# Effects of R&D Subsidies in a Hybrid Model of Endogenous Growth and Semi-Endogenous Growth

Angus C. Chu Xilin Wang

January 15, 2020

#### Abstract

We explore R&D subsidies in a hybrid growth model which may exhibit semiendogenous growth or fully endogenous growth. We consider two types of subsidies on variety-expanding innovation and quality-improving innovation. R&D subsidies on quality-improving innovation only have effects in the fully-endogenous-growth regime, in which more subsidies cause an earlier activation of quality-improving innovation and increase the transitional/steady-state growth rate. R&D subsidies on variety-expanding innovation have contrasting effects in the two regimes. In the semiendogenous-growth regime, more subsidies on variety-expanding innovation increase transitional growth but have no effect on steady-state growth. In the fully-endogenousgrowth regime, more subsidies on variety-expanding innovation continue to increase short-run growth but delay the activation of quality-improving innovation and reduce long-run growth. Increasing subsidies on variety-expanding (quality-improving) innovation makes the semi-endogenous-growth (fully-endogenous-growth) regime more likely to emerge. Finally, we calibrate the model and find that under reasonable parameter values, the fully-endogenous-growth regime is more likely to emerge.

JEL classification: O31, O34

Keywords: R&D subsidies, innovation, endogenous growth regimes

Chu: angusccc@gmail.com. Management School, University of Liverpool, Liverpool, United Kingdom. Wang: 17110680014@fudan.edu.cn. China Center for Economic Studies, School of Economics, Fudan University, Shanghai, China. The authors are grateful to an anonymous Referee for helpful comments. The usual disclaimer applies.

# 1 Introduction

In this study, we provide a growth-theoretic analysis on the effects of R&D subsidies. The novelty of our analysis is that we consider a hybrid growth model in which the economy may exhibit semi-endogenous growth or fully endogenous growth in the long run. The model is based on Peretto (2015), who develops a Schumpeterian growth model of endogenous takeoff. In this model, the economy initially experiences stagnation with zero growth in output per capita. As the market size of the economy becomes sufficiently large due to population growth, the economy starts to experience innovation and growth. Although the economy eventually experiences the development of new products (i.e., variety-expanding innovation), it may or may not experience the quality improvement of products (i.e., quality-improving innovation). If the economy only features variety-expanding innovation in the long run, then the balanced growth path exhibits semi-endogenous growth. If the economy features both variety-expanding innovation and quality-improving innovation, then the balanced growth path exhibits fully endogenous growth. In other words, the model in Peretto (2015) nests the semi-endogenous growth model, in which the long-run growth rate is independent of policies, and the second-generation Schumpeterian growth model, in which the long-run growth rate is fully endogenous, as special cases.

Within the above growth-theoretic framework, we consider two types of R&D subsidies on variety-expanding innovation and quality-improving innovation and obtain the following results. R&D subsidies on quality-improving innovation only have effects in the fullyendogenous-growth regime, in which a higher subsidy rate leads to an earlier activation of quality-improving innovation and increases the transitional and steady-state growth rate of output per capita. Interestingly, R&D subsidies on variety-expanding innovation have contrasting effects in the semi-endogenous-growth regime and the fully-endogenous-growth regime. Specifically, if the economy is in the semi-endogenous-growth regime, then a higher subsidy rate on variety-expanding innovation leads to a higher transitional growth rate of output per capita but has no effect on its steady-state growth rate. If the economy is in the fully-endogenous-growth regime, then a higher subsidy rate on variety-expanding innovation continues to have a positive effect on the growth rate in the short run but leads to a later activation of quality-improving innovation and a lower growth rate in the long run. Increasing R&D subsidies on variety-expanding innovation makes the semi-endogenous-growth regime more likely to emerge in equilibrium, whereas increasing R&D subsidies on quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge. Finally, we calibrate the model and find that under reasonable parameter values, the fully-endogenousgrowth regime is more likely to emerge in equilibrium.

This study relates to the literature on innovation and economic growth. Romer (1990) develops the variety-expanding R&D-based growth model in which innovation is driven by the creation of new products. Aghion and Howitt (1992) develop the Schumpeterian quality-ladder growth model in which innovation is driven by the quality improvement of existing products. Jones (1995) argues that these seminal studies feature a counterfactual scale effect of the population size on economic growth and develops the semi-endogenous growth model,

<sup>&</sup>lt;sup>1</sup>See also Segerstrom et al. (1990) and Grossman and Helpman (1991a).

in which the steady-state growth rate is scale-invariant.<sup>2</sup> Smulders and van de Klundert (1995), Peretto (1998, 1999) and Howitt (1999) combine the two dimensions of innovation and develop a second-generation Schumpeterian model with endogenous market structure that also removes the scale effect.<sup>3</sup> This study explores the effects of R&D subsidies in this vintage of the Schumpeterian growth model and considers their different implications under semi-endogenous growth versus fully endogenous growth.<sup>4</sup>

In the literature, other studies also explore the effects of R&D subsidies in the R&D-based growth model; see for example, Segerstrom (1998), Lin (2002), Zeng and Zhang (2007), Impullitti (2010), Chu and Cozzi (2018), Yang (2018) and Hu, Yang and Zheng (2019). These studies mostly focus on either variety expansion or quality improvement. Only a few studies, such as Segerstrom (2000) and Chu, Furukawa and Ji (2016), explore the effects of R&D subsidies in the Schumpeterian growth model with both dimensions of innovation. However, none of these studies consider how R&D subsides affect the endogenous activation of the two types of innovation.

This study also relates to the literature on endogenous takeoff and economic growth. In this literature, seminal studies include Galor and Weil (2000) and Galor and Moav (2002), who develop unified growth theory.<sup>5</sup> Unified growth theory shows that the quality-quantity tradeoff in childrearing and human capital accumulation allow an economy to escape from the Malthusian trap and experience economic takeoff.<sup>6</sup> While human capital is certainly a crucial engine of economic growth, innovation is another important engine of growth. Therefore, we consider the Schumpeterian growth model in Peretto (2015) in which endogenous takeoff is driven by innovation. This model features both variety-expanding innovation and quality-improving innovation. A novel contribution of our study is to incorporate R&D subsidies into the Peretto model to explore their effects on the endogenous activation of the two types of innovation and the endogenous determination between the semi-endogenous-growth regime and the fully-endogenous-growth regime.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 explores the effects of R&D subsides. Section 4 concludes.

# 2 The model

We consider the Schumpeterian growth model with both variety-expanding innovation and quality-improving innovation in Peretto (2015), in which endogenous growth in the number of products gives rise to a dilution effect that removes the scale effect. In the model, labor is used as a factor input for the production of final good. Final good is consumed by households or used as a factor input for entry, in-house R&D and the production/operation of intermediate

<sup>&</sup>lt;sup>2</sup>See also Grossman and Helpman (1991b, p. 75-76) who anticipated the semi-endogenous growth model.

 $<sup>^3</sup>$ See Laincz and Peretto (2006) and Ha and Howitt (2007) for empirical evidence that supports the second-generation Schumpeterian model.

