Graph colourings using structured colour sets

Thesis

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Graph Colourings using Structured Colour Sets

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ABSTRACT

A natural generalisation of k-tuple colourings are (p,q)-overlap colourings where p distinct colours are assigned to each vertex such that adjacent vertices share exactly q colours. The (p,q)-chromatic number is the smallest number of colours needed for a (p,q)-overlap colouring. Inequalities of the (p,q)-chromatic number are obtained together with analogues of the Attainment and Periodicity theorems of Hilton, Rado & Scott [9].

Classes of subgraphs of the Kneser graphs G_k^n are introduced, by identifying the underlying n-set with Z_n, giving it a circular metric, and considering subgraphs induced by vertices whose k colours are pairwise at least a given distance, d apart. These Schrijver graphs ${}_dS_k^n$ are investigated and their fractional chromatic number is computed. A conjecture that generalises the Erdos-Ko-Rado Theorem [4] in the context of Schrijver graphs is given. The conjecture is proved to be true for the Schrijver graphs ${}_dS_k^n$ with

$d = k = 2$ and for $d = \left\lfloor \frac{n}{k} \right\rfloor$ where $n < (k+1)d$.

The concept of displacement sequence is introduced together with the graphs they induce, the rotation subgraphs. Their independence and fractional chromatic numbers are found. In particular where the colours at each vertex are as far apart as possible and evenly distributed, the resulting rotation subgraph of G_k^n, which has $\frac{n}{\gcd(n,k)}$ vertices, is shown to have the same fractional chromatic number as G_k^n. It is also shown to be star extremal and vertex critical with respect to both the fractional and circular chromatic numbers.

Circular chromatic numbers of the Kneser graphs G_k^n for $n = 2k+1$ and for $n = 2k+2$ is computed.

The relation between n-chromatic number for Z_n-colouring (introduced by Vince [17]) and the k^{th} chromatic number for k-tuple colouring (discussed by Stahl and by Hilton, Rado and Scott) is investigated. These two types of colouring are combined into a single colouring ($Z_{n,k}$-colouring). Inequalities of its respective chromatic number are obtained and a generalisation of Theorem 1 of [17] is given.

The circular distance graph is considered and shown to be isomorphic to a family of rotation subgraphs of G_k^n. Its k^{th} chromatic number is derived.

CONTENTS

CHAPTER 1

Definitions & Introduction

Throughout this thesis, a ***graph*** is assumed to be finite and simple.

Let $I^n = \{x \in Z^+ : x \leq n\}$, and I_k^n denote the family of subsets of I^n of cardinality k. For $k \geq 1$ and $n \geq 2k$ we define the graph G_k^n whose vertex set is I_k^n, and two vertices are adjacent iff they are disjoint as subsets. These graphs are more widely known as ***Kneser graphs*** (see [12]).

An ***k-tuple colouring*** of G is an assignment of k distinct colours to each vertex such that no two adjacent vertices share a colour. The ***k^{th} chromatic number*** of G, denoted by $\chi_k(G)$, is the least number of colours needed for an k-tuple colouring of G (see [16]). Thus $\chi_1(G)$ is the ordinary chromatic number.

Hilton, Rado & Scott [9] studied the ***fractional chromatic number*** (previously known as the multichromatic number) of G, defined as

$$\chi_f(G) = \inf\left\{\frac{\chi_m(G)}{m} : m \in Z^+\right\}.$$

It was shown in [9] that this is also equal to $\lim_{m\to\infty}\left(\frac{\chi_m(G)}{m}\right)$, and furthermore that it is equal to $\left(\frac{\chi_k(G)}{k}\right)$ for some k. This in conjunction with Corollary to Theorem 9 of [16] gives the following result.

Lemma 1.1 $\chi_f(G_k^n) = \frac{n}{k}$ $(n, k \in Z^+, n \geq 2k)$.

A ***graph homomorphism***, θ, is a mapping $\theta : G \to H$ such that $\theta(u)$ and $\theta(v)$ are adjacent in H whenever u and v are adjacent in G. This leads to a reformulation of the concept of a k-tuple colouring of G with n colours as a homomorphism $\theta : G \to G_k^n$. This is definition AF1 of [16].

A k-tuple colouring demands that no two adjacent vertices share a colour. In contrast to this we define for non-negative integers p and q $(p \geq q)$ a ***(p,q)-overlap colouring*** as an assignment of p distinct colours to each vertex so that any pair of adjacent vertices share exactly q colours. The ***(p,q)-chromatic number*** of G, denoted by $\chi_{p,q}(G)$, is the smallest number of colours needed for a (p,q)-overlap colouring. Thus $\chi_{p,0}(G)$ is the p^{th} chromatic number $\chi_p(G)$ as defined by Hilton, Rado & Scott [9]. In particular, $\chi_{1,0}(G)$ is, once again, the ordinary chromatic number.

We begin Chapter 2 by investigating (p,q)-overlap colourings, in particular we shall concentrate on the case where $p = mq$ for some integer $m > 1$, and use the notation: ${}_m\chi_q(G) = \chi_{mq,q}(G)$. These (mq,q)-overlap colourings distribute the colours at each vertex of G such that the ratio of the number of colours assigned to each vertex to that shared by adjacent vertices is $m:1$. In analogy to the fractional chromatic number we define the ***overlap fractional chromatic number*** as

$$_m\chi_f(G) = \inf\left\{\frac{_m\chi_q(G)}{mq} : q \in Z^+\right\}.$$

In Chapter 3 we investigate subgraphs of Kneser graphs and consider their factional chromatic numbers. To define these subgraphs we make use of the ***circular norm*** $|x|_n$ introduced by Bondy & Hell [3] and Vince [17]. Let Z_n denote the set of integers modulo n. If $x \in Z_n$, we denote by $\Gamma_n(x)$ the integer representative of x belonging to I^n; if $x \in Z$, we abbreviate $\Gamma_n(x(\text{mod}\, n))$ to $\Gamma_n(x)$. Thus the circular norm on I^n (or Z_n) may be conveniently characterised as:

$$|x|_n = \min\{\Gamma_n(x), n - \Gamma_n(x)\}.$$

The ***circular distance*** between two elements x, y of I^n (or Z_n) is $|x-y|_n$.

For $1 \leq d \leq \left\lfloor \frac{n}{k} \right\rfloor$, we define the the ***$d^{th}$ Schrijver graph***, denoted by $_dS^n_k$, to be the subgraph of G^n_k induced by the vertex set

$$V(_dS^n_k) = \{v \in I^n_k : |i-j|_n \geq d \;\; (i, j \in v)\}.$$

(Note that if d > $\frac{n}{k}$, the vertex set would be empty).

In 1977 Lovasz [13], showed that

$$\chi(G_k^n) = n - 2k+2.$$

In 1978 Schrijver [15], showed that the 2^{nd} Schrijver graph, ${}_2S_k^n$, is a vertex-critical subgraph of G_k^n also of chromatic number $n - 2k+2$.

It is worth pointing out that Schrijver graphs have some relevance to problems arising from radio communications. For example the problem of allocating sets of channels to mobile telephone providers covering different areas. (For example of recent work on the channel assignment problem, see [2]). Each provider i should be allocated a set S_i of channels with enough mutual separation, say d, to avoid signal interference between their own users that may be physically close, while the sets S_i, S_j allocated to providers i and j respectively should be such that

$$\left| s_i - s_j \right| \geq c_{ij} \qquad (s_i \in S_i,\ s_j \in S_j) \text{ for some parameters } c_{ij}$$

that depend on the separation of the areas. Simplifying this model by setting c_{ij} equal to unity if the corresponding areas are adjacent and imposing no restriction otherwise, the problem reduces to finding a homomorphism from the adjacency graph of the providers to the relevant Schrijver graph. That is we model the channel assignment problem under the assumption that each provider must be allocated k channels with required mutual separation d. Now, since a graph homomorphism does not decrease chromatic number, we require the chromatic number of ${}_dS_k^n$ to be at least equal to that of the adjacency graph of providers. Thus investigation of chromatic properties of Schrijver graphs has a practical application.

In Chapter 4 we introduce graph colourings together with their respective chromatic numbers that involve the use of the circular norm.

For any $n \in Z^+$, a ***Z_n-colouring*** of a graph is a function $\theta : V(G) \rightarrow Z_n$. Assuming G is non-null, i.e. has at least one edge, we define

$$\mu(\theta) = \min|\theta(u) - \theta(v)|_n,$$

where the minimum is taken over all pairs u, v of adjacent vertices.

For any $n,d \in Z^+$ with $n \geq 2d$, a ***(n,d)-colouring*** of a non-null graph G is a Z_n-colouring θ such that $\mu(\theta) \geq d$. Let n be such that there exists at least one proper colouring of G (i.e. $\chi(G) \leq n$). Let

$$d = \max\{\delta : G \text{ has a } (n, \delta)\text{-colouring}\}.$$

Then the ***n-chromatic number*** of G is defined as

$$\eta_n(G) = \frac{n}{d}.$$

This number was introduced by Vince [17] where it was denoted by $\chi_n(G)$.

Vince also introduced the ***circular chromatic number*** $\chi_c(G)$ (known in [17] and [3] as the star chromatic number and denoted there by $\chi^*(G)$), which is defined as

$$\chi_c(G) = \inf\{\eta_n(G) : n \in Z^+\}.$$

It is shown in [17] that $\chi_c(G) = \eta_n(G)$ for some $n \leq |V(G)|$ (Theorem 3) and that it is also equal to $\lim_{n\to\infty} \eta_n(G)$ (Corollary 2).

We define the ***circular distance graph***, denoted by H^n_d, whose vertex set is Z_n and vertices x and y are adjacent iff $|x-y|_n \geq d$. In this context a (n,d)-colouring of G is simply a homomorphism $G \to H^n_d$.

We investigate the relation between n-chromatic numbers for (n,d)-colourings and k^{th} chromatic number for k-tuple colourings. We compute the circular chromatic numbers of certain Kneser graphs of low order. By considering different methods, alternative insight into the circular colouring of Kneser graphs is offered. The circular chromatic numbers of classes of Kneser graphs of the type G_k^{2k+1} and G_k^{2k+2} is obtained.

In Chapter 5 we investigate when both k-tuple colourings and Z_n-colourings are combined into a single colouring. For $n \geq 2k$, a $Z_{n,k}$***-colouring*** of a non-null graph G is a k-tuple colouring of G using colours from Z_n.

We define two distance functions of the colouring θ, one related to adjacent vertices and the other to single vertices, as follows:

$$\mu_1(\theta) = \min\{|u_i - v_j|_n : u_i \in \theta(u),\ v_j \in \theta(v), uv \in E(G)\}$$

$$\mu_2(\theta) = \min\{|u_i - u_j|_n : u_i, u_j \in \theta(u),\ i \neq j, u \in V(G)\}.$$

Let $C_{n,k}$ denote the set of all $Z_{n,k}$-colourings of G. Assume that $2k. \leq \chi_k(G) \leq n$, so that $C_{n,k}$ contains at least one k-tuple colouring.

Let $M_{n,k}(G) = \max_{\theta \in C_{n,k}} \mu_1(\theta)$. Then the ***$n^k$-chromatic number*** of a non-null graph G is defined as

$$\eta_{n,k}(G) = \frac{n}{M_{n,k}(G)}.$$

We note that $\eta_{n,1}(G) = \eta_n(G)$.

$\chi_k^m(G)$, the ***k_m- chromatic number*** is the smallest value of n such that G can be $Z_{n,k}$-coloured with $\mu_1(\theta) \geq m$. Thus $\chi_k^1(G)$ is the k^{th} chromatic number, $\chi_k(G)$ and $\chi_1^1(G)$ is the ordinary chromatic number, $\chi(G)$.

We generalise (n,d)-colourings. For $n \geq 2d_1 k$, a ***(n, d_1, d_2, k)-colouring*** of a non-null graph is a *$Z_{n,k}$-colouring* θ, such that $\mu_1(\theta) \geq d_1$ and $\mu_2(\theta) \geq d_2$.

In Chapter 6 we consider the circular distance graph, H_k^n and show it is always a subgraph of the Kneser graph, G_k^n. We explore certain types of subgraphs of Kneser and in particular we show that there is, in general, a family of subgraphs that are isomorphic to H_k^n. We study the graph H_k^n further and find certain properties, including its m^{th}-chromatic number.

CHAPTER 2

Overlap Colourings

2.1 General Properties of Overlap Colourings.

Stahl [16], established that $\chi_p(G)$ is a sublinear function; that is for all $n, p, r \in Z^+$, $\chi_{np+r}(G) \le n\chi_p(G) + \chi_r(G)$. In analogy we show that (mq,q)-overlap colourings have a similar sublinearity property in this constant ratio sense.

Lemma 2.1 $\quad {}_m\chi_{nq+r}(G) \le n\,{}_m\chi_q(G) + {}_m\chi_r(G)$ for all $m, n, q, r \in Z^+$.

Proof

Let G be (mq,q)-overlap coloured with ${}_m\chi_q(G)$ colours and (mr,r)-overlap coloured with ${}_m\chi_r(G)$ colours disjoint from the other colour set. Then G can be $(m(q+r),q+r)$-overlap coloured by using the union of the ${}_m\chi_q(G)$ and ${}_m\chi_r(G)$ colours. Thus,

$$ {}_m\chi_{q+r}(G) \le {}_m\chi_q(G) + {}_m\chi_r(G). $$

The result now readily follows. ■

Lemma 2.2

Let G be a graph with E edges and of maximum vertex degree D. Let $q \geq \max(E,m)$ and $m > D$. Then any (mq,q)-overlap colouring of G contains an $(m,1)$-overlap colouring.

Proof

Consider an (mq,q)-overlap colouring utilising ${}_m\chi_q(G)$ colours. Let $\{C_v : v \in V(G)\}$ be the family of colour sets involved in this colouring. For each vertex v, let S_v be the set of colours involved in the overlap with its adjacent vertices; then

$$S_v \subseteq C_v \text{ and } |S_v| \leq \deg(v)q \leq Dq.$$

It follows that for each vertex v,

$$|C_v - S_v| \geq (m-D)q \geq q \geq m. \qquad \textbf{(1)}$$

In view that $q \geq E$, there exists a set of E distinct colours $\{c_e : e \in E(G)\}$ such that $c_e \in C_u \cap C_v$ for each edge $e = uv$.

Now for each vertex v, let $T_v = \{c_e : e \text{ is incident to } v\}$. Then

$$|T_v| = \deg(v) \leq D < m. \qquad \textbf{(2)}$$

It follows from **(1)** and **(2)** that there exists, for each vertex v, a subset $U_v \subset C_v - S_v$ of cardinality $m - |T_v|$. Let $R_v = T_v \cup U_v$, then $|R_v| = m$, and it immediately follows that $\{R_v : v \in V(G)\}$ constitutes an $(m,1)$-overlap colouring.

■

Theorem 2.1

Let G be a graph with E edges and of maximum vertex degree D.

Let $q \geq \max(E-1, m-1)$ and $m > D$. Then ${}_m\chi_{q+1}(G) \geq {}_m\chi_q(G)$.

Proof

Let G be $(m(q+1), q+1)$-overlap coloured with ${}_m\chi_{q+1}(G)$ colours. By Lemma 2.2, this colouring contains an $(m,1)$-overlap colouring. At each vertex we remove the colour set involved in the $(m,1)$-overlap colouring. It is clear the remaining colours give G an (mq,q)-overlap colouring using at most as many colours as the original colouring . The result immediately follows.

■

Stahl [16], in Theorem 2, established that $\chi_{p+1}(G) > \chi_p(G)$. Below we give a slight weakened analogue to this result for (p,q)-overlap colourings to non-strict inequality.

Theorem 2.2

Let G be a graph of maximum vertex degree D. If $p \geq qD$, then

$$\chi_{p+1,q}(G) \geq \chi_{p,q}(G).$$

Proof

Let G be $(p+1,q)$-overlap coloured using $\chi_{p+1,q}(G)$ colours. Now for each vertex v, the number of colours involved in the overlap of colours with its adjacent vertices is at most qD. It follows that the number of colours not involved in the overlap is at least

$$p+1-qD \geq qD+1-qD = 1.$$

It follows we can remove one colour from each vertex that is not involved in an overlap of colours with its adjacent vertices and so giving G a (p,q)-overlap colouring using at most $\chi_{p+1,q}(G)$ colours. The result immediately follows.

■

Theorem 2.2 gives the following corollary with regards to (mq,q)-overlap colourings.

Corollary If $m \geq D$, then $\chi_{mq+1,q}(G) \geq {}_m\chi_q(G)$.

Both inequalities of Theorems 2.1 and 2.2 are dependent on reasonably large vertex colour sets. Indeed, a counter example given in [11] shows these inequalities are not true in full generality. This raises a question as to what are the smallest values of m and p that satisfy these two inequalities.

2.2 The Attainment and Periodicity Theorems

As previously pointed out, Hilton, Rado & Scott [9] showed that the fractional chromatic number is equal to $\left(\frac{\chi_k(G)}{k}\right)$ for some k. This, in essence is the Attainment Theorem for k-tuple colourings. Also for such colourings, it was shown in [9] that there exists a positive integer k_1 such that the sequence

$\{\{\chi_k(G) - k\chi_f(G)\} : k = k_1, k_1 + 1, \ldots\}$ is periodic. This is the Periodicity Theorem. In this section we show that both Theorems can be extended to (mq,q)-overlap colourings. The proofs themselves are closely analogous to that of [9].

Let $r = 2^{|V(G)|} - 1$ and let $\{ V_i : i = 1, 2, \ldots, r \}$ be the family of all non-empty subsets of $V(G)$. Consider an (mq,q)-colouring using J colours. For $1 \le i \le r$, let C_i be the set of colours which are received by each vertex in V_i and by no other vertex, and let $y_i = |C_i|$. Then it is clear that each colour belongs to exactly one C_i and it follows that:

$$y_i \ge 0 \qquad (1 \le i \le r) \tag{3}$$

$$\sum_{i=1}^{r} y_i = J \tag{4}$$

$$\sum_{i:v \in V_i} y_i = mq \qquad (v \in V(G)) \tag{5}$$

$$\sum_{i:u,v \in V_i} y_i = q \qquad (uv \in E(G)) \tag{6}$$

Theorem 2.3 **(The Attainment Theorem).**

There exists a positive integer q_0 such that

$$ {}_m\chi_f(G) = \frac{{}_m\chi_{q_0}(G)}{mq_0}. $$

Proof

There exists an an (mq,q)-overlap colouring of G with J colours if and only if there is a sequence $(y_1,y_2,\ldots\ldots,y_r)$ of non-negative integers satisfying (3) – (6).

Let $z_i = \dfrac{y_i}{q}$ $(i = 1, 2,\ldots, r)$. Then $\dfrac{{}_m\chi_q(G)}{q}$ is the smallest value of $\sum_{i=1}^{r} z_i$ which satisfies:

$$ z_i \geq 0 \qquad (1 \leq i \leq r) \qquad \textbf{(7)} $$

$$ \sum_{i:v\in V_i} z_i = m \qquad (v \in V(G)) \qquad \textbf{(8)} $$

$$ \sum_{i:u,v\in V_i} z_i = 1 \qquad (uv \in E(G)). \qquad \textbf{(9)} $$

and each z_i is an integer multiple of $\frac{1}{q}$. Let $\mu(m,G)$ be the minimum value of $\sum_{i=1}^{r} z_i$ such that (7) – (9) are satisfied without this last restriction. Then $\mu(m,G)$ is the smallest value of c for which the hyperplane

$$ \sum_{i=1}^{r} z_i = c $$

meets the convex polytope defined by (7) – (9). At least one vertex $P = (p_1, p_2, \ldots, p_r)$ of the polytope meets the hyperplane

$$\sum_{i=1}^{r} z_i = \mu(m,G).$$

It follows that the point $P = (p_1, p_2, \ldots, p_r)$ satisfies r linearly independent simultaneous equations, each with coefficients 0, 1 or m. By Cramer's Rule all the solutions are rational with denominator equal to the determinant of a non-singular matrix of 0's and 1's. Let q_0 be the modulus of the determinant. Then each solution p_i, is of the form $p_i = \frac{y_i}{q_0}$ for some integer y_i. It follows that with $J = q_0\,\mu(m,G)$ and $q = q_0$, the sequence $(y_1, y_2, \ldots, y_r)$ is a sequence of non-negative integers satisfying (3) – (6). Hence the y_i define an (mq_0, q_0)-overlap colouring of G using $q_0\,\mu(m,G)$ colours. (Note that $\mu(m,G)$ need not be an integer). By the minimality of $\mu(m,G)$, $q_0\,\mu(m,G)$ is the smallest number of colours needed for a (mq_0, q_0)-overlap colouring of G. But by definition, this is precisely ${}_m\chi_{q_0}(G)$. Also since ${}_m\chi_f(G)$ is a lower bound of $\left\{\frac{{}_m\chi_q(G)}{mq} : q \in Z^+\right\}$ it follows that

$$\frac{\mu(m,G)}{m} = \frac{{}_m\chi_{q_0}(G)}{mq_0} \geq {}_m\chi_f(G). \tag{10}$$

Further, since $\mu(m,G)$ is the smallest value of $\sum_{i-1}^{r} z_i$ such that (7) – (9) are satisfied without the restriction that each z_i is a multiple of $\frac{1}{q}$, it follows that

$$\frac{{}_m\chi_q(G)}{q} \geq \mu(m,G) \text{ for all integers } q \in Z^+.$$

Hence, $\frac{\mu(m,G)}{m}$ is a lower bound of $\left\{\frac{{}_m\chi_q(G)}{mq} : q \in Z^+\right\}$. The fact that ${}_m\chi_f(G)$ is the greatest lower bound gives

$${}_m\chi_f(G) \geq \frac{\mu(m,G)}{m}.$$

This together with (10) gives ${}_m\chi_f(G) = \frac{{}_m\chi_{q_0}(G)}{mq_0}$ as required. ■

Theorem 2.4 **(The Periodicity Theorem).**

There exists a positive integer q_1 such that the sequence

$$\{\{{}_m\chi_q(G) - mq\,{}_m\chi_f(G)\} : q = q_1, q_1 + 1, \ldots\}$$

is periodic with period at most q_0 (where q_0 is as in Theorem 2.3).

Proof

Let q be any positive integer, and let n and r be the non-negative integers such that

$q = nq_0 + r$ where $r < q_0$.

Now by Theorem 2.3 ${}_m\chi_f(G) = \frac{{}_m\chi_{q_0}(G)}{mq_0}$, and by Lemma 2.1

$${}_m\chi_{(n+1)q_0+r}(G) \leq {}_m\chi_{nq_0+r}(G) + {}_m\chi_{q_0}(G).$$

Thus,

$$_{m}\chi_{(n+1)q_0+r}(G) - (n+1)\ _{m}\chi_{q_0}(G) \leq\ _{m}\chi_{nq_0+r}(G) - n\ _{m}\chi_{q_0}(G).$$

Letting,

$$S_{n,r} =\ _{m}\chi_{nq_0+r}(G) - n\ _{m}\chi_{q_0}(G) =\ _{m}\chi_{nq_0+r}(G) - nmq_0\ _{m}\chi_{f}(G)$$

Then,

$$S_{n+1,r} \leq S_{n,r}$$

Further, by definition $_{m}\chi_{nq_0+r}(G) \geq m(nq_0+r)\ _{m}\chi_{f}(G) \geq nmq_0\ _{m}\chi_{f}(G)$.

Thus, $S_{n,r} \geq 0$ and so for each $r < q_0$, the sequence $\{S_{n,r} : n \in Z^+\}$ is a monotone decreasing sequence of non-negative integers.

