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Four-dimensional vector multiplets in arbitrary signature (II)

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Following the classification up to isomorphism of $\mathcal{N} = 2$ Poincaré Lie superalgebras in four dimensions with arbitrary signature obtained in a companion paper, we present off-shell vector multiplet representations and invariant Lagrangians realizing these algebras. By dimensional reduction of five-dimensional off-shell vector multiplets we obtain two representations in each four-dimensional signature. In Euclidean and neutral signature these representations can be mapped to each other by a field redefinition induced by the action of the Schur group on the space of superbrackets. In Minkowski signature we show that the superbrackets underlying the two vector multiplet representations belong to distinct open orbits of the Schur group and are therefore inequivalent. Our formalism allows to answer questions about the possible relative signs between terms in the Lagrangian systematically by relating them to the underlying space of superbrackets.

Keywords: Poincaré Lie superalgebras; extended supersymmetry; arbitrary signature.

1. Introduction

Four-dimensional $\mathcal{N} = 2$ Poincaré Lie superalgebras in arbitrary space-time dimension have been classified up to isomorphism in [1]. In this second part of a two-part series on four-dimensional supersymmetry in arbitrary signature, we present off-shell vector multiplet representations of all inequivalent four-dimensional $\mathcal{N} = 2$ supersymmetry algebras, together with the corresponding invariant Lagrangians. We obtain these results by dimensional reduction of the five-dimensional off-shell vector multiplets constructed in [2]. We will assume that the reader is familiar with the approach detailed in [1], in particular with the relation between super-admissible bilinear forms on the complex spinor module $\mathbb{S}$ and Poincaré Lie superalgebras. The structure of this article is as follows. In Section 2 we provide the necessary background on $\mathcal{N}=2$ supersymmetry algebras in five and four dimensions, as far as not already covered in [1]. We review the concept of doubled spinors, which are generalizations of symplectic Majorana spinors, and we show how four-dimensional supersymmetry algebras and the maps between them are expressed in this formalism. In Section 3 we review $\mathcal{N}=2$ vector multiplets in five and four dimensions. Then we present and discuss the four-dimensional off-shell representations and Lagrangians obtained by dimensional reduction. Section 4 gives an outlook on future research. Our conventions for Clifford algebras and γ-matrices are summarized in Appendix A together with a collection of the key formulae needed to carry out the dimensional reduction.

2. Spinors and Poincaré Lie superalgebras in five and four dimensions

2.1. *Super-admissible bilinear forms on* $\mathbb{S}$ *and the associated Schur algebras*

On the complex spinor module $\mathbb{S}$ one can always find matrices A and C which relate the γ-matrices to the Hermitian conjugate and transposed γ-matrices, respectively, as in A.1. The matrices A and C define on $\mathbb{S}$ the *Dirac sesquilinear form*

$$A(\lambda,\chi)=\lambda^\dagger A\chi$$

and the complex *Majorana bilinear form*

$$C(\lambda,\chi)=\lambda^T C\chi\ .$$

Both forms are $\mathrm{Spin}_0(t,s)$-invariant, and their real and imaginary parts define four real admissible bilinear forms $\mathrm{Re}(A)$, $\mathrm{Im}(A)$, $\mathrm{Re}(C)$ and $\mathrm{Im}(C)$. The forms A and C are independent of the representation which we choose for the γ-matrices, up to conventional signs or phase factors which we have fixed for convenience by imposing certain conditions on the γ-matrices, see Appendix A for details. The Dirac sesquilinear form depends on the signature, while the Majorana bilinear form only depends on the dimension.

In even dimensions we can define four additional real admissible bilinear forms by inserting the chirality matrix γ_* into one argument of the above four bilinear forms. For $\mathrm{Re}(C)$ and $\mathrm{Im}(C)$ this is equivalent to replacing the charge conjugation matrix C by the second inequivalent charge conjugation matrix $\gamma_* C$, which has opposite type, $\tau(\gamma_* C)=-\tau(C)$. Therefore there are at most eight linearly independent real admissible bilinear forms on $\mathbb{S}$ that can be built out of A, C, γ_*.

In five dimensions there is a unique real super-admissible bilinear form on $\mathbb{S}$, which can be taken to be $\mathrm{Re}(A)$ for $t=0,1,4,5$ and $\mathrm{Im}(A)$ for $t=2,3$ [2]. In four dimensions the eight bilinear forms constructed above are linearly independent and therefore form a basis for the eight-dimensional space of real $\mathrm{Spin}_0(t,s)$-invariant bilinear forms on $\mathbb{S}$.

On $\mathbb{S}$ we can also define a matrix B, which relates the γ-matrices to the complex-conjugated γ-matrices, (A.1),(A.2). It satisfies $BB^* = \epsilon\mathbb{1}$, with $\epsilon \in \{\pm 1\}$ depending on the signature. Therefore it either defines a $\mathrm{Spin}_0(t,s)$-invariant real structure (for $\epsilon = 1$) or a $\mathrm{Spin}_0(t,s)$-invariant quaternionic structure (for $\epsilon = -1$) on $\mathbb{S}$. Defining the complex anti-linear map

$$J_{\mathbb{S}}^{(\epsilon)(\alpha)}(\lambda) = \alpha^* B^* \lambda^* \ ,$$

where $\alpha \in \mathbb{C}$ is a phase factor, $|\alpha| = 1$, we find

$$(J_{\mathbb{S}}^{(\epsilon)(\alpha)})^2 = \epsilon\mathbb{1} \Leftrightarrow BB^* = \epsilon\mathbb{1} \ .$$

The phase α reflects that the equations (A.2) are invariant under phase transformations $B \mapsto \alpha B$. We have fixed this invariance by the conventional choice $B = (CA^{-1})^T$, but we will find it convenient to adjust reality conditions using the phase factor α.

We denote by I the natural complex structure of $\mathbb{S}$ which acts through multiplication by the imaginary unit i. In the case $\epsilon = -1$ the anti-linear map $J^{(-1)(\alpha)}$ defines a second complex structure on $\mathbb{S}$ which anticommutes with I. Therefore $I, J^{(-1)(\alpha)}$ generate an algebra isomorphic to the quaternion algebra $\mathbb{H}$, and commutes with the $\mathrm{Spin}_0(t,s)$ representation. This explains why one says that $J^{(-1)(\alpha)}$ defines a quaternionic structure on $\mathbb{S}$. Similarly, for $\epsilon = 1$ the real structure $J^{(+1)(\alpha)}$ anticommutes with I, and therefore I and $J^{(+1)(\alpha)}$ generate an algebra isomorphic to $\mathbb{R}(2)$, which can be interpreted as the algebra of para-quaternions, $\mathbb{H}' \cong \mathbb{R}(2)$, see the appendix of [2] for details. Therefore we will say that $J^{(+1)(\alpha)}$ defines a para-quaternionic structure on $\mathbb{S}$, and treating both cases in parallel we will also say that $J^{(\epsilon)(\alpha)}$ defines an ϵ-quaternionic structure on $\mathbb{S}$.

Also note that if we consider $\mathbb{S}$ as a real module, then $J^{(\epsilon)(\alpha)}$ provides it with a complex structure for $\epsilon = -1$ and with a para-complex structure for $\epsilon = 1$.[a] To treat both cases in parallel we will say that $J^{(\epsilon)(\alpha)}$ defines an ϵ-complex structure.

In five dimensions $\mathbb{S}$ is $\mathbb{C}$-irreducible. The natural complex structure I and the $\mathrm{Spin}_0(t,s)$-invariant ϵ-quaternionic structure $J^{(\epsilon)(\alpha)}$ already generate the full Schur algebra

$$\mathcal{C}(\mathbb{S}) = \mathbb{H}_\epsilon := \begin{cases} \mathbb{H}_{-1} := \mathbb{H} \ , \\ \mathbb{H}_{+1} := \mathbb{H}' \cong \mathbb{R}(2) \ , \end{cases}$$

as can be seen by comparison to Table 1. Note that the Schur algebra $\mathcal{C}_{t,s}(\mathbb{S})$ is determined by the pair $(Cl_{t,s}, Cl^0_{t,s})$. The complex spinor module $\mathbb{S}$ is $\mathbb{C}$-irreducible in any odd dimension, and by comparison to the classification of Clifford algebras,

[a]A para-complex structure is a product structure, that is an endomorphism J of the tangent bundle such that $J^2 = \mathbb{1}$, with the additional property that the eigenspaces of J have equal dimension at each point. See [3] for details.

Table 1. The real Clifford algebras in five dimensions, together with their even subalgebras, the Schur algebras $\mathcal{C}(\mathbb{S})$ and $\mathcal{C}(S_{\mathbb{R}})$ of the complex and real spinor module, the R-symmetry groups G_R, and the relations between the complex spinor module $\mathbb{S}$, real spinor module $S_{\mathbb{R}}$ and real semi-spinor modules $S^{\pm}_{\mathbb{R}}$.

Signature	$Cl_{t,s}$	$Cl_0(t,s)$	$\mathcal{C}_{t,s}(\mathbb{S})$	$\mathcal{C}_{t,s}(S_{\mathbb{R}})$	G_R	$\mathbb{S}$
$(0,5)$	$2\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_{\mathbb{R}}$
$(1,4)$	$\mathbb{C}(4)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_{\mathbb{R}}$
$(2,3)$	$2\mathbb{R}(4)$	$\mathbb{R}(4)$	$\mathbb{H}'$	$\mathbb{R}$	SU$(1,1)$	$S_{\mathbb{R}} \otimes \mathbb{C}$
$(3,2)$	$\mathbb{C}(4)$	$\mathbb{R}(4)$	$\mathbb{H}'$	$\mathbb{H}'$	SU$(1,1)$	$S_{\mathbb{R}} = S^{\pm}_{\mathbb{R}} \otimes \mathbb{C}$
$(4,1)$	$2\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_{\mathbb{R}}$
$(5,0)$	$\mathbb{C}(4)$	$\mathbb{H}(2)$	$\mathbb{H}$	$\mathbb{H}$	SU(2)	$S_{\mathbb{R}}$

all types of pairs which are possible already appear in Table 1. Therefore the Schur algebra $\mathcal{C}_{t,s}(\mathbb{S})$ is equal to either $\mathbb{H}$ or to $\mathbb{H}' \cong \mathbb{R}(2)$ in any odd dimension.

In four dimensions $\mathbb{S}$ decomposes into two $\mathbb{C}$-irreducible complex semi-spinor modules $\mathbb{S}_{\pm}$, which are the eigenspaces of γ_*. And there exist two C-matrices $C_{\pm}$ of opposite type $\tau(C_{\mp}) = \pm 1$, which are related though multiplication by γ_*, that is, $C_{\pm} = \gamma_* C_{\mp}$. Associated to these are two B-matrices $B_{\pm}$, which define either two quaternionic structures, two real structures or one quaternionic and one real structure. It is easy to see that the two structures are of the same type if $B_{\pm}$ commutes with γ_* and of opposite type if $B_{\pm}$ anticommutes with γ_*. The relevant relations between $C_{\pm}, B_{\pm}$ and γ_* have been collected in (A.4) – (A.6). We will refer to ϵ-quaternionic structures which commute with γ_* as *Weyl-compatible* and to ϵ-quaternionic structures which anti-commute with γ_* as *Weyl-incompatible*. Table 2 summarizes which invariant structures occur for a given signature. This table

Table 2. Here we list for each signature how many independent invariant real and quaternionic structures exist on $\mathbb{S}$. In all cases these structures generate, together with the natural complex structure of $\mathbb{S}$, the full Schur algebra. Each invariant real (quaternionic structure) allows to define (symplectic) Majorana spinors. In signatures where the structures are Weyl compatible, reality and chirality conditions can be imposed simultanously.

Signature	Quat. strct.	Real strct.	Weyl compatible?	Schur algebra
Euclidean	2	0	Yes	$\mathcal{C}(\mathbb{S}) = \mathbb{H}^{\oplus 2}$
Minkowksi	1	1	No	$\mathcal{C}(\mathbb{S}) = \mathbb{C}(2)$
Neutral	0	2	Yes	$\mathcal{C}(\mathbb{S}) = \mathbb{R}(2)^{\oplus 2} \cong \mathbb{H}'^{\oplus 2}$

together with the results of [1] implies that in Euclidean signature the $\mathcal{N} = 2$ Poincaré Lie superalgebra is minimal, while in Minkowski signature one can obtain an $\mathcal{N}=1$ algebra based on Majorana spinors. In neutral signature Majorana and Weyl conditions are compatible, which potentially allow an '$\mathcal{N}=1/2$' algebra based on Majorana-Weyl spinors. However such an algebra does not appear in our classification [1], and one can check that the superbracket is completely degenerate when restricting it to Majorana-Weyl spinors. We remark that table 2 covers all cases which can occur in any even dimension, as can be verified by inspecting the classification of Clifford algebras.[b]

2.2. *Doubled Spinors*

2.2.1. *General considerations*

As in [2] the vector multiplet representation will not use Dirac spinors (the complex spinor module $\mathbb{S}$) but *doubled spinors*, which are a generalisation of symplectic Majorana spinors. This allows to disentangle the actions of the Spin group and of the Schur group on the fermionic fields of the theory. One advantage is that the supersymmetry variations and Lagrangians can be brought to a universal form, where only certain signs, and, for fermions, certain factors of i, depend on the space-time signature. The idea is to start with two copies $\mathbb{S}\oplus\mathbb{S}\cong\mathbb{S}\otimes\mathbb{C}^2$ of the complex spinor module, from which we can recover $\mathbb{S}$ by imposing a $\mathrm{Spin}_0(t,s)$-invariant reality condition. To specify this reality condition we define on $\mathbb{C}^2$ a pair of complex anti-linear map by

$$J^{(\epsilon)}_{\mathbb{C}^2}\;:\;\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\mapsto\begin{pmatrix} \epsilon z_2^* \\ z_1^* \end{pmatrix}\;,\quad \epsilon=\pm 1\;.$$

These maps satisfy

$$(J^{(\epsilon)}_{\mathbb{C}^2})^2=\epsilon\mathbb{1}\;.$$

so that $J_{\mathbb{C}}$ is a real structure for $\epsilon=1$ and a quaternionic structure for $\epsilon=-1$. Using that on $\mathbb{S}$ the matrix B always either defines an invariant real or an invariant quaternionic structure, we define a $\mathrm{Spin}_0(t,s)$-invariant real structure on $\mathbb{S}\otimes\mathbb{C}^2$, by[c]

$$\rho=\rho^{(\alpha)}=J^{(\epsilon)(\alpha)}_{\mathbb{S}}\otimes J^{(\epsilon)}_{\mathbb{C}^2}\;:\;\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}\mapsto\begin{pmatrix} \epsilon\alpha^* B^*\lambda^{2*} \\ \alpha^* B^*\lambda^{1*} \end{pmatrix}=:(\alpha^* B^*\lambda^{j*}N_{ji})_{i=1,2}$$

[b] The invariant structures and Schur algebras are determined by the pairs $(Cl_{t,s},\,Cl^0_{t,s})$ and one can check that all possible types of combinations are realized in four dimensions.

