

$$\leq k \int_o^{\varepsilon_o} \|\beta_u^1 - \beta_u^2\|\,du \qquad \text{(by (1))}$$

$$\leq k\,\varepsilon_o \|\beta^1 - \beta^2\| \qquad \text{(by (5))}$$

and $k\,\varepsilon_o < 1$. Thus T has a unique fixed point which is the unique local solution of f with initial path θ_o. Q.E.D.

Having proved the above theorem, the way is now paved clear for the main result of this section which says that, under fairly mild conditions on X (§1) and F (Definition 1.4), a unique local solution of the Cauchy problem always exists for arbitrary initial data in ${}_o\mathcal{L}_1^2(J,X)$.

Theorem(1.2):

Let X be a C^p $(p \geq 4)$ Banach manifold without boundary and admitting a C^{p-2} connection (as in §1). Suppose that (F, $[o,K)$,J,X) is a RFDE on X where F is continuous and locally Lipschitz at each $\theta \in \mathcal{L}_1^2(J,X)$. Then for given $\theta \in \mathcal{L}_1^2(J,X)$ F has a unique local solution with initial path θ.

Proof:

Let $\theta \in \mathcal{L}^2_1(J,X)$. We first localize F around $\theta(o)$; indeed by the hypotheses and Definition (1.4), choose a small chart (U,ϕ) at $\theta(o)$ in X s.t. the map

$f_o = p_E \circ T\phi \circ F^U_\theta \circ (id_{[o,\varepsilon)}, \bar{\phi}^{-1})|Y^{\phi(U)}_{\bar{\phi}(\theta^*)}$ is continuous and uniformly Lipschitz in the supremum metric, $\theta^* = \theta|[-\delta,o]$, and we use the notation of Definition (1.4). The continuity of f_o on $Y^{\phi(U)}_{\bar{\phi}(\theta^*)}$ holds because F is continuous (by hypothesis) and $C|Y^U_\theta$ also is continuous (by Lemma 1.1). Therefore by Theorem (1.1) the RFDE $(f_o,[o,\varepsilon), [-\delta,o], \phi(U))$ has a unique local solution at $\bar{\phi}(\theta^*)$ i.e. $\exists\ o < \varepsilon_o < \varepsilon \leq \delta$ and $\bar{\alpha} \in \mathcal{L}^2_1([-\delta,\varepsilon_o],\phi(U))$ s.t. $\bar{\alpha}|[o,\varepsilon_o]$ is C^1 and

$$\left.\begin{aligned} \bar{\alpha}'(t) &= f_o(t,[\bar{\alpha}_t]_{[-\delta,o]}) \qquad \forall\, t \in [o,\varepsilon_o) \\ [\bar{\alpha}_o]_{[-\delta,o]} &= \bar{\phi}(\theta^*) \end{aligned}\right. \tag{1}$$

Define $\bar{\bar{\alpha}} \in \mathcal{L}^2_1([-\delta,\varepsilon_o],U)$ by

$$\bar{\bar{\alpha}} = \phi^{-1} \circ \bar{\alpha} \tag{2}$$

Then it follows that

$$[\bar{\bar{\alpha}}_t]_{[-\delta,o]} = \bar{\phi}^{-1}([\bar{\alpha}_t]_{[-\delta,o]}) \tag{3}$$

We also define $\alpha \in \mathcal{L}^2_1([-r,\varepsilon_o],X)$ by

$$\alpha(t) = \begin{cases} \theta(t) & t \in J \\ \bar{\bar{\alpha}}(t) & t \in [o,\varepsilon_o] \end{cases} \tag{4}$$

Since $\bar{\bar{\alpha}}|[-\delta,o] = \theta^* = \theta|[-\delta,o]$, then by Lemma (1.1)

$$C(t,[\bar{\bar{\alpha}}_t]_{[-\delta,o]}) = [\alpha_t]_{[-r,o]} \qquad t \in [o,\varepsilon_o] \tag{5}$$

The following simple calculation shows that α is indeed a solution of F with initial path θ : if $t \in [o,\varepsilon_o)$.

$$\begin{aligned}\alpha'(t) &= T\phi^{-1}\{\bar{\alpha}'(t)\}\\ &= T\phi^{-1}\{p_E \circ T\phi \circ F_\theta^U \circ (id_{[o,\varepsilon)}, \bar{\phi}^{-1})(t,[\bar{\alpha}_t]_{[-\delta,o]})\} \quad \text{(by (1))}\\ &= F_\theta^U(t,[\bar{\alpha}_t]_{[-\delta,o]}) \quad \text{(by (3))}\\ &= F(t, C(t,[\bar{\alpha}_t]_{[-\delta,o]})) \quad \text{(Definition (1.4))}\\ &= F(t,[\alpha_t]_{[-r,o]}) \quad \text{(by (5))}\end{aligned}$$

Reversing the above argument and using the uniqueness of Theorem (1.1), it is not hard to see that if $\alpha_1 \in \mathcal{L}_1^2([-r,\varepsilon_1),X)$ is also a solution of F with the same initial path θ , then $\alpha(t) = \alpha_1(t)$ for every $t \in [-r,\min(\varepsilon_o,\varepsilon_1))$. Q.E.D.

<u>Remark</u>:

Note that in the above theorem we need both the continuity of F and the local Lipschitz condition, even when F is autonomous. However in the autonomous case a strong local Lipschitz condition would imply continuity (Corollary 1.2.1)

<u>Corollary (1.2.1)</u>

The conclusion of Theorem (1.2) also holds if any of the following conditions are satisfied:

i) F is continuous and strongly locally Lipschitz near each $\theta \in \mathcal{L}_1^2(J,X)$.

ii) F is autonomous and strongly locally Lipschitz (Oliva [38])

iii) F is autonomous and extends to a C^1 map $\mathcal{C}^o(J,X) \to TX$.

<u>Proof</u>:

Clearly (iii)$\Rightarrow$(ii), so that by the above remark we need only show that if F is strongly locally Lipschitz then it is locally Lipschitz.

Let $\theta \in \mathcal{L}^2_1(J,X)$. We use the notation employed in the proof of (iii) of Lemma (1.1). Let $(U,\phi),\varepsilon,\delta$ be as before. Fix $t_o \in [o,\varepsilon)$, $\gamma_o \in Y^U_\theta(t_o)$. Taking a natural chart $(\mathcal{U},\mathcal{Y})$ centred at some $\psi \in \mathcal{C}^p(J,X)$ very close to $C(t_o,\gamma_o)$ in $\mathcal{L}^2_1(J,X)$ and s.t. $\psi(-\delta) = \theta(t_o-\delta)$, we see that in a small neighbourhood of $(t_o,\bar{\phi}(\gamma_o))$ in $Y^{\phi(U)}_{\bar{\phi}(\theta^*)}$

$$(\mathcal{Y} \circ C(t,.)\ \bar{\phi}^{-1})(\tilde{\gamma})(s) = \begin{cases} \exp^{-1}_{\psi(s)}\,(\theta(s+t)) & s \in [-r,-\delta] \\ \left[(\mathrm{Exp}_{\psi|[-\delta,o]})^{-1} \circ (id_{[-\delta,o]}, \bar{\phi}^{-1}(\tilde{\gamma}))\right](s) & s \in [-\delta,o] \end{cases}$$

Now by using the smoothness of ψ and the exponential map it is not hard to see that Lemma (1.2) would then yield that $\mathcal{Y} \circ C(t\ ,.) \circ \bar{\phi}^{-1}$ is Lipschitz in the supremum metric in a neighbourhood of $\bar{\phi}(\gamma_o)$ and locally uniformly wrt t near t_o; indeed $\exists$ a neighbourhood $\mathcal{N}$ of $\bar{\phi}(\gamma_o)$ in $\mathcal{L}^2_1([-\delta,o],\phi(U))$, a neighbourhood I of t_o in $[o,\varepsilon)$ and a constant $C^* > o$ s.t.

$$\sup_{s \in J} |(\mathcal{Y} \circ C(t,.)\ \bar{\phi}^{-1})(\tilde{\gamma}_1)(s) - (\mathcal{Y} \circ C(t,.)\ \bar{\phi}^{-1})(\tilde{\gamma}_2)(s)|$$

$$\leq C^* \sup_{s\in[-\delta,o]} |\tilde{\gamma}_1(s) - \tilde{\gamma}_2(s)|$$

$$\forall\ (t,\tilde{\gamma}_1),(t,\tilde{\gamma}_2) \in Y^{\phi(U)}_{\bar{\phi}(\theta^*)} \cap \{I \times \mathcal{N}\}$$

Since $[-\delta,o]$ is compact, the constant C^* may be chosen independent of $t_o \in [o,\varepsilon)$. But F is strongly locally Lipschitz; hence $p_E \circ \psi \circ F \circ (id_{[o,\varepsilon)}, \mathcal{Y}^{-1})$ is Lipschitz and by composition so is

$$f_o = p_E \circ \psi \circ F \circ (id_{[o,\varepsilon)}, \mathcal{Y}^{-1}) \circ (id_{[o,\varepsilon)}, \mathcal{Y}) \circ \bar{C} \circ (id_{[o,\varepsilon)}, \bar{\phi}^{-1}),$$ Q.E.D.

thus completing the proof of the Corollary.

The following remarks are now in order.

Remarks:

1. The case $r = o$ (i.e. zero retardation) corresponds to F being a time dependent vector field, which is the ODE case; so that the local existence and uniqueness for solutions of vector fields is a special case of Theorem (1.2) (cf. Lang [32]). Needless to say this comment applies to all results in this thesis which are concerned with RFDE's. For $r > o$ the connection between RFDE's and vector fields will be established in due course.

2. In the flat case $X = E$, F is strongly locally Lipschitz iff it is locally Lipschitz.

3. The hypotheses on X are weak enough for X to be a manifold of maps e.g. $X = \mathcal{C}^k(N,M)$ where N is a compact manifold and M is a differential (finite dimensional) manifold admitting a connection. This is the reason for not assuming that X should admit smooth partitions of unity, because these may not exist on manifolds of maps (or even Banach spaces of functions e.g. $\mathcal{C}^o([o,1],R)$) (Eells [12]).

4. Theorem (1.2) and its preceding lemmas are all valid if $\mathcal{L}_1^2(J,X)$ was replaced by the continuous paths $\mathcal{C}^o(J,X)$.

5. If $\dim X < \infty$ and $F \circ \tilde{c}|Y_\theta^U$ is only continuous (not necessarily locally Lipschitz), then a solution still exists, though it may not be unique. To see this, observe that the proof of Theorem (1.1) can be modified so as to apply Schauder's fixed point theorem for the map T. However if X is infinite dimensional, the continuity condition by itself does not guarantee existence (cf. Dieudonne[8], Yorke[33])

6. Suppose that E is a separable Hilbert space and X is C^∞ and separable. Then the definition of the localizing map $C : [o,\epsilon)\times\mathcal{L}_1^2([-\delta,o],U)\to\mathcal{L}_1^2(J,X)$ (Lemma 1.1) can be modified in such a way that C is continuous everywhere

and in particular across the boundaries of the cylinder Y_θ^U. We sketch the construction as follows: if $(t,\gamma) \in Y_\theta^U$, define $C(t,\gamma)$ as in Lemma (1.1); but X can be given a complete Riemannian structure (Eells [12]§5), so that U may be chosen small enough for any two points in U to be joined by a unique geodesic whose length is equal to the distance between the two points; thus if $(t,\gamma) \notin Y_\theta^U$, we join $\theta(t-\delta)$ and $\gamma(-\delta)$ by the geodesic connecting them and which we call $g : [o, \Delta(t,\gamma)] \to U$ where $\Delta(t,\gamma) = d_0(\theta(t-\delta),\gamma(-\delta))$ is the distance between $\theta(t-\delta)$ and $\gamma(-\delta)$. Translate g by $t-\delta$ to get a path $\tilde{g} : [t-\delta, t-\delta + \Delta(t,\gamma)] \to U$, and define $k: [-r,-\delta + \Delta(t,\gamma)] \to X$ by

$$k(s) = \begin{cases} \theta(s+t) & s \in [r,-\delta] \\ \tilde{g}(s+t) & s \in [\delta, -\delta+\Delta(t,\gamma)] \end{cases}$$

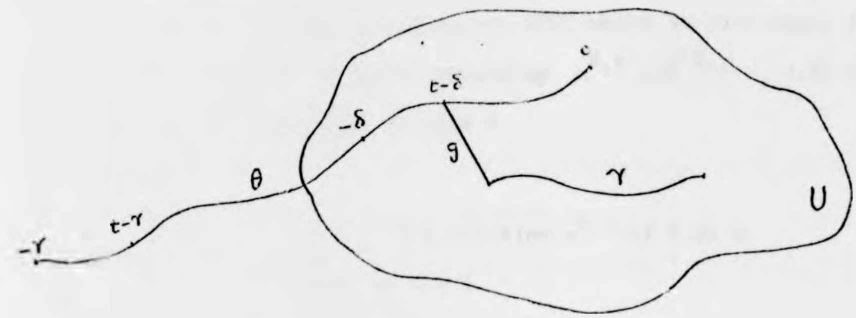
We then re-parameterize k by squashing it back to the interval $[-r,-\delta]$ through a change of variable $w: [-r,-\delta] \to [-r,-\delta + \Delta(t,\delta)]$ where

$$w(s) = \left[\frac{r-\delta + \Delta(t,\gamma)}{r-\delta}\right](s+\delta) - \delta + \Delta(t,\gamma) \qquad \forall\, s \in [-r,-\delta]$$

Finally define $C(t,\gamma) \in \mathcal{L}_1^2(J,X)$ by

$$C(t,\gamma)(s) = \begin{cases} k(w(s)) & s \in [-r,-\delta] \\ \gamma(s) & s \in [-\delta,o] \end{cases}$$

Then a tedious and rather lengthy calculation shows that C admits a continuous extension to a map $[0,\varepsilon) \times \mathcal{C}^0([-\delta,0],U) \to \mathcal{C}^0(J,X)$.



7. Krikorian ([31]) describes a method of placing a differentiable structure on the space of paths $\mathcal{L}^2_1(J,X)$ where X is only $C^p (p \geq 2)$ and not necessarily admitting a differentiable connection. If $\mathcal{L}^2_1(J,X)$ were given this differentiable structure, theorem (1.2) can be shown to hold with X of class $C^p(p \geq 2)$. The proof follows on similar lines to the one presented here but is much more cumbersome because of the complicated nature of the Krikorian structure.Theorem (1.2) as stated is good enough for our future purposes since in most cases we shall be needing some geometric structure on X (e.g. a connection, a Riemannian structure, etc.) during our forthcoming discussions.

8. The assertion (ii) of Corollary (1.2.1) had been proved in the $\mathcal{C}^0$ context by Oliva ([38], 1969) for the special case of compact manifolds; his technique relies heavily on an embedding theorem of Whitney, and his hypotheses are considerably stronger than ours. On the other hand - and as far as I know - Oliva's paper seems to be the only piece of literature which looks at the problem within a global setting.

3. Continuation:

Suppose that X is a manifold satisfying the permanent hypotheses of §1, and let (F,J,X) be an autonomous RFDE which is continuous and locally Lipschitz. For each $\theta \in \mathcal{L}_1^2(J,X)$ denote by $\alpha^{\theta,\epsilon} \in \mathcal{L}_1^2([-r,\epsilon),X)$ the unique local solution of F with initial path θ.

Define the set

$$I(\theta) = \bigcup\{[o,\epsilon): \quad \epsilon > o, \ \exists \text{ a solution } \alpha^{\theta,\epsilon_1} \text{ of } F \text{ at } \theta \text{ with } \epsilon_1 \geq \epsilon\}$$

Then $I(\theta)$ is a half-open interval in R, because it is a union of connected sets having o in common; indeed $I(\theta) = [o,t^+(\theta))$. By uniqueness, a solution $\alpha^\theta \in \mathcal{L}_1^2([-r,t^+(\theta)),X)$ of F at θ is then well-defined. Define the set $\mathcal{D}(F) \subset R^{\geq o} \times \mathcal{L}_1^2(J,X)$ by

$$\mathcal{D}(F) = \{(t,\theta) : \theta \in \mathcal{L}_1^2(J,X), \ o \leq t < t^+(\theta)\}$$

and the solution map $\alpha : \mathcal{D}(F) \to X$ by the property that for each $\theta \in \mathcal{L}_1^2(J,X)$, $\alpha(.,\theta) = \alpha^\theta$.

Our next result tells us that solutions of the RFDE can be continued to the right as long as we are within a maximal interval of existence (cf. ODE case). This result is well-known to hold for vector fields on manifolds and for RFDE's on Euclidean space R^n in the $\mathcal{C}^o$ context (cf. Hale [21]), in all cases the proofs run on parallel lines although the underlying assumptions are different.

Theorem (1.3):

Let (F,J,X) be a continuous locally Lipschitz autonomous RFDE on a manifold X satisfying the hypotheses of §1.

Suppose $\theta \in \mathcal{L}_1^2(J,X)$ and $t_o \in [o,t^+(\theta))$. Then

$$t^+(\alpha_{t_o}^\theta) = t^+(\theta) - t_o$$

and

$$\alpha(t,\alpha_{t_o}^\theta) = \alpha(t+t_o,\theta) \qquad \forall\ o \leq t < t^+(\theta) - t_o.$$

Proof:

The result holds because of uniqueness and maximality of solutions. Indeed, we start with the maximal solution $\alpha^\theta \in \mathcal{L}_1^2([-r,t^+(\theta)),X)$ of F at θ. Since $t_o \in [o,t^+(\theta))$ we can slide α^θ by an amount t_o to get a map $\tilde{\alpha} : [-r,t^+(\theta) - t_o) \to X$ defined by

$$\tilde{\alpha}(t) = \alpha(t + t_o,\theta) \qquad \forall t \in [-r,t^+(\theta) - t_o)$$

It is then obvious that

$$\tilde{\alpha}_t = \alpha_{t+t_o}^\theta \qquad \forall t \in [o,t^+(\theta) - t_o)$$

$$\tilde{\alpha}'(t) = F(\tilde{\alpha}_t) \qquad \forall t \in [o,t^+(\theta) - t_o)$$

and $$\tilde{\alpha}_o = \alpha_{t_o}^\theta \tag{1}$$

Thus $\tilde{\alpha}$ is a solution of F with initial path $\alpha_{t_o}^\theta$. Now $\alpha(.,\alpha_{t_o}^\theta)$ is a maximal solution of F with initial path $\alpha_{t_o}^\theta$, so by maximality

$$t^+(\theta) - t_o \leq t^+(\alpha_{t_o}^\theta) \tag{2}$$

and by uniqueness we must have

$$\alpha(t + t_o,\theta) = \tilde{\alpha}(t) = \alpha(t,\alpha_{t_o}^\theta) \tag{3}$$

To prove equality in (2), we need to show that $\tilde{\alpha}$ cannot be continued to the right of $t^+(\theta) - t_o$. Suppose, if possible, that $\exists t^+(\theta) - t_o < \varepsilon \leq t^+(\alpha_{t_o}^\theta)$ and a solution $\tilde{\tilde{\alpha}}^\varepsilon : [-r,\varepsilon) \to X$ of F extending $\tilde{\alpha}$, viz one s.t. $\tilde{\tilde{\alpha}}^\varepsilon \big| [-r,t^+(\theta) - t_o) = \tilde{\alpha}$. Reparametrize this solution to a map $\beta : [-r, t_0 + \varepsilon) \to X$ where

$$\beta(t) = \begin{cases} \tilde{\tilde{\alpha}}^{\varepsilon}(t - t_0) & t \in [t_0 - r, t_0 + \varepsilon) \\ \alpha^{\theta}(t) & t \in [-r, t_0 - r) \end{cases}$$

It is not hard to see that, because $\tilde{\tilde{\alpha}}^{\varepsilon}$ is a solution of F, then so is β but with initial path θ ; thus by maximality of domain we must have $t_0 + \varepsilon \leq t^+(\theta)$, which is a contradiction. Hence $t^+(\alpha^{\theta}_{t_0}) \leq t^+(\theta) - t_0$, completing the proof. Q.E.D.

It seems that the time is now ripe to introduce a simple but far-reaching idea originally due to Krasovskii([30], 1963): if α^{θ} is the (maximal) solution of F at θ , then by using the memory map we view its <u>orbit</u> through θ as a curve

$$\begin{aligned} [0, t^+(\theta)) &\longrightarrow \mathcal{L}^2_1(J,X) \\ t &\longmapsto m(t, \alpha^{\theta}) = \alpha^{\theta}_t \end{aligned}$$

in the infinite-dimensional manifold of paths $\mathcal{L}^2_1(J,X)$, rather than on the base manifold X. This point of view carries the philosophy that the dynamical properties of F are being faithfully reflected upon the state space $\mathcal{L}^2_1(J,X)$ through the orbits of solutions. One realization of the above idea is the following result which asserts that orbits with finite life-time cannot be imprisoned within compact sets in $\mathcal{L}^2_1(J,X)$ (cf.Hale [21] when $X = R^n$).

<u>Theorem (1.4)</u>:

Let X be a $C^p (p \geq 4)$ Banach manifold without boundary, and admitting a C^{p-2} connection (i.e. as in §1); and suppose (F,J,X) is a continuous locally Lipschitz RFDE on X. Let $\theta \in \mathcal{L}^2_1(J,X)$ be s.t. $t^+(\theta) < \infty$. Then for every compact set $\mathcal{A} \subset \mathcal{L}^2_1(J,X)$ $\exists\ \varepsilon > 0$ with $\alpha^{\theta}_t \notin \mathcal{A}\ \forall\, t > t^+(\theta) - \varepsilon$. (N.B. ε depends on $\mathcal{A}$).

Proof:

This is an adaptation of the proof of the corresponding result for vector fields ($r = o$, Lang [32]). It is sufficient to take $r > o$. Let $\theta \in \mathcal{L}^2_1(J,X)$ be s.t. $t^+(\theta) < \infty$. Suppose the conclusion of the theorem is false. Then there is a compact set $\mathcal{A} \subset \mathcal{L}^2_1(J,X)$ and a sequence $\{t_n\}_{n=1}^{\infty} \subset [o,t^+(\theta))$ s.t. $t_n \to t^+(\theta)$ as $n \to \infty$ and $\alpha^{\theta}_{t_n} \in \mathcal{A}\ \forall n \geq 1$.

Since $\mathcal{A}$ is compact, $\exists$ a subsequence $\{t_{n_i}\}_{i=1}^{\infty}$ of $\{t_n\}$ and a $\theta_o \in \mathcal{A}$ s.t.

$$\lim_{i\to\infty} \alpha^{\theta}_{t_{n_i}} = \theta_o$$

Now since the evaluation map $J \times \mathcal{L}^2_1(J,X) \to X$ is continuous, then

$$\alpha^{\theta}(t^+(\theta) + s) = \lim_{i\to\infty} \alpha^{\theta}(t_{n_i} + s) = \lim_{i\to\infty} \alpha^{\theta}_{t_{n_i}}(s) = \theta_o(s) \qquad \forall s \in [-r,o)$$

Extend α^{θ} by continuity to a map

$\tilde{\alpha} : [-r,t^+(\theta)] \to X$ of class $\mathcal{L}^2_1$. Thus $\tilde{\alpha}_{t^+(\theta)} \in \mathcal{L}^2_1(J,X)$, so that by the local existence theorem (Theorem 1.2) $\exists$ a map $\tilde{\tilde{\alpha}} \in \mathcal{L}^2_1([-r,\varepsilon'),X)$ s.t. $t^+(\theta) < \varepsilon'$, $\tilde{\tilde{\alpha}}$ is C^1 on $[t^+(\theta),\varepsilon')$,

$$\tilde{\tilde{\alpha}}'(t) = F(\tilde{\tilde{\alpha}}_t) \qquad \forall t \in [t^+(\theta),\varepsilon')$$

and

$$\tilde{\tilde{\alpha}}\Big|[-r,t^+(\theta)] = \tilde{\alpha}$$

We claim that $\tilde{\tilde{\alpha}}$ is a solution of F on the whole of $[o,\varepsilon')$; to see this observe that $\tilde{\tilde{\alpha}}$ satisfies the differential equation F on $[o,t^+(\theta))$, and if we denote the right and left hand derivatives of $\tilde{\tilde{\alpha}}$ by + and - respectively, then

$$\begin{aligned}\tilde{\tilde{\alpha}}'_+(t^+(\theta)) &= \lim_{t\to t^+(\theta)+} F(\tilde{\tilde{\alpha}}_t) = F(\tilde{\alpha}_{t^+(\theta)}) \\ &= \lim_{t\to t^+(\theta)-} F(\tilde{\alpha}_t) = \lim_{t\to t^+(\theta)-} \tilde{\alpha}'(t) \\ &= \tilde{\tilde{\alpha}}'_-(t^+(\theta))\end{aligned}$$

, using the continuity of F. Hence $\tilde{\tilde{\alpha}}$ is a solution of F at θ extending the maximal solution α^{θ}; this is a contradiction. Q.E.D

Corollary (1.4.1)

With the hypotheses of Theorem (1.4), let $\theta \in \mathcal{L}_1^2(J,X)$ be s.t. $t^+(\theta) < \infty$. Then the orbit $\{\alpha_t^\theta : t \in [o,t^+(\theta))\}$ is not relatively compact in $\mathcal{L}_1^2(J,X)$.

The above Corollary suggests that orbits with a finite life-time may be highly undesirable because they do not belong to compact sets and are therefore more difficult to control. This provides motivation for studying the case of θ s.t. $t^+(\theta) = \infty$ which corresponds by definition to a full solution $\alpha^\theta \in \mathcal{L}_1^2([-r,\infty)X)$ of F at θ.
Note that Corollary (3.2.1) says that solutions with compact orbits are full. On the other hand, to get full solutions - i.e. $t^+(\theta) = \infty \;\forall\theta \in \mathcal{L}_1^2(J,X)$ or $\mathcal{D}(F) = R \times \mathcal{L}_1^2(J,X)$ - it seems necessary that we place a geometric structure (viz. a Finsler) on X together with topological completeness. We therefore make some definitions.

Definition (1.5): (Eliasson [17], Palais [39], Eells [12]).

The Banach manifold X is said to be a Finsler manifold with Finsler $|.|$ if $|.| : TX \to R$ is a continuous function on its tangent bundle which restricts to an admissible norm $|.|_x : T_xX \to R$, $x \in X$, on each tangent space and is s.t. for each $x \in X$, $\exists$ a chart (U,ϕ) at x in X and constants $k_1, k_2 > o$ s.t.

$$k_2|v|_y \leq |(T_y\phi)(v)|_E \leq k_1|v|_y \qquad \forall y \in U, \quad \forall v \in T_yX$$

Under this assumption X has a canonical metric d, induced by its Finsler structure, and defined by

$$d(x_1,x_2) = \inf \{ \int_o^1 |\sigma'(t)|_{\sigma(t)}dt : \sigma : [o,1] \to X \text{ is piecewise } C^1 \text{ and } \sigma(o) = x_1,\ \sigma(1) = x_2 \}$$

X is a complete Finsler manifold if (X,d) is complete in the Finsler metric d.

A RFDE (F,J,X) on a Finsler manifold is said to be <u>bounded</u> if $\exists\, M > o$ s.t. $|F(\theta)|_{\theta(o)} \leq M \;\forall \theta \in \mathcal{L}_1^2(J,X)$.

<u>Theorem (1.5)</u>

Let X be a complete C^p $(p \geq 4)$ Finsler manifold, admitting a C^{p-2} connection. Suppose (F,J,X) is a continuous locally Lipschitz RFDE which is bounded in the Finsler. Then for every $\theta \in \mathcal{L}_1^2(J,X)$ $t^+(\theta) = \infty$ i.e. each maximal solution of F is full.

<u>Proof</u>:

With the hypotheses of the theorem, suppose $\exists\, \theta \in \mathcal{L}_1^2(J,X)$ and $\alpha^\theta : [-r,t^+(\theta)) \to X$ a maximal solution of F with $t^+(\theta) < \infty$. Take $t_1,t_2 \in [o,t^+(\theta))$. Then $\alpha^\theta|[t_1,t_2]$ is C^1 (Definition 1.3), and by the definition of d it follows that

$$d(\alpha^\theta(t_1),\alpha^\theta(t_2)) \leq \left|\int_{t_1}^{t_2} |\alpha^{\theta\prime}(t)|_{\alpha^\theta(t)}dt\right| \qquad \text{(Definition 3.1)}$$

$$= \left|\int_{t_1}^{t_2} |F(\alpha^\theta{}_t)|_{\alpha^\theta(t)}dt\right|$$

$$\leq \quad M|t_1 - t_2| \qquad \text{(because F is bounded)}$$

$\therefore \alpha^\theta$ is globally Lipschitz on $[o,t^+(\theta))$ wrt the Finsler metric d. By the completeness of X (and the uniform continuity of α^θ), α^θ has a (unique) extension to an $\mathcal{L}_1^2$ path $\tilde{\alpha}^\theta : [-r,t^+(\theta)) \to X$. Thus $\tilde{\alpha}^\theta_{t^+(\theta)} \in \mathcal{L}_1^2(J,X)$ and we can apply the local existence theorem to get $o < \varepsilon < r$ and a solution $\hat{\tilde{\alpha}} : [-r,\varepsilon) \to X$ of F with initial path $\tilde{\alpha}^\theta_{t^+(\theta)}$. Again this gives a solution $\beta : [-r,t^+(\theta) + \varepsilon) \to X$ of F at θ defined by

$$\beta(t) = \begin{cases} \alpha^{\theta}(t) & t \in [-r, t^{+}(\theta)) \\ \bar{\alpha}\,(t - t^{+}(\theta)) & t \in [t^{+}(\theta),\ t^{+}(\theta) + \varepsilon) \end{cases}$$

and which extends the maximal solution α^{θ} to the right of $t^{+}(\theta)$ - a contradiction. Therefore we must have $t^{+}(\theta) = \infty\ \forall \theta \in \mathcal{L}^{2}_{1}(J,X)$. Q.E.D.

<u>Corollary</u> (1.5.1)

Suppose X is a complete C^{p} $(p \geq 4)$ Finsler manifold and F is continuous and locally Lipschitz. Let $\alpha^{\theta} : [-r, t^{+}(\theta)) \to X$ be a maximal solution of F s.t. $\int_{0}^{t^{+}(\theta)} \frac{1}{|F(\alpha^{\theta}_{t})|}\, dt = \infty$. Then $t^{+}(\theta) = \infty$.

<u>Proof</u>:

If $t^{+}(\theta) < \infty$, observe that the hypotheses of the Corollary imply that F is bounded on the orbit $\{\alpha^{\theta}_{t}\}_{t^{+} > t \geq 0}$. Repeat the argument used in the proof of the theorem to get the required result. Q.E.D.

CHAPTER 2

Critical Paths

This chapter is primarily intended to throw some light on the general behaviour of the autonomous RFDE (F,J,X) at a critical path $\theta \in \mathcal{L}_1^2(J,X)$, which is, by definition, one s.t. $F(\theta) = 0 \in T_{\theta(0)} X$.
Our methods will lean heavily upon the following basic observation:

A geometric structure on X, viz a complete Riemannian structure will allow us to give the state space $\mathcal{L}_1^2(J,X)$ a complete Riemannian structure (Refer to §4 of this Chapter). This is a natural setting for a Morse theory. On the other hand, we shall be able to establish strong relationships between RFDE's and vector fields. These considerations, provide motivation for choosing $\mathcal{L}_1^2(J,X)$ as our state space in preference to the continuous paths $\mathcal{C}^0(J,X)$, the latter being only a Finsler manifold with a non-Hilbertable model.

1. Asymptotic Behaviour of Solutions:

Let X be a C^p ($p \geqslant 4$) Banach manifold as in Chapter 1 §1, and (F,J,X) a continuous (locally Lipschitz) RFDE on X. The following theorem describes the connection between the constant critical paths for F and its full solutions. It says that whenever a full solution of F converges asymptotically then it does so by levelling out to a constant critical path for F.

Theorem (2.1):

Suppose that (F,J,X) satisfy the given hypotheses. Let $\alpha\colon [-r,\infty) \to X$ be a full solution of F s.t. $\lim_{t\to\infty} \alpha(t) = x_0 \in X$, where x_0 is some point of X. Define $\tilde{x}_0\colon J \to X$ to be the constant path through x_0 i.e. $\tilde{x}_0(s) = x_0 \ \forall s \in J$. Then $\tilde{x}_0$ is a critical path of F.