<sup>&</sup>lt;sup>4</sup>See also Cozzi (2017a,b) who develops a general innovation specification that may yield semi-endogenous growth or fully endogenous growth in the long run.

<sup>&</sup>lt;sup>5</sup>See also Jones (2001) and Hansen and Prescott (2002) for other early studies on endogenous takeoff.

<sup>&</sup>lt;sup>6</sup>See Galor and Mountford (2008) and Ashraf and Galor (2011) for recent studies and empirical evidence for unified growth theory. Galor (2011) provides a comprehensive overview of unified growth theory.

goods. We extend Peretto (2015) by incorporating two types of R&D subsidies into the model and analyzing their effects on the takeoff, transitional dynamics and the balanced growth path of the economy.

#### 2.1 Household

The representative household has the following utility function:

$$U = \int_0^\infty e^{-(\rho - \lambda)t} \ln c_t dt, \tag{1}$$

where  $c_t \equiv C_t/L_t$  is (per capita) consumption of final good (numeraire) at time t, and  $\rho > 0$  is the subjective discount rate. Population grows at an exogenous rate  $\lambda \in (0, \rho)$ . We normalize the initial population to unity (i.e.,  $L_t = e^{\lambda t}$ ). The household maximizes (1) subject to the following asset-accumulation equation:

$$\dot{a}_t = (r_t - \lambda) a_t + (1 - \tau) w_t - c_t, \tag{2}$$

where  $a_t \equiv A_t/L_t$  is the real value of assets owned by each member of the household, and  $r_t$  is the real interest rate. Each member supplies one unit of labor to earn  $w_t$ , and  $\tau \in (0,1)$  is an exogenous tax rate on labor income. Standard dynamic optimization yields the familiar Euler equation given by

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{3}$$

# 2.2 Final good

Final output  $Y_t$  is produced by competitive firms. The production function is given by

$$Y_{t} = \int_{0}^{N_{t}} X_{t}^{\theta}(i) \left[ Z_{t}^{\alpha}(i) Z_{t}^{1-\alpha} L_{t} / N_{t}^{1-\sigma} \right]^{1-\theta} di, \tag{4}$$

where  $\{\theta, \alpha, \sigma\} \in (0, 1)$ .  $X_t(i)$  is the quantity of non-durable intermediate goods  $i \in [0, N_t]$ . The productivity of  $X_t(i)$  is determined by its own quality  $Z_t(i)$  and also by the average quality of all intermediate goods  $Z_t \equiv \int_0^{N_t} Z_t(j) \, dj/N_t$ , which captures technology spillovers. The parameter  $\alpha$  determines the private return to quality, and  $1 - \alpha$  determines the degree of technology spillovers. The parameter  $1 - \sigma$  captures a congestion effect of variety, and  $\sigma$  determines the social return to variety as we will show.

Profit maximization yields the following conditional demand functions for  $L_t$  and  $X_t$  (i):

$$L_t = (1 - \theta) Y_t / w_t, \tag{5}$$

$$X_t(i) = \left(\frac{\theta}{p_t(i)}\right)^{1/(1-\theta)} Z_t^{\alpha}(i) Z_t^{1-\alpha} L_t / N_t^{1-\sigma}, \tag{6}$$

where  $p_t(i)$  is the price of  $X_t(i)$ . Perfect competition implies that firms in this sector pay  $\theta Y_t = \int_0^{N_t} p_t(i) X_t(i) di$  for intermediate goods.

#### 2.3 Intermediate goods and in-house R&D

Monopolistic firms produce differentiated intermediate goods. The production process is based on a linear technology that requires  $X_t(i)$  units of final good to produce  $X_t(i)$  units of intermediate good  $i \in [0, N_t]$ . The firm in industry i also incurs  $\phi Z_t^{\alpha}(i) Z_t^{1-\alpha}$  units of final good as a fixed operating cost, which is increasing in the level of technology. Furthermore, the firm devotes  $I_t(i)$  units of final good to in-house R&D in order to improve the quality of its products. The innovation process is specified as

$$\dot{Z}_{t}\left(i\right) = I_{t}\left(i\right),\tag{7}$$

and the firm's (before-R&D) profit flow at time t is

$$\Pi_{t}(i) = [p_{t}(i) - 1] X_{t}(i) - \phi Z_{t}^{\alpha}(i) Z_{t}^{1-\alpha}.$$
(8)

The value of the monopolistic firm in industry i is

$$V_t(i) = \int_t^\infty \exp\left(-\int_t^v r_u du\right) \left[\Pi_v(i) - (1 - s_Z)I_v(i)\right] dv, \tag{9}$$

where  $s_Z \in (0,1)$  is the subsidy rate on quality-improving innovation. The monopolistic firm maximizes (9) subject to (6), (7) and (8). We solve this dynamic optimization problem in the proof of Lemma 1 and find that the profit-maximizing markup ratio is  $1/\theta$ . Hence, the equilibrium price is<sup>7</sup>

$$p_t(i) = 1/\theta. (10)$$

We follow previous studies to consider a symmetric equilibrium in which  $Z_t(i) = Z_t$  for  $i \in [0, N_t]$  and the size of each intermediate-good firm is identical across all industries  $X_t(i) = X_t$ . We define the following variable for the quality-adjusted firm size:

$$x_t \equiv \frac{X_t}{Z_t}. (11)$$

Substituting (10) into (6) and imposing symmetry yield

$$x_t = \theta^{2/(1-\theta)} \frac{L_t}{N_t^{1-\sigma}},\tag{12}$$

which is a state variable that determines the dynamics of the economy. In Lemma 1, we derive the rate of return on quality-improving R&D, which is increasing in  $x_t$  and  $s_z$ .

**Lemma 1** The rate of return on quality-improving in-house R&D is

$$r_t^q = \frac{\alpha}{1 - s_Z} \frac{\Pi_t}{Z_t} = \frac{\alpha}{1 - s_Z} \left( \frac{1 - \theta}{\theta} x_t - \phi \right). \tag{13}$$

**Proof.** See the Appendix.

<sup>&</sup>lt;sup>7</sup>Alternatively, one can introduce a patent policy parameter to impose an upper bound on the equilibrium price. See for example Chu *et al.* (2020) for an analysis of patent policy in the Peretto model, but they focus on the fully-endogenous-growth regime.

<sup>&</sup>lt;sup>8</sup>Symmetry also implies  $\Pi_t(i) = \Pi_t$ ,  $I_t(i) = I_t$  and  $V_t(i) = V_t$ .

### 2.4 Entrants

We follow the standard treatment in the literature to assume that entrants have access to aggregate technology  $Z_t$ , which in turn ensures symmetric equilibrium at any time t. A new firm pays  $\beta X_t$  units of final good to develop a new variety of intermediate goods and set up its operation.  $\beta > 0$  is a cost parameter, and  $X_t$  captures the scale of the initial operation. The asset-pricing equation implies that the rate of return on assets is

$$r_t = \frac{\Pi_t - (1 - s_Z)I_t}{V_t} + \frac{\dot{V}_t}{V_t}.$$
 (14)

When entry is positive (i.e.,  $\dot{N}_t > 0$ ), the no-arbitrage condition is given by

$$V_t = (1 - s_N)\beta X_t,\tag{15}$$

where  $s_N \in (0,1)$  is the subsidy rate on variety-expanding innovation. Substituting (7), (8), (10), (12) and (15) into (14) yields the rate of return on entry as

$$r_t^e = \frac{1}{(1 - s_N)\beta} \left[ \frac{1 - \theta}{\theta} - \frac{\phi + (1 - s_Z)z_t}{x_t} \right] + \frac{\dot{x}_t}{x_t} + z_t, \tag{16}$$

where  $z_t \equiv \dot{Z}_t/Z_t$  is the growth rate of aggregate quality.