[c] We use a notation which is adapted to the NW-SE convention for raising and lowering the indices $i,j=1,2$, compare (3.3). The fact that $i,j,\ldots$ occur in anti-lexicographic ordering in several formulae is a consequence of our NW-SE style notation and does *not* indicate matrix transposition.

where

$$(N_{ji}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} = \begin{cases} (\eta_{ji})_{j,i=1,2} \ , \text{ for } \ \epsilon = 1 \ , \\ (\varepsilon_{ji})_{j,i=1,2} \ , \text{ for } \ \epsilon = -1 \ . \end{cases}$$

The real points $(\mathbb{S} + \mathbb{S})^\rho$ with respect to the real structure ρ define a real Clifford module isomorphic to $\mathbb{S}$, which is embedded into $\mathbb{S} \oplus \mathbb{S}$ as the graph of the ϵ-quaternionic structure on $\mathbb{S}$:

$$(\mathbb{S} \oplus \mathbb{S})^\rho \cong \{(\lambda^1, \lambda^2) \in \mathbb{S} \times \mathbb{S} | \lambda^2 = J_{\mathbb{S}}^{(\epsilon)}(\lambda^1)\} \cong \mathbb{S} \ .$$

Given any admissible real bilinear form β on $\mathbb{S}$ we can obtain a super-admissible complex bilinear form $b = \beta \otimes M$ on $\mathbb{S} \otimes \mathbb{C}^2$, by choosing M to be symmetric if $\sigma(\beta)\tau(\beta) = 1$ and antisymmetric if $\sigma(\beta)\tau(\beta) = -1$.

As bilinear form on $\mathbb{C}^2$ we always choose either the standard symmetric complex bilinear form $g_{\mathbb{C}^2}$ or the standard antisymmetric complex bilinear form $\varepsilon_{\mathbb{C}^2}$ on $\mathbb{C}^2$. Using the matrices

$$\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

to represent these bilinear forms, we have $M(\cdot, \cdot) = g_{\mathbb{C}^2}(\cdot, M \cdot)$, where $M = \delta$ or $M = \varepsilon$.

To have an admissible complex bilinear form on $\mathbb{S}$ to start with, we use the one defined by the charge conjugation matrix C, so that $b \propto C \otimes M$, where we allow a normalization factor for which a convenient value will be chosen later. In even dimensions there are two inequivalent charge conjugation matrices $C_\pm$, so that we can define two super-admissible bilinear forms $b_\pm \propto C_\pm \otimes M_\pm$, where $M_\pm$ is chosen such that the vector-valued bilinear form $b_\pm(\gamma^\mu \cdot, \cdot)$ is symmetric. By restricting $b_\pm$ to the real points with respect to the invariant real structure ρ, we obtain super-admissible real bilinear forms $b_{\pm|\rho}$ on $(\mathbb{S} + \mathbb{S})^\rho \cong \mathbb{S}$.

We remark that the doubled spinor module $\mathbb{S} \oplus \mathbb{S}$ can be viewed as the complexification of the complex spinor module $\mathbb{S}$, as follows. Firstly, $\mathbb{S}$ and $\mathbb{S} \oplus \mathbb{S}$ carry by construction a representation of the complex Clifford algebra $\mathbb{C}l_{t+s}$ and of the complex spin group $\mathrm{Spin}(t+s, \mathbb{C})$, and the complex bilinear form $b \propto C \otimes M$ is $\mathrm{Spin}(t+s, \mathbb{C})$ invariant. Since $\mathbb{S}$ carries an invariant ϵ-complex structure $J_{\mathbb{S}}^{(\epsilon)(\alpha)}$, it is self-conjugate as a complex $\mathrm{Spin}(t, s)$ module, $\mathbb{S} \cong \bar{\mathbb{S}}$. Therefore $\mathbb{S} \oplus \mathbb{S}$ is the complexification of $\mathbb{S}$, regarded as a real module:

$$\mathbb{S}_{\mathbb{C}} := \mathbb{S} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{S} + \bar{\mathbb{S}} \cong \mathbb{S} + \mathbb{S} \cong \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^2 \ .$$

The doubled spinor module, equipped with a super-admissible complex bilinear form, defines a complex Poincaré Lie superalgebra $\mathfrak{g}_{\mathbb{C}} = so(V_{\mathbb{C}}) + V_{\mathbb{C}} + \mathbb{S}_{\mathbb{C}}$, where $V_{\mathbb{C}} = V \otimes \mathbb{C}$. If we extend ρ in the obvious way to $\mathfrak{g}_{\mathbb{C}}$, the restriction of $\mathfrak{g}_{\mathbb{C}}$ to the real points of ρ picks a real form $\mathfrak{g}^\rho \cong \mathfrak{so}(V) + V + \mathbb{S} \subset \mathfrak{g}_{\mathbb{C}}$. In [2] this observation was used to construct the vector multiplet theories with $t+s=5$ as real forms of an underlying 'holomorphic master theory.'

So far our discussion of doubled spinors has applied to all dimensions and signatures. We now specialise to the dimensions $t+s=5,4$ which we consider in this paper. In five dimensions the space of super-admissible bilinear forms is one-dimensional [4], see [2] for a detailed account of the material reviewed in the following. The charge conjugation matrix C satisfies $\sigma\tau=-1$, see Table 6, so that $C\otimes\varepsilon$, where $\varepsilon=\varepsilon_{\mathbb{C}^2}$ is the standard anti-symmetric complex bilinear form on $\mathbb{C}^2$, is a super-admissible form on $\mathbb{S}\otimes\mathbb{C}^2$.

The various signatures (t,s), $t+s=5$ can be grouped into two classes, see also Table 1

(1) $t=0,1,4,5$. For these signatures the super-admissible real bilinear form on $\mathbb{S}$ is $\mathrm{Re}A^{(t,s)}$, where $A^{(t,s)}(\psi,\phi)=\psi^\dagger A^{(t,s)}\phi$ is the standard Spin_0-invariant sesquilinear form. Here we use the same notation $A^{(t,s)}$ for the A-matrix in signature (t,s) and the corresponding sesquilinear form. The complex spinor module $\mathbb{S}$ carries a quaternionic structure, and the Schur group $\mathbb{H}^*=\mathbb{R}^{>0}\times\mathrm{SU}(2)$ acts as $\mathbb{R}^{>0}$, that is by rescaling on the one-dimensional space of superbrackets. The R-symmetry group is $\mathrm{SU}(2)$.
(2) $t=2,3$. The super-admissible real bilinear form on $\mathbb{S}$ is $\mathrm{Im}A^{(t,s)}$, the complex spinor module carries a real structure, and the Schur group $\mathbb{R}^{>0}\cdot\mathrm{SU}(1,1)$ acts again by rescalings, so that the R-symmetry group is $\mathrm{SU}(1,1)$.

The real structures used in [2] are $\rho=J_{\mathbb{S}}^{(\alpha)(\epsilon)}\otimes J_{\mathbb{C}^2}^{(\epsilon)}$ with $\epsilon=-1$ for $t=0,1,4,5$ and with $\epsilon=1$ for $t=2,3$. By adopting the normalisation $b:=-\frac{1}{2}C\otimes\varepsilon$ and making a suitable choices for the phases α (see the first column of Table 3), we can arrange that the restriction $b_{|\rho}$ of b to $(\mathbb{S}\oplus\mathbb{S})^\rho\cong\mathbb{S}$ is

$$b_{|\rho}=\begin{cases}\mathrm{Re}(A^{(t,s)})\text{ for }t=0,1,4,5\ ,\\ \mathrm{Im}(A^{(t,s)})\text{ for }t=2,3\ .\end{cases}$$

In four dimensions we have two inequivalent charge conjugation matrices: C_-, which is equal to the five-dimensional charge conjugation matrix, $C_-=C$, and $C_+=\gamma_*C_-$. Their invariants σ (symmetry), τ (type) and ι (isotropy) can be found in Table 6 in Appendix A.1. We choose a representation where γ_* is real and symmetric, and commutes with $C_\pm$, which is possible in four dimensions, see Appendix A.1 .

Note that both bilinear forms $C_\pm$ are orthogonal ($\iota=1$), that is $C_\pm(\mathbb{S}_\pm,\mathbb{S}_\mp)=0$. Since γ_* anticommutes with all γ-matrices this implies that the vector-valued bilinear forms $C_\mp(\gamma^\mu\cdot,\cdot)$ are isotropic, $C_\pm(\gamma^\mu\mathbb{S}_\pm,\mathbb{S}_\pm)=0$. Since $\sigma(C_-)\tau(C_-)=-1$ and $\sigma(C_+)\tau(C_+)=1$, we can construct two super-admissible isotropic vector-valued complex bilinear form on $\mathbb{S}\otimes\mathbb{C}^2$: $(C_-\otimes\varepsilon)(\gamma^m\cdot,\cdot)$, which is the reduction of the five-dimensional complex bilinear form, and $(C_+\otimes\delta)(\gamma^m\cdot,\cdot)$, which does not have a five-dimensional uplift.

Using the formulae collected in Appendix A.2 we can dimensionally reduce the five-dimensional reality conditions chosen in [2] to obtain the corresponding reality conditions in four-dimensions. The resulting reality conditions are listed in Table 3.

Table 3. Relation between five-dimensional and four-dimensional reality conditions through dimensional reduction.

Signature	Reality Condition	Reduction	Reality Condition
$(0,5)$	$(\lambda^i)^* = B\lambda^j\varepsilon_{ji}$	$(0,5)\to(0,4)$	$(\lambda^i)^* = B_-\lambda^j\varepsilon_{ji}$
$(1,4)$	$(\lambda^i)^* = -B\lambda^j\varepsilon_{ji}$	$(1,4)\to(0,4)$	$(\lambda^i)^* = -iB_+\lambda^j\varepsilon_{ji}$
		$(1,4)\to(1,3)$	$(\lambda^i)^* = -B_-\lambda^j\varepsilon_{ji}$
$(2,3)$	$(\lambda^i)^* = iB\lambda^j\eta_{ji}$	$(2,3)\to(1,3)$	$(\lambda^i)^* = B_+\lambda^j\eta_{ji}$
		$(2,3)\to(2,2)$	$(\lambda^i)^* = iB_-\lambda^j\eta_{ji}$
$(3,2)$	$(\lambda^i)^* = -iB\lambda^j\eta_{ji}$	$(3,2)\to(2,2)$	$(\lambda^i)^* = B_+\lambda^j\eta_{ji}$
		$(3,2)\to(3,1)$	$(\lambda^i)^* = -iB_-\lambda^j\eta_{ji}$
$(4,1)$	$(\lambda^i)^* = B\lambda^j\varepsilon_{ji}$	$(4,1)\to(3,1)$	$(\lambda^i)^* = -iB_+\lambda^j\varepsilon_{ji}$
		$(4,1)\to(4,0)$	$(\lambda^i)^* = B_-\lambda^j\varepsilon_{ji}$
$(5,0)$	$(\lambda^i)^* = -B\lambda^j\varepsilon_{ji}$	$(5,0)\to(4,0)$	$(\lambda^i)^* = -iB_+\lambda^j\varepsilon_{ji}$

2.2.2. *Doubled spinor formulation for signature* $(0,4)$

We have shown in [1] that in Euclidean signature all $\mathcal{N}=2$ Poincaré Lie superalgebras are isomorphic to each other. Starting in five dimensions, we can obtain two theories through the reductions $(0,5)\to(0,4)$ and $(1,4)\to(0,4)$, which we will want to relate explicitly by a field redefinition later. Therefore we now investigate how superbrackets formulated using doubled spinors are related to one another in signature $(0,4)$.

In four dimensions, there are two independent charge conjugation matrices $C_\pm$ which define complex bilinear forms, and two independent matrices $B_\pm$ which define reality conditions. In signature $(0,4)$ both $B_\pm$ define quaternionic structures, that is, we can define two types of symplectic Majorana spinors. Combining these choices we can define the following four super-admissible real bilinear form on $\mathbb{S}$:

$$G_R = \mathbb{R}^{>0}\times \mathrm{SU}(2) \quad \begin{cases} C_-\otimes\varepsilon, & (\lambda^i)^* = \alpha B_-\lambda^j\varepsilon_{ji} \leftarrow (0,\cancel{5}) \\ C_-\otimes\varepsilon, & (\psi^i)^* = \beta B_+\psi^j\varepsilon_{ji} \leftarrow (\cancel{1},4) \\ C_+\otimes\delta, & (\phi^i)^* = \gamma B_-\phi^j\varepsilon_{ji} \\ C_+\otimes\delta, & (\xi^i)^* = \delta B_+\xi^j\varepsilon_{ji} \end{cases} \tag{2.1}$$

Here $\alpha, \beta, \gamma, \delta$ are phase factors. Note that the symbols β, δ were used previously for bilinear forms, but it should be clear from context what is meant. $(0, \cancel{5})$ is a shorthand notation for the reduction $(0,5) \to (0,4)$. The bilinear forms based on C_+ cannot be obtained directly from dimensional reduction. In this section we will show that one can independently map the two complex bilinear forms and the two reality conditions to one another, and thus obtain explicit maps between all four real superbrackets within the doubled spinor formalism.