Proof:

The proof proceeds by changing coordinates near the constant path $\tilde{x}_o$ in $\mathcal{L}_1^2(J,X)$ and then examining the situation in a linear space. More precisely, let (U,ϕ) be a chart at x_o in X; denote by $\bar{\phi}: \mathcal{L}_1^2(J,U) \to \mathcal{L}_1^2(J,\phi(U))$ the induced diffeomorphism. Choose the trivialization $\psi = T\phi : TU \to U \times E$ of TU and look at the composition

$$\begin{tikzcd}
 & TU = \pi_o^{-1}(U) \arrow[r, "\psi"] \arrow[d, "\pi_o"] & U \times E \arrow[dd, "p_E"] \\
\mathcal{L}_1^2(J,U) \arrow[ur, "F"] \arrow[r, "\rho_o"] & U & \\
\mathcal{L}_1^2(J,\phi(U)) \arrow[u, "\bar{\phi}^{-1}"] \arrow[rr, dashed, "f"'] & & E
\end{tikzcd}$$

where $f = p_E \circ \psi \circ F \circ \bar{\phi}^{-1}$, and p_E is the projection onto the model E of X. We shall prove that $\phi(x_o)$ gives a critical path of the RFDE $(f,J,\phi(U))$.

Since $\lim_{t\to\infty} \alpha(t) = x_o$, $\exists\, t_o > o$ s.t. $\alpha_t \in \mathcal{L}_1^2(J,U)$ $\forall\, t \geq t_o$. Define $\bar{\alpha} \in \mathcal{L}_1^2([-r,\infty),\phi(U))$ by

$$\bar{\alpha}(t) = \phi(\alpha(t + t_o)) \qquad \forall\, t \in [-r,\infty) \tag{1}$$

Then it is easy to see that

$$\bar{\alpha}_t = \bar{\phi}(\alpha_{t+t_o}) \qquad \forall \quad t \geq 0 \tag{2}$$

Also by the continuity of ϕ and f, we get

$$\lim_{t\to\infty} \bar{\alpha}(t) = \phi(x_o) \tag{3}$$

and

$$\lim_{\substack{t\to\infty \\ t\geq t_o}} f(\bar{\alpha}_t) = f(\widetilde{\phi(x_o)}) \tag{4}$$

because $\bar{\alpha}_t \to \widetilde{\phi(x_0)} = \bar{\phi}(\tilde{x}_0)$ as $t \to \infty$, where $\widetilde{\phi(x_0)}$ is the constant path through $\phi(x_0)$ in E. Now let $\varepsilon > 0$ be given. Then (4) says $\exists\, t_0' \geq t_0$ s.t.

$$|f(\bar{\phi}(\tilde{x}_0)) - f(\bar{\alpha}_u)| < \varepsilon/_2 \quad \forall\, u \geq t_0' \tag{5}$$

By integrating the expression

$$f(\bar{\phi}(\tilde{x}_0)) = \{f(\bar{\phi}(\tilde{x}_0)) - f(\bar{\alpha}_u)\} + f(\bar{\alpha}_u) \quad u \geq t_0'$$

and using the fact that $\bar{\alpha}$ is a solution of f we obtain

$$f(\bar{\phi}(\tilde{x}_0))\,(t - t_0') = \int_{t_0'}^{t} \{f(\bar{\phi}(\tilde{x}_0)) - f(\bar{\alpha}_u)\}\,du$$

$$+ \bar{\alpha}(t) - \bar{\alpha}(t_0') \quad t \geq t_0' \tag{6}$$

By (5),

$$|f(\bar{\phi}(\tilde{x}_0))|(t - t_0') \leq \frac{\varepsilon}{2}(t - t_0') + |\bar{\alpha}(t) - \bar{\alpha}(t_0')|$$

If $K > o$ is an upper bound for $\{|\bar{\alpha}(t)| : t \geq t_0\}$, then

$$|f(\bar{\phi}(\tilde{x}_0) \leq \varepsilon/_2 + \frac{K + |\bar{\alpha}(t_0')|}{t - t_0'} \qquad t > t_0'$$

$$\to \varepsilon/_2 \quad \text{as } t \to \infty$$

Since ε was arbitrarily chosen, then we must have

$$f(\bar{\phi}(\tilde{x}_0)) = 0$$

$$\therefore (p_E \cdot \psi \cdot F)(\tilde{x}_0) = 0$$

Hence

$$F(\tilde{x}_0) = 0$$

because we have chosen $\psi = T\phi$ which is a linear homeomorphism on fibres. The theorem is proved. Q.E.D.

<u>Remark</u>: (2.1)

From the point of view of applications the critical paths of F have the following significance: in a system which evolves with time under a force represented by a FRDE F, the critical paths correspond to those states at which the system is momentarily at rest; a constant critical path $\tilde{x}_0$, $x_0 \in X$, is an <u>equilibrium state</u> i.e. the solution through $\tilde{x}_0$ is constant for all future time and the system is permanently at rest.

2. A Vector Field on $\mathcal{L}^2_1(J,X)$ induced by F:

Let X be a C^p $(p \geq 5)$ Riemanian manifold modelled on a real Hilbert space E. Let $\theta \in \mathcal{L}^2_1(J,X)$. Then the tangent space $T_\theta \mathcal{L}^2_1(J,X)$ can be naturally identified with the topological vector space $\{\beta : \beta \in \mathcal{L}^2_1(J,TX) \pi_0 \circ \beta = \theta\}$ where $\pi_0 : TX \to X$ is the tangent bundle of X (Eliasson [17]). By virtue of the Riemannianstructure on X we have parallel transport along θ given by a family of isometries (i.e. Hilbert space isomorphisms)

$$ {}^{\theta}\tau_{t_1}^{t_2} : T_{\theta(t_1)} X \to T_{\theta(t_2)} X \quad , t_1, t_2 \in J, \quad t_2 \leq t_1 . $$

For the autonomous RFDE (F,J,X) on X, define the path $\xi^F(\theta): J \to TX$ by

$$ \xi^F(\theta)(s) = {}^{\theta}\tau_0^{s} \{F(\theta)\} \qquad \forall s \in J $$

Thus $\xi^F(\theta)(s) \in T_{\theta(s)}X$ $\forall s \in J$, and because of the above identification we get a map $\xi^F : \mathcal{L}^2_1(J,X) \to T\mathcal{L}^2_1(J,X)$ which is in fact a vector field on $\mathcal{L}^2_1(J,X)$. This canonically induced vector field will be used as a lever with a dual purpose : (a) viewing the set of all RFDE's $\zeta(J,X)$ on X as embedded into the algebra of all vectors fields $\Gamma(T\mathcal{L}^2_1(J,X))$ on $\mathcal{L}^2_1(J,X)$, (b) deweloping a Morse theory for a special class of examples of RFDE's. While (a) will presently be investigated, (b) will be dealt with in later sections.

Theorem (2.2):

Let X be a C^p $(p \geq 5)$ Riemannian manifold, and let $0 \leq k \leq p-4$.

i) Each C^k vector field η on $\mathcal{L}^2_1(J,X)$ induces a C^k RFDE $F(\eta): \mathcal{L}^2_1(J,X) \to TX$ on X given by

$$ F(\eta) = T_{\rho_0} \circ \eta $$

ii) F is C^k iff ξ^F is; moreover,

$$F = T_{\rho_0} \circ \xi^F$$

iii) Let $\rho_0^*(TX)$ be the pull-back of $\pi_0 : TX \to X$ over ρ_0, so that we have a commutative diagram

$$\begin{tikzcd}
\rho_0^*(TX) \arrow[r, "\pi_0^*(\rho_0)"] \arrow[d, "\rho_0^*(\pi_0)"'] & TX \arrow[d, "\pi_0"] \\
\mathcal{L}_1^2(J,X) \arrow[r, "\rho_0"'] \arrow[ur, dashed, "F"] & X
\end{tikzcd}$$

Then the Riemannian structure on X gives a canonical C^{p-4} embedding $i : \rho_0^*(TX) \to T\mathcal{L}_1^2(J,X)$ of $\rho_0^*(TX)$ as a subbundle of $\pi_1 : T\mathcal{L}_1^2(J,X) \to \mathcal{L}_1^2(J,X)$ i.e. the sequence $0 \longrightarrow \rho_0^*(TX) \longrightarrow T\mathcal{L}_1^2(J,X)$ is exact (Eells [12], Lang [32]). Each ξ^F is a section of the bundle $i\{\rho_0^*(TX)\} \longrightarrow \mathcal{L}_1^2(J,X)$.

Proof:

(i) This is true because the evaluation $\rho_0 : \mathcal{L}_1^2(J,X) \to X$ is C^{p-3}, and in fact, for each $\theta \in \mathcal{L}_1^2(J,X)$,

$$(T_\theta \rho_0)(\beta) = \beta(0) \quad \forall \beta \in T_\theta \mathcal{L}_1^2(J,X).$$

It therefore follows immediately that, if η is a C^k vector field on $\mathcal{L}_1^2(J,X)$, then $F(\eta)$ is a RFDE of class C^k.

(ii) If $\theta \in \mathcal{L}_1^2(J,X)$, then

$$(T\rho_0 \circ \xi^F)(\theta) = (T_\theta \rho_0)(\xi^F(\theta)) = {}^{\theta}\tau_0^{\,0}(F(\theta)) = F(\theta).$$

By (i) above, ξ^F is $C^k \Rightarrow F$ is C^k.

iii) Observe that for each $\theta \in \mathcal{L}_1^2(J,X)$ the tangent space $T_\theta \mathcal{L}_1^2(J,X)$ splits in the following manner

$$T_\theta \mathcal{L}_1^2(J,X) = H_\theta \mathcal{L}_1^2(J,X) \oplus Q_\theta \mathcal{L}_1^2(J,X)$$

where

$$H_\theta \mathcal{L}_1^2(J,X) = \{\beta : \beta \in T_\theta \mathcal{L}_1^2(J,X), \quad \frac{D\beta(s)}{ds} = o \qquad \text{a.a. } s \in J\}$$

and

$$Q_\theta \mathcal{L}_1^2(J,X) = \{ \beta : \beta \in T_\theta \mathcal{L}_1^2(J,X), \quad \beta(o) = o \}$$

$\frac{D}{ds}$ denotes covariant differentiation wrt $s \in J$ of vector fields along θ, this being in the sense of Milnor ([35]). Define the map $i_\theta : T_{\theta(o)}X \to H_\theta \mathcal{L}_1^2(J,X)$ by

$$i_\theta(v)(s) = {}^\theta\tau_o^s(v) \qquad \forall s \in J$$

Since parallel transport is a linear homeomorphism on fibres, it is easy to see that i_θ is also a linear homeomorphism onto the closed subspace $H_\theta \mathcal{L}_1^2(J,X)$ of $T_\theta \mathcal{L}_1^2(J,X)$. To discuss the smoothness of the map $\theta \longmapsto i_\theta$ it is sufficient to consider the situation locally and then the problem boils down to looking at the solutions of the ODE

$$\frac{dZ}{ds} + \Gamma(\theta(s))\,(\theta'(s), Z(s)) = o \qquad s \in J_o \qquad (1)$$

of parallel transport (Eliasson [17]), where Γ is the local connector associated with the Levi-Civita connection on TX ; J_o is a subinterval of J and θ ranges through an open neighbourhood $\mathcal{U}$ in $\mathcal{L}_1^2(J_o,E)$. There is no loss of generality in taking $J_o = [-\delta,o]$, so that (1) is solved for each $v \in E$ as initial condition $Z(o) = v$. View (1) as a family of time-dependent vectors fields $f_\theta : J_o \times E \to E$ parameterized by $\theta \in \mathcal{U}$, whose solutions $Z(.,\theta) \in \mathcal{L}_1^2(J_o, L(E))$ for each $\theta \in \mathcal{U}$. Now each f_θ is continuous linear in the second variable and we have a map

$$\mathcal{U} \longrightarrow \mathcal{L}^2(J_o, L(E))$$

$$\theta \longmapsto \{J_o \ni s \mapsto f_\theta(s,.)\}$$

where

$$f_\theta(s,v) = -\Gamma(\theta(s))(\theta'(s),v) \qquad s \in J_o,\ v \in E,$$

$L(E)$ is the space of continuous linear maps of E into itself. Since Γ is C^{p-2}, then $\theta \longmapsto f_\theta(.,.)$ is C^{p-3} by composition. As the solutions of (1) depend differentiably on the parameter θ, then the map

$$\mathcal{U} \longrightarrow \mathcal{L}_1^2(J_o, L(E))$$

$$\theta \longmapsto Z(.,\theta)$$

is C^{p-4}. Therefore $i : \rho_o^*(TX) \to T\mathcal{L}_1^2(J,X)$ is of class C^{p-4}. Since F is C^k, $o \leq k \leq p-4$, then $\xi^F = i \circ F$ is also C^k. The proof is complete. Q.E.D.

<u>Corollary (2.2.1)</u>

With the hypotheses of the theorem, let $\zeta^k(J,X)$ stand for the set of all C^k RFDE's on X and $\Gamma^k(T\mathcal{L}_1^2(J,X))$ for the set of all C^k vector fields on $\mathcal{L}_1^2(J,X)$. Then $\zeta^k(J,X)$ is a module over the ring of C^k functions $C^k(\mathcal{L}_1^2(J,X),R)$ on $\mathcal{L}_1^2(J,X)$, and the mapping

$$\zeta^k(J,X) \longrightarrow \Gamma^k(T\mathcal{L}_1^2(J,X)$$

$$F \longmapsto \xi^F$$

is an embedding of modules.

<u>Proof:</u>

Addition in $\zeta^k(J,X)$ is defined in the obvious way, while multiplication

$$C^k(\mathcal{L}_1^2(J,X),R) \times \zeta^k(J,X) \longrightarrow \zeta^k(J,X)$$

$$(f,F) \longmapsto f.F$$

is defined by

$$(f.F)(\theta) = f(\theta)F(\theta) \qquad \forall\, \theta \in \mathcal{L}_1^2(J,X).$$

This is well-defined because since f and F are C^k then so is $f.\xi^F$ (using the theorem and the fact that $\Gamma^k(T\mathcal{L}_1^2(J,X))$ is a module over $C^k(\mathcal{L}_1^2(J,X),R)$, and hence $f.F = T\rho_o \circ (f.\xi^F) \in \zeta^k(J,X)$.

It is an easy matter checking that the map $F \longmapsto \xi^F$ respects the module operations, because of the linearity of parallel transport. Moreover this map is injective since $F \in \zeta^k(J,X)$ and $\xi^F = 0 \Rightarrow {}^{\theta}\tau_o{}^{s}(F(\theta)) = o$ $\forall\, s \in J \Rightarrow F(\theta) = o\ \forall\, \theta \in \mathcal{L}_1^2(J,X)$. Q.E.D.

<u>Corollary (2.2.2)</u>:

Suppose that $k = p = \infty$. Then the algebra $\zeta^\infty(J,X)$ admits a skew-symmetric $C^\infty(\mathcal{L}_1^2(J,X),R)$ - bilinear product $[[\ .,.]] : \zeta^\infty(J,X)\times \zeta^\infty(J,X)\to \zeta^\infty(J,X)$
This bilinear product coincides with the Lie bracket of vector fields when $J = \{o\}$.

<u>Proof</u>:

Define $[[.,.]]$ by the relation

$$[[F,G]] = T\rho_o \circ [\xi^F, \xi^G] \qquad F,G \in \zeta^\infty(J,X)$$

where $[\ .,.\ ]$ is the Lie bracket of vector fields on $\mathcal{L}_1^2(J,X)$. Since $[\ .,.\ ]$ is bilinear and skew-symmetric (Lang[32]), then it follows easily from the definition that $[[.,.]]$ is also bilinear and skew-symmetric viz.
$[[F,G]] = -[[G,F]]\quad F,G \in \zeta^\infty(J,X)$. Q.E.D.

Remark: (2.2)

If the subbundle

$$\begin{array}{c} i\ \{\rho_0^*(TX)\} \\ \Big\downarrow \\ \mathcal{L}_1^2(J,X) \end{array}$$

of Theorem (2.2) is integrable in $T\mathcal{L}_1^2(J,X)$ (Lang [32]), then $\zeta^\infty(J,X)$ becomes a Lie algebra with Lie bracket $[[\cdot,\cdot]]$, indeed the integrability of the subbundle implies that for any $L,F,G \in \zeta^\infty(J,X)$,

$$[\xi^F,\xi^G] = \xi^{[[F,G]]}$$

and so

$$\xi^{\{[[F,[[G,L]]]] + [[G,[[L,F]]]] + [[L,[[F,G]]]]\}}$$

$$= [\xi^F, \xi^{[[G,L]]}] + [\xi^G, \xi^{[[L,F]]}] + [\xi^L, \xi^{[[F,G]]}]$$

$$= [\xi^F, [\xi^G, \xi^L]] + [\xi^G, [\xi^L, \xi^F]] + [\xi^L, [\xi^F,\xi^G]] \quad (*)$$

$$= 0$$

i.e. $[[\cdot,\cdot]]$ satisfies the Jacobi identity, and the mapping $F \longmapsto \xi^F$ is a Lie algebra embedding of $\zeta^\infty(J,X)$ as a sub-Lie algebra of $\Gamma^\infty(T\mathcal{L}_1^2(J,X))$.

The next result provides a link between the trajectories of ξ^F and solutions of the RFDE F.

Proposition (2.1):

Let (F,J,X) be a C^1 RFDE on the $C^p (p \geq 5)$ complete Riemannian manifold X. Suppose F is bounded (Definition 1.5). Then ξ^F has full trajectories belonging to $C^2(R, \mathcal{L}_1^2(J,X))$. Let $M \subset C^0(R, \mathcal{L}_1^2(J,X))$ stand for the subset of all C^0 $\gamma : R \to \mathcal{L}_1^2(J,X)$ with the property that

$$\gamma(t)(s) = \begin{cases} \gamma(t+s)(o) & t+s \geq 0,\ t \in R,\ s \in J \\ \gamma(o)(t+s) & t+s \leq o,\ t \in R,\ s \in J. \end{cases}$$

Then M is a closed subspace of $C^0(R, \mathcal{L}_1^2(J,X))$, and there is a bijection of M onto $\mathcal{L}_1^2(R,X)$ carrying each trajectory of ξ^F in M into a full solution of F with the same initial data and defined on the whole of R; i.e. each trajectory of ξ^F in M is an orbit of F.

Proof:

Since F is C^1, then ξ^F is also C^1 (Theorem 2.2) and hence locally Lipschitz in the sense of Lang ([32]). Then ξ^F admits unique trajectories. To prove that ξ^F has full trajectories we choose a Finsler on $T\mathcal{L}_1^2(J,X)$ which coincides with the one on the subbundle $i\{\rho_o^*(TX)\}$ induced by the Riemanian metric on TX, e.g. define $||\cdot||_\theta$ for each $\theta \in \mathcal{L}_1^2(J,X)$ by

$$||\beta||_\theta = \left[\frac{1}{r} \int_{-r}^{o} |\beta(s)|^2_{\theta(s)} ds + \frac{1}{r} \int_{-r}^{o} \left| \frac{D\beta(s)}{ds} \right|^2 ds \right]^{\frac{1}{2}}$$

$\forall\ \beta \in T_\theta \mathcal{L}_1^2(J,X)$. Then it is easy to see that each $i_\theta : T_{\theta(o)}X \to H_\theta \mathcal{L}_1^2(J,X)$ becomes an isometry, so that $||\xi^F(\theta)||_\theta = |F(\theta)|_{\theta(o)}\ \forall\ \theta \in \mathcal{L}_1^2(J,X)$.

Hence ξ^F is bounded in the above Finsler because F is bounded by hypotheses. By completeness it then follows that ξ^F has full trajectories (cf. Theorem (1.5) for $r = o$).

Using the continuity of the evaluation map, it is easy to see that M is closed in $C^o(R, \mathcal{L}^2_1(J,X))$. Let $\rho_o: \mathcal{L}^2_1(J,X) \to X$ be the evaluation at o, and define a map $\tilde{\rho}_o: M \to \mathcal{L}^2_1(R,X)$ by

$$\tilde{\rho}_o(\gamma) = \rho_o \circ \gamma \qquad \forall \gamma \in M$$

As a consequence of the definition of M and $\tilde{\rho}_o$, we get that $\tilde{\rho}_o$ is a bijection of M onto $\mathcal{L}^2_1(R,X)$ whose inverse is the mapping

$$\mathcal{L}^2_1(R,X) \longrightarrow C^o(R, \mathcal{L}^2_1(J,X))$$

$$\alpha \longmapsto \{R \ni t \longmapsto \alpha_t \in \mathcal{L}^2_1(J,X)\} \quad .$$

Observe that in order to get the inverse we use the continuity of the memory map $t \mapsto \alpha_t$ for $\alpha \in \mathcal{L}^2_1(J,X)$. Now let $\gamma \in M$ be a full trajectory of ξ^F with $\gamma(o) = \theta \in \mathcal{L}^2_1(J,X)$. Let $\alpha = \tilde{\rho}_o(\gamma) = \rho_o \circ \gamma$, then since ξ^F is C^1 and ρ_o is differentiable we see that $\alpha : R \to X$ is C^2; indeed, if $t \in R$,

$$\alpha'(t) = T\rho_o(\gamma'(t)) = T\rho_o\{\xi^F(\gamma(t))\} = F(\gamma(t)) = F(\alpha_t)$$

and $\alpha_o = \gamma(o) = \theta$. Thus α is a solution of F over the whole of R with initial path θ. Q.E.D.

The following trivial but crucial proposition highlights the significane of the vector field ξ^F in studying the critical paths of the RFDE F.

<u>Proposition (2.2)</u>:

The critical paths of F are precisely the critical points of ξ^F in $\mathcal{L}^2_1(J,X)$.

<u>Proof</u>:

The parallel transport is a linear isomorphism. Q.E.D.

<u>Remark</u> (2.3):

Theorem (2.2) says that each vector field on $\mathcal{L}^2_1(J,X)$ projects onto an autonomous RFDE on X. On the other hand the trajectories of the vector field and the associated RFDE do not correspond in a natural way except perhaps on the subset $M \subset \mathcal{C}^0(R, \mathcal{L}^2_1(J,X))$ of Proposition (2.1), and in this case we get backward solutions of the RFDE. Since this is not in general the case, we therefore do not expect to obtain the local existence theorem (Theorem 1.2) as a corollary of the one for vector fields on $\mathcal{L}^2_1(J,X)$.

§3. The Hessians:

Let (F,J,X) be a C^1 RFDE on a C^p ($p \geq 5$) Riemannian manifold X, and let ξ^F be the induced C^1 vector field on $\mathcal{L}^2_1(J,X)$ (§2). Suppose that $\theta \in \mathcal{L}^2_1(J,X)$ is a critical path of F; we define the Hessians of F and ξ^F at θ following very closely the construction of Abraham and Robbin ([1]). Indeed the zero section $(TX)_0$ of the tangent bundle $\pi_0 : TX \to X$ is a closed submanifold of TX diffeomorphic to X; thus the topological vector space $T_{o_{\theta(o)}}(TX)_o$ of horizontal tangent vectors to TX is canonically isomorphic to $T_{\theta(o)}X$, where $F(\theta) = 0_{\theta(o)} \in T_{\theta(o)}X$ is the zero vector in $T_{\theta(o)}X$. Similarly the space $T_{o_{\theta(o)}}(T_{\theta(o)}X)$ of vertical tangent vectors to TX at $o_{\theta(o)}$ is isomorphic to $T_{\theta(o)}X$. Make the identification

$$T_{o_{\theta(o)}}(TX) = T_{o_{\theta(o)}}(TX)_o \oplus T_{o_{\theta(o)}}(T_{\theta(o)}X) \cong T_{\theta(o)} \oplus T_{\theta(o)}X$$

and denote by

$$\pi_v^{\theta(o)} : T_{o_{\theta(o)}}(TX) \to T_{\theta(o)}X$$

the projection onto the second (vertical) factor. The Hessian of F at the critical path θ is denoted by $(dF)_\theta$ and defined by

$$(dF)_\theta = \pi_v^{\theta(o)} \circ T_\theta F : T_\theta \mathcal{L}^2_1(J,X) \to T_{\theta(o)}X.$$

For the vector field ξ^F the Hessian $(d\xi^F)_\theta : T_\theta \mathcal{L}^2_1(J,X) \circlearrowleft$ at θ is defined similarly.

The main result of this section describes the relationship between the two Hessians of F and ξ^F; the proof essentially amounts to differentiating the parallel transport at a critical path, and the following lemma will be needed.

Lemma (2.1):

Let (U,ϕ) be a chart in X, $f : U \to R$ a C^1 function and $\partial : U \to TU$ a C^1 vector field. Define the vector field $f\partial$ on U by

$$(f\partial)(x) = f(x)\partial(x) \qquad \forall\, x \in U$$

Then $f\partial$ is C^1 and for each $x \in U$

$$\{T_x(f\partial)\}(z) = (T_x f)(z).\ \{T_{f(x)\partial(x)}(T\phi)\}^{-1}[T\phi(\partial(x))]$$

$$+ f(x).\ [\{T_{f(x)\partial(x)}(T\phi)\}^{-1} \circ \{T_{\partial(x)}(T\phi)\}]\ ((T_x\partial)(z)) \qquad (1)$$

$$\forall\, z \in T_x X$$

In particular when x is a zero of f, the Hessian is given by

$$[d(f\partial)]_x(z) = (T_x f)(z)\,\partial(x) \qquad \forall z \in T_x X \qquad (2)$$

Proof:

We have $\phi : U \to \phi(U) \subset E$, where E is the Hilbert space model of X, and for $x \in U$, $z \in TU$ the maps $T_x\phi : T_xX \to E$ and $T_z(T\phi) : T_z(TU) \to E \times E$ are linear homeomorphisms. Applying the formula for the Fréchet derivative of a product in E we get for each $x \in U$ and $z \in T_xU$ the expression

$$\{T_x(f\partial)\}(z) = \{T_{f(x)\partial(x)}(T\phi)\}^{-1}\ [D(f\circ\phi^{-1})(\phi(x))((T_x\phi)(z)).$$

$$\cdot\{(T\phi)\circ\partial\circ\phi^{-1}\}(\phi(x))]$$

$$+ \{T_{f(x)\partial(x)}(T\phi)\}^{-1}\ [(f\circ\phi^{-1})(\phi(x)).\ D(T\phi\circ\partial\circ\phi^{-1})(\phi(x))((T_x\phi)(z))]\ .$$

Now this reduces immediately to the required formula (1) once we notice that

$$T_x(f\partial) = \{T_{f(x)\partial(x)}(T\phi)\}^{-1}\circ\ [D\{(T\phi)\circ(f\partial)\circ\phi^{-1}\}(\phi(x))]\ \circ T_x\phi \qquad (3)$$

and

$$T_x\partial = \{T_{\partial(x)}(T\phi)\}^{-1}\circ\ [D\{(T\phi)\circ\partial\circ\phi^{-1})\}(\phi(x))]\ \circ\ T_x\phi \qquad (4)$$

Now suppose $x \in U$ is s.t. $f(x) = o$. Then (1) $\Longrightarrow$

$$\{T_x(f\partial)\}(z) = (T_x f)(z).\{T_{o_x}(T\phi)\}^{-1}[T\phi(\partial(x))] \qquad (5)$$

As before let

$$\pi_v^x : T_{o_x}(TU) \to T_x U = T_x X$$

be the projection onto the vertical tangents to the zero section in TU.

Then

$$\pi_v^x \circ \{T_{o_x}(T\phi)\}^{-1} \circ T\phi\big|_{T_x X} = id \qquad (6)$$

and

$$[d(f\partial)]_x(z) = \{\pi_v^x \circ T_x(f\partial)\}(z) = (T_x f)(z).\partial(x),$$

using (5) and (6). Q.E.D.

Theorem (2.3):

Suppose F is a C^1 RFDE on a $C^p (p \geq 5)$ finite dimensional Riemannian manifold X. Let $\theta_o \in \mathcal{L}_1^2(J,X)$ be a critical path of F. Then

$$(d\xi^F)_{\theta_o}(\beta)(s) = {}^{\theta}\tau_o^s\{(dF)_{\theta_o}(\beta)\} \qquad \forall s \in J, \ \forall \beta \in T_{\theta_o}\mathcal{L}_1^2(J,X) \ (7)$$

Proof:

The idea of the proof is to use a local argument showing that $s \longmapsto (d\xi^F)_{\theta_o}(\beta)(s)$ is a parallel vector field along θ_o for each $\beta \in T_{\theta_o}\mathcal{L}_1^2(J,X)$ which coincides with $(dF)_{\theta_o}(\beta)$ at $s = o$.

First of all we split the tangent space $T_{o_{\theta_o}}(T\mathcal{L}_1^2(J,X))$ in the form

$$T_{o_{\theta_o}}(T\mathcal{L}_1^2(J,X)) \cong T_{o_{\theta_o}}\mathcal{L}_1^2(J,TX)$$

$$= H\,T_{o_{\theta_o}}\mathcal{L}_1^2(J,TX) \oplus VT_{o_{\theta_o}}\mathcal{L}_1^2(J,TX) \qquad (8)$$

where the horizontal and vertical tangent vectors are given by

$$H\,T_{o_{\theta_o}}\mathcal{L}_1^2(J,TX) = \{\gamma \in \mathcal{L}_1^2(J,T^2X) : \gamma(s) \in T_{o_{\theta_o}(s)}(TX)_o \quad \forall s \in J\}$$

and

$$VT_{o_{\theta_o}}\mathcal{L}_1^2(J,TX) = \{\gamma \in \mathcal{L}_1^2(J,T^2X) : \gamma(s) \in T_{o_{\theta(s)}}(T_{\theta(s)}X) \quad \forall s \in J\}$$

We also make the identification

$$VT_{0_{\theta_0}} \mathcal{L}_1^2(J,TX) \cong T_{\theta_0} \mathcal{L}_1^2(J,X) \qquad (9)$$

Therefore the projection of $T_{\theta_0}(T\mathcal{L}_1^2(J,X))$ onto the vertical factor together with the identification (9) will give us the Hessian at θ_o. Indeed by Theorem (2.2),

$$(T_{\theta_0}F)(\beta) = (T_{\theta_0}\xi^F)(\beta)(o) \qquad \forall\ \beta \in T_{\theta_0}\mathcal{L}_1^2(J,X) \qquad (10)$$

Taking "vertical parts" in (10) and applying the identification in (9) and also $T_{0_{\theta_0(o)}}(T_{\theta_0(o)}X) \cong T_{\theta_0(o)}X$, we see that

$$(dF)_{\theta_0}(\beta) = (d\xi^F)_{\theta_0}(\beta)(o) \qquad \forall\ \beta \in T_{\theta_0}\mathcal{L}_1^2(J,X) \qquad (11)$$

We next show that $s \to (d\xi^F)_{\theta_0}(\beta)(s)$ is a parallel field by proving that it is so in a particular coordinate system in X, viz. normal coordinates; then because the definition of the Hessian is intrinsic the result will hold (on any coordinate system). Fix $s_o \varepsilon$ J, and choose normal coordinates (U,ϕ) at $\theta_o(s_o)$. Let dim X = n and take the model $E = R^n$. Let $\Gamma_{ij}^k : U \to R$ and $\partial_i : U \to TU$ i,j,k = 1,...,n be the Christoffel's symbols and the standard vector fields associated with the chart (U,ϕ), where

$$\Gamma_{ij}^k(\theta_o(s_o)) = o \qquad i,j,k = 1,\ldots,n \qquad (12)$$

(Kobayashi and Nomizu [27], Milnor [35]). Write

$$\xi^F(\theta)(s) = \sum_{i=1}^n g_i(\theta(s))\, \partial_i(\theta(s)) \qquad (13)$$

where θ , s are allowed to vary in open neighbourhoods about θ_o and s_o so that $\theta(s)\ \varepsilon$ U, and the $g_i : U \to R$ are C^1 functions. By parallelism of the field $s \to \xi^F(\theta)(s)$ these satisfy the ODE

$$\sum_{k=1}^n \frac{\partial}{\partial s} g_k(\theta(s)).\, \partial_k(\theta(s)) + \sum_{i,j,k=1}^n \frac{\partial}{\partial s}\phi^i(\theta(s))\, \Gamma_{ij}^k(\theta(s))\, g_j(\theta(s)).\partial_k(\theta s)) = 0 \qquad (14)$$

the $\phi^i : U \to R$ being coordinate functions i.e. $\phi^i = p_i \circ \phi$ with $p_i : R^n \to R$ the projection onto the ith factor.