It follows that for each such r there is an integer p_r and a constant A_r such that

$$S_{n,r} = A_r \text{ for all } n \geq p_r.$$

Letting $K_r = A_r - mr\ _{m}\chi_{f}(G)$, and noting that it is also independent of n, gives

$$_{m}\chi_{nq_0+r}(G) - m(nq_0+r)\ _{m}\chi_{f}(G) = K_r \qquad \text{for all } n \geq p_r.$$

Recalling that $q = nq_0 + r$, let

$$p_{\max} = \max\{p_r : 0 \leq r < q_0\},$$

$$q_1 = (p_{\max} + 1)\, q_0,$$

and

$$f(q) =\ _{m}\chi_{nq_0+r}(G) - m(nq_0+r)\ _{m}\chi_{f}(G).$$

Then, $f(q) = K_r$ for all $q \geq q_1$ $(\Rightarrow n > p_{\max} \geq p_r)$

and

$$f(q+q_0) = {}_m\chi_{(n+1)q_0+r}(G) - m((n+1)q_0+r)\, {}_m\chi_f(G) = K_r = f(q),$$

thus proving the Theorem. ∎

Corollary $${}_m\chi_f(G) = \lim_{q\to\infty} \frac{{}_m\chi_q(G)}{mq}.$$

Proof

Let $K_{\max} = \max\{ K_r : 0 \leq r < q_0 \}$. Then

$${}_m\chi_q(G) - mq\, {}_m\chi_f(G) \leq K_{\max} \qquad \text{for all } q \geq q_1.$$

Thus,

$$0 \leq \frac{{}_m\chi_q(G)}{mq} - {}_m\chi_f(G) \leq \frac{K_{\max}}{mq} \qquad (q \geq q_1)$$

and since $K_{\max}$ is independent of q, the result follows. ∎

CHAPTER 3

The Schrijver Graphs and the Theorem of Erdös-Ko-Rado

In this chapter we concentrate on Schrijver graphs, introduced on page 3. We define and introduce the concept of the displacement sequence of a vertex of G_k^n and the subgraphs they induce, the rotation subgraphs.

For any graph G, we define the ***independence number***, $\alpha(G)$, to be the size of the largest independent set of vertices of G. Now let G be any induced subgraph of the Kneser graph G_k^n. Throughout this chapter, for each $i \in I^n$ we denote by $V_i(G)$ the (independent) set of vertices of G containing the element i. Where no confusion can arise, we abbreviate the notation to V_i.

We begin by giving a possible extension to the Erdö s-Ko-Rado Theorem [4] and discuss its implications.

3.1 The Theorem of Erdö s-Ko-Rado

Adopting similar notation to [4], let $S(1,n,k)$ denote the family of subsets $\{a_1, a_2,..., a_m\}$ of I_k^n that are pairwise non-disjoint. That is they have the intersecting property (see [4]).

Erdö s-Ko-Rado (EKR) Theorem

Let $\{a_1, a_2, \ldots, a_m\} \in S(1,n,k)$ and $n \geq 2k$. Then

$$m \leq \binom{n-1}{k-1}.$$

The EKR Theorem does in effect give the size of the largest possible independent set of vertices of G_k^n $(= {}_1S_k^n)$. We state this below in the context of Kneser graphs.

Erdö s-Ko-Rado (EKR) Theorem for Kneser Graphs

If $n > 2k$, then

$$\alpha(G_k^n) = \binom{n-1}{k-1}.$$

Hilton and Milner [8] showed that the maximum independent sets of vertices of G_k^n (that is, the independent sets of maximum size) are exactly the $V_i(G_k^n)$. That is, they are the sets $V_i = \{ v \in V(G_k^n) : i \in v\}$ $(i = 1,2,\ldots, n)$. A natural way to try to extend the EKR theorem and the result in [8] is to investigate the independence numbers of the Schrijver graphs ${}_dS_k^n$ and, in particular, the question of whether the maximum independent sets are again exactly the $V_i({}_dS_k^n)$. Investigation suggests that this is so, resulting in the following conjecture:

Conjecture

Let $n > 2k$. Then, for all $1 \le d \le \left\lfloor \frac{n}{k} \right\rfloor$:

(i) $\alpha({}_dS_k^n) = \binom{n-kd+k-1}{k-1}$;

(ii) the maximum independent vertex sets of ${}_dS_k^n$ are exactly the V_i $(i = 1,2,\ldots, n)$.

Now by proposition 3.1.1 of [14] the EKR Theorem immediately gives $\chi_f(G_k^n) = \frac{n}{k}$.

Hence $\chi_f({}_dS_k^n)$ is bounded above by $\frac{n}{k}$. Thus, if the above EKR analogue (ii) is true for each Schrijver graph, then $\chi_f({}_dS_k^n) = \frac{n}{k}$ would follow easily (see also Lemmas 3.1 & 3.2).

We shall prove the conjecture is true for $d = k = 2$ and for $d = \left\lfloor \frac{n}{k} \right\rfloor$ when $n < (k+1)d$ (Lemma 3.3 and Theorem 3.5).

We explore subgraphs of ${}_dS_k^n$ and observe that the more that can be discovered about their independence numbers, the closer we are to establishing both the chromatic properties of the Schrijver graphs themselves and whether they obey the conjecture.

We start by introducing and defining displacement sequences and using them to show that the number of vertices of ${}_dS_k^n$ containing a particular element is as given in Conjecture (i) above. We also establish the size of $\left|V({}_dS_k^n)\right|$.

3.2 Displacement Sequences

Let $v = \{a_1, a_2, \ldots, a_k\}$ be a vertex of G_k^n. We use the convention that its elements are listed such that they are in the same cyclic order as the cyclic order obtained when they are written in monotone increasing order. (That is, for some p, the list $a_p, a_{p+1}, \ldots, a_k, a_1, \ldots, a_{p-1}$ is in monotone increasing order.)

Now let v be any vertex of G_k^n. Given any $a \in v$, let us list the elements of v starting from a as $a_1, a_2, \ldots, a_k$ where $a_1 = a$ and define the ***displacement sequence*** of v starting from a as the sequence $\mathbf{d} = \{d_1, d_2, \ldots, d_k\}$ where $d_i = \Gamma_n(a_{i+1} - a_i)$ $(1 \le i \le k-1)$ and $d_k = \Gamma_n(a_1 - a_k)$. For each $d = 1, 2, \ldots, \left\lfloor \frac{n}{k} \right\rfloor$, let ${}_d\mathbf{D}$ be the set of all displacement sequences of vertices of ${}_dS_k^n$, that is, the set of all sequences $\mathbf{d} = \{d_1, \ldots, d_k\}$ where

$$d \le d_i \le n-d \quad (1 \le i \le k) \quad \text{and} \quad \sum_{i=1}^{k} d_i = n.$$

Given any $a \in I^n$ and any $\mathbf{d} \in {}_d\mathbf{D}$, we denote by $v_{a,\mathbf{d}}$ the vertex of ${}_dS_k^n$ whose displacement sequence starting from a is $\mathbf{d}$.

Lemma 3.1

$$\left|V({}_dS_k^n)\right| = \frac{n}{k}\left|V_a\right| \text{ for any } a \in I^n.$$

Proof

By symmetery, each set V_a $(a \in I^n)$ contains the same number, say c, of vertices. Moreover each vertex v of ${}_dS_k^n$ is contained in exactly k of these sets. Thus,

$$nc = k\left|V({}_dS_k^n)\right|.$$

■

Lemma 3.2

$$\left|V_a({}_dS_k^n)\right| = \binom{n-kd+k-1}{k-1} \text{ for any } a \in I^n.$$

Proof

For each $a \in I^n$, V_a is just the set of all $v_{a,\mathbf{d}}$ such that $\mathbf{d} \in {}_d\mathbf{D}$. Thus the number of vertices of ${}_dS_k^n$ containing the element a is equal to the number of displacement sequences of ${}_d\mathbf{D}$.

Now any displacement sequence $\mathbf{d} = \{d_1,...,d_k\}$ can be written as

$\mathbf{d} = \{(d-1)+e_1,(d-1)+e_2,....,(d-1)+e_k\}$

where $\sum_{i=1}^{k} e_i = n-(d-1)k,\ \ e_i \geq 1\ \ (i = 1,2,.....,k)$. It follows that the number of displacement sequences of ${}_d\mathbf{D}$ is also equal to the number of displacement sequences of the form $\{\{e_1,e_2,...,e_k\} : \sum_{i=1}^{k} e_i = n-(d-1)k,\ e_i \geq 1\}$. But these are precisely the set of displacement sequences of the vertices of $G_k^{n-(d-1)k}$, and so the number of vertices of ${}_dS_k^n$ containing a is equal to the number of vertices of $G_k^{n-(d-1)k}$ containing a particular element of $I^{n-(d-1)k}$ (note that we are allowing for the case $n-(d-1)k < 2k$; that is when $G_k^{n-(d-1)k}$ is a null graph).

Since $\left|V(G_k^{n-(d-1)k})\right| = \binom{n-(d-1)k}{k}$, it follows from Lemma 3.1 that

$$|V_a| = \frac{k}{n-(d-1)k}\binom{n-(d-1)k}{k} = \binom{n-(d-1)k-1}{k-1}.$$

■

Lemmas 3.1 and 3.2 immediately give the size of the vertex set of Schrijver graphs;

$$\left|V({}_dS_k^n)\right| = \frac{n}{k}\binom{n-(d-1)k-1}{k-1}.$$

We now focus on the Schrijver graphs, ${}_2S_2^n$ $(n > 2)$, and find their independence numbers. By making use of linear programming duality we compute their fractional chromatic numbers.

3.3 The Graphs ${}_2S_2^n$

We show below that $V_i({}_2S_2^n)$ are the largest possible independent sets.

Lemma 3.3

Let C be an independent set of vertices of ${}_2S_2^n$ of largest possible size. Then $C = V_i$ for some $i \in I^n$; indeed, $\alpha({}_2S_2^n) = n-3$.

Proof

The cases for $n = 4$, 5 and 6 can be readily verified. We shall consider the case where $n \geq 7$.

Assume by way of contradiction that no element of I^n is contained in all the vertices of C. It follows that C must contain three vertices, say u, v and w all of which are pairwise non-disjoint and that $u \cap v \cap w = \emptyset$.

Let $u = \{a,b\}$ and $v = \{a,c\}$. Now since both the intersections $u \cap w$ and $v \cap w$ are non-empty, it follows that $w = \{b,c\}$. But any other vertex cannot simultaneously have a non-empty intersection with the vertices u, v and w respectively and so $|C| = 3$. In view of the fact that $|V_i| = n - 3 \geq 4$ for each $i \in I^n$; this gives a contradiction and the result follows.

■

From this it follows that every vertex of ${}_2S_2^n$ is contained in an independent set of largest possible size; an observation we now use.

For a general graph G, and a vertex v of G, let $\lambda_v(G)$ be the size of the largest independent set containing v. We define a graph parameter, $\mu(G)$, that relates to these sets, namely

$$\mu(G) = \sum_{v \in V(G)} \frac{1}{\lambda_v(G)}.$$

Lemma 3.4 $\mu({}_2S_2^n) = \frac{n}{2}$

Proof This follows from Lemmas 3.1 and 3.3.

■

Our next task is to show that $\mu({}_2S_2^n) \le \chi_f({}_2S_2^n)$. We achieve this through the use of linear programming and the duality theorem.

3.4 Linear Programming and Duality

Let G be a graph with m vertices. Following Hilton, Rado & Scott [9], let $\{C_1, C_2, \ldots, C_t\}$ be the set of all the independent sets of vertices of G. Let **A** be the $m \times t$ matrix with 1 in the (i, j) entry if vertex v_i belongs in the set C_j, otherwise 0. Let **c** be the $t \times 1$ column vector with all entries 1 and **b** the $m \times 1$ column vector, also with all entries 1. Then the problem of colouring G with k colours at each vertex using the minimum number of colours can be restated as the integer programming problem:

minimize $\mathbf{c}^T \mathbf{y}$ subject to $\mathbf{Ay} \ge k\mathbf{b}$, $\mathbf{y} \ge \mathbf{0}$,

where **y** is a $t \times 1$ column vector and each y_i is required to be a non-negative integer.

Also following [9], let $z_i = \frac{y_i}{k}$; then the problem becomes:

minimize $\mathbf{c}^T \mathbf{z}$ subject to $\mathbf{Az} \ge \mathbf{b}$, $\mathbf{z} \ge \mathbf{0}$,

where each z_i is now required to be a multiple of $\frac{1}{k}$.

[9] further shows that $\chi_f(G)$ is the value of an optimal solution to this linear programming problem, without the restriction that each z_i is a multiple of $\frac{1}{k}$.

The dual problem to this linear programming problem is:

maximize $\mathbf{b}^T\mathbf{x}$ subject to $\mathbf{A}^T\mathbf{x} \le \mathbf{c}$, $\mathbf{x} \ge \mathbf{0}$, where $\mathbf{x}$ is an $m \times 1$ vector.

Now $\mathbf{x}$ may be considered as a non-negative weighting function on the vertices of G:

$\mathbf{x}(v_i) = x_i \; (v_i \in V(G))$, which maximises the sum $\sum_{i=1}^{m} x_i$ subject to

$$\sum_{i:v_i \in C} x_i \le 1 \text{ for all } C \in \{C_1, C_2, \ldots, C_t\}.$$

Now consider the weighting function:

$$\mathbf{x}(v) = \frac{1}{\lambda_v(G)}.$$

For any independent set C, all the vertices in C have a weight at most $\frac{1}{|C|}$.

Hence, for any such C,

$$\sum_{i:v_i \in C} x_i \le 1,$$

and $\mathbf{x}$ is a 'feasible' weighting function.

But $\mathbf{b}^T\mathbf{x} = \mu(G)$, where μ is the graph parameter introduced in Section 3.3. This immediately leads to:

Lemma 3.5 For any graph G,

$$\mu(G) \leq \chi_f(G).$$

Proof.

By the above, $\mu(G)$ is the value of a feasible solution of the linear programming to maximize $\mathbf{b}^T\mathbf{x}$, while by the duality theorem of linear programming, $\chi_f(G)$ is the value of an optimal solution.

■

Theorem 3.1 $$\chi_f({}_2S_2^n) = \frac{n}{2}$$

Proof.

By Lemmas 1.1, 3.4 and 3.5 we have,

$$\frac{n}{2} = \mu({}_2S_2^n) \leq \chi_f({}_2S_2^n) \leq \chi_f(G_2^n) = \frac{n}{2}.$$

■

We investigate the Schrijver graphs ${}_dS_k^n$ further and obtain an upper bound for their independence numbers. We introduce subgraphs induced by a given displacement sequence of ${}_dS_k^n$. We consider Schrijver subgraphs with certain 'types' of displacement sequence and find their independence number.

3.5 Rotation Graphs

Given any $a \in I^n$ and any $\mathbf{d} \in {}_d\mathbf{D}$, we recall that $v_{a,\mathbf{d}}$ is the vertex of ${}_dS_k^n$ whose displacement sequence starting from a is $\mathbf{d}$. Clearly, if $v_{a_1,\mathbf{d}_1} = v_{a_2,\mathbf{d}_2}$, then $\mathbf{d}_2$ has the same elements as $\mathbf{d}_1$ in the same cyclic order; that is, $\mathbf{d}_1$ and $\mathbf{d}_2$ are ***cyclically equivalent***.

Let $Q = \{x \in Z^+ : x | n \text{ and } x | k\}$. For each $x \in Q$, let

$$_d\mathbf{D}_x = \{\mathbf{d} \in {}_d\mathbf{D} : \mathbf{d} \text{ is periodic with period } \frac{k}{x}\}.$$

Then ${}_d\mathbf{D} = \bigcup_{x \in Q} {}_d\mathbf{D}_x$, since if $x | k$ and $\mathbf{d}$ is periodic with period $\frac{k}{x}$ then

$n = \sum_{i=1}^{k} d_i = x \sum_{i=1}^{k/x} d_i$, and so $x | n$.

Given any displacement sequence $\mathbf{d} \in {}_d\mathbf{D}$, we define the ***rotation subgraph***, ${}_\mathbf{d}R_k^n$, to be the subgraph of ${}_dS_k^n$ induced by the vertices of the form $v_{a,\mathbf{d}}$ for some $a \in I^n$.

Lemma 3.6 Let $\mathbf{d} \in {}_d\mathbf{D}_x$. Then $\left|V({}_\mathbf{d}R_k^n)\right| = \frac{n}{x}$; indeed, $V({}_\mathbf{d}R_k^n) = \{v_{a,\mathbf{d}} : a \in I^{n/x}\}$.

Proof. By definition, $V({}_{\mathbf{d}}R_k^n) = \{v_{a,\mathbf{d}} : a \in I^n\}$. Clearly $v_{a,\mathbf{d}} = v_{b,\mathbf{d}}$ if and only if b is the $(p+1)^{\text{th}}$ element of $v_{a,\mathbf{d}}$ (starting from a) where $p = \frac{mk}{x}$ for some non-negative integer m. Therefore,

$$b - a = m\sum_{i=1}^{k/x} d_i.$$

Thus there are $\sum_{i=1}^{k/x} d_i = \frac{n}{x}$ distinct vertices with displacement sequence $\mathbf{d}$, characterised as in the statement of the lemma.

■

3.6 Independence Numbers of Schrijver Graphs

Lemma 3.7 Let $\mathbf{d} \in {}_d\mathbf{D}_x$. Then $\alpha({}_{\mathbf{d}}R_k^n) \leq \frac{n}{xd}$.

Proof

Let $\mathbf{d} \in {}_d\mathbf{D}_x$ and consider the rotation subgraph ${}_{\mathbf{d}}R_k^n$ of ${}_dS_k^n$. Let $v_{a,\mathbf{d}}$ and $v_{b,\mathbf{d}}$ be two distinct vertices. Now the elements of $v_{b,\mathbf{d}}$ are those of $v_{a,\mathbf{d}}$ displaced by $\Gamma_n(b-a)$. It follows that if $|b-a| < d$ then every element of $v_{b,\mathbf{d}}$ is distant less than d from the corresponding element of $v_{a,\mathbf{d}}$ and vice-versa. But as these are vertices of ${}_dS_k^n$, then they must be disjoint and so cannot be independent.

Now let $\{ v_{a_1,\mathbf{d}}, v_{a_2,\mathbf{d}}, v_{a_3,\mathbf{d}}, \ldots\ldots, v_{a_p,\mathbf{d}} \}$ be an independent set of vertices of ${}_{\mathbf{d}}R_k^n$.

By Lemma 3.6, we may choose $0 < a_1 < a_2 < \ldots\ldots < a_p \leq \frac{n}{x}$. By above argument, independence implies that $a_{i+1} - a_i \geq d$ for $i = 1,2,\ldots, p-1$. Furthermore, by periodicity $v_{a_1,\mathbf{d}} = v_{a_1+n/x,\mathbf{d}}$ and so by independence $a_1 + \frac{n}{x} - a_p \geq d$. Finally, adding these p inequalities gives $p \leq \frac{n}{xd}$ as required.

■

Theorem 3.2 $\alpha({}_dS_k^n) \leq \frac{n}{d} \sum_{x \in Q} \frac{|{}_d\mathbf{D}_x|}{x}$.

Proof

Let V be an independent set of vertices of ${}_dS_k^n$. By Lemma 3.7, for each $\mathbf{d} \in {}_d\mathbf{D}_x$ there are at most $\frac{n}{xd}$ vertices of V with displacement sequence $\mathbf{d}$. The result now follows.

■

Lemma 3.7 gives an upper bound for the rotation subgraphs in general. The previous theorem demonstrates that finding the independence numbers of rotation subgraphs can play an important role towards finding the independence number of Schrijver graphs. We consider the displacement sequences of the type $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \neq D$ and the rotation subgraphs they induce, ${}_{\mathbf{s}}R_k^n$. We find their independence numbers, and consequently their fractional chromatic numbers.

3.7 The Rotation Subgraphs ${}_{s}R_{k}^{n}$ where s = $\{\delta, \delta, \ldots\ldots, \delta, D\}$

Every vertex of ${}_{\mathbf{s}}R_{k}^{n}$ can be written in the form

$v_{a,\mathbf{s}} = \{\, a,\ \Gamma_n(a+\delta),\ \Gamma_n(a+2\delta),\ldots,\ \Gamma_n(a+(k\text{-}1)\delta)\,\}$ for some $a \in I^n$.

Using this representation we say that $\{a,\ \Gamma_n(a+\delta),\ \Gamma_n(a+2\delta),\ldots,\Gamma_n(a+(p\text{-}1)\delta)\}$ and $\{\Gamma_n(a+(k\text{-}p)\delta),\ldots,\Gamma_n(a+(k\text{-}1)\delta)\}$ are the first and last p elements of $v_{a,\mathbf{s}}$ respectively $(1 \le p \le k-1)$.

Lemma 3.8

Let $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \nmid D$ and $V = \{\, v_i : 1 \le i \le q \,\}$ be an independent set of q (≥ 2) vertices of ${}_{\mathbf{s}}R_{k}^{n}$ such that

$\mathrm{B} = \bigcap_{i=1}^{q} v_i \ne \varnothing$, $|B| = p$. Then B occurs as the first p elements of some v_i and the last p elements of another v_j, $(1 \le i, j \le q)$.

Proof

Let $v_{a,\mathbf{s}}$ be a vertex belonging to V. Since $B \subseteq v_{a,\mathbf{s}}$, then B can be expressed in the form

$B = \{\Gamma_n(a+f_1\delta),\ \Gamma_n(a+f_2\delta),\ldots,\ \Gamma_n(a+f_p\delta)\}$ where the elements f_i $(1 \le i \le p)$ are in monotone increasing order.

Now let $v_{b,s}$ be any other vertex of V. Then,

$B = \{\Gamma_n(b+e_1\delta), \Gamma_n(b+e_2\delta), \ldots, \Gamma_n(b+e_p\delta)\}$ where

$$\Gamma_n(a+f_i\delta) = \Gamma_n(b+e_i\delta) \quad (0 \le e_i, f_i \le k-1, 1 \le i \le p) \qquad \textbf{(1)}$$

We first show that the elements $\{e_i : 1 \le i \le p)$ are also in monotone increasing order.

Now (1) gives

$$\Gamma_n((e_{j+1} - e_j)\delta) = \Gamma_n((f_{j+1} - f_j)\delta) = (f_{j+1} - f_j)\delta \qquad (1 \le j \le p-1)$$

If $e_j > e_{j+1}$, then $(e_{j+1} - e_j)\delta + n = (f_{j+1} - f_j)\delta$.

In view of the fact that $n = (k-1)\delta + D$, it follows that

$D = (f_{j+1} - f_j + e_j - e_{j+1} - k + 1)\delta \Rightarrow \delta \mid D$; giving a contradiction.

Hence $e_{j+1} > e_j$ for $j = 1,2,\ldots, p-1$.

We next show that the elements $f_i(1 \le i \le p)$ are indeed consecutive. Suppose by way of contradiction there is some positive integer, r ($<p$) such that $f_r < f_r+1 < f_{r+1}$.

Again invoking (1) gives $\Gamma_n((e_{r+1} - e_r)\delta) = \Gamma_n((f_{r+1} - f_r)\delta)$ and since both the e_i and f_i are strictly in monotone increasing order it follows that

$e_{r+1} - e_r = f_{r+1} - f_r \ge 2$ from which $e_r < e_r+1 < e_{r+1} \le k-1$.

Hence,

$\Gamma_n(a+(f_r+1)\delta) = \Gamma_n(b+(e_r+1)\delta) \in v_{b,s}$.

By the arbitrary choice of the vertex $v_{b,s}$ of V, we conclude that

$\Gamma_n(a+(f_r+1)\delta) \in B$; giving a contradiction.

Thus $f_i (1 \le i \le p)$ are consecutive integers.

Finally, letting $c = \Gamma_n(a+f_1\delta)$, we can express B in the form

$B = \{c, \Gamma_n(c+\delta), \Gamma_n(c+2\delta), \ldots, \Gamma_n(c+(p-1)\delta)\}$ for some $p \le k-1$.

Now c must necessarily be the first element of some v_i (otherwise $\Gamma_n(c-\delta) \in B$)

and similarly $\Gamma_n(c+(p-1)\delta)$ must be the last element of some v_j.

■

Lemma 3.9

Let $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \nmid D$ and $V = \{v_i : 1 \le i \le t\}$ be an independent set of vertices of ${}_{\mathbf{s}}R_k^n$. Then $\bigcap_{i=1}^{t} v_i \ne \varnothing$.