#### 2.5 Government

The government collects income tax  $T_t$  from the representative household. The amount of tax revenue is

$$T_t = \tau w_t L_t = \tau (1 - \theta) Y_t. \tag{17}$$

The balanced-budget condition is given by

$$T_t = G_t + s_Z \int_0^{N_t} I_t(i)di + s_N \dot{N}_t \beta X_t, \tag{18}$$

where  $G_t$  is unproductive government spending. We follow Peretto (2007) to assume that  $G_t$  changes endogenously to balance the fiscal budget.

# 2.6 Equilibrium

The equilibrium is a time path of allocations  $\{A_t, Y_t, C_t, X_t, I_t, G_t\}$  and prices  $\{r_t, w_t, p_t, V_t\}$  such that

- the household maximizes utility taking  $\{r_t, w_t\}$  and the tax rate  $\tau$  as given;
- competitive firms produce  $Y_t$  and maximize profits taking  $\{w_t, p_t\}$  as given;
- incumbents for intermediate goods produce  $X_t$  and choose  $\{p_t, I_t\}$  to maximize  $V_t$  taking  $r_t$  and the subsidy rate  $s_Z$  as given;

- entrants make entry decisions taking  $V_t$  and the subsidy rate  $s_N$  as given;
- the government balances the fiscal budget in (18);
- the value of all existing monopolistic firms adds up to the value of the household's assets such that  $A_t = N_t V_t$ ; and
- the following market-clearing condition of final good holds:

$$Y_{t} = G_{t} + C_{t} + N_{t} (X_{t} + \phi Z_{t} + I_{t}) + \dot{N}_{t} \beta X_{t}.$$
(19)

## 2.7 Aggregation

Substituting (6) and (10) into (4) and imposing symmetry yield aggregate output as

$$Y_t = \theta^{2\theta/(1-\theta)} N_t^{\sigma} Z_t L_t. \tag{20}$$

The growth rate of per capita output  $y_t \equiv Y_t/L_t$  is

$$g_t \equiv \dot{y}_t / y_t = \sigma n_t + z_t, \tag{21}$$

which is determined by the variety growth rate  $n_t \equiv \dot{N}_t/N_t$  and the quality growth rate  $z_t$ .

## 2.8 Dynamics of the economy

The dynamics of the economy is determined by the firm size  $x_t = \theta^{2/(1-\theta)} L_t / N_t^{1-\sigma}$ . Its initial value is  $x_0 = \theta^{2/(1-\theta)} / N_0^{1-\sigma}$ . In the first stage of the economy, there is neither variety expansion nor quality improvement. At this stage,  $x_t$  increases solely due to population growth. When  $x_t$  becomes sufficiently large, innovation occurs. The following inequality ensures the case in which the creation of products (i.e., variety-expanding innovation) occurs prior to the improvement of products (i.e., quality-improving innovation):<sup>9</sup>

$$\alpha < \frac{(1-s_Z)\left[\left(1-\theta\right)/\theta - \left(1-s_N\right)\left(\rho - \lambda\right)\beta\right]}{(1-s_N)\left(\rho - \lambda\right)\beta\phi} \left\{\rho + \frac{\theta^2\left[\left(1-\theta\right)/\theta - \left(1-s_N\right)\left(\rho - \lambda\right)\beta\right]\lambda}{1-\theta^2\left[1/\theta - \left(1-s_N\right)\left(\rho - \lambda\right)\beta\right] - \tau\left(1-\theta\right)}\right\}.$$

Variety-expanding innovation happens (i.e.,  $n_t > 0$ ) when  $x_t$  reaches the first threshold  $x_N$ :

$$x_N \equiv \frac{\phi}{(1-\theta)/\theta - (\rho - \lambda)(1-s_N)\beta},\tag{23}$$

which is decreasing in  $s_N$ . Then, quality-improving innovation also happens (i.e.,  $z_t > 0$ ) if the firm size  $x_t$  reaches the second threshold  $x_Z$  defined as

$$x_Z = \underset{x}{\operatorname{arg solve}} \left\{ \left( \frac{1-\theta}{\theta} x - \phi \right) \left[ \frac{\alpha}{1 - s_Z} - \frac{\sigma}{(1 - s_N) \beta x} \right] = (1 - \sigma) (\rho - \lambda) + \lambda \right\}, \quad (24)$$

which is decreasing in  $s_Z$  and increasing in  $s_N$ . The inequality in (22) implies  $x_N < x_Z$ .

<sup>&</sup>lt;sup>9</sup>Peretto (2015) shows that if quality-improving innovation occurs before variety-expanding innovation, then the model features both types of innovation in the long run and never exhibits semi-endogenous growth.

The firm size  $x_t$  must eventually reach  $x_N$ , at which point variety-expanding innovation occurs. However,  $x_t$  may or may not reach  $x_Z$ . If  $x_t$  never reaches  $x_Z$ , then the economy features only variety-expanding innovation and exhibits semi-endogenous growth in the long run as we will show in the next section. If  $x_t$  reaches  $x_Z$ , then the economy features quality-improving innovation in addition to variety-expanding innovation and exhibits fully endogenous growth in the long run. The following proposition adapted from Peretto (2015) summarizes the dynamics of  $x_t$ .

**Proposition 1** Suppose the initial condition of the economy satisfies<sup>10</sup>

$$\phi\theta/(1-\theta) < x_0 < x_N$$

and the following inequality holds:<sup>11</sup>

$$\min\left\{\frac{1-\theta}{(1-s_N)\beta\theta}, \frac{\phi}{1-s_Z}\right\} > \frac{1}{1-\alpha}\left(\rho + \frac{\sigma\lambda}{1-\sigma}\right). \tag{25}$$

If  $x_Z \geq \bar{x}^*$ , then the dynamics of  $x_t$  is given by

$$\dot{x}_t = \begin{cases} \lambda x_t > 0 & x_0 \le x_t \le x_N \\ \bar{v} \left( \bar{x}^* - x_t \right) \ge 0 & x_N < x_t \le \bar{x}^* \end{cases} , \tag{26}$$

where

$$\overline{v} = \frac{1 - \sigma}{(1 - s_N)\beta} \left[ \frac{1 - \theta}{\theta} - (1 - s_N)\beta \left( \rho + \frac{\sigma \lambda}{1 - \sigma} \right) \right],$$

$$\overline{x}^* \equiv \frac{\phi}{(1 - \theta)/\theta - (1 - s_N)\beta \left[ \rho + \frac{\sigma \lambda}{1 - \sigma} \right]}.$$