Since θ_o is critical and the ∂_i are linearly independent, $g_i(\theta_o(s)) = o \quad \forall\, 1 \leq i \leq n$ and for all s in a neighbourhood of s_o. Now regarding the left hand side of (13) as a function of two variables θ, s and taking the Hessian at $\theta = \theta_o$, Lemma (2.1) gives us:

$$(d\xi^F)_{\theta_o}(\beta)(s) = \sum_{i=1}^{n} (T_{\theta_o(s)}\, g_i)(\beta(s)).\, \partial_i(\theta_o(s)) \qquad \forall\, \beta \in T_{\theta_o}\mathcal{L}_1^2(J,X) \tag{15}$$

for s in a neighbourhood of s_o.

(14) $\Longrightarrow$

$$\frac{\partial}{\partial s}\, g_k(\theta(s)) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial s}\, \phi^i(\theta(s))\; \Gamma^k_{ij}(\theta(s))\; g_j(\theta(s)) \tag{16}$$

Denote by $\frac{\partial}{\partial \theta}$ differentiation wrt θ while s is kept fixed. Then by differentiating the evaluation map $\rho : J \times \mathcal{L}_1^2(J,X) \to X$ wrt θ it is easy to see that

$$(T_{\theta(s)} g_k)(\beta(s)) = \left[\frac{\partial}{\partial \theta}\, g_k(\theta(s))\right](\beta) \qquad \beta \in T_\theta\, \mathcal{L}_1^2(J,X) \tag{17}$$

$$\therefore\ \frac{\partial}{\partial s}\left[(T_{\theta(s)} g_k)(\beta(s))\right] = \frac{\partial}{\partial s}\left[\frac{\partial}{\partial \theta}\, g_k(\theta(s))\right](\beta)$$

$$= \left[\frac{\partial}{\partial \theta}\, \frac{\partial}{\partial s}\, g_k(\theta(s))\right](\beta) \qquad \text{a.a.s.} \tag{18}$$

where the equality in (18) holds for almost all s in the neighbourhood of s_o and for al θ near θ_o; this is because locally we can write $\beta = \frac{d}{dt}\,\theta_t\Big|_{t=o}$ where $t \mapsto \theta_t$ is a C^1 path in $\mathcal{L}_1^2(J,X)$ and then defining the function $f: I_o \times J_o \to R$ by $f(t,s) = g_k(\theta_t(s)) \quad t \in I_o, \quad s \in J_o$

where I_o is a neighbourhood of o and J_o a neighbourhood of s_o, we see that the relation

$$\frac{\partial^2 f(t,s)}{\partial t \partial s} = \frac{\partial^2 f(t,s)}{\partial s \partial t} \qquad \text{a.a.s} \qquad \forall\ t \tag{19}$$

holds by integrating over arbitrary rectangles in $I_o \times J_o$. (19) will then imply (18).

Now differentiate (15) covariantly wrt s to obtain

$$\frac{D}{ds}\left[(d\xi^F)_{\theta_o}(\beta)(s))\right]\Bigg|_{s=s_o} = \sum_{i=1}^{n} \frac{d}{ds}\left[({}^{T}\theta_o(s) g_i)(\beta(s))\right]\Bigg|_{s=s_o} \cdot \partial_i(\theta_o(s_o))$$

$$= -\frac{\partial}{\partial\theta}\sum_{i,j,k=1}^{n}\left[\frac{\partial}{\partial s}\phi^i(\theta(s))\ \Gamma_{ij}^k(\theta(s))\ g_j(\theta(s))\right]\Bigg|_{\substack{\theta=\theta_o\\ s=s_o}} (\beta).\partial_k(\theta_o(s_o))$$

(by (16) and (18))

$$= -\sum_{i,j,k=1}^{n}\left\{\frac{\partial}{\partial\theta}\frac{\partial}{\partial s}\phi^i(\theta(s))\ \Gamma_{ij}^k(\theta(s))\ g_j(\theta(s)) + \right.$$

$$\left. + \frac{\partial}{\partial s}\phi^i(\theta(s))\ \frac{\partial}{\partial\theta}\Gamma_{ij}^k(\theta(s))\ g_j(\theta(s)) + \frac{\partial}{\partial s}\phi^i(\theta(s))\ \Gamma_{ij}^k(\theta(s))\ \frac{\partial g_j(\theta(s))}{\partial\theta}\right\}\Bigg|_{\substack{\theta=\theta_o\\ s=s_o}}$$

$$(\beta).\partial_k(\theta_o(S_o))$$

$= 0$ because of (12) and the fact that $g_j(\theta_o(s_o)) = o$.

Since s_o is arbitrary it follows that $s \to (d\xi^F)_{\theta_o}(\beta)(s)$ is a parallel vector field along θ_o and, by uniqueness of parallel transport, relation (11) dictates that

$$(d\xi^F)_{\theta_o}(\beta)(s) = {}^{\theta}\tau_o{}^s\{(dF)_{\theta_o}(\beta)\} \qquad \forall\ s \in J \qquad \text{Q.E.D.}$$

Remark:(2.9)

In terms of the notation of Theorem (2.2), our last result (Theorem 2.3) says that for each critical θ_o range of $(d\xi^F)_{\theta_o} \subseteq H_{\theta_o}\mathcal{L}_1^2(J,X)$, the fibre at θ_o of the subbundle $i\{\rho_o^*(TX)\} \to \mathcal{L}_1^2(J,X)$. This observation will be used in the forthcoming section to give a satisfactory definition for the index of a critical path $\theta_o \in \mathcal{L}_1^2(J,X)$.

§4. RFDE's of Gradient Type: (GRFDE)

This section is intended to contribute towards isolating a class of RFDE's for which the classical Morse inequalities are valid in the state space $\mathcal{L}^2_1(J,X)$.

X is a C^p $(p \geqslant 5)$ finite dimensional Riemannian manifold. We fix a C^{p-4} Riemannian metric g on $\mathcal{L}^2_1(J,X)$ which coincides with the pull-back onto the subbundle $i\{\rho_0^*(TX)\}$ of the Riemannian metric on the base manifold X.; g may be taken to be either of the following two metrics

$$g_1(\theta)(\beta,\gamma) = \langle\beta(-r),\gamma(-r)\rangle_{\theta(-r)} + \frac{1}{r}\int_{-r}^{0}\left\langle \frac{D\beta(s)}{ds}, \frac{D\gamma(s)}{ds}\right\rangle_{\theta(s)} ds$$

$$g_2(\theta)(\beta,\gamma) = \frac{1}{r}\int_{-r}^{0}\langle\beta(s),\gamma(s)\rangle_{\theta(s)}\, ds + \frac{1}{r}\int_{-r}^{0}\left\langle \frac{D\beta(s)}{ds}, \frac{D\gamma(s)}{ds}\right\rangle_{\theta(s)} ds$$

for $\theta \in \mathcal{L}^2_1(J,X)$, β , $\gamma \in T_\theta\mathcal{L}^2_1(J,X)$. All results may be taken to hold for any of the above metrics unless one of them is explicitly singled out.

A gradient RFDE on X is a 4-tuple (F,Φ,J,X) where (F,J,X) is a (C^1) RFDE and $\Phi : \mathcal{L}^2_1(J,X) \to R$ a C^2 function s.t. $\xi^F = \text{grad}\,\Phi$ in the admissible metric g on $\mathcal{L}^2_1(J,X)$.

Condition (M):

A C^1 RFDE (F,J,X) satisfies condition (M) if for each critical path $\theta \in \mathcal{L}^2_1(J,X)$ the restriction $(d\xi^F)_\theta|_{H_\theta\mathcal{L}^2_1(J,X)} : H_\theta\mathcal{L}^2_1(J,X)$ is a linear homeomorphism. (See Remark (2.3)).

If we denote the set of critical paths of F by $C(F) \subset \mathcal{L}^2_1(J,X)$, condition (M) is a regularity condition on C(F) making it into a submanifold of $\mathcal{L}^2_1(J,X)$ and at the same time expressing "non-degeneracy" in the transverse direction to C(F). Indeed we have

Proposition (2.3):

Let (F,Φ,J,X) be a GRFDE with Φ C^2. Then $C(F)$ coincides with the critical points of Φ in $\mathcal{L}^2_1(J,X)$, and if $T^2_\theta\,\Phi$: $T_\theta\mathcal{L}^2_1(J,X) \times T_\theta\mathcal{L}^2_1(J,X) \to R$ is the Hessian of Φ at $\theta \in C(F)$ (Palais [39] §7) then $(d\xi^F)_\theta$ is a symmetric operator on $T_\theta\mathcal{L}^2_1(J,X)$ s.t.

$$(T^2_\theta\,\Phi)(\beta,\gamma) = g(\theta)((d\xi^F)_\theta(\beta),\gamma) \quad \forall\ \beta\ ,\ \gamma \in T_\theta\ \mathcal{L}^2_1(J,X).$$

Proof:

In what follows we choose a local model of the form $\mathcal{L}^2_1(J,H)$, where H is some Hilbert space and $\mathcal{L}^2_1(J,H)$ is furnished with an inner product which coincides with that of H on the constant paths i.e. think of it as either $\langle\beta,\gamma\rangle_1 = \langle\beta(-r),\gamma(-r)\rangle_H + \frac{1}{r}\int_{-r}^{0}\langle\beta'(s),\gamma'(s)\rangle_H\,ds$ or $\langle\beta,\gamma\rangle_2 = \frac{1}{r}\int_{-r}^{0}\langle\beta(s),\gamma(s)\rangle_H ds + \frac{1}{r}\int_{-r}^{0}\langle\beta'(s),\gamma'(s)\rangle_H ds$ for $\beta,\gamma \in \mathcal{L}^2_1(J,H)$.

It is easy to see from the definition of a GRFDE that

$$\theta \in C(F) \iff T_\theta\,\Phi = 0.$$

Working locally, the Hessian $T^2_\theta\Phi$ at $\theta \in C(F)$ is given by the second Frechet derivative

$$\begin{array}{ccc} \mathcal{L}^2_1(J,H) \times \mathcal{L}^2_1(J,H) & \longrightarrow & R \\ (\beta,\gamma) & \longmapsto & D^2\Phi(\theta_0)(\beta)(\gamma) \end{array}$$

and because Φ is C^2 it follows that $T^2_\theta\,\Phi$ is a continuous symmetric bilinear form on $T_\theta\,\mathcal{L}^2_1(J,X)$. (Dieudonné [8] P.175).

To prove the last assertion of the proposition, we pass to the cotangent bundles T^*X and $T^*\,\mathcal{L}^2_1(J,X)$. Define the 1-form

$$\omega : \mathcal{L}^2_1(J,X) \longrightarrow T^*\ \mathcal{L}^2_1(J,X) \text{ by}$$

$$\omega(\theta) = T_\theta\Phi \qquad \forall\ \theta \in \mathcal{L}^2_1(J,X)$$

Then

$$\omega = (\text{grad } \Phi)^* \tag{1}$$

where * is the dual isomorphism fulfilling the diagram

$$\begin{tikzcd}
T(T^*\mathcal{L}^2_1(J,X)) \arrow[d] & & T(T\mathcal{L}^2_1(J,X)) \arrow[ll, "T(*)"'] \arrow[d] \\
T^*\mathcal{L}^2_1(J,X) & & T\mathcal{L}^2_1(J,X) \arrow[ll, "*"'] \\
& \mathcal{L}^2_1(J,X) \arrow[ul, "\omega"] \arrow[ur, "\text{grad } \Phi = \xi^F"'] &
\end{tikzcd}$$

with * and $T(*)$ are linear isometries on the fibres. In fact $T_{0_\theta}(*) : T_{0_\theta}(T\mathcal{L}^2_1(J,X)) \longrightarrow T_{0^*_\theta}(T\mathcal{L}^2_1(J,X))$ can be identified with

$$T_\theta\mathcal{L}^2_1(J,X) \times T_\theta\mathcal{L}^2_1(J,X) \longrightarrow T_\theta\mathcal{L}^2_1(J,X) \times T^*_\theta\mathcal{L}^2_1(J,X)$$

$$(\beta,\gamma) \longmapsto (\beta,\gamma^*)$$

where $\gamma^*(s) = (\gamma(s))^* \quad \forall\, s \in J$.

Observe that $\theta \in C(F) \Leftrightarrow \omega(\theta) = o$, so that we can define the Hessian of ω at such a θ in the spirit of §3; it then follows easily from (1) and the above observations that for each $\theta \in C(F)$,

$$(d\omega)_\theta = [(d(\text{grad } \Phi))_\theta]^* = [(d\xi^F)_\theta]^*$$

where * denotes the adjoint of the operator $(d\xi^F)_\theta$. Finally, identifying the Hessian $T^2_\theta \Phi$ with the composition

$$T_{0_\theta}(T\mathcal{L}^2_1(J,X)) \xrightarrow{T_{0_\theta}(T\Phi)} R \times R \xrightarrow{pr_2} R$$

we obtain

$$(T^2_\theta \Phi)(\beta,\gamma) = (d\omega)_\theta(\beta)(\gamma) = g(\theta)((d\xi^F)_\theta(\beta),\gamma)$$

for $\beta, \gamma \in T_\theta\mathcal{L}^2_1(J,X)$. Q.E.D.

<u>Proposition (2.4)</u>:

Suppose dim $X = n$ and (F,Φ, J,X) is a C^k GRFDE satisfying condition (M). Then $C(F)$ is a C^k $(o < k \leqslant p-3)$ closed submanifold of $\mathcal{L}^2_1(J,X)$ with codimension n, called the <u>critical manifold of F</u>. Furthermore, the correspondence

$$\mathcal{L}^2_1(J,X) \ni \theta \longmapsto [H_\theta \mathcal{L}^2_1(J,X)]^\perp \subset T_\theta \mathcal{L}^2_1(J,X)$$

defines a C^{p-4} subbundle of $\pi_1 : T\mathcal{L}^2_1(J,X) \longrightarrow \mathcal{L}^2_1(J,X)$ orthogonal to $i\{\rho_0^*(TX)\}$ and tangential to the critical manifold $C(F)$.

<u>Proof:</u>

The zero section $(TX)_0$ of $\pi_0 : TX \to X$ is a C^{p-1} submanifold of TX and

$$C(F) = F^{-1}\{(TX)_0\}$$

Recall that by Theorem (2.3) we have for each $\theta \in C(F)$

$$(d\xi^F)_\theta (\beta)(s) = {}^\theta\tau_0^s \{(dF)_\theta(\beta)\} \qquad \forall s \in J, \forall \beta \in T_\theta \mathcal{L}^2_1(J,X)$$

so that Condition (M) implies that $(dF)_\theta : T_\theta \mathcal{L}^2_1(J,X) \to T_{\theta(o)}X$ is surjective. Moreover $\ker (dF)_\theta$ splits in $T_\theta \mathcal{L}^2_1(J,X)$ because of the Hilbert space structure, it therefore follows that F is transversal to $(TX)_0$ and hence $C(F)$ is a C^k submanifold of $\mathcal{L}^2_1(J,X)$ with tangent space(s)

$$T_\theta C(F) = \ker (dF)_\theta = \ker (d\xi^F)_\theta \qquad \forall \theta \in C(F).$$

But by Proposition (2.3) we know that $(d\xi^F)_\theta$ is self-adjoint in $g(\theta)$ and because of the fact that $(d\xi^F)_\theta | H_\theta \mathcal{L}^2_1(J,X)$ is a linear homeomorphism we must have

$$\ker (d\xi^F)_\theta = [H_\theta \mathcal{L}^2_1(J,X)]^\perp = T_\theta C(F) \qquad \forall \theta \in C(F).$$

The statement about the differentiability of the subbundle $\theta \longmapsto [H_\theta \mathcal{L}^2_1(J,X)]^\perp$ is a direct consequence of the differentiability of the Riemannian metric together with that of parallel transport (Theorem (2.2) (iii)). Q.E.D.

The above proposition exhibits a high degree of degeneracy for the critical paths C(F); and it also suggests that: (a) if we are to develop Morse inequalities for the function Φ , then these will have to involve estimates for the number of components of C(F) rather than the individual critical paths i.e. adopting the viewpoint of R. Bott (Bott [4], Eells [12]), (b) since C(F) is infinite-dimensional for $r > 0$, Φ never satisfies condition (C) of Palais and Smale (Palais and Smale [40], Palais [39]).

However if X is finite-dimensional then, by counting components of C(F), one might be able to drop condition (C) altogether. The components of C(F) are called critical manifolds of Φ , and condition (M) says that these are non-degenerate in the sense of Bott (Eells [12]).

The following result was first proved in [4] for compact non-degenerate critical manifolds.

Proposition (2.5):

Let (F,Φ, J,X) be a GRFDE satisfying condition (M). For each $\theta \in C(F)$ define the index of θ , $\lambda(\theta)$, to be the dimension of the maximal subspace of $H_\theta \mathcal{L}^2_1(J,X)$ on which $(d\xi^F)_\theta | H_\theta \mathcal{L}^2_1$ (or $T^2_\theta\Phi | H_\theta \mathcal{L}^2_1 \times H_\theta \mathcal{L}^2_1$) is negative definite.

Then Φ and the function

$$\lambda : C(F) \longrightarrow Z^{\geq 0}$$
$$\theta \longmapsto \lambda(\theta)$$

are both constant on each critical manifold in C(F).

Proof:

We prove first that $\Phi | C(F)$ is locally constant on C(F). By Proposition (2.4) it is sufficient to show that $T_\theta \Phi$, $\theta \in C(F)$, vanishes on the fibres $[H_\theta \mathcal{L}^2_1(J,X)]^\perp$ tangent to C(F); indeed if $\beta \in [H_\theta \mathcal{L}^2_1(J,X)]^\perp$ then

$$(T_\theta\Phi)(\beta) = g(\theta)\ (\xi^F(\theta),\beta) = 0 \text{ by orthogonality.}$$

Thus Φ is constant on components of C(F).

To show that the index map $\lambda: C(F) \to Z^{\geq 0}$ is locally constant we use the notation of Theorem (2.2). Fix $\theta_o \in C(F)$ and a sufficiently small neighbourhood V of θ_o in C(F) so that $\rho_o(V)$ is contained within a normal chart in X around $\theta_o(o)$. Let H be the real Hilbert space $T_{\theta_o(o)}X$, then for each $\theta \in V$ we have isometries $h_{\theta(o)}: H \xrightarrow{\cong} T_{\theta(o)}X$ given by parallel transport along geodesics in X. Also by choice of the Riemannian metric on $\mathcal{L}^2_1(J,X)$ each map $i_\theta : T_{\theta(o)}X \to T_\theta \mathcal{L}^2_1(J,X)$ is an isometric embedding.

Now define for each $\theta \in V$ a continuous linear map $A_\theta : H \circlearrowleft$ by setting

$$A_\theta = h_{\theta(o)}^{-1} \circ (dF)_\theta \circ i_\theta \circ h_{\theta(o)}$$

Using the symmetry of the Hessian $(d\xi^F)_\theta$ (Proposition 2.3) and applying Theorem (2.3), it is not hard to see that each A_θ is a symmetric linear homeomorphism. $\therefore \forall \theta \in V$, $A_\theta \in GL(H)$, the group of invertible linear operators on H. Therefore we have a continuous map

$$\begin{aligned} V &\longrightarrow GL(H) \\ \theta &\longmapsto A_\theta \end{aligned}$$

For any $A \in GL(H)$, let d(A) be the dimension of the maximal subspace of H on which A is negative definite. Then since the identification maps $h_{\theta(o)}$ and i_θ are isometries it follows from the definition of λ that $\lambda(\theta) = d(A_\theta)$ $\forall \theta \in V$. By virtue of the continuity of the map $\theta \longmapsto A_\theta$ we need only show that the map

$$\begin{aligned} d: GL(H) &\longrightarrow Z^{\geq 0} \\ A &\longmapsto d(A) \end{aligned}$$

is locally constant in the uniform operator topology on GL(H). We proceed to do so by choosing $A \in GL(H)$ and letting $E_A^- \subset H$ be the maximal negative subspace for A. Define the map $\mu : GL(H) \times (E_A^- - \{o\}) \to R$ by

$$\mu(C,v) = \langle Cv,v\rangle \qquad \forall\, C \in GL(H)$$
$$\forall\, v \in E_A^- - \{o\}$$

where $\langle \cdot,\cdot \rangle$ is the inner product on H. Then μ is continuous because the evaluation map and the inner product are. Therefore the set

$$M_A \equiv \{(B,v) : B \in GL(H),\ v \in E_A^- - \{o\},\ \langle Bv,v\rangle < o\}$$
$$= \mu^{-1}(-\infty,o)$$

is open in $GL(H)\times(E_A^- - \{o\})$. Denote by $M_A^1 \subset GL(H)$ the projection of M_A onto GL(H). Then M_A^1 is open and $A \in M_A^1$; so $\exists\, \varepsilon > o$ s.t.

$$B \in GL(H),\quad \|B-A\| < \varepsilon \Rightarrow (B,v) \in M_A \quad \forall\, v \in E_A^- - \{o\}$$
$$\Rightarrow E_A^- \subseteq E_B^- \Rightarrow d(A) \leq d(B) \tag{1}$$

With the above ε (depending on A), replace A by -A and B by -B to get

$$\|B-A\| < \varepsilon \quad \Rightarrow \quad d(-A) \leq d(-B) \tag{2}$$

If dim H = n, then as A and B are linear homeomorphisms, $d(A) + d(-A) = n = d(B) + d(-B)$, and (2) gives $d(A) \geqslant d(B)$ if $\|B-A\| < \varepsilon$. Combining this with (1) it follows that $\|B-A\| < \varepsilon \Rightarrow d(A) = d(B)$. This completes the proof of this proposition. Q.E.D.

Having made the necessary preparation, our study of C(F) culminates in writing down the Morse inequalities in the state space $\mathcal{L}_1^2(J,X)$ for the g_1- GRFDE (F,Φ,J,X) i.e. where $\mathcal{L}_1^2(J,X)$ is being furnished with the metric g_1. The inequalities are intended to point out the relationships between the topology of the state space $\mathcal{L}_1^2(J,X)$ and the number of critical manifolds of Φ with a given index, the latter being well-defined by Proposition (2.5).

We shall borrow our terminology from Palais [39] and Milnor [35]; so fix a field K (e.g. K = Q or R) and for any pair of topological spaces (B,A), $A \subseteq B$, denote by $H_k(B,A;K)$, k = 0,1,2,..., the relative singular homology groups with coefficients in K. Say (B,A) is <u>admissible</u> if each $H_k(B,A;K)$ is finitely generated and $\exists\, n_o \geq o$ s.t. $H_k(B,A;K) = o \quad \forall\, k \geq n_o$. The <u>k-th Betti number</u> $\beta_k(B,A;K)$ of an admissible pair (B,A) wrt K is the rank of $H_k(B,A;K)$ over K, and the <u>Euler characteristic</u> $\chi(B,A;K)$ is defined by

$$\chi(B,A;K) = \sum_{m=o}^{\infty} (-1)^m \beta_m(B,A;K) \quad \text{(a finite sum)}$$

which reduces to a finite sum for admissible pairs. If $a \in R$ define $\Phi_a = \{\theta : \theta \in \mathcal{L}_1^2(J,X),\ \Phi(\theta) \leq a\}$; Call $a \in R$ a <u>regular value of</u> Φ if $\Phi^{-1}(a)$ contains no critical paths of F.

<u>Theorem (2.4)</u>:

Let X be a compact n-dimensional $C^p (p > 5)$ Riemannian manifold and (F,Φ,J,X) a g_1-GRFDE with F of class C^2 and satisfying condition (M). Suppose a, b $\in$ R, a < b, are regular values of Φ . Then the pair (Φ_b,Φ_a) is admissible, and $\Phi^{-1}[a,b] \cap C(F)$ is the union of a finite number of critical manifolds of F. Indeed, if $\mu_m(a,b;\Phi)$ is the number of critical manifolds of F in $\Phi^{-1}[a,b]$ with index m, then

$$\sum_{m=o}^{k} (-1)^{k-m} \beta_m(\Phi_b,\Phi_a;K) \leq \sum_{m=o}^{k} (-1)^{k-m} \mu_m(a,b;\Phi)$$

$\forall\, k \geq o$; equality holds for $k \geq n = \dim X$ i.e.

$$\chi(\Phi_b,\Phi_a;K) = \sum_{m=o}^{n} (-1)^{n-m} \mu_m(a,b;\Phi) \quad .$$

Proof:

Since $\mathcal{L}^2_1(J,X)$ is endowed with the special metric g_1 it is easy to see that for each $\theta \in \mathcal{L}^2_1(J,X)$

$$[H_\theta \ \mathcal{L}^2_1(J,X)]^\perp = \{\beta : \beta \in T_\theta \mathcal{L}^2_1(J,X), \quad \beta(-r) = o\}$$

Thus $\theta \longmapsto [H_\theta \mathcal{L}^2_1(J,X)]^\perp$ is an integrable subbundle of $T\mathcal{L}^2_1(J,X) \to \mathcal{L}^2_1(J,X)$; in fact it is tangent to the fibres of the fibration $\rho_{-r}: \mathcal{L}^2_1(J,X) \to X$ where ρ_{-r} is evaluation at -r i.e.

$$\rho_{-r}(\theta) = \theta(-r) \qquad \forall\, \theta \in \mathcal{L}^2_1(J,X)$$

The fibres of ρ_{-r} are the closed C^{p-3} submanifolds $\rho^{-1}_{-r}(x)$, $x \in X$, of $\mathcal{L}^2_1(J,X)$ and $T_\theta\, \rho^{-1}_{-r}(x) = [H_\theta \mathcal{L}^2_1(J,X)]^\perp \quad \forall\, \theta$ s.t. $\theta(-r) = x$.

Now Φ is constant in each fibre $\rho^{-1}_{-r}(x)$, because if $\theta \in \rho^{-1}_{-r}(x)$ and $\beta \in [H_\theta \mathcal{L}^2_1(J,X)]^\perp$ then

$$(T_\theta \Phi)(\beta) = g_1(\theta)\,(\xi^F(\theta),\beta) = o \text{ by orthogonality.}$$

Define a function $f : X \to R$ by

$$f(x) = \Phi(\tilde{x}) \qquad \forall\, x \in X \tag{1}$$

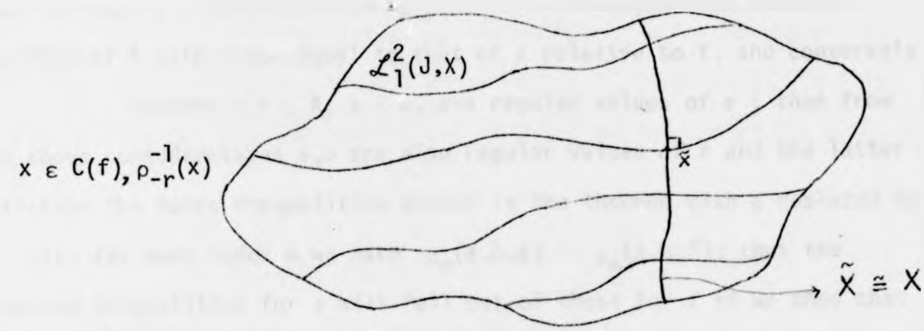
where $\tilde{x} : J \longrightarrow X$ is the constant path at x. Then f is C^3 because Φ is C^3, by hypothesis, and the mapping $\begin{array}{l} X \to \mathcal{L}^2_1(J,X) \\ x \to \tilde{x} \end{array}$ is a C^{p-3} (Riemannian) embedding. Moreover it is easily seen that $\Phi = f \circ \rho_{-r}$, and a simple calculation yields

$$C(F) = \rho^{-1}_{-r}\{C(f)\} \tag{2}$$

where $C(f) \subset X$ is the set of critical points of f in X.

We next show that f is a Morse function on X, i.e. one all of whose critical points are non-degenerate (Milnor [35]).

Let $\tilde{X} \subset \mathcal{L}^2_1(J,X)$ be the set of constant paths $J \to X$. Then $\tilde{X}$ is a closed Riemannian submanifold of $\mathcal{L}^2_1(J,X)$ canonically isometric to X. In fact $\tilde{X}$ lies orthogonally across the fibration ρ_{-r} in the following manner



Define $\tilde{f}: \tilde{X} \to R$ by $\tilde{f} = \Phi|\tilde{X}$; then $\tilde{f}$ is C^3. Now for each $\tilde{x} \in \tilde{X}$ we have

$T_{\tilde{x}}\tilde{X} = H_{\tilde{x}}\mathcal{L}^2_1(J,X)$ and $g_1(\tilde{x})|H_{\tilde{x}}\mathcal{L}^2_1 \times H_{\tilde{x}}\mathcal{L}^2_1 = \langle\cdot,\cdot\rangle_x$,

the inner product on T_xX. Also from the definition of $\tilde{f}$ it is easily checked that

$$\mathrm{grad}\,\tilde{f} = \mathrm{grad}\,\Phi|\tilde{X} \tag{3}$$

where the gradients are taken wrt the Riemannian structures on $\tilde{X}$ and $\mathcal{L}^2_1(J,X)$ respectively. Let $\tilde{x} \in C(\tilde{f})$. Then the Hessian $[d(\mathrm{grad}\,\tilde{f})]_{\tilde{x}}: T_{\tilde{x}}\tilde{X} \circlearrowleft$ is given by the composition.

$$T_{\tilde{x}}(\mathrm{grad}\,\tilde{f}): T_{\tilde{x}}\tilde{X} \to T_{0_{\tilde{x}}}(T\tilde{X}) \cong T_{\tilde{x}}\tilde{X} \oplus T_{\tilde{x}}\tilde{X} \xrightarrow{pr_2} T_{\tilde{x}}\tilde{X} ,$$

therefore, differentiating (3) and using the fact that the Hessian of $\mathrm{grad}\,\Phi = \xi^F$ is the same whether taken wrt the subbundle $\theta \to H_\theta\mathcal{L}^2_1(J,X)$ or wrt the whole tangent bundle $T\mathcal{L}^2_1(J,X)$ (Theorem 2.3) it follows that

$$[d(\mathrm{grad}\,\tilde{f})]_{\tilde{x}} = (d\xi^F)_{\tilde{x}}|H_{\tilde{x}}\mathcal{L}^2_1(J,X) \tag{4}$$

Since F satisfies condition (M), (4) implies that $\tilde{f}$ has all critical points non-degenerate and hence so has f by isometry. (4) also tells us that

$\lambda(\tilde{x})$ = dimension of maximal negative subspace of $[d(\text{grad } \tilde{f})]_{\tilde{x}}$

= index of x wrt f, again by isometry.

Thus for each $x \varepsilon C(f)$ the leaf $\rho_{-r}^{-1}(x)$ is a non-degenerate critical manifold of F with index equal to that of x relative to f, and conversely.