Proof

Suppose by way of contradiction that $\bigcap_{i=1}^{t} v_i = \varnothing$, then there is an integer q such that

$B = \bigcap_{i=1}^{q} v_i \neq \varnothing$, and that $B \cap v_{q+1} = \varnothing$ ($1 < q <$ t). Letting $|B| = p$, then by Lemma 3.8 B occurs as the first p elements of some vertex v_i and the last p elements of another vertex v_j.

Let $B = \{b,\ \Gamma_n(b+\delta),\ \Gamma_n(b+2\delta),\ldots,\Gamma_n(b+(p-1)\delta)\}$ $(p \leq k-1)$, then $v_i = v_{b,s}$ and $v_j = v_{a,s}$ where $a = \Gamma_n(b+(p-1)\delta + D)$.

Consider the intersection, $v_i \cap v_{q+1}$ and let $|v_i \cap v_{q+1}| = m$. Now since B does not contain any element of v_{q+1}, then by Lemma 3.8 the last m elements of v_i occur as the first m elements of v_{q+1}. In particular the last element of v_i, $\Gamma_n(b+(k-1)\delta) \in v_{q+1}$. Considering the intersection $v_j \cap v_{q+1}$, and by a similar argument, the first element of v_j, $\Gamma_n(b+(p-1)\delta + D) \in v_{q+1}$.

Hence, $\Gamma_n(b+(p-1)\delta + D)$ and $\Gamma_n(b+(k-1)\delta)$ both belong to v_{q+1} and so their difference is a multiple of δ. That is $(k-p)\delta - D = u\delta$ for some integer u, from which it immediately follows that $\delta \mid D$; giving a contradiction.

■

Theorem 3.3

Let $s = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \neq D$ and ${}_sR_k^n$ be the rotation subgraph of G_k^n induced by s.

(i) If $\delta \nmid D$ then $\alpha({}_sR_k^n) = k$.

(ii) If $\delta \mid D$, say $D = (m+1)\delta$, then $\alpha({}_sR_k^n) = \begin{cases} k+m & \text{if } m<k \\ k & \text{if } m \geq k. \end{cases}$

Every vertex in both cases is contained in a maximum independent set.

Proof

For (i), let $C = \{ v_i : 1 \leq i \leq t \}$ be an independent set, then by Lemma 3.9 $\bigcap_{i=1}^{t} v_i \neq \varnothing$.

Let $a \in \bigcap_{i=1}^{t} v_i$ be the p^{th} and m^{th} elements of v_i and v_j respectively, then $i \neq j$ iff $m \neq p$. It follows that since each vertex has k elements, the number of vertices cannot exceed k.

Now given any vertex, say $v_{a,s}$ $(a \in I^n)$, we can construct an independent set containing $v_{a,s}$ of size k as follows.

Let $v_i = \{\Gamma_n(a+(1-i)\delta), \Gamma_n(a+(2-i)\delta), \ldots, \Gamma_n(a+(k-i)\delta)\}$.

Then it is clear that $\{ v_i : 1 \leq i \leq k \}$ is a set of k vertices containing $v_{a,s}$ and that this is indeed equal to $V_a({}_sR_k^n)$. This completes the proof of (i).

For (ii), we first assume $m < k$. Since $\delta \mid D$, the minimum circular distance between two elements of a vertex in ${}_sR_k^n$ is δ. Lemma 3.7 gives

$$\alpha({}_{s}R_{k}^{n}) \leq \frac{n}{\delta} = \frac{(k+m)\delta}{\delta} = k+m.$$

Now any vertex with displacement sequence $\mathbf{s} = \{\delta, \delta, ..., \delta, (m+1)\delta\}$, will have all its elements in the same congruence class modulo δ. There are $k + m$ distinct elements of I^n that belong to the same congruence class modulo δ, and since $m < k$, the majority of these are contained in every vertex. It follows that any two vertices using the same congruence class modulo δ must necessary have an element in common and so be independent. Thus we get $k + m$ independent vertices.

Let $m \geq k$ and V be an independent set of vertices. Let $X = \{x \in v : v \in V\}$. By symmetry and without loss of generality assume δ to be the smallest element of X. Then the largest element of X must be less than or equal to

$(2k-1)\delta \leq (k+m-1)\delta < n$ and the elements of every vertex of V are ordered as consecutive multiples of δ, all of which lie between δ and $(2k-1)\delta$. Clearly you cannot 'squeeze' more than k vertices $\Rightarrow$ $|V| \leq k$. Given any vertex, an independent set of size k containing the vertex can be constructed in a similar way to (i).

■

Theorem 3.3 shows that the maximum independent sets of the rotation graph ${}_{\mathbf{s}}R_k^n$ are either of size k or $k+m$.

Lemma 3.10

Let $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \neq D$ and ${}_{\mathbf{s}}R_k^n$ be the rotation subgraph of G_k^n induced by s.

(i) If $\delta \nmid D$ then $\mu({}_{\mathbf{s}}R_k^n) = \dfrac{n}{k}$.

(ii) If $\delta \mid D$, say $D = (m+1)\delta$, then $\mu({}_{\mathbf{s}}R_k^n) = \begin{cases} \delta & \text{if } m < k \\ \dfrac{n}{k} & \text{if } m \geq k. \end{cases}$

Proof

This follows from Lemma 3.6 and Theorem 3.3.

■

Lemma 3.11

Let ${}_{\mathbf{s}}R_k^{(k+m)\delta}$ and ${}_{\mathbf{t}}R_{k+m}^{(k+m)\delta}$ be the rotation subgraphs of $G_k^{(k+m)\delta}$ and $G_{k+m}^{(k+m)\delta}$ induced by the displacement sequences $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, (m+1)\delta\}$ and $\mathbf{t} = \{\delta, \delta, \ldots\ldots, \delta\}$ respectively. If $m < k$ then there exist a homomorphism

$$\Phi : {}_{\mathbf{s}}R_k^{(k+m)\delta} \rightarrow {}_{\mathbf{t}}R_{k+m}^{(k+m)\delta}$$

Proof

For each vertex $v_{a,\mathbf{s}}$ of ${}_{\mathbf{s}}R_k^{(k+m)\delta}$, we define the mapping

$$\Phi(v_{a,\mathbf{s}}) = v_{a,\mathbf{t}} \in V({}_{\mathbf{t}}R_{k+m}^{(k+m)\delta}).$$

Let uv be an edge of ${}_{\mathbf{s}}R_k^{(k+m)\delta}$, then $u \cap v = \varnothing$.

It is sufficient to show that $\Phi(u) \cap \Phi(v) = \varnothing$. Suppose by way of contradiction that there is an element $b \in \Phi(u) \cap \Phi(v)$. Now each vertex of ${}_{\mathbf{t}}R_{k+m}^{(k+m)\delta}$ contains all the available $k+m$ elements of $I^{(k+m)\delta}$ that are in the same congruence class modulo δ. It follows that $\Phi(u) = \Phi(v)$ and in particular the elements of u and v belong to the same congruence class modulo δ. Since $m < k$, the majority of the above $k+m$ elements are contained in each of the vertices u and v and so must necessary have an element in common; giving a contradiction.

■

Theorem 3.4

Let $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \neq D$ and ${}_{\mathbf{s}}R_k^n$ be the rotation subgraph of G_k^n induced by $\mathbf{s}$.

(i) If $\delta \nmid D$ then $\chi_f({}_{\mathbf{s}}R_k^n) = \dfrac{n}{k}$.

(ii) If $\delta \mid D$, say $D = (m+1)\delta$, then $\chi_f({}_{\mathbf{s}}R_k^n) = \begin{cases} \delta & \text{if } m<k \\ \dfrac{n}{k} & \text{if } m \geq k. \end{cases}$

Proof

Lemmas 1.1, 3.5 and 3.10 applied to case (i) and case (ii) with $m \geq k$ give

$$\frac{n}{k} = \mu({}_sR_k^n) \leq \chi_f({}_sR_k^n) \leq \chi_f(G_k^n) = \frac{n}{k}$$

Finally, we consider case (ii) with $m < k$.

Now by Lemma 3.11 and Theorem 3 of [16], $\chi_p({}_sR_k^n) \leq \chi_p\,({}_tR_{k+m}^n)$ for all $p \geq 1$

from which it follows that $\chi_f({}_sR_k^n) \leq \chi_f({}_tR_{k+m}^n)$. This together with Lemmas 3.5

and 3.10 gives

$$\delta = \mu({}_sR_k^n) \leq \chi_f({}_sR_k^n) \leq \chi_f({}_tR_{k+m}^n) \leq \chi_f(G_{k+m}^n) = \delta\,.$$

■

For the Schrijver graph ${}_dS_k^n$ to have any vertices, the maximum value of d is $\left\lfloor \frac{n}{k} \right\rfloor$.

When d takes this value we denote ${}_dS_k^n$ simply by S_k^n, so that $n = kd + r$, where

$0 \leq r < k$.

3.8 The Graphs S_k^n

We first consider the graph, S_k^{kd+1} (the case $r = 1$). Since $\sum_{i=1}^{k} d_i = kd+1$ and $d_i \geq d$

for all $1 \leq i \leq k$ it follows that every vertex of S_k^{kd+1} has displacement sequence

$\mathbf{q} = \{d, d, \ldots\ldots, d, d+1\}$. That is $S_k^{kd+1} = {}_{\mathbf{q}}R_k^{kd+1}$. Invoking Theorems 3.3 and 3.4 gives the following result which we state as a corollary.

Corollary 3.1 $\alpha(S_k^{kd+1}) = k$ and $\chi_f(S_k^{kd+1}) = \dfrac{kd+1}{k}$.

It is worth noting that S_k^{kd+1} is in general a significantly smaller graph than G_k^{kd+1}, also of the same fractional chromatic number. The contrast of their differences in size is illustrated by the example in Figures 3.1 and 3.2 with $k = 2$ and $d = 3$. This raises the question whether we can reduce the number of vertices of S_k^{kd+1} while maintaining the same fractional chromatic number. The answer to this, as will be shown later in section 6.2, is no.

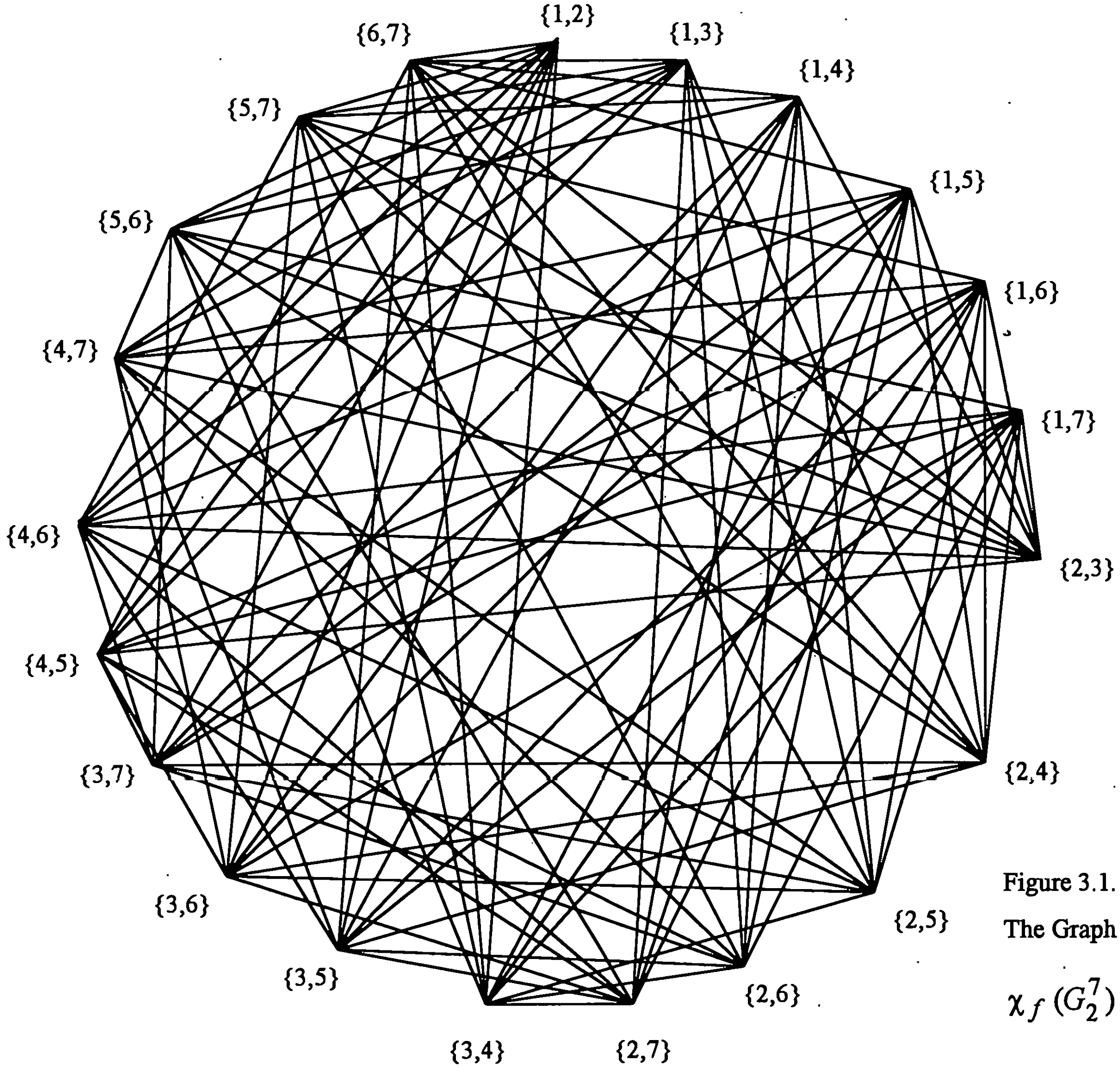

Figure 3.1.
The Graph G_2^7.
$\chi_f(G_2^7) = \frac{7}{2}$

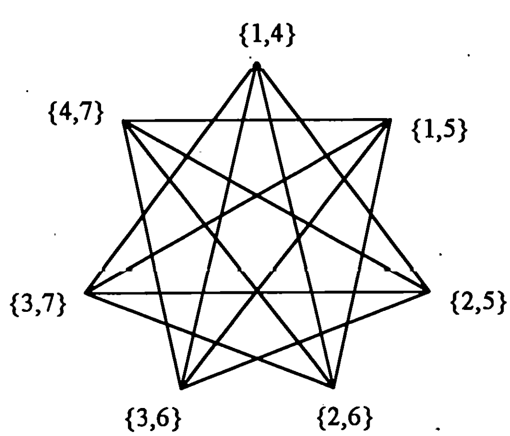

Figure 3.2.
The Graph S_2^7.
$\chi_f(S_2^7) = \frac{7}{2}$

Lemma 3.7 enables us to obtain an equality for the independence number of the graphs

S_k^n when $r < d$.

Theorem 3.5

Let $n = kd + r$ where $0 \le r < \min(k,d)$. Then:

(i) $\alpha(S_k^n) = k \sum_{x \in Q} \frac{|{}_d\mathbf{D}_x|}{x} = \binom{r+k-1}{k-1}$;

(ii) Every vertex is contained in a maximum independent set.

Proof

Consider the subgraph ${}_\mathbf{d}R_k^n$ of S_k^n.

Now by Lemma 3.7, for each displacement sequence, $\mathbf{d} = \{d_1, d_2, \ldots, d_k\} \in {}_d\mathbf{D}_x$

$(\left\lfloor \frac{n}{k} \right\rfloor \le d_i \le n - \left\lfloor \frac{n}{k} \right\rfloor)$,

$$\alpha({}_\mathbf{d}R_k^n) \le \frac{n}{xd} = \frac{k}{x} + \frac{r}{xd} < \frac{k}{x} + 1 \Rightarrow \alpha({}_\mathbf{d}R_k^n) \le \frac{k}{x}.$$

Thus,

$$\alpha(S_k^n) \le k \sum_{x \in Q} \frac{|{}_d\mathbf{D}_x|}{x}.$$

Finally, for each vertex we shall exhibit an independent set of size $k \sum_{x \in Q} \frac{|{}_d\mathbf{D}_x|}{x}$.

For this we shall use the convention that

$\sum_{j=1}^{r-i} d_j = - \sum_{j=1}^{i-r} d_j$ if $r < i$ and that $\sum_{j=1}^{r-i} d_j = 0$ if $r = i$.

Now, given any element $a \in I^{n/x}$, and thereby any vertex, we construct the independent sets of vertices $V_a({}_{\mathbf{d}}R_k^n)$ and $V_a(S_k^n)$ as follows:

For each $\mathbf{d} \in {}_d\mathbf{D}_x$,

$$V_a({}_{\mathbf{d}}R_k^n) = \{\{\Gamma_{n/x}(a + \sum_{j=1}^{1-i} d_j),\ \Gamma_{n/x}(a + \sum_{j=1}^{2-i} d_j), \ldots\ldots, \Gamma_{n/x}(a + \sum_{j=1}^{k/x-i} d_j)\}: 1 \le i \le \frac{k}{x}\}$$

Thus we can express the independent set $V_a(S_k^n)$ as:

$$V_a(S_k^n) = \{V_a({}_{\mathbf{d}}R_k^n) : \mathbf{d} \in {}_d\mathbf{D} = \bigcup_{x \in Q} {}_d\mathbf{D}_x\}.$$

Clearly, $\left|V_a(S_k^n)\right| = k \sum_{x \in Q} \frac{\left|{}_d\mathbf{D}_x\right|}{x}$ and by Lemma 3.2 this is also equal to

$$\binom{n-kd+k-1}{k-1} = \binom{r+k-1}{k-1}.$$

■

We now consider the rotation subgraphs of S_k^n induced by a displacement sequence that spaces out the k 'colours' at each vertex as evenly as possible round I^n. These special rotation subgraphs, denoted by SP_k^n, will be referred to as spaced subgraphs and their properties will be investigated. For these subgraphs, as will be proved, the independence number is $\frac{k}{q}$, where $q = \gcd(k,n)$ (=gcd (k,r)). Ultimately, this will lead

to the fractional chromatic number for all the Schrijver graphs ${}_dS^n_k$ $(1 \le d \le \left\lfloor \frac{n}{k} \right\rfloor)$,

thereby generalising the results of Theorem 3.1 and Corollary 3.1.

3.9 Spaced Subgraphs SP^n_k

Letting $n = kd + r$ where $r < k$, we define the displacement sequence $\mathbf{s}$ as follows:

$$\mathbf{s} = \{d_1,\ldots,d_k\}, \text{ where } d_j = d + \left\lfloor \frac{(j+1)r}{k} \right\rfloor - \left\lfloor \frac{jr}{k} \right\rfloor \quad (j = 1,\ldots\ldots,k).$$

That is,

$$\begin{aligned} v_{a,\mathbf{s}} &= \left\{ \Gamma_n\left(a + \left\lfloor \frac{jn}{k} \right\rfloor \right) : j=1,\ldots.,k \right\} \\ &= \left\{ \Gamma_n\left(a + jd + \left\lfloor \frac{jr}{k} \right\rfloor \right) : j=1,\ldots\ldots,k \right\}. \end{aligned} \tag{2}$$

Throughout the remainder of this section, we set $n' = \frac{n}{q}$ and $k' = \frac{k}{q}$ $(q = \gcd(n,k))$.

Recall that given any displacement sequence $\mathbf{d} \in {}_d\mathbf{D}$, the rotation subgraph, ${}_{\mathbf{d}}R^n_k$, is a subgraph of ${}_dS^n_k$ induced by the vertices of the form $v_{a,\mathbf{d}}$ for some $a \in I^n$. The rotation subgraph induced by this displacement sequence, **s** is referred to as

spaced subgraph and denoted by SP_k^n.

Lemma 3.12 The period of **s** is k', that is $\mathbf{s} \in {}_d\mathbf{D}_q$.

Proof. Recall that $\mathbf{s} = \{d_1, \ldots, d_k\}$. From the definition of the d_j:

$$d_{j+k'} - d_j = \left(\left\lfloor \frac{(j+1+k')r}{k} \right\rfloor - \left\lfloor \frac{(j+1)r}{k} \right\rfloor\right) - \left(\left\lfloor \frac{(j+k')r}{k} \right\rfloor - \left\lfloor \frac{jr}{k} \right\rfloor\right)$$

$$= 0, \text{ as } k'r \text{ is a multiple of } k.$$

Thus the period is a divisor of k'. Suppose that s has period $\dfrac{k}{qx}$. Then this period is repeated qx times, so the number of values of j ($j = 1, \ldots, k$) such that $d_j = d+1$ is a multiple of qx. But this number is also equal to r, so that qx is a common divisor of r and k. Thus, $x = 1$.

■

It follows from Lemma 3.6 that SP_k^n has n' vertices and that

$$V(SP_k^n) = \{v_{a,s} : a \in I^{n'}\}.$$

Now let $r' = \frac{r}{q}$, and assume $r' \geq 1$ (that is, $r \neq 0$). Let $X = \{v_{a_i,s} : 1 \leq i \leq x\}$ be a set of independent vertices of SP_k^n. Let $Y = \{v_{b_j,s} : 1 \leq j \leq r'\}$ be a set of r' vertices of SP_k^n such that $X \cap Y = \varnothing$.

Since the a_i, b_j may be assumed to belong to $I^{n'}$, we now proceed modulo n'. Thus, denote the sequence of clockwise displacements between consecutive 'first' elements $\{b_j\}$ of Y by

$$\delta = \{\delta_1, \ldots, \delta_{r'}\}$$

where $\delta_j = b_{j+1} - b_j \quad (j = 1, \ldots, r' - 1)$ and $\delta_{r'} = \Gamma_{n'}(b_1 - b_{r'}) = n' + b_1 - b_{r'}$.

(For the case $r' = 1$, there is just one 'first' element, b_1. We take its corresponding displacement δ_1 to be the 'full circle' distance $n' = \frac{kd + r}{q} = k'd + 1$).

Now set $\left\lceil \frac{k}{r} \right\rceil = l$, $\left\lfloor \frac{k}{r} \right\rfloor = s$, and suppose that, for each j such that $1 \leq j \leq r'$,

$$\delta_j = ld + 1 \text{ or } \delta_j = sd + 1.$$

(We refer to these displacement lengths, where they differ, as ***long*** and ***short*** respectively.)

Then Y is said to be an ***interlace*** for X, and X is said to ***possess an interlace***, Y.

Example

For the graph SP_9^{22} we have $d = 2$, $r = 4$, $q = 1$, $\left\lceil \frac{k}{r} \right\rceil = 3$. A maximal independent set

$X = \{v_1, \ldots, v_9\}$ and its interlace $Y = \{w_1, \ldots, w_4\}$ are shown below, with $\{v_i\}$ in fine type and $\{w_j\}$ in bold.

s	2	3	2	3	2	3	2	3	2
v_1	1	3	6	8	11	13	16	18	21
v_2	3	5	8	10	13	15	18	20	1
v_3	5	7	10	12	15	17	20	22	3
w_1	**7**	**9**	**12**	**14**	**17**	**19**	**22**	**2**	**5**
v_4	8	10	13	15	18	20	1	3	6
v_5	10	12	15	17	20	22	3	5	8
w_2	**12**	**14**	**17**	**19**	**22**	**2**	**5**	**7**	**10**
v_6	13	15	18	20	1	3	6	8	11
v_7	15	17	20	22	3	5	8	10	13
w_3	**17**	**19**	**22**	**2**	5	7	**10**	**12**	**15**
v_8	18	20	1	3	6	8	11	13	16
v_9	20	22	3	5	8	10	13	15	18
w_4	**22**	**2**	**5**	7	**10**	**12**	**15**	**17**	**20**

Lemma 3.13 Let $r \geq 1$, $q = \gcd(k,r)$, and let X be an independent subset of $V(SP_k^n)$.

If X possesses an interlace, then

$$|X| \leq \frac{k}{q}.$$

Proof. Let $r' = \frac{r}{q}$ and let z be the number of long displacements of $\delta = \{\delta_1, ..., \delta_{r'}\}$. (Where $r \mid k$, there is no distinction between long and short displacements, and the value of z is arbitrary.)