Mapping reality conditions, preserving the bilinear form

Let λ^i be a doubled spinor subject to a reality condition of the form

$$(\lambda^i)^* = \alpha B_- \lambda^j M_{ji} \; , \tag{2.2}$$

where M is a two-by-two matrix.

We would like to find a linear transformation $(\lambda^i) \mapsto (\psi^i)$, such that ψ^i satisfy the reality condition

$$(\psi^i)^* = \beta B_+ \psi^j M_{ji} \; . \tag{2.3}$$

In signature $(0,4)$ $B_\pm$ are related by $\gamma_* B_- = B_- \gamma_* = B_+$.

We make the following ansatz:

$$\lambda^i = \frac{1}{\sqrt{2}}(a\mathbb{1} + b\gamma_*)\psi^i \;\Leftrightarrow \psi^i = \frac{1}{\sqrt{2}}(a^*\mathbb{1} + b^*\gamma_*)\lambda^i \; , \tag{2.4}$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 2$ and $ab^* + ba^* = 0$.[d] Then we compute:

$$\begin{aligned}(\psi^i)^* &= \frac{1}{\sqrt{2}}(a\mathbb{1} + b\gamma_*)(\lambda^i)^* = \frac{1}{\sqrt{2}}(a\mathbb{1} + b\gamma_*)\alpha B_- \lambda^j M_{ji} \\ &= \frac{1}{\sqrt{2}}\alpha B_-(a\mathbb{1} + b\gamma_*)\lambda^j M_{ji} = \frac{1}{\sqrt{2}}\alpha B_+(a\gamma_* + b\mathbb{1})\lambda^j M_{ji} \; .\end{aligned}$$

Comparing to

$$(\psi^i)^* = \beta B_+ \psi^j M_{ji} = \frac{1}{\sqrt{2}}\beta B_+(a^*\mathbb{1} + b^*\gamma^*)\lambda^j M_{ji}$$

we obtain

$$\alpha b = \beta a^* \; , \quad \alpha a = \beta b^* \; ,$$

which implies $|a| = |b|$. The condition $ab^* + ba^* = 0$ can always be solved by taking one of the coefficients to be real, the other purely imaginary. Since $|a|^2 + |b|^2 = 2$, one solution is given by $a = 1$, $b = \frac{\beta}{\alpha}$. In table 3 the phase factors of the reality conditions in signature $(0,4)$ are related by $\beta = -i\alpha$ so that the reality conditions can be mapped by setting $a = 1, b = -i$:

$$\lambda^i = \frac{1}{\sqrt{2}}\left(\mathbb{1} - i\gamma_*\right)\psi^i \Leftrightarrow \psi^i = \frac{1}{\sqrt{2}}\left(\mathbb{1} + i\gamma_*\right)\lambda^i \; . \tag{2.5}$$

[d] The operator $a\mathbb{1} + b\gamma_*$ is invertible for $a \neq \pm b$.

Since $a\mathbb{1} + b\gamma_*$ commutes with γ_*, the chirality of spinors is preserved under the transformation. To see how expressing λ^i in terms of ψ^i acts on the complex bilinear forms, compute

$$(\gamma^m \lambda^i)^T C_\pm \chi^j M_{ji} = \frac{1}{2}(\gamma^m (a\mathbb{1} + b\gamma_*)\psi^i)^T C_\pm (a + b\gamma_*)\Omega^j M_{ji}$$
$$= \frac{1}{2}(\gamma^m \psi^i)^T C_\pm (a\mathbb{1} - b\gamma_*)(a\mathbb{1} + b\gamma_*)\Omega^j M_{ji} = \frac{1}{2}(a^2 - b^2)(\gamma^m \psi^i)^T C_\pm \Omega^j M_{ji} \,.$$

Here we used that γ_* is symmetric and $\gamma_* C_\pm = C_\pm \gamma_*$, as well as $\gamma_* \gamma^m = -\gamma^m \gamma_*$. Thus the four super-admissible bilinear forms are invariant up to a factor, and strictly invariant for the choice $a = 1$, $b = -i$. Thus we can map the two reality conditions to each other while preserving any of the two super-admissible complex bilinear forms.

Mapping bilinear forms, preserving reality conditions

Next we look for a map relating the two complex bilinear forms $C_- \otimes \epsilon$ and $C_+ \otimes \delta$ to one another. For this it is helpful to use the natural embedding $\mathbb{S}_\pm \subset \mathbb{S}$ and to use a notation employing 'twice-doubled spinors':

$$\lambda^I = [\lambda_+^i, \lambda_-^i] = [\lambda_+^1, \lambda_+^2, \lambda_-^1, \lambda_-^2] \in \mathbb{S}_+ \oplus \mathbb{S}_+ \oplus \mathbb{S}_- \oplus \mathbb{S}_- \subset \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{S} \cong \mathbb{S} \otimes \mathbb{C}^4 \,,$$

where $I = 1, 2, 3, 4$ is an index for the extended internal space $\mathbb{C}^4$. We can now use a concise block-matrix type notation for the bilinear form $C_- \otimes \varepsilon$:

$$(C_- \otimes \varepsilon)(\gamma^m \lambda, \chi) = (\gamma^m \lambda_+^i)^T C_- \chi_-^j \varepsilon_{ji} + (\gamma^m \lambda_-^i)^T C_- \chi_+^j \varepsilon_{ji}$$
$$= \left[(\gamma^m \lambda_+^i)^T, (\gamma^m \lambda_-^i)^T\right] C_- \begin{bmatrix} 0 & -\varepsilon_{ij} \\ -\varepsilon_{ij} & 0 \end{bmatrix} \begin{bmatrix} \chi_+^j \\ \chi_-^j \end{bmatrix} \,.$$

Matrices and vectors with respect to the internal space $\mathbb{C}^4$ of the twice-doubled spinor module are indicated by the use of square brackets. We use a 2×2 block matrix solution, with index notation for two-component sub-vectors and two-by-two sub-matrices.

Using that $C_- \lambda_\pm = \pm C_- \gamma_* \lambda_\pm = \pm C_+ \lambda_\pm$ we can rewrite $C_- \otimes \varepsilon$ in terms of C_+:

$$(C_- \otimes \varepsilon)(\gamma^m \lambda, \chi) = \left[(\gamma^m \lambda_+^i)^T, (\gamma^m \lambda_-^i)^T\right] C_+ \begin{bmatrix} 0 & \varepsilon_{ij} \\ -\varepsilon_{ij} & 0 \end{bmatrix} \begin{bmatrix} \chi_+^j \\ \chi_-^j \end{bmatrix} \,.$$

Expressing the complex bilinear form $C_+ \otimes \delta$ in terms of twice-doubled spinors we find

$$(C_+ \otimes \delta)(\gamma^m \Psi, \Omega) = (\gamma^m \Psi_+^i)^T C_+ \Omega_-^j \delta_{ji} + (\gamma^m \Psi_-^i)^T C_+ \Omega_+^j \delta_{ji}$$
$$= \left[(\gamma^m \Psi_+^i)^T, (\gamma^m \Psi_-^i)^T\right] C_+ \begin{bmatrix} 0 & \delta_{ij} \\ \delta_{ij} & 0 \end{bmatrix} \begin{bmatrix} \Omega_+^j \\ \Omega_-^j \end{bmatrix} \,.$$

To relate the two bilinear forms we need a linear transformation $\lambda^I = S^I{}_J \Psi^J$ between twice-doubled spinors such that

$$S^T \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix} S = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} .$$

One solution is given by

$$S = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\varepsilon \end{bmatrix} , \quad \begin{array}{l} \lambda^i_+ = \Psi^i_+ \\ \lambda^i_- = -\varepsilon_{ij}\Psi^j_- = \Psi^j_-\varepsilon_{ji} \Leftrightarrow \Psi^i_- = -\lambda^j_-\varepsilon_{ji} \ . \end{array} \tag{2.6}$$

Note that when using twice-doubled spinors, we only need to consider linear transformations which act on the extended internal index I, but not on spinor indices. This disentangling of spinor and internal indices with respect to the action of the Schur group is an important advantage of the twice-doubled notation. It reflects that while the Schur group only acts on internal space of the doubled spinor formalism in odd dimensions, it can act differently on the chiral components of a spinor in even dimension. This is taken care of in the twice-doubled notation by doubling the auxiliary internal space.

The map defined by (2.6) works for any signature, but whether it preserves reality conditions depends on the signature. In signature $(0,4)$ γ_* commutes with $B_\pm$, and therefore reality conditions can be imposed consistently on the symplectic Majorana-Weyl spinors $\lambda^I_\pm$. We should therefore expect that any of the two reality conditions is preserved. To verify this note first that S is block-diagonal and manifestly preserves chirality. It is also manifest that λ^i_+ and Ψ^i_+ satisfy the same reality condition. Now assume that $(\lambda^i_-)^* = \alpha B_\mp \lambda^j_- \varepsilon_{ji}$. Then

$$(\Psi^i_-)^* = -(\lambda^j_-)^*\varepsilon_{ji} = -\alpha B_\mp \lambda^k_i \varepsilon_{kj}\varepsilon_{ji} = \alpha B_\mp \lambda^i_- = \alpha B_\mp \Psi^j_- \varepsilon_{ji} \ ,$$

and we see that the reality condition is preserved. Thus the map defined by S interchanges the complex vector-valued bilinear forms while preserving any of the two reality conditions in signature $(0,4)$.

The following diagram summarizes the situation. We can independently change the reality condition by (2.5) and the complex bilinear form by (2.6). These two operations are indicated by 'RC' and 'Bil' respectively.

$$\begin{tikzcd}
(C_- \otimes \varepsilon, (\psi^i)^* = \beta B_+ \psi^j \varepsilon_{ji}) \arrow[r, leftrightarrow, "\mathrm{RC}"] \arrow[d, leftrightarrow, "\mathrm{Bil}"'] & (C_- \otimes \varepsilon, (\lambda^i)^* = \alpha B_- \lambda^j \varepsilon_{ji}) \arrow[d, leftrightarrow, "\mathrm{Bil}"] \\
(C_+ \otimes \delta, (\zeta^i)^* = \delta B_+ \zeta^j \varepsilon_{ji}) \arrow[r, leftrightarrow, "\mathrm{RC}"'] & (C_+ \otimes \delta, (\phi^i)^* = \gamma B_- \phi^j \varepsilon_{ji})
\end{tikzcd}$$

2.2.3. *Doubled spinor formulation for signature* $(1,3)$

It was shown in [1] that there in Minkowski signature there are two non-isomorphic supersymmetry algebras distinguishable by their R-symmetry groups, which are

U(2) and U(1, 1) respectively. Moreover, by comparing four- and five-dimensional R-symmetry groups, we know that the first case can be realized through reduction from (1, 4), while the second arises by reduction from (2, 3).

On the complex spinor module $\mathbb{S}$ the B-matrix B_- induced by dimensional reduction defines a quaternionic structure, while B_+ defines a real structure. Therefore theories can be formulated using either symplectic Majorana spinors or Majorana spinors. This leads us to consider the following three reality conditions:

$$(\lambda^i)^* = \alpha B_- \lambda^j \epsilon_{ji} \ , \tag{2.7}$$

$$(\Psi^i)^* = \beta B_+ \Psi^i = \beta B_+ \Psi^j \delta_{ji} \ , \tag{2.8}$$

$$(\varphi^i)^* = \gamma B_+ \varphi^j \eta_{ji} \ . \tag{2.9}$$

The first and second condition are the standard symplectic Majorana and standard Majorana condition, respectively. The third condition is a Majorana condition which couples a pair of spinors through the matrix

$$\eta = (\eta_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

Upon diagonalization this becomes a 'twisted' Majorana condition

$$(\phi^i)^* = \gamma B_+ \phi^j \eta'_{ij} \ , \quad \eta' = (\eta'_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.10}$$

which differs from the standard Majorana condition by a relative sign. Such reality conditions have appeared in [5], where they were used to define the 'twisted' supersymmetry algebras of type-II* string theories. In the terminology of [5] the reality conditions (2.8) and (2.9), (2.10) are referred to as O(2) Majorana and O(1, 1) Majorana, respectively. In our approach it is crucial that the matrices entering into the definition of the (complexified) superbracket and into the reality condition are chosen independently. Since the R-symmetry group is an invariance group of the super-bracket rather than the reality condition, we will call (2.8) the *standard* or *diagonal Majorana condition* and (2.9) the *twisted Majorana* condition. The twisted Majorana condition was used in [2] to formulate five-dimensional vector multiplets in signatures (2, 3) and (3, 2).

From Section 2.1 we know that in signature (1, 3) reality conditions are not compatible with chirality, since complex conjugation flips the chirality of a spinor. Therefore chiral projections of reality conditions take the form

$$(\lambda^i_\pm)^* = \alpha B_- \lambda^j_\mp M_{ji} \ , \quad (\lambda^i_\pm)^* = \alpha B_+ \lambda^j_\mp M_{ji} \ ,$$

with $M_{ji} \in \{\delta_{ji}, \eta_{ji}, \eta'_{ji}, \varepsilon_{ji}\}$. In order to relate reality conditions to one another it is useful to note that $B_\pm \gamma_* = B_\mp$ implies

$$B_+ \lambda^i_+ = B_- \lambda^i_+ \ , \quad B_+ \lambda^i_- = -B_- \lambda^i_- \ . \tag{2.11}$$

Combining the three distinct reality conditions with the two choices for the complex superbrackets we obtain six real superalgebras. We will now show how these fit into the two equivalence classes found in [1].