If  $x_Z < \bar{x}^*$ , 12 then the dynamics of  $x_t$  is given by

$$\dot{x}_t = \begin{cases} \lambda x_t > 0 & x_0 \le x_t \le x_N \\ \bar{v} \left( \bar{x}^* - x_t \right) > 0 & x_N < x_t \le x_Z \\ v \left( x^* - x_t \right) \ge 0 & x_Z < x_t \le x^* \end{cases}$$
 (27)

where

$$v = \frac{1 - \sigma}{(1 - s_N)\beta} \left[ (1 - \alpha) \frac{1 - \theta}{\theta} - (1 - s_N)\beta \left( \rho + \frac{\sigma\lambda}{1 - \sigma} \right) \right],$$
$$x^* = \frac{(1 - \alpha)\phi - (1 - s_Z)(\rho + \sigma\lambda/(1 - \sigma))}{(1 - \alpha)(1 - \theta)/\theta - (1 - s_N)\beta(\rho + \sigma\lambda/(1 - \sigma))}.$$

#### **Proof.** See the Appendix.

<sup>&</sup>lt;sup>10</sup>The inequality  $x_0 > \phi\theta/(1-\theta)$  implies that  $\Pi_0 > 0$ .

<sup>&</sup>lt;sup>11</sup>Together with the initial condition, the inequality in (25) ensures that  $x_N \in (0, \bar{x}^*)$ , where  $\bar{x}^*$  is the steady-state value of  $x_t$  under the semi-endogenous-growth regime.

<sup>&</sup>lt;sup>12</sup>Together with (22), (25) and the initial condition, this inequality implies that  $0 < x_N < x_Z < \bar{x}^* < x^*$ .

# 3 Effects of R&D subsidies

In this section, we explore the different effects of R&D subsidies under the two growth regimes. First, we consider the semi-endogenous-growth regime (i.e.,  $x_Z \geq \bar{x}^*$ ). Then, we consider the fully-endogenous-growth regime (i.e.,  $x_Z < \bar{x}^*$ ). Finally, we consider how R&D subsidies determine which growth regime emerges in equilibrium.

## 3.1 Semi-endogenous growth

When the market size of the economy is not large enough (i.e.,  $x_t \leq x_N$ ), there are insufficient incentives for firms to develop new products. In this case, output per capita is

$$y_t = \theta^{2\theta/(1-\theta)} N_0^{\sigma} Z_0, \tag{28}$$

and the growth rate of  $y_t$  is  $g_t = 0$ . At this stage, an increase in the subsidy rate  $s_N$  on variety-expanding innovation affects neither the level of output per capita nor its growth rate. However, it leads to an earlier takeoff by decreasing  $x_N$ , so that  $x_t$  crosses this threshold at an earlier time. Intuitively, a higher subsidy rate  $s_N$  increases the return  $r_t^e$  to entry in (16), and hence, a smaller firm size  $x_t$  is required for variety-expanding innovation to occur.

When the market size becomes sufficiently large (i.e.,  $x_t > x_N$ ), the economy experiences variety-expanding innovation. In this case output per capita is

$$y_t = \theta^{2\theta/(1-\theta)} N_t^{\sigma} Z_0, \tag{29}$$

and the growth rate of  $y_t$  is  $g_t = \sigma n_t$ . In the Appendix, we show that whenever  $n_t > 0$ , the consumption-output ratio  $c_t/y_t$  always jumps to a steady state. Therefore, we can substitute  $r_t^e$  in (16) into the Euler equation  $r_t = \rho + g_t = \rho + \sigma n_t$  in (3) and also use (12) to derive the variety growth rate as

$$n_t = \frac{1}{(1 - s_N)\beta} \left[ \frac{1 - \theta}{\theta} - \frac{\phi}{x_t} \right] + \lambda - \rho, \tag{30}$$

which is increasing in the subsidy rate  $s_N$  for a given level of  $x_t$ . Intuitively, a higher subsidy rate  $s_N$  increases the return  $r_t^e$  to entry and increases the variety growth rate.

In the semi-endogenous-growth regime (i.e.,  $x_Z \geq \bar{x}^*$ ), the economy never experiences quality-improving innovation because the firm size  $x_t$  reaches its steady state at  $\bar{x}^*$  and stops growing. In this case, the economy only experiences variety-expanding innovation even in the long run. Substituting (12) into (30) yields

$$\frac{\dot{N}_t}{N_t} = \frac{1}{(1 - s_N)\beta} \left[ \frac{1 - \theta}{\theta} - \frac{\phi N_t^{1 - \sigma}}{\theta^{2/(1 - \theta)} L_t} \right] + \lambda - \rho, \tag{31}$$

which shows that the growth rate of  $N_t$  is decreasing in the level of  $N_t$  as in the semiendogenous growth model in Jones (1995). When the economy reaches the balanced growth path, the ratio  $N_t^{1-\sigma}/L_t$  becomes stationary. In this case, the steady-state variety growth rate is  $n^* = \lambda/(1-\sigma)$ , which in turn determines the steady-state growth rate  $g^* = \sigma n^*$  that is independent of  $s_N$ . A higher subsidy rate  $s_N$  on variety-expanding innovation instead increases the balanced growth path of  $N_t$  given by

$$N_t^* = \left[\theta^{2/(1-\theta)} \frac{L_t}{\bar{x}^*}\right]^{1/(1-\sigma)},\tag{32}$$

where

$$\bar{x}^* = \frac{\phi}{(1-\theta)/\theta - (1-s_N)\beta \left[\rho + \sigma\lambda/(1-\sigma)\right]},$$
(33)

which is obtained by setting  $n_t$  in (30) to  $n^* = \lambda/(1 - \sigma)$ . Equation (33) shows that  $\bar{x}^*$  is decreasing in  $s_N$ . These effects of R&D subsidies are quite common in the semi-endogenous growth model; see for example Segerstrom (1998).

Proposition 2 summarizes the effects of R&D subsidies on variety-expanding innovation in the semi-endogenous-growth regime. Figure 1 shows that an increase in the subsidy rate  $s_N$  on variety-expanding innovation leads to an earlier takeoff of the economy and a higher transitional growth rate before converging to the steady-state growth rate  $g^* = \sigma \lambda/(1-\sigma)$ , which is independent of  $s_N$ .<sup>13</sup>

**Proposition 2** In the semi-endogenous-growth regime, an increase in the subsidy rate  $s_N$  on variety-expanding innovation has the following effects. When  $x_t \leq x_N$ , it has no effect on the level of output per capita and its growth rate; however, it leads to an earlier activation of variety-expanding innovation. When  $x_t \in (x_N, \bar{x}^*)$ , it leads to a higher growth rate  $g_t = \sigma n_t$  for a given  $x_t$ . When  $x_t = \bar{x}^*$ , it has no effect on the steady-state growth rate  $g^* = \sigma \lambda/(1-\sigma)$  but increases the balanced growth path of  $N_t$ .