Now since $X \cap Y = \varnothing$, then $\{a_i\} \cap \{b_j\} = \varnothing$. It follows that every a_i must strictly lie between two consecutive b_j's. But, as the displacements of s are all at least d, a necessary condition for $v_{a_i,s}$ and $v_{a_j,s}$ to be independent is that $\left|a_i - a_j\right|_{n'} \geq d$. Thus the number of a_i's between two b_j's is at most l for a long displacement and at most s for a short displacement. (In the case $r' = 1$, the single 'displacement' may be taken as long or short.) It follows that the number of a_i's, and hence the number of vertices in X, is bounded above by

$$|X| \leq r's + z(l - s).$$

Now if $r \mid k$, then $l = s$ and $r's = \frac{k}{q}$, giving the required result. Thus, assume the contrary, so that

$$|X| \leq r's + z \qquad \textbf{(3)}$$

Recall that we are working modulo n', so that $\sum_{j=1}^{r'} \delta_j = n' = \frac{kd + r}{q} = \frac{kd}{q} + r'$.

Thus,

$$z(ld + 1) + (r' - z)(sd + 1) = \frac{kd}{q} + r'.$$

Since $l - s = 1$,

$$zd = \frac{kd}{q} + r' - r'(sd+1)$$

$$= \frac{kd}{q} - r'sd,$$

so that $z = \frac{k}{q} - r's$, and the result follows immediately from (3).

■

Our next aim is to show the existence of an interlace. To do this we shall make use of the following Lemma.

Lemma 3.14

Let $1 \le m \le r$. Let $v_{a,s}, v_{b,s} \in V(SP_k^n)$ be such that $b - a = \left\lceil \frac{mk}{r} \right\rceil d + m - 1$.

Then $v_{a,s} \cap v_{b,s} = \varnothing$. (That is, these vertices are adjacent in SP_k^n.)

Proof. Assume the contrary. By (2), there exist i, j $(1 \le i, j \le k)$ such that

$$\Gamma_n\left(b + id + \left\lfloor \frac{ir}{k} \right\rfloor\right) = \Gamma_n\left(a + jd + \left\lfloor \frac{jr}{k} \right\rfloor\right);$$

that is (choosing a suitably):

$$\Gamma_n\left(\left\lceil \frac{mk}{r} \right\rceil d + m - 1 + id + \left\lfloor \frac{ir}{k} \right\rfloor\right) = jd + \left\lfloor \frac{jr}{k} \right\rfloor.$$

We now have two cases.

Case 1 Suppose that

$$\left\lceil \frac{mk}{r} \right\rceil d + m - 1 + id + \left\lfloor \frac{ir}{k} \right\rfloor \le n,$$

so that

$$\left\lceil \frac{mk}{r} \right\rceil d + m - 1 + id + \left\lfloor \frac{ir}{k} \right\rfloor = jd + \left\lfloor \frac{jr}{k} \right\rfloor; \tag{4}$$

that is,

$$\left\lceil \frac{mk}{r} \right\rceil d = (j-i)d + \left\lfloor \frac{jr}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor + 1 - m. \tag{5}$$

Now if $j \ge i + \frac{mk}{r}$, then

$$j - i = \lceil j - i \rceil \ge \left\lceil \frac{mk}{r} \right\rceil$$

and also

$$\left\lfloor \frac{jr}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor \ge m.$$

Thus from (5),

$$\left\lceil \frac{mk}{r} \right\rceil d \geq \left\lceil \frac{mk}{r} \right\rceil d + m + 1 - m$$

which is absurd.

Hence

$$0 < j - i < \frac{mk}{r} \tag{6}$$

from which

$$0 \leq \left\lfloor \frac{jr}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor \leq m.$$

It follows, again from (5), that

$$\left\lceil \frac{mk}{r} \right\rceil d \leq (j-i)d + m + 1 - m$$

$$= (j-i)d + 1$$

Therefore,

$$\left(\left\lceil \frac{mk}{r} \right\rceil - (j-i) \right) d \leq 1$$

But (6) also implies $j - i < \left\lceil \frac{mk}{r} \right\rceil$, so that

$$0 < \left(\left\lceil \frac{mk}{r} \right\rceil - (j-i) \right) d \leq 1$$

But since $n \geq 2k$ for a Kneser graph, it follows that $d \geq 2$, giving a contradiction.

Case 2 Suppose that $\left\lceil \frac{mk}{r} \right\rceil d + m - 1 + id + \left\lfloor \frac{ir}{k} \right\rfloor > n$. Then (4) becomes

$$\left\lceil \frac{mk}{r} \right\rceil d + m - 1 + id + \left\lfloor \frac{ir}{k} \right\rfloor = jd + \left\lfloor \frac{jr}{k} \right\rfloor + (kd + r);$$

$$\left\lceil \frac{mk}{r} \right\rceil d = (j + k - i)d + \left\lfloor \frac{(j+k)r}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor + 1 - m,$$

and by substituting j + k for j in (5), the argument follows as before.

■

Corollary 3.2 Let X be an independent set of vertices of SP_k^n such that

$$v_{1,s} \in X,$$

and let $Y = \{v_{b_m,s} : 1 \le m \le r'\}$ where

$$b_m = \left\lceil \frac{mk}{r} \right\rceil d + m \qquad (1 \le m \le r').$$

Then Y is an interlace for X.

Proof. By Lemma 3.14, $v_{1,s} \cap v_{b_m,s} = \varnothing$ $(1 \le m \le r')$. Since the point sets constituting the vertices in X all have non-empty intersection with $v_{1,s}$, it follows that $X \cap Y = \varnothing$. Moreover,

$$b_{m+1} - b_m = \left\lceil \frac{(m+1)k}{r} \right\rceil d - \left\lceil \frac{mk}{r} \right\rceil d + 1$$

$$= \left\lceil \frac{k}{r} \right\rceil d + 1 \text{ or } \left\lfloor \frac{k}{r} \right\rfloor d + 1.$$

Thus Y is an interlace, as required.

■

Theorem 3.6

(i) $\alpha(SP_k^n) = \frac{k}{q}$.

(ii) Every vertex of SP_k^n is contained in a maximum independent set.

Proof.

Now for $1 \le r < k$, Lemma 3.13 and Corollary 3.2 give $\alpha(SP_k^n) \le \frac{k}{q}$.

Also, given any $a \in I^{n'}$, we can construct the independent set of vertices, $V_a(SP_k^n)$ of size $\frac{k}{q}$ in a similar way to that given for the subgraph ${}_{\mathbf{d}}R_k^n$ on page 44. Namely the set

$$V_a(SP_k^n) = \{\{\Gamma_{n/q}(a + \sum_{j=1}^{1-i} d_j), \Gamma_{n/q}(a + \sum_{j=1}^{2-i} d_j), \ldots\ldots, \Gamma_{n/q}(a + \sum_{j=1}^{k/q-i} d_j)\} : 1 \le i \le \frac{k}{q}\}.$$

Alternatively,

$$V_a(SP_k^n) = \{v_{a,\mathbf{t}} : \mathbf{t} \text{ is cyclically equivalent to } \mathbf{s}\}.$$

Since the period of **s** is $\frac{k}{q}$, it follows that $\left|V_a(SP_k^n)\right| = \frac{k}{q}$.

The case $r = 0$ is trivial because SP_k^{kd} is isomorphic to the complete graph K_d.

■

Lemma 3.15 $\mu(SP_k^n) = \frac{n}{k}$.

Proof. This follows immediately from Lemmas 3.6, 3.12 and Theorem 3.6.

■

Theorem 3.7 For $1 \le d \le \left\lfloor \frac{n}{k} \right\rfloor$,

$$\chi_f({}_dS_k^n) = \chi_f(SP_k^n) = \frac{n}{k}.$$

Proof.

By Lemmas 1.1, 3.5 and 3.15 and in view of the fact that $SP_k^n \subseteq {}_dS_k^n$ $(1 \le d \le \left\lfloor \frac{n}{k} \right\rfloor)$,

we have $\frac{n}{k} = \mu(SP_k^n) \le \chi_f(SP_k^n) \le \chi_f({}_dS_k^n) \le \chi_f(G_k^n) = \frac{n}{k}$.

■

In section 3.2 we defined displacement sequences and in section 3.7 we investigated the rotation subgraphs induced by a displacement sequence of the form $\mathbf{s} = \{\delta, \delta, \ldots\ldots, \delta, D\}$, where $\delta \neq D$. We shall generalise these sequences and consider the subgraphs they induce.

Definition Let $\varepsilon(n)$, the ***Euler set,*** be the set of positive integers that are less than n and relatively prime to n. For each $d \in \varepsilon(n)$ let we define the k-element sequence

$$x_d = \{d, d, \ldots\ldots\ldots, d, \Gamma_n((n-k+1)d)\} \quad \text{and}$$

$$\mathbf{S} = \{x_d : d \in \varepsilon(\text{n})\}$$

Given any $a \in I^n$ and $x_d \in \mathbf{S}$, let

$v_{a,x_d} = \{a, \Gamma_n(a+d), \Gamma_n(a+2d), \ldots\ldots\ldots, \Gamma_n(a+(k-1)d)\}$.

Any two elements of v_{a,x_d} are distinct. To see this, consider two such elements, say $\Gamma_n(a+pd)$ and $\Gamma_n(a+qd)$ with $0 \le q < p \le k-1$.

Suppose by way of contradiction that $\Gamma_n(a+pd) = \Gamma_n(a+qd)$.

Then $a + pd = a + qd + mn$.

Since gcd(n,d) = 1, it follows that $n|\ (p-q)$, giving a contradiction. Thus, v_{a,x_d} is indeed a vertex of G_k^n.

Note that the sum of elements of x_d is equal to a multiple of n but not necessarily to n. To distinguish this from the displacement sequence, **d** defined on page 21, we shall refer to x_d as the ***difference sequence*** for the vertex v_{a,x_d}. Also, in analogy to the rotation subgraphs we make the following definition.

Given any difference sequence $x_d \in$ S, we define the ***constant-step subgraph***, ${}_{x_d}C_k^n$, to be the subgraph of G_k^n induced by the vertices of the form v_{a,x_d} for some $a \in I^n$.

We now investigate the constant-step subgraphs.

3.10 Constant-Step Subgraphs ${}_{x_d}C_k^n$

Lemma 3.16 If $x_d \in$ S, then $\left|V({}_{x_d}C_k^n)\right| = n$.

Proof

It is sufficient to show that, for $a, b \in I^n$, $v_{a,x_d} = v_{b,x_d} \Rightarrow a = b$.

Let $v_{a,x_d} = v_{b,x_d}$, then a, $\Gamma_n(a+(k-1)d) \in v_{b,x_d}$. It follows that there are integers $0 \le p, q \le k-1$ such that $a = \Gamma_n(b+pd)$ and $\Gamma_n(a+(k-1)d) = \Gamma_n(b+qd)$.

Hence,

$\Gamma_n((k-1)d) = \Gamma_n((q-p)d)$, and so $(k-1)d = (q-p)d + xn$ for some

integer x.

Therefore,

$$(k-1+p-q)d = xn \text{ and } n|\,(k-1+p-q)d.$$

Since gcd(n,d)=1, it follows that $n|\,(k-1+p-q)$. But $0 \le k-1+p-q < n$, from which the only possible conclusion is that $k-1+p-q=0$.

Hence, $q-p=k-1$ with $0 \le p, q \le k-1$, and so $q=k-1$ and $p=0$. Thus,

$a = \Gamma_n(b) = b.$

■

Theorem 3.8

Let n and k be positive integers such that $k \ge 1, n \ge 2k$ and gcd(n,k) = 1.

Then there exists $x_\delta \in$ S such that ${}_{x_\delta}C_k^n$ is a subgraph of S_k^n.

Proof

Let $\delta = \Gamma_n(k^{\phi(n)-1})$ where ϕ is Euler's function. Then by the Euler-Fermat Theorem, (Theorem 5.17 page 113 of [1]) $\Gamma_n(k\delta) = 1$ and so gcd $(n, \delta) = 1$.

Consider the difference sequence $x_\delta = \{\delta, \delta, \ldots\ldots, \delta, \Gamma_n((n-k+1)\delta)\}$ and the constant-step subgraph ${}_{x_\delta}C_k^n$ it induces.

By symmetry it is enough to consider just one vertex of ${}_{x_\delta}C_k^n$, say

$v_{\delta, x_\delta} = \{\delta, \Gamma_n(2\delta), \Gamma_n(3\delta), \ldots\ldots, \Gamma_n(k\delta)\}$ and show the 'cyclic distance' between any two of its elements is at least $\left\lfloor \frac{n}{k} \right\rfloor$. That is, we need to show $|\Gamma_n(p\delta)|_n \geq \left\lfloor \frac{n}{k} \right\rfloor$ for all $1 \leq p \leq k-1$.

We consider $\Gamma_n(p\delta)$. Let q be the the non-negative integer such that

$\Gamma_n(p\delta) = p\delta - qn \in I^n$.

Now $k(p\delta - qn) = pk\delta - qnk = p(1 + mn) - qnk = (pm - qk)n + p$

As $1 \leq p\delta - qn \leq n$ it follows that,

$$k \leq (pm - qk)n + p \leq nk,$$

Hence,

$$1 \leq k - p \leq (pm - qk)n \leq nk - p \leq nk - 1$$

and so,

$$0 < \frac{1}{n} \leq pm - qk \leq k - \frac{1}{n} < k.$$

That is

$$1 \leq pm - qk \leq k - 1.$$

Thus,

$$k(p\delta - qn) = (pm - qk)\,n + p \geq n + p$$

giving
$$p\delta - qn \geq \frac{n}{k} + \frac{p}{k} \geq \left\lfloor \frac{n}{k} \right\rfloor. \qquad \textbf{(7)}$$

Thus,

$$\begin{aligned} nk - k(p\delta - qn) &= nk - (pm - qk)\,n - p \\ &= n(k - (pm - qk)) - p \\ &\geq n(k - (k-1)) - p \\ &= n - p \\ &> n - k \end{aligned}$$

giving

$$n - (p\delta - qn) > \frac{n}{k} - 1 \geq \left\lfloor \frac{n}{k} \right\rfloor. \qquad \textbf{(8)}$$

Finally combining (7) and (8) gives

$$|\Gamma_n(p\delta)|_n = \min\{p\delta - qn, n - (p\delta - qn)\} \geq \left\lfloor \frac{n}{k} \right\rfloor \text{ for all } 1 \leq p \leq k-1.$$

■

The following lemma extends Theorem 3.8 and shows that a constant-step subgraph of S_k^n does exist without the restriction gcd(n,k) = 1.

Lemma 3.17

For any positive integer $c \geq 2$ and $1 \leq d \leq \left\lfloor \frac{n}{k} \right\rfloor$, ${}_dS_k^n$ is isomorphic to a subgraph of ${}_dS_{ck}^{cn}$.

Proof

Using the convention established on page 21, let $v = \{a_1, a_2, \ldots, a_k\}$ be a vertex of ${}_dS_k^n$.

We define a mapping $\theta : V\left({}_dS_k^n\right) \rightarrow V\left({}_dS_{ck}^{cn}\right)$ as follows:

For $v = \{a_1, \ldots, a_k\} \in V\left({}_dS_k^n\right)$,

$$\theta(v) = \{a_1, \ldots, a_k,\ a_1 + n, \ldots, a_k + n,\ \ldots\ldots,\ a_1 + (c-1)n, \ldots, a_k + (c-1)n\}.$$

We first show that θ maps $V\left({}_dS_k^n\right)$ into $V\left({}_dS_{ck}^{cn}\right)$.

That is we show that $\left|a_i + pn - (a_j + qn)\right|_{cn} = \left|a_i - a_j + (p-q)n\right|_{cn} \geq d$ for all $1 \leq i, j \leq k$ and $0 \leq p, q \leq c-1$ such that not both $i = j$ and $p = q$.

We consider two cases, for $i = j$ and $i \neq j$.

Case 1. $i = j$

In this case p cannot equal q. Thus, $\left|a_i - a_j + (p-q)n\right|_{cn} = \left|(p-q)n\right|_{cn} \geq n > d.$

Case 2. $i \neq j$

Let $x \in Z$, then $\Gamma_{cn}(x) = y + pn$ for some integers $0 \leq p \leq c-1$, $y \in I^n$. It follows that

$\Gamma_n(x) = y \leq \Gamma_{cn}(x)$ and that

$$cn - \Gamma_{cn}(x) = cn - y - pn = (c-p)n - y \geq n - y = n - \Gamma_n(x) .$$

Thus,

$$|x|_{cn} = \min\{\Gamma_{cn}(x), cn - \Gamma_{cn}(x)\} \geq \min\{\Gamma_n(x), n - \Gamma_n(x)\} = |x|_n .$$

In particular,

$$\left|a_i - a_j + (p-q)n\right|_{cn} \geq \left|a_i - a_j + (p-q)n\right|_n = \left|a_i - a_j\right|_n \geq d .$$

We next establish that θ is an isomorphism by showing that:

(i) θ is an homomorphism.

(ii) θ is injective.

(iii) Let $u, v \in V\left({}_dS_k^n\right)$. If $\theta(u)\theta(v)$ is an edge of ${}_dS_{ck}^{cn}$, then uv is an edge of ${}_dS_k^n$.

For part (i) we need to show that if $u \cap v = \varnothing$, then $\theta(u) \cap \theta(v) = \varnothing$.

By way of contradiction suppose $\theta(u) \cap \theta(v) \neq \varnothing$, then

$$a_i + rn = b_j + tn \qquad \text{for some integers } 0 \leq r, t \leq c-1$$

where a_i and b_j are the i^{th} and j^{th} elements of the vertices u and v respectively.

Without loss of generality suppose $a_i > b_j$. It follows immediately that

$(t-r)n \leq n-1$ with $t > r$; giving a contradiction.

To prove (ii), we need to show that if $\theta(u) = \theta(v)$, then $u = v$. But if $\theta(u) = \theta(v)$, then $a_i = b_i$ for $1 \leq i \leq k$, and so $u = v$.

Finally, for part (iii), suppose $\theta(u) \cap \theta(v) = \varnothing$. As u and v are subsets of $\theta(u)$ and $\theta(v)$ respectively, it immediately follows that $u \cap v = \varnothing$.

■

In view of Theorem 3.8 and Lemma 3.17, every Kneser graph, G_k^n, contains a constant-step subgraph such that the 'colours' at every vertex are of maximum distance apart.

We now focus our attention on the Kneser graphs, G_k^{2k+1}, letting Γ stand for Γ_{2k+1} for the remainder of the section, and show that every rotation subgraph has independence number always either equal to k or $2k + 1$. We further show that in the case when it is equal to k, the rotation subgraph is equal to some constant-step subgraph.

But first we make the following definitions:

Let $\mathbf{x} = \{d_1, d_2, \ldots\ldots\ldots, d_k\} \in {}_d\mathbf{D}$, be given. We define its ***difference set***, X, as

$$\mathrm{X} = \left\{ \sum_{i=p}^{p+q} d_{\Gamma_k(i)} : \; 1 \le p \le k, \; 0 \le q \le k-1 \right\}.$$

Thus, given any vertex v of G_k^{2k+1} whose displacement sequence is **x**, with difference set X, we have $\mathrm{X} = \{\Gamma(x-y) : x, y \in v\}$.

For example for $\mathbf{x} = \{1, 3, 1, 4\}$ of G_4^9, $\mathrm{X} = \{1, 3, 4, 5, 6, 8, 9\}$ whilst for

$\mathbf{x} = \{1, 1, 2, 5\}$, $\mathrm{X} = I^9$.

If $\mathrm{X} = I^n$, we say that the displacement sequence, **x**, ***spans*** I^n. Thus, $\mathbf{x} = \{1, 1, 2, 5\}$ spans I^9.

It is readily seen that if $a \notin \mathrm{X}$, then $2k + 1 - a \notin \mathrm{X}$. That is, X is invariant under complementation modulo $2k+1$. For example, the displacement sequence $\mathbf{x} = \{1, 3, 1, 4\}$, has $2,7 \notin \mathrm{X}$.

Now if **x** is a displacement sequence of G_k^{2k+1}, then it either spans I^{2k+1} or it does not.

For any k there are displacement sequences that span I^{2k+1} and displacement sequences that do not, as demonstrated below.

For the displacement sequence, $\mathbf{x} = \{1, 1, 1, \ldots\ldots, 1, 1, 2, k+1\}$, it is readily seen that $X = I^{2k+1}$.

By contrast, for the sequence, $\mathbf{x} = \{1, 1, 1, \ldots\ldots, 1, 1, k+2\}$, the elements k, $k+1 \notin X$.

We proceed with the following Theorem.

Theorem 3.9

Let $\mathbf{x}$ be a displacement sequence of G_k^{2k+1} with difference set X and ${}_{\mathbf{x}}R_k^{2k+1}$ be the rotation subgraph it induces.

(i) If $X = I^{2k+1}$, then $\alpha({}_{\mathbf{x}}R_k^{2k+1}) = 2k+1$. Indeed, ${}_{\mathbf{x}}R_k^{2k+1}$ is a null graph.

(ii) If $X \neq I^{2k+1}$, then $\alpha({}_{\mathbf{x}}R_k^{2k+1}) = k$.

Proof

For (i), let $u, v \in V({}_{\mathbf{x}}R_k^{2k+1})$, where $u = \{b_1, b_2, \ldots\ldots\ldots, b_k\}$ and

$v = \{c_1, c_2, \ldots\ldots\ldots, c_k\}$ and where the elements of both the vertices are written in the order defined by the displacement sequence **x**. That is

$$d_i = \Gamma(b_{i+1} - b_i) = \Gamma(c_{i+1} - c_i) \quad \text{for } 1 \le i \le k-1 \text{ and}$$

$$d_k = \Gamma(b_1 - b_k) = \Gamma(c_1 - c_k).$$

Now there is an integer $a \in I^{2k+1}$ which does not depend on i, such that for each $1 \le i \le k$, $\Gamma(b_i + a) = c_i$. Also, since $X = I^{2k+1}$, it follows there are elements $b_i, b_j \in v$ where $\Gamma(b_j - b_i) = a$. Hence,

$\Gamma(b_i + a) = b_j$ and so $c_i = b_j$. Thus, $u \cap v \neq \varnothing$ and the result follows.

For (ii), let $a \in I^{2k+1} - X$ and Q be an independent set of ${}_{\mathbf{x}}R_k^{2k+1}$ with s vertices. Given any vertex of ${}_{\mathbf{x}}R_k^{2k+1}$, say $v = \{b_1, b_2, \ldots\ldots\ldots, b_k\}$, we define the vertex, denoted by $v+a$ as :

$$v+a = \{\Gamma(b_1 + a), \Gamma(b_2 + a), \ldots\ldots\ldots, \Gamma(b_k + a)\}.$$

Clearly, $v+a \in V({}_{\mathbf{x}}R_k^{2k+1})$.

Now, since $a \notin X$, it follows that $\Gamma(b_i + a) \neq b_j$ for all $1 \leq i, j \leq k$ $(i \neq j)$ and hence the elements of v and $v+a$ are disjoint. That is, v and $v+a$ are adjacent.

Consider the set of vertices

$$T = \{v+a : v \in Q\} \quad \text{and}$$

note that $v+a = u+a \Leftrightarrow v = u$. In view of the fact that Q contains s vertices, then so does T.

We next show that Q and T are disjoint.

By way of contradiction suppose $Q \cap T \neq \varnothing$.

Then there are vertices v and $u+a$ of Q and T respectively such that $v = u+a$.

But $u+a$ is adjacent to u, and so u and v are adjacent vertices of Q; contradicting that Q is an independent set. Thus, $Q \cap T = \varnothing$.

Finally, since Q and T each contain s vertices and are disjoint then,

$$2s \leq \left| V({}_{\mathbf{x}}R_k^n) \right| = 2k+1,$$

and so,

$$s \leq k.$$

Moreover, since a maximum independent set of ${}_{\mathbf{x}}R_k^{2k+1}$ has size at least k, the result follows.

■

Theorem 3.9 shows that the independence number of ${}_{\mathbf{x}}R_k^{2k+1}$ is always either equal to k or $2k+1$.