The standard $\mathcal{N}=2$ superalgebra, $G_R \cong \mathrm{U}(2)$

By dimensional reduction from five dimensions, we obtain a representation of the standard $\mathcal{N}=2$ algebra in terms of symplectic Majorana spinors. By comparison to Table 3 we see that the reduction $(1,4) \to (1,3)$ corresponds to the following combination of a bilinear form with a reality condition:

$$\left(C_- \otimes \varepsilon, \quad (\lambda^i)^* = \alpha B_- \lambda^j \epsilon_{ji} \right) , \quad \text{with } \alpha = -1 .$$

In signature $(1,3)$ Majorana spinors are more commonly used. To rewrite symplectic Majorana spinors in terms of Majorana spinors we adapt the map given in the appendix of [3]:[e]

$$\lambda^1 = \frac{1}{\sqrt{2}}(\Psi^1 - i\Psi^2) , \quad \lambda^2 = \frac{\beta}{\sqrt{2}\alpha} B_-^* B_+ (\Psi^1 + i\Psi^2) = -\frac{\beta}{\sqrt{2}\alpha}\gamma_* (\Psi^1 + i\Psi^2) , \tag{2.12}$$

where we used that $(-1)^t \gamma_* B_\pm = B_\pm \gamma_* = B_\mp$. It is straightforward to check that $(\Psi^i)^* = \beta B_+ \Psi^i$, so that this formula exchanges the two reality conditions, and simultaneously exchanges the vector-valued bilinear forms, up to a phase factor:

$$[C_- \otimes \varepsilon](\gamma^\mu \lambda, \chi) = \frac{\beta}{\alpha}[C_+ \otimes \delta](\gamma^\mu \Psi, \Omega) . \tag{2.13}$$

Of course $\frac{\beta}{\alpha}$ must be real, since the restrictions of both vector-valued bilinear forms to their respective real points are assumed to be real-valued. By choosing $\alpha = \beta$ we can adjust the phase factor to unity.

Alternatively, we can work with twice-doubled spinors and use the map Bil defined by (2.6) which exchanges the bilinear forms $C_- \otimes \varepsilon$ and $C_+ \otimes \delta$. In signature $(1,3)$ this map acts non-trivially on the reality conditions, since complex conjugation anti-commutes with chiral projection.

If λ^i are symplectic Majorana spinors, then their chiral projections satisfy

$$(\lambda^i_\pm)^* = \alpha B_- \lambda^j_\mp \varepsilon_{ji} .$$

Using the component form (2.6) of the map Bil we compute

$$\begin{aligned}
(\Psi^i_+)^* &= (\lambda^i_+)^* = \alpha B_- \lambda^j_- \varepsilon_{ji} = -\alpha B_- \Psi^j_- = \alpha B_+ \Psi^i_- , \\
(\Psi^i_-)^* &= -(\lambda^j_-)^* \varepsilon_{ji} = -\alpha B_- \lambda^k \varepsilon_{kj} \varepsilon_{ji} = \alpha B_- \lambda^i_+ = \alpha B_- \Psi^i_+ = \alpha B_+ \Psi^i_+ .
\end{aligned}$$

Thus the map Bil exchanges symplectic Majorana and Majorana spinors in signature $(1,3)$.

[e]Note that there is a typographic mistake in formula (A.13) of [3].

Since we will show in the next section that the other four combinations of complex bilinear forms with reality conditions correspond to the second, non-equivalent $\mathcal{N}=2$ superalgebra, we can summarize this section as follows:

$$G_R = \mathrm{U}(2) \quad \begin{cases} C_- \otimes \varepsilon, & (\lambda^i)^* = \alpha B_- \lambda^j \varepsilon_{ji} \leftarrow (1, \not{4}) \, , \\ C_+ \otimes \delta, & (\Psi^i)^* = \beta B_+ \Psi^i \, . \end{cases} \tag{2.14}$$

Here $(1,\not{4})$ is a short-hand notation to indicate the reduction $(1,4) \to (1,3)$. The second line is the most commonly used formulation of the standard $\mathcal{N}=2$ supersymmetry algebra in terms of Majorana spinors. For comparison, we will give a formulation of the twisted $\mathcal{N}=2$ supersymmetry algebra in terms in Majorana spinors in (2.17).

The twisted $\mathcal{N}=2$ supersymmetry algebra, $G_R \cong \mathrm{U}(1,1)$

We now turn to the second family of $\mathcal{N}=2$ algebras, which have R-symmetry group $\mathrm{U}(1,1)$. This algebra can be realized by reduction from five dimensions with signature $(2,3)$, which can be related to the three remaining combinations of complex bilinear forms and reality conditions:

$$G_R = \mathrm{U}(1,1) \quad \begin{cases} C_- \otimes \varepsilon, & (\lambda^i)^* = \alpha B_+ \lambda^j \eta_{ji} \leftarrow (\not{2}, 3) \, , \\ C_- \otimes \varepsilon, & (\Psi^i)^* = \beta B_+ \Psi^i = \beta B_+ \Psi^j \delta_{ji} \, , \\ C_+ \otimes \delta, & (\varphi^i)^* = \gamma B_+ \varphi^j \eta_{ji} \, , \\ C_+ \otimes \delta, & (\Omega^i)^* = \delta B_- \Omega^j \varepsilon_{ji} \, , \end{cases} \tag{2.15}$$

where $\alpha, \beta, \gamma, \delta$ are phase factors. Transformations which relate these four combinations can easily be found using the twice-doubled notation. Since the computations are similar to previous computations, we only give a summary and add some explanatory comments. The map defined by the matrix S in (2.6) exchanges the two bilinear forms. While it preserves reality conditions in Euclidean signature, it changes them in Minkowski signature. Specifically, if we use S to relate Ψ^I to Ω^I, then S maps the standard Majorana condition to the symplectic Majorana condition. And if we apply S to λ^I, it maps the symplectic Majorana condition to the twisted Majorana condition, but with the off-diagonal matrix (η_{ij}) replaced by its diagonalized form (η'_{ij}). This can be corrected for by an additional linear transformations (represented by the matrix F defined below) which brings η'_{ij} back to the off-diagonal form. The resulting transformations, which exchange bilinear forms are:

$$\lambda^I = T^I{}_J \varphi^J \ , \quad (T^I{}_J) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \ , \quad \Omega^I = S^I{}_J \Psi^J \ , (S^I{}_J) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \ .$$

Note that

$$T = T^{-1} = FS^{-1} = SF^{-1} = SF \ , \quad \text{where} \quad (F^I{}_J) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{2.16}$$

is the matrix which exchanges η_{ij} and η'_{ij}.

To relate all four real bilinear forms to each other, we also need transformations which preserve the complex bilinear forms but exchange the reality conditions. Finding one such transformations is sufficient, because then all further relations between the four algebras are determined by consistency. Picking φ^I and Ω^I for concreteness, it is easy to verify that the transformation

$$\varphi^I = U^I{}_J \Omega^J \ , \quad (U^I{}_J) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

preserves $C_+ \otimes \delta$ and maps the respective reality conditions to one another.

The transformation relating λ^I and Ψ^I is then

$$\Psi^I = V^I{}_J \lambda^J \ , \quad V = S^{-1} U^{-1} T^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & i & i \\ 0 & 0 & 1 & -1 \end{bmatrix} \ .$$

The relations between the four real superbrackets are summarized in the following commuting diagram

$$\begin{tikzcd}
(C_- \otimes \varepsilon, (\lambda^i)^* = \alpha B_+ \lambda^j \varepsilon_{ji}) \arrow[r, "V"] & (C_- \otimes \varepsilon, (\Psi^i)^* = \beta B_+ \Psi^j) \arrow[d, "S"] \\
(C_+ \otimes \delta, (\varphi^i)^* = \gamma B_+ \varphi^j \eta_{ji}) \arrow[u, "T"] & (C_+ \otimes \delta, (\Omega^i)^* = \delta B_- \Omega^j \varepsilon_{ji}) \arrow[l, "U"]
\end{tikzcd}$$

To conclude, we mention a further rewriting which brings the supersymmetry algebra to the same form that is used for the twisted supersymmetry algebras underlying type-II* string theory [6]. We have mentioned that instead of the off-diagonal symmetric matrix (η_{ij}) we can use its diagonalized form $(\eta'_{ij}) = \mathrm{diag}(1, -1)$. The supersymmetry algebra is then given by the complex bilinear form $C_+ \otimes \delta$, together with a reality condition of the form $(\varphi^i)^* = \alpha B_+ \varphi^j \eta'_{ji}$. If we redefine $\varphi^2 \mapsto i\varphi^2$ while keeping φ^1 the same, we obtain the pair

$$(C_+ \otimes \eta', (\varphi^i)^* = \gamma B_+ \varphi^i) \ , \tag{2.17}$$

where the Majorana condition is standard, while the complex bilinear form is $C_+ \otimes \eta'$. This is the form in which twisted supersymmetry algebras in ten dimensions were defined in [6].

2.2.4. *Doubled spinor formulation for signature* $(2,2)$

It was shown in [1] that in neutral signature the Schur group acts with a single open orbit, corresponding to a unique $\mathcal{N}=2$ superalgebra with connected R-symmetry group $G_R \cong \mathbb{R}^{>0} \times \mathrm{SO}_0(1,2)$. In signature $(2,2)$ both B_- and B_+ define a real structure, implying that we there are different types of Majorana spinors, but no symplectic Majorana spinors. As discussed in Section 2.2.3 we can impose either the standard or the twisted Majorana condition on pairs of spinors. Together with the choice of a complex superbracket on the doubled spinor module, we have eight different real superbrackets, corresponding to the following combinations between complex superbrackets and reality conditions:

$$G_R = \mathbb{R}^{>0} \times \mathrm{SL}(2,\mathbb{R}) \quad \begin{cases} C_- \otimes \varepsilon, & (\lambda^i)^* = \alpha_1 B_- \lambda^j \eta_{ji} \leftarrow (2,\cancel{3}) \\ C_- \otimes \varepsilon, & (\psi^i)^* = \beta_1 B_+ \psi^j \eta_{ji} \leftarrow (\cancel{3},2) \\ C_+ \otimes \delta, & (\phi^i)^* = \gamma_1 B_- \phi^j \eta_{ji} \\ C_+ \otimes \delta, & (\xi^i)^* = \delta_1 B_+ \xi^j \eta_{ji} \\ C_+ \otimes \delta, & (\Lambda^i)^* = \alpha_2 B_- \Lambda^i \\ C_+ \otimes \delta, & (\Psi^i)^* = \beta_2 B_+ \Psi^i \\ C_- \otimes \varepsilon, & (\Phi^i)^* = \gamma_2 B_- \Phi^i \\ C_- \otimes \varepsilon, & (\Xi^i)^* = \delta_2 B_+ \Xi^i \end{cases} \tag{2.18}$$

The two theories obtained by dimensional reduction have the vector-valued bilinear form $C_- \otimes \varepsilon$ and the off-diagonal Majorana condition with either B_- or B_+, see table 3. Explicit maps between the eight superbrackets can be worked out using the same methods as for the other signatures, but we will not need to specify these maps explicitly for the following.

3. Four-dimensional $\mathcal{N}=2$ supersymmetric vector multiplets and their Lagrangians

3.1. *General considerations*

We will now present the four-dimensional off-shell supersymmetry transformations and Lagrangians which are obtained by dimensional reduction of the five-dimensional supersymmetry transformations and Lagrangians constructed in [2]. Since the actual computational steps are essentially the same as in [3], where the reductions $(1,4) \to (1,3)$ and $(1,4) \to (0,4)$ were carried out, we will only state the final results. All details required to replicate these results can be found in [3,2,1] and in the preceding sections and appendices of this paper. Compared to [3], one has to manage various factors of -1 and i, which is taken care of by our conventions for dimensionally reducing Clifford algebras and reality conditions.

To make the paper self-contained, we still need to review the relevant properties of five- and four-dimensional vector multiplets. The field content of a theory of n_V five-dimensional off-shell vector multiplets is

$$(\sigma^I, \lambda^{iI}, A^I_\mu, Y^I_{ij}) \ ,$$

where $I = 1, \ldots, n_V$, $i = 1, 2$. The fields σ^I are real scalar fields. All couplings in the five-dimensional Lagrangian are encoded in a real function, the Hesse potential $\mathcal{F}(\sigma^I)$ (sometimes also called the prepotential). The scalar and vector coupling matrices are proportional to the Hessian $\mathcal{F}_{IJ} = \partial^2_{IJ}\mathcal{F}$ of the function $\mathcal{F}$. The theory also contains a Chern-Simons term, with couplings proportional to the third derivatives $\mathcal{F}_{IJK}$ of $\mathcal{F}$. Since gauge invariance (up to boundary terms) requires $\mathcal{F}_{IJK}$ to be constant, the function $\mathcal{F}$ must be a cubic polynomial in σ^I.[f] The resulting geometry is called *affine special real geometry*, see the end of Section 4 of [3] for the precise definition. In short, an affine special real manifold is a pseudo-Riemannian manifold equipped with a flat torsion-free connection ∇, such that the metric can be expressed as the Hessian of a cubic real polynomial when using ∇-affine coordinates. The fields λ^{iI}, $i = 1, 2$, are pairs of spinors, subject to either a symplectic Majorana condition or a twisted Majorana condition:

$$(\lambda^i)^* = \begin{cases} \alpha_{t,s} B \lambda^j \varepsilon_{ji} \ , \ t = 0, 1, 4, 5 \ , \\ \alpha_{t,s} B \lambda^j \eta_{ji} \ , \ t = 2, 3 \ . \end{cases} \ .$$

The unit norm complex coefficients $\alpha_{t,s}$ are chosen according to Table 3 in [2], and have been listed in Table 3. With this convention the brackets on $\mathbb{S}$ and $\mathbb{S} \otimes \mathbb{C}^2$ have both standard form. The fields A^I_μ, $\mu = 1, \ldots, 5$ are vector fields, and Y_{ij} are auxiliary fields, which form a symmetric tensor under the action of the R-symmetry group, which is SU(2) for $t = 0, 1, 4, 5$ and SU(1, 1) for $t = 2, 3$. The auxiliary fields are subject to the following R-symmetry invariant reality condition, which is induced by the reality condition imposed on the spinors:

$$(Y^{ij})^* = \begin{cases} Y^{kl} \varepsilon_{ki} \varepsilon_{lj} \ , \ t = 0, 1, 4, 5 \ , \\ Y^{kl} \eta_{ki} \eta_{lj} \ , \ t = 2, 3 \ . \end{cases}$$

All together, a vector multiplet has $8 + 8$ off-shell degrees of freedom, which reduce to $4 + 4$ on-shell degrees of freedom upon imposing the equations of motion. We refer to [2] for further details.