**Proof.** Use (28) to show that  $y_t$  and  $g_t$  are independent of  $s_N$  when  $x_t \leq x_N$ . Use (23) to show that  $x_N$  is decreasing in  $s_N$ . Use (30) to show that  $g_t = \sigma n_t$  is increasing in  $s_N$  for a given  $x_t$  when  $x_t \in (x_N, \bar{x}^*)$ . Use (32) and (33) to show that  $N_t^*$  is increasing in  $s_N$  when  $x_t = \bar{x}^*$ .

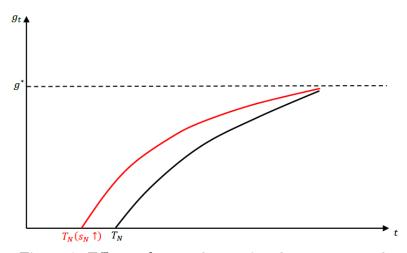


Figure 1: Effects of  $s_N$  under semi-endogenous growth

 $<sup>^{13}</sup>T_N$  is the time when variety-expanding innovation is activated.

As for R&D subsidies on quality-improving innovation, they have no effect on the economy. The reason is that quality-improving innovation is never activated in the semi-endogenous-growth regime. However, they may make the semi-endogenous-growth regime less likely and the fully-endogenous-growth regime more likely to emerge in equilibrium as we will show in Section 3.3.

## 3.2 Fully endogenous growth

In the fully-endogenous-growth regime (i.e.,  $x_Z < \bar{x}^*$ ), the economy eventually experiences quality-improving innovation. At this stage, output per capita is

$$y_t = \theta^{2\theta/(1-\theta)} N_t^{\sigma} Z_t, \tag{34}$$

and the growth rate of  $y_t$  is  $g_t = \sigma n_t + z_t$ . An increase in the subsidy rate  $s_Z$  on quality-improving innovation leads to an earlier activation of quality-improving innovation by decreasing  $x_Z$ , so that  $x_t$  crosses this threshold at an earlier time. Intuitively, a higher subsidy rate  $s_Z$  increases the return  $r_t^q$  to quality improvement in (13), in which case a smaller firm size  $x_t$  is required for quality-improving innovation to occur.

When quality-improving innovation is activated in the economy, we can substitute  $r_t^q$  in (13) into the Euler equation  $r_t = \rho + g_t = \rho + \sigma n_t + z_t$  in (3) to derive the quality-growth rate as

$$z_t = \frac{\alpha}{1 - s_Z} \left( \frac{1 - \theta}{\theta} x_t - \phi \right) - \rho - \sigma n_t.$$
 (35)

Equation (35) shows that for a given level of  $x_t$ , the equilibrium growth rate  $g_t = \sigma n_t + z_t = r_t^q - \rho$  is independent of the variety growth rate  $n_t$  and the variety subsidy rate  $s_N$  but increasing in the quality subsidy rate  $s_Z$ . Intuitively, a higher subsidy rate  $s_Z$  increases the return  $r_t^q$  to quality improvement and leads to a higher rate of quality-improving innovation.<sup>14</sup>

In the long run,  $x_t$  converges to  $x^*$ . Then, the steady-state quality growth rate is

$$z^* = \frac{\alpha}{1 - s_Z} \left( \frac{1 - \theta}{\theta} x^* - \phi \right) - \rho - \sigma n^*, \tag{36}$$

where  $n^* = \lambda/(1-\sigma)$  and

$$x^* = \frac{(1-\alpha)\phi - (1-s_Z)\left[\rho + \sigma\lambda/(1-\sigma)\right]}{(1-\alpha)(1-\theta)/\theta - (1-s_N)\beta\left[\rho + \sigma\lambda/(1-\sigma)\right]},$$
(37)

which is decreasing in the subsidy rate  $s_N$  on variety-improving innovation. Intuitively, raising R&D subsidies on variety-expanding innovation increases the number of products, which in turn reduces the market size of each product. This smaller firm size  $x^*$  decreases the incentives for quality-improving innovation and the steady-state equilibrium growth rate  $g^* = \sigma n^* + z^*$ . This result generalizes the result in Chu et al. (2016), who assume zero social

<sup>&</sup>lt;sup>14</sup>The equilibrium growth rate is also given by  $g_t = r_t^e - \rho$ , but  $r_t^e$  depends on  $z_t$  as (16) shows.

<sup>&</sup>lt;sup>15</sup>See Peretto and Connolly (2007) for a discussion on why quality improvement must be the main engine of innovation in the long run.

return to variety (i.e.,  $\sigma = 0$ ). In contrast, R&D subsidies on quality-improving innovation continue to have a positive effect on quality-improving innovation  $z^*$  and the steady-state equilibrium growth rate  $g^*$ .

Proposition 3 summarizes the effects of R&D subsidies on variety-expanding innovation in the fully-endogenous-growth regime. Figure 2 shows that an increase in the subsidy rate  $s_N$  on variety-expanding innovation leads to an earlier takeoff of the economy but a lower growth rate in the long run.<sup>16</sup>

**Proposition 3** In the fully-endogenous-growth regime, an increase in the subsidy rate  $s_N$  on variety-expanding innovation has the following effects. When  $x_t \leq x_N$ , it has no effect on the level of output per capita and its growth rate; however, it leads to an earlier activation of variety-expanding innovation. When  $x_t \in (x_N, x_Z]$ , it leads to a higher growth rate  $g_t = \sigma n_t$  for a given  $x_t$ . When  $x_t \in (x_Z, x^*)$ , it does not affect the growth rate  $g_t = \sigma n_t + z_t$  for a given  $x_t$ . When  $x_t = x^*$ , it lowers the steady-state growth rate  $g^*$  by reducing  $x^*$ .

**Proof.** Use (28) to show that  $y_t$  and  $g_t$  are independent of  $s_N$  when  $x_t \leq x_N$ . Use (23) to show that  $x_N$  is decreasing in  $s_N$ . Use (30) to show that  $g_t = \sigma n_t$  is increasing in  $s_N$  for a given  $x_t$  when  $x_t \in (x_N, x_Z]$ . Use (35) to show that  $g_t = \sigma n_t + z_t$  is independent of  $s_N$  for a given  $x_t$  when  $x_t \in (x_Z, x^*)$ . Use (36) and (37) to show that  $g^* = \sigma n^* + z^*$  is decreasing in  $s_N$  when  $x_t = x^*$ .

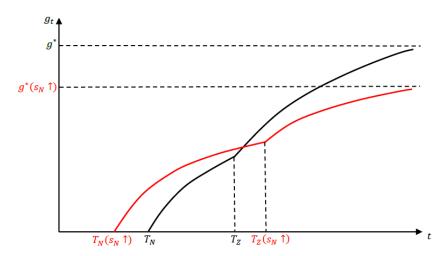


Figure 2: Effects of  $s_N$  under fully endogenous growth

Proposition 4 summarizes the effects of R&D subsidies on quality-improving innovation in the fully-endogenous-growth regime. Figure 3 shows that an increase in the subsidy rate  $s_Z$  leads to an earlier activation of quality-improving innovation in the economy and a higher growth rate in the long run.

 $<sup>^{16}</sup>T_Z$   $(T_N)$  is the time when quality-improving (variety-expanding) innovation is activated.