However, our goal is to also show in the case when $X \neq I^{2k+1}$, that given a vertex, v of ${}_{\mathbf{x}}R_k^{2k+1}$, its elements can be re-ordered so that the underlying difference sequence belongs to **S**. That is every vertex can be written in the form

$$v_{a,x_d} = \{a, \Gamma(a+d), \Gamma(a+2d), \ldots\ldots\ldots, \Gamma(a+(k-1)d)\}.$$

for some positive integer $d \in \varepsilon\,(2k+1)$. This shows that the rotation subgraph, ${}_{\mathbf{x}}R_k^{2k+1}$ is equal to the constant-step subgraph, ${}_{x_d}C_k^{2k+1}$.

Theorem 3.10

Let **x** be a displacement sequence of G_k^{2k+1} with difference set X. If $X \neq I^{2k+1}$, then

${}_{\mathbf{x}}R_k^{2k+1} = {}_{x_d}C_k^{2k+1}$ for some positive integer, d, less than $2k+1$.

Proof

Let $a \in I^{2k+1} - X$. In view of the fact that $a \notin X \Rightarrow 2k+1-a \notin X$; we assume without loss of generality that $a \le k$.

Let $v = \{b_1, b_2, \ldots\ldots\ldots, b_k\}$ be a vertex of ${}_{\mathbf{x}}R_k^{2k+1}$ and consider the vertex

$v+a = \{\Gamma(b_1+a), \Gamma(b_2+a), \ldots\ldots\ldots, \Gamma(b_k+a)\}$.

It was noted that v and $v+a$ are disjoint.

Similarly, the vertices $v+a$ and $v+2a$ $(= \{\Gamma(b_1+2a), \Gamma(b_2+2a), \ldots\ldots\ldots, \Gamma(b_k+2a)\})$ are also disjoint.

For example, the difference sequence $\mathbf{x} = \{1, 2, 3, 3, 2\}$ of G_5^{11}, has $4 \notin X$.

Considering the vertex

$v = \{1, 2, 4, 7, 10\} \in V({}_{\mathbf{x}}R_5^{11})$, it induces the vertices $v+4 = \{5, 6, 8, 11, 3\}$ and $v+8 = \{9, 10, 1, 4, 7\}$; with $v \cap v+4 = v+4 \cap v+8 = \varnothing$.

Now, since $v+2a$ contains k integers, none of which are contained in $v+a$, it follows that at least $k-1$ of these must belong to v. That is v and $v+2a$ must share at least $k-1$ integers.

We now show that in fact they share precisely $k-1$ integers.

By way of contradiction suppose they share k integers.

Then $v = v+2a$ and so it follows that

$$\Gamma(\sum_{i=1}^{k} b_i) = \Gamma(\sum_{i=1}^{k}(b_i+2a)) = \Gamma(\sum_{i=1}^{k} b_i + \sum_{i=1}^{k} 2a) = \Gamma(\sum_{i=1}^{k} b_i + 2ak)$$

Hence,

$$2k+1 = \Gamma(0) = \Gamma(2ak)$$

and so $2k+1 \mid 2ak$

But since $\gcd(2k+1,k) = 1$, it follows that $2k+1 \mid 2a$ with $a \le k$; which is impossible and so giving the required contradiction.

Thus v and $v+2a$ share exactly $k-1$ integers. Now, let b be the only element of v that is not contained in $v+2a$ and similarly let x be the only element of $v+2a$ not contained in v.

We label b as b_1 and for each $i = 2,3,\ldots,k$, we let $b_i = \Gamma(b_1+(i-1)2a)$.

We show that $\{b_1, b_2, \ldots, b_k\}$ is indeed the vertex v. First we establish that these b_j are all distinct. Assume to the contrary that $b_i = b_j$ for some $1 \le i < j \le k$.

Then $\Gamma(b_1+(i-1)2a) = \Gamma(b_1+(j-1)2a)$ and so $\Gamma((j-i)2a) = \Gamma(0) = 2k+1$.

Thus $2k+1 \mid (j-i)2a$ and so (as $\gcd(2k+1,2) = 1$) $2k+1 \mid (j-i)a$ where $1 \le j-i \le k-1$. Now let l ($\le j-i$) be the smallest positive integer such that $2k+1 \mid la$. Then l must be odd, say $l = 2t+1$. Thus $b_{t+1} = \Gamma(b_1+t2a)$ belongs to v, so $\Gamma(b_{t+1}+a)$ belongs to

v+a. But $\Gamma(b_{t+1}+a) = \Gamma(b_1+(2t+1)a) = \Gamma(b_1+la) = \Gamma(b_1) = b_1 \in v$, and so $\Gamma(b_{t+1}+a) \in v \cap v+a$; giving a contradiction.

Next, let us show that each b_j actually belongs to v. Suppose otherwise. Thus for some $j<k$ we have $b_1, b_2, \ldots, b_j \in v$ but $b_{j+1} \notin v$, so that b_{j+1} is the unique element of v+2a not in v, namely x. But then there is some other element, say c, of v that is not one of $b_1, b_2, \ldots, b_j$, and since $\{c, \Gamma(c+a), \Gamma(c+2a), \Gamma(c+4a), \ldots\}$ can only get out of v through one of these being the b_j that we have already used, it must follow that these 'cycle'; but we have already contradicted this possibility.

Thus $\{b_1, b_2, \ldots, b_k\}$ are the elements of v and b_k is the element of v such that $\Gamma(b_k+2a) = x$ and $\Gamma(b_k+3a) = b_1$.

That is, vertex v can be written as

$v = \{b_1, b_2, \ldots, b_k\}$ with clockwise differences

$d_i = \Gamma(b_{i+1} - b_i) = \Gamma(2a)$ for $1 \le i \le k-1$ and

$d_k = \Gamma(b_1 - b_k) = \Gamma(3a)$.

Finally we note that, $\Gamma(\sum_{i=1}^{k} d_i) = \Gamma(2a(k\text{-}1) + 3a) = \Gamma(a(2k+1)) = 2k+1$.

It follows we can express any vertex of ${}_{\mathbf{x}}R_k^{2k+1}$ with an underlying difference sequence

$$x_d = \{d,d,\ldots\ldots\ldots,d,\Gamma_n((n-k+1)d)\} = \{2a,2a,\ldots\ldots\ldots,2a,\Gamma(3a)\} \text{ where } d=2a.$$

We note that $d \in \varepsilon\,(2k+1)$. To see this let $g = \gcd\,(2k+1, 2a)$, and assume by way of contradiction that $g > 1$.

Now, $2k+1 = pg$ and $2a = mg$ for some positive integers p and m.

Thus, $2k+1 = pg \leq 2ap \leq 2pk$ and so $p \geq 1 + \dfrac{1}{2k}$. Therefore $p \geq 2$.

Similarly, $2k+1 = pg \geq 2p$ and so $p \leq \dfrac{2k+1}{2} = k + \dfrac{1}{2}$; giving $p \leq k$.

That is

$$2 \leq p \leq k.$$

Also $2a = mg \Rightarrow \Gamma(2a\,p) = \Gamma(mpg) = \Gamma(m(2k+1)) = 2k+1$.

Thus, considering the element $b_p \in v$, we have

$$\begin{aligned} b_p &= \Gamma(b_1 + 2a(p-1)) \\ &= \Gamma(b_1 - 2a) \\ &= \Gamma(b_k + a) \in v{+}a; \text{ giving a contradiction.} \end{aligned}$$

It follows that, ${}_{\mathbf{x}}R_k^{2k+1} = {}_{x_d}C_k^{2k+1}$ where $d = 2a$.

■

Corollary 3.3

Let **x** be a displacement sequence of G_k^{2k+1} and ${}_{\mathbf{x}}R_k^{2k+1}$ be the induced rotation subgraph. If $\alpha({}_{\mathbf{x}}R_k^{2k+1}) = k$ then ${}_{\mathbf{x}}R_k^{2k+1} = {}_{x_d}C_k^{2k+1}$ for some positive integer, *d*, less than $2k+1$.

Proof

This follows from Theorems 3.9 and 3.10.

■

3.11 The Rotation Subgraphs ${}_{x}R_{k-q}^{2k+1}$ $(1 \leq q \leq k-1)$

Let q be an integer, $1 \leq q \leq k-1$. Given a displacement sequence **x** of the graph G_{k-q}^{2k+1} we investigate the induced subgraphs ${}_{\mathbf{x}}R_{k-q}^{2k+1}$. The maximum possible size of independent set of the induced subgraph, ${}_{\mathbf{x}}R_k^{2k+1}$, is $2k$+1. It is clear that for some value of q, the subgraph, ${}_{\mathbf{x}}R_{k-q}^{2k+1}$ will no longer contain an independent set of size $2k$+1, for any displacement sequence **x**. There seems to be a critical value, say c, such that if $q \geq c$, the size of a maximum independent set of ${}_{\mathbf{x}}R_{k-q}^{2k+1}$ drops down to k or less.

In what follows we find a value, b, such that if $q > b$, then the size of a maximum independent set of ${}_{\mathbf{x}}R_{k-q}^{2k+1}$ is at most k, but such that for certain values of k, the size is $2k+1$ when $q = b$.

Definitions.

Given a displacement sequence, $\mathbf{x} = \{d_1, d_2, \ldots\ldots\ldots, d_k\} \in {}_d\mathbf{D}$, of the general Schrijver graph ${}_dS_k^n$, we define a ***subsequence*** of $\mathbf{x}$ to be a sequence of cyclically 'consecutive' elements of $\mathbf{x}$, that is a sequence of the form $\{d_{\Gamma_k(i)}, d_{\Gamma_k(i+1)}, \ldots\ldots\ldots, d_{\Gamma_k(i+j)}\}$, for some $1 \le i \le k$ and $0 \le j \le k-1$.

We denote the set of subsequences by S:

$$S = \{\{d_{\Gamma_k(i)}, d_{\Gamma_k(i+1)}, \ldots\ldots, d_{\Gamma_k(i+j)}\} : 1 \le i \le k,\ 0 \le j \le k-1\}.$$

It is readily seen that there are k subsequences each of size 1, 2, 3,……, $k-1$ and one of size k ($\mathbf{x}$ itself).

It follows that $|S| = k(k-1) + 1$. **(9)**

For example the displacement sequence $\mathbf{x} = \{1,2,3,4\}$ of G_4^{10} has subsequence set

S =

$\{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{2,3\},\{3,4\},\{4,1\},\{1,2,3\},\{2,3,4\},\{3,4,1\},\{4,1,2\},\{1,2,3,4\}\}$.

We now investigate the rotation subgraphs, ${}_{\mathbf{x}}R_{k-q}^{2k+1}$ and their independence numbers.

We shall use the following result.

Lemma 3.18

Let q be a positive integer less than k such that $\gcd(2k+1, k-q) = 1$ and $\mathbf{x}$ be any displacement sequence of G_{k-q}^{2k+1} with difference set X.

Then

$$X \neq I^{2k+1} \Leftrightarrow k-q \leq \alpha({}_{\mathbf{x}}R_{k-q}^{2k+1}) \leq k.$$

Proof

The proof that $\alpha({}_{\mathbf{x}}R_{k-q}^{2k+1}) \leq k$ is similar to that of Theorem 3.9(ii) with k replaced by $k-q$. The fact that the size of the maximum independent set is at least $k-q$ completes $\Rightarrow$.

Finally, using a similar argument as in the proof of Theorem 3.9(i) proves the case $\Leftarrow$.

■

Theorem 3.11

Let k and q be positive integers with $k \geq 3$, $q < k$ and $\gcd(2k+1, k-q) = 1$. Let $\mathbf{x}$ be any displacement sequence of G_{k-q}^{2k+1}.

$$\text{If } q > \left\lfloor \frac{2k-1-\sqrt{8k+1}}{2} \right\rfloor \text{ then } k-q \leq \alpha({}_{\mathbf{x}}R_{k-q}^{2k+1}) \leq k.$$

Proof

Now each element of X is a result of summing all the integers of some subsequence $s \in S$. Thus from statement (9) above, it follows there are most

$(k - q - 1)(k - q) + 1$ elements of X.

It follows that if $(k - q - 1)(k - q) + 1 < 2k + 1$, then the set X is a proper subset of I^{2k+1} and so X $\neq I^{2k+1}$.

This reduces to the quadratic inequality,

$$q^2 + (1 - 2k)q + k^2 - 3k < 0$$

which gives

$$q > \frac{2k - 1 - \sqrt{8k+1}}{2}$$

Thus,

$$q > \left\lfloor \frac{2k - 1 - \sqrt{8k+1}}{2} \right\rfloor$$

The result now follows from Lemma 3.18.

■

EXAMPLES.

Consider the rotation subgraph ${}_{\mathbf{x}}R_6^{31}$ of G_6^{31} induced by the displacement sequence

$\mathbf{x} = \{1, 3, 2, 7, 8, 10\}$ with $q = 9$, $k = 15$. It can be seen that $X = I^{2k+1} = I^{31}$ and so the size of its maximum independent set is 31. However, from Theorem 3.11 if $q >$ $\left\lfloor \frac{2k-1-\sqrt{8k+1}}{2} \right\rfloor = 9$, the size of a maximum independent set of ${}_{\mathbf{x}}R_{15-q}^{31}$ falls down to at most 15 for any displacement sequence $\mathbf{x}$.

Similarly the Kneser graphs G_5^{17} and G_4^{13} also have the property that

$c = \left\lfloor \frac{2k-1-\sqrt{8k+1}}{2} \right\rfloor$ is the critical value of q in the sense that rotation subgraphs ${}_{\mathbf{x}}R_{k-q}^{2k+1}$ exist (for example those induced by displacement sequences $\mathbf{x} = \{1, 2, 3, 4, 7\}$ and $\mathbf{x} = \{1, 3, 2, 7\}$ respectively) that have independence number $2k+1$ with $q = c$, and that this falls to k or less for all rotation subgraphs ${}_{\mathbf{x}}R_{k-q}^{2k+1}$ when $q > c$.

CHAPTER 4

Circular Colourings

and Kneser Graphs

Theorem 1 of [17] shows that if $n = \chi_1(G)$ then $\eta_n(G) = n$ and so establishes a link between these two types of colouring. Furthermore, if G is bipartite then $\chi_k(G) = 2k$ (Theorem 5 of [16]), while clearly $\eta_{2k}(G) = 2$, and trivially $\chi_k(K_p) = kp$ while $\eta_{kp}(K_p) = k$. That is:

$$\text{if } n = \chi_k(G) \text{ then } \eta_n(G) = \frac{n}{k} \qquad \textbf{(1)}$$

when G is bipartite or a complete graph. This poses a question whether, as an extension to this, there is a more general link between the k^{th} chromatic number of G, $\chi_k(G)$ for k-tuple colourings and the n-chromatic numbers, $\eta_n(G)$ for Z_n-colourings where $n = \chi_k(G)$. The following section establishes a necessary and sufficient condition for such a link in context of homomorphisms. Section 4.2 asserts this link for the odd cycles, C_{2p+1}.

Remark. If G is bipartite, an odd cycle or a complete graph, then Theorems 5 and Corollary 1 of [17], together with Theorems 4, 5 and 6 of [16], give the result

$$\chi_f(G) = \chi_c(G).$$

Graphs having this property are said to be ***star-extremal*** (see [5]).

Indeed any graph which satisfies (1) above is star-extemal. Abbreviating $\chi_k(G)$ as χ_k and $\chi(G)$ as χ, we note that $\{\eta_n = \frac{n}{k} : n = \chi_k, k = 1,2,\ldots\}$ is a subsequence of $\{\eta_n : n = \chi, \chi+1, \chi+2,\ldots\}$. Hence, by Corollary to Theorem 2 of [9] and Corollary 2 of [17] $\chi_f = \lim_{k\to\infty}\left(\frac{\chi_k}{k}\right) = \lim_{n\to\infty}\eta_n = \chi_c$.

4.1 Homomorphisms

We begin by showing that statement (1) above is equivalent to existence of homomorphisms (see Chapter 1 for a reminder of the definitions of *n*-tuple and (*k*,*d*)-colourings in the context of homomorphisms). We give this result as a lemma.

Lemma 4.1

Let $n = \chi_k(G)$ and $\eta_n(G) = \frac{n}{q}$ where *n*, *k* and $q \in Z^+$ with $n \geq 2k$ and $n \geq 2q$. Then

(i) $\eta_n(G) \leq \frac{n}{k} \Leftrightarrow$ there exists a homomorphism $G \to H^n_k$.

(ii) $\eta_n(G) \geq \frac{n}{k} \Leftrightarrow$ there exists a homomorphism $G \to G^n_q$.

Proof

Now the existence of a homomorphism $G \to H^n_k$ defines a (*n*,*k*)-colouring and so $\eta_n(G) \leq \frac{n}{k}$. Now consider $\Rightarrow$. Let $\eta_n(G) = \frac{n}{\nu}$, then $\frac{n}{\nu} \leq \frac{n}{k}$, and so by Proposition 1 of [3] G has a (n,k)-colouring and hence a homomorphism $G \to H_k^n$; thus establishing assertion (i).

To prove (ii) we proceed as follows:

The existence of a homomorphism $G \to G_q^n$ defines a q-tuple colouring of G with n colours and so $\chi_q(G) \leq n = \chi_k(G)$. By Theorem 2 of [16], $\chi_k(G)$ is a strictly increasing function on k; thus $k \geq q$, giving $\eta_n(G) = \frac{n}{q} \geq \frac{n}{k}$.

Conversely if $\eta_n(G) \geq \frac{n}{k}$, then $k \geq q$ and $n \geq \chi_q(G)$. But this defines a q-tuple colouring of G with n colours, and hence a homomorphism $G \to G_q^n$.

■

I am indebted to my supervisor for the following result that asserts the existence of such a homomorphism.

Lemma 4.2

Let $\eta_n(G) = \frac{n}{d}$ where $d \in Z^+$ and $n \geq 2d$. Then there is a homomorphism $G \to G_d^n$.

Proof

Define the mapping $\phi : H_d^n \to G_d^n$ as follows: for each $u \in Z_n$ $(= V(H_d^n))$

$\phi(u) = \{ u, \Gamma_n(u+1), \Gamma_n(u+2), \ldots\ldots, \Gamma_n(u+d-1)\}$. It is clear that, if $|u-v|_n \geq d$, then $\phi(u)$ and $\phi(v)$ are disjoint subsets of I^n, and ϕ is therefore a homomorphism.

Now $\eta_n(G) = \frac{n}{d}$ defines a homomorphism $G \to H_d^n$. Composing this with the homomorphism $\phi : H_d^n \to G_d^n$, we obtain a homomorphism $G \to G_d^n$.

■

Corollary 4.1

(i) If $n = \chi_k(G)$ then $\eta_n(G) \geq \frac{n}{k}$.

(ii) $\chi_f(G) \leq \chi_c(G)$.

Proof

The proof of (i) follows from Lemmas 4.1(ii) and 4.2.

As regards (ii), we have by Theorem 3 of [17], $\chi_c(G) = \eta_n(G)$ for some $n \leq |V(G)|$, and thus by Lemma 4.2 there exists a homomorphism $G \to G_d^n$ where $\chi_c(G) = \frac{n}{d}$. It follows that $\chi_d(G) \leq n$ and so $\chi_f(G) \leq \frac{\chi_d}{d} \leq \frac{n}{d} = \chi_c(G)$.

■

4.2 Odd Cycles

We now show that the property of statement (1) does hold for odd cycles. Namely,

Theorem 4.1

If $n = \chi_k(C_{2p+1})$ then $\eta_n(C_{2p+1}) = \frac{n}{k}$.

Proof

Now $\eta_n(C_{2p+1}) = \frac{n}{d}$ where d is the largest integer for which C_{2p+1} has a (n,d)-colouring. By definition $\chi_c(C_{2p+1}) \leq \frac{n}{d}$. But by Corollary 1 of [17],

$\chi_c(C_{2p+1}) = \frac{2p+1}{p}$ and so d must be the greatest integer such that $d \leq \frac{np}{2p+1}$.

Therefore,

$$d = \left\lfloor \frac{np}{2p+1} \right\rfloor. \qquad (2)$$

Let $q = \left\lfloor \frac{k-1}{p} \right\rfloor$, then by Theorem 6 of [16],

$$n = \chi_k(C_{2p+1}) = 2k + 1 + q. \qquad (3)$$

Now $k - 1 = qp + r \quad (0 \leq r < p)$ **(4)**

and so it follows from (3) and (4) that,

$$np = p(2k + 1 + q)$$
$$= 2pk + p + (k - 1 - r)$$
$$= (2p + 1)k + p - 1 - r$$

so $$\frac{np}{2p+1} = k + \frac{p-1-r}{2p+1}.$$

Since $0 \leq r < p$, then $0 \leq \frac{p-1-r}{2p+1} < 1$, from which it follows that,

$$\left\lfloor \frac{np}{2p+1} \right\rfloor = k.$$

Finally, statement (2) gives $d = k$, from which the result follows.

■

4.3 Kneser Graphs of Low Order

So far we have asserted that if $n = \chi_k(G)$ then $\eta_n(G) = \frac{n}{k}$ when G is bipartite, a complete graph or an odd cycle. At the beginning of the chapter we posed the question as to whether this is generally true for all graphs. The following counter example shows that this is not the case.

Theorem 4.2

(i) Let $k \geq 2$ and $n = \chi_k(G_2^5)$; then $\eta_n(G_2^5) > \frac{n}{k}$.

(ii) $\chi_f(G_2^5) < \chi_c(G_2^5)$.

The proof relies on the following lemma.

Lemma 4.3

There does not exist a homomorphism $G_2^5 \to H_3^8$.

Proof

By way of contradiction suppose such a homomorphism, θ, exists. Since G_2^5 and H_3^8 have 10 and 8 vertices respectively, θ must map two non-adjacent vertices of G_2^5 to the same vertex of H_3^8. We may assume without loss of generality that $\theta(\{1,2\}) = \theta(\{1,3\}) = 1$. Then the five vertices of G_2^5 adjacent to $\{1,2\}$ or $\{1,3\}$ (namely $\{2,4\},\{2,5\},\{3,4\},\{3,5\}$ and $\{4,5\}$) must map to the vertices 4, 5 and 6 of H_3^8. But $\{2,5\}$ and $\{3,4\}$ are adjacent in G_2^5, and so $\theta(\{2,5\})$ and $\theta(\{3,4\})$ must be adjacent in H_3^8; giving a contradiction.

■

Proof of Theorem 4.2

Now by Theorem 7 of [16], $n = 2k + 1 + \left\lfloor \frac{k-1}{2} \right\rfloor$ from which it follows that

$$\frac{n}{k} \leq \frac{8}{3} \qquad \text{for all } k \geq 2.$$

By Corollary 4.1, $\eta_n(G_2^5) \geq \frac{n}{k}$. Suppose $\eta_n(G_2^5) = \frac{n}{k}$; then G_2^5 has a (n,k)-colouring with $\frac{n}{k} \leq \frac{8}{3}$ and so by Proposition 1 of [3] has a (8,3)-colouring. Lemma 4.3 gives a contradiction and completes the proof of (i).

Theorem 7 of [16] also asserts that $\chi_f(G_2^5) = \frac{5}{2}$. In view of the fact that $\frac{5}{2} < \frac{8}{3}$; then once again invoking Lemma 4.3 and Proposition 1 of [3] gives $\chi_c(G_2^5) > \frac{5}{2}$.

■

Lemma 4.3 and Proposition 1 show that the circular chromatic number of G_2^5 is greater than $\frac{8}{3}$. However, by Theorem 7 of [16] $\chi(G_2^5) = 3$. Thus Theorem 4 of [17] gives $\chi_c(G_2^5) \leq 3$. But the largest rational number < 3 with numerator $\leq \left|V(G_2^5)\right| = 10$ is $\frac{8}{3}$, and so Theorem 3 of [17] gives the following result:

Corollary 4.2

$$\chi_c(G_2^5) = 3.$$

We shall next consider the graph G_3^7 and compute its circular chromatic number.

Theorem 4.3

$$\chi_c(G_3^7) = 3.$$

We know from Theorem 7 of [16] and Theorem 4 of [17] that $\chi_c(G_3^7) \leq 3$, and since G_3^7 has 35 vertices and $\frac{35}{12}$ is the largest rational number < 3 with numerator ≤ 35, it is sufficient to prove the non-existence of a homomorphism from G_3^7 to H_{12}^{35}.