Starting from the six possible signatures (t, s), $t + s = 5$ in five dimensions, there are ten different reductions to the five signatures (t', s'), $t' + s' = 4$ in four dimensions. The procedure of reduction is standard and straightforward. We use

[f] To have standard kinetic terms in signature $(1, 4)$ one must impose in addition that $\mathcal{F}_{IJ}$ is positive definite.

the notation and conventions of [3], which allow us to present the final expressions in a concise form. When reducing over the direction labeled by the index $*$, the five-dimensional vector fields A^I_μ decompose into four-dimensional vector fields A^I_m and scalars $b^I = A^I_*$. In the reduction $(1,4) \to (1,3)$ the five-dimensional scalars σ^I combine with the scalars $b^I = A^I_*$ into complex scalars $X^I = \sigma^I + ib^I$. The scalar manifold is an *affine special Kähler* manifold, as required by global $\mathcal{N} = 2$ supersymmetry. For time-like reductions $(1,4) \to (0,4)$, the kinetic terms of the scalars σ^I and b^I come with a relative sign and cannot be combined into a complex scalar. As shown in [3] the scalar geometry of Euclidean four-dimensional rigid vector multiplets is *affine special para-Kähler*, that is the complex structure is replaced by a para-complex structure. One can introduce para-complex scalar fields $X^I = \sigma^I + eb^I$, where the para-complex unit e satisfies $\bar{e} = -e$ and $e^2 = 1$. More generally, the special geometry of rigid and local vector and hypermultiplets in Euclidean signature involves the para-complex analogues of the familiar special Kähler, hyper-Kähler and quaternionic Kähler geometries. We refer to [3,7,8,9] for details.

When carrying out the ten possible reductions from five to four dimensions we find that the target space geometry only depends on the four-dimensional signature, and not on the five-dimensional parent theory. In Lorentz signature the target space is affine special Kähler, in Euclidean and neutral signature it is affine special para-Kähler. The relative signs between the kinetic terms of σ^I and b^I are listed in Table 5, while the types of target space geometries are listed in Table 4. These results are consistent with [10].

Before displaying the supersymmetry transformations and Lagrangians, we explain the ε-complex notation introduced in [3,8]. Depending on context a 'bar' over a scalar X denotes complex or para-complex conjugation:

$$X^I = \sigma^I + ib^I \Rightarrow \bar{X}^I = \sigma^I - ib^I \;, \quad X^I = \sigma^I + eb^I \Rightarrow \bar{X}^I = \sigma^I - eb^I \;.$$

When referring to both the complex and para-complex case simultaneously, we use the term ε-complex, where $\varepsilon = -1$ means complex, and $\varepsilon = 1$ means para-complex, and we define $i_\varepsilon = i, e$, respectively. The field content of a four-dimensional vector multiplet is

$$(X^I, \lambda^{iI}, A^I_m, Y^I_{ij}) \;,$$

where X^I are ε-complex scalars, λ^{iI} are pairs of spinors subject to a reality condition, A^I_m, $m = 1,2,3,4$ are vector fields, and Y^I_{ij} are auxiliary symmetric tensor fields, subject to the reality condition induced by the one imposed on the spinors. Since we construct the four-dimensional theories by the reduction of five-dimensional theories, the reality conditions of λ^{iI} and Y^I_{ij} are inherited from the five-dimensional theory, see Table 3. Note however that because the space of superbrackets is four-dimensional in four dimensions, we can change the superbracket by field redefinitions after the dimensional reduction. In the doubled formalism this changes the reality conditions imposed on λ^{iI} and Y_{ij}.

In four dimensions the supersymmetry transformations and the Lagrangian can be organised into ε-holomorphic and ε-anti-holomorphic terms, which are paired with chiral projections of the spinors. To write expressions uniformly, it is necessary to modify the chiral projection in the para-complex case such that it includes a factor e. In order to see explicitly why this is necessary, recall that since $\gamma_* B_\pm = (-1)^t B_\pm \gamma_*$ in four dimensions, complex conjugation acting on spinors preserves chirality in the signatures $t = 0, 2, 4$ with para-complex scalar geometry, but exchanges chiralities in the signatures $t = 1, 3$ with complex scalar geometry,

$$(\lambda^i_\pm)^* = \alpha \begin{cases} B\lambda^j_\pm M_{ji} \ , \ t = 0, 2, 4 \ , \\ B\lambda^j_\mp M_{ji} \ , \ t = 1, 3 \ , \end{cases}$$

where $\lambda^i_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_*)\lambda^i$. Following [3] we therefore define modified chiral projectors:

$$\Pi_\pm = \frac{1}{2}\left(\mathbb{1} \pm \Gamma_*\right) \ , \quad \Gamma_* = \begin{cases} e\gamma_* \ , \ t = 0, 2, 4 \ , \\ \gamma_* \ , \quad t = 1, 3 \ , \end{cases}$$

and correspondingly $\lambda^i_\pm := \frac{1}{2}(\mathbb{1} \pm \Gamma_*)\lambda^i$. Since $e^2 = 1$, the operators $\Pi_\pm$ are still projection operators. If we define the conjugation $(\lambda^{iI}_\pm)^*$ of the chiral projection of a spinor to include para-complex conjugation, chirality is flipped under $*$ in all signatures:

$$(\lambda^{iI}_\pm)^* = \alpha B \lambda^{jI}_\mp M_{ji} \ .$$

Note that

$$\gamma_* \lambda^i_\pm = \pm e \lambda^i_\pm \Leftrightarrow \Gamma_* \lambda^i_\pm = \pm \lambda^i_\pm \ . \tag{3.1}$$

We also modify the definition of self-dual and anti-self-dual field strength [3]:

$$F^I_{\pm|mn} := \frac{1}{2}\left(F^I_{mn} \pm \frac{1}{i_\varepsilon}\tilde{F}^I_{mn}\right)$$

where

$$\tilde{F}^I_{mn} = \frac{1}{2}\epsilon_{mnpq}F^{pq}$$

is the Hodge dual. These modified self-dual and anti-self-dual field strengths satisfy:

$$(F^I_{\pm|mn})^* = F^I_{\mp|mn} \ ,$$

where $*$ is ε-complex conjugation on the tangent space of the scalar manifold.

Formulas in Euclidean and neutral signature include both factors of i and of e. To avoid confusion, we point out that i corresponds to the action of the complex number on the spinor module, while e corresponds to the action of the para-complex numbers on the para-complexified tangent bundle of the scalar manifold. We refer to [3] for details.

In special ε-Kähler geometry, all couplings are encoded in a single function $\mathcal{F}(X^I)$, which is ε-holomorphic in the ε-complex scalars X^I. When obtaining a four-dimensional vector multiplet theory by dimensional reduction, the prepotential is given by the extension of the cubi Hesse potential $\mathcal{F}(\sigma^I)$ from real to ε-complex values, $\mathcal{F}(X^I) = \mathcal{F}(\sigma^I + i_\varepsilon b^I)$. Without a proportionality factor between Hesse potential and prepotential we obtain a parametrization known as 'old conventions' in the literature. The parametrization according to the 'new conventions' is obtained by setting $\mathcal{F}^{(new)} = \frac{1}{2i_\epsilon}\mathcal{F}^{(old)}$. We will use the old conventions to display our results.

As the Hesse potential is a cubic polynomial, so is any prepotential obtained by dimensional reduction. However, in four dimensions any ε-holomorphic prepotential defines a valid vector multiplet theory as long as the scalar and vector coupling matrices N_{IJ}, which in the old conventions are given by $N_{IJ} = \mathrm{Re}(\mathcal{F}_{IJ})$ are non-degenerate.[g] Since the only term involving the fourth derivative $\mathcal{F}_{IJKL}$ is a four-fermion term, one can take the supersymmetry variations and Lagrangians obtained by dimensional reduction, allow $\mathcal{F}$ to be a general ε-holomorphic function, and obtain the four-fermion term by checking which terms proportional to $\mathcal{F}_{IJKL}$ are generated by supersymmetry, see [3] for details. We will not work out the four-fermion terms in this paper, but write the Lagrangian in a form which remains valid if the prepotential is a general ϵ-holomorphic function. In particular, while $\mathcal{F}_{IJK}$ is a real constant when obtained from dimensional reduction, we will distinguish between $\mathcal{F}_{IJK}$ and $\bar{\mathcal{F}}_{IJK}$ when organising terms into ε-holomorphic and ε-anti-holomorphic components.

In the following sections we present the supersymmetry transformations and Lagrangians for the ten different reductions from five- to four-dimensional vector multiplet theories. Using the ε-complex notation, the ten different reductions can be combined into only four 'types' of supersymmetry transformations and Lagrangians. Table 4 lists for each reduction to which type it corresponds, together with the type of scalar geometry (which is completely determined by the four-dimensional signature, but listed for convenience).

[g] In new conventions the scalar and vector coupling matrices are given by $N_{IJ} = i_\varepsilon(\mathcal{F}^{\mathrm{new}}_{IJ} - \bar{\mathcal{F}}^{\mathrm{new}}_{IJ})$. To have standard kinetic terms for the standard vector multiplet theory in signature $(1,3)$ one must then impose that $\mathrm{Im}\mathcal{F}^{\mathrm{new}}_{IJ}$ is negative definite.

Table 4. The ten possible reductions of five-dimensional theories organise into four types. We also display the target space geometry.

Reduction	Type	Target geometry
$(0,5) \to (0,4)$	Type 2	special para-Kähler
$(1,4) \to (0,4)$	Type 1	special para-Kähler
$(1,4) \to (1,3)$	Type 1	special Kähler
$(2,3) \to (1,3)$	Type 3	special Kähler
$(2,3) \to (2,2)$	Type 3	special para-Kähler
$(3,2) \to (2,2)$	Type 4	special para-Kähler
$(3,2) \to (3,1)$	Type 4	special Kähler
$(4,1) \to (3,1)$	Type 2	special Kähler
$(4,1) \to (4,0)$	Type 2	special para-Kähler
$(5,0) \to (4,0)$	Type 1	special para-Kähler

3.2. *Type 1:* $(1,4) \mapsto (0,4)$ *or* $(1,3)$*, and* $(5,0) \mapsto (4,0)$

Representations

We start with the supersymmetry representations, which are off-shell and thus independent of the specification of a Lagrangian.

$$\begin{aligned}
\delta X^I &= i\bar{\epsilon}_+\lambda^I_+ \ , \qquad \delta\bar{X}^I = i\bar{\epsilon}_-\lambda^I_- \ , \\
\delta A^I_m &= \frac{1}{2}\left(\bar{\epsilon}_+\gamma_m\lambda^I_- + \bar{\epsilon}_-\gamma_m\lambda^I_+\right) \ , \\
\delta Y^I_{ij} &= -\frac{1}{2}\left(\bar{\epsilon}_{+(i}\partial\!\!\!/\lambda^I_{-j)} + \bar{\epsilon}_{-(i}\partial\!\!\!/\lambda^I_{+j)}\right) \ , \\
\delta\lambda^{Ii}_+ &= -\frac{1}{4}\gamma^{mn}F^I_{-mn}\epsilon^i_+ - \frac{i}{2}\partial\!\!\!/X^I\epsilon^i_- - Y^{Iij}\epsilon_{+j} \ , \\
\delta\lambda^{Ii}_- &= -\frac{1}{4}\gamma^{mn}F^I_{+mn}\epsilon^i_- - \frac{i}{2}\partial\!\!\!/\bar{X}^I\epsilon^i_+ - Y^{Iij}\epsilon_{-j} \ .
\end{aligned} \tag{3.2}$$

The supersymmetry variation parameters are doubled spinors denoted $\epsilon = (\epsilon^i)$ and are subject to the same reality conditions as the doubled spinors $\lambda = (\lambda^i)$, which are listed in table 3. For all theories obtained by dimensional reduction the underlying complex superbracket is defined by the complex bilinear form $C_- \otimes \varepsilon$, and therefore indices $i, j = 1, 2$ are raised and lowered using ε^{ij} and ε_{ij}, irrespective of the reality condition, in the same way as in [2], namely

$$\lambda^i = \varepsilon^{ij}\lambda_j \ , \quad \lambda_i = \lambda^j\varepsilon_{ji} \ , \quad \varepsilon^{ik}\varepsilon_{kj} = -\delta^i_j \ , \tag{3.3}$$

which conforms with the NW-SE convention. We use the notation $\partial\!\!\!/ = \gamma^m\partial_m$.

With regard to the splitting into ε-holomorphic and ε-anti-holomorphic parts it is important to keep in mind the following notational conventions: the operation $\bar{\cdot}$

denotes ε-complex conjugation for scalars X^I, but, as before, Majorana conjugation based on the charge conjugation matrix C_- for spinors λ^I. The chiral projectors for spinors include a factor e for those signatures where the target geometry is para-complex. The real structure relating ε-holomorphic and ε-anti-holomorphic expressions is the combined complex/para-complex conjugation $*$, which acts on both the target space and the spinor module. Spinors are Grassmann-valued, and we use a convention where complex conjugation does not reverse the order of factors in monomials.[h] The self-dual and anti-self-dual projections of tensors are defined using projections which include a factor e for signatures where the target space geometry is para-complex. Note that (3.2) agrees with (5.64) of [3], which is the original reference for the reductions $(1,4) \mapsto (0,4)$ and $(1,4) \mapsto (1,3)$.