Suppose $a,b \varepsilon R$, $a < b$, are regular values of Φ ; then from the above considerations a,b are also regular values of f and the latter satisfies the Morse inequalities stated in the theorem with Φ replaced by f. Also for each index m we have $\mu_m(a,b,\Phi) = \mu_m(a,b;f)$; thus the required inequalities for Φ will fall out of those for f if we show that the pair (Φ_b,Φ_a) is homologically the same as (f_b,f_a) in the coefficient field K. We shall prove more than that: viz the homotopy equivalence $(\Phi_b,\Phi_a) \simeq (f_b,f_a)$. Consider the embedding of pairs $j:(f_b,f_a) \to (\Phi_b,\Phi_a)$ defined by

$$j(x) = \tilde{x} \qquad \text{for } x \varepsilon f_b$$

To see that the evaluation $\rho_{-r}: (\Phi_b,\Phi_a) \to (f_b,f_a)$ is a (left and right) homotopy inverse for j, observe that first $\rho_{-r} \circ j = id|f_b$. On the other hand we define a homotopy $h: J \times (\Phi_b,\Phi_a) \to (\Phi_b,\Phi_a)$ of pairs parameterized by J and connecting $j \circ \rho_{-r}$ and $id|\Phi_b$ in the following way:

$$h(t,\theta)(s) = \begin{cases} \theta(s) & s \varepsilon [-r,t] \\ \theta(t) & s \varepsilon [t,o] \end{cases}$$

for $t \varepsilon J$, $\theta \varepsilon \Phi_b$. Then clearly, $h(o,.) = id|\Phi_b$ and $h(-r,.) = j \circ \rho_{-r}$. To prove that h is continuous, fix $t_o \varepsilon J$ and $\theta_o \varepsilon \Phi_b$, use compactness to cover $h(t_o,\theta_o)(J)$ by a finite number of open sets $\{U_k\}_{k=1}^M$ in X and choose closed subintervals $\{J_k\}_{k=1}^M$ of J s.t. $\theta_o(J_k) \subset U_k$ $1 \leq k \leq M$, and $\bigcup_{k=1}^{M-1} J_k \subset [-r,t_o] \subset \bigcup_{k=1}^{M} J_k$. Let $\mathcal{L}_1^2(J_k, U_k)$ stand for the open set

$$\mathcal{L}_1^2(J_k,U_k) = \{\theta : \theta \in \mathcal{L}_1^2(J,X), \quad \theta(J_k) \subset U_k\}$$

in $\mathcal{L}_1^2(J,X)$. By continuity of the evaluation map

$J \times \Phi_b \to X$ $\exists$ an open interval $J_o \subset J_M$ centred at t_o and a neighbourhood V_o of θ_o in $\mathcal{L}_1^2(J,X)$ s.t. $\theta(t) \in U_M$ $\forall\, t \in J_o$, $\forall\, \theta \in V_o$. Then it is not hard to see from the definition of h and the choice of the J_k, U_k, J_o, V_o that

$$(t,\theta) \in J_o \times \{V_o \cap \bigcap_{k=1}^{M} \mathcal{L}_1^2(J_k,U_k)\} \Rightarrow h(t,\theta) \in \bigcap_{k=1}^{M} \mathcal{L}_1^2(J_k,U_k)$$

Since $\bigcap_{k=1}^{M} \mathcal{L}_1^2(J_k,U_k)$ and $J_o \times \{V_o \cap \bigcap_{k=1}^{M} \mathcal{L}_1^2(J_k,U_k)\}$ are open neighbourhoods of $h(t_o,\theta_o),(t_o,\theta_o)$ respectively, it follows that h is the required homotopy.

Hence $\beta_m(\Phi_b,\Phi_a;K) = \beta_m(f_b,f_a;K)$ (Spanier [44] Chapter 4 §4) and the result follows. Q.E.D.

Corollary (2.4.1):

With the hypotheses of the theorem, we have

$$\beta_m(\Phi_b,\Phi_a;K) \leq \mu_m(a,b;\Phi) \qquad \forall\, m.$$

Corollary (2.4.2):

With the hypotheses of the theorem, F has only a finite number of critical manifolds, and $\beta_m(X;K) \leq \mu_m$ $\forall\, m \geq o$, where $\beta_m(X;K)$ is the m-th Betti number of X and μ_m is the number of critical manifolds of F with index m. Also $\chi(X;K) = \sum_{m=o}^{n} (-1)^{n-m} \mu_m$, where $\chi(X;K)$ is the Euler characteristic of X in the field K.

Remark (2.5):

A much wider class of gradient RFDE's can be defined by starting with a gradient field grad Φ on $\mathcal{L}_1^2(J,X)$ wrt an admissible Riemannian metric (§9) where ϕ: $\mathcal{L}_1^2(J,X) \to R$ is a given smooth function. We look at all

RFDE's of the form $F = T\rho_0 \circ \operatorname{grad} \phi$. In the special case when ϕ is the energy function

$$\phi(\theta) = \tfrac{1}{2} \int_{-r}^{0} |\theta'(s)|^2 \, ds \qquad\qquad \theta \in \mathcal{L}_1^2(J,X)$$

the critical set C(F) contains all the geodesics in X.

CHAPTER 3

Linearization of a RFDE, The Stable and Unstable Subbundles

This Chapter is devoted to showing that a C^1 RFDE (F,J,X) on a Riemannian manifold X can be linearized at each $\theta \in \mathcal{L}_1^2(J,X)$, and to the study of the dynamical properties of this linearization which live on the tangent bundle $T\,\mathcal{L}_1^2(J,X)$. The basic idea here is to use covariant differentiation in order to obtain the linearization and then observe that the topology on the Sobolev space $\mathcal{L}_1^2(J,X)$ is flexible enough to make the theory of compact linear semi-groups applicable.

We fix ideas by taking X a $C^p (p \geq 4)$(separable) Riemannian manifold modelled on a real Hilbert space E, and (F,J,X) a C^1 RFDE on X. $\mathcal{L}_1^2(J,X)$ is given a Riemannian structure induced from that of X via the metric g_2 (Chapter 2, §4) viz.

$$g_2(\theta)(\beta,\gamma) = \frac{1}{r}\int_{-r}^{0} \langle \beta(s),\gamma(s)\rangle_{\theta(s)} ds + \frac{1}{r}\int_{-r}^{0} \left\langle \frac{D\beta(s)}{ds}, \frac{D\gamma(s)}{ds} \right\rangle_{\theta(s)} ds$$

$\forall\, \theta \in \mathcal{L}_1^2(J,X)$, $\beta, \gamma \in T_\theta \mathcal{L}_1^2(J,X)$. When X is flat, a Hilbert space H, the corresponding inner product on $\mathcal{L}_1^2(J,H)$ is taken i.e.

$$\langle \beta , \gamma \rangle = \frac{1}{r}\int_{-r}^{0} \langle \beta(s),\gamma(s)\rangle_H ds + \frac{1}{r}\int_{-r}^{0} \langle \beta'(s),\gamma'(s)\rangle_H ds$$

$\forall\, \beta,\gamma \in \mathcal{L}_1^2(J,H)$.

Although all our calculations will be carried out in the metric g_2, the reader may check that everything in this Chapter still works if g_2 were replaced by the metric g_1 of Chapter 2 §4.

Our future discussions will require the following results concerning the smoothness properties of the orbits of F. We draw attention to the fact that these results (viz. Lemma (3.1), Theorem (3.1) below) are evidently independent of the Riemannian structure on X.

Lemma (3.1):

Let $\varepsilon > o$ and $\alpha \in \mathcal{L}_1^2([-r,\varepsilon],X)$.

i) If α is C^1, then the memory map

$$\begin{array}{ccc} [o,\varepsilon] & \xrightarrow{m(.,\alpha)} & \mathcal{C}^o(J,X) \\ t & \longmapsto & \alpha_t \end{array}$$

is C^1 and $(\alpha_t)' = (\alpha')_t \quad \forall t \in [o,\varepsilon]$

ii) If $\alpha' \in \mathcal{L}_1^2([-r,\varepsilon],TX)$, then

$$\begin{array}{ccc} [o,\varepsilon] & \longrightarrow & \mathcal{L}_1^2(J,X) \\ t & \longmapsto & \alpha_t \end{array}$$

is C^1 and also $(\alpha_t)' = (\alpha')_t \quad \forall t \in [o,\varepsilon]$.

Proof:

We only give a proof for (ii). The argument for (i) is completely analogous. First we show that it is sufficient to prove the result in Hilbert (or Banach) space. Since X is separable then by McAlpin's embedding theorem (Eells [12]) we can choose a C^p embedding (or even an immersion) $i : X \to H$ of X into some Hilbert space H. By the hypothesis in (ii) it is clear that $(i \circ \alpha)' \in \mathcal{L}_1^2([-r,\varepsilon],H)$, so that if the result is true in Hilbert space we must have

$$m'(t, i\circ\alpha) = m(t, Ti\circ\alpha') \qquad t \in [o,\varepsilon] \tag{1}$$

Now i induces a C^{p-3} map $\tilde{i} : \mathcal{L}_1^2(J,X) \to \mathcal{L}_1^2(J,H)$ (an embedding) by composition; so it is easy to see that if $\theta \in \mathcal{L}_1^2(J,X)$ and $\beta \in T_\theta\mathcal{L}_1^2(J,X)$ then

$$(T\tilde{i})(\theta)(\beta)(s) = (Ti)\,(\theta(s))(\beta(s)) \qquad s \in J \tag{2}$$

Evaluating each side of (1) at $s \in J$ we get

$$\{m'(t, i \circ \alpha)\}(s) = \{ \frac{d}{dt} \tilde{i}[\tilde{m}(t,\alpha)]\}(s)$$

$$= \{(T\tilde{i})(m(t,\alpha))(m'(t,\alpha))\}(s)$$

$$= (Ti)(\alpha(t+s))(m'(t,\alpha)(s)) \quad \text{by (2))} \tag{3}$$

$$\text{and } \{m(t,Ti \circ \alpha')\}(s) = (Ti)(\alpha(t+s))(\alpha'(t+s)) \tag{4}$$

We then equate the right hand sides of (3) and (4) and use the fact that $(Ti)(\alpha(t+s))$ is injective to get

$$m'(t,\alpha) = m(t,\alpha') \qquad t \in [o,\varepsilon] . \tag{5}$$

Hence without loss of generality assume that $X = H$, a real Hilbert space. Fix $s \in J$, let $t \in [o,\varepsilon]$. Then since α is C^1, Taylor's theorem gives for small enough $h \in R$:

$$\alpha_{t+h}(s) = \alpha(t+s+h)$$

$$= \alpha(t+s) + \alpha'(t+s)h + h \int_0^1 \{\alpha'(t+s+uh) - \alpha'(t+s)\}\, du$$

As the evaluation map is continuous linear, then

$$\alpha_{t+h} = \alpha_t + h.(\alpha')_t + h.R(t,h) \tag{6}$$

where $R(t,h) \in \mathcal{L}^2_1(J,H)$ is given by

$$R(t,h) = \int_0^1 \{(\alpha')_{t+uh} - (\alpha')_t\}du \tag{7}$$

Now since α' is of class $\mathcal{L}^2_1$, it is easily seen that $(t,h) \mapsto R(t,h) \in \mathcal{L}^2_1(J,H)$ is continuous, and $R(t,o) = o$. Therefore we can apply a converse of Taylor's theorem ([1] Chapter 1 §2) to conclude that $t \to \alpha_t \in \mathcal{L}^2_1(J,H)$ is C^1 on (o,ε) and $(\alpha_t)' = (\alpha')_t$ $\forall\, t \in (o,\varepsilon)$. This result can then be extended by continuity of $[o,\varepsilon] \ni t \to (\alpha')_t \in \mathcal{L}^2_1(J,H)$ to include right hand (and left hand) derivatives at $t = o$ $(t = \varepsilon)$. Q.E.D.

Condition $E_1(k)$ $(1 \le k \le p-2)$

Let $\theta \in \mathcal{L}_1^2(J,X)$ and choose a chart (U,ϕ) at $\theta(o)$, $0 < \delta \le r$ and $o < \varepsilon \le \delta$ so that the local representation $F_\theta^U = F \circ C : [o,\varepsilon) \times \mathcal{L}_1^2([-\delta,o],U) \to TX$ of F at θ is well-defined (Cf. Definition (1.4)), where C is the localizing map of Lemma (1.1). Suppose that F_θ^U admits an extension to a C^k map $[o,\varepsilon) \times \mathcal{C}^o([-\delta,o],U) \to TX$, $(1 \le k \le p-2)$.

Note that if F satisfies condition $E_1(1)$ $(k=1)$, then F is locally Lipschitz (Definition 1.4).

The next result exhibits the fact that under condition $E_1(k)$ full solutions (and orbits) of F get smoother and smoother as time goes on.

Theorem (3.1):

Suppose that F satisfies condition $E_1(k)$ $(1 \le k \le p-2)$, and let $\alpha^\theta : [-r,\infty) \to X$, $\theta \in \mathcal{L}_1^2(J,X)$, be a full solution of F at θ. Then $\alpha^\theta | [qr,\infty)$ is C^{q+1} for $o \le q \le k$, and the orbit $t \longmapsto \alpha_t^\theta \in \mathcal{L}_1^2(J,X)$ is C^{q-1} on $[qr,\infty)$, $1 \le q \le k$.

Proof:

The proof proceeds by induction on the integer q. Result is obviously true for $q = o$. Suppose, by induction, that for some $o \le q < k$ $\alpha^\theta | [qr,\infty)$ is C^{q+1}. Fix $t_o \in [(q+1)r,\infty)$ and choose a chart (U,ϕ) at $\alpha^\theta(t_o)$, $o < \delta \le r$ and $o < \varepsilon < \delta$ so that the local representation $F^U_{\alpha^\theta_{t_o}} : [o,\varepsilon) \times \mathcal{L}_1^2([-\delta,o],U) \to TX$ is extendible to a C^k map $[o,\varepsilon) \times \mathcal{C}^o([-\delta,o],U) \to TX$, where $\alpha^\theta\{[t_o-\delta, t_o+\varepsilon)\} \subset U$, $t_o-\delta > qr$. Applying Lemma (3.1)(i) to the C^{q+1} map $\alpha^\theta | [t_o-\delta, t_o+\varepsilon]$ we see that the path $[t_o,t_o+\varepsilon] \to \mathcal{C}^o([-\delta,o],U)$

$$t \longmapsto [\alpha_t^\theta]_{[-\delta,o]}$$

is C^{q+1}.

Now if we look at the proof of Theorem (1.2) (or alternatively, the definition of C: Lemma (1.1)) we get

$$(\alpha^{\theta})'(t) = F^{U}_{\alpha^{\theta}_{t_0}}(t - t_0, [\alpha^{\theta}_{t}]_{[-\delta,0]}) \quad \forall\, t \in [t_0, t_0 + \varepsilon) \qquad (1)$$

As $F^{U}_{\alpha^{\theta}_{t_0}}$ admits a C^k extension to $[0,\varepsilon)\times\ \mathcal{C}^{0}([-\delta,0],U)$, then the right hand side of (1) may be viewed as a composition of C^{q+1} maps, and so $\alpha^{\theta}|[t_0,t_0 + \varepsilon)$ is C^{q+2}. By the arbitrariness of t_0, the inductive hypothesis is valid for q + 1, thus proving the first assertion of the theorem. The second assertion of the theorem follows immediately from the first one and a repeated application of Lemma (3.1)(ii). Q.E.D.

In the notation of §(2.2) recall that the vector field ξ^F on $\mathcal{L}^2_1(J,X)$ is a section of the subbundle $i\{\rho_0^*(T,X)\} \subset T\mathcal{L}^2_1(J,X)$ of the tangent bundle $\pi_1: T\mathcal{L}^2_1(J,X) \to \mathcal{L}^2_1(J,X)$. We use the notation and terminology of Eliasson ([17]§2); let ∇ signify covariant differentation of sections of the Riemannian bundle $i\{\rho_0^*(TX)\} \longrightarrow \mathcal{L}^2_1(J,X)$. Thus for each $\theta \in \mathcal{L}^2_1(J,X)$ we have a continuous linear map $\nabla\xi^F(\theta): T_\theta\mathcal{L}^2_1(J,X) \to H_\theta\,\mathcal{L}^2_1(J,X)$, called the <u>linearization of ξ^F (or F) at θ</u> . Note that when X is a linear space $\nabla\xi^F$ coincides with the ordinary Fréchet derivative, and if θ is a critical path $\nabla\xi^F(\theta)$ is the Hessian $(d\xi^F)_\theta$ (§2.3).

The following lemma is a crucial "bridge-result": allowing us to cross over between the linearization of F on $T\mathcal{L}^2_1(J,X)$ and the classical autonomous linear situation.

<u>Lemma (3.2)</u>:

For each $\theta \in \mathcal{L}^2_1(J,X)$, parallel transport defines a (canonical) Hilbert space isomorphism ${}^{\theta}\tau: T_\theta\mathcal{L}^2_1(J,X) \to \mathcal{L}^2_1(J,T_{\theta(0)}X)$. Indeed if $\dim X < \infty$, then for each $\beta \in T_\theta\mathcal{L}^2_1(J,X)$

$$\frac{D\beta(s)}{ds} = {}^{\theta}\tau_o^s \left[\frac{d}{ds}\{ {}^{\theta}\tau_s^o (\beta(s))\}\right] \qquad \text{a.a.} \quad s \in J \qquad (1)$$

where "$\frac{d}{ds}$" denotes ordinary differentiation of paths on $T_{\theta(o)}X$, and

"$\frac{D}{ds}$" covariant differentiation of vector fields along θ (Milnor [35], Eliasson [17]).

Proof:

Since the identity we want to prove is intrinsic, it suffices to check it locally in X. The linear bijection ${}^{\theta}\tau : T_{\theta}\mathcal{L}_1^2(J,X) \to \mathcal{L}_1^2(J,T_{\theta(o)}X)$ is defined by

$${}^{\theta}\tau(\beta)(s) = {}^{\theta}\tau_s^o(\beta(s)) \qquad \forall\, s \in J.$$

The fact that ${}^{\theta}\tau$ is an isometry - wrt the inner product $g_2(\theta)$ and its flat analogue on $\mathcal{L}_1^2(J,T_{\theta(o)}X)$ - is an easy consequence of the definition, the given identity and the isometric property of parallel transport.

Suppose dim X = n. Let $\beta \in T_{\theta}\mathcal{L}_1^2(J,X)$, $\theta \in \mathcal{L}_1^2(J,X)$, and fix $s_o \in J$. In local coordinates (U,ϕ) at $\theta(s_o)$ in X, write

$$\beta(s) = \sum_{i=1}^{n} h_i(\theta(s))\,\partial_i(\theta(s)) \qquad \text{near } s_o \qquad (2)$$

$$\therefore \frac{D\beta(s)}{ds} = \sum_{k=1}^{n} \frac{d}{ds} h_k(\theta(s))\partial_k(\theta(s)) + \sum_{i,j,k=1}^{n} \frac{d}{ds}\phi^i(\theta(s))\Gamma_{ij}^k(\theta(s))h_j(\theta(s))\partial_k(\theta(s))$$
$$\text{a.e. near } s_o \qquad (3)$$

where $\phi^i: U \to R$ are (C^p) coordinate functions, $\Gamma_{ij}^k: U \to R$ the Christoffel symbols associated with the Levi-Civita connection on X, and the ∂_k are standard vector fields on U. Using the linearity and the group property of parallel transport, it follows from (2) that

$${}^{\theta}\tau_o^s \left[\frac{d}{ds}\{{}^{\theta}\tau_s^o(\beta(s))\}\right] = \sum_{i=1}^{n} \frac{d}{ds} h_i(\theta(s))\,\partial_i(\theta(s)) + \sum_{i=1}^{n} h_i(\theta(s))\,{}^{\theta}\tau_o^s\left(\frac{d}{ds}\left[{}^{\theta}\tau_s^o(\partial_i(\theta(s)))\right]\right) \qquad (4)$$
$$\text{a.e. near } s_o$$

If $\beta(s_o) = o$, then $h_i(\theta(s_o)) = o \ \forall 1 \le i \le n$, and by comparing (3) and (4) we obtain in this case:

$$\left.\frac{D\beta(s)}{ds}\right|_{s=s_o} = {}^{\theta}\tau_o^{s_o} \left(\frac{d}{ds}\{{}^{\theta}\tau_s^{o}(\beta(s))\}\Big|_{s=s_o} \right) \qquad (5)$$

On the other hand if $\beta \in H_\theta \mathcal{L}_1^2(J,X)$ i.e. a parallel vector field along θ then equation (5) is trivially satisfied.

In order to see that (5) actually holds <u>for all</u> $\beta \in T_\theta \mathcal{L}_1^2(J,X)$ write each such β in the form

$$\beta = \beta_1 + \beta_2$$

where $\beta_1(s_o) = o$ and $\beta_2 \in H_\theta \mathcal{L}_1^2(J,X)$. Thus β_1 satisfies (5) and so does β_2; hence by linearity of both sides of (5) in β it follows that the result is valid $\forall \beta \in T_\theta \mathcal{L}_1^2(J,X)$. Q.E.D.

The next theorem contains two well-known classical results: a Sobolev embedding result and Rellich's lemma. We therefore quote them without proof, (Eells [12] §6, Sobolev [45], A. Friedman [19] P.D.E's Part 1 §11).

<u>Theorem (3.2)</u>:

Let H be a real Hilbert space and denote by $\mathcal{L}_2^2(J,H)$ the Hilbert space of all paths $\theta \in \mathcal{L}_1^2(J,H)$ s.t. $\theta' \in \mathcal{L}_1^2(J,H)$, with the inner product

$$\langle\theta,\psi\rangle_{\mathcal{L}_2^2} = \frac{1}{r}\left\{\int_{-r}^{o}\langle\theta(s),\psi(s)\rangle_H\, ds + \int_{-r}^{o}\langle\theta'(s),\psi'(s)\rangle_H ds + \int_{-r}^{o}\langle\theta''(s),\psi''(s)\rangle_H ds\right\}$$

for $\theta, \psi \in \mathcal{L}_2^2(J,H)$. Then the following is true:

i) (<u>Sobolev's embedding theorem</u>): The embedding

$$\mathcal{L}_2^2(J,H) \hookrightarrow \mathcal{C}^o(J,H)$$

is continuous linear.

ii) <u>Rellich's Lemma</u>. The embedding

$$\mathcal{L}^2_2(J,H) \hookrightarrow \mathcal{L}^2_1(J,H)$$

is continuous linear. If, further, dim $H < \infty$ then this embedding is compact.

Our next result draws upon the above theorem and the existence-uniqueness conclusions of Chapter 1, in order to generate a global semi-flow on the fibres of $T\mathcal{L}^2_1(J,X)$; and then explore some of its basic elementary properties.

<u>Theorem (3.3)</u>: Assume dim $X < \infty$.

Suppose that for each $\theta \in \mathcal{L}^2_1(J,X)$ F satisfies <u>Condition E_2</u>: the linearization $\nabla\xi^F(\theta)$ admits an extension to a continuous (linear) map $T_\theta \mathcal{C}^o(J,X) \to T_{\theta(o)}X$.

Then $\exists$ a semi-flow on $T_\theta \mathcal{L}^2_1(J,X)$ given by a strongly continuous semi-group $\{T_t\}_{t\geq o}$ of continuous linear operators on $T_\theta\mathcal{L}^2_1(J,X)$ having the properties:

i) The map $R^{\geq o} \times T_\theta\mathcal{L}^2_1(J,X) \longrightarrow T_0 \mathcal{L}^2_1(J,X)$

$$(t,\beta) \longmapsto T_t(\beta)$$

is continuous, and is C^{q-1} for $t \geq qr$, q(integer) ≥ 1.

ii) $T_{t_1+t_2} = T_{t_1} \circ T_{t_2}$ for $t_1, t_2 \geq o$, $T_o = id$, the identity map on $T_\theta\mathcal{L}^2_1(J.X)$.

iii) $\exists$ constants M, $\mu > o$ (depending only on θ) s.t.

$$\|T_t(\beta)\|_{T_\theta\mathcal{L}^2_1} \leq (Me^{\mu t} + 1)^{\frac{1}{2}} \|\beta\|_{T_\theta\mathcal{L}^2_1} \quad \forall\ t \geq o \quad \forall\ \beta \in T_\theta\mathcal{L}^2_1(J,X). \tag{1}$$

iv) For each $t \geq 2r$ the operator

$$T_t: \quad T_\theta\mathcal{L}^2_1(J,X) \hookleftarrow \quad \text{is compact.}$$

v) For each $\beta \in T_\theta \mathcal{L}^2_1(J,X)$ the vector field $T_t(\beta)$ satisfies the "linear retarded covariant FDE":

$$\frac{D}{\partial s}\{T_t(\beta)(s)\}\Big|_{s=o} = \nabla\xi^F(\theta)(T_t(\beta))(o) \qquad t > o \tag{2}$$

Proof:

Let $\theta \in \mathcal{L}^2_1(J,X)$. Using the isomorphism ${}^\theta\tau$ of Lemma (3.2) we define a continuous linear map $D_\theta F: \mathcal{L}^2_1(J,T_{\theta(o)}X) \to T_{\theta(o)}X$ by $D_\theta F = T_\theta\rho_o \circ \nabla\xi^F(\theta) \circ {}^\theta\tau^{-1}$. Then $(D_\theta F, J, T_{\theta(o)}X)$ is an autonomous linear FDE on the Hilbert space $T_{\theta(o)}X$: because of condition (E_2) it has unique solutions (Theorem 1.2). In fact all solutions of $D_\theta F$ are full: to see this we use successive approximation on an interval $[-r,N]$ where $N > o$ is an arbitrary real number. Let $\gamma \in \mathcal{L}^2_1(J,T_{\theta(o)}X)$ and choose any $\alpha^o \in \mathcal{C}^o([-r,N], T_{\theta(o)}X)$ s.t. $\alpha^o|J = \gamma$. Define the sequence $\{\alpha^n\}_{n=o}^\infty$ in $\mathcal{C}^o([-r,N],T_{\theta(o)}X)$ by

$$\alpha^{n+1}(t) = \begin{cases} \gamma(o) + \int_o^t \widetilde{D_\theta F}(\alpha^n_u)du & o \leq t \leq N \\ \gamma(t) & -r \leq t \leq o \end{cases} \tag{3}$$

for $n \geq o$, where $\widetilde{D_\theta F}$ is an extension of $D_\theta F$ to $\mathcal{C}^o(J,T_{\theta(o)}X)$.

Letting $\|\cdot\|_{\mathcal{C}^o}$ denote the supremum norm, Condition (E_2) implies that $\exists\, K > o$ s.t.

$$|(\widetilde{D_\theta F})(\eta)|_{\theta(o)} \leq K\,\|\eta\|_{\mathcal{C}^o} \qquad \forall\, \eta \in \mathcal{C}^o(J,T_{\theta(o)}X) \tag{4}$$

Hence (3) and (4) imply - by an easy induction argument - that

$$\|\alpha^{n+1}_t - \alpha^n_t\|_{\mathcal{C}^o} \leq \frac{K^n t^n}{n!}\|\alpha^1 - \alpha^o\|_{\mathcal{C}^o} \qquad \forall\, o \leq t \leq N,\ n \geq o \tag{5}$$

Since $\alpha_t^n = \alpha_t^o + (\alpha_t^1 - \alpha_t^o) + \ldots + (\alpha_t^n - \alpha_t^{n-1})$, then it follows from the uniform convergence of the series $\sum_{n=o}^{\infty} \frac{K^n t^n}{n!}$ to e^{Kt}, that $\{\alpha_t^n\}_{n=1}^{\infty}$ converges (uniformly) on $[o,N]$ to an element of $\mathcal{C}^o(J,T_{\theta(o)}X)$ for each $t \in [o,N]$; thus we get a solution $\alpha^\gamma : [-r,N] \to T_{\theta(o)}X$ of $D_\theta F$ at γ. As N was arbitrary α^γ may be considered as a full solution of $D_\theta F$ and we have

$$||\alpha_t^\gamma - \alpha_t^o||_{\mathcal{C}^o} \leqslant ||\alpha^1 - \alpha^o||_{\mathcal{C}^o} e^{Kt} \qquad \forall\, t \geq o.$$

Now it is easy to see that the solution α^γ of $D_\theta F$ at γ satisfies the inequality

$$||\alpha_t^\gamma||_{\mathcal{C}^o} \leqslant ||\gamma||_{\mathcal{C}^o} + K\int_o^t ||\alpha_u^\gamma||_{\mathcal{C}^o} du \qquad t \geq o$$

Therefore by Gronwall's lemma (Coddington and Levinson[5] p. 37 , or Petrovski [41] p.59)

$$||\alpha_t^\gamma||_{\mathcal{C}^o} \leqslant e^{Kt}\, ||\gamma||_{\mathcal{C}^o} \qquad \forall\, t \geq o \tag{6}$$

Employing the above terminology we define for each $t \geqslant o$ a map $T_t : T_\theta \mathcal{L}_1^2(J,X) \hookleftarrow$ by

$$T_t(\beta) = \theta_\tau^{-1}\{\alpha_t^{\theta_\tau(\beta)}\} \qquad \forall\, \beta \in T_\theta\, \mathcal{L}_1^2(J,X) \tag{7}$$

Denote by $||\cdot||_{T_\theta \mathcal{L}_1^2}$ the Hilbert norm on $T_\theta\, \mathcal{L}_1^2(J,X)$ given by the metric g_2, and by $||\ ||_{\mathcal{L}_1^2}$ the corresponding norm on the space of paths $\mathcal{L}_1^2(J,T_{\theta(o)}X)$ viz

$$||\gamma||_{\mathcal{L}_1^2} = [\tfrac{1}{r}\int_{-r}^{o} |\gamma(s)|_{\theta(o)}^2 ds + \tfrac{1}{r}\int_{-r}^{o} |\gamma'(s)|_{\theta(o)}^2 ds]^{\frac{1}{2}} \tag{8}$$

Then by Lemma (3.2) and (7)

$$\|T_t(\beta)\|_{T_\theta \mathscr{L}_1^2} = \| \alpha_t^{\theta_\tau(\beta)} \|_{\mathscr{L}_1^2} \qquad \forall\, \beta \in T_\theta \mathscr{L}_1^2(J,X) \tag{9}$$

Now consider, for $\beta \in T_\theta \mathscr{L}_1^2(J,X)$, the following:

$$\frac{1}{r}\int_{-r}^{0} | \alpha_t^{\theta_\tau(\beta)}(s)|^2\, ds \leq \|\alpha_t^{\theta_\tau(\beta)}\|^2_{\mathcal{C}^0} \leq L^2 e^{2Kt} \|\beta\|^2_{T_\theta \mathscr{L}_1^2} \tag{10}$$

where we have used (6), Sobolev's embedding theorem and the fact that θ_τ is an isometry; L^2 is some constant. Also

$$\frac{1}{r}\int_{-r}^{0} \left| \frac{\partial}{\partial s} \alpha_t^{\theta_\tau(\beta)}(s)\right|^2 ds = \begin{cases} \dfrac{1}{r}\displaystyle\int_{-r}^{0} |D_\theta F(\alpha_{t+s}^{\theta_\tau(\beta)})|^2\, ds, & t \geq r \\ \dfrac{1}{r}\displaystyle\int_{-r}^{-t} |\theta_\tau(\beta)'(t+s)|^2 ds + \dfrac{1}{r}\displaystyle\int_{-t}^{0} |D_\theta F(\alpha_{t+s}^{\theta_\tau(\beta)})|^2\, ds & 0 \leq t \leq r \end{cases} \tag{11}$$

Now estimating each term on the right hand side of the above equation, a simple calculation yields:

$$\frac{1}{r}\int_{-r}^{0} |D_\theta F(\alpha_{t+s}^{\theta_\tau(\beta)})|^2 ds \leq \frac{K}{2r} L^2 \left(e^{2Kt} - e^{2K(t-r)}\right) \|\beta\|^2_{T_\theta \mathscr{L}_1^2} \qquad t \geq r \tag{12}$$

$$\frac{1}{r}\int_{-r}^{-t} |\theta_\tau(\beta)'(t+s)|^2\, ds = \frac{1}{r}\int_{-r}^{-t} \left|\frac{D\beta}{\partial s}(t+s)\right|^2 ds \qquad \text{(Lemma 3.2)}$$

$$\leq \|\beta\|^2_{T_\theta \mathscr{L}_1^2} \qquad 0 \leq t \leq r \tag{13}$$

and

$$\frac{1}{r}\int_{-t}^{o}|D_\theta F(\alpha_{t+s}^{\theta_{\tau}(\beta)})|^2\, ds \leq \frac{K}{2r}\; L^2\,(e^{2Kt}-1)\; \|\beta\|^2_{T_\theta\mathcal{L}_1^2} \qquad 0 \leq t \leq r \tag{14}$$

By combining these estimates we get

$$\|T_t(\beta)\|^2_{T_\theta\mathcal{L}_1^2} = \frac{1}{r}\int_{-r}^{o}|\alpha_t^{\theta_{\tau}(\beta)}(s)|^2\, ds + \frac{1}{r}\int_{-r}^{o}|\frac{\partial}{\partial s}\,\alpha_t^{\theta_{\tau}(\beta)}(s)|^2 ds \text{ (Lemma 3.2)}$$

$$\|T_t(\beta)\|^2_{T_\theta\mathcal{L}_1^2} \leq \left[L^2\;(1+\frac{1}{2r}K)e^{2Kt} + 1\right]\|\beta\|^2_{T_\theta\mathcal{L}_1^2} \qquad \forall\, t \geq 0 \tag{15}$$

If we take $\mu = 2K$ and $M = L^2\,(1+\frac{1}{2r}.K)$, the given result in (iii) follows from (15).