We proceed by assuming that such a homomorphism, θ, does exist and deduce a number of results given as Lemmas that are needed to prove Theorem 4.3.

But first we introduce the following two functions, ω and d. Let $u, v \in V(G_3^7)$, then $u \cap v$ is a subset of I^7 and we define $\omega(u, v) = |u \cap v|$.

Assuming the existence of a homomorphism $\theta : G_3^7 \rightarrow H_{12}^{35}$, we also define $d(u, v) = |\theta(u) - \theta(v)|_{35}$.

Lemma 4.4

If $\omega(u, v) = 2$, then $d(u, v) \leq 11$.

Proof

Without loss of generality let $u = \{1,2,3\}$ and $v = \{2,3,4\}$. Let $x = \{5,6,7\}$, then since θ is a homomorphism $d(u,x) \geq 12$ and $d(v,x) \geq 12$. But there cannot be three vertices mutually of distance ≥ 12 in Z_{35}, and so $d(u, v) \leq 11$.

■

We now define a sector in Z_{35} to be a proper subset of Z_{35} of the form $[a, a+s] = \{a+i:.0 \leq i \leq s\}$, where $s < 34$. Then a and $a+s$ are the **left** and **right ends** respectively of $[a, a+s]$. The **length** of the sector is s.

For any pair P of elements of I^7, we define G_P to be the set of five vertices $\{v_i : 1 \leq i \leq 5\}$ of G_3^7 such that the triple defining v_i contains P. Thus the image set $\theta(G_P)$ is, by Lemma 4.4 , contained in some sector of Z_{35} of length at most ≤ 11. We denote by S(P) the minimal such sector. Thus, although it is not necessarily true that every point in S(P) is the image of some vertex in G_P, it is the case that the left and right ends of S(P) belong to $\theta(G_P)$. We denote these by $\lambda(P)$ and $\rho(P)$ respectively.

Lemma 4.5

Given any pair P of elements of I^7, there is a pair Q disjoint from P, such that $S(P) \cap S(Q) = \varnothing$.

Proof

Without loss of generality let $P = \{1,2\}$. Also let $\lambda(\{1,2\}) = \theta(\{1,2,w\})$,

$\rho(\{1,2\})) = \theta(\{1,2,x\})$ and let $y,z \in I^7$ be distinct from each other and from

1, 2, w, x. We shall show that $S(\{1,2\}) \cap S(\{y,z\}) = \varnothing$.

By way of contradiction suppose the sectors $S(\{1,2\})$ and $S(\{y,z\})$ intersect. Then all the elements of $S(\{y,z\})$ must be of distance ≤ 11 from one of the ends of $S(\{1,2\})$. Now $\theta(\{w,y,z\})$, $\theta(\{x,y,z\}) \in S(\{y,z\})$. But $d(\{1,2,w\}, \{x,y,z\}) \geq 12$ and $d(\{1,2,x\}, \{w,y,z\}) \geq 12$; giving a contradiction.

■

Lemma 4.6

The sectors $\{S(\{1,x\}) : x \in I^7 - \{1\}\}$ mutually intersect non-trivially.

Proof

We merely note that if 1, x and y are distinct, then $\theta(\{1,x,y\}) \in S(\{1,x\}) \cap S(\{1,y\})$.

■

It follows that the union of the sectors $S(\{1,x\})$ must be a sector of length at most 22, and must indeed be the union of two particular sectors, say $S(\{1,w\})$ and $S(\{1,z\})$. We denote this sector by $S(1) = [\lambda(1), \rho(1)]$.

Lemma 4.7

There exist distinct elements a, b, c, $d \in I^7$ such that $\lambda(1) = \theta(\{1,a,b\})$, $\rho(1) = \theta(\{1,c,d\})$.

Proof

Now, there are at least two values of x such that $\lambda(1) = \lambda(\{1,x\})$. Let a be that value that maximises the length of $S(\{1,x\})$. Similarly, of all the values of x such that $\rho(1) = \rho(\{1,x\})$, let d be that which maximises the length of $S(\{1,x\})$. It follows that

$$S(1) = S(\{1,a\}) \cup S(\{1,d\}).$$

And so there are vertices $\{1,a,b\}$, $\{1,c,d\}$ of G_3^7 such that $\lambda(1) = \theta(\{1,a,b\})$ and $\rho(1) = \theta(\{1,c,d\})$; thus $a \neq b$ and $c \neq d$. By Lemma 4.5, there is some pair $\{x,y\}$ such that $S(\{x,y\}) \cap S(\{1,a\}) = \varnothing$; but $\theta(\{1,x,y\}) \in S(1) \cap S(\{x,y\})$, and so $S(\{1,d\})$ cannot be a subset of $S(\{1,a\})$. In particular, $\theta(\{1,c,d\}) \notin S(\{1,a\})$. Also, by our maximising choice of a, $\theta(\{1,c,d\}) \notin S(\{1,b\})$. Thus, c and d are distinct from a and b.

■

Proof of Theorem 4.3

Let a, b, c, d be as in Lemma 4.7, and consider the sector $S(\{e,f\})$ where e and f are distinct from 1, a, b, c, d. Now the vertex $\{a, e, f\}$ is adjacent to $\{1, c, d\}$ and $\{d, e, f\}$ is adjacent to $\{1, a, b\}$. Thus $S(\{e,f\})$ contains a point of distance at least 12 from $\lambda(1)$ and also a point of distance at least 12 from $\rho(1)$. Since its length is at most 11, it follows that neither $\lambda(1)$ nor $\rho(1)$ can belong to $S(\{e,f\})$.

Now $\theta(\{1,e,f\}) \in S(1) \cap S(\{e,f\})$ and so it follows that $S(\{e,f\})$ lies wholly within $S(1)$. But $\theta(\{1,a,d\}) \in S(\{1,a\}) \cap S(\{1,d\})$ and $\theta(\{c,e,f\})$ is a point of $S(\{e,f\})$ that must be of distance of at least 12 from $\theta(\{1,a,d\})$. This is clearly impossible and gives the required contradiction.

■

Corollary 4.2 and Theorem 4.3 show the circular chromatic numbers of the graphs G_2^5 and G_3^7 are equal to 3. This raises the question whether the circular chromatic numbers of all the Kneser graphs of this form, namely G_k^{2k+1}, are also equal to 3. The following section shows this to be the case. The proofs give further insight into these types of graphs. The circular chromatic number of the Kneser graphs G_k^{2k+2} is also computed.

4.4 The Graphs G_k^{2k+1} and G_k^{2k+2}

The following Theorem and proof have been included in [10].

Theorem 4.4

For all $k \geq 1$

(i) $\chi_c(G_k^{2k+1}) = 3$.

(ii) $\chi_c(G_k^{2k+2}) = 4$.

Proof

As already observed, Theorem 4 of [17] gives an upper bound for $\chi_c(G_k^{2k+r})$ as $\chi(G_k^{2k+r})$. However, as pointed out in the introduction Lovasz [13], showed that $\chi(G_k^n) = n - 2k + 2$, giving $\chi(G_k^{2k+r}) = r + 2$. We shall denote this number by χ.

Suppose that $r = 1$ or 2 and that $\chi_c(G_k^{2k+r}) < \chi$; then there is a homomorphism $\theta : G_k^{2k+r} \rightarrow H_d^n$ for some $n, d \in Z^+$ such that $\frac{n}{d} < \chi$.

For any directed edge we define $e = (u,v)$ of G_k^{2k+r}, we define

$$\delta(e) = \frac{\Gamma_n(\theta(v) - \theta(u))}{n}.$$

Now since θ is a homomorphism, then $\Gamma_n(\theta(v) - \theta(u))$ lies between d and $n - d$ and hence:

$$\frac{1}{\chi} < \delta(e) < \frac{\chi - 1}{\chi}. \qquad \textbf{(5)}$$

Moreover, if $e' = (v,u)$, then

$$\delta(e) + \delta(e') = 1. \qquad \textbf{(6)}$$

For any directed cycle C of G_k^{2k+r}, we define the ***winding number***

$$w(C) = \sum_{e \in C} \delta(e).$$

Clearly $w(C) \in Z^+$ and, denoting the order of C by $|C|$, (5) implies:

$$\frac{|C|}{\chi} < w(C) < \frac{(\chi-1)|C|}{\chi}. \qquad \textbf{(7)}$$

Moreover, if C' is the cycle C traversed in the reverse direction, then by (6):

$$w(C) + w(C') = |C|. \qquad \textbf{(8)}$$

For ease of notation we shall use Γ to mean Γ_{2k+1}.

Let $\mathbf{p} = (p_1, p_2, \ldots\ldots, p_{2k+1})$ be any ordering of the elements of I^{2k+1} (in the case $r = 1$), or of all but one of the elements of I^{2k+2} (in the case $r = 2$). Then $\mathbf{p}$ defines a directed cycle $C(\mathbf{p})$ in G_k^{2k+r} of order $2k+1$, the vertices of which are in order:

$$v_1(\mathbf{p}) = \{p_1, p_2, \ldots\ldots, p_k\}, v_2(\mathbf{p}) = \{p_{k+1}, p_{k+2}, \ldots\ldots, p_{2k}\},$$

$$v_3(\mathbf{p}) = \{p_{2k+1}, p_1, p_2 \ldots\ldots, p_{k-1}\}, v_4(\mathbf{p}) = \{p_k, p_{k+1}, \ldots\ldots, p_{2k-1}\}, \ldots\ldots,$$

$$v_i(\mathbf{p}) = \{p_{\Gamma((i-1)k+1)}, p_{\Gamma((i-1)k+2)}, \ldots\ldots, p_{\Gamma(ik)}\}, \ldots\ldots,$$

$$v_{2k}(\mathbf{p}) = \{p_2, p_3 \ldots\ldots, p_{k+1}\},\ v_{2k+1}(\mathbf{p}) = \{p_{k+2}, p_{k+3}, \ldots\ldots, p_{2k+1}\}.$$

The argument now splits into two cases, depending on the value of r.

Case (i). $r = 1$.

In this case $\chi = 3$. Let the ordering **q** differ from **p** by a *transposition*; that is for some s, we have:

$$q_s = p_{s+1},\ q_{s+1} = p_s, \text{ while } q_i = p_i \text{ otherwise.}$$

Assume (as we may do without loss of generality) that $s = 2k$. Then $C(\mathbf{p})$ and $C(\mathbf{q})$ differ as follows:

$$v_i(\mathbf{p}) \neq v_i(\mathbf{q}) \quad (i = 2, 3)$$

all other vertices being in common to both cycles. Thus, $C(\mathbf{p})$ has directed edges

$$e_1 = (v_1(\mathbf{p}), v_2(\mathbf{p})) = (\{p_1, p_2, \ldots\ldots, p_k\}, \{p_{k+1}, p_{k+2}, \ldots\ldots, p_{2k-1}, p_{2k}\})$$

$$e_2 = (v_2(\mathbf{p}), v_3(\mathbf{p})) = (\{p_{k+1}, p_{k+2}, \ldots\ldots, p_{2k}\}, \{p_{2k+1}, p_1, p_2 \ldots\ldots, p_{k-1}\})$$

$$e_3 = (v_3(\mathbf{p}), v_4(\mathbf{p})) = (\{p_{2k+1}, p_1, p_2 \ldots\ldots, p_{k-1}\}, \{p_k, p_{k+1}, \ldots\ldots, p_{2k-1}\})$$

while $C(\mathbf{q})$ has directed edges

$$f_1 = (v_1(\mathbf{q}), v_2(\mathbf{q})) = (\{p_1, p_2, \ldots\ldots, p_k\}, \{p_{k+1}, p_{k+2}, \ldots\ldots, p_{2k-1}, p_{2k+1}\})$$

$$f_2 = (v_2(\mathbf{q}), v_3(\mathbf{q})) = (\{p_{k+1}, p_{k+2}, \ldots\ldots, p_{2k-1}, p_{2k+1}\}, \{p_k, p_1, p_2 \ldots\ldots, p_{k-1}\})$$

$$f_3 = (v_3(\mathbf{q}), v_4(\mathbf{q})) = (\{p_{2k}, p_1, p_2 \ldots\ldots, p_{k-1}\}, \{p_k, p_{k+1}, \ldots\ldots, p_{2k-1}\})$$

all other directed edges being in common.

An example, with $k = 3$ and with $\mathbf{p} = (7,5,4,1,2,3,6)$ and $\mathbf{q} = (7,5,4,1,2,6,3)$, is shown in Figure 4.1.

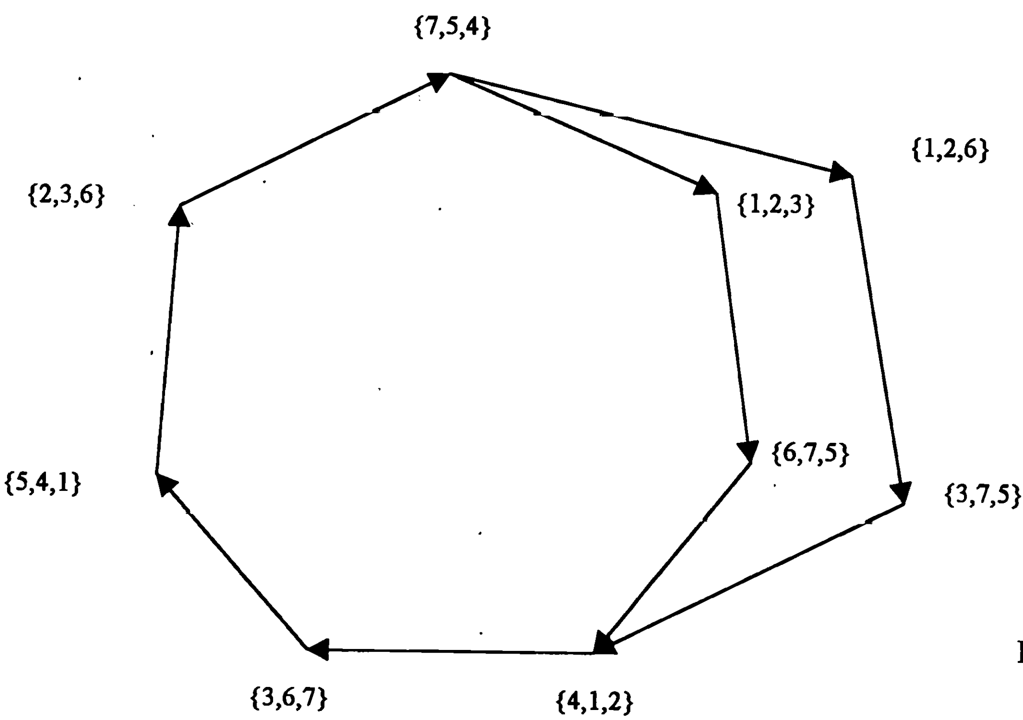

Figure 4.1

Thus,

$$w(C(\mathbf{p})) - w(C(\mathbf{q})) = \sum_{i=1}^{3}\delta(e_i) - \sum_{i=1}^{3}\delta(f_i).$$

But as $\chi = 3$, (5) implies that the sums $\sum_{i=1}^{3}\delta(e_i)$ and $\sum_{i=1}^{3}\delta(f_i)$ each lie strictly between 1 and 2. As $w(C(\mathbf{p}))$ and $w(C(\mathbf{q}))$ are integers, the only possible conclusion is that

$$w(C(\mathbf{p})) = w(C(\mathbf{q})).$$

The above argument is valid for any pair **p**, **q** of orderings of I^{2k+1} that differ by a transposition; but any ordering can be converted into any other by a succession of transpositions, and so $w(C(\mathbf{p}))$ is independent of **p**. In particular,

$$w(C(\mathbf{p})) = w(C(\mathbf{p}'))$$

where $\mathbf{p}'$ is the reversal of **p**. Thus, by (8)

$$w(C(\mathbf{p})) = \frac{|C(\mathbf{p})|}{2} = k + \frac{1}{2},$$

contradicting the fact that $w(C(\mathbf{p}))$ is an integer. Thus our supposition that

$\chi_c(G_k^{2k+1}) < \chi = 3$ is false.

Case (ii). $r = 2$.

In this case $\chi = 4$. Let t be the element of I^{2k+2} not involved in the ordering **p**, and let **q** differ from **p** by a ***switch***; that is, for some s we have

$$q_s = t, \text{ while } q_i = p_i \text{ otherwise.}$$

Assume (as we may do without loss of generality) that $s = 2k+1$.

Comparing the vertices of the cycles $C(\mathbf{p})$ and $C(\mathbf{q})$, we note that

$$v_1(\mathbf{p}) = v_1(\mathbf{q})$$

$$v_{2i+1}(\mathbf{p}) \neq v_{2i+1}(\mathbf{q}) \qquad (i = 1,2,\ldots,k)$$

$$v_{2i}(\mathbf{p}) = v_{2i}(\mathbf{q}) \qquad (i = 1,2,\ldots,k)$$

Thus, $C(\mathbf{p})$ and $C(\mathbf{q})$ have only one edge in common. An example with $k = 3$ and with $t = 6$, $\mathbf{p} = (8,3,5,4,7,1,2)$ and $\mathbf{q} = (8,3,5,4,7,1,6)$, is shown in Figure 4.2.

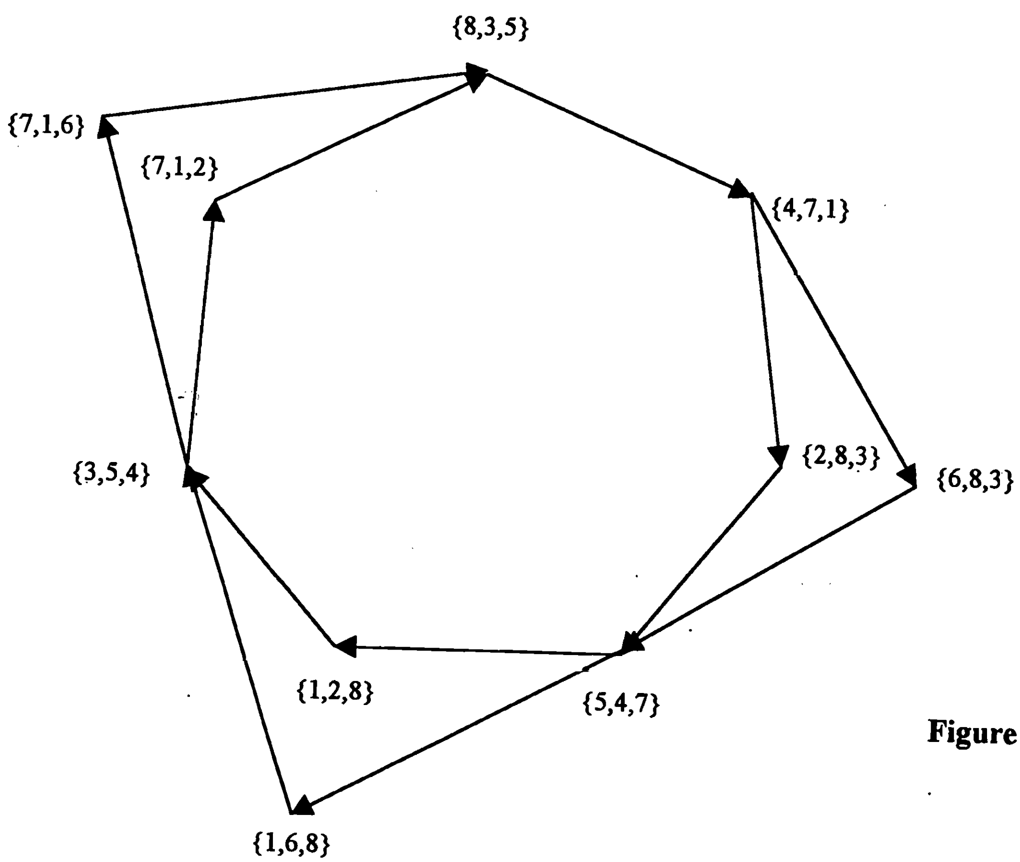

Figure 4.2

We may define a sequence $(C_1, C_2, \ldots\ldots, C_{k+1})$ of $(2k+1)$-cycles, with $C_1 = C(\mathbf{p})$ and $C_{k+1} = C(\mathbf{q})$ and each cycle differing from its predecessor by a switch, as follows: All the cycles have the vertices $v_1(\mathbf{p})$ and $v_{2i}(\mathbf{p})$ $(i = 1,2,\ldots,k)$ in common, while for $j = 2,3,\ldots\ldots,n$, the cycle C_j has vertices $v_i(\mathbf{q})$ $(i = 3,5,\ldots,2j-1)$ and the vertices $v_i(\mathbf{p})$ $(i = 2j+1,\ldots\ldots,2k+1)$.

Thus, for $j = 1,2,\ldots\ldots,k$ the cycles C_j and C_{j+1} differ as follows:

C_j has directed edges

$$e_1 = (v_{2j}(\mathbf{p}), v_{2j+1}(\mathbf{p})), \; e_2 = (v_{2j+1}(\mathbf{p}), v_{2j+2}(\mathbf{p}))$$

while C_{j+1} has directed edges

$$f_1 = (v_{2j}(\mathbf{p}), v_{2j+1}(\mathbf{q})), \; f_2 = (v_{2j+1}(\mathbf{q}), v_{2j+2}(\mathbf{p})).$$

But as $\chi = 4$, (5) implies that $\delta(e_1) + \delta(e_2)$ and $\delta(f_1) + \delta(f_2)$ each lie strictly between $\frac{1}{2}$ and $\frac{3}{2}$. Arguing as in Case (i), we conclude that $w(C_j) = w(C_{j+1})$, and hence,

$$w(C(\mathbf{p})) = w(C(\mathbf{q})).$$

Now any ordering of any $(2k+1)$-sets of I^{2k+2} may be converted to any other by a succession of switches, and so arguing as in Case (i),

$$w(C(\mathbf{p})) = k + \frac{1}{2},$$

giving the same contradiction as in Case (i) and showing that the supposition

$\chi_c(G_k^{2k+2}) < \chi = 4$ is false.

■

I am thankful to my supervisor for the elegant proof of case (i) which I have extended to prove case (ii).

Conclusion

It is shown in [8] that that $\chi_c(G_2^n) = \chi = n - 2$, and this together with Theorem 4.2 shows that the circular chromatic numbers of the Kneser graphs G_k^{2k+1}, G_k^{2k+2} $(k \geq 1)$, and G_2^n $(k \geq 4)$ are equal to their respective chromatic numbers. We conjecture this to be the case for any Kneser graph, namely that:

Conjecture

For every Kneser Graph, G_k^n $(k \geq 1, n \geq 2k)$,

$$\chi_c(G_k^n) = \chi(G_k^n) = n - 2k + 2.$$

CHAPTER 5

Combined k-Tuple and Z_n-Colourings

In Chapter 4 we posed the question whether, as an extension to Theorem 1 of [17], that if $n = \chi_k(G)$ then $\eta_n(G) = \frac{n}{k}$ (statement 1). This was found to be true for bipartite, complete graphs and odd cycles. However, it was found that it did not apply in general to any graph (Theorem 4.2).

In this Chapter we shall show that by combining both k-tuple colourings and Z_n-colourings into a single colouring ($Z_{n,k}$-colouring), a generalisation to Theorem 1 of [17] is obtainable:

Theorem 5.1

Let $k,m \in Z^+$ and $n = \chi_k^m(G)$, then $\eta_{n,k}(G) = \dfrac{\chi_k^m(G)}{m}$.

The proof of this Theorem relies on the following Lemma.

Lemma 5.1

Let $M_{n,k}(G) \geq 2$, then

$$M_{n\text{-}1,k}(G) \geq M_{n,k}(G) - 1.$$

Proof

For some $Z_{n,k}$-colouring, θ, $\mu_1(\theta) = M_{n,k}(G)$.

Define the $Z_{n-1,k}$-colouring θ' as follows:

For $u \in V(G)$,

$$\theta'(u) = \begin{cases} \theta(u) & \text{if } n \notin \theta(u) \\ (\theta(u) - n) \cup \{r\} & \text{if } n \in \theta(u) \end{cases}$$

where $r \notin \theta(u)$ such that $|r - s|_n = 1$ for some $s \in \theta(u)$.