Lagrangians

The following Lagrangians, obtained by dimensional reduction, are by construction invariant under the supersymmetry transformations given in the previous section. With regard to the overall sign of the Lagrangian, we have adopted the convention that the sign of the coefficient of the Maxwell term is always negative. This is motivated by the fact that in Lorentz signature this choice of sign corresponds to positive kinetic energy of the Maxwell field, irrespective of whether we choose the mostly plus or the mostly minus convention.

$$\begin{aligned} L = &-\frac{1}{4}\big(F^I_{-mn}F^{Jmn}_-\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_+\bar{\mathcal{F}}_{IJ}(\bar{X})\big) \\ &-\frac{1}{2}\partial_m X^I \partial^m \bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij}Y^J_{ij}N_{IJ}(X,\bar{X}) \\ &-\frac{1}{2}\big(\bar{\lambda}^I_+ \partial\!\!\!/ \lambda^J_- + \bar{\lambda}^I_- \partial\!\!\!/ \lambda^J_+\big)N_{IJ}(X,\bar{X}) \\ &-\frac{1}{4}\big(\bar{\lambda}^I_- \partial\!\!\!/ \mathcal{F}_{IJ}(X)\lambda^J_+ + \bar{\lambda}^I_+ \partial\!\!\!/ \bar{\mathcal{F}}_{IJ}(\bar{X})\lambda^J_-\big) \\ &-\frac{i}{8}\big(\bar{\lambda}^I_+\gamma^{mn}F^J_{-mn}\lambda^K_+\mathcal{F}_{IJK} + \bar{\lambda}^I_-\gamma^{mn}F^J_{+mn}\lambda^K_-\bar{\mathcal{F}}_{IJK}\big) \\ &-\frac{i}{2}\big(\bar{\lambda}^{Ii}_+\lambda^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} + \bar{\lambda}^{Ii}_-\lambda^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\big)\,. \end{aligned} \tag{3.4}$$

Note that this Lagrangian agrees with (5.70) of [3].[i] Also note that the fermions are symplectic Majorana spinors, while in signature $(1,3)$ one would normally write the theory in terms of Majorana spinors. This can be done using the isomorphism found in Section 2.2.3. In fact it was checked in [3] that upon rewriting the theory in terms

[h]If one converts our expressions to the convention where complex conjugation reverses the order of Grassmann variables, this leads to additional factors of (powers of) i.
[i]We remark that $\bar{Y}^I_{ij}$ in (5.70) of [3] should read Y^I_{ij}, that is, the 'bar' is superfluous. This is easily seen by checking that $(\bar{\lambda}^{Ii}_-\lambda^{Jj}_-Y^K_{ij})^* = -\bar{\lambda}^{Ii}_+\lambda^{Jj}_+Y^K_{ij}$.

of Majorana spinors one obtains supersymmetry transformations and Lagrangians which are consistent with the literature.

3.3. *Type 2:* $(0,5) \to (0,4)$ *and* $(4,1) \to (3,1)$ *or* $(4,0)$

From here on we just list the representations and Lagrangians without comment. The discussion is continued further below.

Representations

$$
\begin{aligned}
\delta X^I &= \bar{\epsilon}_+ \lambda_+^I \ , \qquad \delta \bar{X}^I = \bar{\epsilon}_- \lambda_-^I \ , \\
\delta A_m^I &= \frac{1}{2}\left(\bar{\epsilon}_+ \gamma_m \lambda_-^I + \bar{\epsilon}_- \gamma_m \lambda_+^I\right) \ , \\
\delta Y_{ij}^I &= -\frac{1}{2}\left(\bar{\epsilon}_{+(i} \partial\!\!\!/ \lambda_{-j)}^I + \bar{\epsilon}_{-(i} \partial\!\!\!/ \lambda_{+j)}^I\right) \ , \\
\delta \lambda_+^{Ii} &= -\frac{1}{4}\gamma^{mn} F_{-mn}^I \epsilon_+^i + \frac{1}{2} \partial\!\!\!/ X^I \epsilon_-^i - Y^{Iij} \epsilon_{+j} \ , \\
\delta \lambda_-^{Ii} &= -\frac{1}{4}\gamma^{mn} F_{+mn}^I \epsilon_-^i + \frac{1}{2} \partial\!\!\!/ \bar{X}^I \epsilon_+^i - Y^{Iij} \epsilon_{-j} \ .
\end{aligned}
\tag{3.5}
$$

Lagrangians

$$
\begin{aligned}
L = &-\frac{1}{4}\left(F_{-mn}^I F_-^{Jmn} \mathcal{F}_{IJ}(X) + F_{+mn}^I F_+^{Jmn} \bar{\mathcal{F}}_{IJ}(\bar{X})\right) \\
&+ \frac{1}{2}\partial_m X^I \partial^m \bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij} Y_{ij}^J N_{IJ}(X,\bar{X}) \\
&- \frac{1}{2}\left(\bar{\lambda}_+^I \partial\!\!\!/ \lambda_-^J + \bar{\lambda}_-^I \partial\!\!\!/ \lambda_+^J\right) N_{IJ}(X,\bar{X}) \\
&- \frac{1}{4}\left(\bar{\lambda}_-^I \partial\!\!\!/ \mathcal{F}_{IJ}(X) \lambda_+^J + \bar{\lambda}_+^I \partial\!\!\!/ \bar{\mathcal{F}}_{IJ}(\bar{X}) \lambda_-^J\right) \\
&- \frac{1}{8}\left(\bar{\lambda}_+^I \gamma^{mn} F_{-mn}^J \lambda_+^K \mathcal{F}_{IJK} + \bar{\lambda}_-^I \gamma^{mn} F_{+mn}^J \lambda_-^K \bar{\mathcal{F}}_{IJK}\right) \\
&- \frac{1}{2}\left(\bar{\lambda}_+^{Ii} \lambda_+^{Jj} Y_{ij}^K \mathcal{F}_{IJK} + \bar{\lambda}_-^{Ii} \lambda_-^{Jj} Y_{ij}^K \bar{\mathcal{F}}_{IJK}\right) \ .
\end{aligned}
\tag{3.6}
$$

3.4. *Type 3:* $(2,3) \to (1,3)$ *or* $(2,2)$.

Representations

$$
\begin{aligned}
\delta X^I &= \bar{\epsilon}_+ \lambda_+^I \,, \qquad \delta \bar{X}^I = \bar{\epsilon}_- \lambda_-^I \,, \\
\delta A_m^I &= \frac{1}{2}\big(\bar{\epsilon}_+ \gamma_m \lambda_-^I + \bar{\epsilon}_- \gamma_m \lambda_+^I\big) \,, \\
\delta Y_{ij}^I &= -\frac{i}{2}\big(\bar{\epsilon}_{+(i} \not{\partial} \lambda_{-j)}^I + \bar{\epsilon}_{-(i} \not{\partial} \lambda_{+j)}^I\big) \,, \\
\delta \lambda_+^{Ii} &= -\frac{1}{4}\gamma^{mn} F_{-mn}^I \epsilon_+^i + \frac{1}{2}\not{\partial} X^I \epsilon_-^i + i Y^{Iij} \epsilon_{+j} \,, \\
\delta \lambda_-^{Ii} &= -\frac{1}{4}\gamma^{mn} F_{+mn}^I \epsilon_-^i + \frac{1}{2}\not{\partial} \bar{X}^I \epsilon_+^i + i Y^{Iij} \epsilon_{-j} \,.
\end{aligned}
\tag{3.7}
$$

Lagrangians

$$
\begin{aligned}
L = &-\frac{1}{4}\big(F_{-mn}^I F_-^{Jmn} \mathcal{F}_{IJ}(X) + F_{+mn}^I F_+^{Jmn} \bar{\mathcal{F}}_{IJ}(\bar{X})\big) \\
&+ \frac{1}{2}\partial_m X^I \partial^m \bar{X}^J N_{IJ}(X,\bar{X}) - Y^{Iij} Y_{ij}^J N_{IJ}(X,\bar{X}) \\
&- \frac{1}{2}\big(\bar{\lambda}_+^I \not{\partial} \lambda_-^J + \bar{\lambda}_-^I \not{\partial} \lambda_+^J\big) N_{IJ}(X,\bar{X}) \\
&- \frac{1}{4}\big(\bar{\lambda}_-^I \not{\partial} \mathcal{F}_{IJ}(X) \lambda_+^J + \bar{\lambda}_+^I \not{\partial} \bar{\mathcal{F}}_{IJ}(\bar{X}) \lambda_-^J\big) \\
&- \frac{1}{8}\big(\bar{\lambda}_+^I \gamma^{mn} F_{-mn}^J \lambda_+^K \mathcal{F}_{IJK} + \bar{\lambda}_-^I \gamma^{mn} F_{+mn}^J \lambda_-^K \bar{\mathcal{F}}_{IJK}\big) \\
&+ \frac{i}{2}\big(\bar{\lambda}_+^{Ii} \lambda_+^{Jj} Y_{ij}^K \mathcal{F}_{IJK} + \bar{\lambda}_-^{Ii} \lambda_-^{Jj} Y_{ij}^K \bar{\mathcal{F}}_{IJK}\big) \,.
\end{aligned}
\tag{3.8}
$$

3.5. *Type 4:* $(3,2) \to (3,1)$ *or* $(2,2)$

Representations

$$
\begin{aligned}
\delta X^I &= i\bar{\epsilon}_+ \lambda_+^I \,, \qquad \delta \bar{X}^I = i\bar{\epsilon}_- \lambda_-^I \,, \\
\delta A_m^I &= \frac{1}{2}\big(\bar{\epsilon}_+ \gamma_m \lambda_-^I + \bar{\epsilon}_- \gamma_m \lambda_+^I\big) \,, \\
\delta Y_{ij}^I &= -\frac{i}{2}\big(\bar{\epsilon}_{+(i} \not{\partial} \lambda_{-j)}^I + \bar{\epsilon}_{-(i} \not{\partial} \lambda_{+j)}^I\big) \,, \\
\delta \lambda_+^{Ii} &= -\frac{1}{4}\gamma^{mn} F_{-mn}^I \epsilon_+^i - \frac{i}{2}\not{\partial} X^I \epsilon_-^i + i Y^{Iij} \epsilon_{+j} \,, \\
\delta \lambda_-^{Ii} &= -\frac{1}{4}\gamma^{mn} F_{+mn}^I \epsilon_-^i - \frac{i}{2}\not{\partial} \bar{X}^I \epsilon_+^i + i Y^{Iij} \epsilon_{-j} \,.
\end{aligned}
\tag{3.9}
$$

Lagrangians

$$
\begin{aligned}
L = & -\frac{1}{4}\big(F^I_{-mn}F^{Jmn}_{-}\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_{+}\bar{\mathcal{F}}_{IJ}(\bar{X})\big) \\
& -\frac{1}{2}\partial_m X^I \partial^m \bar{X}^J N_{IJ}(X,\bar{X}) - Y^{Iij}Y^J_{ij}N_{IJ}(X,\bar{X}) \\
& -\frac{1}{2}\big(\bar{\lambda}^I_+ \partial\!\!\!/ \lambda^J_- + \bar{\lambda}^I_- \partial\!\!\!/ \lambda^J_+\big)N_{IJ}(X,\bar{X}) \\
& -\frac{1}{4}\big(\bar{\lambda}^I_- \partial\!\!\!/ \mathcal{F}_{IJ}(X)\lambda^J_+ + \bar{\lambda}^I_+ \partial\!\!\!/ \bar{\mathcal{F}}_{IJ}(\bar{X})\lambda^J_-\big) \\
& -\frac{i}{8}\big(\bar{\lambda}^I_+\gamma^{mn}F^J_{-mn}\lambda^K_+\mathcal{F}_{IJK} + \bar{\lambda}^I_-\gamma^{mn}F^J_{+mn}\lambda^K_-\bar{\mathcal{F}}_{IJK}\big) \\
& -\frac{1}{2}\big(\bar{\lambda}^{Ii}_+\lambda^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} + \bar{\lambda}^{Ii}_-\lambda^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\big)\,.
\end{aligned}
\tag{3.10}
$$

3.6. *(In-)Equivalent theories and the relative signs of scalar and vector kinetic terms*

We now continue our discussion of the properties of the ten vector multiplet representations and Lagrangians that we have obtained in the five distinct signatures. From the classification of four-dimensional $\mathcal{N}=2$ Poincaré Lie superalgebras [1], combined with our knowledge of R-symmetry groups, we already know in which cases the two theories in any given signature must be equivalent.

Thanks to the use of ε-complex notation and of doubled spinors the 10 sets of supersymmetry transformations and Lagrangians take exactly the same form and only differ by relative signs and factors of i between terms. We focus on the bosonic terms in the following. The relative signs between the kinetic terms of the scalars $\sigma^I = \mathrm{Re}X^I$ and $b^I = \mathrm{Im}X^I$ have already been discussed. They are related to whether the target geometry is complex or para-complex, which in turn depends on the signature, or more precisely on the Abelian factor of the R-symmetry group [3], which is U(1) for complex and SO(1,1) for para-complex target geometry. We now turn to the relative sign between the scalar and the vector term (Maxwell term) $F^2 \propto N_{IJ}F^I_{mn}F^{J|mn}$. All relevant signs have been listed in Table 5. As already mentioned our convention for the overall sign is that the vector term always comes with a negative sign. The signature of N_{IJ} depends on the choice of the prepotential and the range of the scalar fields. We focus on the model-independent overall sign between scalar and vector terms.

3.6.1. *Euclidean signature*

The Euclidean signatures $(0,4)$ and $(4,0)$ are equivalent. We discuss the case $(0,4)$ for definiteness. The target space geometry is para-Kähler, and the relative sign between scalar and vector terms is different for the reductions $(0,5) \to (0,4)$ and $(1,4) \to (0,4)$. Since we have shown in [1] that the Euclidean $\mathcal{N}=2$ supersymmetry algebra is unique up to isomorphism, we expect that the two sets of supersymmetry

Table 5. Relative signs between vector kinetic terms and scalar kinetic terms for the ten possible dimensional reductions.