From the linearity of $D_\theta F$ and the isomorphism $^{\theta}\tau$ it is easy to see that each T_t is linear and continuous because of (1). The semi-group property of the T_t is a direct consequence of Theorem (1.3).

We prove the joint continuity of the semi-flow

$R^{\geq o}\times\; T_\theta\mathcal{L}_1^2(J,X) \to T_\theta\mathcal{L}_1^2(J,X) : (t,\beta) \to T_t(\beta)$

by the following argument. If $\beta \in T_\theta\mathcal{L}_1^2(J,X)$, then

$$\|\alpha^{\theta_{\tau}(\beta)}\|^2_{\mathcal{L}_1^2([-r,r],T_{\theta(o)}X)} = \|\beta\|^2_{T_\theta\mathcal{L}_1^2} + \|T_r(\beta)\|^2_{T_\theta\mathcal{L}_1^2}$$

$$\leq (Me^{\mu r}+2)\|\beta\|^2 \tag{16}$$

Thus the map $\beta \longmapsto \alpha^{\theta_{\tau}(\beta)} \in \mathcal{L}_1^2([-r,r],\,T_{\theta(o)}X)$ is bounded linear, and by continuity of the memory map (and the isometry $^{\theta}\tau$) the result follows. The RFDE $D_\theta F$ clearly satisfies Condition $E_1(k)$ for any $k > o$, so by Theorem (3.1) the map $t \to T_t(\beta) \in T_\theta\,\mathcal{L}_1^2(J,X)$ is C^{q-1} for $t \geq qr$,this complete the proof of (i).

To prove the compactness of T_t, $t \geq 2r$, let $V \subset T_\theta \mathcal{L}^2_1(J,X)$ be the unit ball i.e.

$$V = \{\beta : \beta \in T_\theta \mathcal{L}^2_1(J,X), \quad \|\beta\|_{T_\theta \mathcal{L}^2_1(J,X)} \leq 1\}$$

Now ${}^{\theta}\alpha^{\tau(\beta)}|[r,\infty)$ is C^2 (Theorem 3.1), so for each $t \geq 2r$, ${}^{\theta}\alpha_t^{\tau(\beta)} \in \mathcal{C}^2(J,T_{\theta(o)}X)$. Thus in view of Rellich's lemma and the isometry ${}^{\theta}\tau$ it is sufficient to show that the set $\{{}^{\theta}\alpha_t^{\tau(\beta)} : \beta \in V\}$ is bounded in the norm of $\mathcal{L}^2_2(J,T_{\theta(o)}X)$. We proceed to do just that. First differentiate the linear FDE

$$\frac{d}{dt}\ {}^{\theta}\alpha^{\tau(\beta)}(t) = (\widetilde{D_\theta F})({}^{\theta}\alpha_t^{\tau(\beta)}) \tag{17}$$

wrt t, to get the following estimates for $s \in J$, $t \geq 2r$:

$$\left|\frac{\partial^2}{\partial s^2}\{{}^{\theta}\alpha_t^{\tau(\beta)}(s)\}\right|_{T_{\theta(o)}X} = |\widetilde{D_\theta F}(({}^{\theta}\alpha^{\tau(\beta)})'_{t+s})|_{T_{\theta(o)}X} \quad \text{(Lemma 3.1(i))}$$

$$\leq K \sup_{u\in J} |({}^{\theta}\alpha^{\tau(\beta)})'(t+s+u)|_{T_{\theta(o)}X} \quad \text{(by (4))}$$

$$\leq K^2 \sup_{u\in J} \|{}^{\theta}\alpha_{t+s+u}^{\tau(\beta)}\|_{\mathcal{C}^0(J,T_{\theta(o)}X)}$$

$$\leq K^2\ L\ e^{Kt}\ \|\beta\|_{T_\theta \mathcal{L}^2_1} \tag{18}$$

where $L > o$ is a constant defined before, and the last inequality holds because of (6) and the isometry ${}^{\theta}\tau$. Thus for all $\beta \in V$,

$$\|{}^{\theta}\alpha_t^{\tau(\beta)}\|^2_{\mathcal{L}^2_2} = \|{}^{\theta}\alpha_t^{\tau(\beta)}\|^2_{\mathcal{L}^2_1} + \frac{1}{r}\int_{-r}^{o} \left|\frac{\partial^2}{\partial s^2}\ {}^{\theta}\alpha_t^{\tau(\beta)}(s)\right|^2_{T_{\theta(o)}X} ds$$

$$\leq Me^{\mu t} + 1 + K^4L^2\ e^{\mu t} \tag{19}$$

Hence the set $\{\alpha_t^{\theta}\tau(\beta) : \beta \in V\}$ is $\mathcal{L}_2^2$-bounded for each $t \geqslant 2r$; so T_t is compact for all $t \geqslant 2r$.

To prove (V), rewrite (17) in the form

$$\frac{\partial}{\partial s} \left. \alpha_t^{\theta}\tau(\beta)(s)\right|_{s=o} = [T_\theta \rho_o \circ \nabla\xi^F(\theta)] \, ({}^{\theta}\tau^{-1}(\alpha_t^{\theta}\tau(\beta))) \qquad t > o \tag{20}$$

By the definition of T_t it follows from Lemma (3.2) that

$$\frac{D}{\partial s} \left.\{T_t(\beta)(s)\}\right|_{s=o} = \frac{\partial}{\partial s} \left. {}^{\theta}\tau_s^{\,o}(T_t(\beta)(s))\right|_{s=o} = \frac{\partial}{\partial s} \left.\alpha_t^{\theta}\tau(\beta)(s)\right|_{s=o}$$

$$= (\nabla\xi^F)(\theta)(T_t(\beta))(o) \qquad t > o \qquad \text{Q.E.D.}$$

<u>Corollary (3.3.1)</u>:

Suppose $\dim X < \infty$ and F satisfies Condition $E_1(2)$. Let $\alpha : [-r,\infty) \to X$ be a full solution of F. Then $\alpha' | [2r,\infty)$ is a solution of the time-dependent covariant RFDE

$$\frac{D}{dt}(\alpha'(t)) = (T\rho_o \circ \nabla\xi^F)(\alpha_t)(\alpha'_t) \qquad t \geqslant 2r \tag{1}$$

If, in addition, the set

$$\bigcup_{t \geqslant 2r} \sup \{ \| (\nabla\xi^F)(\alpha_t)(\beta) \| \; : \; \beta \in T_{\alpha_t}\mathcal{L}_1^2(J,X), \; \sup_{s\in J} |\beta(s)| \leqslant 1\}$$

is bounded (uniformly in t), then $\exists$ constants $M', \mu' > o$ s.t.

$$\|\alpha'_t\|_{T_{\alpha_t}\mathcal{L}_1^2} \leqslant M' \|\alpha'_{2r}\|_{T_{\alpha_{2r}}\mathcal{L}_1^2} e^{\mu' t} \qquad \forall\, t \geqslant 3r \tag{2}$$

Proof:

Because of Condition $E_1(2)$ and Theorem (3.1), the map $[2r,\infty) \ni t \longmapsto \alpha_t \in \mathcal{L}_1^2(J,X)$ is C^1 with derivative $[2r,\infty) \ni t \mapsto \alpha'_t \in \mathcal{L}_1^2(J,TX)$.

Now observe that α satisfies the equation

$$\alpha'(t) = (T\rho_0 \circ \xi^F)(\alpha_t) \tag{3}$$

This equation can then be differentiated covariantly wrt t on $[2r,\infty)$. To do that remark that if K is the (Levi-Civita) connection map on TX and K^* is the induced connection map on the subbundle $i\{\rho_0^*(TX)\} \to \mathcal{L}_1^2(J,X)$ of parallel fields (Eliasson[17]) then the diagram

$$\begin{tikzcd}
T[i\{\rho_0^*(TX)\}] \arrow[r, "T^2\rho_0"] \arrow[d, "K^*"'] & T(TX) \arrow[d, "K"] \\
i\{\rho_0^*(TX)\} \arrow[r, "T\rho_0"] \arrow[d] & TX \arrow[d, "\pi_0"] \\
\mathcal{L}_1^2(J,X) \arrow[r, "\rho_0"] & X
\end{tikzcd}$$

commutes. Thus equation (3) gives for $t \geqslant 2r$:

$$\frac{D}{dt}(\alpha'(t)) = (K \circ T^2\rho_0 \circ T\xi^F)(\alpha_t)(\alpha'_t)$$

$$= (T\rho_0 \circ K^* \circ T\xi^F)(\alpha_t)(\alpha'_t)$$

$$= (T\rho_0 \circ \nabla\xi^F)(\alpha_t)(\alpha'_t) \qquad t \geqslant 2r$$

The given boundedness condition implies that $\exists$ a constant $\mu' > o$ (independent of t,β) s.t.

$$\|\nabla\xi^F(\alpha_t)(\beta)\| \leq \mu' \|\beta\|_{T_{\alpha_t}\mathcal{C}^o} \quad \forall \beta \in T_{\alpha_t} \mathcal{L}_1^2(J,X) \tag{4}$$

where

$$\|\beta\|_{T_{\alpha_t}\mathcal{C}^o} = \sup_{s\in J} |\beta(s)|$$ is the supremum Finsler on the tangent space $T_{\alpha_t}\mathcal{C}^o(J,X)$ to $\mathcal{C}^o(J,X)$.

We prove the required estimate on the time derivative of the orbit by transporting along the solution using the point $t = 2r$ as reference point. Denote parallel transport along α by ${}^{\alpha}\tau$. Then by (1) it follows that

$$^{\alpha}\tau_t^{2r}(\alpha'(t)) = \alpha'(2r) + \int_{2r}^{t} {}^{\alpha}\tau_u^{2r} \left[(T\rho_o\circ\nabla\xi^F)(\alpha_u)(\alpha'_u)\right] du \qquad t \geq 2r \tag{5}$$

where we have used Lemma (3.2). Therefore because of (4) and the fact that $T\rho_o$ is an isometry on the subbundle $i\{\rho_o^*(TX)\}$ we get from (5)

$$|\alpha'(t)|_{\alpha(t)} \leq |\alpha'(2r)|_{\alpha(2r)} + \mu' \int_{2r}^{t} \|\alpha'_u\|_{T_{\alpha_u}\mathcal{C}^o} du \qquad t \geq 2r \tag{6}$$

Take $t \geq 3r$. Then it follows easily from (6) that

$$\|\alpha'_t\|_{T_{\alpha_t}\mathcal{C}^o} \leq \|\alpha'_{2r}\|_{T_{\alpha_{2r}}\mathcal{C}^o} + \mu' \int_{2r}^{t} \|\alpha'_u\|_{T_{\alpha_u}\mathcal{C}^o} du \tag{7}$$

An application of Gronwall's lemma to (7) gives

$$\|\alpha'_t\|_{T_{\alpha_t}\mathcal{C}^o} \leq \|\alpha'_{2r}\|_{T_{\alpha_{2r}}\mathcal{C}^o} e^{\mu' t} \qquad t \geq 3r \tag{8}$$

By Sobolev's embedding theorem, $\exists$ a constant $L > o$ s.t.

$$\|\alpha'_t\|_{T_{\alpha_t}\mathcal{C}^o} \leq L \|\alpha'_t\|_{T_{\alpha_t}\mathcal{L}_1^2} \quad \forall t \geq 3r \tag{9}$$

The proof is then completed by an argument similar to the one used in the proof of the theorem, viz. using (1), (4), (8) and (9) to obtain

$$\|\alpha'_t\|_{T_{\alpha_t}\mathcal{L}^2_1} = \left[\frac{1}{r}\int_{-r}^{0}|\alpha'_t(s)|^2\, ds + \frac{1}{r}\int_{-r}^{0}\left|\frac{D}{\partial s}\alpha'_t(s)\right|^2 ds\right]^{\frac{1}{2}}$$

$$\leq L\,(\mu'^2 + 1)^{\frac{1}{2}}\ \|\alpha'_{2r}\|_{T_{\alpha_{2r}}\mathcal{L}^2_1}\ e^{\mu' t}, \quad t \geq 3r.$$

Q.E.D.

Remark (3.1):

The above theorem, as well as the remaining results in this Chapter, has largely been motivated by the work of Hale [22], Shimanov [43], Perello [23] in the $\mathcal{C}^0$ context where X is the Eucludian space R^n.

The evolution equation associated with the semi-flow $(t,\beta) \to T_t(\beta)$ is specified by the next result as a family of vector fields on the fibres of $T\mathcal{L}_1^2(J,X)$ whose integral curves are just the paths $t \to T_t(\beta)$.

For the rest of this Chapter take X to be of finite dimension.

Theorem (3.4)

For each $\theta \in \mathcal{L}_1^2(J,X)$, let $A^\theta : \mathcal{D}(A^\theta) \subset T_\theta \mathcal{L}_1^2(J,X) \to T_\theta \mathcal{L}_1^2(J,X)$ be the infinitesimal generator of the semi-group $\{T_t\}_{t \geq o}$ on $T_\theta \mathcal{L}_1^2(J,X)$, (Dunford and Schwartz [11] § VIII. 1). Then

i) $\mathcal{D}(A^\theta)$, the domain of A^θ, is a dense linear subspace of $T_\theta \mathcal{L}_1^2(J,X)$; A^θ is a closed linear operator.

ii) $\mathcal{D}(A^\theta) = \{\beta : \beta \in T_\theta \mathcal{L}_1^2(J,X), \quad s \mapsto \frac{D\beta(s)}{ds}$ is of class $\mathcal{L}_1^2$,

$$\left.\frac{D\beta(s)}{ds}\right|_{s=o} = \nabla\xi^F(\theta)(\beta)(o) \} \quad ;$$

iii) if $\beta \in \mathcal{D}(A^0)$, we have

$$(A^\theta\beta)(s) = \{\lim_{h\to o+} \frac{T_h\beta - \beta}{h}\}(s) = \frac{D\beta(s)}{ds}$$

iv) $T_t \{\mathcal{D}(A^\theta) \subseteq \mathcal{D}(A^\theta)$, A^θ commutes with T_t on $\mathcal{D}(A^\theta)$

i.e.

$$\frac{D}{\partial(.)} T_t(\beta)(.) = T_t \left(\frac{D\beta(.)}{d(.)} \right) \qquad \forall \beta \in \mathcal{D}(A) \quad \forall t \geq o$$

where $\frac{D\beta(.)}{d(.)}$ denotes the vector field $s \to \frac{D\beta(s)}{ds}$.

Proof:

The assertions (i) and (iv) hold as standard results of linear semi-group theory (e.g. as in Dunford and Schwartz [11] § VIII.1). So we only prove (ii) and (iii).

Let $\beta \in \mathcal{D}(A^\theta)$, and denote by $\alpha^{\theta_{\tau(\beta)}}$ the solution of $D_\theta F$ at ${}^\theta\tau(\beta)$, as in the proof of the previous theorem. By definition of A^θ we have

$$A^\theta\beta = \lim_{h\to o+} \frac{T_h\beta - \beta}{h} \tag{1}$$

where the limit is in the norm on $T_\theta \mathcal{L}^2_1(J,X)$ given by the inner product $g_2(\theta)$. Let $s \in [-r,o)$ and think of $h > o$ small enough so that $-r < s + h < o$. Then (1) gives

$$\begin{aligned}(A^\theta\beta)(s) &= \lim_{h\to o+} \frac{1}{h}\{(T_h\beta)(s) - \beta(s)\} \\ &= {}^\theta\tau_o^s \lim_{h\to o+} \frac{1}{h}\{\alpha^{\theta_{\tau(\beta)}}(h+s) - {}^\theta\tau_s^o(\beta(s))\}] \\ &= {}^\theta\tau_o^s \left[\lim_{h\to o+} \frac{1}{h}\{{}^\theta\tau(\beta)(h+s) - {}^\theta\tau(\beta)(s)\}\right] \\ &= \frac{D\beta(s)}{ds} \qquad \text{(Lemma 3.2)}\end{aligned} \tag{2}$$

Since $A^\theta\beta \in T_\theta\mathcal{L}^2_1(J,X)$, then $J \ni s \to (A^\theta\beta)(s) \in TX$ is continuous and so (2) still holds for $s = o$ i.e.

$$\left.\frac{D\beta(s)}{ds}\right|_{s=o} = \lim_{s\to o-} \frac{D\beta(s)}{ds} = (A^\theta\beta)(o) \tag{3}$$

$$= \lim_{h\to o+} \frac{1}{h}\{\alpha^{\theta_{\tau}\beta}(h) - \beta(o)\}$$

$$= \lim_{h\to o+} \frac{1}{h} \int_o^h (D_\theta F)(\alpha_u^{\theta}{}^{\tau(\beta)})\, du$$

$$= (D_\theta F)(\alpha_o^{\theta}{}^{\tau(\beta)})$$

$$= (\nabla\xi^F)(\theta)(\beta)(o) \qquad (4)$$

from the definition of $D_\theta F$. This proves that $\mathcal{D}(A^\theta) \subseteq \{\beta : \beta \in T_\theta \mathcal{L}_1^2(J,X)$, $\frac{D(.)}{d(.)}$ is of class $\mathcal{L}_1^2$, β satisfies (4)$\}$. (5)

To prove the opposite inclusion, let β belong to the set on the right hand side of (5). We contend that

$$\frac{D\beta(.)}{d(.)} = \lim_{h\to o+} \frac{T_h \beta - \beta}{h} \qquad (6)$$

For simplicity of notation take $\alpha \equiv \alpha^{\theta}{}^{\tau(\beta)}$. Then it is easy to see from the definition of T_h that (6) is equivalent to the pair of equations

$$\lim_{h\to o+} \int_{-r}^{o} \left| \frac{\alpha(h+s) - \alpha(s)}{h} - \alpha'(s)\right|^2_{T_{\theta(o)}X} ds = o \qquad (a)$$

and

$$\lim_{h\to o+} \int_{-r}^{o} \left| \frac{\alpha'(h+s) - \alpha'(s)}{h} - \alpha''(s)\right|^2 ds = o \qquad (b)$$

(7)

We prove (7)(b) by appealing to Lebesgue's dominated convergence theorem (Halmos [24], p.110); the proof of (7)(a) is similar. From the hypothesis on β and Lemma (3.2), ${}^\theta\tau(\beta) \in \mathcal{L}_2^2(J,J_{\theta(o)}X)$ the space of all paths $J \to T_{\theta(o)}X$ with $\mathcal{L}_1^2$ first derivative. Working over $[-r,o)$ with $h > o$ small enough, we get

$$\lim_{h \to o+} \frac{\alpha'(h+s) - \alpha'(s)}{h} = \lim_{h \to o+} \frac{1}{h} \int_s^{s+h} \alpha''(u)du$$

$$= \alpha''(s) \qquad \text{for a.a. } s \in [-r,o) \tag{8}$$

Choose any $\varepsilon > o$. Then

$$\left|\frac{\alpha'(h+s) - \alpha'(s)}{h}\right| = \frac{1}{h} \left|\int_s^{h+s} \alpha''(u)du\right|$$

$$\leq \left(\int_s^{h+s} |\alpha''(u)|^2 du\right)^{\frac{1}{2}} \qquad \text{(Hölder's inequality)}$$

$$< \left(\int_{-r}^{\varepsilon} |\alpha''(u)|^2 du\right)^{\frac{1}{2}} \qquad \forall\, o < h < \varepsilon \tag{9}$$

by using Hölder's inequality. Thus in view of (8) and (9) we can apply Lebesgue's dominated convergence theorem to obtain (7)(b). This proves our contention. Q.E.D.

Remark (3.2):

We see from the above theorem that the values of the generator A^θ are independent of the RFDE F, but depend only on the Riemannian structure on X, $\mathcal{D}(A^\theta)$ however depends heavily on F and the Riemannian structure.

Corollary (3.4.1):

Suppose F satisfies Condition $E_1(2)$, and let $\{\alpha_t\}_{t \geq o}$ be the orbit of a full solution of F. Then for each $t \geq 3r$, $\alpha'_t \in \mathcal{D}(A^{\alpha_t})$; α_t is a geodesic segment (on X) iff $\alpha'_t \in \ker A^{\alpha_t}$.

Proof:

The result follows trivially from the theorem and the fact that

$$\frac{D}{\partial s} \alpha'_t(s)\Big|_{s=o} = \nabla\xi^F(\alpha_t)(\alpha'_t)(o) \qquad t > 2r$$

(Corollary 3.3.1). Q.E.D.

The next step in our study of the semi-flow $\{T_t\}_{t\geq o}$ on $T\mathcal{L}_1^2(J,X)$ is to construct a splitting of $T\mathcal{L}_1^2(J,X)$ as a (Whitney) direct sum of two subbundles : the <u>unstable</u> and the <u>stable</u> one. Both of these subbundles are invariant under the semi-flow $\{T_t\}_{t\geq o}$. The unstable subbundle is always finite-dimensional and on it the semi-flow $\{T_t\}_{t\geq o}$ can be continued backwards in time to give a genuine flow which is defined for all time; within the stable subbundle the semi-flow is "asymptotically small" - in a sense to be specified later.

First of all we strengthen Condition (E_2) by supposing - till further notice - that F satisfies <u>Condition (E_3)</u>:

For each $\theta \in \mathcal{L}_1^2(J,X)$ let $D_\theta F = T_\theta \rho_o \circ \nabla\xi^F(\theta) \circ {}^\theta\tau^{-1}$ (i.e. as in the proof of Theorem 3.3). Denote by $L(T_{\theta(o)}X)$ the space of all bounded linear operators equipped with the operator norm $\|\cdot\|$. Suppose that $D_\theta F$ can be represented as

$$(D_\theta F)(\gamma) = \int_{-r}^{0} E(s)(\gamma(s))\, ds \qquad \forall\, \gamma \in \mathcal{L}_1^2(J,T_{\theta(o)}X)$$

where $E : J \to L(T_{\theta(o)}X)$ is s.t. $\int_{-r}^{0} \|E(s)\|^2\, ds < \infty$.

The decomposition of $T_\theta \mathcal{L}_1^2(J,X)$ for each $\theta \in \mathcal{L}_1^2(J,X)$ is achieved by an analysis of the spectrum of the generator A^θ viewed as a subset of the complex plane $\mathbb{C}$. We are therefore forced to complexify the objects we have been working with so far. Following Halmos ([25] §77. pp. 150-153) we adopt the following terminology: If H is a real Hilbert space, denote the complexification $H_\mathbb{C}$ of H by $H_\mathbb{C} = H \oplus iH$, $i = \sqrt{-1}$. An element in $H_\mathbb{C}$ is symbolized by $u + iv$, $u, v \in H$; and H is always identified with a real vector subspace of $H_\mathbb{C}$ (considered as a vector space over R). $H_\mathbb{C}$ is a complex Hilbert space whose norm $|\cdot|_\mathbb{C}$ satisfies

$$|u + iv|^2_{\mathbb{C}} = |u|^2 + |v|^2$$

where $|\cdot|$ is the norm on H. If G is another real Hilbert space and $K : H \to G$ is a linear map, then its complexification $K_{\mathbb{C}}: H_{\mathbb{C}} \to G_{\mathbb{C}}$ is a complex linear map extending K and defined by

$$K_{\mathbb{C}}(u + i v) = K(u) + i K(v) \quad , \quad u, v \in H.$$

K is bounded iff $K_{\mathbb{C}}$ is, and then $\|K\| = \|K_{\mathbb{C}}\|$.

Using this notation the entities $T_{\theta(o)}X$, $\mathcal{L}^2_1(J,T_{\theta(o)}X)$, $T_\theta \mathcal{L}^2_1(J,X)$, T_t, A^θ, $\nabla\xi^F(\theta)$, $D_\theta F$, E, Condition E_3 are complexified to yield the corresponding ones:

$(T_{\theta(o)}X)_{\mathbb{C}}$, $[\mathcal{L}^2_1(J,T_{\theta(o)}X)]_{\mathbb{C}} = \mathcal{L}^2_1(J,(T_{\theta(o)}X)_{\mathbb{C}})$, $[T_\theta \mathcal{L}^2_1(J,X)]_{\mathbb{C}}$,

$T^{\mathbb{C}}_t$, $A^\theta_{\mathbb{C}}$, $(\nabla\xi^F(\theta))_{\mathbb{C}}$, $(D_\theta F)_{\mathbb{C}}$, $E_{\mathbb{C}}(s) = (E(s))_{\mathbb{C}}$, Condition $(E_3)_{\mathbb{C}}$, defined in the obvious way. It is easy to see that Condition E_3 implies Condition $(E_3)_{\mathbb{C}}$, viz. $(D_\theta F)_{\mathbb{C}}$ admits a representation of the form

$$(D_\theta F)_{\mathbb{C}}(\gamma) = \int_{-r}^{0} E_{\mathbb{C}}(s)(\gamma(s)) \qquad \forall \gamma \in \mathcal{L}^2_1(J,(T_{\theta(o)}X)_{\mathbb{C}}).$$

where $E_{\mathbb{C}} : J \to L((T_{\theta(o)}X)_{\mathbb{C}})$ is square integrable.

Under these conditions we prove the following theorem about the spectrum $\sigma(A^\theta_{\mathbb{C}})$ of $A^\theta_{\mathbb{C}}$ which was first proved by Hale ([22], [21]) in the flat case $X = R^n$ with F autonomous linear and $\mathcal{C}^0(J,R^n)$ as the state space.

<u>Theorem (3.5)</u>:

Define the map $B : \mathbb{C} \to L((T_{\theta(o)}X)_{\mathbb{C}})$ by

$$B(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda s} E_{\mathbb{C}}(s)\, ds$$

where $I: (T_{\theta(o)}X)_{\mathbb{C}} \circlearrowleft$ is the identity operator. Then the resolvent set $\sigma(A^\theta_{\mathbb{C}}) = B^{-1}\{GL((T_{\theta(o)}X)_{\mathbb{C}})\}$, where $GL((T_{\theta(o)}X)_{\mathbb{C}})$ is the general linear group of all linear homeomorphisms of $(T_{\theta(o)}X)_{\mathbb{C}}$ onto itself. $\sigma(A^\theta_{\mathbb{C}})$ is discrete, has real parts bounded above, with no accumulation points and consists entirely of eigenvalues of $A^\theta_{\mathbb{C}}$.

Proof: This is an adaptation of an argument by Hale [21]. First note that it is a pure formality checking that the complexified versions of Theorem (3.3) and Theorem (3.4) hold true. In particular $\{T_t^{\mathbb{C}}\}_{t\geq 0}$ is a strongly continuous semi-group of bounded linear operators on $(T_\theta \mathcal{L}_1^2(J,X))_{\mathbb{C}}$ with generator $A_{\mathbb{C}}^\theta$ and $\mathcal{D}(A_{\mathbb{C}}^\theta) = \mathcal{D}(A^\theta) \oplus i\, \mathcal{D}(A^\theta)$. Parallel transport is also complexified (being a linear map) in the obvious way and the resulting complexification will be denoted - for the sake of simplicity - by the same symbol ${}^\theta\tau_s^o : (T_{\theta(s)}X)_{\mathbb{C}} \to (T_{\theta(o)}X)_{\mathbb{C}}$. The covariant derivative of vector fields along θ may be treated similarly so that the complexification of Lemma (3.2) is valid.

Now let $\lambda \in \mathbb{C}$ be s.t. $B(\lambda)$ is a linear homeomorphism. We prove that $\lambda \in \rho(A_{\mathbb{C}}^\theta)$ by showing that for each $\eta \in [T_\theta \mathcal{L}_1^2(J,X)]_{\mathbb{C}}$ the equation

$$(\lambda I - A_{\mathbb{C}}^\theta)\beta = \eta \tag{1}$$

has a unique solution $\beta \in \mathcal{D}(A_{\mathbb{C}}^\theta)$ which, depends continuously on η wrt the $\mathcal{L}_1^2$-norm on $[T_\theta \mathcal{L}_1^2(J,X)]_{\mathbb{C}}$. Because of Theorem (3.4), (1) is equivalent to the covariant ODE (unretarded) problem

$$\left.\begin{aligned} \lambda\beta(s) - \frac{D\beta(s)}{ds} &= \eta(s) \qquad s \in J \\ \left.\frac{D\beta(s)}{ds}\right|_{s=o} &= [(\nabla\xi^F)(\theta)]_{\mathbb{C}}(\beta)(o) \end{aligned}\right\} \tag{2}$$

Note that each solution β of (2) must necessarily have $\dfrac{D\beta(.)}{d\,(.)}$ of class $\mathcal{L}_1^2$.

Try a solution of (2) in the form

$$\beta(s) = e^{\lambda s}\, {}^\theta\tau_o^s(v) + \int_s^o e^{\lambda(s-u)}\, {}^\theta\tau_u^s(\eta(u))du \tag{3}$$

where $v \in [T_{\theta(o)}X]_{\mathbb{C}}$ is to be determined so that the right hand side of (3) satisfies the second equation of (2). Using the complexified version of Lemma (3.2), (3) gives

$$\frac{D\beta(s)}{ds} = \lambda e^{\lambda s}\, {}^{\theta}\tau_o^s(v) + \lambda e^{\lambda s} \int_s^o e^{-\lambda u}\, {}^{\theta}\tau_u^s(\eta(u))\, du \; - \eta(s)$$

$$= \lambda\beta(s) - \eta(s) \qquad s \in J \qquad (4)$$

Since F satisfies condition $(E_3)_{\mathbb{C}}$ then an easy calculation shows that β satisfies the second equation of (2) iff

$$B(\lambda)(v) = \eta(o) + \int_{-r}^{o} \int_s^o e^{\lambda(s-u)} E_{\mathbb{C}}(s)({}^{\theta}\tau_u^o(\eta(u)))\, du \; ds \qquad (5)$$

Thus taking

$$v = [B(\lambda)]^{-1} \{\eta(o) + \int_{-r}^{o} \int_s^o e^{\lambda(s-u)} E_{\mathbb{C}}(s)({}^{\theta}\tau_u^o(\eta(u)))\, du \; ds\} \qquad (6)$$

gives a solution of the problem (2) . To prove uniqueness of solutions of (1) , it is sufficient to show that when $\eta = o$ then (2) has no non-trivial solutions. Supposing $\eta = o$, then by Lemma (3.2) any solution β_o of (2) must have the form

$$\beta_o(s) = e^{\lambda s}\, {}^{\theta}\tau_o^s(v) \qquad s \in J$$

for some $v \in [T_{\theta(o)}X]_{\mathbb{C}}$. Since β_o must satisfy the second equation of (2) then $B(\lambda)(v) = o$; thus $v = o$ and hence $\beta_o = o$.

Now in (1) it is clear that β depends linearly on η ; we then have to prove that the map

$$[T_\theta \mathcal{L}_1^2(J,X)]_{\mathbb{C}} \longrightarrow [T_\theta \mathcal{L}_1^2(J,X)]_{\mathbb{C}}$$

$$\eta \longmapsto \beta \qquad (7)$$

is continuous. We make use of the equations (3), (4) and (6) to obtain the following estimates , where $\|\cdot\|$ denotes both the operator norm and the norm on $[T_\theta \mathcal{L}_1^2(J,X)]_{\mathbb{C}}$ and $|\cdot|$ stands for the norms on the complexified tangent spaces $(T_{\theta(s)}X)_{\mathbb{C}}$:

$$|v| \leq \|[B(\lambda)]^{-1}\| \{K_1 \|\eta\| + e^{2r|\mathrm{Re}\lambda|}\left(\int_{-r}^{0} \|E_{\mathbb{C}}(s)\|^2 ds\right)^{1/2}\left(\int_{-r}^{0} |\eta(u)|^2 du\right)^{1/2}\}$$

(by Hölder's inequality)

$$\leq \|[B(\lambda)]^{-1}\| K_2 \|\eta\| \tag{8}$$

K_1 is some positive constant and

$$K_2 = K_1 + e^{2r|\mathrm{Re}\lambda|}\left(\int_{-r}^{0} \|E_{\mathbb{C}}(s)\|^2 ds\right)^{1/2} > 0 .$$

Since parallel transport is an isometry, then (8) and (3) give a constant $K_3 > 0$ s.t.

$$|\beta(s)| \leq K_3 \|\eta\| \qquad \forall \; s \in J \tag{9}$$

To estimate the L.H.S. of (4) we use the inequality

$$(a + b)^2 \leq 2(a^2 + b^2) \qquad a,b \in \mathbb{R} \tag{10}$$

to get

$$\left|\frac{D\beta(s)}{ds}\right|^2 \leq 2(|\lambda|^2|\beta(s)|^2 + |\eta(s)|^2) \tag{11}$$

Therefore (9), (11) imply that $\exists \; K_4 > 0$ s.t.

$$\|\beta\| \leq K_4 \|\eta\| \tag{12}$$

Thus the map (7) is bounded linear and $\lambda \in \rho(A_{\mathbb{C}}^{\theta})$.