We show that $\mu_1(\theta') \geq M_{n,k}(G) - 1$.

Consider

$\mu_1(\theta') = |x - y|_{n-1}$ for some $x \in \theta'(u)$, $y \in \theta'(v)$ and $uv \in E(G)$.

There are two cases to consider depending whether neither x nor y is equal to r, or one of x or $y = r$. Note that it is impossible for both x and y to equal r, since at most one of u and v can have n as a colour.

Suppose neither x nor y is equal to r. In this case $x \in \theta(u)$ and $y \in \theta(v)$. Thus,

$\mu_1(\theta') = |x-y|_{n-1} = \min\{\, |x-y|, n-1- |x-y|\} \geq \mu_1(\theta) - 1 = M_{n,k}(G) - 1.$

If one of x or $y = r$, say $y = r$ then $x \in \theta(u)$ and $n \in \theta(v)$. Thus,

$\mu_1(\theta') = |x-r|_{n-1} = \min\{\, |x-r|, n-1- |x-r|\}.$

Now $|x-r| \geq |x-s| - |r-s| = |x-s| - 1 \geq \mu_1(\theta) - 1 = M_{n,k}(G) - 1$.

Next consider $n - 1 - |x-r|$.

Now if $x < r$ then $|x-r| \leq |n-x|$ and so $n - 1 - |x-r| \geq \mu_1(\theta) - 1 = M_{n,k}(G) - 1$,

whereas if $x > r$ then $|x-r| \leq |x|$ and so $n - 1 - |x-r| \geq \mu_1(\theta) - 1 = M_{n,k}(G) - 1$.

Thus both cases yield $\mu_1(\theta') \geq M_{n,k}(G) - 1$.

Finally, we have $M_{n\text{-}1,k}(G) \geq \mu_1(\theta') \geq M_{n,k}(G) - 1$, from which the result follows.

■

Proof of Theorem 5.1

Recalling that $\chi_k^m(G)$ is the smallest value of n such that G can $Z_{n,k}$-coloured with $\mu_1(\theta) \geq m$, we have $\eta_{n,k}(G) = \dfrac{n}{M_{n,k}(G)} \leq \dfrac{n}{m} = \dfrac{\chi_k^m(G)}{m}$.

We now proceed and establish the reverse inequality. Assume to the contrary that there is some $Z_{n,k}$-colouring, θ such that

$\dfrac{n}{\mu_1(\theta)} = \dfrac{n}{M_{n,k}(G)} < \dfrac{n}{m}$. Then $M_{n,k}(G) > m$ and so by Lemma 5.1 $M_{n\text{-}1,k}(G) \geq m$.

It follows that there is some $Z_{n-1,k}$-colouring, θ' such that $\mu_1(\theta') \geq m$. Thus, $\chi_k^m(G) \leq n - 1$; giving a contradiction.

■

The case $m = 1$ gives the following corollary.

Corollary 5.1

If $n = \chi_k(G)$, then $\eta_{n,k}(G) = \chi_k(G)$.

Thus n^k-chromatic numbers, $\eta_{n,k}(G)$, generalise the k^{th}-chromatic numbers, $\chi_k(G)$.

Theorem 5.2

Let n, k and $m \in Z^+$ such that , $n \geq \chi_k(G)$ and $m \leq k-1$. Then,

$$\eta_{n,m}(G) \leq \eta_{n,m+1}(G).$$

Proof

For some $Z_{n,m+1}$-colouring, θ, $\mu_1(\theta) = M_{n,m+1}(G) = |x-y|_n$ for some $x \in \theta(u)$, $y \in \theta(v)$ and $uv \in E(G)$. Removing any one of the colours of θ from each vertex defines a $Z_{n,m}$-colouring, say θ'. Thus,

$\mu_1(\theta) = |x-y|_n \leq \mu_1(\theta') \leq M_{n,m}(G)$ from which the result follows.

■

In Chapter 1 we defined (k, d_1, d_2, n)-colouring of a graph to be a $Z_{n,k}$-colouring θ, such that $\mu_1(\theta) \geq d_1$ and $\mu_2(\theta) \geq d_2$. A (k, 1, 1 ,n)-colouring of a graph is simply a k-tuple colouring ($n \geq \chi_k(G)$). Using the alternative reformulation (AF1 of [16]), it is a homomorphism $\theta : G \to G_k^n$. Similarly a (n, 1, d ,k)-colouring of G is a homomorphism $\theta : G \to {}_dS_k^n$.

Lemma 5.2

Let $\frac{n}{d} \leq \frac{n'}{d'}$ and $x, y \in Z_n$. If $|x-y|_n \geq d$ then

$$\left\| \left\lfloor \frac{n'}{n}x \right\rfloor - \left\lfloor \frac{n'}{n}y \right\rfloor \right\|_{n'} \geq d'.$$

Proof

The proof is analogous to that of Proposition 1 of [3].

Let $a = \left\lfloor \frac{n'}{n}x \right\rfloor$ and $b = \left\lfloor \frac{n'}{n}y \right\rfloor$.

Without loss of generality assume $x > y$; then $d \leq x - y \leq n - d$.

Therefore,

$$b + d' = \left\lfloor \frac{n'}{n}y \right\rfloor + d' \leq \left\lfloor \frac{n'}{n}y \right\rfloor + \left\lfloor \frac{n'}{n}d \right\rfloor \leq \left\lfloor \frac{n'}{n}(y+d) \right\rfloor \leq \left\lfloor \frac{n'}{n}x \right\rfloor = a.$$

Also,

$$a = \left\lfloor \frac{n'}{n}x \right\rfloor \leq \left\lfloor \frac{n'}{n}(y+n-d) \right\rfloor \leq \left\lfloor \frac{n'}{n}y + n' - d' \right\rfloor = \left\lfloor \frac{n'}{n}y \right\rfloor + n' - d' = b + n' - d'.$$

Combining the inequalities gives

$$b + d' \le a \le b + n' - d'$$

and so

$$d' \le a - b \le n' - d'; \quad \text{giving } |a-b|_{n'} \ge d'$$

■

We now give an analogue to Proposition 1 of [3] in this more general footing of (n, d_1, d_2, k)-colourings.

Theorem 5.3

Let n, d_1, d_2, n', d_1', d_2', and $k \in Z^+$ such that G has a (n, d_1, d_2, k)-colouring, θ, where $\frac{n}{d_1} \le \frac{n'}{d_1'}$ and $\frac{n}{d_2} \le \frac{n'}{d_2'}$; then G also has a (n', d_1', d_2', k)-colouring.

Proof

Let θ be a (n, d_1, d_2, k)-colouring of G. Define the mapping

$\theta' : V(G) \rightarrow k$-element subsets of $Z_{n'}$

$$\theta'(u) = (\left\lfloor \frac{n'}{n} u_1 \right\rfloor, \left\lfloor \frac{n'}{n} u_2 \right\rfloor, \ldots\ldots\ldots, \left\lfloor \frac{n'}{n} u_k \right\rfloor) \quad (u_i \in \theta(u)).$$

Now $\left\lfloor \frac{n'}{n} u_i \right\rfloor \le \left\lfloor \frac{n'}{n} n \right\rfloor = n'$ for all $1 \le i \le k$. As will be shown later $\mu_2(\theta') \ge d_2'$.

And so the elements of $\theta'(u)$ are distinct. Thus, θ' is indeed a mapping into k-element subsets of $Z_{n'}$.

We next show that $\mu_1(\theta') \geq d_1'$ and $\mu_2(\theta') \geq d_2'$.

For the former we have for some $x \in \theta(u)$, $y \in \theta(v)$ and $uv \in E(G)$,

$\mu_1(\theta') = \left| \left\lfloor \frac{n'}{n}x \right\rfloor - \left\lfloor \frac{n'}{n}y \right\rfloor \right|_{n'}$. By definition $|x-y|_n \geq \mu_1(\theta) \geq d_1$ and it follows

by Lemma 5.2 that $\mu_1(\theta') \geq d_1'$. Similarly, for the latter we have for some

$e, f \in \theta(u)$ and $u \in V(G)$, $\mu_2(\theta') = \left| \left\lfloor \frac{n'}{n}e \right\rfloor - \left\lfloor \frac{n'}{n}f \right\rfloor \right|_{n'}$ and since $|e-f|_n \geq d_2$,

the result again follows by Lemma 5.2.

■

In a similar way as with the circular chromatic number, $\chi_c(G)$ for (n,d)-colourings, we define the ***k-circular chromatic number***, ${}_{k,d_2}\chi_c(G)$, for (n, d_1, d_2, k)-colourings :

$${}_{k,d_2}\chi_c(G) = \inf\{\ \eta_{n,k}(G) : n \in Z^+\} = \inf\{\ \frac{n}{d_1} :\ G \text{ has a } (n,\ d_1, d_2, k)\text{-colouring }\}.$$

Finding analogues to Theorem 3 of [17] and Corollary 2 of [3] for these numbers is complex in this general form. However, if $d_1 = d_2$ such an analogue is obtainable. For these (n, d, d, k)-colourings, we define the ***k1-circular chromatic number***, ${}_k^1\chi_c(G)$,

$${}_k^1\chi_c(G) = \inf\{\ \frac{n}{d} :\ G \text{ has a } (n,\ d,\ d,\ k)\text{-colouring }\}.$$

Let $G[K_k]$ be the lexicographic product of G with K_k. That is, $G[K_k]$ is the graph obtained by replacing each vertex of the graph G with the complete graph K_k, such that whenever vertices u and v are adjacent in G, then every vertex of each of the two copies of K_k are adjacent (see [5] and [6]).

Lemma 5.3

Let $b = |V(G)|$, then

$$ {}^{1}_{k}\chi_c(G) = \min_{1 \le n \le kb} \{ \frac{n}{d} : G \text{ has a } (n, d, d, k)\text{-colouring} \}. $$

Proof

Clearly G has a (n, d, d, k)-colouring iff $G[K_k]$ has a (n,d)-colouring. Now $G[K_k]$ has kb vertices and the result immediately follows from Corollary 2 of [3].

■

We generalise (n, d, d, k)-colourings and consider (n, pd, d, k)-colourings ($p \ge 1$). For these (n, pd, d, k)-colourings, we similarly define ***kp-circular chromatic number***, ${}^{p}_{k}\chi_c(G)$,

$$ {}^{p}_{k}\chi_c(G) = \inf\{ \frac{n}{d} : G \text{ has a } (n, pd, d, k)\text{-colouring} \}. $$

$$ {}^{p}_{k}\chi_c(G) = \inf\{ \frac{n}{d} : G \text{ has a } (n, pd, d, k)\text{-colouring} \}. $$

Theorem 5.4

Let G be a connected graph. If G has a (n, pd, d, k)-colouring then $G[K_{k+p-1}]$ has a (n,d)-colouring.

Proof

Let u be a vertex of G and K^u_{k+p-1} be the corresponding complete graph in $G[K_{k+p-1}]$. Similar to the construction on page 87 of section 4.4, we define a sector of Z_n to be a proper subset of I^n of the form $[a, \Gamma_n(a+s)] = \{\Gamma_n(a+i) : 0 \le i \le s\}$, where $a \in I^n$ and $s < n$. Then a and $\Gamma_n(a+s)$ are the left and right ends respectively of $[a, \Gamma_n(a+s)]$ and the length of the sector is s.

Now let θ be a (n, pd, d, k)-colouring of G. Let $\{T_u^{(1)}, T_u^{(2)}, \ldots\ldots, T_u^{(q)}\}$ (where q depends on u) be the set of all maximal sectors of length at least $2(pd-1)$ and containing no element of $\theta(u)$. (That is, each such sector is long enough to contain at least one element distant $\ge pd$ from any point in $\theta(u)$). Since G does not contain an isolated vertex, then there must always be at least one such sector, or the colouring can't be done. Assuming that $T_u^{(1)}, T_u^{(2)}, \ldots\ldots, T_u^{(q)}$ are in cyclic order, then there are unique sectors in between, $S_u^{(1)}, S_u^{(2)}, \ldots\ldots, S_u^{(q)}$, such that:

(i) $\{T_u^{(1)}, S_u^{(1)}, T_u^{(2)}, S_u^{(2)}, \ldots\ldots, T_u^{(q)}, S_u^{(q)}\}$, in cyclic order, are a set of sectors that partition Z_n.

(ii) Each $S_u^{(i)} = [a_u^{(i)}, b_u^{(i)}]$ is such that $a_u^{(i)},\ b_u^{(i)} \in \theta(u)$.

(iii) No $[a_u^{(i)}, b_u^{(i)}]$ contains any element of $\theta(v)$ for any v adjacent to u.

Define a Z_n-colouring θ', of $G[K_{k+p-1}]$ as follows:

We take any one of the $S_u^{(i)}$, say $S_u^{(1)}$, and colour k vertices of K_{k+p-1}^u with those of $\theta(u)$, and the remaining $p - 1$ vertices with

$$\Gamma_n(b_u^{(1)} + d),\ \Gamma_n(b_u^{(1)} + 2d), \ldots\ldots\ldots, \Gamma_n(b_u^{(1)} + (p-1)d).$$

We need to establish that $|\theta'(x) - \theta'(y)|_n \geq d$ for all $xy \in E(G[K_{k+p-1}])$.

By construction of θ' it is sufficient to show that $|\theta'(x) - \theta'(y)|_n \geq d$ for all $x \in V(K_{k+p-1}^u)$, $y \in V(K_{k+p-1}^v)$ and $uv \in E(G)$.

Since all the sectors $\{S_u^{(i)}\}$ and $\{S_v^{(i)}\}$ must be of distance $\geq pd$ from each other, we only need to consider the sectors

$$S_u = S_u^{(1)} \cup [\Gamma_n(b_u^{(1)} + d),\ \Gamma_n(b_u^{(1)} + (p-1)d)] = [a_u^{(1)}, \Gamma_n(b_u^{(1)} + (p-1)d)]$$

and

$$S_v = S_v^{(1)} \cup [\Gamma_n(b_v^{(1)} + d),\ \Gamma_n(b_v^{(1)} + (p-1)d)] = [a_v^{(1)}, \Gamma_n(b_v^{(1)} + (p-1)d)]$$

where we have chosen $a_v^{(1)}$ to be the first element contained in $\theta(v)$ and to the right of the sector S_u. (Clearly, since v is adjacent to u the sector S_u does not contain any elements of $\theta(v)$).

By symmetry it is enough to consider the distance between the right end of S_u and the left end of S_v. That is we need to prove that $\left|a_v^{(1)} - \Gamma_n(b_u^{(1)} + (p-1)d)\right|_n \geq d.$

Without loss of generality we may assume,

$$a_v^{(1)} \geq \Gamma_n(b_u^{(1)} + (p-1)d) = b_u^{(1)} + (p-1)d.$$

In view of the fact that θ is a (n, pd, d, k)-colouring and u and v are adjacent then

$$pd \leq a_v^{(1)} - b_u^{(1)} \leq n - pd.$$

It follows that

$$d \leq a_v^{(1)} - (b_u^{(1)} + (p-1)d) \leq n - (2p-1)d \leq n - d.$$

■

Investigation into examples of (n,d)-colourings of $G[K_{k+p-1}]$ suggest the converse to Theorem 5.4, namely 'If $G[K_{k+p-1}]$ has a (n,d)-colouring then G has a (n, pd, d, k)-colouring' does also hold. This together with Corollary 2 of [3] gives the following conjecture.

Conjecture

Let $b = |V(G)|$, then

$$ {}^{p}_{k}\chi_c(G) = \min_{1 \le n \le b(k+p-1)} \left\{ \frac{n}{d} : G \text{ has a } (n, pd, d, k)\text{-colouring} \right\}. $$

Indeed Lemma 5.3 asserts this to be the case for $p = 1$.

CHAPTER 6

Circular Distance Graphs

and Subgraphs of Kneser Graphs

We begin this Chapter by showing that the circular distance graph, H_k^n is a subgraph of the Kneser graph G_k^n.

Lemma 6.1

For all $k \geq 1$, $n \geq 2k$, $H_k^n \subseteq G_k^n$.

Proof

Consider the constant-step subgraph ${}_{x_1}C_k^n$, induced by the displacement sequence $x_1 = \{1,1,1,\ldots\ldots,n-k+1\} \in \mathbf{S}$ (see section 3.10 for a reminder of constant-step subgraphs). Now two vertices v_{a,x_1} and v_{b,x_1} $(b > a)$ are adjacent if and only if $b - a$ is at least k and at most $n - k$. Hence ${}_{x_1}C_k^n$ is isomorphic to the graph formed by its 'first' elements where two such elements $a,b \in I^n$ are adjacent according to adjacency in H_k^n. Thus ${}_{x_1}C_k^n \cong H_k^n$.

■

6.1 Relation Between ${}_{x_d}C_k^n$, SP_k^n and H_k^n

In section 3.9 we defined and studied spaced subgraphs, SP_k^n. The following result asserts that these subgraphs are in fact isomorphic to the circular distance graph, $H_{k'}^{n'}$ (where from section 3.9 $n' = \frac{n}{q}$, $k' = \frac{k}{q}$ and $q = \gcd(n,k)$).

Theorem 6.1

For all $k \geq 1$, $n \geq 2k$, $SP_k^n \cong H_{k'}^{n'}$.

Before giving the proof, we introduce the mapping $\theta : V\left(SP_k^n\right) \to V\left(H_{k'}^{n'}\right)$ defined as follows:

for each $u = \{a_1, a_2, \ldots, a_k\} \in V(SP_k^n) = \{v_{a,s} : a \in I^{n'}\}$, $\theta(u) = \Gamma_{n'}\left(\sum_{i=1}^{k'} a_i\right)$.

The proof relies mostly on the following two Lemmas.

Lemma 6.2

The mapping, θ, is a homomorphism.

Proof

Let u and v be adjacent in SP_k^n. We need to show that $\theta(u)$ and $\theta(v)$ are adjacent in $H_{k'}^{n'}$. That is if $u \cap v = \varnothing$ then $|\theta(v) - \theta(u)|_{n'} \geq k'$.

Without loss of generality let u and v be distance 'a' apart, $u = \{x_1, x_2, ..., x_k\}$ where $x_i = id + \left\lfloor \frac{ir}{k} \right\rfloor$, and $v = \{\Gamma_n(x_1 + a), \Gamma_n(x_2 + a), ..., \Gamma_n(x_k + a)\}$.

Then $\theta(u) = \Gamma_{n'}\left(\sum_{i=1}^{k'} x_i\right)$ and $\theta(v) = \Gamma_{n'}\left(k'a + \sum_{i=1}^{k'} x_i\right)$, and so

$\theta(v) - \theta(u) = \Gamma_{n'}(k'a)$. Now let p be the non-negative integer such that

$k'a - pn' \in I^{n'}$. Then,

$\Gamma_{n'}(k'a) = k'a - pn'$, and $|\theta(v) - \theta(u)|_{n'} = \min\{(k'a - pn'), n' - (k'a - pn')\}$.

Now since $k'a - pn' \in I^{n'}$ and $a \neq n'$ (by Lemma 3.12, $a = n'$ would imply $u = v$), it follows that:

(i) $p < \frac{ka}{n} < \frac{k}{q}$ and so $p \leq k' - 1$.

(ii) $ka > pn = kpd + pr$ and so $a > pd + \frac{pr}{k}$.

(iii) $ka - pn < n$ and so, $ka < (p+1)n = (p+1)kd + (p+1)r$

giving $a < (p+1)d + (p+1)\frac{r}{k}$.

Now $\frac{pr}{k} = \frac{pr'}{k'}$ is not an integer because gcd $(k',r') = 1$ and from (i) $p < k'$. **(1)**

By way of contradiction, suppose that $|\theta(v) - \theta(u)|_{n'} < k'$, then either

$k'a - pn' < k'$ or $n' - (k' - pn') < k'$. That is $ka - pn < k$ or $n - (ka - pn) < k$.

Case 1. If $ka - pn < k$.

This gives $ka < k + pn = k + kpd + pr$ and so $a < 1 + pd + \frac{pr}{k}$. Combining this with (ii), gives

$$pd + \frac{pr}{k} < a < 1 + pd + \frac{pr}{k}.$$

But from (1), $\frac{pr}{k}$ is not an integer. Hence, $a = 1 + pd + \left\lfloor \frac{pr}{k} \right\rfloor$.

Consider the element $x_{k'-p} + a$ of vertex v, where $1 \le p \le k' - 1$.

$$x_{k'-p} + a = (k'-p)d + \left\lfloor \frac{(k'-p)r}{k} \right\rfloor + 1 + pd + \left\lfloor \frac{pr}{k} \right\rfloor$$

$$= k'd - pd + \left\lfloor r' - \frac{pr}{k} \right\rfloor + 1 + pd + \left\lfloor \frac{pr}{k} \right\rfloor$$

$$= k'd + r' + \left\lfloor -\frac{pr}{k} \right\rfloor + 1 + \left\lfloor \frac{pr}{k} \right\rfloor$$

$$= n' - \left\lfloor \frac{pr}{k} \right\rfloor - 1 + 1 + \left\lfloor \frac{pr}{k} \right\rfloor \qquad \text{(since } \frac{pr}{k} \text{ is not an integer)}$$

$$= n'.$$

But $n' = x_{k'} = k'^{th}$ element of u. It follows that $x_{k'} \in u \cap v$ and so giving the required contradiction.

Case 2. If $n - (ka - pn) < k$.

This gives $ka > pn + n - k = (p+1)kd + (p+1)r - k$, and so

$(p+1)d + (p+1)\frac{r}{k} - 1 < a$. Combining this with (iii), gives

$$(p+1)d + (p+1)\frac{r}{k} - 1 < a < (p+1)d + (p+1)\frac{r}{k}$$

from which it follows that

$$(p+1)d + \left\lfloor (p+1)\frac{r}{k} \right\rfloor \le a \le (p+1)d + \left\lfloor (p+1)\frac{r}{k} \right\rfloor.$$

Hence $a = (p+1)d + \left\lfloor (p+1)\frac{r}{k} \right\rfloor = x_{p+1}$, where $0 \le p \le k' - 1$.

Consider the k^{th} element, $\Gamma_n(x_k + a)$ of vertex v. Now

$\Gamma_n(x_k + a) = \Gamma_n(n + x_{p+1}) = x_{p+1}$.

That is the k^{th} element of v is precisely the $(p+1)^{th}$ element of u. Hence, $x_{p+1} \in u \cap v$, giving the required contradiction.

■

Lemma 6.3

(i) The mapping, θ , is injective.

(ii) Let $u, v \in V\left(SP_k^n\right)$. If $\theta(u)\theta(v)$ is an edge of $H_{k'}^{n'}$, then uv is an edge of SP_k^n.

Proof

Without loss of generality, let u and v be as in the proof of Lemma 6.2. To prove (i), we need to show that if $\theta(u)=\theta(v)$ then $u = v$. Now $\theta(v) - \theta(u) = \Gamma_{n'}(k'a) = k'a - pn'$.

It follows that since $\theta(u)=\theta(v)$, then $k'a = pn'$ and so $n' \mid k'a$. But $\gcd(n',k')=1$, hence $n' \mid a$. In view that $a \in I^{n'}$, it follows that $a = n'$ and so $u = v$.

For part (ii) we need to show that if $\left|\theta(v) - \theta(u)\right|_{n'} \geq k'$ then $u \cap v = \varnothing$.

Assume the contrary. Then there exist i, j $(1 \leq i, j \leq k)$ such that $x_j = \Gamma_n(x_i + a)$.

There are two cases to consider.