Reduction	F^2	$(\partial\sigma)^2$	$(\partial b)^2$
$(0,5)\to(0,4)$	$-$	$+$	$-$
$(1,4)\to(0,4)$	$-$	$-$	$+$
$(1,4)\to(1,3)$	$-$	$-$	$-$
$(2,3)\to(1,3)$	$-$	$+$	$+$
$(2,3)\to(2,2)$	$-$	$+$	$-$
$(3,2)\to(2,2)$	$-$	$-$	$+$
$(3,2)\to(3,1)$	$-$	$-$	$-$
$(4,1)\to(3,1)$	$-$	$+$	$+$
$(4,1)\to(4,0)$	$-$	$+$	$-$
$(5,0)\to(4,0)$	$-$	$-$	$+$

transformations and Lagrangians are related by a field redefinition, which we will now identify explicitly. The relation between the two supersymmetry algebras in the doubled spinor formulation was found in Section 2.2.2, see formula (2.4). In the following we denote the spinors resulting from the Type 2 reduction $(0,5)\to(0,4)$ by λ^i and the spinors resulting from the Type 1 reduction $(1,4)\to(0,4)$ by $\tilde{\lambda}^i$. Then (2.5) becomes

$$\lambda^i = \frac{1}{\sqrt{2}}\left(1 - i\gamma_*\right)\tilde{\lambda}^i = \frac{1}{\sqrt{2}}\left(1 - ie\Gamma_*\right)\tilde{\lambda}^i \,, \tag{3.11}$$

where we expressed the standard chirality matrix γ_* in terms of the matrix $\Gamma_* = e\gamma_*$, which we use in the para-holomorphic formalism. The inverse transformation is

$$\tilde{\lambda}^i = \frac{1}{\sqrt{2}}\left(1 + i\gamma_*\right)\lambda^i = \frac{1}{\sqrt{2}}\left(1 + ie\Gamma_*\right)\lambda^i \,.$$

The chiral projections are related by:

$$\lambda^i_\pm = \frac{1}{\sqrt{2}}(1 \pm ie)\tilde{\lambda}^i_\pm \,.$$

Note that the positive and negative chirality terms transform with a relative sign. We will also need the relations between the following spinor bilinears:

$$\bar{\epsilon}\lambda = -i\bar{\tilde{\epsilon}}\gamma_*\tilde{\lambda} = -ie\bar{\tilde{\epsilon}}\Gamma_*\tilde{\lambda} \Rightarrow \bar{\epsilon}_\pm\lambda_\pm = \mp ie\bar{\tilde{\epsilon}}_\pm\tilde{\lambda}_\pm \,,$$
$$\bar{\epsilon}\gamma^m\lambda = \bar{\tilde{\epsilon}}\gamma^m\tilde{\lambda} \Rightarrow \bar{\epsilon}_\pm\lambda_\mp = \bar{\tilde{\epsilon}}_\pm\tilde{\lambda}_\mp \,,$$
$$\bar{\epsilon}\gamma^{mn}\lambda = -i\bar{\tilde{\epsilon}}\gamma^{mn}\gamma_*\tilde{\lambda} = -ie\bar{\tilde{\epsilon}}\gamma^{mn}\Gamma_*\tilde{\lambda} \Rightarrow \bar{\epsilon}_\pm\gamma^{mn}\lambda_\pm = \mp ie\bar{\tilde{\epsilon}}_\pm\gamma^{mn}\tilde{\lambda}_\pm \,.$$

Note that the vector bilinear remains the same, as it must since the vector bilinear defines the complex supersymmetry algebra, which remains unchanged. The scalar and tensor bilinear transform non-trivially, and with a relative sign between terms of positive and negative chirality.

Substituting (3.11) into the supersymmetry transformations (3.5), and using the above relations, we obtain

$$
\begin{aligned}
\delta X^I &= -\mathbf{ie}\bar{\tilde{\epsilon}}_+\tilde{\lambda}^I_+ \;, \qquad \delta \bar{X}^I = \mathbf{ie}\bar{\tilde{\epsilon}}_-\tilde{\lambda}^I_- \;, \\
\delta A^I_m &= \frac{1}{2}\left(\bar{\tilde{\epsilon}}_+\gamma_m\tilde{\lambda}^I_- + \bar{\tilde{\epsilon}}_-\gamma_m\tilde{\lambda}^I_+\right) \;, \\
\delta Y^I_{ij} &= -\frac{1}{2}\left(\bar{\tilde{\epsilon}}_{+(i}\partial\!\!\!/\tilde{\lambda}^I_{-j)} + \bar{\tilde{\epsilon}}_{-(i}\partial\!\!\!/\tilde{\lambda}^I_{+j)}\right) \;, \\
\delta\tilde{\lambda}^{Ii}_+ &= -\frac{1}{4}\gamma^{mn}F^I_{-mn}\tilde{\epsilon}^i_+ + \mathbf{ie}\frac{1}{2}\partial\!\!\!/ X^I\tilde{\epsilon}^i_- - Y^{Iij}\tilde{\epsilon}_{+j} \;, \\
\delta\tilde{\lambda}^{Ii}_- &= -\frac{1}{4}\gamma^{mn}F^I_{+mn}\tilde{\epsilon}^i_- - \mathbf{ie}\frac{1}{2}\partial\!\!\!/ \bar{X}^I\tilde{\epsilon}^i_+ - Y^{Iij}\tilde{\epsilon}_{-j} \;,
\end{aligned}
\tag{3.12}
$$

where changes of relative factors have been indicated in boldface. Comparing to the supersymmetry variations (3.2) for the reduction $(1,4)\to(0,4)$ we see that they agree up to factors of e which can be aborbed by setting $\tilde{X}^I = -eX^I$. Thus we have identified a field redefinition which maps the two vector multiplets to each other. Turning our attention to the Lagrangian we find that applying (3.11) to (3.6) gives

$$
\begin{aligned}
L = &-\frac{1}{4}\left(F^I_{-mn}F^{Jmn}_-\mathcal{F}_{IJ}(X) + F^I_{+mn}F^{Jmn}_+\bar{\mathcal{F}}_{IJ}(\bar{X})\right) \\
&+\frac{1}{2}\partial_m X^I\partial^m\bar{X}^J N_{IJ}(X,\bar{X}) + Y^{Iij}Y^J_{ij}N_{IJ}(X,\bar{X}) \\
&-\frac{1}{2}\left(\bar{\tilde{\lambda}}^I_+\partial\!\!\!/\tilde{\lambda}^J_- + \bar{\tilde{\lambda}}^I_-\partial\!\!\!/\tilde{\lambda}^J_+\right)N_{IJ}(X,\bar{X}) \\
&-\frac{1}{4}\left(\bar{\tilde{\lambda}}^I_-\partial\!\!\!/\mathcal{F}_{IJ}(X)\tilde{\lambda}^J_+ + \bar{\tilde{\lambda}}^I_+\partial\!\!\!/\bar{\mathcal{F}}_{IJ}(\bar{X})\tilde{\lambda}^J_-\right) \\
&-\frac{1}{8}\left(-\mathbf{ie}\bar{\tilde{\lambda}}^I_+\gamma^{mn}F^J_{-mn}\tilde{\lambda}^K_+\mathcal{F}_{IJK} + \mathbf{ie}\bar{\tilde{\lambda}}^I_-\gamma^{mn}F^J_{+mn}\tilde{\lambda}^K_-\bar{\mathcal{F}}_{IJK}\right) \\
&-\frac{1}{2}\left(-\mathbf{ie}\bar{\tilde{\lambda}}^{Ii}_+\tilde{\lambda}^{Jj}_+Y^K_{ij}\mathcal{F}_{IJK} + \mathbf{ie}\bar{\tilde{\lambda}}^{Ii}_-\tilde{\lambda}^{Jj}_-Y^K_{ij}\bar{\mathcal{F}}_{IJK}\right)
\end{aligned}
\tag{3.13}
$$

This has to match with (3.4) upon setting $\tilde{X}^I = -eX^I$. To see that this is indeed true we simply note that the prepotential is a para-holomorphic function and transforms as a scalar: $\tilde{\mathcal{F}}(\tilde{X}) = \mathcal{F}(X)$, which implies that para-holomorphic derivatives transform as $\tilde{\mathcal{F}}_I = -e\mathcal{F}_I$, $\tilde{\mathcal{F}}_{IJ} = \mathcal{F}_{IJ}$, $\tilde{\mathcal{F}}_{IJK} = -e\mathcal{F}_{IJK}, \ldots$ and anti-para-holomorphic derivatives transform as $\bar{\tilde{\mathcal{F}}}_I = e\bar{\mathcal{F}}_I$, $\bar{\tilde{\mathcal{F}}}_{IJ} = \bar{\mathcal{F}}_{IJ}$, $\bar{\tilde{\mathcal{F}}}_{IJK} = e\bar{\mathcal{F}}_{IJK}, \ldots$. Note that the second derivatives of $\mathcal{F}$, and therefore the tensor N_{IJ} which enters into defining the scalar metric, do not change. The only bosonic term affected by the transformation is the scalar sigma model term, where the overall sign flips:

$$
\partial_m X^I\partial^m X^J N_{IJ} = (-e)(-\bar{e})\partial_m\tilde{X}^I\partial^m\bar{\tilde{X}}^J N_{IJ} = -\partial_m\tilde{X}^I\partial^m\bar{\tilde{X}}^J N_{IJ} \;,
$$

where we used that $e\bar{e} = -e^2 = -1$[j]. Thus changing the vector multiplet represen-

[j]This conclusion remains, of course, unchanged when using real instead of para-complex coordinates. See [3] for a discussion of so-called adapted real coordinates.

tation from one Euclidean $\mathcal{N}=2$ superalgebra to a different, but isomorphic one flips the relative sign between scalar and vector terms.

We remark that our transformation is different from the one advocated in [11], which is a duality-like rotation of the field equations combined with multiplying the vector (X^I, F_I) by e. This transformation flips the sign of the vector term, while the extra factor e has the effect of keeping the sign of the scalar term the same. While the net effect on the bosonic Lagrangian differs from our transformation only by an overall sign, their transformation is non-local, and was interpreted as a strong-weak coupling duality. In contrast, our transformation is local, works for the off-shell representation and the Lagrangian, includes fermionic terms, and is induced by an isomorphism between two Euclidean $\mathcal{N}=2$ superalgebras that arise from dimensionally reducing five-dimensional supersymmetry algebras.

When listing our Lagrangians we have fixed the overall sign of the Lagrangian by the convention that the sign of the vector term is always negative, so that relative signs show up in front of the scalar term. The two four-dimensional Euclidean supergravity theories discussed in [11] by the sign of the vector term, while the scalar and Einstein-Hilbert term have the same sign. While the full treatment of supergravity in the superconformal approach requires working out the Weyl multiplet in arbitrary signature, we remark that the Einstein-Hilbert term will have a prefactor $-e(X^I\bar{\mathcal{F}}_I - \mathcal{F}_I\bar{X}^I)$, which is then fixed to a constant value by imposing the so-called D-gauge. This term changes sign under the redefinition $\tilde{X}^I = -eX^I$, thus giving rise to the same pattern of relative signs as in [11].

3.6.2. *Neutral signature*

Neutral signature can be realized by the reductions $(2,3)\to(2,2)$ and $(3,2)\to(2,2)$, which are of Type 3 and of Type 4, respectively. Since the five-dimensional theories are related by going from a mostly plus to a mostly minus convention for the metric, we expect them to be equivalent. In fact, we have proved in [1] that there is a unique neutral signature $\mathcal{N}=2$ superalgebra up to isomorphism, and therefore both theories must be related by a field redefinition, which can be worked out using the same methods as for Euclidean signature.

3.6.3. *Minkowski signature*

Here we have to consider the reductions $(1,4)\to(1,3)$, $(2,3)\to(1,3)$ and $(4,1)\to(3,1)$, $(3,2)\to(3,1)$. The four-dimensional signatures $(1,3)$ and $(3,1)$ are related by going from a mostly plus to a mostly minus convention from the metric, and from [1] we know that there are two classes of non-isomorphic $\mathcal{N}=2$ superalgebras: the standard one with compact R-symmetry U(2) and the twisted (or type-*) one with non-compact R-symmetry group U(1,1). Since the five-dimensional theories in signature $(1,3)$, $(3,1)$ have R-symmetry SU(2), while those in signature $(2,3)$, $(3,2)$ have R-symmetry SU(1,1), we see that while reductions from

Minkowski signature to Minkowski signature give (of course) a realization of the standard supersymmetry algebra, we can obtain the twisted Minkowski signature supersymmetry algebra by reducing five-dimensional theories with two time-like directions. Looking at the respective Lagrangians we see that this time the relative sign between scalar and vector terms immediately distinguishes both cases. Since in Minkowski signature these signs are tied to the kinetic energy of scalar and vector fields being positive or negative, it is clear that they have invariant physical meaning. In contrast, in Euclidean and neutral signature we have seen that these signs can be changed by local field redefinitions relating representations of distinct but isomorphic supersymmetry algebras.

In [10] the bosonic Lagrangians and Killing spinor equations of two $\mathcal{N} = 2$ Lorentzian supergravity theories differing by the sign of the vector term relative to the scalar term and also relative to the Einstein-Hilbert term were obtained by dimensional reduction of five dimensional supergravity with one or two time-like dimensions. This is consistent with our results, and we expect that the theory with inverted sign for the vector term realizes the $\mathcal{N} = 2$ supersymmetry algebra with R-symmetry group $\mathrm{U}(1,1)$. In particular, we expect that upon coupling to supergravity the Einstein Hilbert term will have the same sign relative to the vector term as the scalar term, because within the superconformal formalism the Einstein-Hilbert term arises from a term of the form $D_m X^I D^m \bar{X}^J N_{IJ}(X, \bar{X})$, where D_m is the covariant derivative with respect to superconformal transformations. Since the Einstein-Hilbert term also obtains a contribution from the superconformal hypermultiplet sector, a full derivation will require reformulating hypermultiplets and the Weyl multiplet in arbitrary signature, which we leave to future work.