Conversely, suppose $\lambda \in \rho(A^{\theta}_{\mathbb{C}})$. By virtue of the Open Mapping Theorem, $B(\lambda) \in GL((T_{\theta(o)}X)_{\mathbb{C}})$ if it is a bijection of $(T_{\theta(o)}X)_{\mathbb{C}}$ onto itself. So let $v \in (T_{\theta(o)}X)_{\mathbb{C}}$ be s.t.

$$B(\lambda)(v) = o \tag{13}$$

Define $\beta \in [T_{\theta}\, {}^{2}_{1}(J,X)]_{\mathbb{C}}$ by

$$\beta(s) = e^{\lambda s}\, {}^{\theta}\tau^{s}_{o}(v) \quad \forall s \in J \tag{14}$$

Then it is easy to see that β satisfies the first equation of (2) ; also (13) , the definition of $B(\lambda)$ and Condition $(E_3)_{\mathbb{C}}$ imply that β must necessarily satisfy the second equality in (2) . Hence $\beta \in D(A^{\theta}_{\mathbb{C}})$ and $A^{\theta}_{\mathbb{C}}\beta = \lambda\beta$. Therefore β must be zero, as $\lambda \notin \sigma(A^{\theta}_{\mathbb{C}})$.
Thus $v = o$ and $B(\lambda)$ is injective. Moreover, by consulting the right hand side of (5) it is easy to see that $B(\lambda)$ is surjective if the map

$$[T_{\theta}\mathcal{L}^{2}_{1}(J,X)]_{\mathbb{C}} \longrightarrow (T_{\theta(o)}X)_{\mathbb{C}}$$

$$\eta \longmapsto \eta(o) + \int_{-r}^{o}\int_{s}^{o} e^{\lambda(s-u)}E_{\mathbb{C}}(s)({}^{\theta}\tau^{o}_{u}(\eta(u)))\, du\, ds$$

is ; the surjectivity of the latter map will follow as a direct consequence of the lemma below (Lemma 3.3) :

<u>Lemma (3.3)</u>:

Suppose V is a finite dimensional (complex) Hilbert space and $K : J \times J \longrightarrow L(V)$ a continuous map. If $w \in V$, then $\exists$ a C^1 map $\eta : J \to V$ s.t.

$$\eta(t) = w - \int_{-r}^{t}\int_{s}^{o} K(s,u)(\eta(u))\, du\, ds \qquad t \in J$$

<u>Proof of Lemma (3.3)</u>:

Define the map $U : \mathcal{C}^{o}(J,V) \hookleftarrow$ by

$$(Ux)(t) = w - \int_{-r}^{t}\int_{s}^{o} K(s,u)(x(u))\, du\, ds \qquad t \in J \tag{15}$$

for each $x \in \mathcal{C}^o(J,V)$. Then U is a compact map, for observe that for each $x \in \mathcal{C}^o(J,V)$, $Ux \in \mathcal{C}^1(J,V)$ and the map U considered as $U : \mathcal{C}^o(J,V) \to \mathcal{C}^1(J,V)$ is easily seen from (15) to be continuous wrt the C^1 norm on $\mathcal{C}^1(J,V)$ viz.

$$||x||_{\mathcal{C}^1} = \sup_{s \in J} \{|x(s)| + |x'(s)|\}, x \in \mathcal{C}^1(J,V).$$

Therefore by the compactness of the inclusion map

$\mathcal{C}^1(J,V) \hookrightarrow \mathcal{C}^o(J,V)$ (Ascoli's Theorem)

it follows that $U : \mathcal{C}^o(J,V) \to \mathcal{C}^o(J,V)$ is compact. Thus U has a fixed point η which satisfies the lemma. The lemma is proved.

<u>Continuation of Proof of Theorem (3.5)</u>:

$\sigma(A^\theta_{\mathbb{C}})$ consists entirely of eigenvalues of $A^\theta_{\mathbb{C}}$ because of the following reason: If $\lambda \in \sigma(A_{\mathbb{C}})$, then by the finite-dimensionality of $(T_{\theta(o)}X)_{\mathbb{C}}$ $B(\lambda)$ is not injective ie.

$\exists\, o \neq v \in (T_{\theta(o)}X)_{\mathbb{C}}$ s.t. $B(\lambda)(v) = o$. Define $\beta \neq o$ as in (14), then $\beta \in D(A^\theta_{\mathbb{C}})$ and $A^\theta_{\mathbb{C}}\beta = \lambda\beta$.

Also since $A^\theta_{\mathbb{C}}$ is the infinitesinal generator of the strongly continuous semi-group $\{T^{\mathbb{C}}_t\}_{t \geq o}$, then the set $\{Re\lambda : \lambda \in \sigma(A^\theta_{\mathbb{C}})\}$ is bounded from above (Dunford and Schwartz [11]) . To see that $\sigma(A^\theta_{\mathbb{C}})$ is discrete and without accumulation points, observe that for a fixed $t \geq 2r$ $\sigma(T^{\mathbb{C}}_t) = \{e^{t\lambda} : \lambda \in \sigma(A^\theta_{\mathbb{C}})\}$ is discrete without accumulation points except possibly zero because $T^{\mathbb{C}}_t$ is compact

(Hille and Philips [26] p.467, Dunford and Schwartz [11]), and for a given $\mu \in \sigma(T_t^{\mathbb{C}})$ the equation $e^{t\lambda} = \mu$ has countably many solutions $\lambda \in \mathbb{C}$. This ends the proof of Theorem (3.5). Q.E.D.

We thus finally arrive at the following main result giving the existence of the stable and unstable subbundles of $T\mathcal{L}_1^2(J,X)$:

<u>Theorem (3.6)</u>:(The Stable Bundle Theorem)

Suppose X is finite dimensional and F satisfies Condition (E_3). Then $\exists$ subbundles U, S of $T\mathcal{L}_1^2(J,X)$ over $\mathcal{L}_1^2(J,X)$ with the following properties: for each $\theta \in \mathcal{L}_1^2(J,X)$:

i) $T_\theta\mathcal{L}_1^2(J,X) = U_\theta \oplus S_\theta$

ii) U_θ is finite-dimensional, S_θ is a closed linear subspace of $T_\theta\mathcal{L}_1^2(J,X)$. Both U_θ and S_θ are invariant wrt the semi-flow $\{T_t\}_{t\geq o}$ and the generator A^θ, with $U_\theta \subset D(A^\theta)$.

iii) $T_t|U_\theta : U_\theta \circlearrowleft$ is a linear homeomorphism $\forall$ $t\geq o$, and the semi-group $\{T_t|U_\theta\}_{t\geq o}$ extends to a 1-parameter group $\{\tilde{T}_t\}_{t\in R}$ on U_θ (ie. a flow) defined by

$$\tilde{T}_t = \begin{cases} T_t|U_\theta & t \geq o \\ (T_{-t}|U_\theta)^{-1} & t \leq o \end{cases} \qquad (2)$$

and satisfying

$$\frac{D}{\partial s}\{\tilde{T}_t(\beta)(s)\}\Big|_{s=o} = (\nabla\xi^F)(\theta)(\tilde{T}_t(\beta))(o) \quad \forall\ -\infty < t < \infty \qquad (3)$$
$$\forall\ \beta \in U_\theta$$

iv) $\exists$ constants $K, \mu > o$ (depending on θ) s.t.

$$\|T_t(\beta)\|_{T_\theta \mathcal{L}_1^2} \leq K e^{-\mu t} \|\beta\|_{T_\theta \mathcal{L}_1^2} \qquad \forall \beta \in S_\theta \quad \forall t \geq o \tag{4}$$

Proof:

Again the situation is amenable to linear semi-group analysis on $[T_\theta \mathcal{L}_1^2(J,X)]_{\mathbb{C}}$ in the spirit of Hale [21], so we follow him closely.

Theorem (3.5) says that $\sigma(A_{\mathbb{C}}^\theta)$ is the set of zeros of the function

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & R \\ \lambda & \overset{\det B(.)}{\longmapsto} & \det B(\lambda) \end{array}$$

Therefore it is not hard to see from the definition of $B(\lambda)$ that there are only finitely many $\lambda \in \sigma(A_{\mathbb{C}}^\theta)$ on each vertical line in $\mathbb{C}$; in fact for each $x \in \mathbb{R}$ the set $\{\lambda : \lambda \in \mathbb{C} , \det B(\lambda) = 0 , \operatorname{Re}\lambda = x\}$ is bounded. Since $\sigma(A_{\mathbb{C}}^\theta)$ has real parts bounded above and is discrete with no accumulation points, it follows that there are only finitely many $\lambda \in \sigma(A_{\mathbb{C}}^\theta)$ with $\operatorname{Re} \lambda \geq o$.

Since the zeros of the entire function $\det B(.)$ have finite multiplicity it follows from the spectral properties of the closed operator $A_{\mathbb{C}}^\theta$

that for each $\lambda \in \sigma(A^\theta_{\mathbb{C}})$ $\exists$ an integer $k(\lambda) > o$ with the property that $\ker(\lambda I - A^\theta_{\mathbb{C}})^{p-1} \subsetneq \ker(\lambda I - A^\theta_{\mathbb{C}})^p$ $\forall\ 1 \leq p \leq k(\lambda)$,

$\ker(\lambda I - A^\theta_{\mathbb{C}})^{k(\lambda)} = \ker(\lambda I - A^\theta_{\mathbb{C}})^m$ $\forall\ m \geq k(\lambda)$ and

$$[T_\theta \mathcal{L}^2_1(J,X)]_{\mathbb{C}} = \ker(\lambda I - A^\theta_{\mathbb{C}})^{k(\lambda)} \oplus \text{range}\,(\lambda I - A^\theta_{\mathbb{C}})^{k(\lambda)} \tag{5}$$

where range $(\lambda I - A^\theta_{\mathbb{C}})^{k(\lambda)}$ is closed ([46] Theorem 5.8 - A p.306). Also $\ker(\lambda I - A^\theta_{\mathbb{C}})^{k(\lambda)}$ is finite dimensional because it is a subspace of the finite-dimensional space $\ker(e^{\lambda t} I - T^{\mathbb{C}}_t)^{k(\lambda)}$, $t \geq 2r$, ($T^{\mathbb{C}}_t$ is compact for $t \geq 2r$, [21] Lemma 22.1 p.112). Since $T^{\mathbb{C}}_t$ commutes with $A^\theta_{\mathbb{C}}$ it is easy to see that the splitting (5) is invariant wrt $T^{\mathbb{C}}_t$, $t \geq o$.

Now let $\{\lambda_j\}^m_{j=1}$ be the set of all $\lambda \in \sigma(A^\theta_{\mathbb{C}})$ s.t. Re $\lambda \geq o$. Define the finite dimensional subspace $\tilde{U}_\theta$ and the closed subspace $\tilde{S}_\theta$ in $[T_\theta \mathcal{L}^2_1(J,X)]_{\mathbb{C}}$ by $\tilde{U}_\theta = \ker(\lambda_1 I - A^\theta_{\mathbb{C}})^{k(\lambda_1)} \oplus \{ \ker(\lambda_2 I - A^\theta_{\mathbb{C}})^{k(\lambda_2)} \cap \text{range}\,(\lambda_1 I - A^\theta_{\mathbb{C}})^{k(\lambda_1)}\}$

$$\oplus \ldots\ldots \oplus \{\ker(\lambda_m I - A)^{k(\lambda_m)} \cap \text{range}\,(\lambda_{m-1} I - A^\theta_{\mathbb{C}})^{k(\lambda_{m-1})} \cap \ldots \cap \text{range}\,(\lambda_1 I - A^\theta_{\mathbb{C}})^{k(\lambda_1)}\} \tag{6}$$

and

$$\tilde{S}_\theta = \bigcap_{j=1}^{m} \text{range}\,(\lambda_j I - A^\theta_{\mathbb{C}})^{k(\lambda_j)} \tag{7}$$

Therefore both $\tilde{U}_\theta$ and $\tilde{S}_\theta$ are invariant wrt $T^{\mathbb{C}}_t$, $t \geq o$, and

$$[T_\theta \mathcal{L}^2_1(J,X)]_{\mathbb{C}} = \tilde{U}_\theta \oplus \tilde{S}_\theta \tag{8}$$

Also from (6) $\tilde{U}_\theta \subset \mathfrak{D}(A^\theta_{\mathbb{C}})$ and is invariant under $A^\theta_{\mathbb{C}}$. We now intersect (8) with the real vector space $T_\theta \mathcal{L}^2_1(J,X)$ viewed as a subspace of $[T_\theta \mathcal{L}^2_1(J,X)]_{\mathbb{C}}$, obtaining

$$T_\theta \mathcal{L}^2_1(J,X) = U_\theta \oplus S_\theta$$

where

$$U_\theta = \tilde{U}_\theta \cap T_\theta \mathcal{L}^2_1(J,X) \ , \ S_\theta = \tilde{S}_\theta \cap T_\theta \mathcal{L}^2_1(J,X) \qquad (9)$$

Since $T^{\mathbb{C}}_t$ is an extension of T_t , it follows that U_θ and S_θ are invariant under T_t , $t \geq o$. Also $U_\theta \subset \mathfrak{D}(A^\theta)$ and $A^\theta(U_\theta) \subset U_\theta$. As U_0 is finite-dimensional, then $A^\theta | U_\theta$ is bounded linear and $T_t | U_\theta = e^{tA|U_\theta}$ is therefore a linear homeomorphism giving a group $\{\tilde{T}_t\}_{t\in\mathbb{R}}$ as defined. The differential equation (3) is satisfied because if $\beta \in U_\theta$ and $t \in \mathbb{R}$ then $\tilde{T}_t(\beta) \in U_0 \subset \mathfrak{D}(A^\theta)$ and therefore (3) must hold according to Theorem (3.4) (ii).

Finally since $T^{\mathbb{C}}_t$ is completely reduced by the splitting (8), then $\sigma(T^{\mathbb{C}}_t | \tilde{S}_\theta) = \{e^{\lambda t} : \lambda \in \sigma(A^\theta_{\mathbb{C}})\ ,\ \operatorname{Re}\lambda < o\}$.
But $\operatorname{Re}\lambda < o \Rightarrow |e^{\lambda t}| < 1$; hence the spectral radius of $T^{\mathbb{C}}_t | \tilde{S}_\theta$ is less than 1 and by Lemma 22.2 of ([21] p.112) it follows that $\exists\ K\ ,\ \mu > o$ s.t.

$$\|T^{\mathbb{C}}_t(\beta)\| \leq K e^{-\mu t} \|\beta\| \qquad \forall\ t \geq o \quad \forall\ \beta \in \tilde{S}_\theta \qquad (10)$$

(10) implies the required estimate (4). Q.E.D.

CHAPTER 4

Examples

In Chapter 2 we have shown that a RFDE on a Riemannian manifold X can be canonically pulled back into a vector field on the state space $\mathcal{L}_1^2(J,X)$. The present Chapter looks at the situation from a different angle, although it still draws heavily on the "vector field" point of view. In fact we shall start with vector field(s) on the ground manifold X and use the Riemannian structure on X to construct various examples of RFDE's on X. Some of these examples will be touched upon sparingly without going much deeper beyond the elementary properties, while the rest of the examples are investigated in some detail with reference to the general theory developed in the previous chapters.

§1. The ODE:

This example is well-known and has been thoroughly discussed in the subject of vector field theory or ODE's; we only mention it very briefly for the sake of completeness. Let X be a C^p manifold and $\eta : (-K,K) \times X \to TX$ a (time dependent) vector field on X, $K > o$. Define the RFDE $(F,(-K,K),J,X)$ by

$$F(t,\theta) = \eta(t, \theta(o)) \qquad \forall\, t \in (-K,K)$$
$$\qquad\qquad\qquad\qquad \forall\, \theta \in \mathcal{L}_1^2(J,X)$$

Then each solution of η is a solution of F and conversely. The initial state of the system in this case is essentially the "present" $\theta(o)$, and with suitable smoothness conditions on the vector field η solutions can be defined on the whole of the line R for any initial data, (See Lang [32] , Coddington and Levinson [5]).

§2. Delayed Development

Let X be a smooth Riemannian manifold, and let $\rho_o^{-1}(x) = \{\theta : \theta \in \mathcal{L}_1^2(J,X),\ \theta(o) = x\}$. Denote by $\mathcal{D}_x : \rho_o^{-1}(x) \longrightarrow \mathcal{L}_1^2(J,T_xX)$, $x \in X$, Cartan's development i.e.

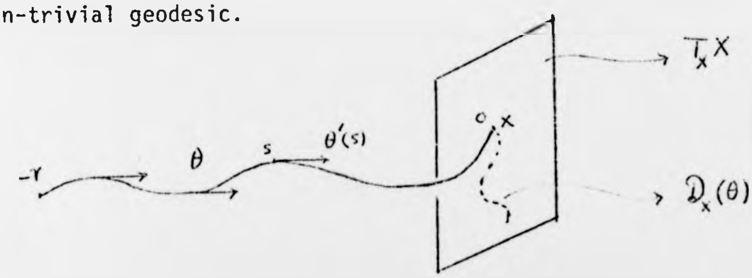
$$\mathcal{D}_x(\theta)(s) = \int_s^o {}^{\theta}\tau_u^o(\theta'(u))du \qquad s \in J,\ \theta \in \rho_o^{-1}(x)$$

(Kobayashi and Nomizu [29], Eells-Elworthy [15])

Define $F : \mathcal{L}_1^2(J,X) \to TX$ by

$$F(\theta) = \mathcal{D}_{\theta(o)}(\theta)(-r) \qquad \forall\ \theta \in \mathcal{L}_1^2(J,X)$$

Then by the smoothness of the development and the evaluation map, it follows that F is a smooth RFDE on X. It is also easy to check that no critical path of F is a non-trivial geodesic.

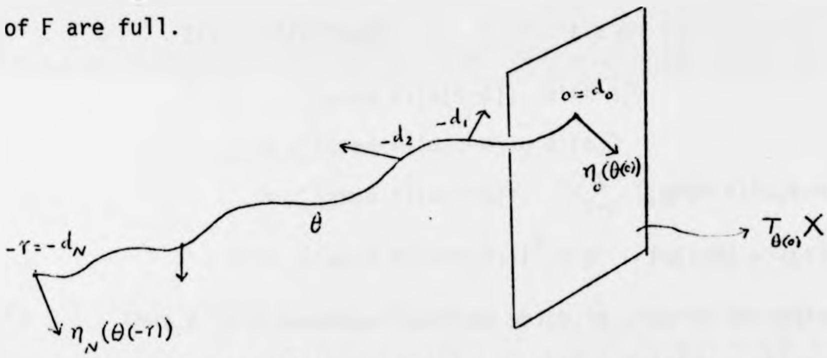


§3. The Differential Delay Equation (with Several Constant delays) (DDE)

Let X be a C^p Riemannian manifold modelled on a real Hilbert space, with $p \geqslant 5$. Take $N + 1$ real numbers $o = d_o < d_1 < d_2 < \ldots < d_N = r$ and $N + 1$ vector fields $\{\eta_i\}_{i=o}^N$ on X. Define the RFDE F by

$$F(\theta) = \sum_{i=o}^{N} {}^{\theta}\tau_{-d_i}^{o}\ \{\overleftrightarrow{\eta}_i(\theta(-d_i))\} \qquad \forall\ \theta \in \mathcal{L}_1^2(J,X) \qquad (1)$$

F is said to be a <u>differential delay equation</u> (DDE) with several constant delays $\{d_i\}_{i=o}^{N}$. Note that if $d_i = o \; \forall 1 \leq i \leq N$ or if $\eta_i = o \; \forall 1 \leq i \leq N$, then F reduces to Example §1 of an ODE. In the general case when the η_i are continuous and F is locally Lipschitz, F has unique local solutions by virtue of Theorem (1.2). If X is complete and each η_i is furthermore bounded on X (wrt the Riemannian Finsler on TX), then each maximal solution of F is full; this follows from the fact that in this case F is bounded and so the conditions of Theorem (1.5) are satisfied. In particular if X is compact, then the η_i are bounded if they are continuous and so all solutions of F are full.



When $X = R^n$ (or a Hilbert space), F reduces to the classical differential difference (or delay) equation (Bellman and Cooke [3]).

We further specialize F to be a single delay equation of the form

$$F(\theta) = {}^{\theta}\tau^{o}_{-d}\ [(\text{grad } f)(\theta(-d))] \qquad \theta \in \mathcal{L}^2_1(J,X) \qquad (2)$$

where $f: X \to R$ is C^1 and $o \leq d \leq r$. Then there are no non-trivial periodic solutions of (2) with least period equal to the delay d. To see this, let $\alpha: [-r,\infty) \to X$ be such a solution and define the function $Z : R^{\geq o} \to R$ by

$$Z(t) = f(\alpha(t-d)) \qquad t \geq o \qquad (3)$$

$$\begin{aligned} Z'(t) &= \langle (\text{grad } f)(\alpha(t-d)),\ \alpha'(t-d)\rangle \\ &= \langle (\text{grad } f)(\alpha(t-d)),\ \alpha'(t)\rangle \\ &= \langle (\text{grad } f)(\alpha(t-d)),\ {}^{\alpha}\tau^{t}_{t-d}\ [(\text{grad } f)(\alpha(t-d))\rangle \\ &= |(\text{grad } f)(\alpha(t-d))|^2 \geq o \qquad (\alpha(t-d) = \alpha(t)) \end{aligned}$$

$\forall\, t \geq o$. Thus Z is a monotone function which is clearly periodic with period d because of (3). Therefore we must have $Z'(t) = o$ which contradicts the assumption that α is a non-constant solution.

§4. Integro-Differential Equations:

The Levin-Nohel Equation:

This equation was first studied in the one-dimensional case. $X = R$, by J.J.Levin and J.Nohel ([34]).

More generally, let X be a $C^p (p \geq 4)$ complete Riemannian manifold, $a: [o,r] \to R$ a C^2 function and $\eta : X \to TX$ a continuous vector field on X.

Define the RFDE (F,J,X) by

$$F(\theta) = \int_{-r}^{o} a(-s)\ {}^{\theta}\tau_s^{o}\ \{\eta(\theta(s))\}\ ds \qquad \theta \in \mathcal{L}_1^2(J,X) \tag{1}$$

Suppose that a satisfies the hypotheses

$$a(r) = o \tag{2}$$

$$a'(t) \leq o \qquad \forall\ t \in [o,r] \tag{3}$$

$$a''(t) \geq o \quad \forall t \in [o,r] \quad \text{and} \quad \exists\ t_o \in [o,r] \quad \text{s.t. } a''(t_o) > o \tag{4}$$

Assume that F is locally Lipschitz and η is bounded in the Riemannian Finsler on X. Therefore under these hypotheses we have

Theorem (4.1):

With the above assumptions, each maximal solution of the Levin-Nohel equation (1) is full. Moreover let η be a gradient field and α: $[-r,\infty) \to X$ a solution of F in (1). Then

either (i) α is not periodic

or (ii) α is constant on $[o,\infty)$ with $\alpha(o)$ a critical point of η and $\widetilde{\alpha(o)}$ a constant critical path for F.

Proof:

Since η is bounded, it follows from (1) that F is also bounded and so by completeness of X all maximal solutions of F are full (Theorem 1.5).

To prove the second assertion of the theorem suppose that η = grad f where f : X $\to$ R is a C^1 function. Let α : $[-r,\infty) \to X$ be a full solution of F and define the function V : $R^{\geq o} \to R$ by

$$V(t) = \int_o^t \langle \eta(\alpha(s)),\alpha'(s)\rangle ds - \tfrac{1}{2}\int_{t-r}^{t} a'(t-v) \left| \int_v^t \tau_w^t\ \{\eta(\alpha(w))\} dw \right|^2_{\alpha(t)} dv \tag{5}$$

where τ stands for parallel transport along α . For simplicity of notation we call

$$K(t,v) = \left|\int_v^t \tau_w^t \,[n(\alpha(w))]dw\right|^2_{\alpha(t)} \qquad t \geq o \qquad (6)$$
$$t-r \leq v \leq t$$

Differentiating (5) wrt t and using (6) we get

$$V'(t) = \langle n(\alpha(t)),\alpha'(t)\rangle_{\alpha(t)} + \tfrac{1}{2}\, a'(r)\, K(t,t-r) - \tfrac{1}{2}\int_{t-r}^{t} a''(t-v)K(t,v)dv$$

$$- \langle n(\alpha(t)), \int_{t-r}^{t} a'(t-v)\{\int_v^t \tau_w^t\, [n(\alpha(w))]dw\}\, dv\rangle_{\alpha(t)} \qquad (7)$$

As α is a solution of F, then

$$\alpha'(t) = \int_{t-r}^{t} a(t-v)\, \tau_v^t\, [n(\alpha(v))]dv \qquad t \geq o \qquad (8)$$

$$= - \int_{t-r}^{t} a'(t-u)\{\int_u^t \tau_s^t\, [n(\alpha(s))]ds\}\, du \qquad (8)'$$

using integration by parts and the fact that $a(r) = o$. Thus (7) and (8)' give

$$V'(t) = \tfrac{1}{2}\, a'(r)K(t,t-r) - \tfrac{1}{2}\int_{t-r}^{t} a''(t-v)K(t,v)dv \qquad (9)$$

$$\leq o \qquad \forall \;\; t \geq o \qquad (9)'$$

because a satisfies (3) and (4), and K is non-negative.

Thus V is non-increasing and in particular

$$V(t) \leq V(o) = -\tfrac{1}{2}\int_{-r}^{o} a'(-v)\left|\int_v^o \tau_w^o\, [n(\alpha(w))]dw\right|^2_{\alpha(o)} dv$$

$$\leq -\frac{r}{2}\int_{-r}^{o} a'(-v)\{\int_v^o |n(\alpha(w))|^2_{\alpha(w)}\, dw\}\, dv \quad \text{(Hölder's inequality)}$$

$$= \frac{r}{2}\int_{-r}^{o} a(-v)\, |n(\theta_o(v))|^2\, dv \qquad (10)$$

where $\theta_o = \alpha|J$ is the initial path of α. Now $\eta = \text{grad } f$, so

$$\langle \eta(\alpha(s)),\alpha'(s)\rangle = \frac{d}{ds} f(\alpha(s)) \tag{11}$$

and

$$V(t) = f(\alpha(t)) - f(\alpha(o)) - \tfrac{1}{2}\int_{t-r}^{t} a'(t-v)\left|\int_{v}^{t} \tau_w^t\,[(\text{grad } f)(\alpha(w))]dw\right|^2_{\alpha(t)} dv \tag{12}$$

Now suppose that α is periodic on $[-r,\infty)$ with period p; then it is easy to check that

$$\int_{t+p-r}^{t+p} a'(t+p-v)\left|\int_{v}^{t+p} \tau_w^{t+p}[(\text{grad } f)(\alpha(w))]dw\right|^2_{\alpha(t+p)} dv$$

$$= \int_{t-r}^{t} a'(t-u)\left|\int_{u}^{t} \tau_w^t[(\text{grad } f)(\alpha(w'))]dw'\right|^2_{\alpha(t)} du \tag{13}$$

Therefore

$$V(t+p) = V(t) \qquad \forall\, t \geq o \tag{14}$$

But V is non-increasing, so (14) implies that V must be constant on $[o,\infty)$ i.e. $V'(t) = o\ \forall\, t \geq o$. Since both terms on the right hand side of (9) are non-positive, we must have

$$\int_{t-r}^{t} a''(t-v)K(t,v)dv = o \qquad \forall\, t \geq o \tag{15}$$

Now $a'' \geq o$ and is strictly positive on some sub-interval of $[o,r]$; hence (15) implies that for each t, $\exists\ \delta_1 > \delta_2 > o$ s.t. $K(t,v) = o$ $\forall\, v \in (t-\delta_1, t-\delta_2)$. Therefore by (6),

$$\int_{v}^{t} \tau_w^t\,[\eta(\alpha(w))]dw = o \qquad \forall\, v \in (t-\delta_1, t-\delta_2)$$

$$\therefore\ \tau_v^t\,[\eta(\alpha(v))] = o \qquad \forall\, v \in (t-\delta_1, t-\delta_2)$$

Consequently

$$\eta(\alpha(t)) = o \qquad \forall\ t \geq o$$

because t was arbitrary. Thus (8) gives $\alpha'(t) = o \quad \forall\ t \geq o$.

i.e. $\alpha(t) = \theta_o(o)\ \forall\ t \geq o$ with $\eta(\theta_o(o)) = o$ and

$F\ (\widetilde{\theta_o}(o)) = o$ (by (1)). Q.E.D.

The following conjecture - if true - may give an estimate on the time derivative of the orbits of solutions of the RFDE (1).

Conjecture:

Suppose X is compact and η is a C^1 vector field on X. Let $\alpha : [-r,\infty) \to X$ be a solution of F. Then $\exists$ constants $M', \mu' > o$ s.t.

$$\|\alpha'_t\|_{T_{\alpha_t}\mathcal{L}^2_1} \leq M' \ \|\alpha'_r\|_{T_{\alpha_r}\mathcal{L}^2_1} \ e^{\mu' t} \qquad \forall\ t \geq 2r \tag{16}$$

Idea of Proof:

Use the method of proof of Corollary (3.3.1). Note also that because

$$\alpha'(t) = \int_{t-r}^{t} a(t-v)\ \tau_v^t\ [\eta(\alpha(v))]dv \qquad t \geq o \tag{17}$$

and the parallel transport is of class $\mathcal{L}^2_1$ in t, it follows that α' is $\mathcal{L}^2_1$ on $[o,\infty)$. Therefore by Lemma (3.1) $[r,\infty) \ni t \to \alpha_t \in \mathcal{L}^2_1(J,X)$ is C^1 and

$$\frac{D}{dt}\ (\alpha'(t)) = (T\rho_o \circ \nabla\xi^F)(\alpha_t)(\alpha'_t) \qquad t \geq r \tag{18}$$

as in the proof of Corollary (3.3.1).

But

$$\xi^F(\theta)(s) = \int_{-r}^{o} a(-u)\ {}^{\theta}\tau_u^s[\eta(\theta(u))]du \tag{19}$$

Denote by $\nabla^X\eta(x): T_xX \circlearrowleft$ the covariant derivative of η at x wrt the Riemannian connection on TX. Then for each $\beta \in T_\theta \mathcal{L}_1^2(J,X)$

$$\nabla\xi^F(\theta)(\beta)(s) = \int_{-r}^{0} a(-u)\ {}^\theta\tau_u^s\ [(\nabla^X\eta)(\theta(u))(\beta(u))]du \tag{20}$$

Incidentially, (20) implies that F satisfies Condition (E_3) of Chapter 3 and hence all the relevant results there apply to F e.g. Theorem (3.4), the conclusion of Corollary (3.4.1), Theorem (3.5), Theorem (3.6). Moreover the map

$X \ni x \longrightarrow \|\nabla^X\eta(x)\| \in R$ is continuous (η is C^1) and so by compactness of X the set

$$\bigcup_{t\geq r} \sup \{\|\nabla\xi^F(\alpha_t)(\beta)\| : \beta \in T_{\alpha_t}\mathcal{L}_1^2(J,X),\ \sup_{s\in J} |\beta(s)| \leq 1\}$$

is bounded. Therefore the conjecture will follow by the proof of Corollary (3.3.1).