**Proposition 4** In the fully-endogenous-growth regime, an increase in the subsidy rate  $s_Z$  on quality-improving innovation has the following effects. When  $x_t \leq x_N$ , it has no effect on the level of output per capita and its growth rate; furthermore, it does not affect the activation date of variety-expanding innovation. When  $x_t \in (x_N, x_Z]$ , it does not affect the growth rate  $g_t = \sigma n_t$  for a given  $x_t$ ; however, it leads to an earlier activation of quality-improving innovation. When  $x_t \in (x_Z, x^*)$ , it increases the growth rate  $g_t = \sigma n_t + z_t$  for a given  $x_t$ . When  $x_t = x^*$ , it raises the steady-state growth rate  $g^*$  by increasing  $z^*$ .

**Proof.** Use (28) to show that  $y_t$  and  $g_t$  are independent of  $s_Z$  when  $x_t \leq x_N$ . Use (23) to show that  $x_N$  is independent of  $s_Z$ . Use (30) to show that  $g_t = \sigma n_t$  is independent of  $s_Z$  for a given  $x_t$  when  $x_t \in (x_N, x_Z]$ . Use (24) to show that  $x_Z$  is decreasing in  $s_Z$ . Use (35) to show that  $g_t = \sigma n_t + z_t$  is increasing in  $s_Z$  for a given  $x_t$  when  $x_t \in (x_Z, x^*)$ . Use (36) and (37) to show that  $g^* = \sigma n^* + z^*$  is increasing in  $s_Z$  when  $x_t = x^*$ .

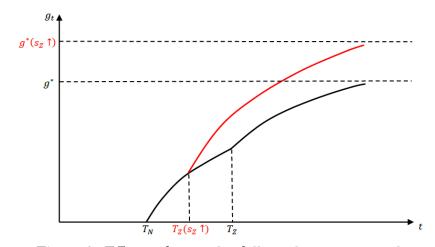


Figure 3: Effects of  $s_Z$  under fully endogenous growth

# 3.3 Endogenous switching between the growth regimes

Whether the semi-endogenous-growth regime or the fully-endogenous-growth regime emerges in equilibrium depends on the relative value of  $x_Z$  and  $\bar{x}^*$ . Specifically, if  $x_Z \geq \bar{x}^*$ , then the semi-endogenous-growth regime emerges in equilibrium. If  $x_Z < \bar{x}^*$ , then the fully-endogenous-growth regime emerges in equilibrium. Therefore, an increase in  $x_Z/\bar{x}^*$  makes the semi-endogenous-growth regime more likely to emerge in equilibrium, whereas a decrease in  $x_Z/\bar{x}^*$  makes the fully-endogenous-growth regime more likely to emerge in equilibrium.

An increase in the subsidy rate  $s_Z$  on quality-improving innovation reduces  $x_Z$  but does not affect  $\bar{x}^*$ . Therefore, increasing R&D subsidies on quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge in equilibrium. Intuitively, the fully-endogenous-growth regime depends on the *presence* of quality-improving innovation.

Therefore, an increase in the subsidy rate  $s_Z$  that raises the return to quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge.

An increase in the subsidy rate  $s_N$  on variety-expanding innovation reduces  $\bar{x}^*$  and raises  $x_Z$ . Therefore, increasing R&D subsidies on variety-expanding innovation makes the semi-endogenous-growth regime more likely to emerge. Intuitively, the semi-endogenous-growth regime depends on the *absence* of quality-improving innovation. Therefore, an increase in the subsidy rate  $s_N$  that raises the return to variety-expanding innovation ends up crowding out resources for quality-improving innovation and making the semi-endogenous-growth regime more likely to emerge. Proposition 5 summarizes these results.

**Proposition 5** An increase in the subsidy rate on quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge in equilibrium. An increase in the subsidy rate on variety-expanding innovation makes the semi-endogenous-growth regime more likely to emerge in equilibrium.

**Proof.** One can use (24) and (33) to show that  $x_Z < \bar{x}^*$  can be expressed as

$$\frac{\alpha\phi}{1-s_Z} > \frac{1-\theta}{(1-s_N)\beta\theta} - \left(\rho + \frac{\sigma\lambda}{1-\sigma}\right),\,$$

which is equivalent to  $z^* > 0$  in (36). This inequality holds if and only if  $s_Z$  is sufficiently large or  $s_N$  is sufficiently small.

# 3.4 Quantitative analysis

We calibrate the model to quantitatively examine which growth regime is more likely to emerge. The model features the following parameters:  $\{\rho, \alpha, \sigma, \lambda, \tau, s_N, s_Z, \theta, \beta, \phi\}$ . We set the discount rate  $\rho$  to 0.05. We follow Iacopetta et al. (2019) to set the degree of technology spillovers  $1 - \alpha$  to 0.833 and the degree of congestion  $1 - \sigma$  to 0.75. We consider a long-run population growth rate  $\lambda$  of 1%. According to the OECD, the average tax rate  $\tau$  on wage income in the US is 23.8%. Given that the US has a uniform rate of R&D subsidies, we consider  $s_N = s_Z$  and follow Impullitti (2010) to set the rate of subsidies to 18.8%. Then, we calibrate  $\{\theta, \beta\}$  by matching the following moments: (a) the ratio of labor income to output is 60%; and (b) the ratio of consumption to output is 60%. Finally, we compute a range of values for the operating cost  $\phi$  under which the quality growth rate  $z^*$  is positive and examine whether this range of values for  $\phi$  is empirically plausible.

Table 1: Calibration and simulation										
ρ	$\alpha$	σ	λ	τ	$s_N$	$s_Z$	$\theta$	β	$\phi$	$z^*$
0.050	0.167	0.250	0.010	0.238	0.188	0.188	0.400	27.478	0.068	0%
0.050	0.167	0.250	0.010	0.238	0.188	0.188	0.400	27.478	0.070	1%
0.050	0.167	0.250	0.010	0.238	0.188	0.188	0.400	27.478	0.072	2%

<sup>&</sup>lt;sup>17</sup>The equilibrium expressions for  $\{wL/Y, C/Y\}$  are the same across the two growth regimes.

Table 1 shows that the quality growth rate  $z^*$  is increasing in the operating cost  $\phi$ . Intuitively, a larger operating cost  $\phi$  increases the average firm size  $x^*$  in (37), which in turn increases the incentives for quality-improving innovation. Also, when  $\phi > 0.068$ , the quality growth rate  $z^*$  is positive (i.e., the fully-endogenous-growth regime emerges in equilibrium). Ferraro et al. (2019) estimate the operating cost parameter  $\phi$  and find that its mean estimate is 0.125 with a standard error of 0.027. Therefore, we can conclude that under reasonable parameter values, the fully-endogenous-growth regime with both quality-improving innovation and variety-expanding innovation is more likely to emerge than the semi-endogenous growth regime. Although this finding depends on the specific structure of our model, it is consistent with empirical studies, such as Laincz and Peretto (2006), Ha and Howitt (2007), Madsen (2008, 2010) and Ang and Madsen (2011), which also find supportive evidence for endogenous growth in the second-generation Schumpeterian model.