4. Outlook

In this paper we have provided off-shell vector multiplet representations and Lagrangians for the four-dimensional $\mathcal{N} = 2$ supersymmetry algebras classified in [1]. We have shown that the relative sign between scalar and vector terms is conventional in Euclidean and neutral signature, but discriminates between the two inequivalent supersymmetry algebras in Lorentz signature. Since the vector spaces of superbrackets have been constructed in [4] for all dimensions and signatures, carrying out a full classification appears feasible along the lines of [1] and the present paper. This would then also include the case of signature $(1,1)$, which was excluded from Theorem 1 in [1]. Such a classification should also list the corresponding R-symmetry groups and BPS extensions, the latter based on the results of [12]. Moreover, it is desirable to more directly relate the formalism used in [4,12] to the language used in the physics literature. This would include a description of the basis of super-admissible forms using the matrices A, B, C and relating spinor modules to doubled spinor modules, as we have done in this paper for four-dimensional $\mathcal{N} = 2$ supersymmetry algebras.

Part of this programme will be addressed in an upcoming paper [13] which will develop an extension of the doubled spinor formalism to provide realizations of $\mathcal{N}$-extended supersymmetry algebras in arbitrary dimension and signature, and for any $\mathcal{N}$, with explicit separation of the actions of the Lorentz and of the R-symmetry group, thus making R-symmetry manifest. Regarding physical applications, further steps will include hypermultiplets, and Weyl multiplets, thus facilitating the coupling to supergravity. So far off-shell formulations of five- and four-dimensional $\mathcal{N}=2$ supergravity within the superconformal approach are available in signature $(1,3)$,$(1,4)$ and $(0,4)$ [14,15,16]. This formalism allows one to include higher curvature terms through explicit dependence of the prepotential on the Weyl multiplet. Following the strategy of [2] and of the present paper, it should be possible to extend existing results to arbitrary signature. This would allow one to extend the study of BPS solutions with higher derivative terms to arbitrary signature. In the past years there has been work on the classification of four-dimensional BPS solutions both in Euclidean signature, see for example [17], [18], and in neutral signature, see for example [19,20], and as well on so-called phantom solutions of Lorentzian signature theories with flipped gauge kinetic terms [21,22,23].

More generally, we expect that further developing the approach used in [2,1] and in the present paper will be useful for exploring the extended network of string and M-theories across dimensions and signatures. In particular it should provide a new perspective on generalized Killing spinor equations and non-standard supergravity theories, which have been discussed under names such as 'fake-/pseudo-Killing spinor equations' and 'fake-/pseudo-supergravity,' following [24,25,26], see also [27] for an overview and more references. It seems clear that fake-/pseudo-supersymmetry is related to existence of de Sitter and type-* superalgebras, non-compact R-symmetries and their gaugings, and time-like T-duality [6,28,29], the common feature being the analytic continuation of 'conventional' theories and Killing spinor equations. Therefore a more unified picture requires a systematic way of dealing with complexification and reality conditions. In [30,31] it was shown that all maximal supergravities in ten and eleven dimensions arise from contractions of different real forms of a complex ortho-symplectic Lie superalgebra. Our approach is similar in spirit but instead of ortho-symplectic Lie superalgberas it works directly with Poincaré Lie superalgebras, it allows one to study the space of all possible superbrackets, and it provides a new way of dealing with complexification and reality conditions through the doubled spinor formalism.

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Appendix A. Clifford algebras

A.1. *Conventions for γ-matrices*

The real Clifford algebra $Cl_{t,s}$ is represented by matrices γ^μ, $\mu = 1,\dots t+s = n$ satisfying

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}\ , \quad (\eta^{\mu\nu}) = \mathrm{diag}(-1,\dots,-1,1,\dots 1)\ .$$

This is the same convention as in [3,2], which differs from [32] by a relative sign in the defining relation of the Clifford algebra, and a relative sign in the definition of $\eta^{\mu\nu}$. The net effect is that $Cl_{t,s}$ refers to the same real associative algebra.

The γ-matrices are chosen such that they are either Hermitian or anti-Hermitian matrices:

$$(\gamma^\mu)^\dagger = \begin{cases} -\gamma^\mu\ , \ \mu = 1,\dots t, \\ \gamma^\mu\ , \quad \mu = t+1,\dots t+s\ . \end{cases}$$

We will refer to the anti-Hermitian γ-matrices as time-like and to the Hermitian γ-matrices as space-like, though for physics purposes we take $\min\{t,s\}$ to be the number of dimensions interpreted as time. This reflects that we conventionally prefer the ‘mostly plus’ convention for Minkowski signature.

There exist matrices A, B, C which relate the γ-matrices to the Hermitian conjugate, complex conjugate and transposed matrices [33,3]:

$$\begin{aligned} (\gamma^\mu)^\dagger &= (-1)^t A\gamma^\mu A^{-1}\ , \\ (\gamma^\mu)^* &= (-1)^t \tau B\gamma^\mu B^{-1}\ , \\ (\gamma^\mu)^T &= \tau C\gamma^\mu C^{-1}\ , \end{aligned} \tag{A.1}$$

where $\sigma,\tau \in \{\pm 1\}$. The parameters σ,τ are related to the parameters ε,η used in [33] by $\sigma = -\varepsilon$ and $\tau = -\eta$. Note that $\sigma = \sigma(C)$ and $\tau = \tau(C)$ are the symmetry and type of the $\mathrm{Spin}_0(t,s)$-invariant complex bilinear form (‘Majorana bilinear form’)

$$C(\lambda,\chi) = \lambda^T C\chi$$

on $\mathbb{S}$ defined by the *charge conjugation matrix* C. We choose a representation where C is Hermitian and unitary, which is always possible [33]:

$$C^{-1} = C^\dagger = C\ .$$

The matrix A defines the $\mathrm{Spin}_0(t,s)$-invariant sesquilinear form (‘Dirac sesquilinear form’)

$$A(\lambda,\chi) = \lambda^\dagger A\chi$$

on $\mathbb{S}$. The matrix A is chosen to be the product of all time-like γ-matrices, with index in the lower position:

$$A = \gamma_1 \cdots \gamma_t \ ,$$

where $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$. For signature $(0,n)$ we take $A = \mathbb{1}$. We note that

$$A^\dagger = (-1)^{t(t+1)/2} A = A^{-1} = (-1)^t \gamma_t \cdots \gamma_1 \ .$$

We choose the matrix B as $B := (CA^{-1})^T$. It satisfies

$$BB^\dagger = \mathbb{1} \ , \quad BB^* = \pm\mathbb{1} \tag{A.2}$$

and therefore defines either a real structure or a quaternionic structure on the complex spinor module $\mathbb{S}$.[k]

The volume element $\omega = \gamma_1 \cdots \gamma_{t+s}$ of the real Clifford algebra $Cl_{t,s}$ satisfies

$$\omega^2 = \begin{cases} (-1)^t \mathbb{1} \ , & \text{for } \ t+s = 1,4 \bmod 4 \ , \\ (-1)^{t+1}\mathbb{1} \ , & \text{for } \ t+s = 2,3 \bmod 4 \ , \end{cases} \quad \text{and} \ \ \gamma_\mu \omega = \omega\gamma_\mu (-1)^{t+s+1} \ . \tag{A.3}$$

One can therefore define a matrix γ_* with $\gamma_*^2 = \mathbb{1}$ by setting $\gamma_* = \pm\omega$ or $\gamma_* = \pm i\omega$, depending on (A.3).

In odd dimensions, $\mathbb{S}$ is irreducible and γ_* commutes with all γ-matrices, therefore $\gamma_* \propto \mathbb{1}$. In this case γ_* distinguishes the two inequivalent representations of the complex Clifford algebra $\mathbb{C}l_{t,s}$. From the physics point of view the choice of a representation is conventional because both Clifford representations induce equivalent representations of $\mathrm{Spin}(t,s)$. In even dimensions γ_* anticommutes with all γ-matrices. The complex spinor module $\mathbb{S}$ is reducible and decomposes into complex semi-spinor modules $\mathbb{S}_\pm$, which are irreducible $\mathbb{C}l_{t+s}$-modules with projection operators

$$\Pi_\pm = \frac{1}{2}\left(\mathbb{1} \pm \gamma_*\right) \ : \mathbb{S} \to \mathbb{S}_\pm \ .$$

The *chirality matrix* γ_* generalises the 'γ_5'-matrix of four-dimensional Minkowski space to arbitrary dimension and signature.

In odd dimensions, the charge conjugation matrix C is unique up to equivalence, while in four dimensions there are always two inequivalent charge conjugation matrices $C_\pm$ which are distinguished by their type τ. Following physicist conventions [33] we use the notation $C_\pm = C_{\mp\tau}$, i.e. $\tau(C_\pm) = \mp\tau$. The existence of at least two inequivalent charge conjugation matrices follows from the observation that if C is a charge conjugation matrix, so is $\gamma_* C$, which has the opposite value of τ. In five dimension the charge conjugation matrix C has invariants $\sigma = -1$ and $\tau = +1$. In four dimensions we choose $C_- := C$, with $\sigma_- = \sigma = -1$ and $\tau_- = \tau = +1$ and $C_+ = \gamma_* C_-$ with $\sigma_+ = -1$ and $\tau_+ = -\tau = -1$ as the two inequivalent charge conjugation matrices.

[k]We note that if we multiply B by a phase $\alpha \in \mathbb{C}$, $|\alpha| = 1$, the matrix $B_\alpha = \alpha B$ still satisfies (A.2), and defines a real or quaternionic structure.

Since we have two inequivalent charge conjugation matrices $C_\pm$ in even dimensions, we also have two inequivalent 'B-matrices', $B_\pm := (C_\pm A^{-1})^T$. The matrices $C_\pm$ and $B_\pm$ are related to each other through multiplication by γ_*. To obtain explicit relations, we use that in dimensions divisible by four we can choose a representation where $C_\pm$ commute with γ_*, and where γ_* is real and symmetric [33]. In such a representation it is straightforward to verify the following relations:[1]

$$C_\pm\gamma_* = \gamma_* C_\pm = C_\mp \ , \ \ C_\pm^T\gamma_* = \sigma_+\sigma_- C_\mp^T \ , \tag{A.4}$$

$$B_\pm\gamma_* = \sigma_+\sigma_- B_\mp \ , \ \ \gamma_* B_\pm = (-1)^t\sigma_+\sigma_- B_\mp \Rightarrow \gamma_* B_\pm = (-1)^t B_\pm\gamma_* \ , \tag{A.5}$$

$$B_\pm^*\gamma_* = B_\mp^* \ , \quad \gamma_* B_\pm^* = (-1)^t B_\mp^* \ . \tag{A.6}$$

We remark that in a representation where γ_* commutes with $C_\pm$ it is manifest that $C_\pm$ commutes with the projectors $\Pi_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_*)$ onto the complex semi-spinor modules and therefore has isotropy $\iota_\pm = 1$. For reference we summarise the invariants of the five-dimensional charge conjugation matrix C and of the four-dimensional charge conjugation matrices $C_\pm$ in Table 6.

	σ	τ
C	$-$	$+$

,

	σ	τ	ι
C_-	$-$	$+$	$+$
C_+	$-$	$-$	$+$

Table 6. Invariants of five- and four-dimensional charge conjugation matrices.

A.2. *Dimensional reduction of the matrices A, B and C*

Here we summarize how the dimensional reduction of γ-matrices and fermionic terms has been carried out and list the relations for the matrices A, B, C explicitly. Let Γ_i, $i = 1, \ldots, 5$ be Hermitian matrices generating an irreducible representation of $Cl_{0,5}$ with $\Gamma_1 \cdots \Gamma_5 = \mathbb{1}$. To obtain matrix generators for $Cl_{1,4}$ we replace Γ_1 by $\Gamma_1' = -i\Gamma_1$ and proceed accordingly for other signatures. Spacelike dimensional reductions are carried out along the direction labeled by $i = 5$. For example matrix generators for signature $(0, 4)$ are $\gamma_i = \Gamma_i$, for $i = 1, \ldots, 4$, while the extra generator becomes the chirality matrix, $\gamma_* = \Gamma_5$. Timelike dimensional reductions are carried out along the direction labeled by $i = 1$. For example for the reduction $(1, 4) \to (0, 4)$ we take $\gamma_i = \Gamma_{i+1}$, $i = 1, \ldots, 4$, and the chirality matrix is $\gamma_* = i\Gamma_1' = \Gamma_1 = \gamma_1 \cdots \gamma_4$.

For any reduction from 5 to 4 dimensions we take the four-dimensional charge conjugation matrix C_- to be equal to the five-dimensional charge conjugation ma-

[1]In even dimensions not divisible by four there are similar relations which differ from those given here at most by signs. In this paper we only need explicit expressions in four dimensions.

trix C:

$$C = C_- \ .$$

We choose a representation where $C_\pm = \gamma_* C_\mp = C_\mp \gamma_*$,

The relation between A-matrices is:[m]

$$A^{(t,s)} = \Gamma'_1 \cdots \Gamma'_t = A^{(t,s-1)} = \Gamma'_1 A^{(t-1,s)} \ ,$$

which implies

$$(A^{(t,s)})^{-1} = (-1)^t \Gamma'_t \cdots \Gamma'_1 = (A^{(t,s-1)})^{-1} = (-1)^t \Gamma'_1 (A^{(t-1,s)})^{-1} \ .$$

In four dimensions we have two B-matrices. Using that $\sigma_- = \sigma_+ = -1$ we have $\gamma_* B_\pm = (-1)^t B_\mp$ and $B_\pm \gamma_* = B_\mp$. We choose $\gamma_* = \Gamma_5$ for space-like and $\gamma_* = i\Gamma'_1$ for time-like reductions. Then the space-like reduction of the five-dimensional B-matrix is B_-,

$$B^{(t,s)} = (C(A^{(t,s)})^{-1})^T = (C_-(A^{(t,s)})^{-1})^T = B_-^{(t,s-1)} \ ,$$

while the time-like reduction of the five-dimensional B-matrix is proportional to B_+:

$$\begin{aligned} B^{(t,s)} &= (C_-(-1)^t \Gamma'_1 A^{(t-1,s)})^T = -i(-1)^t (C_- \gamma_* A^{(t-1,s)})^T \\ &= -i(-1)^t (C_+ A^{(t-1,s)})^T = -(-1)^t i B_+^{(t-1,s)} \ . \end{aligned}$$

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