§5. Retarded Parobolic Functional Differential Equations:

i) The General Problem

Let X be a smooth Riemannian manifold of finite dimension, (with or) without boundary. Suppose

$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

is a finite-dimensional smooth vector bundle over X with a smooth connection. Then we have a smooth vector bundle $\mathcal{L}_1^2(J,E) \to X$ over X whose fibre at x is the Banach space $\mathcal{L}_1^2(J,E_x)$. Construct the linear map bundle $L(\mathcal{L}_1^2(J,E),E) \to X$ whose fibres are

$[L(\mathcal{L}_1^2(J,E),E)]_x = L(\mathcal{L}_1^2(J,E_x),E_x)$, the space of all continuous linear maps (FDE's) $\mathcal{L}_1^2(J,E_x) \to E_x$. A section of the latter bundle is a map

$$X \ni x \longmapsto F_x : \mathcal{L}_1^2(J,E_x) \to E_x$$

where F_x is an autonomous linear RFDE on the fibre E_x of E. Given such a section F, we let $A : \Gamma(E) \circlearrowleft$ be an elliptic operator on the space $\Gamma(E)$ of smooth sections of E (Eells [14]) and consider the differential equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= A\{F_x(u_t(.,x))\} \qquad x \in X,\ t \geqslant 0 \quad \text{(a)} \\ u(s,x) &= \theta(s,x) \qquad s \in J, \quad x \in X \quad \text{(b)} \end{aligned} \tag{1}$$

where a solution of (1) is a map.
$u: [-r,\infty) \times X \to E$ s.t. for each $t \in [-r,\infty)$
$u(t,.) \in \Gamma(E)$, for each $x \in X$ $u(.,x) \in \mathcal{L}_1^2([-r,\infty),E_x)$
and $u_t(.,x) \in \mathcal{L}_1^2(J,E_x)$, $t \geqslant 0$, is defined by

$$u_t(.,x)(s) = u(t+s,x) \qquad \forall\ s \in J$$

θ: $J \times X \to E$ is a given initial condition s.t.
$\theta(.,x) \in \mathcal{L}_1^2(J,E_x)$ for each $x \in X$, and $\theta(s,.) \in \Gamma(E)$ for each $s \in J$.

Alternatively the differential equation (1) - referred to as a Retarded Parabolic FDE, RPFDE - may be viewed as a linear autonomous RFDE on the Frechet space of sections $\Gamma(E)$:

$$\begin{aligned} \frac{d\tilde{u}(t)}{dt} &= \tilde{F}(\tilde{u}_t) \\ \tilde{u}_0 &= \tilde{\theta} \end{aligned} \tag{2}$$

where

$\tilde{F}$: $\mathcal{L}_1^2(J,\Gamma(E)) \longrightarrow \Gamma(E)$ is defined by

$\tilde{F} = A \circ F$, $\tilde{u}(t)(x) = u(t,x)$, $\tilde{\theta}(s)(x) = \theta(s,x)$

for $x \in X$, $t \in [-r,\infty)$, $s \in J$. Observe that $\tilde{F}$ is never continuous but is a closed map.

When X has a boundary ∂X a homogeneous boundary condition

$$u(t,x) = o \qquad \forall t \in [-r,\infty), \forall x \in \partial X \qquad (3)$$

may be attached to the initial value problem (1).

A solution to the general problem (1) is so far unknown; but in the special case when E is a trivial line bundle $X \times R$ and A is a second order elliptic operator the problem can be solved satisfactorily with fairly mild conditions on F. This fact follows as a corollary of the discussion in the next example.

ii) <u>The Retarded Heat Equation</u> (RHE)

Here X is a compact smooth Riemannian manifold of dimension $m \geqslant 1$. Let $F: \mathcal{L}_1^2(J,R^n) \to R^n$ be a linear RFDE on R^n which admits an extension to a continuous map $\mathcal{L}^2(J,R^n) \to R^n$ i.e. F satisfies Condition (E_3) of Chapter 3. Let $\Delta : \mathcal{C}^\infty(X,R) \circlearrowleft$ be the Laplacian of X operating on the Fréchet space of smooth real functions on X. X is without boundary and we are given a map $\theta : J \times X \to R^n$ which is of class $\mathcal{L}_1^2$ in the first (time) variable and is C^∞ in the second (space) variable. We seek a solution

$u: [-r,\infty) \times X \to R^n$ of the retarded heat equation.

$$\left.\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta\{F(u_t(.,x))\} \qquad x \in X \qquad &(a)\\ u(s,x) &= \theta(s,x) \qquad s \in J,\ x \in X \qquad &(b)\end{aligned}\right\} \qquad (4)$$

Give X the canonical measure dx associated with its Riemannian structure, and the space $\mathcal{L}^2(X,R)$ is furnished with a Hilbert space structure through the inner product

$$\langle\phi,\psi\rangle = \int_X \phi(x)\,\psi(x)dx \qquad (5)$$

for $\phi,\psi \in \mathcal{L}^2(X,R)$ (Ref: Eells [14]).

It is known that we can choose an orthonormal system $\{\phi_i\}_{i=0}^{\infty}$ in $\mathcal{L}^2(X,R)$ and real numbers $\{\lambda_i\}_{i=0}^{\infty} \subset R^{\geq 0}$ s.t.

$$\Delta \phi_i + \lambda_i \phi_i = 0 \qquad \forall\ i \geq 0 \tag{6}$$

The λ_i's are ordered increasingly i.e. $\lambda_i \leq \lambda_{i+1}$, $i = 0,1,2,\ldots,$ each ϕ_i is C^∞ and the system $\{\phi_i\}$ is complete in $\mathcal{L}^2(X,R)$.

We attempt to find a unique solution of the RHE (4) by using the classical Fourier method which essentially separates the time and space variables in (4), thus reducing the original problem to the eigenvalue problem (6) coupled with a retarded linear FDE on R^n which can then be treated by the techniques of Chapter 3.

By completeness of the ϕ_i's we can write

$$\theta(s,x) = \sum_{i=0}^{\infty} \theta_i(s)\,\phi_i(x) \qquad \forall\ s \in J \tag{7}$$

where the convergence - at the moment - is $\mathcal{L}^2$ in x, and

$$\theta_i(s) = \int_X \theta(s,x)\phi_i(x)\,dx \qquad \forall\ s \in J \tag{8}$$

To study the uniform convergence of the series (7) in both s and x, we view the left hand side of (7) as a map $X \ni x \to \theta(.,x) \in \mathcal{L}^2_1(J,R^n)$ and consider

$$\theta(.,x) = \sum_{i=0}^{\infty} \theta_i \phi_i(x) \qquad x \in X \tag{9}$$

Assume without loss of generality that $\lambda_i > 0$, $i = 1,2,\ldots,$ and let $k > 0$ be any integer. Then by working on each coordinate of R^n and using the symmetry of the Laplacian we get

$$\theta_i(s) = \frac{(-1)^k}{\lambda_i^k} \int_X \theta(s,x) \; \Delta^k \phi_i(x)dx \qquad i \geq 1$$

$$= \frac{(-1)^k}{\lambda_i^k} \int_X \Delta^k\theta(s,x) \; \phi_i(x)dx \qquad s \in J, \; i \geq 1 \tag{10}$$

where $\Delta^k \theta(s,x)$ means, for each $s \in J$, the value of $\Delta^k\theta(s,.)$ at x. If $J \ni s \to \Delta^k\theta(s,x) \in R^n$ is $\mathcal{L}_1^2$, then by the smoothness of θ in x and the compactness of X $\exists$ constants $K_1(\theta,k), K_2(\theta,k) > o$ s.t.

$$\int_{-r}^{o} |\Delta^k\theta(s,x)|^2 ds < K_1(\theta,k), \; \int_{-r}^{o} |\frac{\partial}{\partial s} \; \Delta^k\theta(s,x)|^2 ds < K_2(\theta,k) \tag{11}$$

$\forall$ $x \in X$, where R^n is given its standard Hilbert space structure with Euclidean norm $|.|$. Now using the fact that $\int_X |\phi_i(x)|^2 dx = 1$ and Hölder's inequality, (10) and (11) give for each integer $k > o$:

$$\|\theta_i\|_{\mathcal{L}_1^2} = \left(\frac{1}{r} \int_{-r}^{o} |\theta_i(s)|^2 ds + \frac{1}{r} \int_{-r}^{o} |\theta_i'(s)|^2 ds \right)^{\frac{1}{2}} < \frac{K_3(\theta,k)}{\lambda_i^k} \qquad i > 1 \tag{12}$$

where $K_3(\theta,k) = (K_1(\theta,k) + K_2(\theta,k))^{\frac{1}{2}}$ depends on θ , k, but independent of i = 1,2,... . We now need the following lemma concerning the series (9) and its space derivatives.

Lemma (4.1):

i) $\exists$ an integer $p > o$ s.t. for each $k > p$ the series $\sum_{i=1}^{\infty} \frac{\phi_i(x)}{\lambda_i^k}$ converges to an element of R uniformly and absolutely wrt $x \in X$.

ii) For each integer $k \geq o$, $\sum_{i=1}^{\infty} \lambda_i^{\,k}\, \theta_i \phi_i(x)$ converges to an element of $\mathcal{L}_1^2(J,R^n)$ uniformly and absolutely wrt $x \in X$.

Also $$\Delta^k \theta(.,x) = (-1)^k \sum_{i=1}^{\infty} \lambda_i^{\,k}\, \theta_i \phi_i(x) \qquad \forall\, x \in X,\ k \geq 1 \tag{13}$$

Proof of Lemma (4.1):

If $q > o$ is an integer, denote by $\mathcal{L}_{2q}^2(X,R)$ the space of all functions $\phi : X \to R$ s.t. ϕ has square integrable j-th derivative for $o \leq j \leq 2q$. Give $\mathcal{L}_{2q}^2(X,R)$ the norm

$$||\phi||_{\mathcal{L}_{2q}^2} = \left[\sum_{j=o}^{q} \int_X |\Delta^j \phi(x)|^2 dx \right]^{\frac{1}{2}} \tag{14}$$

Then by Sobolev's embedding theorem (Eells [12]) the inclusion $\mathcal{L}_{2q}^2(X,R) \hookrightarrow \mathcal{C}^o(X,R)$ is continuous for $2q > \frac{m}{2} = \frac{1}{2}\dim X$; thus if $q > m/4$ $\exists$ a constant $C > o$ s.t.

$$||\phi||_{\mathcal{C}^o} \leq C \;||\phi||_{\mathcal{L}_{2q}^2} \qquad \forall\, \phi \in \mathcal{L}_{2q}^2(X,R) \tag{15}$$

In particular for the eigen functions ϕ_i we have

$$||\phi_i||_{\mathcal{C}^o} \leq C \left[\sum_{j=o}^{q} \int_X |\Delta^j \phi_i(x)|^2 dx \right]^{\frac{1}{2}} = C\left(\sum_{j=o}^{q} \lambda_i^{2j} \right)^{\frac{1}{2}} \tag{16}$$

$\forall\, q > m/4$ and for all $i = 1, 2, \ldots$

Now since the λ_i's are monotone increasing to ∞ there is no loss of generality in assuming that $\lambda_i \geq 1$ $\forall\, i$. By a Corollary of Ikehara's theorem we have a constant $K > o$ s.t. $N(\lambda_i < T) \sim K\, T^{m/2}$ as $T \to \infty$, where $N(\lambda_i < T)$ is the number of eigenvalues $\lambda_i < T$ (S.Minakshisundaram, Å. Pleijel [36]). Taking $T = \lambda_{i+1}$, we get

$i \sim K\, \lambda_{i+1}^{m/2}$ as $i \to \infty$ i.e. the limit

$$\lim_{i\to\infty} \frac{(i-1)^2}{\lambda_i^{\,m}} = K^2 \qquad (17)$$

exists. Hence the series

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{\,q'}} \qquad (18)$$

converges for $q' \geq m$. Using (16) and comparing with the series (18) it is easy to see that the series $\sum_{i=1}^{\infty} \frac{\phi_i(x)}{\lambda_i^{\,k}}$ converges uniformly and absolutely for $x \in X$ provided that $k > \frac{5m}{4}$. This proves (i) of the lemma.

To prove (ii) it suffices to observe that as (12) holds for <u>any</u> (large) integer k, then by fixing $k' > \frac{5m}{4}$ we get

$$\|\lambda_i^{\,k}\, \theta_i \phi_i(x)\|_{\mathcal{L}_1^2} \leq \lambda_i^{\,k} \frac{K_3(\theta,k+k')}{\lambda_i^{\,k+k'}} |\phi_i(x)|$$

$$\leq K_3(\theta,k+k') . \frac{|\phi_i(x)|}{\lambda_i^{\,k'}}$$

Therefore the uniform convergence of the series $\sum_{i=1}^{\infty} \lambda_i^{\,k}\, \theta_i \phi_i(x)$, for arbitrary $k > 0$, follows from that of the series $\sum_{i=1}^{\infty} \frac{\phi_i(x)}{\lambda_i^{\,k'}}$. The proof of the lemma is completed through term by term differentiations of the series (9).

We now resume our study of the RHE (4), by letting $T_t : \mathcal{L}_1^2(J,R^n) \circlearrowleft$, $t \geq o$, be the semi-flow of the RFDE $-F$. Then for each i, $\{T_{\lambda_i t}\}_{t \geq o}$ is the semi-flow of $-\lambda_i F$ because $-\lambda_i F$ has unique solutions by virtue of Condition (F_3) and the map

$$\begin{aligned} [o,\infty) \ni t &\longmapsto T_{\lambda_i t}(\theta_o)(o) \\ J \ni s &\longrightarrow \theta_o(s) \end{aligned} \tag{19}$$

is the solution of $-\lambda_i F$ at $\theta_o \in \mathcal{L}_1^2(J,R^n)$.

We next construct an appropriate state space for our RHE (4), as follows. For each integer $k > o$ define $\mathcal{L}_{2k}^2(X,R^n)$ to be the Banach space of all functions $\phi : X \to R^n$ with square integrable j-th derivatives, $o \leq j \leq 2k$, and with norm

$$\|\phi\|_{\mathcal{L}_{2k}^2(X,R^n)} = \{\sum_{j=o}^{k} \|\Delta^j\phi\|^2_{\mathcal{L}^2(X,R^n)}\}^{\frac{1}{2}} = \{\sum_{j=o}^{k} \int_X |\Delta^j\phi(x)|^2 dx\}^{\frac{1}{2}} \tag{20}$$

Then the Fréchet space $\mathcal{C}^\infty(X,R^n)$ is the inverse limit of the decreasing sequence

$$\ldots\ldots \subset \mathcal{L}^2_{2(k+1)}(X,R^n) \subset \mathcal{L}^2_{2k}(X,R^n) \subset \mathcal{L}^2_{2(k-1)}(X,R^n) \subset \ldots \subset \mathcal{L}^2(X,R^n)$$

of Hilbert spaces (with increasing norms). Thus the sequence $\{\mathcal{L}_1^2(J, \mathcal{L}_{2k}^2(X,R^n))\}_{k=o}^{\infty}$ is a decreasing sequence of Banach (Hilbert) spaces which forms an inverse limit system with continuous inclusions $\mathcal{L}_1^2(J, \mathcal{L}^2_{2(k+1)}(X,R^n)) \subset \mathcal{L}_1^2(J,\mathcal{L}^2_{2k}(X,R^n))$ $k = o,1,2,\ldots$ denote its inverse limit by

$$\mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)) = \lim_{\leftarrow k} \mathcal{L}_1^2(J,\mathcal{L}^2_{2k}(X,R^n)) \tag{21}$$

$\mathcal{L}^2_1(J, \mathcal{C}^\infty(X,R^n))$ shall be our state space, it is a Fréchet space, viz a locally convex complete metrizable topological vector space (which is not Banachable (Horvárth [27])).

Let $\theta = \sum_{i=o}^{\infty} \theta_i \phi_i$ (as before) belong to $\mathcal{L}^2_1(J, \mathcal{C}^\infty(X,R^n))$ and try a formal solution of (4) by setting

$$\left.\begin{aligned} u(t,x) &= \sum_{i=1}^{\infty} T_{\lambda_i t}(\theta_i)(o)\phi_i(x) && x \in X,\ t \in R && \text{(a)} \\ u(s,x) &= \theta(s)(x) \qquad s \in J, && x \in X && \text{(b)} \end{aligned}\right\} \quad (22)$$

The question of convergence of the series (22)(a) is basic and shall be dealt with by constructing - via Theorem (3.6) - a splitting of the state space $\mathcal{L}^2_1(J, \mathcal{C}^\infty(X,R^n))$ in the Fréchet category as a direct sum of two closed subspaces which are both invariant under the heat flow. On the one subspace the series (22)(a) will converge for $t \geq o$ to a <u>forward solution</u> of (4), while on the other subspace it converges for $t \leq o$ to a <u>backward solution</u> of (4).

Indeed $\mathcal{L}^2_1(J,R^n)$ splits as a direct sum

$$\mathcal{L}^2_1(J,R^n) = \mathcal{U} \oplus \mathcal{S} \tag{23}$$

where the unstable subspace $\mathcal{U}$ is finite-dimensional, the stable subspace $\mathcal{S}$ is closed, $T_{\lambda_i t}(\mathcal{U}) \subseteq \mathcal{U}$, $T_{\lambda_i t}(\mathcal{S}) \subseteq \mathcal{S}$ $\forall t \geq o$, $\forall$ $i = o,1,2,\ldots.$, $\{T_{\lambda_i t}| \mathcal{U}\}_{t \geq o}$ is a group of linear homeomorphisms and $\exists$ constants K, $\mu > o$ (independent of $i = o,1,2,\ldots$) s.t.

$$\|T_{\lambda_i t}(\theta_o)\|_{\mathcal{L}^2_1(J,R^n)} \leq K e^{-\mu\lambda_i t} \|\theta_o\|_{\mathcal{L}^2_1(J,R^n)} \quad \forall t \geq o \tag{24}$$

and $\forall$ $\theta_o \in \mathcal{S}$.

Define the linear subspaces $\mathcal{F}, \mathcal{B} \subset \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n))$ by

$$\mathcal{F} = \{\theta : \theta \in \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)), \quad \theta = \sum_{i=0}^{\infty} \theta_i \phi_i \ , \ \theta_i \in \mathcal{S} \ \forall i \geq 0\} \tag{25}$$

$$\mathcal{B} = \{\theta : \theta \in \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)), \quad \theta = \sum_{i=1}^{\infty} \theta_i \phi_i \ , \ \theta_i \in \mathcal{U} \ \forall i \geq 0\} \tag{26}$$

Because of the direct sum in (23) and the orthonormality of the ϕ_i's it is easy to obtain the algebraic direct sum

$$\mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)) = \mathcal{B} \oplus \mathcal{F} \tag{27}$$

To see that (27) is also a topological sum use the continuity of the projections $p_{\mathcal{U}} : \mathcal{L}_1^2(J,R^n) \longrightarrow \mathcal{U}$, $p_{\mathcal{S}} : \mathcal{L}_1^2(J,R^n) \to \mathcal{S}$ to prove that the induced projections $p_{\mathcal{B}} : \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)) \to \mathcal{B}$, $p_{\mathcal{F}} : \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)) \to \mathcal{F}$ are also continuous, remembering that the space $\mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n))$ is generated by the increasing sequence of norms:

$$\|\phi\|_{\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2)} = \left[\frac{1}{r}\int_{-r}^{0} \sum_{j=0}^{k} \{\|\Delta^j\theta(s)(.)\|^2_{\mathcal{L}^2(X,R^n)} + \|\tfrac{\partial}{\partial s}\Delta^j\theta(s)(.)\|^2_{\mathcal{L}^2(X,R^n)}\}ds\right]^{1/2}$$

$$= \left[\sum_{j=0}^{k} \|\Delta^j \circ \theta\|^2_{\mathcal{L}_1^2(J,\mathcal{L}^2)}\right]^{1/2}, \qquad k = 1,2,\ldots. \tag{28}$$

$\Delta^j \circ \theta$ stands for the map $J \ni s \longmapsto \Delta^j\{\theta(s)(.)\} \in \mathcal{C}^\infty(X,R^n)$.

If $\theta \in \mathcal{F}$, define the map $H_t(\theta) \in \mathcal{L}_1^2(J,\mathcal{L}^2(X,R^n))$ for $t \geq 0$ by

$$H_t(\theta)(.)(x) = \sum_{i=0}^{\infty} T_{\lambda_i t}(\theta_i)\phi_i(x) \qquad t \geq 0,\ x \in X \tag{29}$$

By (24) we have for each $i \geq 0$

$$\|T_{\lambda_i t}(\theta_i)\|_{\mathcal{L}_1^2(J,R^n)} \leq K e^{-\mu\lambda_i t} \|\theta_i\|_{\mathcal{L}_1^2(J,R^n)} \leq K \|\theta_i\| \tag{30}$$

$$\forall t \geq 0$$

so that by comparison with the absolutely uniformly convergent series $\sum_{i=0}^{\infty} \theta_i \phi_i(x)$ it follows that (29) is also absolutely and uniformly convergent for $x \in X$. Also because of the estimate

$$\begin{aligned} \left| \frac{d}{ds} T_{\lambda_i t}(\theta_i)(s) \right| &= \left| F(T_{\lambda_i t+s}(\theta_i)) \right| \qquad t > o,\ s \text{ near } o \\ &\leq \|F\| \|T_{\lambda_i t+s}(\theta_i)\| \\ &\leq \|F\| \ K \|\theta_i\| \end{aligned} \tag{31}$$

if $t > o$ and $s \in [-\varepsilon, o]$ for sufficiently small $\varepsilon > o$ we see from Lemma (4.1)(ii) that the series $\sum_{i=0}^{\infty} T_{\lambda_i t}(\theta_i)(s)\phi_i(x)$ can be differentiated term by term wrt $s \in [-\varepsilon, o]$. To check that $u(t,x) = H_t(\theta)(o)(x)$ is indeed a forward solution of the RHE at $\theta \in \mathfrak{F}$ consider the following

$$\begin{aligned} \Delta\{F(u_t(\cdot,x))\} &= \sum_{i=0}^{\infty} F(T_{\lambda_i t}(\theta_i))\Delta\phi_i(x) \qquad \text{(Continuity of } F \text{ and Lemma 4.1)} \\ &= \sum_{i=0}^{\infty} -\lambda_i F(T_{\lambda_i t}(\theta_i))\phi_i(x) \\ &= \sum_{i=0}^{\infty} \frac{\partial}{\partial t} T_{\lambda_i t}(\theta_i)(o) \quad \phi_i(x) \\ &= \frac{\partial}{\partial s} H_t(\theta)(s)(x) \Big|_{s=o} \\ &= \frac{\partial}{\partial t} H_t(\theta)(o)(x) = \frac{\partial u(t,x)}{\partial t} \qquad t > o \end{aligned} \tag{32}$$

From (29), clearly $H_o(\theta) = \theta$ and so u is the required solution of the RHE for $t > o$. It is also clear from (29) and the invariance of $\mathfrak{F}$ under $T_{\lambda_i t}$, $t \geq o$, that $\mathfrak{F}$ is invariant under the forward heat semi flow $\{H_t\}_{t \geq o}$. Observe that $x \to H_t(\theta)(.)(x)$ is C^∞ because of (30) and Lemma (4.1)(ii).

Furthermore the equation (4) has unique solutions with orbits in $\mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n))$; for let u^o : $[-r,\infty)\times X \to R^n$ be also a solution of (4) at θ s.t. $u_t^o \in \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n))$ $\forall$ $t > o$. Expand

$$u_t^o(.,x) = \sum_{i=o}^{\infty} f_i(t)\phi_i(x) \tag{33}$$

where

$$f_i(t) = \int_X u_t^o(.,x)\phi_i(x)dx \qquad i=o,1,2,\ldots.. \tag{34}$$

$$f_i(o) = \theta_i \qquad i=o,1,2,\ldots. \tag{35}$$

Using (34) and the smoothness of $x \to u_t^o(.,x)$ it is easy to see that for each integer $h > o$ $\exists$ $K(t,h) > o$ s.t.

$$\|f_i(t)\|_{\mathcal{L}_1^2(J,R^n)} \leq \frac{K(t,h)}{\lambda_i^h} \qquad i = 1,2,\ldots. \tag{36}$$

(36) implies that the series (33) and all its term by term space derivatives converge uniformly and absolutely for $x \in X$. Moreover for any integer $h > o$, and for $s \in [-\varepsilon,o]$, ε small:

$$\begin{aligned}
\left|\frac{\partial}{\partial s} f_i(t)(s)\right| &= \frac{1}{\lambda_i^h}\left|\int_X \frac{\partial}{\partial s} u^o(t+s,x)\,\Delta^h\phi_i(x)dx\right| \\
&= \frac{1}{\lambda_i^h}\left|\int_X \Delta^{h+1}\{F(u^o_{t+s}(.,x))\}\,\phi_i(x)\,dx\right| \\
&\leq \frac{C(t,h)}{\lambda_i^h} \qquad i \geq 1 \quad \text{(Hölder)}
\end{aligned} \tag{37}$$

$$C(t,h) = \sup_{\substack{x\in X \\ s\in[-\varepsilon,o]}} \left[\int_X \|\Delta^{h+1}\{F(u^o_{t+s}(.,x))\}\|^2 dx\right]^{\frac{1}{2}} \tag{38}$$

The estimate (37) together with Lemma (4.1)(i) permit differentiation of the series $\sum_{i=1}^{\infty} f_i(t)(s)\phi_i(x)$ term by term wrt $s \in [-\varepsilon, o]$.

Then a simple calculation starting with

$$\left.\frac{\partial u_t^o(s,x)}{\partial s}\right|_{s=o} = \Delta\{F(u_t^o(.,x)\} \qquad t > o$$

gives

$$\sum_{i=o}^{\infty} \frac{\partial}{\partial s} f_i(t)(s)\Big|_{s=o} \phi_i(x) = - \sum_{i=o}^{\infty} \lambda_i F(f_i(t))\phi_i(x) \tag{39}$$

$$\therefore \left.\frac{\partial}{\partial s} f_i(t)(s)\right|_{s=o} = -\lambda_i F(f_i(t)) \qquad t > o \tag{40}$$

By uniqueness of solutions of $-\lambda_i F$, we must have

$$f_i(t) = T_{\lambda_i t}(\theta_i) \qquad \forall\, t > o \tag{41}$$

and $u^o = u$.

$\{H_t\}_{t \geq o}$ is clearly a semi-group of linear operators on the subspace $\mathfrak{F}$.

$\mathfrak{F}$ is invariant under composition with the Laplacian in the sense that for each $\theta \in \mathfrak{F}$, $\Delta \circ \theta \in \mathfrak{F}$ and in fact

$$H_t(\Delta^j \circ \theta) = \Delta^j \circ H_t(\theta) \qquad t \geq o,\ \theta \in \mathfrak{F} \quad j = 1,2,\ldots.. \tag{42}$$

To discuss the continuity and smoothness properties of solutions of (4) on the closed subspace $\mathfrak{F}$ we consider the semi-flow

$$H : R^{\geq o} \times \mathfrak{F} \to \mathfrak{F}$$

$$(t,\theta) \longmapsto H_t(\theta)$$

Fix $\theta \in \mathcal{F}$ and rewrite (29) in the form

$$u(t,x) = \sum_{i=0}^{\infty} \alpha^{\theta_i}(\lambda_i t)\phi_i(x) \qquad t \geq 0, \quad x \in X \tag{43}$$

$$H_t(\theta)(.)(x)=u_t(.,x)= \sum_{i=0}^{\infty} \alpha^{\theta_i}_{\lambda_i t}\, \phi_i(x) \tag{44}$$

where $\alpha^{\theta_i} : [-r,\infty) \to R^n$ is the solution of -F at θ_i. We look at the smoothness properties of (43), (44) in t by viewing them as maps $R^{\geq 0} \ni t \mapsto u(t,.) \in \mathcal{L}^2_{2k}(X,R^n)$, $R^{\geq 0} \ni t \to u_t = H_t(\theta) \in \mathcal{L}^2_1(J, \mathcal{L}^2_{2k}(X,R^n))$ for every integer k > o. We need the following lemma.

Lemma (4.2):

Let $q \geq 1$ be an integer. Then $\exists$ constants K_q, $\tilde{K}_q$, $\mu > 0$ independent of t, i s.t.

$$\left|\frac{d^q}{dt^q} \alpha^{\theta_i}(\lambda_i t)\right| \leq K_q\, \lambda_i^q\, \|\theta_i\|_{\mathcal{C}^0} \qquad \forall\ t \geq \frac{(q-1)r}{\lambda_1} \tag{45}$$

$$\|(\alpha^{\theta_i})^{(q)}_{\lambda_i t}\|_{\mathcal{C}^0} \leq \tilde{K}_q\, e^{-\mu(\lambda_i t-qr)}\, \|\theta_i\|_{\mathcal{C}^0} \qquad \forall\ t \geq \frac{qr}{\lambda_1} \tag{46}$$

for each i = 1,2,...

Proof of Lemma (4.2):

Use induction on q. Suppose that both (45) and (46) are valid for some q. Then if $t \geq \frac{qr}{\lambda_1}$ we have

$$\left|\frac{d^{q+1}}{dt^{q+1}} \alpha^{\theta_i}(\lambda_i t)\right| = \lambda_i^{q+1} \left| \tilde{F}((\alpha^{\theta_i})^{(q)}_{\lambda_i t})\right|$$

by Lemma (3.1), where $\tilde{F} : \mathcal{C}^o(J,R^n) \to R^n$ is a continuous linear extension of F. Hence

$$\left| \frac{d^{q+1}}{dt^{q+1}} \alpha^{\theta_i}(\lambda_i t) \right| \leq \lambda_i^{q+1} \|\tilde{F}\|_{\mathcal{C}^o} \|(\alpha^{\theta_i})^{(q)}_{\lambda_i t}\|_{\mathcal{C}^o}$$

$$\leq \lambda_i^{q+1} \|\tilde{F}\|_{\mathcal{C}^o} \tilde{K}_q e^{-\mu(\lambda_i t - qr)} \|\theta_i\|_{\mathcal{C}^o} \quad \text{(inductive hypotheses)}$$

$$\leq \tilde{K}_{q+1} \lambda_i^{q+1} \|\theta_i\|_{\mathcal{C}^o}, \quad \tilde{K}_{q+1} = \|\tilde{F}\|_{\mathcal{C}^o} \tilde{K}_q,$$

since $\lambda_i t - qr \geq \lambda_1 t - qr \geq o$. Similarly if $t \geq \frac{(q+1)r}{\lambda_1}$, then

$$|(\alpha^{\theta_i})^{(q+1)}_{\lambda_i t}(s)| = |\tilde{F}((\alpha^{\theta_i})^{(q)}_{\lambda_i t+s})| \qquad s \in J$$

$$\leq \|\tilde{F}\|_{\mathcal{C}^o} \tilde{K}_q e^{-\mu\{(\lambda_i t+s)-qr\}} \|\theta_i\|_{\mathcal{C}^o}$$

$$\leq \tilde{K}_{q+1} e^{-\mu\{\lambda_i t-(q+1)r\}} \|\theta_i\|_{\mathcal{C}^o} \qquad \forall s \in J.$$

Thus (45) and (46) both hold for q + 1. The Lemma is easily seen to be true for q = 1, and hence true generally.

Using the lemma we see that for each k > o

$$\left\| \frac{d^q}{dt^q} \alpha^{\theta_i}(\lambda_i t)\, \phi_i \right\|_{\mathcal{L}^2_{2k}(X,R^n)}$$

$$\leq K_q \lambda_i^q \|\theta_i\|_{\mathcal{C}^o} \left(\sum_{j=o}^{k} \lambda_i^{2j} \right)^{\frac{1}{2}}$$

$$\leq \tilde{\tilde{K}}_q \lambda_i^q \left(\sum_{j=o}^{k} \lambda_i^{2j} \right)^{\frac{1}{2}} \|\theta_i\|_{\mathcal{L}^2_1} \qquad \forall\, t \geq \frac{(q-1)r}{\lambda_1} \quad (47)$$

some $\tilde{\tilde{K}}_q > o$.

By the convergence of the series $\sum_{i=1}^{\infty} \lambda_i^q \left(\sum_{j=o}^{k} \lambda_i^{2j} \right)^{\frac{1}{2}} \|\theta_i\|_{\mathcal{L}_1^2}$

it follows from (43) that the map

$$\left[\frac{(q-1)r}{\lambda_1}, \infty\right) \ni t \longmapsto u(t,.) \in \mathcal{L}_{2k}^2(X, R^n) \quad \text{is } C^q$$

for each integer $k > o$. Thus $\left[\frac{(q-1)r}{\lambda_1}, \infty\right) \ni t \longmapsto u(t,.) \in \mathcal{C}^\infty(X, R^n)$

is C^q. Also by Lemma (3.1), the map

$$\left[\frac{qr}{\lambda_1}, \infty\right) \ni t \longmapsto u_t = H_t(\theta) \in \mathcal{F} \subset \mathcal{L}_1^2(J, \mathcal{C}^\infty(X, R^n))$$

is C^{q-1}.