# 4 Conclusion

This study explores the effects of R&D subsidies in a hybrid growth model that may exhibit semi-endogenous growth or fully endogenous growth in equilibrium. Whether the semi-endogenous-growth regime or the fully-endogenous-growth regime emerges in equilibrium is endogenously determined. Within this growth-theoretic framework, we obtain the following novel results. First, R&D subsidies have different effects on the endogenous activation of variety-expanding innovation and that of quality-improving innovation. Second, R&D subsidies have different effects on economic growth in the semi-endogenous-growth regime and in the fully-endogenous-growth regime. Finally, R&D subsidies determine which growth regime emerges in equilibrium. Therefore, previous studies that restrict their analysis to either growth regime may not capture the complete effects of R&D subsidies.

 $<sup>^{-18}</sup>$ See also Ferraro and Peretto (2020) and Iacopetta et al. (2019) in which the calibrated values of  $\phi$  are 0.262 and 0.715, respectively.

# References

- [1] Aghion, P., and Howitt, P., 1992. A model of growth through creative destruction. *Econometrica*, 60, 323-351.
- [2] Ang, J., and Madsen, J., 2011. Can second-generation endogenous growth models explain the productivity trends and knowledge production in the Asian miracle economies?. Review of Economics and Statistics, 93, 1360-1373.
- [3] Ashraf, Q., and Galor, O., 2011. Dynamics and stagnation in the Malthusian epoch. *American Economic Review*, 101, 2003-41.
- [4] Chu, A., and Cozzi, G., 2018. Effects of patents versus R&D subsidies on income inequality. *Review of Economic Dynamics*, 29, 68-84.
- [5] Chu, A., Furukawa, Y., and Ji, L., 2016. Patents, R&D subsidies and endogenous market structure in a Schumpeterian Economy. *Southern Economic Journal*, 82, 809-825.
- [6] Chu, A., Kou, Z., and Wang, X., 2020. Effects of patents on the transition from stagnation to growth. *Journal of Population Economics*, forthcoming.
- [7] Cozzi, G., 2017a. Endogenous growth, semi-endogenous growth... or both? A simple hybrid model. *Economics Letters*, 154, 28-30.
- [8] Cozzi, G., 2017b. Combining semi-endogenous and fully endogenous growth: A generalization. *Economics Letters*, 155, 89-91.
- [9] Ferraro, D., Ghazi, S., and Peretto, P., 2019. Lessons for tax reform from an equilibrium model of innovation. Economic Research Initiatives at Duke (ERID) Working Paper No. 282.
- [10] Ferraro, D., and Peretto, P., 2020. Innovation-led growth in a time of debt. *European Economic Review*, 121, 1-32.
- [11] Galor, O., 2011. Unified Growth Theory. Princeton University Press.
- [12] Galor, O., and Moav, O., 2002. Natural selection and the origin of economic growth. Quarterly Journal of Economics, 117, 1133-1192.
- [13] Galor, O., and Mountford, A., 2008. Trading population for productivity: Theory and evidence. *Review of Economic Studies*, 75, 1143-1179.
- [14] Galor, O., and Weil, D., 2000. Population, technology and growth: From the Malthusian regime to the demographic transition. *American Economic Review*, 110, 806-828.
- [15] Grossman, G., and Helpman, E., 1991a. Quality ladders in the theory of growth. *Review of Economic Studies*, 58, 43-61.
- [16] Grossman, G., and Helpman, E., 1991b. Innovation and Growth in the Global Economy. The MIT Press.

- [17] Ha, J., and Howitt, P., 2007. Accounting for trends in productivity and R&D: A Schumpeterian critique of semi-endogenous growth theory. *Journal of Money, Credit, and Banking*, 33, 733-74.
- [18] Hansen, G., and Prescott, E., 2002. Malthus to Solow. *American Economic Review*, 2002, 92, 1205-1217.
- [19] Howitt, P., 1999. Steady endogenous growth with population and R&D inputs growing. Journal of Political Economy, 107, 715-730.
- [20] Hu, R., Yang, Y., and Zheng, Z., 2019. Optimal subsidies in a quality-ladder growth model with elastic labor. MPRA Paper 96801.
- [21] Iacopetta, M., Minetti, R., and Peretto, P., 2019. Financial markets, industry dynamics and growth. *Economic Journal*, 129, 2192-2215.
- [22] Impullitti, G., 2010. International competition and U.S. R&D subsidies: A quantitative welfare analysis. *International Economic Review*, 51, 1127-1158.
- [23] Jones, C., 1995. R&D-based models of economic growth. *Journal of Political Economy*, 103, 759-784.
- [24] Jones, C., 2001. Was an industrial revolution inevitable? Economic growth over the very long run. B.E. Journal of Macroeconomics (Advances), 1, 1-45.
- [25] Laincz, C., and Peretto, P., 2006. Scale effects in endogenous growth theory: An error of aggregation not specification. *Journal of Economic Growth*, 11, 263-288.
- [26] Lin, H., 2002. Shall the Northern optimal R&D subsidy rate inversely respond to Southern intellectual property protection?. Southern Economic Journal, 69, 381-397.
- [27] Madsen, J., 2008. Semi-endogenous versus Schumpeterian growth models: Testing the knowledge production function using international data. *Journal of Economic Growth*, 13, 1-26.
- [28] Madsen, J., 2010. The anatomy of growth in the OECD since 1870. *Journal of Monetary Economics*, 57, 753-767.
- [29] Peretto, P., 1998. Technological change and population growth. *Journal of Economic Growth*, 3, 283-311.
- [30] Peretto, P., 1999. Cost reduction, entry, and the interdependence of market structure and economic growth. *Journal of Monetary Economics*, 43, 173-195.
- [31] Peretto, P., 2007. Corporate taxes, growth and welfare in a Schumpeterian economy. Journal of Economic Theory, 137, 353-382.
- [32] Peretto, P., 2015. From Smith to Schumpeter: A theory of take-off and convergence to sustained growth. *European Economic Review*, 78, 1-26.

- [33] Peretto, P., and Connolly, M., 2007. The Manhattan metaphor. *Journal of Economic Growth*, 12, 329-50.
- [34] Romer, P., 1990. Endogenous technological change. *Journal of Political Economy*, 98, S71-S102.
- [35] Segerstrom, P., 1998. Endogenous growth without scale effects. *American Economic Review*, 88, 1290-1310.
- [36] Segerstrom, P., 2000. The long-run growth effects of R&D subsidies. *Journal of Economic Growth*, 5, 277-305.
- [37] Segerstrom, P., Anant, T., and Dinopoulos, E., 1990. A Schumpeterian model of the product life cycle. *American Economic Review*, 80, 1077-91.
- [38] Smulders, S. and van de Klundert T., 1995. Imperfect competition, concentration and growth with firm-specific R&D. European Economic Review, 39, 139-160.
- [39] Yang, Y., 2018. On the optimality of IPR protection with blocking patents. *Review of Economic Dynamics*, 27, 205-230.
- [40] Zeng, J., and Zhang, J., 2007. Subsidies in an R&D growth model with elastic labor. Journal of Economic Dynamics and Control, 31, 861-886.