Case 1. If $x_i + a \leq n$.

$$\text{Then } a = x_j - x_i = jd + \left\lfloor \frac{jr}{k} \right\rfloor - id - \left\lfloor \frac{ir}{k} \right\rfloor$$

$$= (j-i)d + \left\lfloor \frac{jr}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor \tag{2}$$

$$\geq (j-i)d + \left\lfloor (j-i)\frac{r}{k} \right\rfloor. \tag{3}$$

Combining (3) with that of inequality (iii), gives

$$(j-i)d + \left\lfloor (j-i)\frac{r}{k} \right\rfloor \le a \le (p+1)d + \left\lfloor (p+1)\frac{r}{k} \right\rfloor. \qquad \textbf{(4)}$$

Hence, $\qquad j - i \le p + 1.$

Now if $j - i = p + 1$, then from (4),

$$a = (p+1)d + \left\lfloor (p+1)\frac{r}{k} \right\rfloor > (p+1)d + (p+1)\frac{r}{k} - 1, \text{ giving}$$

$ka > (p+1)d + (p+1)r - k = pn + n - k$. Thus $n - (ka - pn) < k$, and

$$|\theta(v) - \theta(u)|_{n'} = \min\{(k'a - pn'), n' - (k'a - pn')\} \le n' - (k'a - pn') < k',$$

giving a contradiction.

It follows that

$$j - i \le p. \qquad \textbf{(5)}$$

Now,

$$a = (j-i)d + \left\lfloor \frac{jr}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor$$

$$\le (j-i)d + \left\lfloor (j-i)\frac{r}{k} \right\rfloor + 1$$

and from (5) this gives

$$a \le pd + \left\lfloor \frac{pr}{k} \right\rfloor + 1$$

$$< pd + \frac{pr}{k} + 1 \qquad \text{(since } \frac{pr}{k} \text{ is not an integer).}$$

This gives

$$ka - pn < k.$$

Thus

$$\left|\theta(v) - \theta(u)\right|_{n'} = \min\{(k'a - pn'), n' - (k'a - pn')\} \le k'a - pn' < k',$$

giving a contradiction.

Case 2. If $x_i + a > n$ then (2) becomes

$$a = kd + r + (j-i)d + \left\lfloor \frac{jr}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor$$

$$= (k+j-i)d + \left\lfloor (k+j)\frac{r}{k} \right\rfloor - \left\lfloor \frac{ir}{k} \right\rfloor$$

and by substituting $j + k$ for j in (4), the argument follows as before.

■

Proof of Theorem 6.1

In view of the fact that both the graphs SP_k^n and $H_{k'}^{n'}$ contain the same number of vertices, the result follows from Lemmas 6.2 and 6.3.

■

In the proof of Lemma 6.1 we showed that the constant-step subgraph, ${}_{x_1}C_k^n$ is isomorphic to H_k^n. This raises a question whether all constant-step subgraphs are likewise isomorphic to H_k^n. The following confirms this to be the case.

Theorem 6.2

$$ {}_{x_d}C_k^n \cong H_k^n \text{ for all } x_d \in \mathbf{S}. $$

Proof

We first show $\{\Gamma_n(ad) : 1 \le a \le n\} = I^n$. Let $\Gamma_n(ad) = \Gamma_n(bd)$ with $1 \le a < b \le n$. Then $bd = ad + mn$ $(m < d)$. Since $\gcd(n,d) = 1$, it follows that $n \backslash b - a$ and so $b = a$.

In view of this we let $w_a = v_{\Gamma_n(ad), x_d}$ so that $\{w_a : 1 \le a \le n\} = V({}_{x_d}C_k^n)$.

We assume the vertices are placed around the circle in this cyclic order.

We next show for each $1 \le a \le n$, the set of vertices adjacent to w_a are.:

$$ X = \{w_{a+i} : k \le i \le n-k\} = \{w_{a+k}, w_{a+k+1}, \ldots\ldots\ldots, w_{a+n-k}\} $$

From this it would follow that ${}_{x_d}C_k^n \cong H_k^n$.

By symmetry it is enough to consider one vertex, say w_a and show that its 'neighbours' are precisely the vertices of X.

By way of contradiction suppose $w_a \cap w_{a+i} \neq \varnothing$ for some $k \leq i \leq n-k$.

Then $\Gamma_n(ad+pd) = \Gamma_n(ad+id+qd)$ for some $0 \leq p, q \leq k-1$, $(p \neq q)$.

It follows that $\Gamma_n(pd) = \Gamma_n(id+qd)$.

Now

$$p < i+q \leq n-k+k-1 < n .$$

Hence

$$pd + mn = id + qd \qquad \text{for some } m < d,$$

from which

$$nm = (i+q-p)d$$

and so $\qquad d \mid nm.$

But since $\qquad \gcd(n,d) = 1$, then $d \mid m$; giving a contradiction.

Thus w_a is adjacent to each of the n-$2k$+1 vertices of X.

Now for $1 \leq b \leq k-1$, the $(k+1-b)^{th}$ element of $w_{a+b+n-k}$ is $\Gamma_n(ad)$, the first element of w_a, whilst the $(k-b)^{th}$ element of w_{a+b} is $\Gamma_n((a+k-1)d)$, the last element of w_a.

It follows that for $1+n-k \leq i \leq n-1$ and for $1 \leq i \leq k-1$ that $w_a \cap w_{a+i} \neq \emptyset$, and so the vertices in **X** are the only 'neighbours' of w_a.

■

Theorems 6.1, 6.2 and 3.8 show that ${}_{x_\delta}C_k^n$ (of Theorem 3.8) and SP_k^n are both isomorphic to the circular distance graph $H_{k'}^{n'}$, whilst their 'colours' at every vertex are of maximum distance apart; that is they are also subgraphs of the Schrijver graph S_k^n. On the strength of this and consideration of examples we make the following conjecture.

Conjecture

(i) There exists a $x_\delta \in \mathbf{S}$ such that ${}_{x_\delta}C_k^n = SP_k^n$.

(ii) Let H be a subgraph of G_k^n isomorphic to H_k^n; then $H = {}_{x_d}C_k^n$ for some $x_d \in \mathbf{S}$.

(iii) The size of the family of subgraphs of G_k^n isomorphic to H_k^n is $\frac{\phi(n)}{2}$

(where ϕ is Euler's function).

Example

Figure 6.1 shows 3 'copies' of H_2^7 contained in G_2^7 together with their respective difference sequences.

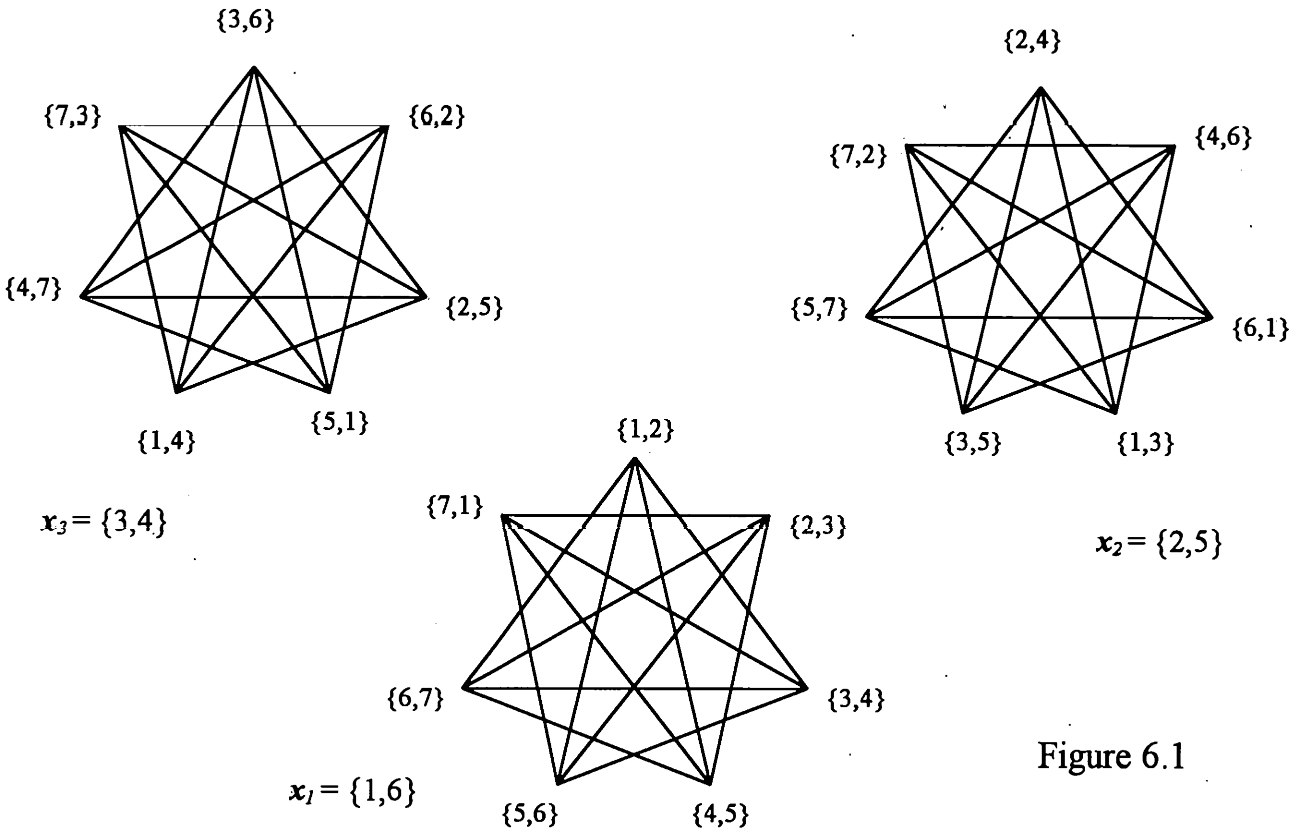

Figure 6.1

6.2 Properties of H_k^n

Since $H_{k'}^{n'} \cong SP_k^n$, Theorem 3.6 shows the independence number of $H_{k'}^{n'}$ is also equal to k'. This result readily extends to all integers $k \geq 1$, $n \geq 2k$.

Theorem 6.3 For all $k \geq 1$, $n \geq 2k$,

(i) $\alpha(H_k^n) = k$.

(ii) Every vertex of H_k^n is contained in a maximum independent set.

Proof.

Let X be an independent set, and a and b be the smallest and largest elements of X respectively. Now, since a and b are independent vertices of H_k^n, it follows that $|b-a|_n < k$. By symmetry, we can assume without loss of generality that $|b-a|_n = |b-a| = b-a$. Since all the vertices must lie between a and b, it follows there cannot be more than k vertices.

Finally, let a be any vertex of H_k^n, then $V_a = \{1_n(a+i) : 0 \leq i \leq k-1\}$ is an independent set containing a of cardinality k.

■

Since H_k^n has n vertices, Theorem 6.3 immediately gives the following result.

Corollary 6. 1 $\mu(H_k^n) = \frac{n}{k}$.

Theorem 6.4 For all $k \geq 1$, $n \geq 2k$,

$$\chi_f(H_k^n) = \frac{n}{k}.$$

Proof

By Lemmas 1.1, 3.5, 6.1 and Corollary 6.1, we have

$$\frac{n}{k} = \mu(H_k^n) \leq \chi_f(H_k^n) \leq \chi_f(G_k^n) = \frac{n}{k}.$$

■

By Theorem 6 of [17], the circular chromatic number of H_k^n is also $\frac{n}{k}$. Recalling that a graph G is star extremal if $\chi_f(G) = \chi_c(G)$, it follows that:

Corollary 6.2 The graphs H_k^n, $x_d C_k^n$ and SP_k^n are star extremal.

Theorem 6.5

Let n and k be positive integers such that $k \geq 1$, $n \geq 2k$. Then, $\chi_m(H_k^n) = \left\lceil \frac{mn}{k} \right\rceil$.

Proof

For convenience, we write χ_m to mean $\chi_m(H_k^n)$, and similarly for χ_f.

The method of proof is to show that $\chi_m \geq \left\lceil \frac{mn}{k} \right\rceil$ and then exhibit an m-tuple colouring using $\left\lceil \frac{mn}{k} \right\rceil$ colours.

By way of contradiction suppose $\chi_m < \left\lceil \frac{mn}{k} \right\rceil$.

Now if k divides mn, then $\chi_m < \frac{mn}{k}$. In view of the fact that $\chi_f = \frac{n}{k}$, it follows that $\frac{n}{k} = \chi_f \leq \frac{\chi_m}{m} < \frac{mn}{km} = \frac{n}{k}$; giving a contradiction. Whilst, if k does not divide mn, then

$$\chi_m \leq \left\lceil \frac{mn}{k} \right\rceil - 1 = \left\lfloor \frac{mn}{k} \right\rfloor < \frac{mn}{k}, \text{ and so } \frac{n}{k} = \chi_f \leq \frac{\chi_m}{m} < \frac{mn}{km} = \frac{n}{k}; \text{ again}$$

giving a contradiction.

Thus we conclude $\chi_m \geq \left\lceil \frac{mn}{k} \right\rceil$.

To complete the proof it is sufficient to demonstrate an m-tuple colouring of H_k^n using $\left\lceil \frac{mn}{k} \right\rceil$ colours.

To achieve this we use the reformulation of an m-tuple colouring given as definition AF1 of [16]. Letting $p = \left\lceil \frac{mn}{k} \right\rceil$, we show the existence of a homomorphism.

$$\phi : H_k^n \to G_m^p.$$

Now $\frac{n}{k} \leq \frac{p}{m}$ and so by Proposition 1 of [3] there is a homomorphism from H_k^n to H_m^p. Composing this with the homomorphism H_m^p to G_m^p constructed in the proof of Lemma 4.2 gives us the required homomorphism ϕ.

■

Of course, the result that the subgraphs ${}_{x_d}C_k^n$ and SP_k^n also possess this m^{th}-chromatic number follows immediately.

Criticality

We next show that when gcd(n,k) = 1, then H_k^n is both χ_f-critical and χ_c-critical in the sense that removing any vertex and its incident edges reduces their respective chromatic numbers.

Theorem 6.6

Let n and k be positive integers such that $n \geq 2k$ and gcd $(n,k) = 1$, then the graph H_k^n is χ_c-critical.

Proof

Let H be the subgraph of H_k^n, obtained by removing a vertex $v \in V\left(H_k^n\right) = I^n$ and its incident edges.

Now by remark 5 of [3] $\chi_c(H) \leq \chi_c\left(H_k^n\right) = \frac{n}{k}$. Also as H has $n - 1$ vertices, then there exist positive integers a and b, such that

$$\chi_c(H) = \frac{a}{b} \text{ where } a \leq n-1$$

Suppose that $\chi_c(H) = \frac{a}{b} = \frac{n}{k}$. But since gcd $(n,k) = 1$, then $a \geq n$; giving a contradiction.

■

Corollary 6.3

Let n and k be positive integers such that $n \geq 2k$ and gcd $(n,k) = 1$, then the graph H_k^n is χ_f-critical.

Proof

By Corollary 4.1(ii) $\chi_f(H_k^n) \leq \chi_c(H_k^n)$, and so the result immediately follows from Theorem 6.6.

■

Note that if gcd $(n,k) > 1$, then H_k^n contains a subgraph isomorphic to $H_{k'}^{n'}$. It follows that the condition gcd $(n,k) = 1$ in Theorem 6.6 and its Corollary is essential.

The subgraph of Figure 6.1 induced by the difference sequence $x_3 = \{3,4\}$, is also the subgraph SP_2^7 and that of S_2^7 of section 3.8. In that section we posed the question whether we can reduce the number of vertices of $S_k^{kd+1} (= {}_{x_d}C_k^{kd+1} = SP_k^{kd+1})$ while maintaining the same fractional number. As this graph is isomorphic to H_k^{kd+1}, then Corollary 6.3 indeed asserts the answer is no.

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GLOSSARY

circular chromatic number is defined as $\chi_c(G) = \inf\{\eta_n(G) : n \in Z^+\}$

circular distance between two elements *x*, *y* of I^n is $|x-y|_n$.

circular distance graph denoted by H_d^n, has vertex set Z_n and vertices *x* and *y* are adjacent iff $|x-y|_n \geq d$.

circular norm - Given $x \in Z_n$, we denote by $\Gamma_n(x)$ the integer representative of *x* belonging to I^n $(= \{x \in Z^+ : x \leq n\})$; if $x \subset Z$, we abbreviate $\Gamma_n(x(\text{mod}\, n))$ to $\Gamma_n(x)$. The circular norm is then defined as $|x|_n = \min\{\Gamma_n(x), n - \Gamma_n(x)\}$.

constant-step subgraph – Given a difference sequence x_d, the constant-step subgraph ${}_{x_d}C_k^n$, is the subgraph of G_k^n induced by the vertices of the form v_{a,x_d} for some $a \in I^n$.

cyclically equivalent – Given any $a \in I^n$ and any $\mathbf{d} \in {}_d\mathbf{D}$, we recall that $v_{a,\mathbf{d}}$ is the vertex of ${}_dS_k^n$ whose displacement sequence starting from a is $\mathbf{d}$. If $v_{a_1,\mathbf{d}_1} = v_{a_2,\mathbf{d}_2}$, then $\mathbf{d}_2$ has the same elements as $\mathbf{d}_1$ in the same cyclic order; that is, $\mathbf{d}_1$ and $\mathbf{d}_2$ are cyclically equivalent.

difference sequence is the k-sequence $x_d = \{d, d, \ldots\ldots\ldots, d, \Gamma_n((n-k+1)d)\}$ where $d \in \varepsilon(n)$.

difference set - Let $\mathbf{x} = \{d_1, d_2, \ldots\ldots\ldots, d_k\}$ be a displacement sequence. Its difference set is defined as $\mathrm{X} = \left\{ \sum_{i=p}^{p+q} d_{\Gamma_k(i)} : \ 1 \le p \le k, \ 0 \le q \le k-1 \right\}$

displacement sequence - Let $v = \{a_1, a_2, \ldots, a_k\}$ be a vertex of G_k^n. We use the convention that its elements are listed such that they are in the same cyclic order as the cyclic order obtained when they are written in monotone increasing order. Given any $a \in v$, list the elements of $v \in V(G_k^n)$, starting from a as $a_1, a_2, \ldots, a_k$ where $a_1 = a$. The displacement sequence of v starting from a is defined as the sequence $\mathbf{d} = \{d_1, d_2, \ldots, d_k\}$ where $d_i = \Gamma_n(a_{i+1} - a_i)$ $(1 \le i \le k-1)$ and $d_k = \Gamma_n(a_1 - a_k)$.

Euler set $\varepsilon(n)$ is the set of positive integers that are less than n and relatively prime to n.

fractional chromatic number of G, denoted by $\chi_f(G)$ is the $\inf\left\{\frac{\chi_m(G)}{m} : m \in Z^+\right\}$.

graph homomorphism, θ, is a mapping $\theta : G \to H$ such that $\theta(u)$ and $\theta(v)$ are adjacent in H whenever u and v are adjacent in G.

independence number, $\alpha(G)$, is the size of the largest independent set of vertices of G.

k^{th} chromatic number of G, denoted by $\chi_k(G)$, is the least number of colours needed for an k-tuple colouring of G.

k-circular chromatic number for (n, d_1, d_2, k)-colourings is defined as

$_{k,d_2}\chi_c(G) = inf\{ \eta_{n,k}(G) : n \in Z^+ \} = \inf\{ \frac{n}{d_1} : G$ has a (n, d_1, d_2, k)-colouring $\}$.

k1-circular chromatic number is defined as $^1_k\chi_c(G) = \inf\{ \frac{n}{d} : G$ has a (n, d, d, k)-colouring $\}$

k_m- chromatic number $\chi_k^m(G)$, is the smallest value of n such that G can be $Z_{n,k}$-coloured with $\mu_1(\theta) \geq m$, where $\mu_1(\theta) = \min\{ \left| u_i - v_j \right|_n : u_i \in \theta(u),\ v_j \in \theta(v),\ uv \in E(G)\}$.

Kneser graphs - Let $I^n = \{x \in Z^+ : x \leq n\}$, and I_k^n denote the family of subsets of I^n of cardinality k. For $k \geq 1$ and $n \geq 2k$ we define the Kneser graph, G_k^n whose vertex set is I_k^n, and two vertices are adjacent iff they are disjoint as subsets.

kp-circular chromatic number is defined as ${}_k^p\chi_c(G) = \inf\{ \frac{n}{d} : G$ has a (n, pd, d, k)-colouring $\}$

k-tuple colouring of G is an assignment of k distinct colours to each vertex such that no two adjacent vertices share a colour.

n-chromatic number of G - Let n be such that there exists at least one proper colouring of G (i.e. $\chi(G) \leq n$) and $d = \max\{ \delta : G$ has a (n, δ)-colouring$\}$. Then the n-chromatic number is defined as $\eta_n(G) = \frac{n}{d}$.

(n,d)-colouring of G is a Z_n-colouring θ such that $\mu(\theta) = \min|\theta(u)-\theta(v)|_n \geq d$ (where the minimum is taken over all pairs u, v of adjacent vertices).

(n, d_1, d_2, k)-colouring of a graph is a $Z_{n,k}$-colouring θ, such that $\mu_1(\theta) \geq d_1$ and $\mu_2(\theta) \geq d_2$, where $\mu_1(\theta) = \min\{|u_i - v_j|_n : u_i \in \theta(u),\ v_j \in \theta(v),\ uv \in E(G)\}$, $\mu_2(\theta) = \min\{|u_i - u_j|_n : u_i, u_j \in \theta(u),\ i \neq j,\ u \in V(G)\}$and $n \geq 2d_1 k$.

n^k-chromatic number of G - Let $C_{n,k}$ denote the set of all $Z_{n,k}$-colourings of G. Assume that $2k \leq \chi_k(G) \leq n$, so that $C_{n,k}$ contains at least one k-tuple colouring. Let $\mu_1(\theta) = \min\{|\theta(u_i) - \theta(v_j)|_n : u_i \in \theta(u),\ v_i \in \theta(v),\ uv \in E(G)\}$and

$M_{n,k}(G) = \max_{\theta \in C_{n,k}} \mu_1(\theta)$. The n^k-chromatic number of G is defined as

$$\eta_{n,k}(G) = \frac{n}{M_{n,k}(G)}.$$

overlap fractional chromatic number for (mq,q)-overlap colourings (for some integer $m > 1$) is defined as ${}_m\chi_f(G) = \inf\left\{\frac{{}_m\chi_q(G)}{mq} : q \in Z^+\right\}$ where

${}_m\chi_q(G) = \chi_{mq,q}(G)$, the smallest number of colours needed for a (mq,q)-overlap colouring.

(p,q)-chromatic number of G, denoted by $\chi_{p,q}(G)$, is the smallest number of colours needed for a (p,q)-overlap colouring.

(p,q)-overlap colouring is an assignment of p distinct colours to each vertex so that any pair of adjacent vertices share exactly q colours.

rotation subgraph - Given any displacement sequence, $\mathbf{d}$ of a vertex of the Schrijver graph, ${}_dS_k^n$, the rotation subgraph, ${}_\mathbf{d}R_k^n$, is defined to be the subgraph of ${}_dS_k^n$ induced by the vertices of the form $v_{a,\mathbf{d}}$ for some $a \in I^n$.

Schrijver graph – For $1 \le d \le \left\lfloor \frac{n}{k} \right\rfloor$, we define the d^{th} Schrijver graph as the induced subgraph, ${}_dS_k^n$ of G_k^n whose vertex set is

$$V({}_dS_k^n) = \{ v \in I_k^n \; : \; |i-j|_n \ge d \;\; (i,j \in v) \}.$$

spaced subgraph , SP_k^n is the rotation subgraph induced by the displacement sequence,

$\mathbf{s} = \{d_1,...,d_k\}$, where $d_j = d + \left\lfloor \frac{(j+1)r}{k} \right\rfloor - \left\lfloor \frac{jr}{k} \right\rfloor \quad (j = 1,.....,k)$.

star-extremal – A graph, G is star-extremal if its fractional and circular chromatic numbers are equal: $\chi_f(G) = \chi_c(G)$

subsequence of a displacement sequence , $\mathbf{x} = \{d_1, d_2,.........., d_k\}$ is a sequence of clockwise and 'consecutive' elements of **x**. That is a sequence of the form $\{d_{\Gamma(i)}, d_{\Gamma(i+1)},.........., d_{\Gamma(i+j)}\}$, for some $1 \le i \le k$ and $0 \le j \le k-1$. Here Γ is taken to mean Γ_k .

Z_n-colouring of a graph is a function $\theta : V(G) \rightarrow Z_n$.

$Z_{n,k}$-colouring of a non-null graph G is a k-tuple colouring of G using colours from Z_n $(n \ge 2k)$.