Fix $t \geqslant o$ and consider the estimate

$$\|H_t(\theta)\|^2_{\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2)} \leqslant K^2 \sum_{i=o}^{\infty} \sum_{j=o}^{k} \lambda_i^{2j} \|\theta_i\|^2_{\mathcal{L}_1^2} = K^2 \|\theta\|^2_{\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2)} \quad (48)$$

$\forall\, \theta \in \mathcal{F}$, $k = 1, 2, \ldots$. This estimate is easy to prove using the definitions and the inequality (30). Therefore $H_t: \mathcal{F} \circlearrowleft$ is continuous linear. Note that the compactness of the maps H_t for $t \geqslant \frac{2r}{\lambda_1}$ is not very interesting because $\mathcal{F}$ is a Montel space (Horvárth [27]). On the other hand, denote by $(.,\phi_o)$ the map $\mathcal{L}_1^2(J,\mathcal{L}^2) \ni \theta \longmapsto (\theta,\phi_o) = \theta_o \in \mathcal{L}_1^2(J,R^n)$; then, if $t \geqslant \frac{2r}{\lambda_1}$, H_t admits an extension to a unique linear map $H_t^{\#}: \mathcal{F}_k^{\#}$ s.t. $H_t^{\#} - (.,\phi_o)\phi_o$ is compact, where $\mathcal{F}_k^{\#}$ is the closure of $\mathcal{F}$ in $\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2(X,R^n))$ wrt the norm $\|\cdot\|_{\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2)}$ for each $k > o$. To see this, extend H_t by virtue of (48) to a uniquely determined continuous linear map

$H_t^{\#} : \mathfrak{F}_k^{\#} \circlearrowleft$. We approximate $H_t^{\#} - (\cdot,\phi_o)\phi_o$ in the uniform operator topology by compact operators: let $\varepsilon > o$ be given, then by (30) $\exists$ an integer $N > o$ s.t.

$$\|T_{\lambda_i t}\| < \varepsilon \quad \forall\ i \geq N \tag{49}$$

Define the operator $H_t^N : \mathfrak{F}_k^{\#} \circlearrowleft$ by

$$H_t^N(\theta) = \sum_{i=1}^{N} T_{\lambda_i t}(\theta_i)\,\phi_i$$

Each $T_{\lambda_i t}$ is compact (for $i \geq 1$) because $t \geq \frac{2r}{\lambda_1}$ (Theorem 3.3 iv); thus H_t^N is compact. Now if $\theta \in \mathfrak{F}_k^{\#}$, then

$$\|H_t^{\#}(\theta) - (\theta,\phi_o)\phi_o - H_t^N(\theta)\|^2_{\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2)}$$

$$= \sum_{i=N+1}^{\infty} \|T_{\lambda_i t}(\theta_i)\|^2_{\mathcal{L}_1^2} \sum_{j=o}^{k} \lambda_i^{2j}$$

$$\leq \varepsilon^2 \|\theta\|^2_{\mathcal{L}_1^2(J,\mathcal{L}_{2k}^2)} \tag{50}$$

Since $\varepsilon > o$ is arbitrarily chosen, (50) gives the compactness of $H_t - (\cdot,\phi_o)\phi_o$ for $t \geq \frac{2r}{\lambda_1}$.

The subspace $\mathfrak{F}$ is stable wrt the semi-flow $\{H_t\}_{t \geq o}$ in the sense that for each $\theta \in \mathfrak{F}$, $\lim_{t \to o} H_t(\theta) = \theta_o \phi_o$, where ϕ_o is the constant harmonic function $\phi_o(x) = (\int_X 1\, dx)^{-\frac{1}{2}}$ $\forall\, x \in X$. Indeed, for each $k > o$ we have

$$\|H_t(\theta) - \theta_o\phi_o\|_{\mathcal{L}^2_1(J,\mathcal{L}^2_{2k})} \leq K e^{-\mu\lambda_1 t} \|\theta\|_{\mathcal{L}^2_1(J,\mathcal{L}^2_{2k})} \qquad \forall\ t \geq o \quad (51)$$

where k, $\mu >$ o are as in (30). Therefore the closed subspace

$$\mathcal{A} = \{\theta_o\phi_o \quad : \quad \theta_o \in \mathfrak{S} \ \} \subset \mathcal{F}$$

is an attractor for the semi-flow $\{H_t\}_{t\geq o}$.

$\mathcal{A}$ is infinite-dimensional.

One can obtain solutions of the RHE on the subspace $\mathcal{B}$ by looking at the following cases:

i). <u>The Hyperbolic Case</u>:

Let A be the infinitesimal generator of the semi-flow $\{T_t\}_{t\geq o}$ of -F. Assume that the complexified generator $A_{\mathbb{C}}$ has no eigenvalues on the imaginary axis in $\mathbb{C}$.

Suppose that $\theta = \sum_{i=o}^{\infty} \theta_i\phi_i \in \mathcal{B}$ i.e.

$\theta_i \in \mathcal{U}\ \forall\ i \geq o$. Referring to the proof of Theorem (3.6), we have a splitting

$$[\mathcal{L}^2_1(J,R^n)]_{\mathbb{C}} = \tilde{\mathcal{U}} \oplus \tilde{\mathfrak{S}}$$

where $A_{\mathbb{C}}|\tilde{\mathcal{U}} : \tilde{\mathcal{U}} \circlearrowleft$ is bounded linear and each $e^{-tA_{\mathbb{C}}|\tilde{\mathcal{U}}}$ for $t > o$ has spectral radius < 1 because all the eigenvalues of $A_{\mathbb{C}}|\tilde{\mathcal{U}}$ have strictly positive real parts. Therefore $\exists$ constants $\tilde{K}, \tilde{\mu} >$ o s.t.

$$\|e^{-tA|\mathcal{U}}\| \leq \tilde{K} e^{-\tilde{\mu}t} \qquad t \geq o \qquad (52)$$

Define the backward semi-flow $\{B_t\}_{t \leq o}$ on $\mathcal{B}$ by

$$B_t(\theta) = \sum_{i=o}^{\infty} e^{\lambda_i tA|\mathcal{U}} (\theta_i)\phi_i \qquad -\infty < t \leq o \tag{53}$$

The uniform convergence of the series (53) and its term by term derivatives are studied in the same manner as we did for the forward semi-flow $\{H_t\}_{t \geq o}$ taking into account the basic estimate (52). By a similar calculation to the one used in obtaining (32), we get

$$\frac{\partial}{\partial s} B_t(\theta)(s)(x)\Big|_{s=o} = \Delta\{F(B_t(\theta)(.)(x))\} \qquad t < o \tag{54}$$

i.e. $\{B_t(\theta)\}_{t \leq o}$ generates a backward solution of the RHE starting at $\theta \in \mathcal{B}$.

As before each $B_t : \mathcal{B} \circlearrowleft$ $t \leq o$ is a continuous linear map, and term by term differentiations of (53) wrt t (together with Lemma 4.1) imply that the map $(-\infty, o] \ni t \to B_t(0) \in \mathcal{B}$ is C^∞.

Since $\mathcal{U}$ is finite-dimensional, then for each $t < o$, B_t is the uniform limit of a sequence of operators of finite rank; hence B_t is compact $\forall\, t < o$ wrt the norms $\|.\|_{\mathcal{L}_1^2(J, \mathcal{L}_{2k}^2)}$, $k = o,1,2,\ldots$

Using (52) we obtain as before $\lim_{t \to -\infty} B_t(\theta) = \theta_o\phi_o$ for each $0 \in \mathcal{B}$, and the finite-dimensional repelling subspace

$$\mathcal{R} = \{\theta_o\phi_o : \theta_o \in \mathcal{U}\} \subset \mathcal{B}$$

for the backward semi-flow $\{B_t\}_{t \leq o}$.

ii) The Delayed Case:

Here we specialize F by taking it to be a delay equation of the form

$$F(\gamma) = \sum_{j=1}^{N} L_j(\gamma(-d_j)) \qquad \forall\, \gamma \in \mathcal{L}^2_1(J,R^n) \tag{55}$$

with finite delays $o < d_1 < d_2 \ldots < d_N = r$; each $L_j : R^n \circlearrowleft$ is a linear map. Thus (4) becomes the delayed heat equation (DHE):

$$\left.\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \sum_{j=1}^{N} \Delta\{L_j(u(t-d_j,x))\} && x \in X \\ u(s,x) &= \theta(s)(x) \qquad s \in J\ , && x \in X \end{aligned}\right\} \tag{56}$$

A forward solution of (56) can be defined for $\theta \in \mathcal{B}$ again by the same formula (29). The main idea here is that - because of the delay- the series (29) is made to converge for each fixed $t > o$. This is attained through

Lemma (4.3):

Let $\{T_t\}_{t\geqslant o}$ be the semi-flow of -F in (55), and $p > o$ be any integer. Then $\exists$ a constant $M > o$ (independent of t, i) s.t.

$$\|T_{\lambda_i t}(\theta_i)\|_{\mathcal{L}^2_1} \leqslant \{K(\lambda_i)\}^p \|\theta_i\|_{\mathcal{L}^2_1} \qquad \forall\, o \leqslant t \leqslant pd_1 \tag{57}$$

where

$$K(\lambda_i) = \{M\,[1 + \lambda_i d_1 \sum_{j=1}^{N} \|L_j\|\,]^2 + \frac{1}{\lambda_i^2} + M\,(\sum_{j=1}^{N} \|L_j\|\,)^2\}^{\frac{1}{2}} \tag{58}$$

$i = 1,2,\ldots.$

<u>Proof</u>:

Starting from the equation

$$T_{\lambda_i t}(\theta_i)(s) = \begin{cases} \theta_i(o) - \lambda_i \sum_{j=1}^{N} \int_o^{t+\frac{s}{\lambda_i}} L_j(\theta_i(u-d_j))du & o \le t+\frac{s}{\lambda_i} \le t \le d_1 \\ & s \in J \\ \theta_i(t+\frac{s}{\lambda_i}) & -r \le t+\frac{s}{\lambda_i} \le o, \ s \in J \\ & o \le t \le d_1 \end{cases} \tag{59}$$

it is easy to see that

$$|T_{\lambda_i t}(\theta_i)(s)| \le [1 + d_1\lambda_i \sum_{j=1}^{N} \|L_j\|] \ \|\theta_i\|_{\mathcal{C}^o} \quad \begin{matrix} s \ J \\ o \le t \le d_1 \end{matrix} \tag{60}$$

$$\left|\frac{\partial}{\partial s} T_{\lambda_i t}(\theta_i)(s)\right| \le \begin{cases} \{\sum_{j=1}^{N} \|L_j\|\} \ \|\theta_i\|_{\mathcal{C}^o} & t+\frac{s}{\lambda_i} \ge o \\ & s \in J \\ \frac{1}{\lambda_i} | \theta_i'(t+\frac{s}{\lambda_i})| & t+\frac{s}{\lambda_i} \le o \end{cases} \tag{61}$$

for all $s \in J$, $o \le t \le d_1$. Therefore $\exists$ a constant $M > o$ s.t.

$$\|T_{\lambda_i t}(\theta_i)\|_{\mathcal{L}_1^2} \le K(\lambda_i) \ \|\theta_i\|_{\mathcal{L}_1^2} \qquad o \le t \le d_1 \tag{62}$$

where $K(\lambda_i)$ is given by (58). Thus the lemma holds for p = 1; for arbitrary p (57) follows by an easy induction argument which makes use of (62). Q.E.D.

In this way the semi-flow $\{H_t\}_{t \ge o}$ for the DHE (56) is defined on the <u>whole</u> of the state space $\mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n))$. Each $H_t : \mathcal{L}_1^2(J, \mathcal{C}^\infty(X,R^n)) \circlearrowleft$ is a continuous linear map leaving the closed subspaces $\mathcal{F}$, $\mathcal{B}$ invariant. If furthermore we are in the hyperbolic situation (i) above, then $H_t|\mathcal{B} : \mathcal{B} \circlearrowleft$ is a linear homeomorphism for $t \ge o$; indeed $H_t|\mathcal{B}$ is injective and has a continuous inverse $(H_t|\mathcal{B})^{-1}$ given by

$$(H_t|\mathcal{B})^{-1}(\theta) = \sum_{i=o}^{\infty} e^{-\lambda_i tA}(\theta_i)\phi_i \qquad \theta \in \mathcal{B}.$$

$$t \ge o$$

As in the ordinary case (Theorem 3.6), the semi-group $\{H_t|\mathcal{B}\}_{t\geq o}$ extends naturally to a 1-parameter group $\{\tilde{H}_t|\mathcal{B}\}_{t\varepsilon R}$ which solves the DHE (56) on the whole of R.

By checking on the subspace $\mathcal{B}$, similar arguments to the ones before - but exploiting the estimate (57) - give the following smoothness properties of the solution u of the DHE (56): for any $\theta \varepsilon \mathcal{L}^2_1(J, \mathcal{C}^\infty(X,R^n))$ and any integer $q \geqslant 1$ the map $[\frac{(q-1)r}{\lambda_1}, \infty) \ni t \to u(t,.) \varepsilon \mathcal{C}^\infty(X,R^n)$ is C^q, and the map

$$[\frac{qr}{\lambda_1}, \infty) \ni t \longmapsto u_t = H_t(\theta) \varepsilon \mathcal{L}^2_1(J, \mathcal{C}^\infty(X,R^n)) \text{ is } C^{q-1}.$$

Remarks

1. The case $d_1 = o$ is not covered by the above analysis and Lemma (4.3) fails to give any information on the existence of a solution of (56) on the subspace $\mathcal{B}$ for $t > o$. If $d_1 = o$ we do not know whether a (unique) solution of (56) exists for $t \geqslant o$ and with initial path $\theta \varepsilon \mathcal{B}$.

2. The hyperbolic situation (i) is largely typical (i.e. "generic" in some sense) among the class of all RHE's, because the underlying assumption on F is known to be generic (Oliva [38]). The usual heat equation $\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x)$ does not represent generic behaviour - even among the non-retarded ones; but instead it shows the totally stable case: $\mathcal{B} = \{o\}$, $\mathcal{F} = \mathcal{L}^2_1(J, \mathcal{C}^\infty(X,R^n))$.

3. Replace Δ by a second order elliptic self adjoint operator on X.

CHAPTER 5

Generalizations and Suggestions for Further Research

Here we make some suggestions and conjectures which may be of some significance in the future. The treatment shall be sketchy in most cases.

We start by assuming in this Chapter that all RFDE's admit unique local solutions which depend continuously on initial data.

§1. Smooth Dependence on Initial data:

Let (F,J,X) be a C^1 RFDE on a Banach manifold X. Define its semi-flow $S_t : \mathcal{L}_1^2(J,X) \hookleftarrow$ $t \geq o$ by

$$S_t(\theta) = \alpha_t^\theta \ , \quad \theta \in \mathcal{L}_1^2(J,X) \tag{1}$$

where α^θ is the solution through θ.

Conjecture (5.1):

$S_t : \mathcal{L}_1^2(J,X) \hookleftarrow$ is of class C^1 for each t.

Sketch of Proof:

By localization, it is sufficient to consider the case when $X = E$, a real Banach space. Then the conjecture will hold because of the following Lemma which is proved via the Implicit Function Theorem (Lang [32]).

Lemma (5.1):

Let $\theta_o \in \mathcal{L}_1^2(J,E)$ and $F : [o,K) \times \mathcal{L}_1^2(J,E) \to E$ a C^1 (time-dependent) RFDE. Then $\exists$ $\varepsilon > o$, a neighbourhood

V of θ_o in $\mathcal{L}_1^2(J,E)$ and a unique C^o map $\phi : [o,\varepsilon] \times V \to \mathcal{L}_1^2(J,E)$ s.t.

$$\frac{\partial \phi\ (t,\theta)\ (o)}{\partial t} = F(t,\phi(t,\theta)) \qquad \text{a.a. } t \in [o,\varepsilon]$$
$$\forall\ \theta \in V \qquad (12)$$
$$\phi(o,\theta) = \theta \qquad \forall\ \theta \in V$$

Moreover $\phi(t,.) : V \to \mathcal{L}_1^2(J,E)$ is C^1 for each $t \in [o,\varepsilon]$.

<u>Proof</u>:

Use the implicit function theorem.

Assume without loss of generality that $\theta_o = o$.

Let $I = [o,K]$. Denote by $G_{\mathcal{L}_1^2} \subset \mathcal{C}^o(I,\mathcal{L}_1^2(J,E))$ the set of all continuous maps $\gamma : [o,K] \longrightarrow \mathcal{L}_1^2(J,E)$ s.t.

$$\gamma(t)(s) = \begin{cases} \gamma(o)(t+s) & t+s \leq o \\ \gamma(t+s)(o) & o \leq t+s \leq \varepsilon \end{cases}$$

Then $G_{\mathcal{L}_1^2}$ is a Banach space with the norm

$$\|\gamma\|_{\mathcal{C}^o(I,\mathcal{L}_1^2)} = \sup_{t \in I} \|\gamma(t)\|_{\mathcal{L}_1^2}$$

Take a neighbourhood U of $\theta_o = o$ in $\mathcal{L}_1^2(J,E)$ and define

$$G_U = \{\gamma : \gamma \in G_{\mathcal{L}_1^2}\ ,\ \gamma(I) \subset U\}$$

Define the map $g : [o,K) \times U \times G_u \to \mathcal{L}^2([-r,K],E)$ by

$$g(a,\theta,\gamma)(t) = \begin{cases} \frac{d}{dt}\gamma(t)(o) - aF(at,\gamma(t)) & t \in I = [o,K) \\ \frac{d}{dt}\gamma(o)(t) - \frac{d\,\theta(t)}{dt} & \text{a.a. } t \in J \end{cases} \qquad (3)$$

for $a \in [o,K)$, $\theta \in U$, $\gamma \in G_U$. Then g is C^1 and $g(o,\theta_o,o) = o$.
Also if $\delta \in G_{\mathcal{L}^2_1}$, then

$$\{D_3 g(o,\theta_o,o)\}(\delta)(t) = \begin{cases} \frac{d}{dt}\delta(t)(o) & \text{a.a. } t \in [o,K] \\ \frac{d}{dt}\delta(o)(t) & \text{a.a. } t \in J \end{cases} \qquad (4)$$

where D_3 denotes Frechét differentiation wrt the third variable γ.

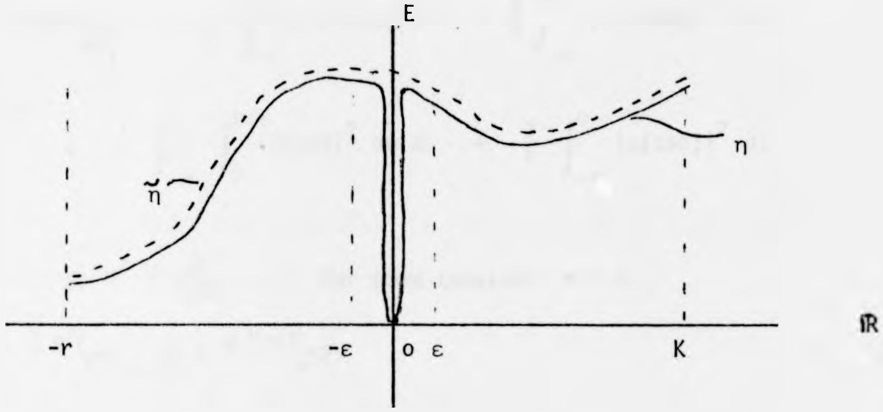
It is easy to see that $D_3 g(o,\theta_o,o) : G_{\mathcal{L}^2_1} \to \mathcal{L}^2([-r,k],E)$ is a continuous linear injection. To prove that it is actually a linear homeomorphism, it is sufficient to show that we can invert it continuously on a dense set in $\mathcal{L}^2([-r,K],E)$. Consider the linear subspace

$$V^{\#} = \{\eta : \eta \in \mathcal{C}^1([-r,K], E) \ , \ \eta(o) = o\}$$

of $\mathcal{L}^2([-r,K], E)$. It is not hard to see that $V^{\#}$ is dense in $\mathcal{L}^2([-r,K], E)$ wrt the $\mathcal{L}^2$-norm

$$\|\tilde{\eta}\|_{\mathcal{L}^2} = \left[\int_{-r}^{K} |\tilde{\eta}(t)|^2\,dt\right]^{1/2} \quad , \quad \tilde{\eta} \in \mathcal{L}^2([-r,K], E) \quad ,$$

by looking at the picture



where the dotted curve $\tilde{\eta}$ is in $\mathcal{L}^2([-r,K],E)$ and $\eta \in V^{\#}$ is an $\mathcal{L}^2$ - approximation of $\tilde{\eta}$ for any given $\varepsilon > o$.

The inverse of $D_3g(o,\theta_o,o)$ on the subspace $V^{\#}$ is given as follows: Let $\eta \in V^{\#}$ and define $\alpha \in \mathcal{L}_1^2([-r,K], E)$ by

$$\alpha(t) = \int_o^t \eta(u)\, du \qquad \forall\, t \in [-r,K]$$

Define $\delta \in \mathcal{C}^o(I,\mathcal{L}_1^2(J,E))$ by

$$\delta(t) = \alpha_t \quad , \quad t \in [o,K]$$

Then $\delta \in G_{\mathcal{L}_1^2}$ and $D_3g(o,\theta_o,o)(\delta) = \eta$. Also for each $t \in I$,

$$\|\delta(t)\|^2_{\mathcal{L}^2_1} = \frac{1}{r}\int_{-r}^{0} |\alpha\ (t+s)|^2 ds + \frac{1}{r}\int_{-r}^{0} |\alpha'(t+s)|^2\ ds$$

$$\leq \frac{1}{r}\int_{-r}^{0}\int_{0}^{t+s} |\eta(u)|^2\ du\ ds + \frac{1}{r}\int_{-r}^{0} |\eta(t+s)|^2\ ds$$

$$\leq M^2 \|\eta\|^2_{\mathcal{L}^2}, \quad \text{for some constant } M > 0 .$$

$$\therefore \|\delta\|_{\mathcal{C}^0(I,\mathcal{L}^2_1)} \leq M\|\eta\|_{\mathcal{L}^2}$$

Thus $D_3 g(o,\theta_o,o) : G_{\mathcal{L}^2_1} \to \mathcal{L}^2([-r,K], E)$ is a linear homeomorphism, so by the implicit function theorem $\exists\ \varepsilon > o$, a neighbourhood V of θ_o in $\mathcal{L}^2_1(J,E)$ and a unique C^1 map $h : [o,\varepsilon) \times V \longrightarrow G_u$ s.t.

$$g(a,\theta,\ h(a,\theta)) = o \qquad \forall\ a \in [o,\varepsilon],\ \forall\ \theta \in V \tag{5}$$

The map h is unique among the <u>continuous</u> ones which satisfy (5) for small enough ε and V. Now define the continuous map $\phi : [o,\varepsilon] \times V \longrightarrow U$ by

$$\phi\ (t,\theta) = h(\varepsilon,\theta)(t/\varepsilon) \qquad t \in [o,\varepsilon],\quad \theta \in V \tag{6}$$

It follows immediately from (5), (6) that

$$\frac{\partial\phi(t,\theta)(o)}{\partial t} = F\ (t,\ \phi(t,\theta)) \qquad t \in [o,\varepsilon],\quad \theta \in V$$

Also

$$\frac{d}{dt} h(\varepsilon,\theta)(o)(t) = \frac{d\theta(t)}{dt} \qquad \text{a.a.} \quad t \in J$$

gives

$$\phi(o,\theta) = \theta \qquad \forall \ \theta \in V \tag{7}$$

Since the semi-flow $(t,\theta) \longmapsto \alpha_t^\theta$ gives a continuous map satisfying (5), it follows that

$$\phi(t,\theta) = \alpha_t^\theta \qquad t \in [o,\varepsilon], \quad \theta \in V$$

Hence $\theta \longmapsto \alpha_t^\theta$ is C^1 because $\phi(t,.)$ is .

<u>Remarks</u>:

1. We feel that the differentiability of $F : \mathcal{L}_1^2(J,X) \to TX$ is sufficient to guarantee that the maps $S_t : \mathcal{L}_1^2(J,X) \circlearrowleft$ are C^1 . This may probably be done by a modification of the above lemma to bypass the assumption concerning continuous dependence on initial paths. This may then yield a new and short proof of the basic existence,uniqueness and smooth dependence on initial data in the Cauchy problem for differentiable RFDE's (See Graves [20], Robbin[42]) .

2. If dim $X < \infty$ an implicit function argument can also be applied to prove the existence and smoothness of local stable and unstable manifolds through a hyperbolic equilibrium path of a C^1 RFDE $F : \mathcal{L}_1^2(J,X) \to TX$. We shall not give details of this argument here.

The following conjecture is a corollary of continuous dependence:

<u>Proposition (5.2)</u>:

Suppose X is compact and has Euler characteristic $\chi(X) \neq o$. Then for each $t \geq o$, $\exists$ $x_o \epsilon X$ s.t. $S_t(\tilde{x}_o)(o) = x_o$, where $\tilde{x}_o : J \to X$ is the constant path through x_o.

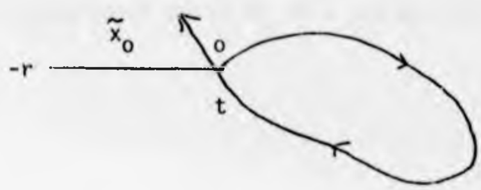
<u>Proof</u>:

Consider the continuous map

$$\begin{array}{ccc} X & \longrightarrow & X \\ x & \xmapsto{S_t(\tilde{\cdot})(o)} & S_t(\tilde{x})(o) \end{array}$$

This is homotopic to the identity $id_X : X \circlearrowleft$ on X because $S_o(\tilde{x})(o) = x$ and $t \longmapsto S_t(\tilde{x})(o)$ is continuous. Hence the Lefschetz number of $S_t(\tilde{\cdot})(o)$ is equal to $\chi(X)$ and is therefore non-zero, by hypothesis. Therefore by Lefschetz fixed point theorem ([44]), the map $S_t(\tilde{\cdot})(o)$ has a fixed point for each t i.e. $\exists$ $x_o \epsilon X$ s.t. $S_t(\tilde{x}_o)(o) = x_o$. Q.E.D.

The above result proposes to give a criterion for the existence of solutions which loop back upon themselves after any finite time:



The next proposition is "dual" to the last one in the sense that it says that when X is finite dimensional, then each point of X is attainable by the semi-flow at any future time.

Proposition (5.3):

Suppose $\dim X < \infty$. Then for every $t \geq o$ and each $x \in X$, $\exists$ $\theta \in \mathcal{L}_1^2(J,X)$ s.t. $S_t(\theta)(o) = x$.

Proof:

Suppose $\dim X = n$, and look at the continuous map

$$\begin{array}{ccc} \mathcal{L}_1^2(J,X) & \xrightarrow{S_t(.)(o)} & X \\ \theta & \longmapsto & S_t(\theta)(o) \end{array}$$

We claim that this map is surjective. Suppose not, then $\exists$ $x_o \in X$ s.t. $S_t(.)(o)$ induces a map $[S_t(.)(o)]_*$ of the n-th homology groups

$$H_n(X) \xrightarrow[{[S_t(.)(o)]_*}]{} H_n(X - x_o) \hookrightarrow H_n(X)$$

Now in the punctured manifold $X - x_o$, n-dimensional cycles retract onto lower dimensional parts of X, so we must have $H_n(X - x_o) = o$;

hence $[S_t(.)(o)]_* = o$.

Since $\mathcal{L}_1^2(J,X)$ is homotopically equivalent to X , then

$H_n\{\mathcal{L}_1^2(J,X)\} = H_n(X)$. Also the map $S_t(.)(o)$ is homotopic to the evaluation $\rho_o : \mathcal{L}_1^2(J,X) \to X$ and $(\rho_o)_* = id_* : H_n(X) \hookleftarrow$.

Thus

$$[S_t(.)(o)]_* = id_* \quad , \text{ a contradiction.}$$

This proves the proposition. Q.E.D.

§2. Some General Properties of the non-linear Semi-flow:

Use the notation of the last section to denote by $\{S_t\}_{t\geq o}$ the semi-flow of an autonomous C^∞ RFDE (F,J,X) .

The generator B of $\{S_t\}_{t\geq o}$ is a vector field

$B : D(B) \subset \mathcal{L}_1^2(J,X) \longrightarrow T\mathcal{L}_1^2(J,X)$ defined by

$$B(\theta) = \frac{d}{dt} S_t(\theta)\Big|_{t=o}$$

whenever the right hand side exists (See Theorem (3.4)) .

The conjecture below generalizes Theorem (3.4) of the linear case:

Conjecture (5.4):

With the above notation,

i) $D(B) = \{\theta : \theta \in \mathcal{L}_1^2(J,X)\ ,\ \theta' \in \mathcal{L}_1^2(J,TX)\ ,\ F(\theta) = \theta'(o)\}$,

and is dense in $\mathcal{L}_1^2(J,X)$.

ii) $B(\theta) = \theta' \quad \forall\ \theta \in D(B)$, and $B(\theta)(o) = F(\theta)$

for all $\theta \in D(B)$.

iii) $S_t\{D(B)\} \subseteq D(B) \quad \forall\ t \geq o$. For each $\theta \in D(B)$,

$$\begin{array}{rcl} [o,\varepsilon) & \longrightarrow & D(B) \\ t & \longmapsto & S_t(\theta) \end{array}$$

is the unique solution of vector field B starting at θ . Also

$$B(S_t(\theta)) = (T_\theta S_t)(B(\theta)) \qquad \begin{array}{l} \forall\ \theta \in D(B) \\ \forall\ t \geq o \end{array}$$

iv) B has closed graph.

<u>Proof</u>:

Appeal to general properties of non-linear semi-groups as in Chernoff - Marsden (Properties of ∞-dimensional Hamiltonian systems [6])

<u>Remarks</u>:

1. We do not know whether the semi-flow $\{S_t\}_{t\geq o}$ extends to a group of bijections on a dense subset of $\mathcal{L}_1^2(J,X)$, so that the RFDE

$$\alpha'(t) = F(\alpha_t) \quad t\geq o\ , \quad \alpha_o = \theta$$

can be solved backwards on this dense set.

2. It seems plausible that the tangent semi-flow $\{TS_t\}_{t\geq o}$ on $T\mathcal{L}_1^2(J,X)$ corresponds to a RFDE on TX ; if so, then we can say something about the compactness of the linear maps

$T_\theta S_t : T_\theta \mathcal{L}_1^2(J,X) \longrightarrow T_{S_t(\theta)}\mathcal{L}_1^2(J,X)$ for each $\theta \in \mathcal{L}_1^2(J,X)$.

Can $D(B)$ have a manifold structure - in a natural way - so that B is C^∞ and TB is the generator of $\{TS_t\}_{t\geq o}$?

§3. g_2 - Gradient RFDE's

In connection with the discussion in Chapter 2, recall that $\mathcal{L}_1^2(J,X)$ can be given the metric

$$g_2(\theta)(\beta,\gamma) = \frac{1}{r}\int_{-r}^{o}\langle\beta(s),\gamma(s)\rangle_{\theta(s)}\, ds + \frac{1}{r}\int_{-r}^{o}\left\langle\frac{D\beta(s)}{ds}, \frac{D\gamma(s)}{ds}\right\rangle_{\theta(s)} ds$$

for $\beta,\gamma \in T_0\mathcal{L}_1^2(J,X)$. It is an open question whether the Morse inequalities can be developed for a g_2- GRFDE .

§4. Stochastic Retarded Integral Equations:

We have already seen that the deterministic techniques - particularly those involving parallel transport - fail to yield any information if we are working with $\mathcal{C}^{0}(J,X)$ as state space. On the other hand a stochastic versionof parallel transport along continuous paths is made available to us through the work of Itô ([28]), and one may use this idea in looking for stochastic analogues of the major results of Chapters 2 and 3. The RFDE is replaced by a stochastic integral equation and the Cauchy problem may be examined in the spirit of Eells - Elworthy ([16]). An interesting problem here is to prove a stochastic parallel of the stable-bundle Theorem (Theorem 3.6) of Chapter 3.

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