#### Appendix

**Proof of Lemma 1.** The current-value Hamiltonian for monopolistic firm i is

$$H_t(i) = \Pi_t(i) - (1 - s_Z)I_t(i) + \eta_t(i)\dot{Z}_t(i),$$
 (A1)

where  $\eta_t(i)$  is the multiplier on  $\dot{Z}_t(i) = I_t(i)$ . Substituting (6)-(8) into (A1), we can derive

$$\frac{\partial H_t(i)}{\partial p_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial p_t(i)} = 0, \tag{A2}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \eta_t(i) = 1 - s_Z, \tag{A3}$$

$$\frac{\partial H_t(i)}{\partial Z_t(i)} = \alpha \left\{ \left[ p_t(i) - 1 \right] \left[ \frac{\theta}{p_t(i)} \right]^{1/(1-\theta)} \frac{L_t}{N_t^{1-\sigma}} - \phi \right\} Z_t^{\alpha-1}(i) Z_t^{1-\alpha} = r_t \eta_t(i) - \dot{\eta}_t(i). \quad (A4)$$

First,  $\partial \Pi_t(i) / \partial p_t(i) = 0$  in (A2) yields

$$p_t(i) = 1/\theta. \tag{A5}$$

Then, substituting (A3), (A5) and (12) into (A4) and imposing symmetry yield

$$r_t^q = \frac{\alpha}{1 - s_Z} \frac{\Pi_t}{Z_t} = \frac{\alpha}{1 - s_Z} \left( \frac{1 - \theta}{\theta} x_t - \phi \right), \tag{A6}$$

which is the rate of return on quality-improving in-house R&D.

Before we prove Proposition 1, we first derive the dynamics of the consumption-output ratio  $C_t/Y_t$  when  $n_t > 0$ .

**Lemma 2** When  $n_t > 0$ , the consumption-output ratio always jumps to

$$C_t/Y_t = (1 - s_N) (\rho - \lambda) \beta \theta^2 + (1 - \tau) (1 - \theta).$$
 (A7)

**Proof.** The total value of assets owned by the household is

$$A_t = N_t V_t. (A8)$$

When  $n_t > 0$ , the no-arbitrage condition for entry in (15) holds. Then, substituting (15) and  $X_t N_t = \theta^2 Y_t$  into (A8) yields

$$A_t = N_t (1 - s_N) \beta X_t = (1 - s_N) \beta \theta^2 Y_t, \tag{A9}$$

which implies that the asset-output ratio  $A_t/Y_t$  is constant. Substituting (A9), (2), (3) and (5) into  $\dot{A}_t/A_t = \dot{a}_t/a_t + \lambda$  yields

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} = r_t + (1 - \tau) \frac{w_t L_t}{A_t} - \frac{C_t}{A_t} 
= \rho + \frac{\dot{C}_t}{C_t} - \lambda + \frac{(1 - \tau)(1 - \theta)}{(1 - s_N)\beta\theta^2} - \frac{1}{(1 - s_N)\beta\theta^2} \frac{C_t}{Y_t},$$
(A10)

which can be rearranged as

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{Y}_t}{Y_t} = \frac{1}{(1 - s_N)\beta\theta^2} \frac{C_t}{Y_t} - \frac{(1 - \tau)(1 - \theta)}{(1 - s_N)\beta\theta^2} - (\rho - \lambda). \tag{A11}$$

Therefore, the dynamics of  $C_t/Y_t$  is characterized by saddle-point stability, such that  $C_t/Y_t$  jumps to its steady-state value in (A7).

**Proof of Proposition 1.** Using (12), we can derive the growth rate of  $x_t$  as

$$\frac{\dot{x}_t}{x_t} = \lambda - (1 - \sigma) n_t. \tag{A12}$$

When  $x_0 \le x_t \le x_N$ , we have  $n_t = 0$  and  $z_t = 0$ . In this case, the dynamics of  $x_t$  is given by

$$\dot{x}_t = \lambda x_t. \tag{A13}$$

When  $x_N < x_t \le x_Z$ , we have  $n_t > 0$  and  $z_t = 0$ . In this case, Lemma 2 implies that  $C_t/Y_t$  is constant and  $\dot{c}_t/c_t = \dot{y}_t/y_t$ . Therefore, we can substitute  $r_t^e$  in (16) and (A12) into  $r_t = \rho + \sigma n_t$  in (3) to obtain (30). Substituting (30) into (A12) yields the dynamics of  $x_t$  as

$$\dot{x}_t = \frac{1 - \sigma}{(1 - s_N)\beta} \left\{ \phi - \left[ \frac{1 - \theta}{\theta} - (1 - s_N)\beta \left( \rho + \frac{\sigma}{1 - \sigma} \lambda \right) \right] x_t \right\}.$$
 (A14)

Defining  $\overline{v} \equiv \frac{1-\sigma}{(1-s_N)\beta} \left[ \frac{1-\theta}{\theta} - (1-s_N)\beta \left(\rho + \frac{\sigma\lambda}{1-\sigma}\right) \right]$  and  $\overline{x}^* \equiv \frac{\phi}{(1-\theta)/\theta - (1-s_N)\beta[\rho + \sigma\lambda/(1-\sigma)]}$ , we can express (A14) as

$$\dot{x}_t = \overline{v}(\overline{x}^* - x_t). \tag{A15}$$

If  $\overline{x}^* < x_Z$ , then  $x_t$  reaches its steady state at  $x_t = \overline{x}^*$ .

However, it is also possible for  $x_Z < \overline{x}^*$ . In this case, when  $x_t > x_Z$ , we have  $n_t > 0$  and  $z_t > 0$ . Given  $n_t > 0$ ,  $C_t/Y_t$  is constant, and  $\dot{c}_t/c_t = \dot{y}_t/y_t$ . Then, substituting  $r_t^e$  in (16) and (A12) into  $r_t = \rho + \sigma n_t + z_t$  in (3) yields

$$n_t = \frac{1}{(1 - s_N)\beta} \left[ \frac{1 - \theta}{\theta} - \frac{\phi + (1 - s_Z)z_t}{x_t} \right] - \rho + \lambda.$$
 (A16)

We substitute (35) into (A16) to derive

$$n_{t} = \frac{\left[ (1 - \alpha) (1 - \theta) / \theta - (1 - s_{N}) (\rho - \lambda) \beta \right] x_{t} - (1 - \alpha) \phi + (1 - s_{Z}) \rho}{(1 - s_{N}) \beta x_{t} - (1 - s_{Z}) \sigma}.$$
 (A17)

Substituting (A17) into (A12) yields the dynamics of  $x_t$  as

$$\dot{x}_t = v\left(x^* - x_t\right),\tag{A18}$$

where

$$v \equiv \frac{1 - \sigma}{(1 - s_N)\beta - (1 - s_Z)\sigma/x_t} \left[ (1 - \alpha)\frac{1 - \theta}{\theta} - (1 - s_N)\beta \left(\rho + \frac{\sigma}{1 - \sigma}\lambda\right) \right]$$
(A19)

and  $x^*$  is in (37). Finally, we approximate  $(1 - s_Z) \sigma/x_t \cong 0$  for  $x_t > x_Z$ , so v becomes a constant.